Chapter 8

Generalised Koch Type Sets and Relative Centralisation of Sets

We turn now to the generalisation of the sets A_ε and Γ_ε which in our generalisations turn out to be two examples of the same sort of set. As already hinted at in Definition 6.2.2 the generalisation can be seen as increasing the freedom with which the base angles of the triangular caps $\theta^A_{n,i}$ for a set A. We allow this freedom in two differing strengths. Firstly that $\theta_{n,i} = \theta_{n,j}$, $i, j \in \{1, ..., 2^n\}$ as in the construction of A_ε. Secondly that $\theta_{n,j}$ are allowed to vary freely over n and j. A common restriction to the two variations is that $T_{n,i} \subset T_{m,j} \Rightarrow \theta_{n,i} \leq \theta_{m,j}$. That is, as we take triangular caps inside of previously constructed ones, the base angles reduce. The rate of reduction in separate triangular caps may of course vary.

It is clear that the second variation is a direct generalisation of the first. We keep them separate, however, since the first allows fewer complications than the second so that some of the results are able to be proved in a stronger and cleaner form for the first variation.

The original motivation for this investigation stems from an interest in the dimension of these sets. Γ_ε and A_ε are both examples of the first variation where for Γ_ε, $\theta^{\Gamma_\varepsilon}_{n,i}$ is constant over n and i, and for A_ε $\theta^{A_\varepsilon}_{n,i}$ varies by strictly decreasing to 0 in n. The motivating question asks whether higher dimensions than (in this case) 1 can only arise from constructions with a constant base angle as found in Γ_ε. The answer turns out to be no. Along with a presentation of this answer for both variations of our generalisation we present various other results concerning measure and rectifiability relating to our generalisations.

In this chapter we present the two main definitions of the sets in question and show their equivalence (both definitions are necessary as they will both be used due to the fact that which definition is more convenient to complete a proof varies with the results that we will prove). We further show another characterisation of these sets in terms of a bijection from $\mathbb{R}$. We then present some general lemmas and background results necessary to present the main results concerning measure, rectifiability and dimension. The main results are then presented in the next and final chapter.

8.1 Equivalent Constructions of Koch Type Sets

We start, quite naturally with definitions, equivalences and characterisations. First of all with a formal definition of the first variation of the generalisations.

Definition 8.1.1.
Suppose we can construct a set B as follows:

Let $A_{0,1}$ be a base (a line segment in $\mathbb{R}^2$) and $T_{0,1}$ be a triangular cap on $A_{0,1}$ with vertical height $\varepsilon\mathcal{H}^1(A_{0,1})$ with $\varepsilon < 1/100$. Let $\psi(0,\varepsilon)$ be the base angles of $T_{0,1}$ and the two shorter sides of $T_{0,1}$ be named $A_{1,1}$ and $A_{1,2}$. We then construct two new triangular caps $T_{1,1}$ and $T_{1,2}$ on $A_{1,1}$ and $A_{1,2}$ with base angles $\psi(1,\varepsilon) \leq \psi(0,\varepsilon)$. We define

$$A_0 = T_{0,1} \text{ and } A_1 = \bigcup_{i=1}^{2} T_{1,i}.$$

Then suppose we have $A_n = \cup_{i=1}^{2^n} T_{n,i}$ a union of 2^n triangular caps with base angles $\psi(n,\varepsilon)$ and 2^{n+1} shorter sides labelled $A_{n,i}$, $i \in \{1, ..., 2^{n+1}\}$. Then construct a triangular cap $T_{n+1,i}$ on each $A_{n+1,i}$ such that the base angles $\psi(n+1,\varepsilon)$ satisfy $\psi(n+1,\varepsilon) \leq \psi(n,\varepsilon)$. Define $A_{n+1} = \cup_{i=1}^{2^{n+1}} T_{n,i}$. Finally define

$$B = \bigcap_{n=0}^{\infty} A_n.$$

*We then call a set A an A_ε-**type** set whenever $A \in \{B, B \sim E(B)\}$.*

Remark: We see that the above is a generalisation as there is no pattern to the size of $\psi(n,\varepsilon)$ provided that it is non-increasing in n. In Γ_ε $\psi(n,\varepsilon)$ was constant in n, and in A_ε $\psi(n,\varepsilon)$ reduced to zero in a prescribed way. By comparison to Constructions 4.1.1 and 4.2.1 it is clear that Γ_ε is an A_ε-type set but not clear that A_ε is an A_ε-type set. However, through the second form of the definition and the proof of their equivalence it will be proven that both Γ_ε and A_ε are A_ε-type sets.

Then immediately we define the second variation.

Definition 8.1.2.
Suppose we can construct a set B as follows:

Let $A_{0,1}$ be a base (a line of positive finite length in $\mathbb{R}^2$) (without loss of generality we will generally assume that $A_{0,1} = [0,1] \subset \mathbb{R}$) and $T_{0,1}$ be a triangular cap on $A_{0,1}$ with vertical height $\varepsilon\mathcal{H}^1(A_{0,1})$ with $\varepsilon < 1/100$. Let θ_0 be the base angle of $T_{0,1}$ and the two shorter sides of $T_{0,1}$ be denoted $A_{1,1}$ and $A_{1,2}$. We then construct two new triangular caps $T_{1,1}$ and $T_{1,2}$ on $A_{1,1}$ and $A_{1,2}$ with base angles $\theta_{1,1}, \theta_{1,2} \leq \theta_0$. We define

$$A_0 = T_{0,1} \text{ and } A_1 = \bigcup_{i=1}^{2} T_{1,i}.$$

Then suppose we have $A_n = \cup_{i=1}^{2^n} T_{n,i}$ a union of 2^n triangular caps with base angles $\theta_{n,i}$ and 2^{n+1} "shorter sides" (two per triangular cap) labelled $A_{n+1,i}$, $i \in \{1, ..., 2^{n+1}\}$. Then construct a triangular cap $T_{n+1,i}$ on each $A_{n+1,i}$ such that the base angles $\{\theta_{n+1,i}\}_{i=1}^{2^{n+1}}$ satisfy for each $i \in \{1, ..., 2^n\}$

$$\theta_{n,i} \geq \begin{cases} \theta_{n+1,2i-1} \\ \theta_{n+1,2i} \end{cases}$$

(i.e. the new base angles for each triangular cap are bounded by the base angle of the nth level that the new triangular cap is contained in).

Define $A_{n+1} = \cup_{i=1}^{2^{n+1}} T_{n+1,i}$. *Finally define*

$$B = \bigcap_{n=0}^{\infty} A_n.$$

We then call a set A *a* **Koch type set** *whenever* $A \in \{B, B \sim E(B)\}$. *We denote the set of all such sets by* $\mathcal{K}$.

Remark:
(1) By choosing $\theta_{n,i}$ to be constant in i for each n it is clear that Definition 8.1.2 is a direct generalisation of Definition 8.1.1. As noted in the remark following Definition 8.1.1 In general any notation that can be considered in relation to some set $A \in \mathcal{K}$, for example $\theta_{n,j}$, $T_{n,j}$, etc., the superscript A will denote association with the set A when it may be unclear which set we are talking about. That is $T_{n,j}^A$ will denote the triangular cap $T_{n,j}$ associated with the construction of A.

Definition 8.1.3.
Let $A \in \mathcal{K}$. *Then*

$$\tilde{A}_n^A := \bigcup_{i=1}^{2^n} A_{n,i}^A,$$

The second round of definitions for the two variations of generalisation are directly analogous to the original construction of A_ε in that we consider $\tilde{A}_{n,j}$ sets instead of the $T_{n,j}$ sets.

Definition 8.1.4.
Suppose we can construct a set B *as follows:*

Let $A_{0,1}$ *be a base (a line in* $\mathbb{R}^2$ *of positive finite length) and* $T_{0,1}$ *be a triangular cap on* $A_{0,1}$ *with vertical height* $\varepsilon\mathcal{H}^1(A_{0,1})$ *with* $\varepsilon < 1/100$. *Let* $\psi(0,\varepsilon)$ *be the base angles of* $T_{0,1}$ *and the two shorter sides of* $T_{0,1}$ *be named* $A_{1,1}$ *and* $A_{1,2}$. *We then construct two new triangular caps* $T_{1,1}$ *and* $T_{1,2}$ *on* $A_{1,1}$ *and* $A_{1,2}$ *with base angles* $\psi(1,\varepsilon) \le \psi(0,\varepsilon)$. *We define*

$$A_0 = T_{0,1} \text{ and } A_1 = \bigcup_{i=1}^{2} T_{1,i}.$$

Then suppose we have $A_n = \cup_{i=1}^{2^n} T_{n,i}$ *a union of* 2^n *triangular caps with base angles* $\psi(n,\varepsilon)$ *and* 2^{n+1} *shorter sides labelled* $A_{n,i}$, $i \in \{1, ..., 2^{n+1}\}$. *Then construct a triangular cap* $T_{n+1,i}$ *on each* $A_{n+1,i}$ *such that the base angles* $\psi(n+1,\varepsilon)$ *satisfy* $\psi(n+1,\varepsilon) \le \psi(n,\varepsilon)$. *Define* $\tilde{A}_{n+1} = \cup_{i=1}^{2^{n+1}} A_{n,i}$. *Finally define*

$$B = \overline{\bigcup_{n=0}^{\infty} \tilde{A}_n} \sim \bigcup_{n=0}^{\infty} \tilde{A}_n.$$

We then call a set A *an* A_ε**-type** *set whenever* $A \in \{B, B \sim E(B)\}$.

Then immediately we define the second variation.

Definition 8.1.5.
Suppose we can construct a set B *as follows:*

Let $A_{0,1}$ *be a base (a line of positive finite length in* $\mathbb{R}^2$*) (without loss of generality we will generally assume that* $A_{0,1} = [0,1] \subset \mathbb{R}$*) and* $T_{0,1}$ *be a triangular cap on* $A_{0,1}$ *with vertical height*

$\varepsilon\mathcal{H}^1(A_{0,1})$ with $\varepsilon < 1/100$. Let θ_0 be the base angle of $T_{0,1}$ and the two shorter sides of $T_{0,1}$ be denoted $A_{1,1}$ and $A_{1,2}$. We then construct two new triangular caps $T_{1,1}$ and $T_{1,2}$ on $A_{1,1}$ and $A_{1,2}$ with base angles $\theta_{1,1}, \theta_{1,2} \leq \theta_0$. We define

$$A_0 = T_{0,1} \text{ and } A_1 = \bigcup_{i=1}^{2} T_{1,i}.$$

Then suppose we have $A_n = \cup_{i=1}^{2^n} T_{n,i}$ a union of 2^n triangular caps with base angles $\theta_{n,i}$ and 2^{n+1} "shorter sides" (two per triangular cap) labelled $A_{n+1,i}$, $i \in \{1, ..., 2^{n+1}\}$. Then construct a triangular cap $T_{n+1,i}$ on each $A_{n+1,i}$ such that the base angles $\{\theta_{n+1,i}\}_{i=1}^{2^{n+1}}$ satisfy for each $i \in \{1, ..., 2^n\}$

$$\theta_{n,i} \geq \begin{cases} \theta_{n+1,2i-1} \\ \theta_{n+1,2i} \end{cases}$$

(i.e. the new base angles for each triangular cap are bounded by the base angle of the nth level that the new triangular cap is contained in).

Define $\tilde{A}_{n+1} = \cup_{i=1}^{2^{n+1}} A_{n+1,i}$. Finally define

$$B = \overline{\bigcup_{n=0}^{\infty} \tilde{A}_n} \sim \bigcup_{n=0}^{\infty} \tilde{A}_n.$$

We then call a set A a **Koch type set** *whenever $A \in \{B, B \sim E(B)\}$. We denote the set of all such sets by $\mathcal{K}$.*

Definition 8.1.6.
Let $A \in \mathcal{K}$ we then define the edge points of A, $E(A)$ by

$$E(A) := \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^n} E(T_{n,i}^A)$$

where $E(T_{n,i}^A)$ is as defined in Definition 4.1.2.

Before going on to show that these definitions are equivalent we need the following simple but important fact.

Lemma 8.1.1.
Let $A \in \mathcal{K}$. Then for any sequence $\{n, i(n)\}_{n \in \mathbb{N}}$ such that $T_{n,i(n)} \subset T_{n-1,i(n-1)}$ for each $n \in \mathbb{N}$

$$\lim_{n \to \infty} \mathcal{H}^1(A_{n,i(n)}) = 0.$$

Proof:
Since, by assumption $\theta_{0,1} < \pi/32$ and by construction $\theta_{n,i(n)}$ is decreasing in n. It follows from the inductive definition of the $A_{n,i(n)}$'s that

$$\mathcal{H}^1(A_{n,i(n)}) = (cos\theta_{n-1,i(n-1)})^{-1}\mathcal{H}^1(A_{n-1,i(n-1)}) \leq (cos\theta_{0,1})^{-1}\mathcal{H}^1(A_{n-1,i(n-1)}) = C\mathcal{H}^1(A_{n-1,i(n-1)})$$

where $C = (cos\theta_{0,1})^{-1} < 1$. It follows inductively that $\mathcal{H}^1(A_{n,i(n)}) \leq C^n\mathcal{H}^1(A_{0,1})$. Since $\mathcal{H}^1(A_{0,1}) < \infty$ by construction, the result follows. ♢

We now show that these definitions are equivalent.

Proposition 8.1.1.
Definition 8.1.1 *is equivalent to Definition* 8.1.4. *Definition* 8.1.2 *is equivalent to Definition* 8.1.5.

Proof:
We show these equivalences by showing that should $\mathcal{A}_2$ be defined as in Definition 8.1.2 and $\mathcal{A}_1$ be defined as in Definition 8.1.5 with the same $T_{n,i}$, $A_{n,i}$, $\theta_{n,i}$ etc. then

$$\mathcal{A}_2 \sim E(\mathcal{A}_2) \subset \mathcal{A}_1 = \overline{\bigcup_{n=0}^{\infty} \tilde{A}_n} \sim \bigcup_{n=0}^{\infty} \tilde{A}_n \subset \left(\bigcap_{n=0}^{\infty} \bigcup_{i=1}^{2^n} T_{n,i} \right) = \mathcal{A}_2.$$

That $E(\mathcal{A}_1) = E(\mathcal{A}_2)$ follows from Definition 8.1.6 and the fact that the $T_{n,i}$ used for $\mathcal{A}_1$ and $\mathcal{A}_2$ are the same. We thus denote $E(A) := E(\mathcal{A}_1) = E(\mathcal{A}_2)$. This will complete the proof since $E(A)$ is countable and thus $\mathcal{H}^1(E(A)) = 0$.

As in Lemma 4.2.1 we see that $\mathcal{A}_1 \cup E(A)$ is closed. Let $x \in (\mathcal{A}_2 \sim E) \sim \mathcal{A}_1$. then $d_x := d(x, \mathcal{A}_1 \cup E) > 0$.

Now, for each $n \in \mathbb{N}$, $x \in T_{n,i}$ for some i so that $d(x, \mathcal{A}_1 \cup E) < diam(T_{n,i}) = \mathcal{H}^1(A_{n,i})$. From Lemma 8.1.1 we have $\lim_{n\to\infty} \mathcal{H}^1(A_{n,i}) = 0$. Hence there is an $n_0 \in \mathbb{N}$ such that $diam(T_{n_0,j}) = \mathcal{H}^1(A_{n_0,j}) < d_x$ which implies $d(x, \mathcal{A}_1 + E) < \mathcal{H}^1(A_{n_0,i}) < d_x = d(x, \mathcal{A}_1 + E)$. This contradiction implies

$$\bigcap_{n=0}^{\infty} \bigcup_{i=1}^{2^n} T_{n,i} \subset \mathcal{A}_1 \cup E \text{ and thus } \left(\bigcap_{n=0}^{\infty} \bigcup_{i=1}^{2^n} T_{n,i} \right) \sim E(A) = \mathcal{A}_2 \sim E(\mathcal{A}_2) \subset \mathcal{A}_1.$$

Next, we note that for each $m \in \mathbb{N} \cup \{0\}$

$$\overline{\bigcup_{n=0}^{\infty} \tilde{A}_n} = \overline{\bigcup_{n=m+1}^{\infty} \tilde{A}_n} \cup \bigcup_{n=0}^{m} \overline{\tilde{A}_n}.$$

Noting that for each $m \in \mathbb{N}$, $\cup_{i=1}^{2^m+1} T_{m+1,i}$ is closed and that

$$\bigcup_{n=m+1}^{\infty} \tilde{A}_n \subset \bigcup_{i=1}^{2^{m+1}} T_{m+1,i}$$

we have

$$\overline{\bigcup_{n=m+1}^{\infty} \tilde{A}_n} \subset \bigcup_{i=1}^{2^{m+1}} T_{m+1,i}.$$

Also, we note that for each $n \in \mathbb{N} \cup \{0\}$, $\overline{\tilde{A}_n} = \tilde{A}_n$ so that for each $m \in \mathbb{N} \cup \{0\}$

$$\bigcup_{n=0}^{m} \overline{\tilde{A}_n} = \bigcup_{n=0}^{m} \tilde{A}_n.$$

It follows that for each $m \in \mathbb{N} \cup \{0\}$

$$\overline{\bigcup_{n=0}^{\infty} \tilde{A}_n} \sim \bigcup_{n=0}^{m} \tilde{A}_n = \left(\overline{\bigcup_{n=m+1}^{\infty} \tilde{A}_n} \cup \bigcup_{n=0}^{m} \tilde{A}_n \right) \sim \bigcup_{n=0}^{m} \tilde{A}_n = \overline{\bigcup_{n=m+1}^{\infty} \tilde{A}_n} \sim \bigcup_{n=0}^{m} \tilde{A}_n \subset \bigcup_{i=1}^{2^{m+1}} T_{m+1,i}.$$

Thus

$$\mathcal{A}_2 \sim E(\mathcal{A}_2) \subset \mathcal{A}_1 = \overline{\bigcup_{n=0}^{\infty} \tilde{A}_n} \sim \bigcup_{n=0}^{\infty} = \bigcap_{m=0}^{\infty} \left(\overline{\bigcup_{n=0}^{\infty} \tilde{A}_n} \sim \cup_{n=0}^{m} \tilde{A}_n \right) \subset \bigcap_{m=0}^{\infty} \bigcup_{i=1}^{2^{m+1}} T_{m+1,i} = \mathcal{A}_2.$$

◇

8.2 Bijective Characterisation of Koch Type Sets

We now show that sets in $\mathcal{K}$ can be characterised by a bijection from $\mathbb{R}$ into $\mathbb{R}^2$. Since some sets in $\mathcal{K}$ do not have dimension 1 it may seem odd at first glance that such a bijection exists. By quoting the fact that there is a bijection between $\mathbb{R}$ and the Cantor set, however, we see that the concept is neither new nor foreign in mathematics.

