

# Chapter 1

## Introduction

Geometric flows, have, since their inception into the field of mathematics, inspired a lot of interest. Not only because of their aesthetic value to a mathematician in providing many interesting and challenging problems, but also to physicists, engineers, and the general public. The study of what happens to a surface when allowed to move (flow) when a force, or field of forces is applied to it clearly has many applications: in the study of space time sheets, world sheets and other astrophysical phenomena, in the observation of soap films, the flow of heated metal, or indeed the movement of the junction lines between crystallising cooling metals.

Of course, evolution equations of the surfaces and inducted measures are of great interest, but the problem of most interest, not least of all because of its difficulty, is the behaviour of surfaces around singularities.

A surface in some metric space (among those regularly considered are Euclidean, Riemannian, Minkowski or Lorenzian spaces) is moving under a geometric flow if there exists some pointwise function/rule by which the surface moves at each point in time. A singularity occurs when the flow under which we would like to allow our surface to move is no longer well defined. It is around such points that the interesting phenomena occur.

We shall restrict our attention to variations of mean curvature flow, namely, mean curvature flow, mean curvature flow with zero curvature (minimal surfaces) and mean curvature flow with boundary (we happen to examine the particular boundary setting of the Neumann free boundary conditions.) A surface, in general terms, is said to be moving by its mean curvature if at each point the surface moves in the normal direction at that point with speed equal to the mean curvature at that point. A singularity in mean curvature flow occurs when, under this flow, either the mean curvature is no longer defined, or the surface can no longer be locally described by a graph, losing the structure in which we are interested.

The study of mean curvature flow singularities has three main parts. The study of the structure of the singularity in a set theoretic sense, the study of conditions under which we can decide if a point is or will shortly be a singular point or not, and finally the study of the size (mass) of the entirety of the singular set. In all of these studies there is still a lot unknown.

In this thesis, we make an attack on all three problems in some sense or other.

In the sense of structure, amazingly little is known about singularity sets. In fact, the entirety

of our definite knowledge can be summed up in the form of a few specific examples and the proof that the singularity set of a special form of minimal surface can be described as the finite union of rectifiable sets in the dimension in which the singularity occurs. As rectifiability is the weakest form of definite structure we have, although the said result (due to Simon [25]) is certainly an excellent breakthrough result, we still know, in the larger picture, very little about the structure of singular sets.

In Part I of this thesis, we look at a result coming from Simon's work that compares singularity sets to a plane. The type of comparison used by Simon is sometimes known as having the Reifenberg property. We exploit this result to prove further heavy restrictions on possibly extant poorly behaved singularity sets. We also show that in the realm of possibilities there still exist extremely poorly behaved singularity sets, including purely unrectifiable sets. The background to this work and a more detailed description of the aims and structure of the first part of the thesis is given in the introduction to that part of the thesis.

The study of the remaining two forms of singularities has had somewhat more success, although there are still a lot of unanswered questions. The theory at present can tell us at least that the measure of the singularity set in the dimension of the surface being allowed to flow is zero. This type of regularity result has been looked at by many, including Brakke [5], Ecker [7] and White [32]. Also, there are several results giving conditions which imply the regularity of a given point. Fundamental work in this area was carried out by Allard [1] as well as Almgren [3] in the minimal surfaces setting. Brakke continued the work to the flow setting.

In both cases, the ground breaking monotonicity formula of Huisken [14] in the general study of mean curvature flows allowed for simpler and more elegant proofs of the regularity results, as shown by Ecker [7].

Although there are also similar regularity results with boundary in the fixed boundary case (White [32] or Allard [2]) and in the minimal surface case, regularity results for Neumann free boundary conditions (see Grüter and Jost [13]) there are no such results for flows with Neumann free boundary conditions. Following the development of a general monotonicity formula for mean curvature flow with Neumann free boundary conditions (see Buckland [6]) the supporting theory is available to fill this gap.

Our addition in this thesis, to the study of the second and third types of regularity for mean curvature flow is to fill exactly this gap in the regularity theory of mean curvature flows. We present our results in Part II of the thesis, showing regularity results for both conditions ensuring regularity and a regularity theory on the measure of the singularity set. Exactly what is shown, and how it is presented is explained in more detail in the introduction to part II of the thesis.

Throughout the thesis, we present, at the conclusion of each chapter, a Notes section providing origins and/or sources for all of the material presented within the chapter and, where appropriate, further reading can be found.

As a final note, we point out that since the literature considering the structure of singularities is, to an extent, disjoint from that of the other forms of regularity, the types of singularity considerations that we deal with in Part II of this thesis, there are occasionally definitions using the same name, but not having the same meaning. To avoid confusion we therefore apply a strict rule of scope to the two parts of the thesis. That is, no result or definition presented in Part I is a priori valid in Part II nor is any result or definition in Part II assumed to be valid in Part I.