Chapter 7

Miscellaneous Results

In this section we present some further interesting and relevant results found in association with the study leading to the classification that we have presented, but that were not directly necessary in the classification. In particular we show that our present counter examples would not be sufficient for a δ-fine version of property (v) and that A_ε does not satisfy (vii), showing that there is no direct strength ranking of the 8 definitions in Definition 3.2 since Λ_δ which satisfies (vii) does not satisfy (iii) which is satisfied by A_ε. Further, in the proof that A_ε does not satisfy (vii) we see that the sets A_ε do infact spiral infinitely finely in a sense that will become clearer.

We also discuss how to extend the presented counter examples into higher dimensional counter examples. We show one such extension since the process of extending to higher dimensions remains the same for each of the sets.

7.1 The Existence of Spiralling

We start by showing that A_ε does not satisfy (v) for each $\delta > 0$. Similarly, but oppositely to Lemma 6.5.1 we show that there is also a function $\delta_1 : \mathbb{R} \to \mathbb{R}$ such that for each $\varepsilon > 0$, for each $\delta < \delta_1(\varepsilon)$, A_ε does not satisfy (v) for δ. Thus, although for each δ there is an A_ε that fits, there is no ε such that A_ε satisfies (v) for every δ, thus showing that A_ε and indeed Γ_ε are not sufficient as counter examples to any δ-fine version of (v).

Proposition 7.1.1.
There is a function

$$\delta_1 : \mathbb{R} \to \mathbb{R}^+$$

with $\delta_1(x) > 0$ for all $x > 0$ such that for each $\varepsilon > 0$ and all $\delta < \delta(\varepsilon)$ A_ε does not satisfy (v) with respect to δ.

Proof:
First, we take

$$y \in A_\varepsilon \cap B_{\frac{\varepsilon}{32}}\left(\left(\frac{1}{2}, 2\varepsilon\right)\right)$$

and $\rho > \frac{3}{8}$, say $\rho = \rho_0 = \frac{1}{2}$ ($\rho_0 = \frac{1}{2}$ as $B_{1/2}((1/2, 0)) \supset A$. It is not hard to see that we then have

$$\partial B_\rho(y) \cap int(T_{3,1}) \neq \emptyset \text{ and } \partial B_\rho(y) \cap int(T_{3,4}) \neq \emptyset$$

so that, more particularly

$$B_\rho(y) \cap T_{3,1} \cap A_\varepsilon \neq \emptyset \text{ and } B_\rho(y) \cap T_{3,4} \cap A_\varepsilon \neq \emptyset.$$

We now note that $\sup_{x \in T_{3,i}} \pi_y(x) < \varepsilon$ for $i = 1, 4$ and that clearly $\inf_{x \in B_{\frac{\varepsilon}{32}}((\frac{1}{2}, 2\varepsilon))} \pi_y(x) > \frac{63\varepsilon}{32}$. It follows that a vertical gap between y and points in $A \cap B_\rho(y)$ of at least $\frac{31\varepsilon}{32}$ exists both "to the left" of y (that is points $z \in T_{3,1}$ where we must have $\pi_x(z) < \pi_x(y)$) and "to the right" of y (similarly to above, that is points $z \in T_{3,4}$ where we must have $\pi_x(z) > \pi_x(y)$).

Similarly clearly, we know that $\pi_x(z) > 0$ for all $x \in T_{3,1}$ (and also in fact $T_{3,4}$) and conversely we have $\pi_x(z) < 1$ for all $z \in T_{3,4}$ (and in fact, but unimportantly $T_{1,4}$). Also we have, since $\varepsilon < 1/4$

$$\pi_x(y) \in \left(\frac{1}{2} - \frac{\varepsilon}{32}, \frac{1}{2} + \frac{\varepsilon}{32}\right) \subset \left(\frac{63}{128}, \frac{65}{128}\right)$$

This means that in the best case any cone has less than a horizontal length of $\frac{65}{128}$ to spread out to meet a set of vertical distance $\frac{31\varepsilon}{32}$ away from its center.

Supposing, to begin with, that $L_{y,\rho} || \mathbb{R}_x$ (that is $L_{y,\rho}$ is parallel to the horizontal axis) then the cone angle must be, to cover the "best case mentioned above" at least

$$tan^{-1}\left(\frac{\left(\frac{31\varepsilon}{32}\right)}{\left(\frac{65}{128}\right)}\right).$$