We show also immediately that a certain level of control of the preimage can be retained. To this end we need the following two definitions.

Definition 8.2.1.
Let A be a Koch type set. We denote the vertices of the triangular cap $T_{m,i}$ $a_{m,i}, l_{m,i}$, and $r_{m,i}$ chosen such that

$$\pi_x \circ O_{A_{m,i}}(a_{m,i}) = 0, \quad \pi_x \circ O_{A_{m,i}}(l_{m,i}) < 0, \text{ and } \pi_x \circ O_{A_{m,i}}(r_{m,i}) > 0.$$

That is a denotes the "top" vertex as we have previously defined, and l and r denote the identical "left" and "right" base angles.

Definition 8.2.2.
Let $A \in \mathcal{K}$ and $n \in \mathbb{N}$, then a dyadic interval of order n in $A_{0,0}$ (or simply, a dyadic intervals of order n) is defined as an interval D_n of the form

$$D_n = [l_{0,0} + i2^{-n}(r_{0,0} - l_{0,0}), l_{0,0} + (i+1)2^{-n}(r_{0,0} - l_{0,0})]$$

for some $i \in \{0, ..., 2^n - 1\}$. For some chosen $j \in \{0, ..., 2^n - 1\}$, the particular interval $D^A_{n,j}$ is defined by

$$D^A_{n,j} = [l_{0,0} + j2^{-n}(r_{0,0} - l_{0,0}), l_{0,0} + (j+1)2^{-n}(r_{0,0} - l_{0,0})].$$

As per usual the superscript A is dropped when the set is understood.

Remark: Note that should $A_{0,0}$ be $[0,1]$ on the real line, then the dyadic intervals in $A_{0,0}$ are simply the usual dyadic intervals.

Proposition 8.2.1.
Let $A \in \mathcal{K}$. Then there exists a sequence of Lipschitz functions $F_n : \mathbb{R} \mapsto \mathbb{R}^2$ (these functions will, in general, be denoted by F^A_n, when the set $A \in \mathcal{K}$ being refered to is not clear) such that

$$F^A_n(A_{0,1}) = \tilde{A}_{n-1}.$$

Further there exists a bijection $\mathcal{F}$ (which will, in general, be denoted by $\mathcal{F}^A$, when the set $A \in \mathcal{K}$ being refered to is not clear) such that

$$\mathcal{F}(A_{0,1}) = A.$$

Additionally, denoting the relatively dyadic points of $A_{0,1}$ *by* D*;*
that is, for $\{x_1, x_2\} = E(A_{0,1})$*,* $x_1 < x_2$*,*

$$D := \{y : y = x_1 + (x_2 - x_1)j2^{-n}, n \in \mathbb{N}, j \in \{0, ..., 2^n\}\};$$

we have $\mathcal{F}(D) = E(A)$*. Finally for each dyadic interval* $D_{n,i}$ *in* $A_{0,0}$*,*

$$F_n(D_{n,i}) = A_{n,i} \text{ and } \mathcal{F}(D_{n,i}) \subset T_{n,i}.$$

Proof:
Since the proof is the same for any $A_{0,1}$, we assume for notational convenience that $A_{0,1} = [0, 1]$. In this case D is also exactly the set of dyadic rationals in $[0, 1]$. That is $D = \{j2^{-n} : n \in \mathbb{N}, j \in \{0, ..., 2^n\}\}$.

We will define $\mathcal{F}$ as the limit of the F_n functions, and then show that it is well defined and has the required properties. Firstly, we define $f_0 : A_0 \to \mathbb{R}^2$ as

$$f_0(y) = \begin{cases} (y, tan\theta_{0,1}y) & y \in [0, 1/2) \\ (y, tan\theta_{0,1}(1 - y)) & y \in [1/2, 1] \end{cases}.$$

We see clearly that f_0 is a Lipschitz bijection between A_0 and A_1 (since the graph of the function draws out the triangular cap $T^A_{0,1}$) with Lipschitz constant (and Jacobian) $Lipf_0 = Jf_0 \equiv cos\theta^{-1}_{0,1}$. We then similarly define for each $n \in \mathbb{N}$ $f_{n,i} : A_{n,i} \to \mathbb{R}^2$ by

$$f_{n,i}(y) = \begin{cases} O^{-1}_{A_{n,i}}(\pi_x(O_{A_{n,i}}(y)), tan\theta_{n,i}(\pi_x(O_{A_{n,i}}(y)) + \mathcal{H}^1(A_{n,i})/2)) & y \in I_1 \\ O^{-1}_{A_{n,i}}(\pi_x(O_{A_{n,i}}(y)), tan\theta_{n,i}(\mathcal{H}^1(A_{n,i})/2 - \pi_x(O_{A_{n,i}}(y)))) & y \in I_2 \end{cases},$$

where $I_1 = O^{-1}_{A_{n,i}}([-\mathcal{H}^1(A_{n,i})/2, 0))$ and $I_2 = O^{-1}_{A_{n,i}}([0, \mathcal{H}^1(A_{n,i})/2])$. (Note that the $(1 - y)$ factor in the definition of f_0 would change to some other appropriate constant should $A_{0,1} \neq [0, 1]$.) We note in particular that $f_{n,i}(A_{n,i}) \subset T_{n,i}$. Noting also that the two end points of $A_{n,i}$ stay fixed we can define $f_n : A_n \to \mathbb{R}^2$ by

$$f_n(y) = f_{n,i}(y) \quad y \in A_{n,i}.$$

We see then that similarly to the f_0 situation f_n is a Lipschitz bijection between A_n and A_{n+1} with Lipschitz constant (and Jacobian in the case A is an A_ε type set) $Lipf_n(= Jf_n) = \max_{1 \leq i \leq 2^n} cos\theta^{-1}_{n,i}$.

By writing for a collection of functions $\{g_i\}^n_{i=0}$

$$\circ^n_{i=0} g_i = g_n \circ g_{n-1} \circ ... \circ g_0$$

we can then define the Lipschitz bijection between A_0 and A_{n+1}, $F_n : A_0 \to \mathbb{R}^2$ by

$$F_n = \circ^n_{i=0} f_i$$

which will then have Lipschitz constant (and Jacobian in the A_ε type set case)

$$LipF_n = JF_n = \prod_{i=1}^{n} (\cos \theta_i)^{-1}.$$

This demonstrates the first claim.

We can then propose a definition for $\mathcal{F}$ and indeed we propose the definition of $\mathcal{F} : A_0 \to \mathbb{R}^2$ to be

$$\mathcal{F}(y) = \lim_{n\to\infty} F_n(y).$$

We need first of all to show that this function is well defined. To do this we suppose first of all that

$$F_n(y) \in A_{n+1,i} \subset T_{n+1,i},$$

for some $i \in \{1, ..., 2^{n-1}\}$. Then

$$F_{n+1}(y) = f_{n+1,i}(y) \subset T_{n+1,i}.$$

Thus by induction, for each $n, k \in \mathbb{N}$

$$F_n(y) \in T_{n+1,i} \Rightarrow F_{n+k}(y) \in T_{n+1,i}.$$

Then, from Lemma 8.1.1, since $diam(T_{n,i}) = \mathcal{H}^1(A_{n,i})$, $diam(T_{n,i(n)}) \to 0$ as $n \to \infty$ for any sequence $\{n, i(n)\}_{n\in\mathbb{N}}$ and thus by setting the sequence $\{i(y,n)\}_{n\in\mathbb{N}}$ to be the sequence such that $y \in T_{n,i(y,n)}$ for each $n \in \mathbb{N}$ (so that it is always well defined, we choose arbitrarily $i(n)$ to be chosen such that $y = l_{n,i}$ for each n for which y is an edge point) it follows that for any $\varepsilon > 0$ there is an $n_0 > 0$ such that for all $n, m = n + k > n_0$,

$$d(F_n(y), F_m(y)) < diam(T_{n_0+1,i(y,n)}) < \varepsilon$$

so that $\{F_n(y)\}$ is a Cauchy sequence in $\mathbb{R}^2$ and thus converges. It follows that $\mathcal{F}$ is well defined.

We still need to show that $\mathcal{F}$ is a bijective function between A_0 and A.

We note firstly that for any $y \in A_0$ $F_n(y) \in A_n$ so that $\mathcal{F}(y) \in \overline{\cup_{n=0}^{\infty} A_n}$ and thus

$$\mathcal{F}(A_0) = \bigcup_{y\in A_0} \mathcal{F}(y) \subset \overline{\bigcup_{n=0}^{\infty} A_n}.$$

Now, since new edge points $a_{n,i}$ are by the definition of triangular caps always directly over the center of the base of the triangular cap, it follows that for all $e \in E$, $e = a_{n,i}$ for some $n \in \mathbb{N}$ and $i \in \{1, ..., 2^n\}$ and thus $e = F_n((2i-1)/2^{n+1})$. Since the set $\{(2i-1)/2^{n+1})\}_{n\in\mathbb{N}, i\in\{1,...,2^n\}} = D$ the set of dyadic rationals, it follows that $\mathcal{F}(D) = E$ which is a claim in our Proposition.

Further, for all $y \in A_0 \sim D$, $F_{n+k}(y) \cap A_n \subset (A_{n+1+k} \sim E) \cap A_n = \emptyset$ for each $k \geq 0$ so that $\mathcal{F}(y) \notin A_n$ for all $n \in \mathbb{N}$. It thus follows that

$$\mathcal{F}(A_0 \sim D) \subset \overline{\bigcup_{n=0}^{\infty} A_n} - \bigcup_{n=0}^{\infty} A_n \in \{A, A - E(A)\} \subset A$$

and thus that

$$\mathcal{F}(A_0) = \mathcal{F}(A_0 \sim D) \cup \mathcal{F}(D) \subset A \cup E = A.$$

We therefore have $\mathcal{F} : A_0 \to A$. We now need to show that it is bijective. We first show, however, the final two claims that refer to the relationship of $\mathcal{F}$ to the dyadic intervals of $A_{0,0}$.

We quickly mention a sketch of a proof and motivation of the last two claims which will be more rigorously proven in the following result.

From the above comment on the image of the dyadic rationals and the definition of F_n for an $n \in \mathbb{N}$ it follows that for each $i \in \{1, ..., 2^{n+1}\}$

$$F_n\left(\left[\frac{i-1}{2^{n-1}}, \frac{i}{2^n}\right]\right) = A_{n+1,i}.$$

This proves also our second last claim. Since, we have from definition that from each $n \in \mathbb{N}$ and any $x \in A_{0,0}$, $F_{n+1}(x)$ is in the same triangular cap $T_{n,i}$ as $F_n(x)$. It follows by induction that $\mathcal{F}(x) \in T_{n,i}$. Since this is true for each x_0 such that $F_n(x_0) \in T_{n,i}$ and from the above this set is equal to $D_{n,i}$. It follows that $\mathcal{F}(D_{n,i}) \subset T_{n,i}$ which is our final claim in the Theorem.

Continuing with the proof of the bijective property we use the above proven important facts as follows.

Firstly, that should $x, z \in A_0$ with $x \neq z$ we then have that there is an $n \in \mathbb{N}$ such that $2^{1-n} \geq |x - z| > 2^{-n}$ and thus there exist $i, j \in \{1, ..., 2^{n+2}\}$ with $4 \geq |i - j| \geq 2$ and the property that $x \in [(i-1)2^{-n-2}, i2^{n-2}]$ and $z \in [(j-1)2^{-n-2}, j2^{n-2}]$.

It then follows that $F_n(x) \in T_{n+2,i}$ and thus, as above, $\mathcal{F}(x) \in T_{n+2,i}$. Similarly $\mathcal{F}(j) \in T_{n+2,j}$.

Since from Lemma 6.2.1 we know that for any $n \in \mathbb{N}$ $T_{n+2,i} \cap T_{n+2,j} = \emptyset$ whenever $4 \geq |i - j| \geq 2$ it follows that $\mathcal{F}(x) \neq \mathcal{F}(z)$ and therefore that $\mathcal{F}$ is injective.

For surjectivity, we consider an arbitrary element $y \in A$. For all $n \in \mathbb{N}$, $y \in T_{n,i(y,n)}$ for some $i(y, n) \in \{1, ..., 2^n\}$. Then, again from $F_n\left(\left[\frac{i-1}{2^{n-1}}, \frac{i}{2^n}\right]\right) = A_{n+1,i}$ we see that it is instructive to consider the intervals

$$\mathcal{F}^{-1}(A_{n,i(y,n)}) = [(i(y, n) - 1)2^{-n}, i(y, n)2^{-n}] =: D_{n,i(y,n)}.$$

Since $T_{n+1,i(y,n)} \subset T_{n,i(y,n)}$ for each n it follows that $D_{n+1,i(y,n+1)} \subset D_{n,i(y,n)}$ for each n. We now observe $y_0 = \cap_{n=0}^{\infty} D_{n,i(y,n)}$. For this y_0

$$F_n(y_0) \subset F_n(D_{n,i(y,n)}) \subset A_{n,i(y,n)} \subset T_{n,i(y,n)}$$

for each n. Thus for each $n \in \mathbb{N}$, $|F_n(y_0) - y| \leq diam(T_{n,i(y,n)})$. Since this diameter goes to zero as n approached infinity it follows that

$$\mathcal{F}(y_0) = \lim_{n\to\infty} F_n(y_0) = y.$$

From well definedness and the arbitrariness of y the surjectivity and thus bijectivity of $\mathcal{F}$ follows. $\diamondsuit$

We now show some results on the structure of $\mathcal{F}$ which expand on the last two points of the previous results, as well as embellishing the proof somewhat. We show that the function can be looked at as a function on each dyadic interval. A in any given triangular cap is a bijection between A in this cap and a dyadic interval in $A_{0,0}$. These results make it much easier to track images and pre-images and thus also to track how much measure has come from, or gone to where.

Proposition 8.2.2.
Let $A \in \mathcal{K}$ be constructed from a base $[0, 1]$. Suppose that $\{F_n\}_{n=0}^{\infty}$ are the Lipschitz functions such that

$$\mathcal{F}^A := \lim_{n\to\infty} F_n$$

pointwise on $A^A_{0,0}$,

$$F_n = \circ^n_{i=0} f_i$$

and writing l^A_{ni}, a^A_{ni} *and* r_{ni} *as the edge point of* $A^A_{n,i}$ *adjoining* $A^A_{n,i-1}$ *(or (0,0) should* $i = 1$*), the centerpoint of* $A^A_{n,i}$ *and the edge point of* $A^A_{n,i}$ *adjoining* $A^A_{n,i+1}$ *(or (1,0) should* $i = 2^n$*) respectively.*

Then for $n \in \mathbb{N}$ *and* $i \in \{1, ..., 2^n\}$ *we have*

$$A^A_{n,i} = F_{n-1}([(i-1)2^{-n}, i2^{-n}]), F^{-1}_{n-1}(A^A_{n,i}) = [(i-1)2^{-n}, i2^{-n}],$$

$$l_{ni} = F_{n-1}(2^{-n}(i-1))$$

$$r_{ni} = F_{n-1}(2^{-n}i)$$

and that F_{n-1} *preserves relative distances. That is for each* $x, y \in [(i-1)2^{-n}, i2^{-n}]$

$$|F_{n-1}(x) - F_{n-1}(y)| = p_{n,i}|x - y|$$

for some $p_{n,i} \in \mathbb{R}$.

Remark: Of the claims stated we are most interested in and thus emphasise

$$A^A_{n,i} = F_{n-1}([(i-1)2^{-n}, i2^{-n}]), F^{-1}_{n-1}(A^A_{n,i}) = [(i-1)2^{-n}, i2^{-n}],$$

which gives us in essence a trace of the movement of a dyadic interval as it approaches the limit set A. With this we can follow the track either forward or backwards to identify which parts of A or $A_{0,0}$ have positive measure given information about the measure of the other of A and $A_{0,0}$. The other claims are stated here as an aid to proving the inductive step which is the key to the proof.

Proof:
We prove the statement by induction on n.

