Chapter 10

Introduction to Regularity Theory

In this part of the thesis we continue to deal with the singularity sets of Geometric flows related to the Mean Curvature Flow. Specifically speaking, Mean Curvature Flow with Neumann Free Boundary Conditions. As was previously mentioned (in the general introduction), our intent is to address all three main directions from which to attack singularity sets. In the first part on the thesis we addressed the 'hardest' (in the sense that the least is known about it), the 'shape', that is, structure of the singularity set. In this part we address the other two. That is the 'size', or more formally, measure of the set and point tests of regularity, that is, conditions by which we can decide if a given point at a given time is regular (other than by referring to the definition). These two approaches are often (see, for e.g. [7]) and will be here referred to respectively as global regularity and local regularity.

We recall, loosely speaking, that a surface in Euclidean space is said to be moving by mean curvature flow if at all times each point on the surface is moving along the normal vector to the surface at a velocity equal to its mean curvature. Also loosely speaking, the singularity set that we are studying at a time T is the set of points in the surface at time T where the definition of mean curvature flow no longer makes sense, or where the surface has collapsed upon itself.

Unlike the problem of structure, the two approaches we address here already have lead to results and there exists a significant literature. Our work mostly stems from that of Buckland [6] and Ecker [7] (which of course in turn, draw from other sources). Allard [1] and Almgren [3] addressed the general form of local regularity theory for minimal varifolds. Brakke [5] addressed global regularity for varifolds that move by mean curvature. A breakthrough result for mean curvature flow was Huiskens [14] monotonicity formula in the classical flow (which has been extended, see Ilmanen [17], to weak formulations without boundary). The localised form of this formula has been used, see [7], to give simpler and more elegant proofs (at least in the classical formulation) of local and global regularity theorems for mean curvature flow.

In order to extend this theory we allow for additional obstacles in the flow. In particular we allow for Neumann Free Boundary Conditions. Again speaking approximately for now, the Neumann Free Boundary Condition provides a fixed support surface along which the flowing surface is free to move provided that wherever the flowing and support surfaces meet, they meet perpendicularly. The flowing surface, in this problem continues to move by its mean curvature including on its boundary (where it meets the support surface). Related problems in physics would include a soap film sitting on a fixed surface (for example, a bath tub) or the boundary lines between cooling plates in a pot of molten steel.

Minimal varifolds satisfying boundary conditions have been observed by Allard [2]. The particular problem of minimal varifolds satisfying Neumann free boundary conditions was dealt with by Grüter and Jost in [13]. Although Grüter and Jost's work was for minimal surfaces it shows that the theory in general could be expected to be extendable. A particular contribution of Grüter and Jost is a reflection function that becomes important in our work. The analog to Huiskens Monotonicity Formula, together with some important supporting technical theory was developed by Buckland [6] in his Doctoral Thesis. Bucklands work being dependent on the grounding work of Stahl [29] who proved the existence of mean curvature flows with Neumann free boundary conditions in the classical sense along with providing interior derivative estimates.

We provide, firstly a localised monotonicity formula in the classical sense for mean curvature flow with Neumann free boundary conditions, and then, under conditions to be discussed later, both local and global regularity theorems for mean curvature flow with Neumann free boundary conditions. We further provide, as an appendix, an analogy to Ilmanens monotonicity for weak flows. It is provided as an appendix as it is dependent on a weak formulation of mean curvature flow with Neumann free boundary conditions that has yet to be shown to be well defined.

The structure of this part of the thesis is as follows.

In Chapter 11 we give a formal definition of the framework which we will be dealing with. We also introduce the results already existing in the literature, mostly from Stahl and Buckland, that will be necessary for the present work.

Chapter 12 is a short but important chapter. In it we note that on the interior of our surface, or rather away from the support surface, all relevant localised results for mean curvature flow without boundary hold. An explanation of why and an example theorem are given.

Chapter 13 introduces what we mean in this part by singularities and regular points. We state the aims and main results sought here. We go on to describe the further assumptions we make on the surfaces that we allow to flow, and also properties of the limit surface that allow us to be able to prove our main theorems.

Chapter 14 introduces the Local Monotonicity Formula for mean curvature flows with Neumann free boundary conditions, analogous to that of Ecker in the boundaryless case.

Chapter 15 and 16 are the technical heart of this part of the thesis. They provide area, and boundary area bounds. Contained within are the characterisations and properties of the Gaussian density for mean curvature flow with Neumann free boundary conditions and provides various area ratio bounds including the appropriate form of the now well known Clearing Out Lemma of Brakke [5].

Chapter 17 proves our Local Regularity Theorem, first showing that under certain conditions we can provide a local bound on the second fundamental form. This, in turn, allows us to prove local regularity.

Chapter 18 then addresses good points (a set of points defined later that satisfy certain integral inequalities allowing for global regularity theorems and that can be shown to be present almost everywhere in a sufficient sense) and provides the proofs of a Global regularity Theorem under the two sets of assumptions found to be sufficient to ensure global regularity. Finally in Appendix B (Appendix A being related to Part I) a weak formulation of mean curvature flow with Neumann free boundary conditions is proposed and a Monotonicity Formula is shown to hold under the proposed formulation.

As in Part I each chapter is concluded with a section providing an explanation of the source (or originality) of each result and where further reading can be found.

As a final note, we point out that we write Part II separately from Part I in the sense of scope of definitions. That is, in order to prevent confusion, we point out that definitions and notations used in one part will not be understood as holding in the other, and so we define common terms again. Any definition made in Part I is no longer taken as a definition in Part II, and any definition made in Part II is not taken as a definition in Part I. This means, unfortunately, that some definitions, such as that of singular sets will be doubly stated.