Now, should $L_{y,\rho}$ not be horizontal, we have that it is either positively or negatively sloped, but in either case, it continues to go through y. In the former case, we have that the cone angle estimate is improved for points in $T_{3,1}$, however, continuing to observe the $y = (63/128, 63\varepsilon/32)$ case with a $z \in T_{3,4}$, it is clear that the resulting required cone angle for this z can be no better than the cone angle required to include $z = (1, \varepsilon)$. We must therefore have that the minimum cone angle is no better than

$$\theta = tan^{-1}\left(\frac{\left(\frac{31\varepsilon}{32}\right)}{\left(\frac{65}{128}\right)}\right) + ||(L_{y,\rho} - y) - \mathbb{R}_x||_{G(1,2)} > tan^{-1}\left(\frac{\left(\frac{31\varepsilon}{32}\right)}{\left(\frac{65}{128}\right)}\right)$$

where $|| \cdot ||_{G(1,2)}$ denotes the norm on the grassman manifold of 1-planes in $\mathbb{R}^2$. A similar argument works considering points in $T_{3,1}$ in the case that $L_{y,\rho}$ is negatively sloped possibly improving the estimate for points in $T_{3,4}$. We therefore have that the cone angle $tan^{-1}\left(\frac{\left(\frac{31\varepsilon}{32}\right)}{\left(\frac{65}{128}\right)}\right)$ cannot be improved on, so that for any $\delta < \frac{\left(\frac{31\varepsilon}{32}\right)}{\left(\frac{65}{128}\right)}$ A_ε cannot satisfy (v) with respect to δ.

Thus the function δ_1 defined by

$$\delta_1(x) := \frac{\left(\frac{31x}{32}\right)}{\left(\frac{65}{128}\right)}$$

satisfies the requirements for the Proposition. ◇

To prove that A_ε (and indeed Γ_ε) cannot satisfy (vii) irrespective of δ, we have to show that although no spiralling occurs in the vicinity of any given point in A_ε at a given approximation level, spiralling does indeed occur.

This means that for any point $x \in A_\varepsilon$, any radius $r > 0$ and any potential approximating affine space, there exists a (smaller) triangular cap in $B_r(x)$ whose base is arbitrarily close to perpendicular to the approximating affine space.

It then follows that an appropriate choice of testing points and testing radius smaller than or equal to r in such a triangular cap will allow us to show that for any $\delta < 1$ A_ε and indeed Γ_ε cannot possibly satisfy (vii).

Proposition 7.1.2.
For each $0 < \delta < 1$, A_ε does not satisfy Property (vii) with respect to δ.

Proof:
Let $\delta \in (0,1)$, $\varepsilon > 0$ and $y \in A_\varepsilon$; then should A_ε satisfy the definition then for each $\rho_y > 0$ there would exist an affine space L_{y,ρ_y} such that for all $x \in A_\varepsilon \cap B_{\rho_y}(y)$ and all $\rho < \rho_y$ $B_\rho(x) \cap A_\varepsilon \subset L^{\delta\rho}_{y,\rho} + x$.

Now, since we are assuming that A_ε satisfies the definition there must be a function,

$$\phi : (0,1) \to (0, 2\pi)$$

dependent only on δ which describes the cone outside of which no boundary points of a ball around a point $x \in B_{\rho_y}(y)$ are in A_ε. Specifically, by defining

$$E_{\phi,\rho,x} := \left\{ x \in \partial B_\rho(x) : tan^{-1}\left(\frac{\pi^{\perp}_{L_{y,\rho_y}}(x)}{\pi_{L_{y,\rho_y}}(x)} \right) \geq \phi(\delta) \right\}$$

we have

$$A_\varepsilon \cap E_{\phi(\delta),\rho,x} = \emptyset,$$

for all $x \in A \cap B_{\rho_0}(y)$ and also that there is a $\eta(\delta) > 0$ such that for all $x \in A \cap B_{\rho_0}(y)$, $\rho \in (0, \rho_0]$ and all $z \in E_{\frac{\pi}{2} - \frac{\pi - \psi(\delta)}{2},\rho,x}$

$$B_{\rho\eta(\delta)}(z) \cap A = \emptyset. \tag{7.1}$$

That is, around points in the central part of $E_{\phi,\rho,x}$ we can put a ball depending only on ρ and δ that will be completely empty of A_ε.

We observe that y must be in some triangular cap of the construction of A, $T_{n,i}$, for some n and i, also such that $T_{n,i} \subset B_{\rho_0}(y)$. We make the nomenclatorial choice to call the vertices of the triangular cap $T_{m,i}$ $a_{m,i}, l_{m,i}$, and $r_{m,i}$ chosen such that

$$\pi_x \circ O_{A_{m,i}}(a_{m,i}) = 0, \quad \pi_x \circ O_{A_{m,i}}(l_{m,i}) < 0, \text{ and } \pi_x \circ O_{A_{m,i}}(r_{m,i}) > 0.$$

That is a denotes the "top" vertex as we have previously defined, and l and r denote the identical "left" and "right" base angles.