From the definition of $A^A_{1,1}$, $A^A_{1,2}$ and the definition

$$f_0(y) = \begin{cases} (y, tan\theta_0 y) & y \in [0, 1/2) \\ (y, tan\theta_0(1-y)) & y \in [1/2, 1] \end{cases}.$$

It follows that $A^A_{1,1} = F_0([0, 1/2]), A^A_{1,2} = F_0([1/2, 1])$, that $F_0^{-1}(A^A_{1,1}) = [0, 1/2], F_0^{-1}(A^A_{1,2}) = [1/2, 1]$, that $F_0(0) = (0, 0) = l_{11}$, that $F_0(1) = (1, 0) = r_{12}$, and hence that $F_0(1/2) = r_{11} = l_{12}$.

We see also that the preservation of relative distances holds with $p_{1,i} \equiv tan\theta^A_{0,1}$ for $i = 1, 2$. The claim thus holds for $n = 1$.

Now suppose that the claim is true for each $n \leq m$ for some $m \in \mathbb{N}$.

We note that for any arbitrary $i \in \{1, ..., 2^{m+1}\}$ there is a $j \in \{1, ..., 2^m\}$ such that $i \in \{2j-1, 2j\}$. Now since $A^A_{m,j} = F_{m-1}([(j-1)2^{-m}, j3^{-1}])$

$$F_m([(j-1)2^{-m}, j2^{-m}]) = f_m \circ F_{m_1}([(j-1)2^{-m}, j2^{-m}]) = f_m(A_{m,j}).$$

Since $F_{m-1}((j-1)2^{-m}) = l^A_{lmj}$, $F_m(j2^{-m}) = l^A_{rmj}$ and F_m preserves relative distances we also have

$$F_{m-1}((j-1)2^{-m} + 2^{-m-1}) = l^A_{mj},$$

and thus $\pi_x(O_{A^A_{n,j}}(F_{m-1}((j-1)2^{-m}+2^{-m-1}))) = 0$. Thus, again from relative distance preservation

$$\pi_x(O_{A^A_{n,j}}(F_{m-1}([(j-1)2^{-m},(j-1)2^{-m}+2^{-m-1}]))) = [-\mathcal{H}^1(A^A_{n,j})/2, 0] =: I_1$$

and

$$\pi_x(O_{A^A_{n,j}}(F_{m-1}([(j-1)2^{-m}+2^{-m-1}, j2^{-m}]))) = [0, \mathcal{H}^1(A^A_{n,j})/2] =: I_2.$$

It follows then from the definition of $f_m|_{[(j-1)2^{-m},j2^{-m}]}$

$$f_{n,i}(y) = \begin{cases} O^{-1}_{A_{n,i}}(\pi_x(O_{A_{n,i}}(y)), tan\theta_n(\pi_x(O_{A_{n,i}}(y)) + \mathcal{H}^1(A_{n,i})/2)) & y \in I_1 \\ O^{-1}_{A_{n,i}}(\pi_x(O_{A_{n,i}}(y)), tan\theta_n(\mathcal{H}^1(A_{n,i})/2 - \pi_x(O_{A_{n,i}}(y)))) & y \in I_2 \end{cases},$$

and from the definition of $A^A_{m+1,k}$, $k \in \{1, ..., 2^{-m-1}\}$ that

$$A^A_{m+1,2j-1} = F_m([(2j-2)2^{-m-1},(2j-1)2^{-m-1}])$$

$$A^A_{m+1,2j} = F_m([(2j-1)2^{-m-1}, 2j2^{-m-1}])$$

and since we know F_m is a bijection that

$$F_m^{-1}(A^A_{m+1,2j-1}) = [(2j-2)2^{-m-1},(2j-1)2^{-m-1}], \text{ and}$$

$$F_m^{-1}(A^A_{m+1,2j}) = [(2j-1)2^{-m-1}, 2j2^{-m-1}].$$

Further,

$$F_m((2j-1)2^{-m-1}), F_m((2j-2)2^{-m-1}) \in E(A^A_{m+1,2j-1})$$

and

$$F_m((2j-1)2^{-m-1}), F_m(2j2^{-m-1}) \in E(A^A_{m+1,2j})$$

from which it must therefore follow that

$$F_m((2j-1)2^{-m-1}) = r^A_{(m+1)(2j-1)} = l^A_{l(m+1)(2j)},$$

$$F_m((2j-2)2^{-m-1}) = l^A_{(m+1)(2j-1)} \text{ and that}$$

$$F_m(2j2^{-m-1}) = r^A_{(m+1)(2j)}.$$

Further, since $F_{m-1}|_{[(2j-1)2^{-m},2j2^{-m}]}$ preserves relative distance with $|F_{m-1}(x)-F_{m-1}(y)| = p_{m-1,j}|x-y|$ for all $x, y \in [(2j-1)2^{-m}, 2j2^{-m}]$, from the definition of $f_m(y)$ and $F_m = f_m \circ F_{m-1}$ it follows that F_m preserves relative distances on $[(2j-2)2^{-m-1},(2j-1)2^{-m-1}]$ and $[(2j-1)2^{-m-1}, 2j2^{-m-1}]$ with

$$p_{m,2j-1} = p_{m,2j} = (tan\theta^A_{n,j})p_{m-1,j}.$$

By substituting in i for $2j-1$ or $2j$ as necessary it follows that all required properties are satisfied for $m+1$ with the choice of $i \in \{1, ..., 2^{m+1}\}$. Since the choice of i was arbitrary this completes the inductive step and thus the proof. ♢

8.3 Further Characterisations and Properties of Sets in $\mathcal{K}$

Equiped with these results we are able to give a list of nomenclaturial definitions that will be instrumental in describing our results.

Definition 8.3.1.
Let $A \in \mathcal{K}$. We introduce $i(n,x) := \mathbb{N} \times A_0 \to \mathbb{N}$ defined by

$$i(n,x) := \{i \in \{1, ..., 2^n\} : x \in T^A_{n,i}\}.$$

We write

$$\tilde{\theta}^A_{x_0} = \tilde{\theta}^A_x = \lim_{n\to\infty} \theta^A_{n,i(n,x)}$$

and define the functions $\tilde{\Pi}^A, \tilde{\Pi}^A_n, \tilde{\Pi}^A_{n,i} : \mathbb{R} \to \mathbb{R}$ for and $x \in A \cap T^A_{n,i}$ by

$$\tilde{\Pi}^A(x) = \prod_{i=0}^{\infty}(cos\theta^A_{n,i(n,x)})^{-1}$$

$$\tilde{\Pi}^A_n(x) = \prod_{i=0}^{n}(cos\theta^A_{n,i(n,x)})^{-1}, \text{ and}$$

$$\tilde{\Pi}^A_{n,i} = \tilde{\Pi}^A_n(x),$$

for any $x \in A \cap T^A_{n,i}$ which will be independent of which $x \in T^A_{n,i}$ is used. The superscript A is dropped when the set A is understood.

Further

$$\Lambda_m := \{x \in A : \tilde{\Pi}(x) \le m\}$$
$$\Lambda^{-1}_m := \mathcal{F}^{-1}(\Lambda_m)$$
$$\Lambda_{m+} := \{x \in A : \tilde{\Pi}(x) \ge m\}$$
$$\Lambda^{-1}_{m+} := \mathcal{F}^{-1}(\Lambda_{m+})$$
$$\Lambda_\infty := \{x \in A : \tilde{\Pi}(x) = \infty\}$$
$$\Lambda^{-1}_\infty := \mathcal{F}^{-1}(\Lambda_\infty).$$

Also, for each $a \in \mathbb{R}$ we define

$$\Upsilon^{-1}_a := \{x \in A_0 : \tilde{\theta}^A_x \le a\}$$
$$\Upsilon_a := \mathcal{F}(\Upsilon^{-1}_a)$$
$$\Upsilon^{-1}_{a+} := \{x \in A_0 : \tilde{\theta}^A_x \ge a\}, \text{ and}$$
$$\Upsilon_{a+} := \mathcal{F}(\Upsilon^{-1}_{a+}).$$

As with the other notations, when the $A \in \mathcal{K}$ we are referring to is unclear we add a superscript A, for example $(\Lambda^{-1}_\infty)^A$.

Two further definitions relating to sets being used will now be presented. Firstly a variant of the angle between sets, and then a generalisation of the $i(n,x)$ notation.

Definition 8.3.2.
Let L_1, L_2 be any two straight lines in $\mathbb{R}^2$ and L^1, L^2 be the extensions of these lines to simply connected lines of infinite length in both directions. We then denote the smaller of the two types of angles that occur at the intersection of L^1 and L^2 by $\psi^{L_1}_{L_2}$.

Definition 8.3.3.
Let $A \in \mathcal{K}$ and $B \subset A_{0,0}$. Suppose that for some $n \in \mathbb{N}$, $i(n,x)$ is uniform for all $x \in B$. Then we will sometimes for convenience denote this common value $i(n,B)$.

Further notations will occasionally be used, but not regularly and so will be defined as they are used. We continue now with further definitions and properties relating to the previous two definitions. These properties will be necessary in dealing with the main results in the next chapter concerning the sets in $\mathcal{K}$.

Definition 8.3.4.
We define, for $r \in \mathbb{R}$, the collection A^r by

$$A^r := \{A : A \textit{ is an } A_\varepsilon \textit{ type set and } \tilde{\theta}^A_{\cdot} = r\}.$$

We now state formally, to connect to the previous work, the A^r sets to which our previous sets Γ_ε and A_ε belong.

Proposition 8.3.1.
$\Gamma_\varepsilon \in A^{tan^{-1}(2\varepsilon)}$ *and* $A_\varepsilon \in A^0$.

Proof:
That $\Gamma_\varepsilon \in A^{tan^{-1}(2\varepsilon)}$ follows from the definition of Γ_ε since we can calculate from the construction that $\theta^{\Gamma_\varepsilon}_{n,\cdot} \equiv tan^{-1}(2\varepsilon)$. Since $\theta^A_{n,\cdot}$ is constant and from the proof of Lemma 6.4.1 $\lim_{n\to\infty} \theta^A_{n,\cdot} = 0$ it follows that $A \in A^0$. $\diamondsuit$

We now wish to investigate some of the properties possessed by $\mathcal{F}$ and resultant from the definitions that we have just made. We first look at two results concerning the $\theta_{n,i}$. We see that the stretch (and when $\mathcal{F}$ has appropriate properties the Jacobian) that occurs to each $\tilde{A_n}$ is described by a product of the base angles. Secondly, we consider a convergence equivalence of this stretch factor to a convergence of the sum, which can be thought of as a test of whether a set $A \in \mathcal{K}$ spirals infinitely or not. We also would like to have a result concerning the density of A around the image of a considered point in $A_{0,0}$. An appropriate result does exist. It is, however, most conveniently in the next section as a Corollary of results presented there. With these two results we conclude the section.

Lemma 8.3.1.
For any $A \in \mathcal{K}$, $n \in \mathbb{N}$ and $i \in \{1, ..., 2^n\}$

$$\mathcal{H}^1(A^A_{n,i}) = \frac{\mathcal{H}^1(A^A_{0,0})}{2^n} \prod_{j=1}^{n} \frac{1}{cos(\theta^A_{j,D_{j,i}})}.$$

Proof:
By considering the right angled triangle consisting of $A^A_{n,i}$, half of the base $A^A_{n-1,j}$ of the triangular cap in which $A^A_{n,i}$ arises and the line connecting the ends that don't meet, we see that

$$\frac{\mathcal{H}^1(A^A_{n-1,j})}{2\mathcal{H}^1(A^A_{n,i})} = cos(\theta^A_{n,D_{n,i}}) \text{ so that } \mathcal{H}^1(A^A_{n,i}) = \frac{\mathcal{H}^1(A^A_{n-1,j})}{2cos(\theta^A_{n,D_{n,i}})}.$$

Thus repeating this step inductively we get

$$\mathcal{H}^1(A^A_{n,i}) \quad = \quad \frac{\mathcal{H}^1(A^A_{n-1,j})}{2cos(\theta^A_{n,D_{n,i}})}$$

$$
\begin{aligned}
&= (cos(\theta^A_{n,D_{n,i}})^{-1})(cos(\theta^A_{n-1,D_{n-1,i}}))^{-1}\frac{1}{4}\mathcal{H}^1(A^A_{n-2,\cdot}) \\
&= \ldots \\
&= \left(\prod_{j=0}^{n-1}(cos(\theta^A_{j,D_{j,i}}))^{-1}\right)2^{-n}\mathcal{H}^1(A^A_{0,1})
\end{aligned}
$$

as required. ◊

Proposition 8.3.2.
Let $A \in A^0$ and $x \in A$. Then

$$\prod_{n=0}^{\infty}(cos(\theta^A_{n,i(n,x)}))^{-1} < \infty \Longleftrightarrow \sum_{n=0}^{\infty}(\theta^A_{n,i(n,x)})^2 < \infty.$$

Proof:
We note that $\theta^A_{n,i(n,x)} < \pi/32$ for all $A \in A^0$, $n \in \mathbb{N}$, and $x \in A$. Further we write $M := \prod_{n=0}^{\infty}(cos(\theta^A_{n,i(n,x)}))^{-1}$ and calculate

$$ln(M) = ln\left(\prod_{n=0}^{\infty}(cos(\theta^A_{n,i(n,x)}))^{-1}\right) = \sum_{n=0}^{\infty} ln((cos(\theta^A_{n,i(n,x)}))^{-1}).$$

We also use the relatively simple inequality, easily verifiable, that there exists a real $\infty > C > 1$ such that for all $\theta \in [0, \pi/32]$

$$\frac{1}{C}\theta^2 \le ln((cos\theta)^{-1}) \le C\theta^2.$$

(This can be proven either by using a Taylor series expansion or by comparing the second and (at zero disappearing) first derivatives of e^{θ^2} and $(cos\theta)^{-1}$ with respect to θ for θ sufficiently close to zero.) We can now calculate

$$
\begin{aligned}
\sum_{n=0}^{\infty}(\theta^A_{n,i(n,x)})^2 &\ge \frac{1}{C}\sum_{n=0}^{\infty} ln((cos(\theta^A_{n,i(n,x)}))^{-1})\left(=\frac{1}{C}ln(M)\right) \\
&\ge \frac{1}{C^2}\sum_{n=0}^{\infty}(\theta^A_{n,i(n,x)})^2.
\end{aligned}
$$

It follows that

$$M < \infty \Longleftrightarrow ln(M) < \infty \Longleftrightarrow \sum_{n=0}^{\infty}(\theta^A_{n,i(n,x)})^2 < \infty,$$

which completes the proof. ◊

8.4 Properties of The Bijective Functions

We now examine some important properties of the functions F_n and the function $\mathcal{F}$. In order to work with $\mathcal{F}$ properly we must first check that it has some basic properties. We show that the function $\mathcal{F}$ is continuous and measurable. We show that images of compact sets are compact. We show that positive measure is preserved. A less well behaved, but nonetheless important, property is that, under conditions on Λ^{-1}_{∞}, sets of positive measure have images of infinite measure. We additionally prove the $A \in \mathcal{K}$ version of Corollary 8.4.1. First of all, however, we prove that parts of the limit function $\mathcal{F}$ can be expressed as Lipschitz functions. Recalling that $\tilde{\Pi}^A_{(\cdot)}$ can be seen as the stretching (or Jacobian) factor of $\mathcal{F}$ it would seem sensible that when this is bounded, we are actually looking at a Lipschitz function. We show that this is true after defining how to bound the Jacobian. We make bounds by simply looking at the restriction of the function to pre-image sets on which $\tilde{\Pi}^A$ is bounded.

Definition 8.4.1.
Let $A \in \mathcal{K}$, then we define $\mathcal{F}_m := \mathcal{F}|_{\Lambda_m^{-1}}$.

Lemma 8.4.1.
For $m \in \mathbb{R}$, $\mathcal{F}_m := \mathcal{F}|_{\Lambda_m^{-1}}$ is Lipschitz with $Lip\mathcal{F}_m \leq Cm^2$.

Proof:
Let $x, y \in \Lambda_m^{-1}$, and without loss of generality let $y < x$. There are then two cases to consider

1. $\{ty + (1-t)x : t \in [0,1]\} \subset \Lambda_m^{-1}$, and
2. otherwise.

Case 1 is the simpler. In this case we have $\mathcal{F}_m|_{[y,x]} = \mathcal{F}|_{[y,x]}$ and $\tilde{\Pi}(z) \in [y,x]$. It follows from the construction of the F_n from which $\mathcal{F}$ is defined as a limit that

$$d(F_n(y), F_n(x)) \leq md(y,x), \text{ for all } n \in \mathbb{N},$$

which implies

$$d(\mathcal{F}(y), \mathcal{F}(x)) \leq \limsup_{n\to\infty} d(F_n(y), F_n(x)) \leq \limsup_{n\to\infty} md(y,x) = md(y,x).$$

For case 2 we know that there must exist a $z \in (y,x)$ such that $\tilde{\Pi}(x) > m$ and therefore there is an $n_0 \in \mathbb{N}$ such that $y, x \notin T^A_{n_0,i(n_0,z)}$ and indeed $i(n_0,y) < i(n_0,z) < i(n_0,x)$.

It follows that we can find a minimum such n_0 and therefore an $n_1 \in \mathbb{N}$ such that $i(n_1,x) - 3 \leq i(n_1,y) \leq i(n_1,x) - 2$ and such that for all $n < n_1$ $i(n,y) \in \{i(n,x) - 1, i(n,x)$.

