Chapter 2

Linear and related arrangements

Let V be an $(n+1)$–dimensional vector space over $\mathbb{K}$ with $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. We consider a linear arrangement $\mathcal{A}$ in V, that is a finite set of proper linear subspaces of V. We define Q to be the intersection poset of $\mathcal{A}$, that is the set $\{\bigcap C \colon C \subset \mathcal{A}\}$ ordered by inclusion.

2.0.23 Definition. We denote the category of finite dimensional vector spaces over $\mathbb{K}$ and linear monomorphisms by $\mathfrak{V}$. We define $D^{\mathcal{A}} \in \mathfrak{V}^{Q^o}$ by $D^Q(q) := q$ and letting $D^{\mathcal{A}}(p \leftarrow q) \colon D^{\mathcal{A}}(p) \to D^{\mathcal{A}}(q)$ be the inclusion map.

2.0.24 Notation. For $u \in Q$, we set $d(u) := \dim_{\mathbb{K}} u - 1$. For $S \subset \mathbb{N}$, we set $Q_S := \{q \in Q \colon d(q) \in S\}$.

2.1 Homology and Homotopy

ZŽ–maps

We will subsequently construct ZŽ–maps involving arrangements associated with the linear arrangement $\mathcal{A}$. As a basis for these we will now construct a set of maps that could be called a linear ZŽ–map. All of these constructions will depend on choices of points in V with certain properties. The most fundamental case is the following.

2.1.1 Proposition. *There is a function x assigning to each $u \in Q$ a system $(x^u_j)_{0 \leq j \leq d(u)}$ of $d(u)+1$ vectors in u such that for all $k \in \mathbb{N}$, $u_0, \dots, u_k \in Q$, with $u_0 < u_1 < \dots < u_r$ and $\lambda = (\lambda_0, \dots, \lambda_r) \in \Delta^r$, the system of vectors*

$$\left(\sum_{i=0}^{r} \lambda_i x^{u_i}_j \right)_{0 \leq j \leq d(u_0)}$$

is linearly independent.

Proof. We give a simple recursive construction, because similar ones will be important later on. Let Λ be a linear functional on V that vanishes on no element of $Q_{[0,n]}$ and set $H := \ker \Lambda$. By induction there are $(x^u_j)_{0 \leq j < d(u)}$ with $x^u_j \in u \cap H$ such that for all $k, r \in \mathbb{N}$, $u_0 < u_1 < \dots < u_r$, $\lambda \in \Delta^r$, the system

$\left(\sum_i \lambda_i x_j^{u_i}\right)_{0\le j<d(u_0)}$ is linearly independent. (The case $u_0 = 0$ is trivial.) We now choose $x^u_{d(u)} \in u \setminus H$ for all u. Now, if $\mu \in \mathbb{K}^{d(u_0)+1}$ with $\sum_j \mu_j \sum_i \lambda_i x_j^{u_i} = 0$, then $\mu_{d(u_0)}\lambda_0 x^{u_0}_{d(u_0)} \in H$ and therefore $\mu_{d(u_0)}\lambda_0 = 0$. If $\lambda_0 = 0$ then $\mu = 0$ by linear independence of $\left(\sum_{i=1}^r \lambda_i x_j^{u_i}\right)_{0\le j\le d(u_0)}$. If $\mu_{d(u)} = 0$, then $\mu = 0$ by linear independence of $\left(\sum_{i=0}^r \lambda_i x_j^{u_i}\right)_{0\le j<d(u_0)}$. □

It will be important whether the space of possible choices of points is connected. One such case is the following.

2.1.2 Proposition. *For $\mathbb{K} = \mathbb{C}$, the set of all functions x as in Proposition 2.1.1, considered as a subspace of the affine space $\prod_{u\in Q} u^{d(u)+1}$ contains a non-empty Zariski-open set and is hence path-connected.*

Proof. The complement of the set is contained in the union of the sets

$$\left\{ x\colon \text{Ex. } 0 \ne \lambda \in \mathbb{C}^{r+1} \text{ and } 0 \ne \mu \in \mathbb{C}^{d(u_0)+1} \text{ s.t. } \sum_{i=0}^{k} \sum_{j=0}^{d(u_0)} \lambda_i \mu_j x_j^{u_i} = 0 \right\}$$

for all chains $u_0 < u_1 < \cdots < u_r$. Since the defining equations are homogenous in λ and μ, these sets are algebraic by the main theorem of elimination theory[Sha94, I.5, Thm 3]. Since the affine space is irreducible, it will be sufficient to show that each of these sets has non-empty complement. So we fix a chain $u_0 < u_1 < \cdots < u_k$. We choose a basis $(e_l)_{l=0,\dots,n}$ of V such that $e_l \in u_i$ for $l \le d(u_0) + l$ and set $x_j^i := e_{i+j}$. Now if $\sum_j \sum_i \mu_j \lambda_i x_j^i = 0$ then $\sum_i \lambda_i \mu_{s-i} = 0$ for all s, and it follows that $\lambda = 0$, $\mu = 0$. □

2.1.3 Definition. Given a function x as in Proposition 2.1.1, we define functions f^p for $p \in Q$ by

$$\begin{aligned} f^p\colon \mathbb{K}^{d(p)+1} \times \Delta[p, V] &\to V,\\ \left((\mu_0, \dots, \mu_{d(p)}), \sum_{i=0}^{r} \lambda_i u_i\right) &\mapsto \sum_{j=0}^{d(p)} \sum_{i=0}^{r} \mu_j \lambda_i x_j^{u_i}. \end{aligned} \tag{2.1}$$

2.1.4 Proposition. *For any $y \in \Delta[p, q]$, the map $f^p(\bullet, y)$ is a linear injection into $q \subset V$ (and so necessarily an isomorphism, if $p = q$), and for $p \le q$, the maps f^p and f^q agree on $\mathbb{K}^{d(p)+1} \times \Delta[q, V]$ where we identify $\mathbb{K}^{d(p)+1}$ with $\left\{\mu \in \mathbb{K}^{d(q)+1}\colon \mu_i = 0 \text{ for } i > d(p)\right\}$.* □

This proposition can be formulated as the maps f^p forming a commutative diagram as (1.6) with V for X and the functor F^Q now to be defined for E. With this definition we follow [WZŽ99, p. 141]

2.1.5 Definition. We define $F^Q \in \mathfrak{V}^{Q^o}$ by $F^Q(q) := \mathbb{K}^{d(q)+1}$ and letting $F^Q(p \leftarrow q)\colon \mathbb{K}^{d(p)+1} \to \mathbb{K}^{d(q)+1}$ be the standard inclusion.

Spherical arrangements

We now define and study the *spherical arrangement* $\mathbb{S}\mathcal{A}$ associated to the linear arrangement $\mathcal{A}$. This is done in some detail mainly to demonstrate the methods at our hands. The homological results will be obtained again later on as special cases of affine arrangements.

2.1.6 Definition. For a finite dimensional $\mathbb{K}$–vector space u, we set $\mathbb{S}(u) := (u \setminus \{0\})/\sim$, where $\sim$ is the equivalence relation identifying x and y if there exists $\lambda > 0$ with $x = \lambda y$. If u is equipped with a norm, we can alternatively set $\mathbb{S}(u) := \{x \in u \colon |x| = 1\}$. For a linear injection $f \colon u \to v$ we define $\mathbb{S}(f) \colon \mathbb{S}(u) \to \mathbb{S}(v)$ by $[x] \mapsto [f(x)]$, making $\mathbb{S}$ into a functor from $\mathfrak{V}$ to $\mathfrak{T}op$.

2.1.7 Definition. We define the *spherical arrangement* $\mathbb{S}\mathcal{A}$ derived from $\mathcal{A}$ by $\mathbb{S}\mathcal{A} := \{\mathbb{S}A \colon A \in \mathcal{A}\}$. The union $\bigcup \mathbb{S}\mathcal{A}$ of $\mathbb{S}\mathcal{A}$ is usually called the *link of* $\mathcal{A}$ and the complement $\mathbb{S}V \setminus \bigcup \mathbb{S}\mathcal{A}$ is homotopy equivalent the the complement $V \setminus \bigcup \mathcal{A}$ of $\mathcal{A}$.

We will describe the homotopy type and the homology groups of the link of $\mathcal{A}$, as well as the cohomology groups of the complement, which are isomorphic to $H_*(\mathbb{S}V, \bigcup \mathbb{S}\mathcal{A})$ via Poincaré duality.

2.1.8 Definition. From the functions $f^p \colon \mathbb{K}^{d(p)+1} \times \Delta[p, V] \to V$ we derive functions $\mathbb{S}(f^p) \colon \mathbb{S}(\mathbb{K}^{d(p)+1}) \times \Delta[p, V] \to \mathbb{S}(V)$ by $\mathbb{S}(f^p)(x, y) := \mathbb{S}(f^p(\bullet, y))(x)$. More explicitly

$$\begin{aligned} &\mathbb{S}(f^p) \colon \mathbb{S}(\mathbb{K}^{d(p)+1}) \times \Delta[p, V] \to \mathbb{S}(V) \\ &\left([(\mu_0, \ldots, \mu_{d(p)})], \sum_{i=0}^{r} \lambda_i u_i \right) \mapsto \left[\sum_{j=0}^{d(p)} \sum_{i=0}^{r} \mu_j \lambda_i x_j^{u_i} \right]. \end{aligned} \tag{2.2}$$

We note that $\mathbb{S}(\mathbb{R}^{d(p)+1}) \approx \mathbb{S}^{d(p)}$ and $\mathbb{S}(\mathbb{C}^{d(p)+1}) \approx \mathbb{S}^{2d(p)+1}$.

2.1.9 Proposition. *The maps $\mathbb{S}(f^p)$ induce a homotopy equivalence*

$$\operatorname{hcolim} \mathbb{S}(F^{Q_{[0,n)}}) \xrightarrow{\simeq} \bigcup \mathbb{S}\mathcal{A}.$$

Proof. By Proposition 2.1.4 the conditions of Proposition 1.3.5 are satisfied. □

This shows that the homotopy type of $\bigcup \mathbb{S}\mathcal{A}$ depends on the intersection poset Q equipped with the dimension function d only. However, a different description of this homotopy type is more usual.

2.1.10 Proposition. *Let $M \in \mathfrak{T}op^{Q^o{}_{[0,n)}}$ be defined by $M(q) := \mathbb{S}(\mathbb{K}^{d(q)+1})$ and all non-identity morphisms being constant to the base-point. Then* $\operatorname{hcolim} M \simeq \bigcup \mathbb{S}\mathcal{A}$.

Proof. We want to compare the diagrams M and $\mathbb{S}(F^{Q_{[0,n)}})$. We have $M(q) = \mathbb{S}(F^{Q_{[0,n)}})(q)$ for all q, and the inclusion maps in $\mathbb{S}(F^{Q_{[0,n)}})$ are homotopic to the constant maps in M. These homotopies can be arranged to form a ZŽ–map.

Let $H\colon \mathbb{S}^\infty \times I \to \mathbb{S}^\infty$ be a base-point preserving homotopy from the identity to the constant map satisfying $H[\mathbb{S}^k \times I] \subset \mathbb{S}^{k+1}$ for all k. The latter condition can be achieved for example by cellular approximation. We define maps f_p^q for $p \le q$, $p, q \in Q_{[0,n)}$ by

$$\begin{aligned} f_p^q\colon M(p) \times \Delta[p,q] &\to \mathbb{S}(F^{Q_{[0,n)}})(q) \\ \left(x, \sum_{i=0}^{r} \lambda_i u_i\right) &\mapsto \begin{cases} H(x, 1-\lambda_0), & u_0 = p, \\ *, & u_0 > p. \end{cases} \end{aligned}$$

For $p < p'$, we have $f_{p'}^q(M(p \leftarrow p')(x), t) = H(*, t) = * = H(x, 1) = f_p^q(x, t)$. For $q < q'$, $f_p^{q'}(x,t) = F^Q(f_p^q(x,t))$. Therefore, the maps f_p^q induce a map of ${Q^o}_{[0,n)}$-diagrams $M \times_Q EQ \to F^{Q_{[0,n)}}$. Since f_p^p is a homeomorphism, this map of diagrams is a ZŽ–map. By Proposition 1.1.20 it induces a homotopy equivalence $\operatorname{hcolim} M \xrightarrow{\simeq} \operatorname{hcolim} \mathbb{S}(F^{Q_{[0,n)}})$. □

2.1.11 Remark. Of course, it would have been just as easy to give a ZŽ–map $M \times_Q EQ \to \mathbb{S}(D^{\mathcal{A}})$ directly.

We turn to homology calculations.

2.1.12 Proposition. *The map*

$$\begin{aligned} S(\mathbb{S}(F^Q)) \otimes_Q B(Q) &\to S(\mathbb{S}(D^{\mathcal{A}})) \\ c \times \left(p \leftarrow q_0 \leftarrow \ldots \leftarrow q_n \leftarrow p'\right) &\mapsto \mathbb{S}(f^p)_*(c \times \langle q_0, \ldots, q_n \rangle) \end{aligned} \tag{2.3}$$

is a ZŽ–map and therefore induces isomorphisms

$$H(S(\mathbb{S}(F^Q)) \otimes_Q B(Q) \otimes_Q K^u) \xrightarrow{\cong} H_*\left(\bigcup \mathbb{S}\mathcal{A}\right) \tag{2.4}$$

and

$$H(S(\mathbb{S}(F^Q)) \otimes_Q B(Q) \otimes_Q K^p) \xrightarrow{\cong} H_*\left(\mathbb{S}V, \bigcup \mathbb{S}\mathcal{A}\right). \tag{2.5}$$

Proof. This is an application of Proposition 1.3.12. The conditions of that Proposition are met because of Proposition 2.1.4. □

This argument is not specific to the functor $\mathbb{S}\colon \mathfrak{V} \to \mathfrak{T}op$ and we will investigate similar functors later on.

Proposition 2.1.12 already describes the homology of the link and, via Poincaré duality, the cohomology of the complement in terms of the intersection poset Q and the dimension function d. We will now simplify this description.

2.1.13 Definition and Proposition. *Let* $e_n = [c_n]$ *be a generator of* $\tilde{H}_n(\mathbb{S}^n)$ *and* $b_n \in S_{n+1}(\mathbb{S}^{n+1})$ *with* $\mathfrak{d} b_n = c_n$. *Also let* $a := \langle 1 \rangle \in C_0(S^0)$. *The map*

$$\begin{aligned} H(S(\mathbb{S}(F^Q))) \otimes_Q B(Q) &\to S(\mathbb{S}(F^Q)) \\ [a] \otimes \big(p \leftarrow q_0 \leftarrow \ldots \leftarrow q_n \leftarrow p'\big) &\mapsto \begin{cases} a, & n = 0 \\ 0, & n > 0 \end{cases} \\ e_k \otimes \big(p \leftarrow q_0 \leftarrow \ldots \leftarrow q_n \leftarrow p'\big) &\mapsto \begin{cases} c_k, & n = 0, p = q_0 \\ (-1)^{k+1} b_k, & n = 1, p = q_0 < q_1 \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{2.6}$$

is a ZŽ–map.