We now note that for each $k \in \mathbb{N}$ we have

$$\psi^{T_{n+k,2^k i+4^{k-1}+2}}_{T_{n,i}+z(i,k)} = \sum_{i=n}^{n+k} \psi(i,\varepsilon)$$

for some appropriate point $z(i,k) \in \mathbb{R}^2$.

We now need some properties of the sequence $\{\psi(i,\varepsilon)\}_{i=1}^{\infty}$. First of all we recall that

$$\lim_{i\to\infty} \psi(i,\varepsilon) = 0. \tag{7.2}$$

and that we can specifically write

$$\psi(n,\varepsilon) = tan^{-1}\left(\frac{8\varepsilon}{(1+16n\varepsilon^2)^{1/2}}\right)$$

so that using the facts that $\frac{dtan}{dx}(x) \geq 0$, and $\frac{dtan}{dx}(x)|_{x=0} = 1$ (and hence for sufficiently large n, $tan^{-1}(1/\varepsilon n) > 1/(2\varepsilon n)$), we get for any $n_0 \in \mathbb{N}$

$$\sum_{n=n_0}^{\infty} \psi(i,\varepsilon) = \sum_{n=n_0}^{\infty} tan^{-1}\left(\frac{8\varepsilon}{(1+16n\varepsilon^2)^{1/2}}\right) \geq \sum_{n=n_0}^{\infty} \frac{8\varepsilon}{8\varepsilon(\frac{1}{\varepsilon^2}+n)^{1/2}} = \infty,$$

where E denotes the smallest integer greater than or equal to $1/\varepsilon^2$. It follows that there exists a sequence, $\{n_k\}$, such that for each $k \in \mathbb{N}$

$$\sum_{i=n}^{n_k-1} \psi(i,\varepsilon) < \frac{2k\pi - \pi}{2} < \sum_{i=n}^{n_k+1} \psi(i,\varepsilon).$$

So that there is a triangular cap $T_{n_k,i(k)}$ (for the appropriate i depending on k) such that

$$tan^{-1}\left(\frac{\pi^{\perp}_{L_{y,\rho_0}}(r_{n_k,i(k)} - l_{n_k,i(k)})}{\pi_{L_{y,\rho_0}}(r_{n_k,i(k)} - l_{n_k,i(k)})}\right) - \frac{2k\pi - \pi}{2} < \psi(n_k-1,\varepsilon) + \psi(n_k,\varepsilon).$$

Thus, by (7.2) there exists a $k \in \mathbb{N}$ such that

$$tan^{-1}\left(\frac{\pi^{\perp}_{L_{y,\rho_0}}(r_{n_k,i(k)} - l_{n_k,i(k)})}{\pi_{L_{y,\rho_0}}(r_{n_k,i(k)} - l_{n_k,i(k)})}\right) - \frac{2k\pi - \pi}{2} < \psi(n_k-1,\varepsilon) + \psi(n_k,\varepsilon) < \frac{\frac{\pi}{2} - \phi(\delta)}{2}.$$

That is the triangular cap, $T_{n_k,i(k)}$ has the property that

$$r_{n_k,i(k)} \in E_{\phi(\delta)+\frac{\frac{\pi}{2}-\phi(\delta)}{2}, |r_{n_k,i(k)} - l_{n_k,i(k)}|, l_{n_k,i(k)}}.$$

The endpoints themselves are not in A, however, we can choose $x_l, z_r \in A$ such that

$$|x_l - r_{n_k,i(k)}| < \frac{\eta(\delta)|r_{n_k,i(k)} - l_{n_k,i(k)}|}{2} \text{ and } |z_r - l_{n_k,i(k)}| < \frac{\eta(\delta)|r_{n_k,i(k)} - l_{n_k,i(k)}|}{2}$$

so that there is a $x_r \in \partial B_{|r_{n_k,i(k)} - l_{n_k,i(k)}|}(x_l)$ such that $z_r \in B_{\eta(\delta)|r_{n_k,i(k)} - l_{n_k,i(k)}|}(x_r)$. Since, by our choice of triangular cap, $T_{n,i}$, $x_l \in B_{\rho_0}(y)$ and $|r_{n_k,i(k)} - l_{n_k,i(k)}| < \rho_0$ this contradicts (7.1), proving the proposition since ε and δ were chosen arbitrarily. ◊

7.2 Higher Dimensional Analogies of Γ_ε, A_ε and $\mathcal{A}_\varepsilon$

We now come to the higher dimensional generalisations of the counter examples.

It is unfortunately straightforward - unfortunate from the view of finding interesting mathematics - to generalise our counter examples to higher dimensions so that we obtain no further insight into how the structures of sets work. In each case we simply cross each set with either an interval or simply the plane of the required dimension, depending on whether or not we need the set to be bounded (as we do for ρ_0 uniformity properties). We show, as an example, how A_ε is extended, and demonstrate how it continues to satisfy Property (iii).