From this, it follows firstly that for each $n < n_1$ $[y,x] \subset T^A_{n,i(n,y)} \cup T^A_{n,i(n,x)}$ which implies that $F_{n_0}|_{[y,x]}$ has Lipschitz constant

$$LipF_{n_0}|_{[y,x]} \leq \max\{\tilde{\Pi}^A_{n_0}(x), \tilde{\Pi}^A_{n_0}(y)\} \leq m$$

so that

$$d(F_{n_0}(y), F_{n_0}(x)) \leq md(y,x).$$

It also follows from the choice of n_1 that

$$d(\mathcal{F}(y), \mathcal{F}(x)) < 2 \max_{w\in\{x,y\}} \mathcal{H}^1(A^A_{n_0-1,i(n_0-1,w)}).$$

Now, using Lemma 6.2.1 we know

$$\pi_x(O_{A^A_{n_0,i(n_0,y)}}(T^A_{n_0,i(n_0,y)} \cap T^A_{n_0,i(n_0,x)})) \cap O_{A^A_{n_0,i(n_0,y)}}(A^A_{n_0,i(n_0,y)}) = \emptyset$$

and thus

$$d(F_{n_0}(y), F_{n_0}(x)) \geq \mathcal{H}^1(A^A_{n_0,i(n_0,y)+1}) \geq \frac{1}{2} \min_{w\in y,x} \mathcal{H}^1(A^A_{n_0-1,i_w(n_0-1)})$$

the latter following since $A^A_{n_0,i(n_0,y)+1}$ is a shorter side of either $A^A_{n_0-1,i(n_0-1,y)}$ or $A^A_{n_0-1,i(n_0-1,x)}$.

Since

$$y \in T^A_{n_0,i(n_0,y)+1} \Rightarrow 1 \leq \tilde{\Pi}^A_{n_0-1}(y) \leq m$$

and

$$x \in T^A_{n_0,i(n_0,x)+1} \Rightarrow 1 \leq \tilde{\Pi}^A_{n_0-1}(x) \leq m$$

it follows that

$$\max_{w\in\{x,y\}} \mathcal{H}^1(A^A_{n_0-1,i(n_0-1,w)}) \leq m \min_{w\in\{x,y\}} \mathcal{H}^1(A^A_{n_0-1,i(n_0-1,w)}).$$

Thus

$$d(F_{n_0}(y), F_{n_0}(x)) \geq \frac{1}{2m} \max_{w\in\{x,y\}} \mathcal{H}^1(A^A_{n_0-1,i(n_0-1,w)}).$$

Hence

$$d(\mathcal{F}(y), \mathcal{F}(x)) \leq 2 \max_{w\in\{x,y\}} \mathcal{H}^1(A^A_{n_0-1,i(n_0-1,w)}) \leq 4md(F_{n_0}(y), F_{n_0}(x)) \leq 4m^2 d(y,x).$$

Combining the two cases gives us, using $m \geq 1$

$$d(\mathcal{F}(y), \mathcal{F}(x)) \leq \max\{m, 4m^2\} d(x,y) = 4m^2 d(x,y)$$

for each $x, y \in \Lambda^{-1}_m$. ♢

Proposition 8.4.1.
Let $A \in \mathcal{K}$ and let $\mathcal{F}$ be the function related to A. Then

1. *$\mathcal{F}$ is continuous,*
2. *should $B \subseteq A_0$ be closed, then $\mathcal{F}(B) \subseteq A$ is compact,*
3. *if $B \subset A_0$ is such that $\mathcal{H}^1(B) > 0$ then $\mathcal{H}^1(\mathcal{F}(B)) > \mathcal{H}^1(B)/6 > 0$,*
4. *if $\mathcal{H}^1(\Lambda^{-1}_\infty) > 0$ then $\mathcal{H}^1(\Lambda_\infty) = \infty$,*
5. *if $\Theta^1(\mathcal{H}^1, \Lambda^{-1}_\infty, x) > 0$ then $\Theta^1(\mathcal{H}^1, \Lambda_\infty, \mathcal{F}(x)) = \infty$, and*
6. *$\mathcal{F}$ is $\mathcal{H}^1$-measurable.*

Proof:
As we are considering only one A we shall omit the A superscripts.

For (1), since for all constructions A that we consider we have $\theta_{0,0} \leq \pi/32$ we see that

$$diam(T_{n,\cdot}) = \mathcal{H}^1(A_{n,\cdot}) \leq \frac{(cos(\pi/32))^{-1}}{2} \mathcal{H}^1(A_{n-1,\cdot})$$

which inductively gives us

$$diam(T_{n,i}) \leq \left(\frac{(cos(\pi/32))^{-1}}{2}\right)^n \mathcal{H}^1(A_0).$$

Since $cos(\pi/32) > 1/2$, $(cos(\pi/32))^{-1}/2 < 1$ so that

$$\lim_{n\to\infty} diam(T_{n,\cdot}) = 0.$$

It follows that for all $\varepsilon > 0$, $diam(T_{n,\cdot}) < \varepsilon/2$ for all n greater than some sufficiently large n_0. Consider $x_1, x_2 \in A_0$ such that $|x_1 - x_2| < 2^{-n_0}$. Then

$$x_1, x_2 \in \left[\frac{i-1}{2_0^n}, \frac{i}{2^n}\right] \cup \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right]$$

for some $i \in \{1, ..., 2^n - 1\}$, so that since $\mathcal{F}([(i-1)2^{-n}, i2^{-n}]) \subset T_{n,i}$ for each $n \in \mathbb{N}$ and each $i \in \{1, ..., 2^n\}$ $\mathcal{F}(x_1), \mathcal{F}(x_2) \in T_{n_0,i} \cup T_{n_0,i+1}$, which implies

$$|\mathcal{F}(x_1) - \mathcal{F}(x_2)| \leq diam(T_{n_0,i-1}) + diam(T_{n_0,i}) < \varepsilon.$$

For (2), since A_0 is bounded, so to is any closed subset of A_0, thus should B be a closed subset of A_0 it is also compact. It then follows from the fact that $\mathcal{F}$ is continuous that $\mathcal{F}(B)$ is closed and indeed bounded since $\mathcal{F}(A_0) \subset [0,1] \times [0,1]$ and thus also compact.

For (3), let our set, for convenience be denoted K. Let $\mathcal{H}^1(K) > 0$, say $\mathcal{H}^1(K) =: \beta$. It follows that there is a $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ $\mathcal{H}^1_\delta(K) > \frac{\beta}{2}$.

Now, let $\delta < \delta_0$ and $\{B_\delta\}$ be a δ-cover of $\mathcal{F}(K)$ and consider a $B \in B_\delta$. By Lemma 6.2.1 we see that there is an $n(B) \in \mathbb{N}$ such that whether or not $center(B) \in \mathcal{F}(K)$ $B \cap \mathcal{F}(B) \subset T^A_{n(B),i(B)-1} \cup T^A_{n(B),i(B)} \cup T^A_{n(B),i(B)+1}$ for some $\text{ß}(B) \in \{2, ..., 2^{n(B)} - 1\}$ with

$$diam(T^A_{n(B),\cdot}) = length(A^A_{n(B),\cdot}) \in \left(\frac{diam(B)}{2}, diam(B)\right)$$

so that

$$diam(B) > \sum_{j=i(B)-1}^{i(B)+1} diam(T^A_{n(B),\cdot}).$$

In this case we also have

$$\mathcal{F}(B \cap \mathcal{F}(K)) \subset \bigcup_{j=i(B)-1}^{i(B)+1} \mathcal{F}^{-1}(T_{n(b),j}) = \bigcup_{j=1(B)-1}^{i(B)+1} F^{-1}_{n(b)}(A_{n(B),j})$$

which, since $F_{n(B)}$ is an expansion map, gives three intervals $I_{B,j}$, $j = 1, 2, 3$ with

$$diam(I_{B,j}) = length(I_{B,j}) \leq length(A_{n(B),j}) < diam(B) < \delta.$$

It follows that

$$\sum_{j=i(B)-1}^{i(B)+1} diam(I_{B,j}) < 3diam(B).$$

Since

$$\mathcal{F}(K) \subset \bigcup_{B \in B_\delta} (B \cap \mathcal{F}(K))$$

it follows that

$$K \subset \bigcup_{B \in B_\delta} \bigcup_{j=i(B)-1}^{i(B)+1} I_{B,j}$$

which implies that $\{\{I_{B,j}\}_{B\in B_\delta}\}_{j=i(B)-1}^{i(b)+1}$ is a δ cover of K and thus that

$$\sum_{B\in B_\delta}\sum_{j=i(B)-1}^{i(B)+1} I_{B,j} \geq \mathcal{H}^1_\delta(K) > \frac{\beta}{2}$$

and therefore

$$\sum_{B\in B_\delta} diam(B) > \frac{1}{3}\sum_{B\in B_\delta}\sum_{j=i(B)-1}^{i(B)+1} I_{B,j} > \frac{1}{3}\frac{\beta}{2} = \frac{\beta}{6}.$$

Since this is true for any such δ-cover of $\mathcal{F}(K)$ we see that $\mathcal{H}^1_\delta(\mathcal{F}(K)) > \frac{\beta}{6}$ for any $\delta < \delta_0$ and therefore that

$$\mathcal{H}^1(\mathcal{F}(K)) = \lim_{\delta\to 0}\mathcal{H}^1_\delta(\mathcal{F}(K)) \geq \lim_{\delta\to 0}\frac{\beta}{6} > 0.$$

For (4), let $M > 0$. Then since $\mathcal{H}^1_{A_0}$ is Radon and

$$\Lambda^{-1}_\infty = \bigcup_{n\in\mathbb{N}}\{x\in\Lambda^{-1}_\infty : \tilde{\Pi}_n(x) > M\}$$

it follows that there is an $n_0\in\mathbb{N}$ with

$$\mathcal{H}^1(\{x\in\Lambda^{-1}_\infty : \tilde{\Pi}_{n_0}(x) > M\}) > \frac{\mathcal{H}^1(\Lambda^{-1}_\infty)}{2} > 0.$$

We set $\Lambda^{-1}_{\infty,n_0} := \{x\in\Lambda^{-1}_\infty : \tilde{\Pi}_{n_0}(x) > M\}$. It follows that with $X := \{i\in\{1,...,2^{n_0}\} : T_{n_0,i}\cap F_{n_0}(\Lambda^{-1}_{\infty,n_0}) \neq \emptyset\}$

$$\begin{aligned}
\mathcal{H}^1(F_{n_0}(\Lambda^{-1}_{\infty,n_0})) &= \sum_{i\in X}\mathcal{H}^1(F_{n_0}(\Lambda^{-1}_{\infty,n_0})\cap T_{n_0,i}) \\
&> M\sum_{i\in X}\mathcal{H}^1(\Lambda^{-1}_{\infty,n_0}\cap[2^{-n_0}(i-1), 2^{n_0}i]) \\
&= M\mathcal{H}^1(\Lambda^{-1}_{\infty,n_0}) \\
&> M\frac{\mathcal{H}^1(\Lambda^{-1}_\infty)}{2}.
\end{aligned}$$

We then apply (3) to each set $A^{n_0,i}\in\mathcal{K}$ defined as the subconstruction (and subset) of A starting with $A_0^{n_0,i} = A_{n_0,i}$ to find

$$\mathcal{H}^1(\mathcal{F}\circ F_{n_0}^{-1}(F_{n_0}(\Lambda^{-1}_{\infty,n_0})\cap T_{n_0,i})) > \frac{\mathcal{H}^1(F_{n_0}(\Lambda^{-1}_{\infty,n_0})\cap T_{n_0,i})}{6}$$

and thus that

$$\begin{aligned}
\mathcal{H}^1(\mathcal{F}(\Lambda^{-1}_{\infty,n_0})) &= \sum_{i\in X}\mathcal{H}^1(\mathcal{F}\circ F_{n_0}^{-1}(F_{n_0}(\Lambda^{-1}_{\infty,n_0})\cap T_{n_0,i})) \\
&> \frac{1}{6}\sum_{i\in X}\mathcal{H}^1(F_{n_0}(\Lambda^{-1}_{\infty,n_0})\cap T_{n_0,i}) \\
&= \frac{1}{6}\mathcal{H}^1(F_{n_0}(\Lambda^{-1}_{\infty,n_0})).
\end{aligned}$$

We therefore now have

$$\mathcal{H}^1(\mathcal{F}(\Lambda^{-1}_{\infty,n_0})) > \frac{1}{6}\mathcal{H}^1(F_{n_0}(\Lambda^{-1}_{\infty,n_0})) > \frac{M\mathcal{H}^1(\Lambda^{-1}_\infty)}{12}.$$

Since this is true for each $M > 0$ it follows that

$$\mathcal{H}^1(\mathcal{F}(\Lambda^{-1}_\infty)) > \mathcal{H}^1(\mathcal{F}(\Lambda^{-1}_{\infty,n_0})) = \infty.$$

For (5), suppose $x \in \Lambda^{-1}_\infty$ is such that $\Theta^1(\mathcal{H}^1, \Lambda^{-1}_\infty, x) > 0$.

Consider $\mathcal{F}(x)$ and let $\rho > 0$. We know firstly from definition that there is an $n_0 > 0$ such that $F_n(x) \in B_{\rho/2}(\mathcal{F}(x))$ for all $n > n_0$ and thus, since from the proof of (1) $diam(T_{n,i}) \to 0$ as $n \to \infty$, there is an $n_1 \geq n_0$ such that $diam(T_{n_1,\cdot}) < \rho/4$ and thus $\cup_{j=-1}^{1} T_{n_1,i(x,n_1)+j} \subset B_\rho(\mathcal{F}(x))$.

For the remainder of (5) we write $i := i(n_1, x)$ We now, temporarily have two cases to consider, namely CASE I that $F_n(x) \in E(T_{n_1,i})$ and CASE II that $F_n(x) \in T_{n_1,i} - E(T_{n_1,i})$.

CASE I:
In this case $F_{n_1}(x) \in T_{n_1,i} \cap T_{n_1,i-1}$ or $F_{n_1}(x) \in T_{n_1,i} \cap T_{n_1,i+1}$, without loss of generality let us suppose that it is the latter case. Then

$$x = i2^{-n_1} \in [(i-1)2^{-n_1}, (i+1)2^{-n_1}] = F_n^{-1}(A_{n,i} \cup A_{n,i+1}).$$

Since $\Theta^1(\mathcal{H}^1, \Lambda^{-1}_\infty, x) > 0$ it follows that $\mathcal{H}^1(\Lambda^{-1}_\infty \cap [(j-1)2^{-n_1}, j2^{-n_1}] > 0$ for at least one $j \in \{i, i+1\}$. Without loss of generality let us assume that $j = i$. Then

$$\begin{aligned}
\mathcal{H}^1(F_{n_1}^{-1}(\Lambda^{-1}_\infty) \cap A_{n_1,i}) &= \tilde{\Pi}_{n_1}(x)\mathcal{H}^1(\Lambda^{-1}_\infty \cap [(i-1)2^{-n_1}, i2^{-n_1}]) \\
&\geq \mathcal{H}^1(\Lambda^{-1}_\infty \cap [(i-1)2^{-n_1}, i2^{-n_1}]) \\
&> 0.
\end{aligned}$$

CASE II:
In this case $F_{n_1}(x) \in T_{n_1,i} - E(T_{n_1,i})$ so that

$$x \in ((i-1)2^{-n_1}, i2^{-n_1}) \subset [(i-1)2^{-n_1}, i2^{-n_1}] = F_{n_1}^{-1}(A_{n,i}).$$

Thus since $\Theta^1(\mathcal{H}^1, \Lambda^{-1}_\infty, x) > 0$ it follows that $\mathcal{H}^1(\Lambda^{-1}_\infty \cap [(i-1)2^{-n_1}, i2^{-n_1}]) > 0$ and therefore

$$\begin{aligned}
\mathcal{H}^1(F_{n_1}^{-1}(\Lambda^{-1}_\infty) \cap A_{n_1,i}) &= \tilde{\Pi}_{n_1}(x)\mathcal{H}^1(\Lambda^{-1}_\infty \cap [(i-1)2^{-n_1}, i2^{-n_1}]) \\
&\geq \mathcal{H}^1(\Lambda^{-1}_\infty \cap [(i-1)2^{-n_1}, i2^{-n_1}]) \\
&> 0.
\end{aligned}$$

That is, in either case there is a $n \in \mathbb{N}$ and $i \in \{1, ..., 2^n\}$ such that $T_{n,i} \subset B_\rho(\mathcal{F}(x))$ and $\mathcal{H}^1(F_n^{-1}(\Lambda^{-1}_\infty) > 0$. Applying (iv) to the $A_1 \in \mathcal{K}$ resulting from the subconstruction of A on $T_{n,i}$ it follows that $\mathcal{H}^1(\Lambda_\infty \cap T_{n,i}) = \infty$ and thus that $\mathcal{H}^1(B_\rho(\mathcal{F}(x)) \cap \Lambda_\infty) = \infty$. Since this is true for all $\rho > 0$ it follows that

$$\Theta^1(\mathcal{H}^1, \Lambda_\infty, \mathcal{F}(x)) = \infty,$$

completing the proof of (5).

For (6), we note that the open sets of A with respect to $\mathcal{H}^1$ measure are $U \cap A$ for U open in the usual sense in $\mathbb{R}^2$. Now consider an open set in A, $V := U \cap A$ for some U open in $\mathbb{R}^2$.

Let $\mathcal{T}_1 := \cup\{T_{1,i} : T_{1,i} \subset U\}$ and in general $\mathcal{T}_n := \cup\{T_{n,i} : T_{n,i} \subset U\}$. We claim that

$$V = \bigcup_{n=1}^{\infty} (\mathcal{T}_n \cap A).$$

Clearly $\mathcal{T}_n \cap A \subset U \cap A$ for all $n \in \mathbb{N}$ and thus $\cup_{n=1}^{\infty}(\mathcal{T}_n \cap A) \subset U \cap A = V$.