Proof. The map $h_r \colon H(\mathbb{S}(\mathbb{K}^{r+1})) \to S(\mathbb{S}(\mathbb{K}^{r+1}))$ mapping e_k to c_k and $[a]$ to a is a chain homotopy equivalence. If $r < s$ and $i \colon \mathbb{S}(\mathbb{K}^{r+1}) \to \mathbb{S}(\mathbb{K}^{s+1})$ is the inclusion map, then $e_k \mapsto b_k$ is a chain homotopy from $h_s \circ H(i)$ to $S(i) \circ h_r$. The map (2.6) can now be seen as a special case of that from Proposition 1.2.13 □

2.1.14 Proposition. *There are isomorphisms*

$$H(H(S(\mathbb{S}(F^Q))) \otimes_Q B(Q) \otimes_Q K^u) \xrightarrow{\cong} H_*\left(\bigcup \mathbb{S}\mathcal{A}\right) \tag{2.7}$$

and

$$H(H(S(\mathbb{S}(F^Q))) \otimes_Q B(Q) \otimes_Q K^p) \xrightarrow{\cong} H_*\left(\mathbb{S}V, \bigcup \mathbb{S}\mathcal{A}\right). \tag{2.8}$$

Proof. By Proposition 1.2.15, the ZŽ–maps from Proposition 2.1.12 and (2.6) combine to yield ZŽ–maps inducing the desired isomorphisms. □

2.1.15 Proposition. *For* $\mathbb{K} = \mathbb{R}$ *the above maps induce isomorphisms*

$$H_*\left(\bigcup \mathbb{S}\mathcal{A}\right) \cong H_*(\Delta(Q_{[0,n)})) \oplus \bigoplus_{q \in Q_{[0,n)}} H_*(\Delta[q, V), \Delta(q, V))[-d(q)],$$

$$\begin{aligned} H_*\left(\mathbb{S}V, \bigcup \mathbb{S}\mathcal{A}\right) &\cong H_*(\Delta Q_{[0,n]}, \Delta Q_{[0,n)}) \\ &\oplus \bigoplus_{q \in Q_{[0,n]}} H_*(\Delta[q, V], \Delta[q, V) \cup \Delta(q, V])[-d(q)]. \end{aligned}$$

Proof. The Q^o–diagram $H(\mathbb{S}(F^Q))$ decomposes as

$$H(\mathbb{S}(F^Q)) = P \oplus \bigoplus_{q \in Q_{[0,n]}} O_q \tag{2.9}$$

with P corresponding to the class of a 0-simplex and O_q to a generator of $\tilde{H}_{d(q)}(\mathbb{S}^{d(q)})$, i.e.

$$P(q) \cong \begin{cases} R, & 0 \le d(q) \le n, \\ 0, & d(q) = -1, \end{cases} \tag{2.10}$$

and all morphism the identity where possible, and

$$O_q(p) \cong \begin{cases} R[-d(q)], & p = q, \\ 0, & p \neq q. \end{cases} \tag{2.11}$$

Now

$$\begin{aligned} H(P \otimes_Q B(Q) \otimes_Q K^u) &\cong H_*(\Delta Q_{[0,n)}), \\ H(P \otimes_Q B(Q) \otimes_Q K^p) &\cong H_*(\Delta Q_{[0,n]}, \Delta Q_{[0,n)}), \\ H(Q_q \otimes_Q B(Q) \otimes_Q K^u) &\cong H_*(\Delta[q,V), \Delta(q,V))[-d(q)], \\ H(Q_q \otimes_Q B(Q) \otimes_Q K^p) &\cong H_*(\Delta[q,V], \Delta[q,V) \cup \Delta(q,V])[-d(q)] \end{aligned}$$

proving the proposition. □

2.1.16 Proposition. *For* $\mathbb{K} = \mathbb{R}$

$$\tilde{H}_*\left(\bigcup \mathbb{S}\mathcal{A}\right) \cong \bigoplus_{q\in[\perp,V)} \tilde{H}_*(\Delta(q,V))[-(d(q)+1)], \tag{2.12}$$

$$H_*\left(\mathbb{S}V, \bigcup \mathbb{S}\mathcal{A}\right) \cong \mathbb{Z}[-n] \oplus \bigoplus_{q\in[\perp,V)} \tilde{H}_*(\Delta(q,V))[-d(q)-2]. \tag{2.13}$$

Proof. Since $[q,V)$ has, for $q < V$, the minimum q, $\Delta[q,V)$ is acyclic and $H_k(\Delta[q,V), \Delta(q,V)) \cong \tilde{H}_{k-1}(\Delta(q,V))$. If $\perp \in Q_{[0,n)}$, then $Q_{[0,n)}$ has $\perp$ as its minimum and $\tilde{H}_*(\Delta Q_{[0,n)}) = 0$. Otherwise, $\tilde{H}_*(\Delta Q_{[0,n)}) = \tilde{H}_*(\Delta(\perp,V))$. This proves the first isomorphism.

$H_k(\Delta Q_{[0,n]}, \Delta Q_{[0,n)}) \cong \tilde{H}_{k-1}(\Delta_{[0,n)})$ and for $q < V$

$$H_k(\Delta[q,V], \Delta[q,V) \cup \Delta(q,V]) \cong H_{k-1}(\Delta[q,V), \Delta(q,V)),$$

and these have been dealt with in the preceding paragraph. Also

$$H_*(\Delta[V,V], \Delta[V,V) \cup \Delta(V,V])[-d(V)] = H_*(\Delta\{V\})[-n] \cong \mathbb{Z}[-n],$$

completing the proof of the second isomorphism. □

Projective arrangements

We now come to our main object of study, projective arrangements.

For the functor P associating to a $\mathbb{K}$–vector–space u the projective space Pu we proceed just as for the functor $\mathbb{S}$ before.

2.1.17 Proposition. *The maps* $P(f^p)$ *induce a homotopy equivalence*

$$\operatorname{hcolim} P(F^{Q_{[0,n)}}) \xrightarrow{\simeq} \bigcup P\mathcal{A}.$$

Proof. Again, by Proposition 2.1.4 the conditions of Proposition 1.3.5 are satisfied. □

2.1.18 Remark. This is the same description of the homotopy type of a projective arrangement as in [WZŽ99, Prop. 5.9]. Our methods are very similar and also applicable to the other cases considered there. Our approach which makes ZŽ–maps more central has the advantage of explicitly constructing a homotopy equivalence.

2.1.19 Proposition. *The map*

$$\begin{aligned} S(P(F^Q)) \otimes_Q B(Q) &\to S(P(D^{\mathcal{A}})) \\ c \times \left(p \leftarrow q_0 \leftarrow \ldots \leftarrow q_n \leftarrow p'\right) &\mapsto P(f^p)_*(c \times \langle q_0, \ldots, q_n \rangle) \end{aligned} \tag{2.14}$$

is a ZŽ–map and therefore induces isomorphisms

$$H(S(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^u) \xrightarrow{\cong} H_* \left(\bigcup P\mathcal{A} \right) \tag{2.15}$$

and

$$H(S(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^p) \xrightarrow{\cong} H_* \left(PV, \bigcup P\mathcal{A} \right). \tag{2.16}$$

Proof. Just as Proposition 2.1.12 □

We simplify this description, starting with the most difficult case. The following result will actually also hold in the other cases treated afterwards.

2.1.20 Proposition. *Let $\mathbb{K} = \mathbb{R}$ and $R = \mathbb{Z}$. Then*

$$H_* \left(\bigcup P\mathcal{A} \right) \cong H(H(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^u) \tag{2.17}$$

and

$$H_* \left(PV, \bigcup P\mathcal{A} \right) \cong H(H(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^p). \tag{2.18}$$

Proof. Let $M \in \mathfrak{d}\mathfrak{A}b^{Q^o}$ be defined by

$$M(q)_k = \begin{cases} \mathbb{Z}, & 0 \le k \le d(q) \\ 0, & \text{otherwise,} \end{cases} \qquad M(q)_{k+1} \xrightarrow{\mathfrak{d}} M(q)_k = \begin{cases} 2, & 0 \le k < d(q),\ k \text{ odd,} \\ 0, & \text{otherwise,} \end{cases}$$

and the morphisms in M the identity where possible. It follows from the usual cell decomposition of $\mathbb{R}P^k$ with one cell per dimension that there exists a chain map of Q^o-diagrams $M \to F^Q$ inducing an isomorphism in homology. M allows a direct sum decomposition $M = \bigoplus_i M^i$ with $M^i_k := M_k$ for $k \in \{2i-1, 2i\}$ and $M^i_k := 0$ otherwise. The homology of M^i is nonzero in a single dimension only, hence Proposition 1.2.18 is applicable and yields the result together with Proposition 2.1.19. □

Possibly looking less attractive, these are as explicit combinatorial formulas as those derived in Proposition 2.1.24 for other cases. Nevertheless, for calculating the groups in question, the following may be helpful.

2.1.21 Proposition. *There are isomorphisms*

$$H(H_0(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^u) \cong H_*(\Delta Q_{[0,n)}),$$
$$H(H_{2i}(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^u) \cong 0 \quad \textit{for } i > 0,$$

and there is a long exact sequence

$$\cdots \to H_k(\Delta Q_{[2i+2,n)}) \xrightarrow{2j_*} H_k(\Delta Q_{[2i+1,n)}) \to H_k(H_{2i+1}(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^u) \to H_{k-1}(\Delta Q_{[2i+2,n)}) \xrightarrow{2j_*} \cdots$$

for $i \geq 0$*, where* j *is the inclusion map. Similarly, there are isomorphisms*

$$H(H_0(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^p) \cong H_*(\Delta Q_{[0,n]}, \Delta Q_{[0,n)}),$$
$$H(H_{2i}(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^p) \cong 0 \quad \textit{for } i > 0,$$

and there is a long exact sequence

$$\cdots \to H_k(\Delta Q_{[2i+2,n]}, \Delta Q_{[2i+2,n)}) \xrightarrow{2j_*} H_k(\Delta Q_{[2i+1,n]}, \Delta Q_{[2i+1,n)}) \to H_k(H_{2i+1}(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^p) \to H_{k-1}(\Delta Q_{[2i+2,n]}, \Delta Q_{[2i+2,n)}) \xrightarrow{2j_*} \cdots$$

for $i \geq 0$.

Proof. The parts dealing with $H(H_{2i+1}(P(F^Q)) \otimes_Q B(Q) \otimes_Q K)$ for $K = K^u$ or $K = K^p$ are the interesting ones. With $M_k \in \mathfrak{Ab}^{Q^o}$ as in the preceding proof, there is a short exact sequence $0 \to M_{2i+2} \xrightarrow{2} M_{2i+1} \to H_{2i+1}(P(F^Q)) \to 0$. Since $B(Q) \otimes_Q K$ is free, this induces a short exact sequence

$$0 \to M_{2i+2} \otimes_Q B(Q) \otimes_Q K \xrightarrow{2} M_{2i+1} \otimes_Q B(Q) \otimes_Q K \to H_{2i+1}(P(F^Q)) \otimes_Q B(Q) \otimes_Q K \to 0$$

and the long exact sequences stated in the proposition. □

The following cases are easier.

2.1.22 Definition and Proposition. *Let* $\mathbb{K} = \mathbb{C}$*, or let* $\mathbb{K} = \mathbb{R}$ *and* $R = \mathbb{Z}_2$*. Let* $e_n = [c_n]$ *be a generator of* $H_{2n}(\mathbb{C}P^n)$ *or* $H_n(\mathbb{R}P^n; \mathbb{Z}_2)$ *respectively. For* $i \geq n$ *we define* $e_n^i \in H_*(\mathbb{K}P^i; R)$ *by* $e_n^i = [c_n]$*. The maps*

$$\begin{aligned} H(P(F^Q(u))) &\to S(P(F^Q(u))), \\ e_n^i &\mapsto c_n \end{aligned} \tag{2.19}$$

for $u \in Q$ *induce isomorphisms in homology and form a chain map of* Q^o*–diagrams* $H(S(P(F^Q))) \to S(P(F^Q))$*.* □

2.1.23 Proposition. *For $\mathbb{K} = \mathbb{C}$, as well as for $\mathbb{K} = \mathbb{R}$ and $R = \mathbb{Z}_2$, there are isomorphisms*

$$H(H(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^u) \xrightarrow{\cong} H_* \left(\bigcup P\mathcal{A} \right) \tag{2.20}$$

and

$$H(H(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^p) \xrightarrow{\cong} H_* \left(PV, \bigcup P\mathcal{A} \right). \tag{2.21}$$

Proof. The composition of (2.19) with the ZŽ–map (2.14) again yields a ZŽ–map and can therefore be substituted for (2.14) in Proposition 2.1.19. □

2.1.24 Proposition. *The above maps induce isomorphisms*

$$H_* \left(\bigcup P\mathcal{A}; \mathbb{Z}_2 \right) \cong \bigoplus_{k=0}^{n} H_*(\Delta Q_{[k,n)}; \mathbb{Z}_2)[-k], \tag{2.22}$$

$$H_* \left(PV, \bigcup P\mathcal{A}; \mathbb{Z}_2 \right) \cong \bigoplus_{k=0}^{n} H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}; \mathbb{Z}_2)[-k] \tag{2.23}$$

for $\mathbb{K} = \mathbb{R}$, and

$$H_* \left(\bigcup P\mathcal{A} \right) \cong \bigoplus_{k=0}^{n} H_*(\Delta Q_{[k,n)})[-2k], \tag{2.24}$$

$$H_* \left(PV, \bigcup P\mathcal{A} \right) \cong \bigoplus_{k=0}^{n} H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})[-2k] \tag{2.25}$$

for $\mathbb{K} = \mathbb{C}$.