Suppose that we are taking j-dimensional approximations in $\mathbb{R}^{j+k}$. We take

$$S_\varepsilon = A_\varepsilon \times \mathbb{R}^{j-1} =\subset \mathbb{R}_A \times \mathbb{R}^{j-1} \times \mathbb{R}^{A_c},$$

where $\mathbb{R}_A = \mathbb{R}_{Ac} = \mathbb{R}$ but have been given names for notational convenience. $\mathbb{R}_A$ and $\mathbb{R}_{A_c}$ are identified with $\mathbb{R}$ and $\mathbb{R}^2/\mathbb{R}$ as we have been considering in the preceeding sections so that $A_\varepsilon \subset \mathbb{R}_A \times \mathbb{R}_{A_c}$. Further S_ε is constructed inside of

$$\mathbb{R}^{j+k} = \mathbb{R}_A \times \mathbb{R}^{j-1} \times R_{A,c} \times \mathbb{R}^{k-1}.$$

We can thus see S_ε as

$$\begin{aligned} S_\varepsilon &= \{y = (y_1, x_2, ..., x_{n-1}, y_2, 0, ..., 0) : (y_1, y_2) \in A_\varepsilon \subset \mathbb{R}_A \times \mathbb{R}_{Ac}, x_i \in \mathbb{R}\} \\ &\subset \mathbb{R}_A \times \mathbb{R}_x^{j-1} \times R_{A_c} \times \mathbb{R}_z^{k-1}, \end{aligned}$$

where $\mathbb{R}_x^{j-1} \cong \mathbb{R}^{j-1}$ and $\mathbb{R}_z^{k-1} \cong \mathbb{R}^{k-1}$ are again notational conveniences denoting the dimensions along which the extension of A_ε into S_ε exist ($\mathbb{R}_x^{j-1}$), and the additional codimensions ($\mathbb{R}_z^{k-1}$).

Proposition 7.2.1.
S_ε shows that the answer to (iii) (2) is no for arbitrary j.

Proof:
There are two properties that we need to show that S_ε has. That it has the fine weak j-dimensional δ-approximation property, and that for each $x \in S_\varepsilon$ and $R > 0$, $B_R^{j+k}(x) = +\infty$.

First, to show that S_ε has property (iii). We take arbitrary $y \in S_\varepsilon$ and $\delta > 0$. We now need only show that there exists a j-dimensional affine space, $L_{y,\rho}$ for each $\rho > 0$, such that $S_\varepsilon \cap B_\rho(y) \subset L_{y,\rho}^{\delta\rho}$. We note that since A_ε has property (iii), there exists for the chosen δ and y a 1-dimensional affine space $L_{(y_1,y_2),\rho}$ such that $A_\varepsilon \cap B_\rho^2(y_1, y_2) \subset L_{(y_1,y_2),\rho}^{\delta\rho}$. It is therefore reasonable to take and test $L_{y,\rho} = L_{(y_1,y_2),\rho} \times \mathbb{R}_x^{j-1}$ as our affine space. Clearly

$$S_\varepsilon \cap B_\rho(y) = (A_\varepsilon \cap \pi_{\mathbb{R}_A \times \mathbb{R}_{A_c}}(B_\rho(y))) \times \mathbb{R}_x^{j-1} \subset L_{(y_1,y_2),\rho}^{\delta\rho} \times \mathbb{R}_x^{j-1} = L_{y,\rho}^{\delta\rho},$$

which gives us that S_ε has the appropriate property.

To show that there is infinite measure in each ball $B_{R_1}(y)$ we take an $R_1 > 0$ and a $y \in S_\varepsilon$. Let $R = \min\{R, d(y, \partial S_\varepsilon)\}$. We then get that

$$\begin{aligned} \mathcal{H}^j(S_\varepsilon \cap B_{R_1}(y)) &\geq \mathcal{H}^j(S_\varepsilon \cap B_R(y)) \\ &\geq \mathcal{H}^j(S_\varepsilon \cap ([-R/4, R/4]^{j+k} + (y_1, 0, ..., y_2, 0, ..., 0))) \\ &= \mathcal{H}^1(A \cap ([-R/4, R/4]^2 + (y_1, y_2)))\mathcal{H}^{j-1}(\pi_{R_x^{j-1}}) \\ &= \mathcal{H}^1(A \cap ([-R/4, R/4]^2 + (y_1, y_2))) \left(\frac{R}{4}\right)^{j-1} \\ &= +\infty, \end{aligned}$$

showing that S_ε is not weak locally $\mathcal{H}^j$-finite. ♢

7.3 Notes

The results presented in this chapter are all our own.