Conversely, let $x \in V$. Then $x \in A$ and there exists $\rho > 0$ such that $B_\rho(x) \subset U$. Since we know that for any $A \in \mathcal{K}$, and $x \in A$

$$\lim_{n\to\infty} diam(T_{n,i(n,x)}) = 0$$

there exists $n_\rho \in \mathbb{N}$ such that $diam(T_{n_\rho,i(n_\rho,x)}) < \rho/2$. Then $T_{n_\rho,i(n_\rho,x)} \subset B_\rho(x) \subset U$ thus $T_{n_\rho,i(n_\rho,x)} \subset \mathcal{T}_{n_\rho}$ and thus $x \in \mathcal{T}_{n_\rho}$.

Since $x \in A$ we have $x \in \mathcal{T}_{n_\rho} \cap A$ and thus $x \in \bigcup_{n=1}^{\infty}(\mathcal{T}_{n_{rho}} \cap A)$. It follows that $V \subset \bigcup_{n=1}^{\infty}(\mathcal{T}_{n_{rho}} \cap A)$. Now, for each $n \in \mathbb{N}$ $\mathcal{T}_n \cap A = \bigcup_{i\in I_n} T_{n,i} \cap A$ for some (possibly empty) index $I_n \subset \{0, 1, ..., 2^n - 1\}$. Thus $\mathcal{F}^{-1}(\mathcal{T}_n \cap A) = \bigcup_{i\in I_n} D_{n,i}$ where $D_{n,i}$ is the i-th dyadic interval of order n. Thus

$$\mathcal{F}^{-1}(V) = \mathcal{F}^{-1}\left(\bigcup_{n=1}^{\infty}(\mathcal{T}_n \cap A)\right) = \bigcup_{n=1}^{\infty}\mathcal{F}^{-1}(\mathcal{T}_n \cap A) = \bigcup_{n=1}^{\infty}\bigcup_{i\in I_n} D_{n,i}$$

which is a Borel set in $A_{0,0}$ and thus $\mathcal{H}^1$-measurable. It follows that for any Borel set $B \in A$ $\mathcal{F}^{-1}(B)$ is a Borel set in $A_{0,0}$. Thus, finally, if B is a $\mathcal{H}^1$-measurable set in A, $\mathcal{F}^{-1}(B)$ is a $\mathcal{H}^1$-measurable set in $A_{0,0}$. The fact that the measurability of the inverse images of measurable sets follows from the measurability of the inverse images of open sets is standard measure theory and is discussed in, for example, Rudin [24] or Bartle [4]. ♢

As a corollary we can now present the result mentioned in the previous section addressing the density of points in A_ε type sets. The density is important as it will be the key to the existence or non-existence of approximate tangent spaces to A, and therefore an essential ingredient in discussing the rectifiability of sets in $\mathcal{K}$.

Corollary 8.4.1.
Let A be an A_ε type set. Then either A is the image of a Lipschitz function or for each $y \in A$

$$\Theta^1(\mathcal{H}^1, A, y) = \infty.$$

Proof:
Since $\tilde{\Pi}^A(x)$ is constant in x either $\tilde{\Pi}^A \equiv \infty$ or $\tilde{\Pi}^A \equiv C < \infty$. In the first case Proposition 8.4.1 (5) gives immediately that $\Theta^1(\mathcal{H}^1, A, y) = \infty$ for all $y \in A$. In the latter case we see that $\Lambda_C^{-1} = A_0$, so that by Lemma 8.4.1 $\mathcal{F} = \mathcal{F}|_{A_0} = \mathcal{F}|_{\Lambda_C^{-1}} = \mathcal{F}_C$ is a Lipschitz function. ♢

We are now in a position to present the omitted proofs from Chapter 6 regarding the measure of A_ε and $\mathcal{A}_\varepsilon$. The result for A_ε can be presented directly. For convenience we restate the Lemma before providing it's proof.

Lemma 6.3.1
A_ε is not weakly locally $\mathcal{H}^1$-finite.

Proof:
By Proposition 6.3.1

$$(\Lambda_{\infty}^{-1})^{A_\varepsilon} = \{x \in A_0^{A_\varepsilon} : \prod_{n=0}^{\infty} (cos\theta_{n,i(n,x)}^{A_\varepsilon})^{-1} = \infty\} = A_0^{A_\varepsilon}$$

and thus $\Lambda_\infty = A_\varepsilon$. It then follows from Proposition 8.4.1 (4) that

$$\mathcal{H}^1(A_\varepsilon) = \mathcal{H}^1(\Lambda_\infty) = \infty. \tag{8.1}$$

Now, for each $y \in A_\varepsilon$ and each $\rho > 0$ there is an $n \in \mathbb{N}$ such that $y \in T_{n,i(n,y)} \subset B_\rho(y)$. This can be ensured since, by Lemma 4.3.1, $diam(T_{n,i(n,y)}) \to 0$ as $n \to \infty$. Since, by the symmetry of construction $A_\varepsilon \cap T_{n,i}$ is a $\mathcal{H}^1(A_{n,i(n,y)})$ scale copy of A_ε it follows that $\mathcal{H}^1(B_\rho(y) \cap A_\varepsilon) \geq \mathcal{H}^1(A \cap T_{n,i(n,y)}) = \mathcal{H}^(A_{n,i(n,y)})\mathcal{H}^(A_\varepsilon)$.

By (8.1) it follows that $\mathcal{H}^1(B_\rho(y) \cap A_\varepsilon) \not< \infty$. ♢

Before proving the necessary result concerning $\mathcal{A}_\varepsilon$ we need to complete it's construction in the sense that we need be more explicit about the selection of the set $\mathcal{B}^{\mathcal{A}_\varepsilon}$. We do this in the following definition.

Definition 8.4.2.
Let $\mathcal{F}^{A_\varepsilon}$ be the bijective function associated with A_ε. Let $\{d_i\}_{i=1}^{\infty}$ be an ordering of the dyadic points, D, in $[0,1] = A_0^{A_\varepsilon}$. Take $\tilde{B} := \{B_{\psi(,\varepsilon)i}(d_i)\}_{i=1}^{\infty}$ to be an open covering in $[0,1]$ of D such that $\sum_{i=1}^{\infty} 2p_i < 1/2$. We know this is possible since the dyadic points are countable and hence $\mathcal{H}^1(D) = 0$.

Define $B_1^{\mathcal{A}_\varepsilon} := A_0^{A_\varepsilon} \sim \cup_{B \in \tilde{B}} B$ so that $\mathcal{H}^1(B_1^{\mathcal{A}_\varepsilon}) > 0$, $B_1^{\mathcal{A}_\varepsilon}$ is closed and thus $B_1^{\mathcal{A}_\varepsilon}$ is compact.

Define then $B_2^{\mathcal{A}_\varepsilon} := \mathcal{F}^{A_\varepsilon}(B_1^{\mathcal{A}_\varepsilon})$. From Proposition 8.4.1 $B_2^{\mathcal{A}_\varepsilon}$ is compact. Since $B_1^{\mathcal{A}_\varepsilon} \cap D = \emptyset$, $E(A_\varepsilon) = \mathcal{F}^{A_\varepsilon}(D)$ and $\mathcal{F}^{A_\varepsilon}$ is a bijection it also follows that $B_2^{\mathcal{A}_\varepsilon} \cap E(A_\varepsilon) = \emptyset$. Further, since $B_2^{\mathcal{A}_\varepsilon}$ is compact $\mathbb{R}^2 \sim B_2^{\mathcal{A}_\varepsilon}$ is open. Combining these two fact we see that for each $x \in E(A_\varepsilon)$ there exists a $\rho_x > 0$ such that $B_{\rho_x}(x) \cap B_2^{\mathcal{A}_\varepsilon} = \emptyset$. Take any such collection of balls $B_{\rho_x}(x)$. We then define

$$\mathcal{B}^{\mathcal{A}_\varepsilon} := \bigcup_{x \in E(A_\varepsilon)} B_{\rho_x}(x).$$

Note that $B_2^{\mathcal{A}_\varepsilon} \subset A_\varepsilon \sim \mathcal{B}^{\mathcal{A}_\varepsilon} = \mathcal{A}_\varepsilon$.

Remark: Certainly there are many possible choices for $\mathcal{B}^{\mathcal{A}_\varepsilon}$ and in this sense we have still not given a definite definition of $\mathcal{A}_\varepsilon$. For our purposes, however, it does not matter which set of admissible radii ρ_x are chosen. Any set of radii can thus be chosen and then fixed, providing a definite definition of $\mathcal{A}_\varepsilon$.

Lemma 6.3.2
$\mathcal{H}^1(\mathcal{A}_\varepsilon) = \infty$.

Proof:
Since $\mathcal{A}_\varepsilon \supset B_2^{\mathcal{A}_\varepsilon}$ we see that $(\mathcal{F}^{A_\varepsilon})^{-1}(\mathcal{A}_\varepsilon) \supseteq B_1^{\mathcal{A}_\varepsilon}$. Now, by definition $B_2^{\mathcal{A}_\varepsilon} = \mathcal{F}^{A_\varepsilon}(B_2^{\mathcal{A}_\varepsilon})$ and $\mathcal{H}^1(B_1^{\mathcal{A}_\varepsilon}) > 0$. Further, by Proposition 6.3.1 $\tilde{\Pi}^{\mathcal{A}_\varepsilon}(x) = \infty$ for all $x \in \mathcal{A}_\varepsilon$ and thus for all $x \in B_1^{\mathcal{A}_\varepsilon}$. Hence $(\Lambda_\infty^{-1})^{\mathcal{A}_\varepsilon} \supseteq B_1^{\mathcal{A}_\varepsilon}$ and thus using Definition 8.4.2 $\mathcal{H}^1((\Lambda_\infty^{-1})^{\mathcal{A}_\varepsilon}) > 0$. It then follows from Proposition 8.4.1 (4) that $\mathcal{H}^1(\mathcal{A}_\varepsilon) \geq \mathcal{H}^1((\Lambda_\infty)^{\mathcal{A}_\varepsilon}) = \infty$. ♢

To complete the preliminary results required for our study of measure and rectifiability of sets in $\mathcal{K}$ we have one more lemma concerning density to consider. It is this final general density lemma that will be applied in the proof of non-rectifiability of those Koch sets which are not rectifiable (which ones they are will be made clear later). It shows the presence of infinite density almost everywhere in the image of any measurable subset of Λ_∞^{-1} of positive measure. In order to prove this Lemma, however, we first need a couple of general measure theoretic results showing that the set of points density one are sufficiently large in a set of positive measure in $A_{0,0}$. The second is a condition of non-rectifiability.

Proposition 8.4.2.
Let $B \subset A_0$ be $\mathcal{H}^1$-measurable, then

$$\mathcal{H}^1(\{x \in B : \Theta^1(\mathcal{H}^1, B, x) = 1\}) = \mathcal{H}^1(B).$$

Proof:
Since B is $\mathcal{H}^1$-measurable we know that for all $\rho > 0$

$$1 = (2\rho)^{-1}\mathcal{H}^1(B_\rho(x)) = (2\rho)^{-1}(\mathcal{H}^1(B_\rho(x) \cap B) + \mathcal{H}^1(B_\rho(x) \cap B^c))$$

so that

$$1 = \lim_{\rho \to 0}(2\rho)^{-1}(\mathcal{H}^1(B_\rho(x) \cap B) + \mathcal{H}^1(B_\rho(x) \cap B^c)) = \Theta^1(\mathcal{H}^1, B, x) + \Theta^1(\mathcal{H}^1, B^c, x).$$

From standard theory (see for example [Simon3] Theorem 3.5) we know $\Theta^1(\mathcal{H}^1, C, x) = 0$ for $\mathcal{H}^1$-almost all $x \in C^c$ for any $\mathcal{H}^1$-measurable set C with $\mathcal{H}^1(C) < \infty$. Hence $\Theta^1(\mathcal{H}^1, B^c, x) = 0$ for $\mathcal{H}^1$-almost all $x \in B$ and thus

$$\Theta^1(\mathcal{H}^1, B, x) = 1 - \Theta^1(\mathcal{H}^1, B^c, x) = 1$$

for $\mathcal{H}^1$-almost all $x \in B$. The result follows. ♢

Proposition 8.4.3.
Let $A \subset \mathbb{R}^2$. Let θ be an $L^1(\mathcal{H}^1, \mathbb{R}^2, \mathbb{R})$ positive function on A. Suppose that B is a subset of A of positive measure that satisfies $\theta(x) \geq r > 0$ for all $x \in B$. Let $x \in B$ satisfy

$$\Theta^1(\mathcal{H}^1, A, x) \geq \Theta(\mathcal{H}^1, B, x) = \infty.$$

Then A does not have a 1-dimensional approximate tangent plane for A at x with respect to θ.

Proof:
Let P be any potential approximate tangent plane for A at x with respect to any potential multiplicity function θ and let $\phi \in C_c^0(\mathbb{R}^2; \mathbb{R})$ be radially symmetric with $\chi_{B_1(0)} \leq \phi \leq \chi_{B_2(0)}$ and $\theta(x)\int_P \phi d\mathcal{H}^1 = C_\phi$ for each $P \in G(1, 2)$. Then we note

$$\begin{aligned}
C_\phi &= \lim_{\lambda \to 0} \lambda^{-1} \int_A \phi(\lambda^{-1}(z - x))\theta(z) d\mathcal{H}^1(z) \\
&\geq \lim_{\lambda \to 0} \lambda^{-1} \int_{B \cap B_\lambda(x)} \theta(z) d\mathcal{H}^1 \\
&= r \lim_{\lambda \to 0} \lambda^{-1} \frac{\mathcal{H}^1(B \cap B_\lambda(x))}{\lambda} \\
&= 2r\Theta^1(\mathcal{H}^1, A, x) \\
&= \infty.
\end{aligned}$$

It is therefore impossible that A have an approximate tangent plane at x with respect to θ. ♢

Lemma 8.4.2.
Let $A \in \mathcal{K}$ and $\mathcal{H}^1(B \cap \Lambda_\infty^{-1}) > 0$ for some measurable subset $B \subset A_{0,0}$. Then there exists

$$B_1 \subset B \cap \Lambda_\infty^{-1},$$

$$\mathcal{H}^1(B_1) = \mathcal{H}^1(B \cap \Lambda_\infty^{-1})$$

such that

$$\Theta^1(\mathcal{H}^1, \mathcal{F}(B_1), \mathcal{F}(x)) = \infty$$

for all $x \in B_1$.

In particular, if $A \in \mathcal{K}$ and $\mathcal{H}^1(\Lambda_\infty^{-1}) > 0$, then for $\mathcal{H}^1$-a.e. $x \in \Lambda_\infty^{-1}$

$$\Theta^1(\mathcal{H}^1, A, \mathcal{F}(x)) \geq \Theta^1(\mathcal{H}^1, \Lambda_\infty, \mathcal{F}(x)) = \infty.$$

Proof:
We note from Proposition 8.4.2 that $\Theta^1(\mathcal{H}^1, B \cap \Lambda_\infty^{-1}, x) = 1$ for $\mathcal{H}^1$-a.e. $x \in B \cap \Lambda_\infty^{-1}$. We thus choose

$$B_1 := \{x \in B \cap \Lambda_\infty^{-1} : \Theta^1(\mathcal{H}^1, B \cap \Lambda_\infty^{-1}, x) = 1\},$$

noting that $\mathcal{H}^1(B_1) = \mathcal{H}^1(B \cap \Lambda_\infty^{-1})$ as required.

Choose $y \in B_1$ arbitrarily. We then note that from the definition of Θ^1 there must exist an $r_0 > 0$ so that for all $r \leq r_0$, $(2r)^{-1}\mathcal{H}^1(B_r(y) \cap B_1) > 7/8$.

We now claim that for any dyadic interval $D \ni y$ with $|D| := \mathcal{H}^1(D) < r_0/2$ $\mathcal{H}^1(D \cap B_1) > 3/4|D|$. We see this by selecting $\gamma := \max\{d(y,z) : z \in E(D)\}$ (where $E(D)$ as elsewhere denotes the endpoints of D). Then $\gamma < |D| < r_0$ and $D \subset \overline{B_\gamma(y)}$. Thus

$$\frac{\mathcal{H}^1(B_1^c \cap B_\gamma(y))}{2\gamma} = \frac{1 - \mathcal{H}^1(B_1 \cap B_\gamma(y))}{2\gamma} < \frac{1}{8}$$

which implies

$$\frac{\mathcal{H}^1(B_1^c \cap D)}{|D|} \leq \frac{2\mathcal{H}^1(B_1^c \cap B_\gamma(y))}{2\gamma} < \frac{1}{4}$$

and thus

$$\frac{\mathcal{H}^1(B_1 \cap D)}{|D|} = \frac{|D| - \mathcal{H}^1(B_1^c \cap D)}{|D|} > (|D| - |D|/4)|D|^{-1} = \frac{3}{4},$$

proving the claim.