Proof. For $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$ and $R = \mathbb{Z}_2$, there is a direct sum decomposition

$$H(P(F^Q)) = \bigoplus_{k=0}^{n} O_k, \qquad O_k(q) \cong \begin{cases} R[-mk], & d(q) \geq k, \\ 0, & d(q) < k \end{cases} \tag{2.26}$$

with $m = \dim_{\mathbb{R}} \mathbb{K}$, and $O_k(q \leftarrow q') = \mathrm{id}$ for $d(q) \geq k$. We choose the isomorphism in such a way that $1 \in O_k(q)$ corresponds to the canonical generator of $H_{mk}(\mathbb{K}P^{d(q)})$. Now

$$H(O_k \otimes_Q B(Q) \otimes_Q K^u) \cong H_*(\Delta Q_{[k,n)})[-mk], \tag{2.27}$$

$$H(O_k \otimes_Q B(Q) \otimes_Q K^p) \cong H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})[-mk], \tag{2.28}$$

and the result follows. □

We assemble the different maps to get the more direct handle on these isomorphisms that we will need when discussing intersection products. For Section 2.3 this, together with the preceding proposition, is the main result of this section.

2.1.25 Definition and Proposition. *We define*

$$\begin{aligned} f^k \colon \mathbb{K}P^k \times (\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) &\to \left(PV, \bigcup P\mathcal{A}\right) \\ \left([(\mu_0,\dots,\mu_k)], \sum_{i=0}^{r} \lambda_i u_i\right) &\mapsto \left[\sum_{j=0}^{k}\sum_{i=0}^{r} \mu_j \lambda_i x_j^{u_i}\right] \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} h_k \colon H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) &\to H_*\left(PV, \bigcup P\mathcal{A}\right), \\ H_*(\Delta Q_{[k,n)}) &\to H_*\left(\bigcup P\mathcal{A}\right) \\ c &\mapsto f^k_*([\mathbb{K}P^k] \times c), \end{aligned} \tag{2.30}$$

where $[\mathbb{K}P^k]$ is the orientation class of $\mathbb{K}P^k$, over $\mathbb{Z}_2$ in case $\mathbb{K} = \mathbb{R}$. Then all of the isomorphisms from Proposition 2.1.24 are given by $\sum_{k=0}^{n} h_k$.

For $\mathbb{K} = \mathbb{C}$, the maps h_k do not depend on the choice of the function x.

Proof. It is trivial to check that $\sum_k h_k$ is indeed the composition of the maps (2.14), (2.19), and (2.26). The non-dependence on the choice of x follows from Proposition 2.1.2. □

2.1.26 Remark. The calculations regarding the homology of the projective arrangements could equally well been carried out using the power set of $\mathcal{A}$ instead of the intersection poset, i.e., in the words of Remark 1.3.8, using naive resolutions instead of economical ones. This is because the key fact that all maps in the diagram $H(P(F^Q))$ are injective (assuming $\mathbb{K} = \mathbb{C}$ or $R = \mathbb{Z}_2$) still holds when Q denotes the power set of $\mathcal{A}$. For spherical arrangements however, naive resolutions would have been less appropriate, because the fact that all non-identity homomorphisms in the diagram $H_k(\mathbb{S}(F^Q))$ are zero for $k > 0$ would be lost, because there are non-identity isomorphisms in F^Q when Q denotes the power set of $\mathcal{A}$, so extra care has to be taken.

Affine arrangements

Let H be a hyperplane in V. We will investigate the relationship between the projective arrangement $P\mathcal{A}$, the projective arrangement induced on PH, and the arrangement induced on the affine space $PV \setminus PH$.

We set $\mathcal{A}^H := \{A \cap H \colon A \in \mathcal{A}\}$ and denote the intersection poset of $\mathcal{A}^H$ by Q^H.

We also set $\bar{Q} := Q \setminus Q^H = \{u \in Q \colon u \not\subset H\}$. This is the poset of non-empty intersections of the affine arrangement in $PV \setminus PH$.

2.1.27 Definition. We call the arrangement $\mathcal{A}$ a *≥ 2–arrangement*, if $u < v$ implies $d(v) - d(u) \geq 2$ for all $u, v \in Q$.

2.1.28 Notation. Let X be a compact m-manifold and $A \subset X$ a closed subset. If $X \setminus A$ is orientable and such an orientation is chosen, we denote by $[X, A]$ the corresponding orientation class in $H_m(X, A)$.

2.1.29 Definition and Proposition. *We consider for $u \in \bar{Q}$ systems of vectors $b_0^u, \ldots, b_{d(u)-1}^u \in u \cap H$, $x_u^u \in u$, $x_v^u \in v \cap H$ for $v > u$, such that*

$$\begin{aligned} f^u \colon \mathbb{K}P^{d(u)} \times \Delta[u, V] &\to PV \\ \left([\mu_0 : \cdots : \mu_{d(u)}], \sum_{j=0}^{r} \lambda_j v_j \right) &\mapsto \left[\sum_{i=0}^{d(u)-1} \mu_i b_i^u + \mu_{d(u)} \sum_{j=0}^{r} \lambda_j x_{v_j}^u \right] \end{aligned} \tag{2.31}$$

is well defined, i.e. the term on the right hand side never equals zero. In particular the system of vectors $b_0^u, \ldots, b_{d(u)-1}^u, x_u^u$ determines, in case $\mathbb{K} = \mathbb{R}$, an orientation of $(Pu, P(u \cap H))$ via the homeomorphism

$$\begin{aligned} \bar{f}^u \colon (\mathbb{K}P^{d(u)}, \mathbb{K}P^{d(u)-1}) &\to (Pu, P(u \cap H)) \\ [\mu_0 : \cdots : \mu_{d(u)}] &\mapsto \left[\sum_{i=0}^{d(u)-1} \mu_i b_i^u + \mu_{d(u)} x_u^u \right] \end{aligned} \tag{2.32}$$

which is obtained by restricting f^u.

Such vectors exist, and the induced orientation can be prescribed in case $\mathbb{K} = \mathbb{R}$. There are induced maps

$$\begin{aligned} h_u \colon H_*(\Delta[u, V], \Delta[u, V) \cup \Delta(u, V]) &\to H_* \left(PV, PH \cup \bigcup P\mathcal{A} \right), \\ H_*(\Delta[u, V), \Delta(u, V)) &\to H_* \left(PV, PH \cup \bigcup P\mathcal{A} \right), \\ c &\mapsto f_*^u([\mathbb{K}P^{d(u)}, \mathbb{K}P^{d(u)-1}] \times c). \end{aligned} \tag{2.33}$$

For $\mathbb{K} = \mathbb{C}$ these are independent of the choice of b^u and x^u. For $\mathbb{K} = \mathbb{R}$ and if $\mathcal{A}$ is a ≥ 2–arrangement, they depend only on the induced orientation of $(Pu, P(u \cap H))$. The maps

$$\bigoplus_{u \in \bar{Q}} H_*(\Delta[u, V], \Delta[u, V) \cup \Delta(u, V])[-md(u)] \xrightarrow{\sum_u h_u} H_* \left(PV, PH \cup \bigcup P\mathcal{A} \right), \tag{2.34}$$

$$\bigoplus_{u \in \bar{Q}} H_*(\Delta[u, V), \Delta(u, V))[-md(u)] \xrightarrow{\sum_u h_u} H_* \left(\bigcup P\mathcal{A}, PH \cap \bigcup P\mathcal{A} \right) \tag{2.35}$$

with $m = \dim_{\mathbb{R}} \mathbb{K}$ are isomorphisms.

2.1.30 Remark. If $d(\perp) \geq 0$, that is if $\perp \in \bar{Q}$, the affine arrangement is a (central) linear arrangement and $\bigcup P\mathcal{A} / (PH \cap \bigcup P\mathcal{A})$ is homeomorphic to the suspension of the link of that arrangement. Therefore, these isomorphisms generalize those of Proposition 2.1.15. The proof of Proposition 2.1.16 explains how to deal with the first summands in Proposition 2.1.15 when comparing it with the current proposition.

Proof. For the vectors b_i^u and x_v^u to define a map f^u, it suffices that for every chain $u \leq v_0 < \cdots < v_r$ the vectors $b_0^u, \ldots, b_{d(u)-1}^u, x_{v_0}^u, \ldots, x_{v_r}^u$ are linearly independent. This will be the case, if $(b_i^u)_i$ is a basis of $u \cap H$, $x_u^u \in u \setminus H$ and $x_v^u \in v \setminus w$ for $u \leq w < v$. The systems of vectors with this property form a non-empty Zariski-open set, which is therefore dense in the set of all allowed systems, and it is path-connected for $\mathbb{K} = \mathbb{C}$. For $\mathbb{K} = \mathbb{R}$ and if $\mathcal{A}$ is a ≥ 2-arrangement, it has at most four components, distinguished by the orientation of $u \cap H$ that $(b_i^u)_i$ defines and the component of $u \setminus H$ that contains x_u^u. Since replacing all of the b_i^u and x_v^u by their negatives does not change f^u, we may restrict x_u^u to one of the components of $u \setminus H$, and we see that the homotopy type of f^u depends only on the orientation induced on $(Pu, P(u \cap H))$.

We set $A(u) := (Pu, P(u \cap H))$ for $u \in \bar{Q}$. By Proposition 1.3.13 there are isomorphisms

$$H(S(A(\bar{D}^{\mathcal{A}})) \otimes_{\bar{Q}} K^p) \xrightarrow{\cong} H_*\left(PV, PH \cup \bigcup P\mathcal{A}\right) \tag{2.36}$$

and

$$H(S(A(\bar{D}^{\mathcal{A}})) \otimes_{\bar{Q}} K^u) \xrightarrow{\cong} H_*\left(\bigcup P\mathcal{A}, PH \cap \bigcup P\mathcal{A}\right). \tag{2.37}$$

Let $O_u \in \mathfrak{dAb}^{\bar{Q}^o}$ be defined by

$$O_u(q) := \begin{cases} R[-md(u)], & u = q, \\ 0, & u \neq q \end{cases}$$

and $g_u \in \mathrm{Hom}_{\mathfrak{C}^o}(O_u \otimes_{\bar{Q}} B(\bar{Q}), S(A(\bar{D}^{\mathcal{A}})))$ by

$$1_R \otimes (u \leftarrow v_0 \leftarrow \cdots \leftarrow v_r \leftarrow u') \mapsto f_*^u(o_{d(u)} \times \langle v_0, \ldots, v_r \rangle),$$

where o_k is a relative cycle representing the orientation class of $(\mathbb{K}P^k, \mathbb{K}P^{k-1})$. This is well defined, because $f_*^u(o_{d(u)} \times \langle v_0, \ldots, v_r \rangle) = 0$ for $v_0 > u$. It is a chain map, because $f_*^u(\mathfrak{d} o_{d(u)} \times \langle v_0, \ldots, v_r \rangle) = 0$. We have

$$H(O_u \otimes_{\bar{Q}} B(\bar{Q}) \otimes_{\bar{Q}} K^p) \cong H_*(\Delta[u, V], \Delta[u, V) \cup \Delta(u, V])[-md(u)],$$
$$H(O_u \otimes_{\bar{Q}} B(\bar{Q}) \otimes_{\bar{Q}} K^u) \cong H_*(\Delta[u, V), \Delta(u, V))[-md(u)],$$

and the maps h_u equal $H(g_u \otimes \mathrm{id})$ composed with the above isomorphisms. The map $\sum_u \colon \bigoplus_u O_u \otimes_{\bar{Q}} B(\bar{Q}) \to S(A(\bar{D}^{\mathcal{A}}))$ is a ZŽ–map to a free diagram, therefore $\sum_u h_u$ is an isomorphism. □

That the direct sum decomposition $\bigoplus_u H_*(\Delta[u, V], \Delta[u, V) \cup \Delta(u, V])[-md(u)]$ is finer than the decomposition $\bigoplus_k H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})[-mk]$ can be regarded as a reason why products in affine arrangements are technically simpler than products in projective arrangements. In particular, there seems to be no direct generalization of the proof of Proposition 2.2.12 to a proof of Proposition 2.3.13.

We illustrate the connection between projective and affine arrangements by giving another description of the homomorphisms h_u. For simplicity, we only cover the case $\mathbb{K} = \mathbb{C}$ completely.

2.1.31 Definition and Proposition. *There is a function x as in Proposition 2.1.1 with the additional property that $x_j^u \in H$ for $j < \dim_{\mathbb{K}}(u \cap H)$. If such a function is used in definition of the map f^k from Definition 2.1.25, then f^k maps the subspace $\mathbb{K}P^k \times \Delta Q_{(k,n]} \cup \mathbb{K}P^{k-1} \times \Delta Q_{[k,n]}$ to PH and therefore induces maps*

$$\begin{aligned} \bar{h}_k\colon H_*(\Delta\bar{Q}_{[k,n]}, \Delta\bar{Q}_{[k,n)} \cup \Delta\bar{Q}_{(k,n]}) &\to H_*\left(PV, PH \cup \bigcup P\mathcal{A}\right), \\ H_*(\Delta\bar{Q}_{[k,n)}, \Delta\bar{Q}_{(k,n)}) &\to H_*\left(\bigcup P\mathcal{A}, PH \cap \bigcup P\mathcal{A}\right), \\ c &\mapsto f_*^k([\mathbb{K}P^k, \mathbb{K}P^{k-1}] \times c). \end{aligned} \tag{2.38}$$

Denoting the inclusion map $\Delta[u,V] \to \Delta Q_{[d(u),n]}$ by i^u, the maps

$$\sum_{u\in\bar{Q}_{\{k\}}} i_*^u\colon \bigoplus_{u\in\bar{Q}_{\{k\}}} H_*(\Delta[u,V], \Delta[u,V) \cup \Delta(u,V]) \xrightarrow{\cong} H_*(\Delta\bar{Q}_{[k,n]}, \Delta\bar{Q}_{[k,n)} \cup \Delta\bar{Q}_{(k,n]}) \tag{2.39}$$

and

$$\sum_{u\in\bar{Q}_{\{k\}}} i_*^u\colon \bigoplus_{u\in\bar{Q}_{\{k\}}} H_*(\Delta[u,V), \Delta(u,V)) \xrightarrow{\cong} H_*(\Delta\bar{Q}_{[k,n)}, \Delta\bar{Q}_{(k,n)}) \tag{2.40}$$

are isomorphisms.

For $\mathbb{K} = \mathbb{C}$ we have $h_u = \bar{h}_k \circ i_*^u$ and therefore the maps

$$\sum \bar{h}_k\colon \bigoplus_k H_*(\Delta\bar{Q}_{[k,n]}, \Delta\bar{Q}_{[k,n)} \cup \Delta\bar{Q}_{(k,n]})[-2k] \to H_*\left(PV, PH \cup \bigcup P\mathcal{A}\right), \tag{2.41}$$

$$\bigoplus_k H_*(\Delta\bar{Q}_{[k,n)}, \Delta\bar{Q}_{(k,n)})[-2k] \to H_*\left(\bigcup P\mathcal{A}, PH \cap \bigcup P\mathcal{A}\right) \tag{2.42}$$

are isomorphisms.

