Chapter 2

Introduction to Singularity Structure Results

In the fields of Geometric Measure Theory and Differential Geometry we find that the study of surfaces (Minimal Surfaces, Stationary Surfaces, Energy Minimising Surfaces, etc.) and the flows of surfaces (Mean Curvature Flow, Ricci Flow, Brakke Flow, etc.) play a central role. Geometric flows are not in general well behaved in that they have initially, or develop in finite time, singularities. Simply speaking, these can be thought of as holes, edges, corners, or in general points around which the object can in no neighbourhood be described by a graph. It is natural that an understanding of the structure of such sets would be desired.

In this part of the Thesis, we look at the shape, or structure of singularity sets. In the present state of knowledge surprisingly little is known about these sets. Although, particularly in weak formulations, regularity theorems are relatively standard in studies of these objects (See Allard [2] [1], White [32], Simon [26], Brakke [5], Ecker [7]),present regularity results tell us more about how much of the surface we may consider as being smooth (or regular) than about the structure or measure of the singularity set itself.

Some important results on the structure of the singularity sets themselves are due to White, whose stratification results show that the dimension of the singularity set is no more than the dimension of the surface minus 1, and Simon, who has shown that in a particular class of minimal surfaces, the multiplicity one class, the singularity set is always a finite union of countably rectifiable sets in the dimension of the singularity.

What is not known is anything at all about the shape of a singularity sets. We do indeed have examples of singularity sets but they are all simple, (i.e. the subset of a line, or a point) which leaves a lot of space between examples and generally provable results.

In his paper showing the rectifiability of singularity sets of the multiplicity one class of minimal surfaces, Simon shows that singularity sets can be approximated by planes in the dimension in which they occur. This tells us that the properties of sets that are approximately planes of some dimension are worth considering to see what properties we can get "for free" and what sort of potential problems does one need to be wary of when considering the singularity sets.

As a model for what is meant when we say that a set is approximately a j-dimensional plane or indeed that a set is approximately j-dimensional we use the 'plane like' properties shown by Simon to be possessed by singularity set approximations.

We isolate these properties to construct an ordering of eight strengths of j-dimensional plane approximation of which the property combination specifically used by Simon is the fourth. We classify these definitions in terms of whether or not they ensure actual j-dimensionality, whether or not they ensure locally $\mathcal{H}^j$-finite measure in either a strong or a weak sense and whether or not they ensure countable j-rectifiability.

The definitions allow for the full spectrum of possibilities. The strongest definition implying that the set is locally a finite union of Lipschitz graphs and the weakest two do not even ensure that the set be j-dimensional.

The most interesting case, however, is that of the complications of our fourth definition, intriguingly the same as that arising in Simon's work. This definition ensures j-dimensionality, but what makes this case interesting is that while locally finite j-dimensional measure is not ensured, any counter examples are necessarily exotic. We show that while satisfying 'approximately j-dimensional' properties such sets have points of infinite $\mathcal{H}^j$-density but that no piece of any Lipschitz graph may pass through such a point. This rules out any vaguely well behaved sets (or countable unions of vaguely well behaved sets) from both satisfying our fourth definition and failing to have locally finite $\mathcal{H}^j$-measure.

Since our classification is complete it follows that we can (and indeed do) provide a set satisfying this fourth definition that also does not have locally finite j-dimensional measure. The set is a variation on the fractal known as the Koch set. Since all singularity sets are closed we go on to show that a closed version of this counter example exists which implies that in principle singularity sets could be as terribly behaved as the counter example.

Especially since, at least in the multiplicity one class minimal surface case, singularity sets are known to be finite unions of countably j-rectifiable sets (see [26]) the question of whether such sets as these counter examples are finite unions of countably j-rectifiable sets (and so continue to, potentially, be singularity sets) becomes of particular interest.

The answer to this question for the particular examples initially given turns out to be no, they are not rectifiable without considering the measure conditions differentiating between countably rectifiable and finite unions of countably rectifiable sets and so cannot be finite unions of countably j-rectifiable sets. However, since the explicitly constructed counter examples are members of a family of constructions this by no means rules out the possibility of very poorly behaved singularity sets.

The latter chapters in this part, Part I, of the thesis then define generalisations of the construction of the constructed counter examples. We call these sets, due to their similarity to the Koch sets, Koch type sets. We then concentrate on giving dimension, measure and rectifiability conditions for these generalised sets.

We find, encouragingly for the study of singularity sets that should such a set be first of all rectifiable then it can also be written as a single Lipschitz graph.

This would immediately imply, since we need to remove the 'corners' of the sets in order to satisfy our fourth definition that any singularity set that may be of a Koch type set form should also be a subset of a single Lipschitz graph.

The structure of this part of the thesis is as follows:

In chapter 3 we present a more precise formulation of the motivating mathematics including some particularly relevant standard general geometric measure theoretic definitions and results, provide the list of definitions as well as the results already known in terms of our classification aims and results from which sought classification results are a short corollary.

In chapter 4 we construct the specific counter examples that will be used in our classifications including the explicit examples of Koch type sets mentioned above. We go on to prove some important properties of these sets. Some properties, for example dimension, follow from some relatively general previous results of Hutchinson (see [15]). Since it is often instructive to see the direct proof for explicit examples we provide direct proofs for these results as well.

We will of course need to show that the constructed counter examples satisfy the properties required of them. Before doing this, however, we show that the level of complexity in the counter examples (particularly in A_ε and $\mathcal{A}_\varepsilon$) is necessary. This necessity of complexity is showon in Chapter 5. It is shown that no 'simple' example could possibly suffice. Further, we show that it is possible to demonstrate that singularity sets have locally finite measure than previously thought in that it is only necessart to demonstrate that the set is a graph infinitely dense at all points in the graph. This is shorter than previously thought since such a propertyis weaker. It does not even require that the set be weakly locally countably rectifiable.

In chapter 6 we fit the counter examples to their respective definitions and complete the task of classifying the definitions.

Chapter 7 gathers a few other miscellaneous relevant results and describes dimension generalisation of the explicit counter examples which are constructed to satisfy approximations to dimension 1 (though, of course, some are actually of fractal dimension between 1 and 2.)

Finally, in Chapters 8 and 9, we deal with the question of dimension, measure and rectifiability of the generalisations of the explicit counter examples given. These generalisations are divided into two levels of generalisation, first and second degree variation. We keep the two levels of generalisation distinct since, although first degree variation generalisations are also second degree variation generalisations, they allow for stronger results. This is because much more can 'go wrong' in the second degree variation case.

At the conclusion of each chapter we provide a Notes section providing origins and/or sources for all of the material presented within the chapter and, where appropriate, further reading can be found.

Since the literature considering the structure of singularities is, to an extent, disjoint from that of the other forms of regularity, the types of singularity considerations that we deal with in Part II of this thesis, there are occasionally definitions using the same name, but not having the same meaning. To avoid confusion we therefore apply a strict rule of scope to the two parts of the thesis. That is, no result or definition presented in Part I is a priori valid in Part II nor is any result or definition in Part II assumed to be valid here in Part I.