In particular, the claim holds for any dyadic interval $D_m \ni y$ of order $m \geq m_0$ where m_0 is chosen such that $2^{-m_0} \leq r_0$. Then, selecting, independently from one another, $1 > p > 0$ and $M \in \mathbb{R}$ with

$$\mathcal{H}^1(A_{m_0,i(y,m_0)}) = diam(T_{m_0,i(y,m_0)}) > p.$$

We choose $m \geq m_0$ such that $diam(T_{m,i(y,m)}) \in (p/2, 2p)$ and $T_{m,i(y,m)} \subset B_p(y)$. Note that $F_m^{-1}(A_{m,i(y,m)}) = A \cap T_{m,i(y,m)}$. That is, defining $\mathcal{B}_1 := \mathcal{F}(B_1)$

$$\mathcal{H}^1(B_p(y) \cap \mathcal{B}_1) \geq \mathcal{H}^1(\mathcal{B}_1 \cap T_{m,i(y,m)}) = \mathcal{H}^1(\mathcal{F}(D_m) \cap \mathcal{B}_1).$$

Since, for all $x \in B_1 \cap D_m$, $\prod_{m=0}^{\infty}(cos\theta_{n,i(x,n)})^{-1} = \infty$ there exists a $q_0 \in \mathbb{N}$ such that for each $q \geq q_0$ and defining

$$B^q := \left\{ x \in B_1 \cap D_m : \prod_{n=m+1}^{q} (cos\theta_{n,i(x,n)})^{-1} > M \right\}$$

we have

$$\mathcal{H}^1(B^q) > \mathcal{H}^1(B_1 \cap D_m)/2.$$

If this were not true then since $B^q \subset B^{q+1}$ for each q it would follow that

$$\mathcal{H}^1\left(\bigcup_{q=m+1}^{\infty} B^q\right) \leq \frac{\mathcal{H}^1(B_1 \cap D_m)}{2}$$

and thus there would exist $x \in B_1 \cap D_m$ such that $\prod_{n=m+1}^{\infty}(cos\theta_{n,i(x,n)})^{-1} < M < \infty$. This contradiction confirms the above statement.

We then note

$$\mathcal{H}^1(B_1) > \frac{1}{2}\mathcal{H}^1(B_1 \cap D_m) > \frac{3}{8}|D_m|$$

and that since for all $x \in D_m$, for all $x \in B_1$

$$\prod_{n=0}^{\infty}(cos\theta_{n,i(x,n)})^{-1} \geq \prod_{n=0}^{m}(cos\theta_{n,i(x,n)})^{-1} > \frac{p}{|D_m|}.$$

Denoting dyadic intervals of order q by D_q, it then follows that for $\tilde{y} := \mathcal{F}(y)$

$$\begin{aligned}
\mathcal{H}^1(\mathcal{B}_1 \cap B_p(\tilde{y})) &\geq \mathcal{H}^1(\mathcal{B}_1 \cap T_{m,i(y,m)}) \\
&\geq \mathcal{H}^1\left(\mathcal{B}_1 \cap \bigcup_{D_{q_0} \cap B^{q_0} \neq \emptyset} T_{q_0,i(D_{q_0},q_0)}\right) \\
&= \sum_{D_{q_0} \cap B^{q_0} \neq \emptyset} \mathcal{H}^1(\mathcal{B}_1 \cap T_{q_0,i(D_{q_0},q_0)}) \\
&\geq \sum_{D_{q_0} \cap B^{q_0} \neq \emptyset} \mathcal{H}^1(F_{q_0}(D_{q_0} \cap B_1)) \\
&= \sum_{D_{q_0} \cap B^{q_0} \neq \emptyset} \prod_{n=0}^{q_0}(cos\theta_{q_0,i(D_{q_0},q_0)})^{-1}\mathcal{H}^1(D_{q_0} \cap B_1) \\
&> \frac{p}{|D_m|} \sum_{D_{q_0} \cap B^{q_0} \neq \emptyset} \prod_{n=m+1}^{q_0}(cos\theta_{q_0,i(D_{q_0},q_0)})^{-1}\mathcal{H}^1(D_{q_0} \cap B_1) \\
&> \frac{Mp}{|D_m|} \sum_{D_{q_0} \cap B^{q_0} \neq \emptyset} \mathcal{H}^1(D_{q_0} \cap B_1) \\
&= \frac{Mp}{|D_m|}\mathcal{H}^1\left(\bigcup_{D_{q_0} \cap B^{q_0} \neq \emptyset} D_{q_0} \cap B_1\right) \\
&= \frac{Mp}{|D_m|}\mathcal{H}^1(B^{q_0}) \\
&> \frac{Mp}{2|D_m|}\mathcal{H}^1(B^1 \cap D_m) \\
&> \frac{3Mp}{8|D_m|}|D_m| \\
&= \frac{3Mp}{8}.
\end{aligned}$$

Since this is true for any $p < diam(T_{m_0,i(y,m_0)})$

$$\Theta^1(\mathcal{H}^1, \mathcal{B}_1, \tilde{y}) = \lim_{p \searrow 0} \frac{\mathcal{H}^1(\mathcal{B}_1 \cap B_p(\tilde{y}))}{2p} \geq \frac{3M}{16}.$$

Since this is true for each $M \in \mathbb{R}$ we have

$$\Theta^1(\mathcal{H}^1, \mathcal{B}_1, \tilde{y}) = \infty.$$

As this is true for any $y \in B_1$ it follows that

$$\Theta^1(\mathcal{H}^1, \mathcal{F}(B_1), \mathcal{F}y) = \infty$$

for each $y \in B_1$ completing the first part of the proof.

For the final part of the proof we note that $A_{0,0}$ is itself measurable and that $A_{0,0} \cap \Lambda_\infty^{-1} = \Lambda_\infty^{-1}$. It follows from the above that there is a set $B \subset \Lambda_\infty^{-1}$ with $\mathcal{H}^1(B) = \mathcal{H}^1(\Lambda_\infty^{-1})$ so that

$$\Theta^1(\mathcal{H}^1, \mathcal{F}(B), \mathcal{F}(x)) = \infty$$

for all $x \in B$. Since $A_{0,0} \supset \Lambda_\infty^{-1} \supset B$, $\mathcal{F}(A_{0,0}) = A$, $\mathcal{F}(\Lambda_\infty^{-1}) = \Lambda_\infty$ and $\mathcal{H}^1$-a.e. $x \in \Lambda_\infty^{-1}$, $x \in B$ it follows that for all $x \in B$ and thus $\mathcal{H}^1$-a.e. in Λ_∞^{-1}

$$\Theta^1(\mathcal{H}^1, A, \mathcal{F}(x)) \geq \Theta^1(\mathcal{H}^1, \Lambda_\infty, \mathcal{F}(x)) \geq \Theta^1(\mathcal{H}^1, \mathcal{F}(B), \mathcal{F}(x)) = \infty,$$

which completes the proof. ♢

This completes the preliminary results that we need for the rectifiability and measure results on sets in $\mathcal{K}$.

8.5 Relative Centralisation of Semi-Self-Similar Sets

We now look at some preliminary results that we will need for results on dimension. We will reduce all of our questions to an application of the results of Hutchinson [15] to get our dimension results. We do this, in essence, by a comparison principle. We show that sets in $\mathcal{K}$ depending on properties of $\tilde{\theta}^A$ can be dimension invariantly rearranged so that they are supersets of some sets to which Hutchinsons results apply and subsets of others. By considering sequences of such rearrangements we can deduce the dimension of our sets from the dimensions of the sets to which we are comparing.

It is in fact true that we could, in principle, apply Hutchinson's results directly. However, the parameters of the sets and "self-similarity" functions cannot be (at least not easily) extracted from sets in $\mathcal{K}$. Thus actually giving an explicit dimension directly is not possible.

Our comparison principle, or rearrangement involves separating each triangular cap in a particular approximation to some $A \in \mathcal{K}$, T_n and moving each separately by an orthogonal transformation in such a way that each of the newly positioned triangular caps remain disjoint. We do this by placing each inside of a triangular cap of another, larger, T_n from some other $A' \in \mathcal{K}$. Since all Hausdorff measures are translation invariant it follows that Hausdorff dimension is also translation invariant and thus the union of the replaced triangular caps is the same dimension as the original caps. We can in this way compare the dimension of A to that of each $T_n^{A'}$ and thus of A'. It will be by selecting appropriate A' that we will prove our dimension results.

We start by defining the transformation process, which, due to the placing of one set into parts of another, we call centering. That is one set is centered in the bigger one.

Definition 8.5.1.
Let $A_1, A_2 \subset \mathbb{R}^2$. We say that we can center A_1 in A_2 (or that A_1 can be centered in A_2) written $A_1 \hookrightarrow^c A_2$ if for each $m \in \mathbb{N}$ there exists sets A_{1m} and A_{2m} such that

$$\bigcap_{m=1}^{\infty} A_{2m} \subset A_2, \quad A_{2m} \subset A_{2(m-1)} \quad \textit{for all} \quad m \in \mathbb{N},$$

$$A_1 \subset \bigcap_{m=1}^{\infty} A_{1m}, \quad A_{1m} \subset A_{1(m-1)} \quad \textit{for all} \quad m \in \mathbb{N};$$

that for each $m \in \mathbb{N}$ there exists $n_1(m), n_2(m) \in \mathbb{N}$, $n_1(m) \leq n_2(m)$, disjoint sets $\{A_{1mj}\}_{j=1}^{n_1(m)}$ and disjoint sets $\{A_{2mj}\}_{j=1}^{n_2(m)}$ such that

$$\bigcup_{j=1}^{n_2(m)} A_{2mj} \subseteq A_{2m} \textit{ and } A_{1m} \subseteq \bigcup_{j=1}^{n_1(m)} A_{1mj};$$

that the sets A_i, A_{im} and A_{imj} are all $\mathcal{H}^a$-measurable for $i = 1, 2$ each $a \in \mathbb{R}$ and appropriate $m, j \in \mathbb{N}$ and that there exist orthogonal transformations $\mathcal{T}_{m,j}^{A_1,A_2} : \mathbb{R}^2 \to \mathbb{R}^2$ for $j = 1, ..., n_1(m)$ such that

$$\mathcal{T}_{mj}^{A_1,A_2}(A_{1mj}) \subseteq A_{2mj}.$$

If $A_1 \hookrightarrow^c A_2$ we write

$$C_n^{A_1,A_2} := \bigcup_{j=1}^{n_1(m)} \mathcal{T}_{m,j}^{A_1,A_2}(A_{1mj}).$$

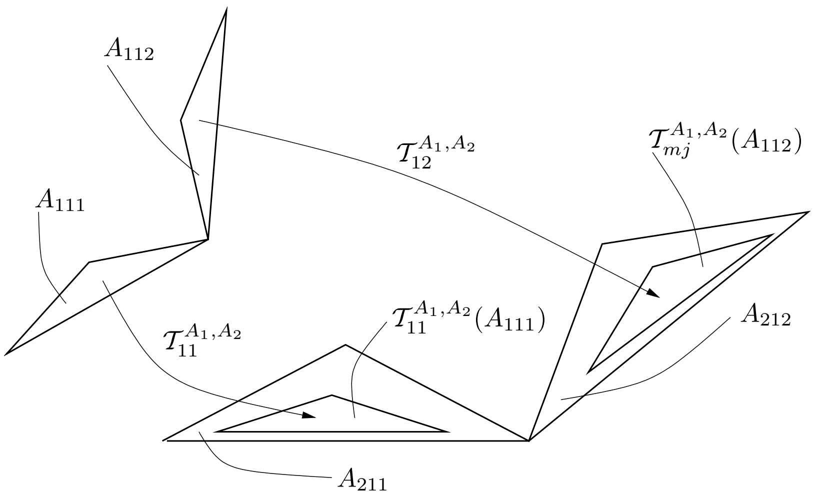

Figure 8.1: Centralisation of semi-self-similar sets

Remark:
For any $A_1, A_2 \in \mathcal{K}$ we can set $n_1(m) = n_2(m) = 2^m$ and for each $i \in \{1, 2\}$ $A_{im} := T_m := \cup_{j=1}^{2^m} T_{m,j}$ and $A_{imj} = T_{m,j}$. In this case, as we shall see, if $\theta_{n,i}^{A_1} \leq \theta_{n,i}^{A_2}$ for each n and i, we have, ignoring the negligible set of edge points E, $A_1 \hookrightarrow^c A_2$.

It would have been a simpler statement of definition to restrict to the case $A_1, A_2 \in \mathcal{K}$. However, as we shall see we will need to apply the definition where A_1 and A_2 are subsets of elements of $\mathcal{K}$ where certain triangular caps have been simply removed in the construction of A_1 and A_2. In any case, to make the definition intuitively easier to understand we may always think of each A_i as an element of $\mathcal{K}$ with triangular caps removed, each A_{im} as a union of a subcollection of the $T_{m,j}^{A_i}$ and each A_{imj} as a $T_{m,j}^{A_i}$.

In the case that A_1 and A_2 are actually in $\mathcal{K}$ we can restate the definition as follows:

Definition 8.5.2. $\mathcal{K}$ **version**
Let A_1 and A_2 be A_ε type sets. We say that we can center A_1 in A_2 (or that A_1 can be centered in A_2) written $A_1 \hookrightarrow^c A_2$ if for each $n \in \mathbb{N}$ and $i \in \{1, .., 2^n\}$ there are orthogonal transformations $\mathcal{T}_{n,i}^{A_1,A_2}$ such that $\mathcal{T}_{n,i}^{A_1,A_2}(T_{n,i}^{A_1}) \subset T_{n,i}^{A_2}$.

If $A_1 \hookrightarrow^c A_2$ then we write

$$C_n^{A_1,A_2} := \bigcup_{i=1}^{2^n} \mathcal{T}_{n,i}^{A_1,A_2}(T_{n,i}^{A_1}).$$

We now look at some properties relating to centering. The first is more a property of A_ε type sets that gives a condition allowing one A_ε type set to be centered into another. The second is a version of the important Lemma 6.2.1 for general sets $A \in \mathcal{K}$. A corollary makes explicit the property for which this second result is necessary. Thirdly we show that any fixed portion of Γ_ε, no matter how small, is sufficient as a comparison set with dimension equal to that of Γ_ε. Finally we show a more general result showing that the dimension comparison works (at least in a sense sufficient for our needs), thus justifying the use of centering.

Proposition 8.5.1.
Let A_1 and A_2 be A_ε type sets. Let $\theta_{n,\cdot}^{A_1}$ be denoted by $\theta_n^{A_1}$ and $\theta_{n,\cdot}^{A_2}$ be denoted by $\theta_n^{A_2}$ for each $n \in \mathbb{N}$. (The $\theta_{n,\cdot}^{A}$ denots $\theta_{n,i}^{A_1}$ for any choice of i. We can drop the i in this case because for A_ε type sets $\theta_{n,\cdot}$ is independent of i.) Then, if $T_{0,1}^{A_1} \subseteq T_{0,1}^{A_2}$ and $\theta_n^{A_1} \leq \theta_n^{A_2}$ for each $n \in \mathbb{N}$ then $A_1 \hookrightarrow^c A_2$.

Proof:
We know that $T_{0,1}^{A_1} \subseteq T_{0,1}^{A_2}$ so that by denoting the identity transformation by ι we have $\mathcal{T}_{0,1}^{A_1,A_2} \equiv \iota$ and thus

$$\mathcal{T}_{0,1}^{A_1,A_2}(T_{0,1}^{A_1}) \subset T_{0,1}^{A_2}.$$

We then continue the proof by induction on n. Assume that

$$\mathcal{T}_{n,i}^{A_1,A_2}(T_{n,i}^{A_1}) \subset T_{n,i}^{A_2}$$

for some $n \in \mathbb{N}_0$ and each $i \in \{1, ..., 2^n\}$. Consider some arbitrarily chosen $j \in \{1, ..., 2^n\}$ with $\mathcal{T}_{n,j}^{A_1,A_2}(T_{n,j}^{A_1}) \subset T_{n,j}^{A_2}$ and therefore since $diam(T_{n,i}^{A}) = \mathcal{H}^1(A_{n,i}^{A})$ for each A_ε type set A it follows that

$$\mathcal{H}^1(A_{n,j}^{A_1}) \leq \mathcal{H}^1(A_{n,j}^{A_2}).$$

Now, $\theta^{A_1}_{n+1} \leq \theta^{A_2}_{n+1}$ by hypothesis and thus also, by Lemma 8.1.1

$$\mathcal{H}^1(A^{A_1}_{n+1,2j+k}) = \frac{1}{2}(cos(\theta^{A_1}_{n+1}))^{-1}\mathcal{H}^1(A^{A_1}_{n,j}) \leq \frac{1}{2}(cos(\theta^{A_2}_{n+1}))^{-1}\mathcal{H}^1(A^{A_2}_{n,j}) = \mathcal{H}^1(A^{A_2}_{n+1,2j+p})$$

for each $k, p \in \{-1, 0\}$.