Proof. Slightly modifying the construction in the proof of Proposition 2.1.1, we first choose $(x_j^u)_{0\le j<\dim(u\cap H)}$ (applying the proposition to $\mathcal{A}^H$) and then choose $x_{d(u)}^u \in u \setminus H$ for all $u \in \bar{Q}$. As in that proof, x has the desired property.

The maps in (2.39) and (2.39) are easily seen to be isomorphisms either directly at the chain level or by a Mayer-Vietoris argument using $\Delta\bar{Q}_{[k,n]} = \bigcup_{u\in\bar{Q}_{\{k\}}} \Delta[u,V]$ and $\Delta[u,V] \cap \Delta[v,V] \subset \Delta\bar{Q}_{(k,n]}$ for $u, v \in \bar{Q}_{\{k\}}$, $u \neq v$.

Let $u \in \bar{Q}$. Any system of points (y_j^v) for $v \ge u$ and $0 \le j \le d(u)$ defines maps

$$\begin{aligned} g\colon (\mathbb{K}P^{d(u)}, \mathbb{K}P^{d(u)-1}) \times (\Delta[u,V], \Delta[u,V) \cup \Delta(u,V]) &\to \left(PV, PH \cup \bigcup P\mathcal{A}\right) \\ (\mathbb{K}P^{d(u)}, \mathbb{K}P^{d(u)-1}) \times (\Delta[u,V), \Delta(u,V)) &\to \left(\bigcup P\mathcal{A}, PH \cap \bigcup P\mathcal{A}\right) \\ \left([(\mu_0, \dots, \mu_k)], \sum_{i=0}^r \lambda_i u_i\right) &\mapsto \left[\sum_{j=0}^k \sum_{i=0}^r \mu_j \lambda_i x_j^{u_i}\right], \end{aligned}$$

if the right hand side is always well defined. For $\mathbb{K} = \mathbb{C}$ the space of such systems of points is again path connected and hence the homotopy type of g independent of the choice of such a system.

Setting $y_j^v := x_j^v$ gives such a system, and for this choice $g = f^k \circ i^u$. Setting $y_j^v := b_j^u$ for $j < d(u)$ and $y_{d(u)}^v := x_v^u$ with b and x the systems from Definition 2.1.29 gives another such system, and for this one $g = f^u$. In case $\mathbb{K} = \mathbb{C}$ it follows that $f^k \circ i^u \simeq f^u$ and $h_u = \bar{h}_k \circ i_*^u$. □

2.1.32 Remark. For a complex affine arrangement the isomorphism

$$H_*\left(PV, PH \cup \bigcup P\mathcal{A}\right) \cong \bigoplus_k H_*(\Delta\bar{Q}_{[k,n]}, \Delta\bar{Q}_{[k,n)} \cup \Delta\bar{Q}_{(k,n]})[-2k],$$

and hence the isomorphism (2.34), can also be deduced combinatorially from the isomorphism (2.25) for projective arrangements by considering the arrangement $\mathcal{A}^+ := \mathcal{A} \cup \{H\}$. This is called an 'interesting exercise' in [GM88]. We sketch how to do this. To this end we denote the intersection poset of $\mathcal{A}^+$ by Q^+. For $0 \leq k < n$, the simplical complex $\Delta(Q^+_{[k,n)} \setminus \bar{Q}_{\{k\}})$ is acyclic, since it contains the cone $\Delta\{q\colon q \leq H, d(q) \geq k\}$ as a deformation retract. Therefore the first map in

$$\begin{aligned} H_*\big(\Delta Q^+_{[k,n]}, \Delta Q^+_{[k,n)}\big) &\xrightarrow{\cong} H_*\big(\Delta Q^+_{[k,n]}, \Delta Q^+_{[k,n)} \cup \Delta(Q^+_{[k,n]} \setminus \bar{Q}_{\{k\}})\big) \\ &\xleftarrow{\cong} H_*\big(\Delta\bar{Q}_{[k,n]}, \Delta\bar{Q}_{[k,n)} \cup \Delta\bar{Q}_{(k,n]}\big), \end{aligned}$$

which is induced by inclusion, is an isomorphism. The second map is also induced by inclusion and is an isomorphism by excision. The isomorphisms also hold in the trivial case $k = n$. This yields $H_*(PV, PH \cup \bigcup P\mathcal{A}) = H_*(PV, \bigcup P\mathcal{A}^+) \cong \bigoplus H_*\big(\Delta Q^+_{[k,n]}, \Delta Q^+_{[k,n)}\big)[-2k] \cong \bigoplus H_*\big(\Delta\bar{Q}_{[k,n]}, \Delta\bar{Q}_{[k,n)} \cup \Delta\bar{Q}_{(k,n]}\big)[-2k]$.

2.1.33 Remark. If the arrangement $\mathcal{A}$ is in general position with respect to H, then $\bar{Q} = Q_{[0,n]}$ and $q \mapsto q \cap H$ induces isomorphisms $\eta\colon Q_{k+1} \to Q_k^H$. For $\mathbb{K} = \mathbb{C}$ it follows from the construction in Proposition 2.1.31 that the diagram

$$\begin{array}{ccc} H_i(\Delta Q^H_{[k,n-1]}, \Delta Q^H_{[k,n-1)}) & \xrightarrow{h_k^H} & H_{i+2k}(PH, \bigcup P\mathcal{A}^H) \\ \downarrow{\scriptstyle (\eta^{-1})_*} & & \downarrow{\scriptstyle \mathrm{incl}_*} \\ H_i(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) & \xrightarrow{h_k} & H_{i+2k}(PV, \bigcup P\mathcal{A}) \\ \downarrow{\scriptstyle \mathrm{incl}_*} & & \downarrow{\scriptstyle \mathrm{incl}_*} \\ H_i(\Delta\bar{Q}_{[k,n]}, \Delta\bar{Q}_{(k,n]} \cup \Delta\bar{Q}_{[k,n)}) & \xrightarrow{\bar{h}_k} & H_{i+2k}(PV, PH \cup \bigcup P\mathcal{A}) \end{array}$$

commutes, since $f^k \circ (\mathrm{id}_{\mathbb{C}P^k} \times \eta^{-1})$ is a suitable function f_H^k for the definition of h_k^H. Both columns are part of a long exact sequence of the form

$$H_j(B, A \cap B) \to H_j(X, B) \to H_j(X, A \cup B) \xrightarrow{\partial} H_{j-1}(B, A \cap B)$$

for an excisive triad $(X;A,B)$. Because of the naturality of the connecting homomorphism, the diagram

$$\begin{array}{ccc} H_i(\Delta\bar{Q}_{[k,n]},\Delta\bar{Q}_{(k,n]}\cup\Delta\bar{Q}_{[k,n)}) & \xrightarrow{\bar{h}_k} & H_{i+2k}(PV,PH\cup\bigcup P\mathcal{A}) \\ \big\downarrow{\scriptstyle \eta_*\circ\partial} & & \big\downarrow{\scriptstyle \partial} \\ H_{i-1}(\Delta Q^H_{[k,n-1]},\Delta Q^H_{[k,n-1)}) & \xrightarrow{h^H_k} & H_{i+2k-1}(PH,\bigcup P\mathcal{A}^H) \end{array}$$

also commutes. Analogous commutative diagrams exist for the long exact sequence of the pair $(\bigcup P\mathcal{A},\bigcup P\mathcal{A}^H)$. If the arrangement $\mathcal{A}$ is not in general position with respect to H, then the same commutative diagrams exist, but the construction of the left column requires more care.

2.1.34 Remark (Gysin sequence). We continue the preceding remark. We set $A := PV\setminus PH$ and denote by $\mathcal{A}^A := PB\setminus PH\colon B\in\mathcal{A}$ the induced arrangement in this affine space. Using Poincaré duality, which we denote by D, we switch to cohomology and obtain the following commutative diagram with exact columns.

$$\begin{array}{ccc} \bigoplus_{k\geq 0} H_{2(n-k-1)-i}(\Delta Q^H_{[k,n-1]},\Delta Q^H_{[k,n-1)}) & \xrightarrow[\cong]{D\circ\sum_k h^H_k} & H^i(PH\setminus\bigcup P\mathcal{A}^H) \\ \big\downarrow{\scriptstyle (\eta^{-1})_*} & & \big\downarrow{\scriptstyle i^!} \\ \bigoplus_{k\geq 0} H_{2(n-k-1)-i}(\Delta Q_{[k,n]},\Delta Q_{[k,n)}) & \xrightarrow[\cong]{D\circ\sum_k h_k} & H^{i+2}(PV\setminus\bigcup P\mathcal{A}) \\ \big\downarrow{\scriptstyle \mathrm{incl}_*} & & \big\downarrow{\scriptstyle \mathrm{incl}^*} \\ \bigoplus_{k\geq 0} H_{2(n-k-1)-i}(\Delta\bar{Q}_{[k,n]},\Delta\bar{Q}_{(k,n]}\cup\Delta\bar{Q}_{[k,n)}) & \xrightarrow[\cong]{D\circ\sum_k \bar{h}_k} & H^{i+2}(A\setminus\bigcup\mathcal{A}^A) \\ \big\downarrow{\scriptstyle \eta_*\circ\partial} & & \big\downarrow \\ \bigoplus_{k\geq 0} H_{2(n-k-1)-i-1}(\Delta Q^H_{[k,n-1]},\Delta Q^H_{[k,n-1)}) & \xrightarrow[\cong]{D\circ\sum_k h^H_k} & H^{i+1}(PH\setminus\bigcup P\mathcal{A}^H) \end{array}$$

If the arrangement $\mathcal{A}^A$ is central, i.e. $\bigcap\mathcal{A}^A\neq\emptyset$, then there is a deformation retraction $\pi\colon PV\setminus\bigcup P\mathcal{A}\xrightarrow{\simeq} PH\setminus\bigcup P\mathcal{A}^H$ homotopy inverse to the inclusion i. Also $\bar{Q}=Q$. From the above we therefore obtain the following diagram.

$$\begin{array}{ccc} \bigoplus_{k\geq 0} H_{2(n-k-1)-i}(\Delta Q_{[k+1,n]},\Delta Q_{[k+1,n)}) & \xrightarrow[\cong]{D\circ\sum h^H_k\circ\eta_*} & H^i(PH\setminus\bigcup P\mathcal{A}^H) \\ \big\downarrow{\scriptstyle \mathrm{incl}_*} & & \big\downarrow{\scriptstyle \smile\alpha} \\ \bigoplus_{k\geq 0} H_{2(n-k-1)-i}(\Delta Q_{[k,n]},\Delta Q_{[k,n)}) & \xrightarrow[\cong]{D\circ i_!\circ\sum_k h_k} & H^{i+2}(PH\setminus\bigcup P\mathcal{A}^H) \\ \big\downarrow{\scriptstyle \mathrm{incl}_*} & & \big\downarrow{\scriptstyle \pi^*} \\ \bigoplus_{k\geq 0} H_{2(n-k-1)-i}(\Delta Q_{[k,n]},\Delta Q_{(k,n]}\cup\Delta Q_{[k,n)}) & \xrightarrow[\cong]{D\circ\sum\bar{h}_k} & H^{i+2}(A\setminus\bigcup\mathcal{A}^A) \\ \big\downarrow{\scriptstyle \partial} & & \big\downarrow \\ \bigoplus_{k\geq 0} H_{2(n-k-1)-i-1}(\Delta Q_{[k+1,n]},\Delta Q_{[k+1,n)}) & \xrightarrow[\cong]{D\circ\sum h^H_k\circ\eta_*} & H^{i+1}(PH\setminus\bigcup P\mathcal{A}^H) \end{array}$$

Here $\alpha\in H^2(PH)$ is the canonical generator. Also note $H_*(\Delta Q_{[0,n]},\Delta Q_{[0,n)})=0$, since $Q_{[0,n]}$ has a minimal element, which explains the missing of a summand in the first row. The map $\pi\colon A\setminus\bigcup\mathcal{A}^A\to PH\setminus\bigcup P\mathcal{A}^H$ is a fibre bundle with fibre $\mathbb{C}\setminus\{0\}$, α is its Thom class, and the right column of the diagram is its Gysin sequence.

If we start with a linear arrangement $\mathcal{A}$ in V, pass to the arrangement $\mathcal{A}\times\mathbb{C}$ in $V\times\mathbb{C}$, and set $H:=V\times\{0\}$, then $P(\mathcal{A}\times\mathbb{C})^H$ will be $P\mathcal{A}$ and $(\mathcal{A}\times\mathbb{C})^A$ will be isomorphic to the original arrangement $\mathcal{A}$. The map π constructed above will correspond to the quotient map $V\setminus\bigcup\mathcal{A}\to PV\setminus\bigcup P\mathcal{A}$, so that we have just described its cohomology Gysin sequence completely combinatorially. Just the isomorphism in the second row is possibly not as explicit as we would like it. This, however, will be remedied in a minute, when we study the intersection with the hyperplane H to obtain $i_!\circ h_k=h^H_{k-1}\circ\eta_*$.

Intersecting with a hyperplane

We will describe the map on the homology of an affine or a projective arrangement given by intersecting with a hyperplane, that is the transfer map of the inclusion of the hyperplane. In contrast to the material covered so far in this chapter, with the exception of the results mentioning cohomology, this depends on the projective or affine space in which the arrangement is contained being a manifold. The same is of course true for the intersection products treated later on, and the results obtained here will be the basis for the inductive step in the proof of the product formulas.

Let $\Lambda^H\colon V\to\mathbb{K}$ be a linear functional that vanishes on no element of $Q_{[0,2n]}$ and $H:=\ker\Lambda^H$. $\mathcal{A}$ induces an arrangement $\mathcal{A}^H:=\{A\cap H\colon A\in\mathcal{A}\}$ in H. If we denote the intersection poset of $\mathcal{A}^H$ by Q^H,

$$\begin{aligned}\eta\colon Q_{(0,n]}&\to Q^H_{[0,n-1]}\\ q&\mapsto q\cap H\end{aligned}$$

is an isomorphism lowering dimensions by one.

We consider the inclusion map $i\colon\big(PH,\bigcup P\mathcal{A}^H\big)\to(PV,\bigcup P\mathcal{A})$.

2.1.35 Proposition. *Let* $\mathbb{K}=\mathbb{C}$. *For* $c\in H_*(\Delta Q_{[k,n]},\Delta Q_{[k,n)})$ *we have*

$$i_!(h_k(c))=\begin{cases}h^H_{k-1}(\eta_*(c)), & k>0,\\ 0, & k=0\end{cases}$$

and in particular $\ker i_!=\operatorname{im}h_0$.

Proof. We first choose $(x^u_j)_{0\le j<k}$ with $x^u_j\in u\cap H$ satisfying the conditions of Definition 2.1.25 and therefore defining functions f^{k-1}_H and h^H_{k-1}. Now for each $u\in Q_{[k,n]}$ we choose $x^u_k\in u$ with $\Lambda^H(x^u_k)=1$. $(x^u_j)_{0\le j\le k}$ then also satisfies the conditions of Definition 2.1.25 and can be used to define f^k and h_k.

Indeed we calculate

$$\Lambda^H\left(\sum_{j=0}^{k}\sum_{i=0}^{r}\mu_j\lambda_i x_j^{u_i}\right)=\sum_{i=0}^{r}\mu_k\lambda_i=\mu_k.$$

First this implies that $f^k(x,y)\in H$ iff $x\in\mathbb{C}P^{k-1}\subset\mathbb{C}P^k$. In particular f^0 misses H, which proves that part of the proposition, and we now assume $k>0$. The equation also implies that for $x=[x_0:\cdots:x_{k-1}]\in\mathbb{C}P^{k-1}$ and $y\in\Delta(Q_{[k,n]})$ the map $\mu\mapsto f^k([x_0:\cdots:x_{k-1}:\mu],y)$ meets H transversally. Furthermore

$$\begin{aligned}f^k\left([\mu_0:\cdots:\mu_{k-1}:0],\sum_{i=0}^{r}\lambda_i u_i\right)&=\left[\sum_{j=0}^{k-1}\sum_{i=0}^{r}\mu_j\lambda_i x_j^{u_i}\right]\\&=f_H^{k-1}\left([\mu_0:\cdots:\mu_{k-1}],\sum_{i=0}^{r}\lambda_i\eta(u_i)\right),\end{aligned}$$

which proves the proposition as we will now show in more detail.

Let $\vartheta\in H^2(PV,PV\setminus PH)$ be the Thom class of (the normal bundle of) PH in PV, i.e. the class satisfying $\theta\frown[PV]=[PH]$. By the above calculations $(f^k)^*(\vartheta)\in H^2\left(\mathbb{C}P^k\times\Delta Q_{[k,n]},(\mathbb{C}P^k\setminus\mathbb{C}P^{k-1})\times\Delta Q_{[k,n]}\right)$ is the Thom class of $\mathbb{C}P^{k-1}\times\Delta Q_{[k,n]}$ in $\mathbb{C}P^k\times\Delta Q_{[k,n]}$ which equals the class $\alpha\times 1$ where $\alpha\in H^2(\mathbb{C}P^k,\mathbb{C}P^k\setminus\mathbb{C}P^{k-1})$ which is again a Thom class and maps to the canonical generator of $H^*(\mathbb{C}P^k)$. We finally calculate

$$\begin{aligned}i_!(h_k(c))&=\vartheta\frown h_k(c)\\&=\vartheta\frown f_*^k([\mathbb{C}P^k]\times c)\\&=f_*^k\left((f^k)^*(\vartheta)\frown([\mathbb{C}P^k]\times c)\right)\\&=f_*^k\left((\alpha\times 1)\frown([\mathbb{C}P^k]\times c)\right)\\&=f_*^k\left((\alpha\frown[\mathbb{C}P^k])\times(1\frown c)\right)\\&=f_*^k\left([\mathbb{C}P^{k-1}]\times c\right)\\&=h_{k-1}(\eta_*(c))\end{aligned}$$

as claimed. □

Turning to affine arrangements we let $I\subset V$ be another hyperplane and consider the inclusion map $i\colon\left(PH,P(I\cap H)\cup\bigcup P\mathcal{A}^H\right)\to(PV,PI\cup\bigcup P\mathcal{A})$.

2.1.36 Proposition. *Let $\mathcal{A}$ be a complex arrangement or a real ≥ 2-arrangement. Let $I\subset V$ be a hyperplane (at infinity) and $H\subset V$ a hyperplane in general position with respect to $\mathcal{A}\cup\{I\}$. Let $u\in\bar{Q}$, $c\in H_*(\Delta[u,V],\Delta[u,V)\cup\Delta(u,V])$ and*

$$h_u\colon H_*(\Delta[u,V],\Delta[u,V)\cup\Delta(u,V])\to H_*\left(PV,PI\cup\bigcup P\mathcal{A}\right)$$

be the homomorphism from Definition 2.1.29. In case $d(u) > 0$*, we also consider the homomorphism*

$$h^H_{\eta(u)}\colon H_*(\Delta[\eta(u),\eta(V)],\Delta[\eta(u),\eta(V))\cup\Delta(\eta(u),\eta(V)])$$
$$\to H_*\left(PH,(PH\cap PI)\cup\bigcup P\mathcal{A}^H\right).$$

In case $\mathbb{K}=\mathbb{R}$*, we assume the orientations of* $(Pu, PI\cap Pu)$ *and* $(P(u\cap H), P(u\cap H\cap I))$ *which are used in the definitions of* h_u *and* $h^H_{\eta(u)}$ *to be related by*

$$(\bar{f}^{\eta(u)}_H)_*([\mathbb{R}P^{d(u)-1},\mathbb{R}P^{d(u)-2}]) = i_!\left(\bar{f}^u_*([\mathbb{R}P^{d(u)},\mathbb{R}P^{d(u)-1}])\right), \tag{2.43}$$

see (2.32)*.*

Considering the inclusion map $i\colon \left(PH, P(I\cap H)\cup\bigcup P\mathcal{A}^H\right)\to(PV, PI\cup\bigcup P\mathcal{A})$ *we have*

$$i_!(h_u(c)) = \begin{cases} h^H_{\eta(u)}(\eta_*(c)), & d(u)>0,\\ 0, & d(u)=0.\end{cases}$$

In particular $\ker i_! = \bigoplus_{u\in\bar{Q}_{\{0\}}} h_u$*.*

Proof. Let $\Lambda^I\colon V\to\mathbb{K}$ be a linear functional with $\ker\Lambda^I = I$.

For $d(u)=0$ we may assume that f^u and hence h_u is defined via points satisfying $\Lambda^H(x^u_v)=1$ for all $v\geq u$. Then

$$\Lambda^H\left(\sum_{i=0}^{d(u)-1}\mu_i b^u_i + \mu_{d(u)}\sum_{j=0}^{r}\lambda_j x^u_{v_j}\right) = \Lambda^H\left(\sum_{j=0}^{r}\lambda_j x^u_{v_j}\right) = 1 \tag{2.44}$$

and f^u misses H so that $i_!(h_u(c))=0$.

Now let $d(u)>0$. We may assume that f^u and hence h_u is defined via points satisfying $b^u_0,\dots,b^u_{d(u)-2}\in u\cap H\cap I$, $b^u_{d(u)-1}\in u\cap I$, $\Lambda^H(b^u_{d(u)-1})=1$, $x^u_u\in u\cap H$, $\Lambda^I(x^u_u)=1$, $x^u_v\in v\cap H\cap I$ for $v>u$. For these

$$\Lambda^H\left(\sum_{i=0}^{d(u)-1}\mu_i b^u_i + \mu_{d(u)}\sum_{j=0}^{r}\lambda_j x^u_{v_j}\right) = \mu_{d(u)-1}. \tag{2.45}$$

Hence, for $y\in\Delta[u,V]$ and $(\mu_0,\dots,\mu_{d(u)-2},\mu_{d(u)})\in\mathbb{K}^{d(u)}\setminus\{0\}$ the map $\mu\mapsto f^u\left([\mu_0:\dots:\mu_{d(u)-2}:\mu:\mu_{d(u)}],y\right)$ meets H transversally in $0\in\mathbb{K}$. Furthermore

$$\Lambda^I\left(\sum_{i=0}^{d(u)-1}\mu_i b^u_i + \mu_{d(u)}\sum_{j=0}^{r}\lambda_j x^u_{v_j}\right) = \begin{cases}0, & v_0>u,\\ \mu_{d(u)}\lambda_0, & v_0=u.\end{cases} \tag{2.46}$$

So $f^u(x,y)\in I$, iff $x\in\mathbb{K}P^{d(u)-1}$ or $y\in\Delta(u,V]$. We set $m:=\dim_{\mathbb{R}}\mathbb{K}$ and define $\vartheta\in H^m(PV\setminus PI, PV\setminus(PI\cup PH))$ to be the Thom class of $PH\setminus PI$ in $PV\setminus PI$, satisfying $\vartheta \frown [PV, PI] = [PH, PI \cap PH]$. It follows that $(f^u)^*(\vartheta) = \varepsilon\alpha \times 1$ with

$$\varepsilon \in \{+1, -1\},$$
$$\alpha \in H^m\left(\mathbb{K}P^{d(u)} \setminus \mathbb{K}P^{d(u)-1}, \mathbb{K}P^{d(u)} \setminus (\mathbb{K}P^{d(u)-1} \cup j[\mathbb{K}P^{d(u)-1}])\right),$$
$$1 \in H^0((\Delta[u, V] \setminus \Delta(u, V))),$$

where α satisfies $\alpha \frown [\mathbb{K}P^{d(u)}, \mathbb{K}P^{d(u)-1}] = [j[\mathbb{K}P^{d(u)-1}], j[\mathbb{K}P^{d(u)-2}]]$.

Ignoring $b^u_{d(u)-1}$, these points also define a map $f_H^{\eta(u)} : \mathbb{K}P^{d(u)-1} \times \Delta[\eta(u), \eta(V)] \to PH$ that induces h_u^H with

$$\begin{aligned} f^u\left([\mu_0 : \cdots : \mu_{d(u)-2} : 0 : \mu_{d(u)}], \sum_{i=0}^r \lambda_i u_i\right) &= \\ &= \left[\sum_{i=0}^{d(u)-2} \mu_i b_i^u + \mu_{d(u)} \sum_{j=0}^r \lambda_j x_{v_j}^u\right] = \\ &= f_H^{\eta(u)}\left([\mu_0 : \cdots : \mu_{d(u)-2} : \mu_{d(u)}], \sum_{i=0}^r \lambda_i u_i\right), \end{aligned} \tag{2.47}$$

i.e. $f^u(j(x), y) = f_H^{\eta(u)}(x, \eta(y))$. We can now calculate

$$\begin{aligned} i_!(h_u(c)) &= \vartheta \frown h_u(c) \\ &= \vartheta \frown f_*^u([\mathbb{K}P^{d(u)}, \mathbb{K}P^{d(u)-1}] \times c) \\ &= f_*^u\left((f^u)^*(\vartheta) \frown ([\mathbb{K}P^{d(u)}, \mathbb{K}P^{d(u)-1}] \times c)\right) \\ &= f_*^u\left((\varepsilon\alpha \times 1) \frown ([\mathbb{K}P^{d(u)}, \mathbb{K}P^{d(u)-1}] \times c)\right) \\ &= \varepsilon f_*^u\left([j[\mathbb{K}P^{d(u)-1}], j[\mathbb{K}P^{d(u)-2}]] \times c\right) \\ &= \varepsilon (f_H^{\eta(u)})_*([\mathbb{K}P^{d(u)-1}, \mathbb{K}P^{d(u)-2}] \times \eta_*(c)) \\ &= \varepsilon h^H_{\eta(u)}(\eta_*(c)). \end{aligned}$$

Similarly we find $i_!\left(\bar{f}_*^u([\mathbb{R}P^{d(u)}, \mathbb{R}P^{d(u)-1}])\right) = \varepsilon(\bar{f}_H^{\eta(u)})_*([\mathbb{R}P^{d(u)-1}, \mathbb{R}P^{d(u)-2}])$ and hence $\varepsilon = 1$ by comparison with (2.43) □

2.1.37 Remark. For affine ≥ 2-arrangements, Proposition 2.1.36 already almost solves the problem of describing the intersection product in combinatorial terms, as we will sketch now.

If $\mathcal{A}$ is an arrangement in X and $\mathcal{B}$ an arrangement in Y then the product $(X, \bigcup\mathcal{A}) \times (Y, \bigcup\mathcal{B})$ equals $(X \times Y, \bigcup(\mathcal{A} \times \mathcal{B}))$, where $\mathcal{A} \times \mathcal{B}$ is the arrangement in $X \times Y$ defined as $\{A \times Y : A \in \mathcal{A}\} \cup \{X \times B : B \in \mathcal{B}\}$. Combinatorial descriptions of the homology of $(X, \bigcup\mathcal{A})$ and of $(Y, \bigcup\mathcal{B})$ easily lead to combinatorial descriptions of the homology of the product and the cross product map. For projective arrangements we will carry this out partially in Proposition 2.3.2. For linear arrangements, see [dLS01, Prop. 4.1].

Having obtained a description of the cross product, to describe intersection products it then remains to describe the intersection with the diagonal, i.e. the transfer map of the diagonal map. The product of two affine arrangements is again an affine arrangement, hence the diagonal is an affine plane itself, and this can be done by applying Proposition 2.1.36 n times. The only hindrance would be that the diagonal is not in general position with respect to the product arrangement, since the original arrangement is not in general position with respect to itself. This problem could be dealt with using methods similar to those we will apply in the proof of Proposition 2.3.13 on page 67.

However, the product of two projective spaces is not again a projective space. Therefore, the description of intersection products in projective arrangements requires additional techniques.

2.2 Products

Statement of results

We remind the reader of the product $\hat{\times}$ from Definition 1.3.14.

2.2.1 Theorem. *Let $\mathcal{A}$ be a complex arrangement. For all $k, l \geq 0$ and all $c \in H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$, $d \in H_*(\Delta Q_{[l,n]}, \Delta Q_{[l,n)})$ we have*

$$h_k(c) \bullet h_l(d) = \begin{cases} h_{k+l-n}(c \mathbin{\hat{\times}} d), & k+l \geq n, \\ 0, & k+l < n. \end{cases} \tag{2.48}$$

This is the main result of this work. Its proof will take up Section 2.3 and be completed on p. 68.

The statement of the corresponding result for affine arrangements will require some preparations, because we do not restrict ourselves to complex arrangements in this case.

The definition of the homomorphisms h_u in Definition 2.1.29 depends on the choice of a basis $(b_i^u)_i$ for $u \in \bar{Q}$. This also orients all of the u. In the case of a ≥ 2-arrangement this orientation determines h_u. Orientations for u and v with $u+v=V$ determine (together with the orientation of V) an orientation of $u \cap v$. Depending on whether this agrees with the orientation of $u \cap v$ defined by $(b_i^{u\cap v})_i$ or not, we set $\varepsilon_{u,v} = 1$ or $\varepsilon_{u,v} = -1$. For complex arrangements, every $u \in \bar{Q}$ has a canonical orientation, and all of the $\varepsilon_{u,v}$ will equal 1. We define these numbers more formally in the form in which we will use them.