Combining these, it follows that $T^{A_1}_{n+1,2j+k}$ can be mapped into $T^{A_2}_{n+1,2j+k}$ by placing $A^{A_1}_{n+1,2j+k}$ in the center of $A^{A_2}_{n+1,2j+k}$ for $k \in \{-1, 0\}$. By defining $\mathcal{T}^{A_1,A_2}_{n+1,2j+k}$ to be the orthogonal transformation that does this it follows that

$$\mathcal{T}^{A_1,A_2}_{n+1,2j+k}(T^{A_1}_{n+1,2j+k}) \subset T^{A_2}_{n+1,2j+k}$$

for $k \in \{-1, 0\}$. Since j was arbitrary we have $\mathcal{T}^{A_1,A_2}_{n+1,i}$ such that

$$\mathcal{T}^{A_1,A_2}_{n+1,i}(T^{A_1}_{n+1,i}) \subset T^{A_2}_{n+1,i}$$

for all $i \in \{1, ..., 2^{n+1}\}$, which completes the inductive step in n. $\diamondsuit$

Lemma 8.5.1.
Suppose that $A \in \mathcal{K}$ and that $\Gamma_\varepsilon \hookrightarrow^c A$ for some $\varepsilon > 0$. Then

1. *should two neighbouring caps $T^A_{n,i}$ and $T^A_{n,i+1}$ be contained in another necessarily earlier triangular cap $T^A_{m,j(i)}$ $(m < n)$ then*

$$\psi^{T^A_{n,i}} T^A_{n,i+1} \leq 2\theta^A_{m,j(i)} \leq 2\theta^A_{0,1},$$

2. *As in Lamme 6.2.1 should $2 \leq |i - j| \leq 3$ then*

$$\pi_x \left(O_{A^A_{n,i}} \left(\bigcup_{j:|i-j|<2} T^A_{n,j} \right) \right) \cap \pi_x(O_{A^A_{n,i}}(T^A_{n,j}) - \{z_{n,i-2}, z_{n,i+1}\}) = \emptyset;$$

similarly should $|i - j| = 1$ then

$$\pi_x \left(O_{A^A_{n,i}}\left(T^A_{n,j}\right) \right) \cap \pi_x(O_{A^A_{n,i}}(T^A_{n,j}) - \{z_{n,i-1}, z_{n,1}\}) = \emptyset,$$

and

3. *for each $n \in \mathbb{N}$ and $i \in \{1, ..., 2^n\}$*

$$T^A_{n,j} \cap N^{\mathcal{H}^1(A^{\Gamma_\varepsilon}_{n,i})}(T^A_{n,i}) = \emptyset$$

for each j such that $|i - j| \geq 2$.

Proof:
Since by definition (Definition 8.1.2) we see that $\theta^A_{0,1} < \pi/32$ and should $T^A_{n,i} \subset T^A_{n,j}$ then $\theta^A_{n,i} \leq \theta^A_{m,j}$. Since these are the only properties of triangular caps and their related angles necessary to prove claims (1) and (2) for Γ_ε and A_ε in Lemma 6.2.1 we see that the proof holds here in an identical manner.

For (3) we see that the claim is trivial for $A^A_{0,1}$ and $A^A_{1,i}$ $(i = 1, 2)$ since there is nothing to check in that there is no i, j pairing in either A_0^A or A_1^A such that $|i-j| \geq 2$.

We now prove the result by induction on n. That is, we assume that for some $p \in \mathbb{N}$ and for each $i \in \{1, ..., 2^p\}$

$$T_{p,j}^A \cap N^{\mathcal{H}^1(A_{p,i}^{\Gamma_\varepsilon})}(T_{p,i}^A) = \emptyset$$

for each j such that $|i-j| \geq 2$, and show that the same holds for $p+1$. Take any $i \in \{1, ..., 2^{p+1}\}$ and note that $T_{p+1,i}^A \subset T_{p,i_1}^A$ (for i_1 equal to the integer element of $\{i/2, (i+1)/2\}$). Since $\mathcal{H}^1(A_{p+1,i}^{\Gamma_\varepsilon}) < \mathcal{H}^1(A_{p,i_1}^{\Gamma_\varepsilon})$ it follows that

$$N^{\mathcal{H}^1(A_{p+1,i}^{\Gamma_\varepsilon})}(T_{p+1,i}^A) \subset N^{\mathcal{H}^1(A_{p+1,i_1}^{\Gamma_\varepsilon})}(T_{p,i_1}^A).$$

It follows that $N^{\mathcal{H}^1(A_{p+1,i}^{\Gamma_\varepsilon})}(T_{p+1,i}^A) \cap T_{p+1,j}^A = \emptyset$ for each j such that $T_{p+1,j}^A \not\subset T_{p,k}^A$ for some $k \in \{i_1-1, i_1, i_1+1\}$.

Without loss of generality we can assume that the affected $j \in \mathbb{N}$ are those satisfying $j \notin \{i-2, ..., i+3\}$. It therefore only remains to show that

$$N^{\mathcal{H}^1(A_{p+1,i}^{\Gamma_\varepsilon})}(T_{p+1,i}^A) \subset N^{\mathcal{H}^1(A_{p+1,i_1}^{\Gamma_\varepsilon})}(T_{p,i_1}^A)$$

for $j \in \{i-2, i+2, i+3\}$.

We note now that for each such j, $|i-j| \leq 3$ and in particular

$$|(i-1)-(i-2)| \leq 3, \quad |(i+1)-(i+2)| \leq 3, \quad |(i+1)-(i+3)| \leq 3 \text{ and } |(i \pm 1) - i| \leq 3.$$

We also note that since $\Gamma_\varepsilon \hookrightarrow^c A$ and that since $\mathcal{H}^1(A_{p+1,i}^{\Gamma_\varepsilon})$ is constant in i it follows that $\mathcal{H}^1(A_{p+1,i}^A) \geq \mathcal{H}^1(A_{p+1,j}^{\Gamma_\varepsilon})$ for each $i, j \in \{1, ..., 2^{p+1}\}$.

It now follows from (2) in the hypothesis of this lemma that

$$O_{A_{p+1,i-1}^A}(T_{p+1,i_2}^A) \subset (-\infty, \mathcal{H}^1(A_{p+1,i-1}^A)/2] \times \mathbb{R} \subseteq (-\infty, -\mathcal{H}^1(A_{p+1,i}^{\Gamma_\varepsilon})/2] \times \mathbb{R}$$

and that

$$O_{A_{p+1,i-1}^A}(T_{p+1,i}^A) \subset [\mathcal{H}^1(A_{p+1,i-1}^A)/2, \infty) \times \mathbb{R} \subseteq [\mathcal{H}^1(A_{p+1,i}^{\Gamma_\varepsilon})/2, \infty) \times \mathbb{R}.$$

Thus

$$d(T_{p+1,i_2}^A, T_{p+1,i}^A) = d(O_{A_{p+1,i-1}^A}(T_{p+1,i-2}^A), O_{A_{p+1,i-1}^A}(T_{p+1,i}^A)) \geq \mathcal{H}^1(A_{p+1,i}^{\Gamma_\varepsilon}).$$

It follows that

$$T_{p,i-2}^A \cap N^{\mathcal{H}^1(A_{p+1,i}^{\Gamma_\varepsilon})}(T_{p+1,i}^A) = \emptyset.$$

The same principles apply to give essentially identical arguments showing that

$$T_{p,i+j}^A \cap N^{\mathcal{H}^1(A_{p+1,i}^{\Gamma_\varepsilon})}(T_{p+1,i}^A) = \emptyset$$

for $j \in \{2, 3\}$ completing the proof. $\diamond$

The above lemma can also be seen as a restriction on spiralling for sets in $\mathcal{K}$. The particular application of the above lemma that we will be needing is expressed in the following corollary.

Corollary 8.5.1.
Let $A \in \mathcal{K}$ be such that $\Gamma_\varepsilon \hookrightarrow^c A$ for some $\varepsilon > 0$. Then for any $n \in \mathbb{N} \cup \{0\}$, $i \in \{1, ...2^n\}$ and any ball $B_p(x)$ satisfying

1. $p \in (\mathcal{H}^1(A^{\Gamma_\varepsilon}_{n,i})/2, \mathcal{H}^1(A^{\Gamma_\varepsilon}_{n,i})]$, *and*

2. $B_p(x) \cap T^A_{n,i} \neq \emptyset$,

then $B_p(x) \cap T^A_{n,j} \neq \emptyset$ *for maximally 3 different* $j \in \{1, ..., 2^n\}$.

Proof:
From (1) and (2) in the hypothesis of this corollary it follows that $B_p(x) \subset N^{\mathcal{H}^1(A^{\Gamma_\varepsilon}_{n,i})}(T^A_{n,i})$. Thus a triangular cap, $T^A_{n,j}$, can only meet $B_p(x)$ if it meets $N^{\mathcal{H}^1(A^{\Gamma_\varepsilon}_{n,i})}(T^A_{n,i})$.

Since, from Lemma 8.5.1 $T^A_{n,j} \cap N^{\mathcal{H}^1(A^{\Gamma_\varepsilon}_{n,i})}(T^A_{n,i}) = \emptyset$ for $|i-j| \geq 2$ it follows that the only triangular caps, $T^A_{n,j}$, that can meet $N^{\mathcal{H}^1(A^{\Gamma_\varepsilon}_{n,i})}(T^A_{n,i})$ and thus $B_p(x)$ are $T^A_{n,j}$ for $j \in \{i-1, i, i+1\}$. Since $|\{i-1, i, i+1\}| = 3$ the proof is complete. ◊

Proposition 8.5.2.
Suppose

$$\tilde{\Gamma_\varepsilon} = \bigcap_{n=0}^{\infty} \bigcup_{j \in I_n} T^{\Gamma_\varepsilon}_{n,j}$$

where $I_n \subset \{1, ..., 2^n\}$ *and* $|I_n| \geq 2^{n-2n_0}$ *for some fixed* $n_0 \in \mathbb{N}$ *and each* $n \in \mathbb{N} \cup \{0\}$. *Then*

$$dim\tilde{\Gamma_\varepsilon} = dim\Gamma_\varepsilon.$$

Proof:
Since $\tilde{\Gamma_\varepsilon} \subset \Gamma_\varepsilon$ it is clear that $\dim\tilde{\Gamma_\varepsilon} \leq \dim\Gamma_\varepsilon$. We need to show that $\dim\Gamma_\varepsilon \leq \dim\Gamma_\varepsilon$.

Let $\eta \in \mathbb{R}$ be such that $\mathcal{H}^\eta(\Gamma_\varepsilon) = \infty$. There there exists a $\delta_0 > 0$ such that for each $\delta < \delta_0$ $\mathcal{H}^\eta_\delta(\Gamma_\varepsilon) > M > 0$.

Let now $\{B^\delta_j\}_{j=1}^\infty$ be a $\delta/5$ cover of $\tilde{\Gamma_\varepsilon}$. Since $\tilde{\Gamma_\varepsilon}$ is a countable intersection of compact sets it is a compact set and thus for some $Q \in \mathbb{N}$, $\{B^\delta_j\}_{j=1}^Q$ is a $\delta/5$ cover of $\tilde{\Gamma_\varepsilon}$. Without loss of generality we can further assume that $\{diam(B^\delta_j)\}_{j=1}^Q$ is decreasing in j.

For each $j \in \{1, .., Q\}$ there exists $n_j \in \mathbb{N} \cup \{0\}$ such that

$$diam(B^\delta_j) \in (diam(T^{\Gamma_\varepsilon}_{n_j,i})/2, diam(T^{\Gamma_\varepsilon}_{n_j,i})].$$

By selecting δ_0 small enough, we can assume that $n_Q - 2n_0 > 1$ which makes the remaining argument more æsthetically pleasing. We also note that since $\{diam(B^\delta_j)\}_{j=1}^Q$ is decreasing in j $\{n_j\}_{j=1}^Q$ is increasing in j.

Note that since $|I_{n_Q}| \geq 2^{n_Q - 2n_0}$ it follows that

$$\left|\left\{ i : T^{\Gamma_\varepsilon}_{n_Q,i} \cap \bigcup_{i=1}^{Q} B^\delta_j \neq \emptyset \right\}\right| \geq 2^{n_Q - 2n_0}.$$

Next, for each $j \in \{1, ..., Q\}$ there exists a $T^{\Gamma_\varepsilon}_{n,j,i_j}$ such that $T^{\Gamma_\varepsilon}_{n_j,i_j} \cap B^\delta_j \neq \emptyset$. By Corollary 8.5.1 $T^{\Gamma_\varepsilon}_{n_j,k} \cap B^\delta_j = \emptyset$ for $k \notin \{i_j - 1, i_j n, i_j + 1\}$ and thus

$$|\{i : T^{\Gamma_\varepsilon}_{n_Q,i} \cap B^\delta_j| \leq \left|\left\{ i : T^{\Gamma_\varepsilon}_{n_Q,i} \cap \bigcup_{k=i_j-1}^{i_j+1} T^{\Gamma_\varepsilon}_{n_j,k} \right\}\right| = 3 \cdot 2^{n_Q - n_j}.$$

It follows that

$$2^{n_Q-2n_0} \leq \left|\left\{i : T^{\Gamma_\varepsilon}_{n_q,i} \cap \bigcup_{i=1}^{Q} B^\delta_j \neq \emptyset\right\}\right| \leq \sum_{j=1}^{Q} |\{i : T^{\Gamma_\varepsilon}_{n_Q,i} \cap B^\delta_j \neq \emptyset\}| \leq \sum_{j=1}^{Q} 3 \cdot 2^{n_Q-n_j}. \tag{8.2}$$

Further, by the selection of n_j and Corollary 8.5.1

$$\bigcup_{k=i_j-1}^{i_j+1} T^{\Gamma_\varepsilon}_{n_j,k} \subset N^{\mathcal{H}^1(A^{\Gamma_\varepsilon}_{n_j,k})}(T^{\Gamma_\varepsilon}_{n_j,i_j}) \subset B^{\delta,5}_j$$

where $B^{\delta,5}_j$ is the ball of 5 times the radius of B^δ_j of common centre with B^δ_j. It follows that $|\{i : T^{\Gamma_\varepsilon}_{n_Q,i} \subset B^{\delta,5}_j\}| \geq 3 \cdot 2^{n_Q-n_j}$.

Define now $f^{i,j}_n$ to be the orthogonal transformation on $\mathbb{R}^2$ satisfying $f^{i,j}_n(T^{\Gamma_\varepsilon}_{n,i}) = T^{\Gamma_\varepsilon}_{n,j}$. Define then

$$B^{\delta,k}_j := f^{i_j,m(j,k)}_n(B^{\delta,5}_j)$$

for $k \in \{1, ..., 2^{2n_0}\}$ where

$$m(j,k) := \min\left\{2^{n_j}, 3k - 1 + 3 \cdot 2^{2n_0} \sum_{l=1}^{j-1} 2^{n_j-n_l}\right\}.$$

Then,

$$\bigcup_{i=1}^{\min\{2^{n_1}, 3\cdot 2^{2n_0}\}} T^{\Gamma_\varepsilon}_{n_1,i} \subset \bigcup_{k=1}^{2^{2n_0}} N^{\mathcal{H}^1(A^{\Gamma_\varepsilon}_{n_1,i})}(T^{\Gamma_\varepsilon}_{n_1,m(j,k)}) \subset \bigcup_{k=1}^{2^{2n_0}} B^{\delta,k}_1$$

which, transfering to an expression in terms of triangular caps in the n_Qth level of approximation gives, by construction, that

$$\bigcup_{i=1}^{\min\{2^{n_Q}, 3\cdot 2^{2n_0} 2^{n_Q-n_1}\}} T^{\Gamma_\varepsilon}_{n_Q,i} \subset B^{\delta,k}_1.$$

In the same way, for general $j \in \{1, ..., Q\}$ we have

$$\bigcup_{i=\min\{2^{n_j}, 1+3\cdot 2^{2n_0} 2^{n_j-n_{j-1}} \sum_{l=1}^{j-1} 2^{n_j-n_l}\}}^{\min\{2^{n_j}, 3\cdot 2^{2n_0} \sum_{l=1}^{j} 2^{n_j-n_l}\}} T^{\Gamma_\varepsilon}_{n_j,i} \subset \bigcup_{k=1}^{2^{2n_0}} B^{\delta,k}_j$$

which implies

$$\bigcup_{i=\min\{2^{n_Q}, 1+3\cdot 2^{2n_0} 2^{n_Q-n_{j-1}} \sum_{l=1}^{j-1} 2^{n_j-n_l}\}}^{\min\{2^{n_Q}, 3\cdot 2^{2n_0} 2^{n_Q-n_j} \sum_{l=1}^{j} 2^{n_j-n_l}\}} T^{\Gamma_\varepsilon}_{n_Q,i} \subset \bigcup_{k=1}^{2^{2n_0}} B^{\delta,k}_j.$$

Therefore

$$\begin{aligned}\bigcup_{j=1}^{Q} \bigcup_{k=1}^{2^{2n_0}} B^{\delta,k}_j \quad &\supset \quad \bigcup_{j=1}^{Q} \bigcup_{i=\min\{2^{n_Q}, 1+3\cdot 2^{2n_0} 2^{n_Q-n_{j-1}} \sum_{l=1}^{j1} 2^{n_j-n_l}\}}^{\min\{2^{n_Q}, 3\cdot 2^{2n_0} 2^{n_Q-n_j} \sum_{l=1}^{j} 2^{n_j-n_l}\}} T^{\Gamma_\varepsilon}_{n_Q,i} \\ &= \quad \bigcup_{i=1}^{\min\{2^Q, 3\cdot 2^{2n_0} \sum_{l=1}^{Q} 2^{n_Q-n_l}\}} T^{\Gamma_\varepsilon}_{n_Q,i}.\end{aligned}$$

Since from (8.2)

$$3\cdot 2^{2n_0}\sum_{l=1}^{Q}2^{n_Q-n_l}=2^{2n_0}\sum_{j=1}^{Q}3\cdot 2^{n_Q-n_j}\geq 2^{n_Q}$$

we have

$$\min\left\{2^{n_Q},3\cdot 2^{2n_0}\sum_{l=1}^{Q}2^{n_Q-n_l}\right\}=2^{n_Q}$$

and thus

$$\Gamma_\varepsilon\subset\bigcup_{i=1}^{2^{n_Q}}T^{\Gamma_\varepsilon}_{n_Q,i}\subset\bigcup_{j=1}^{Q}\bigcup_{k=1}^{2^{2n_0}}B_j^{\delta,k}.$$

$\{\{B_j^{\delta,k}\}_{j=1}^{Q}\}_{k=1}^{2^{2n_0}}$ is thus a δ-cover of Γ_ε for $\delta<\delta_0$ so that we can calculate

$$M<\sum_{j=1}^{Q}\sum_{k=1}^{2^{2n_0}}\alpha(\eta)\left(\frac{diam(B_j^{\delta,k})}{2}\right)^\eta=2^{2n_0}5^\eta\sum_{j=1}^{Q}\alpha(\eta)\left(\frac{diam(B_j^{\delta})}{2}\right)^\eta\leq 2^{2n_0}5^\eta\sum_{j=1}^{\infty}\alpha(\eta)\left(\frac{diam(B_j^{\delta})}{2}\right)^\eta.$$

Since this is true for each δ-cover of $\tilde{\Gamma_\varepsilon}$ it follows that $\mathcal{H}^\eta_\delta(\tilde{\Gamma_\varepsilon})>M2^{-2n_0}5^{-\eta}>0$. Since this is true for all sufficiently small δ $\mathcal{H}^\eta(\tilde{\Gamma_\varepsilon})>M2^{-2n_0}5^{-\eta}>0$. Thus

$$dim\tilde{\Gamma_\varepsilon}=\sup\{\eta:\mathcal{H}^\eta(\tilde{\Gamma_\varepsilon})>0\}\geq\sup\{\eta:\mathcal{H}^\eta(\Gamma_\varepsilon)=\infty\}=dim\Gamma_\varepsilon.$$

$\diamondsuit$

We now prove the crucial step for the result we need to get our desired dimension results, saying that if one set can be centered in another then the expected result that it has a smaller dimension than the other holds.