2.2.2 Definition. Let $\mathcal{A}$ be a real arrangement and functions f^u for all $u \in \bar{Q}$ chosen as in Definition 2.1.29. Let $\bar{f}^u$ be the function defined in (2.32). For $u, v \in \bar{Q}$ with $u+v=V$ and $u \cap v \not\subset H$ we define $\varepsilon_{u,v} \in \{+1, -1\}$ by

$$\bar{f}^u_*([\mathbb{R}P^{d(u)}, \mathbb{R}P^{d(u)-1}]) \bullet \bar{f}^v_*([\mathbb{R}P^{d(v)}, \mathbb{R}P^{d(v)-1}]) = \\ = \varepsilon_{u,v} \bar{f}^{u\cap v}_*([\mathbb{R}P^{d(u\cap v)}, \mathbb{R}P^{d(u\cap v)-1}]), \tag{2.49}$$

where the intersection product is defined by dualizing

$$H^*(PV \setminus PH, PV \setminus (PH \cup Pu)) \otimes H^*(PV \setminus PH, PV \setminus (PH \cup Pv)) \\ \xrightarrow{\smile} H^*(PV \setminus PH, PV \setminus (PH \cup P(u \cap v))).$$

2.2.3 Theorem. *Let $\mathcal{A}$ be a real ≥ 2–arrangement. For all $u, v \in \bar{Q}$ and all $c \in H_r(\Delta[u,V], \Delta[u,V) \cup \Delta(u,V])$, $d \in H_s(\Delta[v,V], \Delta[v,V) \cup \Delta(v,V])$ we have*

$$h_u(c) \bullet h_v(d) = \begin{cases} (-1)^{r(n-d(u))} \varepsilon_{u,v} h_{u \wedge v}(c \mathbin{\hat{\times}} d), \\ \qquad \textit{if } u \wedge v \in \bar{Q} \\ \qquad \textit{and } d(u) + d(v) = d(u \wedge v) + n, \\ 0, \qquad \textit{otherwise.} \end{cases} \tag{2.50}$$

The proof of this theorem will be completed at the end of this section, p. 57. This theorem has first been proved in [dLS01] (for central linear arrangements only) and in [DGM00], where it is stated in a somewhat different form.

2.2.4 Remark. If a complex arrangement is regarded as a real arrangement of double dimension, the bases can be chosen derived from complex bases, in which case all of the $\varepsilon_{u,v}$ still equal 1. For a general real arrangement this cannot always be achieved. The dependence of the cup product in the cohomology of a linear arrangement on the numbers $\varepsilon_{u,v}$ has first been shown in [Zie93], where it is used to construct two real arrangements with equalling intersection posets and dimension functions, one of them being a complex arrangement regarded as a real arrangement, with non-isomorphic cohomology rings.

2.2.5 Remark. Continuing Remark 2.1.32, the complex case of Theorem 2.2.3 can be seen as a special case of Theorem 2.2.1, since all of the isomorphisms in that remark are induced by inclusions and therefore respect the product $\hat{\times}$.

Graded formulas

We first prove graded versions of the theorems stated in the preceding section. These follow more or less for free from the algebraic machinery set up in Chapter 1. They state that the equations from the theorems we aim to prove hold at least up to error terms in higher degrees of the direct sum decompositions of the homology groups of the arrangements. They are therefore typical for the kind of result obtainable by spectral sequence arguments.

The graded versions will be an import part of the proofs of the exact versions. They hold in greater generality and in particular are true for arbitrary real arrangements, for which the exact versions fail. Our proof of the exact version for complex projective arrangements will need further more geometric arguments.

2.2.6 Proposition. *Let $\mathbb{K} = \mathbb{C}$ or let $\mathbb{K} = \mathbb{R}$ and coefficients be in $\mathbb{Z}_2$. For all $k + l \geq n$ and $c \in H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$, $d \in H_*(\Delta Q_{[l,n]}, \Delta Q_{[l,n)})$ we have*

$$h_k(c) \bullet h_l(d) - h_{k+l-n}(c \mathbin{\hat{\times}} d) \in \bigoplus_{i > k+l-n} \operatorname{im} h_i. \tag{2.51}$$

Proof. We want to apply Proposition 1.3.20. As the map ζ there we take the map $H(P(F^Q)) \otimes_Q B(Q) \to S(P(D^{\mathcal{A}}))$ obtained as the composition of (2.19) and (2.14). We set $m := \dim_{\mathbb{R}} \mathbb{K}$. The isomorphism $\bigoplus_k H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})[-mk] \cong H(H(P(D^{\mathcal{A}})) \otimes_Q B(Q) \otimes_Q K^p)$ obtained by composing (2.28) with the first arrow from (1.8) is given explicitly by

$$\begin{aligned} C_r(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) &\to H_{mk}(P(D^{\mathcal{A}})) \otimes_Q B(Q)_r \otimes_Q K^p \\ \langle q_0, \ldots, q_r \rangle &\mapsto e_k^{q_0} \otimes (q_0 \leftarrow q_0 \leftarrow \cdots \leftarrow q_r \leftarrow \top) \otimes 1 \end{aligned}$$

with $1 \in K^p(\top) = R$ and $e_k^q \in H_{mk}(Pq)$ the canonical generator. By Proposition 2.1.25 the map h_k agrees with the composition of this isomorphism and the map ϕ_{mk} from Proposition 1.3.20. We note that the dimension n there equals mn in our current notation. To see that the product in that proposition agrees with $\hat{\times}$ under the above isomorphism we only have to note that $e_k^p \bullet e_l^q = e_{k+l-n}^{p \wedge q}$ for $k + l \geq n$. □

2.2.7 Proposition. *Let $u, v \in \bar{Q}$, $c \in H_*(\Delta[u, V], \Delta[u, V) \cup \Delta(u, V])$, $d \in H_*(\Delta[v, V], \Delta[v, V) \cup \Delta(v, V])$. If $u \wedge v \in \bar{Q}$, then*

$$h_u(c) \bullet h_v(d) \in \bigoplus_{w \geq u \wedge v} \operatorname{im} h_w. \tag{2.52}$$

If additionally $d(u) + d(v) = d(u \wedge v) + n$, then

$$h_u(c) \bullet h_v(d) - (-1)^{|c|(n-d(u))} \varepsilon_{u,v} h_{u \wedge v}(c \mathbin{\hat{\times}} d) \in \bigoplus_{w > u \wedge v} \operatorname{im} h_w \tag{2.53}$$

for $\mathbb{K} = \mathbb{R}$ or

$$h_u(c) \bullet h_v(d) - h_{u \wedge v}(c \mathbin{\hat{\times}} d) \in \bigoplus_{w > u \wedge v} \operatorname{im} h_w \tag{2.54}$$

for $\mathbb{K} = \mathbb{C}$. If $u \cap v \subset I$, then

$$h_u(c) \bullet h_v(d) \in \bigoplus_{w \in \bar{Q}_{(d(u)+d(v)-n,n]}} \operatorname{im} h_w. \tag{2.55}$$

Proof. We treat the real case only, as the easier case of $\mathbb{K} = \mathbb{C}$ can be proved in the same way or derived from the case $\mathbb{K} = \mathbb{R}$.

Considering the arrangement $\{q \in Q \colon q \geq u \text{ or } q \geq v\}$, the intersection poset of which can be considered as a subset of the interval $[u \wedge v, V]$ in Q, we obtain (2.52) by naturality of the intersection product with respect to inlusion maps.

The rest of the proof proceeds as the preceding one, using a relative version of Proposition 1.3.20. We take up the notation from the proof of Proposition 2.1.29. The isomorphism

$$\bigoplus_{u \in \bar{Q}} H_*(\Delta[u, V], \Delta[u, V) \cup \Delta(u, V])[-d(u)] \cong H(H(A(D^{\mathcal{A}})) \otimes_Q B(Q) \otimes_Q K^p)$$

is induced by

$$C_r(\Delta[u,V], \Delta[u,V) \cup \Delta(u,V]) \to H_{d(u)}(A(D^{\mathcal{A}})) \otimes_Q B(Q)_r \otimes_Q K^p$$
$$\langle q_0, \dots, q_r \rangle \mapsto e^u \otimes (u \leftarrow q_0 \leftarrow \dots \leftarrow q_r \leftarrow V) \otimes 1.$$

with $e^u := \bar{f}^u_*([\mathbb{R}P^{d(u)}, \mathbb{R}P^{d(u)-1}]) \in H_{d(u)}(Pu, P(u \cap I))$, $u \in \bar{Q}$. Under this isomorphism the map ϕ_k from Proposition 1.3.20 corresponds to $\sum_{u \in \bar{Q}_{\{k\}}} h_u$.

For $d(u) + d(v) = d(u \wedge v) + n$ we have $e^u \bullet e^v = \varepsilon_{u,v} e^{u \wedge v}$ by definition of $\varepsilon_{u,v}$ and hence

$$h_u(c) \bullet h_v(d) - (-1)^{|c|(n-d(u))} \varepsilon_{u,v} h_{u \wedge v}(c \hat{\times} d) \in \bigoplus_{w \in \bar{Q}_{(k+l-n,n]}} \operatorname{im} h_w.$$

Since

$$\bigoplus_{w \in \bar{Q}_{(k+l-n,n]}} \operatorname{im} h_w \cap \bigoplus_{w \geq u \wedge v} \operatorname{im} h_w = \bigoplus_{w > u \wedge v} \operatorname{im} h_w,$$

this proves (2.53).

For $u \cap v \subset I$, $e^u \bullet e^v = 0$ by necessity, since $H(A(u \wedge v)) = 0$, and the above argument yields (2.55). □

2.2.8 Remark. Proposition 2.2.7 is proved in [dLS01] as Theorem 7.5. While there it is more of an afterthought to the exact version, it appears here as a very natural result in its own right and a possible basis to a proof of the exact version.

2.2.9 Remark. If n is odd, then $\mathbb{R}P^n$ is orientable and Proposition 1.3.20 is applicable to $H_*(PV, \bigcup P\mathcal{A})$ also for $\mathbb{K} = \mathbb{R}$, $R = \mathbb{Z}$, since in the proof of Proposition 2.1.20 the needed ZŽ–map is constructed. In the product $H(Pu) \otimes H(Pv) \xrightarrow{\bullet} H(P(u \wedge v))$ the product of two generators is a generator whenever possible and to determine the sign of the product of two generators of $\mathbb{Z}$-summands, orientation information is needed as in the affine case.

Inductive proofs of product formulas

We now reduce Theorem 2.2.1 and Theorem 2.2.3 to the cases where they state that products be zero because of their degrees. This is done using the graded versions presented in the preceding section. The inductive step is made possible by the results on intersections with a hyperplane presented in Section 2.1.

2.2.10 Proposition. *If it is true for all complex arrangements $\mathcal{A}$ of all dimensions that $h_k(c) \bullet h_l(d) = 0$ for $c \in H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$, $d \in H_*(\Delta Q_{[l,n]}, \Delta Q_{[l,n)})$, whenever $k + l < n$, then Theorem 2.2.1 is true.*

Proof. We have to show that $h_k(c) \bullet h_l(d) = h_{k+l-n}(c \hat{\times} d)$, where we use the convention that $h_i = 0$ for $i < 0$.

We proceed by induction on the dimension of V.

For $k+l<n$ the conjecture is covered by the assumption. For $k+l\geq n$ and $\dim V=1$ it is covered by Proposition 2.2.6 (and trivial anyway). For $k+l\geq n$ and $\dim V>1$ we choose a hyperplane in general position with respect to the arrangement and adopt the notation of Section 2.1. By induction and Proposition 2.1.35 (for $k+l>n$) or the assumption (for $k+l=n$)

$$\begin{aligned}i_!(h_k(c)\bullet h_l(d)) &= i_!(h_k(c))\bullet i_!(h_l(d)) = h^H_{k-1}(\eta_*(c))\bullet h^H_{l-1}(\eta_*(d)) = \\ &= h^H_{k+l-n-1}(\eta_*(c)\,\hat{\times}\,\eta_*(d)) = h^H_{k+l-n-1}(\eta_*(c\,\hat{\times}\,d)) = i_!(h_{k+l-n}(c\,\hat{\times}\,d)).\end{aligned}$$

Again by Proposition 2.1.35, this implies $h_k(c)\bullet h_l(d)-h_{k+l-n}(c\,\hat{\times}\,d)\in\ker i_! = \operatorname{im} h_0$. But $h_k(c)\bullet h_l(d)-h_{k+l-n}(c\,\hat{\times}\,d)\in\bigoplus_{i>k+l-n}\operatorname{im} h_i$ by Proposition 2.2.6. Therefore $h_k(c)\bullet h_l(d)-h_{k+l-n}(c\,\hat{\times}\,d)=0$. □

2.2.11 Proposition. *If it is true for all real ≥ 2-arrangements $\mathcal{A}$ of all dimension and all $u,v\in\bar{Q}$ that $h_u(c)\bullet h_v(d)=0$ for $c\in H_*(\Delta[u,V],\Delta[u,V)\cup\Delta(u,V])$ and $d\in H_*(\Delta[v,V],\Delta[v,V)\cup\Delta(v,V])$ whenever $d(u)+d(v)<n$, then Theorem 2.2.3 is true.*

Proof. The proof proceeds by induction on the dimension of V, parallel to the preceding one.

Let $u,v\in\bar{Q}$, $d(u)+d(v)\geq n$.

If $\dim V=1$, then Theorem 2.2.3 holds trivially for V. We assume $\dim V>1$ and let $H\subset V$ be a hyperplane in general position with respect to $\mathcal{A}$ and I. For simplicity, we also assume $d(u),d(v)>0$, because otherwise $u=V$ or $v=V$ and these cases are easily dealt with directly.

If $u\cap v\subset I$, then $\eta(u)\cap\eta(v)\subset I\cap H$. If $u\wedge v\in\bar{Q}$ and $d(u)+d(v)-d(u\wedge v)<n$ then $\eta(u)\cap\eta(v)\subset I\cap H$ or $d(\eta(u))+d(\eta(v))-d(\eta(u)\wedge\eta(v))<n-1$. In these cases

$$i_!(h_u(c)\bullet h_v(d)) = i_!(h_u(c))\bullet i_!(h_v(d)) = h^H_{\eta(u)}(\eta_*(c))\bullet h^H_{\eta(v)}(\eta_*(d)) = 0$$

and hence $h_u(c)\bullet h_v(d)\in\bigoplus_{w\in\bar{Q}_{\{0\}}} h_w$ by induction and Proposition 2.1.36, and $h_u(c)\bullet h_v(d)\in\bigoplus_{w\in\bar{Q}_{(0,n]}} h_w$ by Proposition 2.2.7. Therefore $h_u(c)\bullet h_v(d)=0$.

Now let $u+v=V$, $u\cap v\not\subset I$. If $d(u)+d(v)=n$, then

$$\begin{aligned}&i_!\big(h_u(c)\bullet h_v(d)-(-1)^{|c|(n-d(u))}\varepsilon_{u,v}h_{u\wedge v}(c\,\hat{\times}\,d)\big) = \\ &\qquad = h^H_{\eta(u)}(\eta_*(c))\bullet h^H_{\eta(v)}(\eta_*(d)) = 0\end{aligned}$$

by induction and Proposition 2.1.36. If $d(u)+d(v)>n$, we make sure that $h^H_{\eta(u)}$, $h^H_{\eta(v)}$, $h^H_{\eta(u\wedge v)}$ are defined with orientations compatible with those underlying h_u, h_v, $h_{u\wedge v}$ as in Proposition 2.1.36. Then

$$\begin{aligned}&(\bar{f}_H^{\eta(u)})_*([\mathbb{R}P^{d(\eta(u))},\mathbb{R}P^{d(\eta(u))-1}])\bullet(\bar{f}_H^{\eta(v)})_*([\mathbb{R}P^{d(\eta(v))},\mathbb{R}P^{d(\eta(v))-1}] \\ &\quad = i_!\big(\bar{f}^u_*([\mathbb{R}P^{d(u)},\mathbb{R}P^{d(u)-1}])\bullet\bar{f}^v_*([\mathbb{R}P^{d(v)},\mathbb{R}P^{d(v)-1}])\big) \\ &\quad = i_!(\varepsilon_{u,v}(\bar{f}_H^{u\wedge v})_*([\mathbb{R}P^{d(u\wedge v)},\mathbb{R}P^{d(u\wedge v)-1}]) \\ &\quad = \varepsilon_{u,v}(\bar{f}_H^{\eta(u)\wedge\eta(v)})_*([\mathbb{R}P^{d(\eta(u)\wedge\eta(v))},\mathbb{R}P^{d(\eta(u)\wedge\eta(v))-1}])\end{aligned}$$

and hence $\varepsilon^H_{\eta(u),\eta(v)} = \varepsilon_{u,v}$. Again, this yields

$$\begin{aligned} i_!\big(h_u(c) \bullet h_v(d) - (-1)^{|c|(n-d(u))}\varepsilon_{u,v} h_{u\wedge v}(c \hat{\times} d)\big) &= \\ h^H_{\eta(u)}(\eta_*(c)) \bullet h^H_{\eta(v)}(\eta_*(d)) - (-1)^{|c|(n-1-d(\eta(u)))}\varepsilon^H_{\eta(u),\eta(v)} h^H_{\eta(u)\wedge\eta(v)}(c \hat{\times} d) & \\ &= 0. \end{aligned}$$

In both cases, we combine this with Proposition 2.1.36 and Proposition 2.2.7 to get $h_u(c) \bullet h_v(d) - (-1)^{|c|(n-d(u))}\varepsilon_{u,v} h_{u\wedge v}(c \hat{\times} d) \in \bigoplus_{w\in \bar{Q}_{\{0\}}} h_w \cap \bigoplus_{w>u\wedge v} h_w = 0$. □

Vanishing for affine arrangements

For affine ≥ 2-arrangements the vanishing of intersection products in the cases required by Proposition 2.2.11 is easily proved, since it is indeed possible to find chains representing the involved hology classes that do not intersect geometrically.

2.2.12 Proposition. *Let $\mathcal{A}$ be a ≥ 2–arrangement, $u, v \in \bar{Q}$, $d(u) + d(v) < n$, $c \in H_*(\Delta[u,V], \Delta[u,V) \cup \Delta(u,V])$, $d \in H_*(\Delta[u,V], \Delta[u,V) \cup \Delta(u,V])$. Then $h_u(c) \bullet h_v(d) = 0$.*

Proof. Let Λ be a linear functional on V with $\ker\Lambda = H$.

There is a linear functional $F \neq 0$ on V with $u \subset \ker F$ and $v \cap H \subset \ker F$. We choose b^u, x^u to define f^u and $\tilde{b}^v$, $\tilde{x}^v$ to define $\tilde{f}^v$ as in Definition 2.1.29. This can be done in such a way that $F(x^u_w) \leq 0$ for all $w \geq u$, $F(x^v_w) \geq 0$ for all $w \geq v$, $F(x^u_V) < 0$, $F(x^v_V) > 0$ and $\Lambda(x^u_u) = \Lambda(\tilde{x}^v_v) = 1$.

We set

$$\begin{aligned} X^+ &:= \big\{[z] \in PV : \Lambda(z) \neq 0, \Lambda(z)^{-1}F(z) \geq 0\big\} \cup PH, \\ X^- &:= \big\{[z] \in PV : \Lambda(z) \neq 0, \Lambda(z)^{-1}F(z) \leq 0\big\} \cup PH. \end{aligned}$$

Then

$$\begin{aligned} \operatorname{im} f^u \subset X^-, &\qquad (f^u)^{-1}[X^+ \cap X^-] \subset \mathbb{K}P^k \times \Delta[u,V) \cup \mathbb{K}P^{k-1} \times \Delta[u,V], \\ \operatorname{im} \tilde{f}^v \subset X^+, &\qquad (\tilde{f}^v)^{-1}[X^+ \cap X^-] \subset \mathbb{K}P^l \times \Delta[v,V) \cup \mathbb{K}P^{l-1} \times \Delta[v,V]. \end{aligned}$$

Therefore $\operatorname{im} f^u \cap \operatorname{im} \tilde{f}^v \subset \bigcup P\mathcal{A} \cup PH$ and $h_u(c) \bullet h_v(d) = 0$. □

Proof of Theorem 2.2.3. The theorem follows directly from Proposition 2.2.12 and Proposition 2.2.11. □

2.3 Products in projective arrangements

In this section we will complete the proof of Theorem 2.2.1, the product formula for complex projective arrangements. We have already reduced this in Proposition 2.2.10 to the case of $h_k(c) \bullet h_l(d)$ with $k + l < n$, which we have to show to be zero. The corresponding fact for affine arrangements could be proved by representing the homology classes by chains which do not intersect geometrically. A proof along these lines seems not to be available for projective arrangements. We will first consider an example of a real projective arrangement where an intersection product of this kind is indeed not zero. Doing this we will also try to gain some intuition on why in the projective case the intersection of the chains should not make a homological contribution, even if existing geometrically. We will then develop the techniques necessary to transform this intuition into a proof.

An example of real projective arrangements

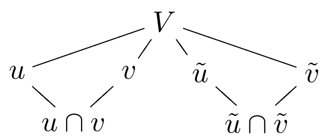
We will see how the product formula of Theorem 2.2.1 fails for real projective arrangements and sketch the difference between real and complex arrangements that will allow us to prove the formula for complex projective arrangements.

Let $k, l \geq 0$, $n := k + l + 1$. We consider the following subspaces of $\mathbb{R}^{n+1} = \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^2$.

$$u := \mathbb{R}^k \times \{0\} \times (\mathbb{R} \cdot (0,1)), \qquad \tilde{u} := \{0\} \times \mathbb{R}^l \times (\mathbb{R} \cdot (1,1)),$$
$$v := \mathbb{R}^k \times \{0\} \times (\mathbb{R} \cdot (4,1)), \qquad \tilde{v} := \{0\} \times \mathbb{R}^l \times (\mathbb{R} \cdot (5,1)),$$

In the arrangement $\check{\mathcal{A}} := \{u, v, \tilde{u}, \tilde{v}\}$ we will find classes $c \in H(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$, $d \in H(\Delta Q_{[l,n]}, \Delta Q_{[l,n)})$ with $h_k(c) \bullet h_l(d) \neq 0$, although $k + l < n$.

The combinatorial data of $\check{\mathcal{A}}$ are given by the intersection poset

$\check{Q}_{[0,n]} =$ 

with $u \cap v$ and $\tilde{u} \cap \tilde{v}$ only present for $k > 0$ and $l > 0$ respectively, and the dimensions $d(u) = d(v) = k$, $d(u \cap v) = k - 1$, $d(\tilde{u}) = d(\tilde{v}) = l$, $d(\tilde{u} \cap \tilde{v}) = l - 1$, $d(V) = n$. In case of $k = l$, we can, if we want to, avoid the intersections $u \cap v$ and $\tilde{u} \cap \tilde{v}$ by a small change of u and $\tilde{u}$ without substantially affecting the calculations below. This shows that, in contrast to the case of affine arrangements, a simple condition on the occuring codimensions will not be enough for the product formula to extend from complex to real arrangements.

To simplify the pictures below and to have the notation parallel that of Section 2.3, we consider $\check{A}$ to be the union of the two arrangements $\mathcal{A} := \{u, v\}$ and $\tilde{A} := \{\tilde{u}, \tilde{v}\}$. The arrangement $\tilde{\mathcal{A}}$ is in general position with respect to $\mathcal{A}$ as in Definition 2.3.4.

Denoting the intersection posets of $\mathcal{A}$ and $\tilde{\mathcal{A}}$ by Q and $\tilde{Q}$ respectively, we have $H_1(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}; \mathbb{Z}_2) \cong \mathbb{Z}_2$, generated by $c := [\langle u, V\rangle + \langle v, V\rangle]$, and $H_1(\Delta \tilde{Q}_{[l,n]}, \Delta \tilde{Q}_{[l,n)}; \mathbb{Z}_2) \cong \mathbb{Z}_2$, generated by $d := [\langle \tilde{u}, V\rangle + \langle \tilde{v}, V\rangle]$. For the definition of

$$\begin{aligned} f^k &\colon \mathbb{R}P^k \times (\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \to (V, \bigcup P\mathcal{A}),\\ \tilde{f}^l &\colon \mathbb{R}P^l \times (\Delta \tilde{Q}_{[k,n]}, \Delta \tilde{Q}_{[k,n)}) \to (V, \bigcup P\tilde{\mathcal{A}}), \end{aligned}$$

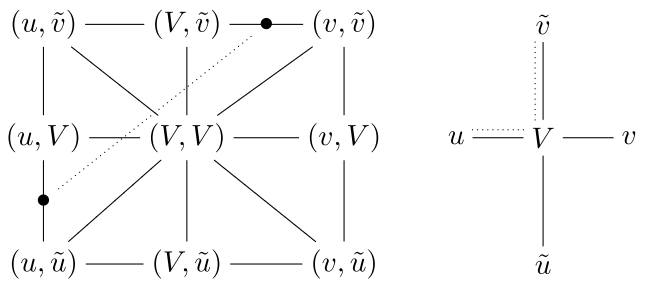
and hence of

$$\begin{aligned} h_k &\colon H_r(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}; \mathbb{Z}_2) \to H_{r+k}(V, \bigcup P\mathcal{A}; \mathbb{Z}_2),\\ \tilde{h}_l &\colon H_r(\Delta \tilde{Q}_{[k,n]}, \Delta \tilde{Q}_{[k,n)}; \mathbb{Z}_2) \to H_{r+l}(V, \bigcup P\tilde{\mathcal{A}}; \mathbb{Z}_2) \end{aligned}$$

we set, with $(e_0, \dots, e_n)$ the standard basis of $V = \mathbb{R}^{n+1}$,

$$\begin{aligned} x_j^u &:= \begin{cases} e_j, & j < k,\\ e_{k+l+1}, & j = k, \end{cases} & \tilde{x}_j^{\tilde{u}} &:= \begin{cases} e_{k+j}, & j < l,\\ e_{k+l} + e_{k+l+1}, & j = l, \end{cases}\\ x_j^v &:= \begin{cases} e_j, & j < k,\\ 4e_{k+l} + e_{k+l+1}, & j = k, \end{cases} & \tilde{x}_j^{\tilde{v}} &:= \begin{cases} e_{k+j}, & j < l,\\ 5e_{k+l} + e_{k+l+1}, & j = l, \end{cases}\\ x_j^V &:= \begin{cases} e_j, & j < k,\\ 2e_{k+l} + e_{k+l+1}, & j = k, \end{cases} & \tilde{x}_j^V &:= \begin{cases} e_{k+j}, & j < l,\\ 3e_{k+l} + e_{k+l+1}, & j = l. \end{cases} \end{aligned}$$

To determine $h_k(c) \bullet h_l(d)$, we first have a look at the geometric intersection $S := f^k\left[\mathbb{R}P^k \times \Delta Q_{[k,n]}\right] \cap f^l\left[\mathbb{R}P^l \times \Delta \tilde{Q}_{[l,n]}\right]$. For $x \in \Delta(Q_{[k,n]})$, $y \in \Delta(\tilde{Q}_{[l,n]})$, the intersection $f\left[\mathbb{R}P^k \times \{x\}\right] \cap f\left[\mathbb{R}P^l \times \{y\}\right]$ is either empty or consists of a single point. The left of the following two pictures shows the two dimensional simplicial complex $\Delta Q_{[k,n]} \times \Delta \tilde{Q}_{[l,n]} = \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})$.



(2.56)

The dotted line depicts the set $\bar{S}$ of those points (x, y) for which the intersection is nonempty. S is a connected 1–dimensional manifold with boundary, and we can see from the picture that one boundary point lies in $u \cap V = u$ and the other one in $V \cap \tilde{v} = \tilde{v}$. A closer look at S, which is the intersection of two manifolds that meet transversely, shows that indeed $h_k(c) \bullet \tilde{h}_l(d) = \check{h}_0([\langle u, V\rangle + \langle \tilde{v}, V\rangle])$. $[\langle u, V\rangle + \langle \tilde{v}, V\rangle]$ is a generator of $H_1(\Delta \check{Q}_{[0,n]}, \Delta \check{Q}_{[0,n)}; \mathbb{Z}_2)$, therefore $h_k(c) \bullet \tilde{h}_l(d) \neq 0$.

We equip $Q \times \tilde{Q}$ with a dimension function $d(p, q) := d(p) + d(q)$. The map $Q \times \tilde{Q} \to \check{Q}$, $(p, q) \mapsto p \cap q$, sends $((Q \times \tilde{Q})_{[n,2n]}, (Q \times \tilde{Q})_{[n,2n)})$ to $(\check{Q}_{[0,n]}, \check{Q}_{[0,n)})$.

In the picture, the border of the square is $\Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,2n)}$ and the four vertices at the corners are $\Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,n)}$. Under the composition of maps

$$\begin{array}{c} \big(\Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]}) \setminus \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,n)}, \\ \qquad \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,2n)} \setminus \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,n)}\big) \\ \downarrow \\ (\Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[n,2n]}, \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[n,2n)}) \\ \downarrow \\ (\Delta\check{Q}_{[0,n]}, \Delta\check{Q}_{[0,n)}), \end{array} \tag{2.57}$$

the first being a deformation retraction and the second given by inclusion, in our example $(\bar{S}, \mathfrak{d}\bar{S})$ is mapped to the dotted line in the picture on the right. This set carries the relative cycle $\langle u, V\rangle + \langle \tilde{v}, V\rangle$ representing $h_k(c) \bullet \tilde{h}_l(d)$. We will see in Section 2.3 that this is not just a coincidence.