Lemma 8.5.2.
Suppose

$$A=\bigcap_{n=0}^{\infty}\bigcup_{j\in I_n}T^{A'}_{n,j}$$

where $I_n\subset\{1,...2^n\}$ *for each* $n\in\mathbb{N}$ *and* $A'\in\mathcal{K}$. *Then*

1. *if* $A\hookrightarrow^c\Gamma_\varepsilon$ *for some* $\varepsilon>0$ $dimA\leq dim\Gamma_\varepsilon$.
2. *if* $|I_n|\geq 2^{n-2n_0}$ *for some* $n_0\in\mathbb{N}\cup\{0\}$ *and each* $n\in\mathbb{N}\cup\{0\}$ *and*

$$\bigcap_{n=0}^{\infty}\bigcup_{j\in I_n}T^{\Gamma_\varepsilon}_{n,j}\hookrightarrow^c A$$

for some $\varepsilon>0$, *then* $dim\Gamma_\varepsilon\leq dimA$.

Proof:
For notational convenience we will write $T^{A'}_{n,j}$ as $T^A_{n,j}$.

For (1) we note first that for each $n\in\mathbb{N}$ and $j\in I_n$ there are at most two $T^A_{n+1,i}$ with $i\in I_{n+1}$ satisfying $T^A_{n+1,i}\subset T^A_{n,j}$. It is also true, by construction, that for ech $n\in\mathbb{N}$ and $j\in\{1,...,2^n\}$ that there are exactly two $i\in\{1,...,2^{n+1}\}$ such that $T^{\Gamma_\varepsilon}_{n+1,i}\subset T^{\Gamma_\varepsilon}_{n,j}$.

Since also for each $n \in \mathbb{N}$ and $i, j \in \{1, ..., 2^n\}$ $T^{\Gamma_\varepsilon}_{n,i}$ is the image of an orthogonal transformation of $T^{\Gamma_\varepsilon}_{n,j}$ it can always be arranged that should $T^A_{n+1,i} \subset T^A_{n,j}$ then, with $T^{\Gamma_\varepsilon}_{n+1,k}$ and $T^{\Gamma_\varepsilon}_{n,l}$ selected such that

$$\mathcal{T}^{A,\Gamma_\varepsilon}_{n+1,i}(T^A_{n+1,i}) \subset T^A_{n+1,k} \quad \text{and} \quad \mathcal{T}^{A,\Gamma_\varepsilon}_{n,j}(T^A_{n,j}) \subset T^{\Gamma_\varepsilon}_{n,l}$$

$$T^{\Gamma_\varepsilon}_{n+1,k} \subset T^{\Gamma_\varepsilon}_{n,l}. \tag{8.3}$$

Now, let η be such that $\mathcal{H}^\eta(\Gamma_\varepsilon) = 0$. Then $\mathcal{H}^\eta_\delta(\Gamma_\varepsilon)) = 0$ for all $\delta > 0$. Now take any $\delta > 0$, $\zeta > 0$ and any collection of balls $\{B^\delta_j\}_{j=1}^\infty$ with $\Gamma_\varepsilon \subset \cup_{j=1}^\infty B^\delta_j$, $diam(B^\delta_j) < \delta$ for all $j \in \mathbb{N}$ and

$$\sum_{j=1}^\infty \alpha(\eta) \left(\frac{diam(B^\delta_j)}{2} \right)^\eta < \zeta.$$

We can assume without loss of generality that $B^\delta_j \cap \Gamma_\varepsilon \neq \emptyset$ for each j.

Now, for each B^δ_j there exists $n_j \in \mathbb{N}$ such that

$$diam(B^\delta_j) \in (\mathcal{H}^1(T^{\Gamma_\varepsilon}_{n_j,i})/2, \mathcal{H}^1(T^{\Gamma_\varepsilon}_{n_j,i})]$$

and from Lemma 6.2.1 there exists $i_j \in \{1, 2, ..., 2^{n_j}\}$ such that $B^\delta_j \cap T^{\Gamma_\varepsilon}_{n_j,k} = \emptyset$ for each $k \notin \{i_j - 1, i_j, i_j + 1\}$. Further, for each $T^{\Gamma_\varepsilon}_{n_j,k}$ such that $B^\delta_j \cap T^{\Gamma_\varepsilon}_{n_j,k} \neq \emptyset$, $T^{\Gamma_\varepsilon}_{n_j,k} \subset \tilde{B}^\delta_j$ where $\tilde{B}^\delta_j$ is the ball of equal radius to B^δ_j but with 4 times the radius.

We then consider the collection $\{\{B^\delta_{jk}\}_{k=1}^3\}_{j=1}^\infty$ defined by

$$B^\delta_{jk} := (\mathcal{T}^{A,\Gamma_\varepsilon}_{n_j,i_j-2+k})^{-1}(\tilde{B}^\delta_j).$$

Now, let $x \in A$ and take the sequence $i(n,x)$ such that $x \in T^A_{n,i(n,x)}$ for each $n \in \mathbb{N}$. Further, let k_n be the sequence satisfying $\mathcal{T}^{A,\Gamma_\varepsilon}_{n,i(n,x)}(T^A_{n,i(n,x)}) \subset T^{\Gamma_\varepsilon}_{n,k_n}$.

Then, by (8.3) $T^{\Gamma_\varepsilon}_{n,k_n}$ is a decreasing sequence of triangular caps in the construction of Γ_ε and therefore there exists a $y \in \mathbb{R}^2$ such that

$$\bigcap_{n=0}^\infty T^{\Gamma_\varepsilon}_{n,k_n} \ni y \neq \emptyset.$$

Then $y \in \Gamma_\varepsilon$ and $y \in B^\delta_j$ for some $j \in \mathbb{N}$. Further, since $diam(T^{\Gamma_\varepsilon}_{n,k_n}) \to 0$ as $n \to \infty$ and

$$diam(T^{\Gamma_\varepsilon}_{n,k_n}) > diam(T^{\Gamma_\varepsilon}_{n+1,k_{n+1}} > diam(T^{\Gamma_\varepsilon}_{n,k_n})/2$$

for each $n \in \mathbb{N} \cup \{0\}$ there exists $m \in \mathbb{N}$ such that

$$T^{\Gamma_\varepsilon}_{m,k_m} \cap B^\delta_j \neq \emptyset \quad \text{and} \quad diam(B^\delta_j) \in (diam(T^{\Gamma_\varepsilon}_{m,k_m})/2, diam(T^{\Gamma_\varepsilon}_{m,k_m})].$$

Thus $T^{\Gamma_\varepsilon}_{m,k_m} \subset \tilde{B}^\delta_j$ and hence

$$x \in T^A_{m,i(m,x)} \subset (\mathcal{T}^{A,\Gamma_\varepsilon}_{m,i(m,x)})^{-1}(T^{\Gamma_\varepsilon}_{m,k_m}) \subset (\mathcal{T}^{A,\Gamma_\varepsilon}_{m,i(m,x)})^{-1}(\tilde{B}^\delta_j) = B^\delta_{jk}$$

for some $k \in \{1,2,3\}$.

Since this is true for each $x \in A$,

$$A \subset \bigcup_{j=1}^{\infty}\bigcup_{k=1}^{3} B_{jk}^{\delta}.$$

Thus

$$\mathcal{H}_{\delta}^{\eta}(A) \leq \sum_{j=1}^{\infty}\sum_{k=1}^{3}\alpha(\eta)\left(\frac{diam(B_{jk}^{\delta})}{2}\right)^{\eta} = 3\cdot 2^{\eta}\sum_{j=1}^{\infty}\alpha(\eta)\left(\frac{diam(B_{j}^{\delta})}{2}\right)^{\eta} < 12\zeta.$$

Since this is true for each $\delta, \zeta > 0$ it follows that $\mathcal{H}^{\eta}(A) = 0$. Since this is true for each η such that $\mathcal{H}^{\eta}(\Gamma_{\varepsilon}) = 0$ it follows that $\dim A \leq \dim\Gamma_{\varepsilon}$.

For (2) we follow the same proof idea. We need, however, to make some important changes. Firstly, we note that since both

$$\tilde{\Gamma_{\varepsilon}} := \bigcap_{n=0}^{\infty}\bigcup_{j\in I_n} T_{n,j}^{\Gamma_{\varepsilon}}$$

and A have the same indexing and since the $T_{n,j}^{\Gamma_{\varepsilon}}$'s are orthogonal transformations of each other we can always assume that

$$\mathcal{T}_{n,j}^{\tilde{\Gamma_{\varepsilon}},A}(T_{n,j}^{\tilde{\Gamma_{\varepsilon}}}) \subset T_{n,j}^{A}$$

for each $n \in \mathbb{N}$ and $j \in I_n$ and in particular we can assume that

$$x \in \bigcap_{n=0}^{\infty} T_{n,i(n,x)}^{\tilde{\Gamma_{\varepsilon}}}$$

implies that $T_{n,i(n,x)}^{A} \supset T_{n+1,i(n+1,x)}^{A}$ for each $n \in \mathbb{N}\cup\{0\}$ and thus, using the construction of A'

$$\bigcap_{n=0}^{\infty} T_{n,i(n,x)}^{A} \ni y \neq \emptyset \tag{8.4}$$

for some $y \in \mathbb{R}^2$.

Let now $\eta \in \mathbb{R}$ be such that $\mathcal{H}^{\eta}(A) = 0$ so that $\mathcal{H}_{\delta}^{\eta}(A) = 0$ for each $\delta > 0$. Now take any $\delta, \zeta > 0$ and a collection of balls $\{B_j^{\delta}\}_{j=1}^{\infty}$ with $A \subset \cup_{j=1}^{\infty}$, $diam(B_j^{\delta}) < \delta$ for each $j \in \mathbb{N}$ and

$$\sum_{j=1}^{\infty}\alpha(\eta)\left(\frac{diam(B_{j}^{\delta})}{2}\right)^{\eta} < \zeta.$$

Again, without loss of generality, we may assume that $B_j^{\delta} \cap A \neq \emptyset$ for each j.

Now, for each B_j^{δ} there exists $x \in A \cap B_j^{\delta}$ and thus $T_{n,i(n,x)}^{A} \cap B_j^{\delta} \neq \emptyset$ for each $n \in \mathbb{N}\cup\{0\}$. Since $diam(T_{n,i(n,x)}^{\Gamma_{\varepsilon}}) \to 0$ as $n \to \infty$ and

$$diam(T_{n,i(n,x)}^{\Gamma_{\varepsilon}}) > diam(T_{n+1,i(n+1,x)}^{\Gamma_{\varepsilon}}) \geq \frac{1}{2}diam(T_{n,i(n,x)}^{\Gamma_{\varepsilon}})$$

for each $n \in \mathbb{N}\cup\{0\}$, there exists an $n_j \in \mathbb{N}\cup\{0\}$ such that

$$T_{n_j,i(n_j,x)}^{A} \cap B_j^{\delta} \neq \emptyset \quad \text{and} \quad diam(B_j^{\delta}) \in (diam(T_{n_j,i(n_j,x)}^{\Gamma_{\varepsilon}})/2, diam(T_{n_j,i(n_j,x)}^{\Gamma_{\varepsilon}})]. \tag{8.5}$$

Using now Corollary 8.5.1 we see that

$$B_j^\delta \cap T_{n_j,k}^A = \emptyset \quad \text{for} \quad k \notin \{i(n_j,x)-1, i(n_j,x), i(n_j,x)+1\}.$$

Therefore there exist orthogonal transformations T_{j1}, T_{j2}, T_{j3} such that

$$T_{ji}(\tilde{B}_j^\delta) \supset \mathcal{T}_{n_j,i(n_j,x)-1+i}^{\tilde{\Gamma_\varepsilon},A}(T_{n_j,i(n_j,x)-1+i}^{\tilde{\Gamma_\varepsilon}}) \quad \text{for} \quad i \in \{0,1,2\} \tag{8.6}$$

where $\tilde{B}_j^\delta$ is the ball of identical centre to B_j^δ but with 4 times the radius.

We then consider the collection $\{\{B_{jk}^\delta\}_{k=0}^2\}_{j=1}^\infty$ defined by

$$B_{jk}^\delta := (\mathcal{T}_{n_j,i+k}^{\tilde{\Gamma_\varepsilon},A})^{-1}(T_{jk}(\tilde{B}_j^\delta))$$

where i is chosen such that $B_j^\delta \cap T_{n_j,i+k}^A = \emptyset$ for $k \notin \{0,1,2\}$ which we know exists from Corollary 8.5.1.

Now let $y \in \tilde{\Gamma_\varepsilon}$ and take the sequence $i(n,y)$ such that $y \in T_{n,i(n,y)}^{\tilde{\Gamma_\varepsilon}}$. Then by (8.4) we see that

$$\emptyset \neq z \in \bigcap_{n=0}^{\infty} T_{n,i(n,y)}^A = \bigcap_{n=0}^{\infty} \mathcal{T}_{n,i(n,y)}^{\tilde{\Gamma_\varepsilon},A}(T_{n,i(n,y)}^{\tilde{\Gamma_\varepsilon}})$$

for some $z \in \mathbb{R}^2$. Then $z \in B_j^\delta$ for some $j \in \mathbb{N}$. Further, as calculated in obtaining (8.5) we see that there exists an $m \in \mathbb{N} \cup \{0\}$ such that

$$T_{m,i(m,y)}^A \cap B_j^\delta \neq \emptyset \quad \text{and} \quad diam(B_j^\delta) \in (diam(T_{m,i(m,y)}^{\tilde{\Gamma_\varepsilon}})/2, diam(T_{m,i(m,y)}^{\tilde{\Gamma_\varepsilon}})].$$

Thus, by (8.6) $\mathcal{T}_{m,i(m,y)}^{\tilde{\Gamma_\varepsilon},A}(T_{m,i(m,y)}^{\tilde{\Gamma_\varepsilon}}) \subset T_{jk}^\delta$. for some $k \in \{0,1,2\}$ and hence

$$y \in T_{m,i(m,y)}^{\tilde{\Gamma_\varepsilon}} \subset (\mathcal{T}_{m,i(m,y)}^{\tilde{\Gamma_\varepsilon},A})^{-1}(T_{jk}(\tilde{B}_j^\delta)) = B_{jk}^\delta.$$

Since this is true for each $y \in \tilde{\Gamma_\varepsilon}$

$$\tilde{\Gamma_\varepsilon} \subset \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{3} B_{jk}^\delta.$$

Thus

$$\mathcal{H}_\delta^\eta(\tilde{\Gamma_\varepsilon}) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{3} \alpha(\eta) \left(\frac{diam(B_{jk}^\delta)}{2}\right)^\eta = 3 \cdot 2^\eta \sum_{j=1}^{\infty} \alpha(\eta) \left(\frac{diam(B_j^\delta)}{2}\right)^\eta < 12\zeta.$$

Since this is true for each $\delta, \zeta > 0$ it follows that $\mathcal{H}^\eta(\tilde{\Gamma_\varepsilon}) = 0$. Since this is true for each η such that $\mathcal{H}^\eta(A) = 0$ it follows that $\dim\tilde{\Gamma_\varepsilon} \leq \dim A$.

Now, since $|I_n| \geq 2^{n-2n_0}$ for some $n_0 \in \mathbb{N} \cup \{0\}$ and for each $n \in \mathbb{N} \cup \{0\}$, it follows from Proposition 8.5.2 that $\dim\tilde{\Gamma_\varepsilon} = \dim\Gamma_\varepsilon$ and thus that $\dim\Gamma_\varepsilon \leq \dim A$. $\diamondsuit$

This completes the presentation of the necessary preliminary results and thus the chapter. In the following chapter we look at the theorems proving various results about the actual measure, rectifiability and dimension of A_ε type sets and Koch type sets.

8.6 Notes

Dyadic intervals, Definition 8.2.2, are by no means here an original concept, being simply a natural family of intervals. Discussions of dyadic intervals can be found in Koeller [19] and [20]. The remainder of the material presented in this chapter is our own.