When considering complex arrangements we will see that in the above situation we gain one dimension compared to real arrangements, and $\bar{S}$ will miss the cone with top the vertix (V, V) and base $\Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,n)}$. The map of $(\bar{S}, \mathfrak{d}\bar{S})$ to $(\Delta\check{Q}_{[0,n]}, \Delta\check{Q}_{[0,n)})$ will therefore miss the vertix V and be homotopic to a map with image in $\Delta\check{Q}_{[0,n)}$. This is the idea behind the proof of Proposition 2.3.14, although it will be technically a bit different.

The product of two arrangements and general position

The intersection of two sets can be identified with the intersection of their cartesian product with a diagonal. Similarly the intersection product of two homology classes equals the image of their cross product under the image of the transfer map associated with the diagonal map. We therefore study the products of two arrangements.

Already in the real example we have discussed, it was useful to assume the homology classes of which the product was to be determined to be carried by different arrangements. So we now assume to be given a second arrangement $\tilde{A}$ in V with intersection poset $\tilde{Q}$. For the first part of this section the arrangement could be in a vector space different from V, but we will have no use for this generality later on.

We equip the poset $Q \times \tilde{Q}$ with a dimension function d by $d(u,v) := d(u) + d(v)$. The counterpart of h for $\tilde{A}$ will be denoted by $\tilde{h}$ and so on.

As noted above, we will be interested in cross products.

2.3.1 Definition and Proposition. *Any choice of $(y_i^{u,v})_{i=0,\dots,k}$, $(z_i^{u,v})_{i=0,\dots,l}$ for $(u,v) \in Q_{[k,n]} \times \tilde{Q}_{[l,n]}$ with $y_i^{u,v} \in u$, $z_i^{u,v} \in v$ and such that for all $(u_0, v_0) <$*

$\cdots < (u_m, v_m)$ and $\lambda \in \Delta^m$ the system $(\sum_j \lambda_j y_i^{u_j,v_j})_i$ as well as the system $(\sum_j \lambda_j z_i^{u_j,v_j})_i$ is linearly independent, yields a map

$$g\colon \mathbb{C}P^k \times \mathbb{C}P^l \times \big(\Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]}), \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,2n)}\big) \to (PV, \bigcup P\mathcal{A}) \times (PV, \bigcup P\tilde{\mathcal{A}})$$

$$\left([\mu_0 : \cdots : \mu_k], [\nu_0 : \cdots : \nu_l], \sum_j \lambda_j (u_j, v_j)\right) \mapsto \left(\left[\sum_{i,j} \lambda_j \mu_i y_i^{u_j,v_j}\right], \left[\sum_{i,j} \lambda_j \nu_i z_i^{u_j,v_j}\right]\right).$$

As in Definition 2.1.25, any two such maps are homotopic for $\mathbb{K} = \mathbb{C}$. □

2.3.2 Proposition. *Let $c \in H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$, $d \in H_*(\Delta \tilde{Q}_{[l,n]}, \Delta \tilde{Q}_{[l,n)})$, and $\mathbb{K} = \mathbb{C}$. Then $h_k(c) \times \tilde{h}_l(d) = g_*\big([\mathbb{C}P^k] \times [\mathbb{C}P^l] \times (c \times d)\big)$.*

Proof. Since $\mathbb{K} = \mathbb{C}$, the homomorphisms h_k, $\tilde{h}_l$ and g_* do not depend on the choices made in defining them. For the choice $y_i^{u,v} = x_i^u$, $z_i^{u,v} = \tilde{x}_i^v$, we just get the map $f^k \times \tilde{f}^l$ up to identification of

$$\mathbb{C}P^k \times \mathbb{C}P^l \times \Big(\Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]}), \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,2n)}\Big)$$

with $\mathbb{C}P^k \times (\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \times \mathbb{C}P^l \times (\Delta \tilde{Q}_{[l,n]}, \Delta \tilde{Q}_{[l,n)})$. Again since $\mathbb{K} = \mathbb{C}$, no sign is introduced by the interchange of factors made in this identification. □

As noted after discussing the real example, it will be important to control the codimension of a set corresponding to the dotted line in (2.56). We will now work towards this and start with an algebraic lemma.

2.3.3 Lemma. *Let $\mathbb{K} = \mathbb{C}$. Let u, v be subspaces of V in general position with respect to each other, $\dim u = r \geq k+1$, $\dim v = s \geq l+1$, $\dim V = n+1$, $k + l < n$. Let O be the open subspace of the affine space $u^{k+1} \times v^{l+1}$ defined by*

$$O := \big\{(y_0, \dots, y_k, z_0, \dots, z_l) \colon \dim(\operatorname{span}\{y_i\}) = k+1,\ \dim(\operatorname{span}\{z_i\}) = l+1\big\}$$

and algebraic subsets $\cdots \subset S_1 \subset S_0 \subset O$ defined by

$$S_m := \{(y_0, \dots, y_k, z_0, \dots, z_l) \colon \dim(\operatorname{span}(\{y_i\} \cup \{z_j\})) < k + l + 2 - m\}$$

Then $S_m \setminus S_{m+1}$ is a complex submanifold of codimension $(1+m)(n-k-l+m)$.

Proof. We consider $(y_0, \dots, y_k, z_0, \dots, z_l) \in S_m \setminus S_{m+1}$. This implies $u + v = V$. We set $Y := \operatorname{span}\{y_i\}$, $t := n - k - s + \dim(Y \cap v)$, and choose a basis $(e_0, \dots, e_n)$ of V such that $\operatorname{span}\{e_0, \dots, e_{r-1}\} = u$, $\operatorname{span}\{e_{n-s+1}, \dots, e_n\} = v$, $\operatorname{span}\{e_t, \dots, e_{k+1}\} = Y$. Let A be the $(n+1) \times (k+l+2)$–matrix with columns $(y_0, \dots, y_k, z_0, \dots, z_l)$ expressed using this basis. Elements of O are represented by matrices $A' = (a'_{ij})$ with $a'_{ij} = 0$ for $r \leq i \leq n$, $0 \leq j \leq k$ and for $0 \leq i \leq n-s$, $k+1 \leq j \leq k+l+1$ such that the first $k+1$ and the last $l+1$ columns are linearly independent, and $A = (a_{ij})$ has the additional property that the first t rows are zero.

There are sets I and J with $\{t, \dots, k+t\} \subset I \subset \{t, \dots, n\}$, $\{0, \dots, k\} \subset J \subset \{0, \dots, k+l+1\}$ and $|I| = |J| = k+l+1-m$ such that the matrix $B :=$

$(a_{ij})_{i \in I, j \in J}$ is regular. Similarly, there exist I', J' with $I' \subset \{t, \dots, n\} \setminus M$, $\{k+1, \dots, k+l+1\} \subset J' \subset \{0, \dots, k+l+1\}$ and $|I|' = |J|' = k+l+1-m$ such that the matrix $C := (a_{ij})_{i \in I', j \in J'}$ is regular.

Let $U \subset O$ be a neighbourhood of A such that for every $A' = (a'_{ij}) \in U$ the matrices $(a'_{ij})_{i \in I, j \in J}$ and $(a'_{ij})_{i \in I', j \in J'}$ are regular. Then an $A' \in U$ is in S_m if and only if the equations

$$f_{i_0 j_0}(A') := \det(a'_{ij})_{\substack{i \in I \cup \{i_0\} \\ j \in J \cup \{j_0\}}} = 0 \qquad \text{for all } i_0 \in I_0 := \{n+1-s, \dots, n\} \setminus I,$$
$$j_0 \in J_0 := \{0, \dots, k+l+1\} \setminus J$$

and

$$g_{i_0 j_0}(A') := \det(a'_{ij})_{\substack{i \in I' \cup \{i_0\} \\ j \in J' \cup \{j_0\}}} = 0 \qquad \text{for all } i_0 \in \{0, \dots, t-1\},$$
$$j_0 \in J'_0 := \{0, \dots, k+l+1\} \setminus J'$$

hold. To see this, assume $A' \notin S_m$, i.e. $\operatorname{rk} A' > k+l+1-m$. If the rank of the matrix A' with the first t rows deleted is greater than $k+l+1-m$, one of the functions $f_{i_0 j_0}$ becomes non-zero, otherwise one of the functions $g_{i_0 j_0}$.

Finally we compute for $(i_1, j_1) \in I_0 \times J_0$

$$\left|\frac{\partial f_{i_0 j_0}(A)}{\partial a_{i_1 j_1}}\right| = \begin{cases} |\det B|, & (i_0, j_0) = (i_1, j_1), \\ 0, & (i_0, j_0) \neq (i_1, j_1), \end{cases} \qquad (i_0, j_0) \in I_0 \times J_0,$$
$$\frac{\partial g_{i_0 j_0}(A)}{\partial a_{i_1 j_1}} = 0, \qquad (i_0, j_0) \in \{0, \dots, t-1\} \times J'_0$$

and for $(i_1, j_1) \in \{0, \dots, t-1\} \times J'_0$

$$\frac{\partial f_{i_0 j_0}(A)}{\partial a_{i_1 j_1}} = 0, \qquad (i_0, j_0) \in I_0 \times J_0,$$
$$\left|\frac{\partial g_{i_0 j_0}(A)}{\partial a_{i_1 j_1}}\right| = \begin{cases} |\det C|, & (i_0, j_0) = (i_1, j_1), \\ 0, & (i_0, j_0) \neq (i_1, j_1), \end{cases} \qquad (i_0, j_0) \in \{0, \dots, t-1\} \times J'_0$$

and $|I_0 \times J_0 \cup \{0, \dots, t-1\} \times J'_0| = (n+1-|I|)(m+1) = (n-k-l+m) \cdot (m+1)$. □

2.3.4 Definition. We say that the arrangement $\tilde{\mathcal{A}}$ is in general position with respect to the arrangement $\mathcal{A}$, if for all $u \in Q$ and $v \in \tilde{Q}$, we have $u \cap v = 0$ whenever $d(u) + d(v) < n$ and $d(u \cap v) = d(u) + d(v) - n$ otherwise.

2.3.5 Proposition. *Let $\mathbb{K} = \mathbb{C}$, $k + l < n$, $D \subset PV \times PV$ be the diagonal and $S \subset \Delta(Q_{[k,n]} \times Q_{[l,n]})$ be defined as the set of all points x such that $g\left[\mathbb{C}P^k \times \mathbb{C}P^l \times \{x\}\right] \cap D \neq \emptyset$. For a generic choice of the points $y_i^{u,v}$ and $z_i^{u,v}$ defining g, the set S intersects every open simplex of $\Delta(Q_{[k,n]} \times Q_{[l,n]})$ in an algebraic set of real codimension $2(n-k-l)$.*

Proof. In regard of Lemma 2.3.3 all that is required is that for each chain $(u_0, v_0) < \cdots < (u_t, v_t)$ the affine plane in $u_t^{k+1} \times v_t^{k+1}$ spanned by the $t+1$ points $(y^{u_0,v_0}, z^{u_0,v_0}), \ldots, (y^{u_t,v_t}, z^{u_t,v_t})$ meets the algebraic set S_0 transversely. Assuming that the affine plane spanned by the first t of these points already meets S_0 transversely, this will be fulfilled for a generic choice of $(y^{u_t,v_t}, z^{u_t,v_t}) \in u_t^{k+1} \times v_t^{k+1}$. □

Recovering the direct sum decomposition

When discussing the real example, it seemed plausible that a certain subset of the order complex of the intersection poset *should* carry the inverse image of the considered intersection product under the isomorphism $\sum_k h_k$. We now develop tools that allow to actually *prove* this kind of proposition.

More generally, given a class in $H_*\left(PV, \bigcup P\mathcal{A}\right)$ we want to determine the corresponding element of $\bigoplus_k H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$. Because of Proposition 2.1.35 it will suffice to identify the part in the summand $H_*(\Delta Q_{[0,n]}, \Delta Q_{[0,n)})$. The key to this will be to not only consider the map $f^0 \colon \Delta Q_{[0,n]} \to PV$, but also a map $PV \to Q_{[0,n]}$, where the poset Q is topologized in an appropriate way yielding the *space of strata*. While we have up to this point used only the former map, in [DGM00] a description of the cohomology ring of the complement of an affine arrangement is obtained using exclusively the latter map. Here the interplay of both maps will be important.

2.3.6 Definition. Let P be a poset. We make P into a topological space by calling a set $O \subset P$ open, iff $x \in O$ implies $y \in O$ for all $y \geq x$.

2.3.7 Lemma. *Let X be a space, P a poset, $A \subset X$, $R \subset P$. If $f, g \colon (X, A) \to (P, R)$ are continuous maps with $f(x) \geq g(x)$ for all $x \in X$, then $f \simeq g$.*

Proof. The desired homotopy is given by

$$\begin{aligned} H \colon (X, A) \times I &\to (P, R) \\ (x, t) &\mapsto \begin{cases} f(x), & t < 1, \\ g(x), & t = 1. \end{cases} \end{aligned}$$

This map is continuous, since $g^{-1}[O] \subset f^{-1}[O]$ for open $O \subset X$, and therefore $H^{-1}[O] = f^{-1}[O] \times [0,1) \cup g^{-1}[O] \times \{1\} = f^{-1}[O] \times [0,1) \cup g^{-1}[O] \times I$. □

2.3.8 Lemma. *If P has a minimum or a maximum, then P is contractible.*

Proof. By the preceding lemma, the constant map to the minimum respectively the maximum is homotopic to the identity. □

2.3.9 Definition. Let X be a space equipped with a covering $\mathfrak{C}$ by closed sets and let P be the poset $P := \{\bigcap M : \emptyset \neq M \subset \mathfrak{C}, \bigcap M \neq \emptyset\}$, odered by inclusion. We define a continous map

$$\begin{aligned} s\colon X &\to P \\ x &\mapsto \min\{p \in P\colon x \in P\} = \bigcap\{C \in \mathfrak{C}\colon x \in C\}\,. \end{aligned}$$

In particular we consider the following two kinds of maps. For our arrangement $\mathcal{A}$ we consider the map $s^{\mathcal{A}}\colon PV \to Q_{[0,n]}$ corresponding to the covering $P\mathcal{A} \cup \{PV\}$ of PV. For a poset P which has unique minima in the sense that for $M \subset P$, $M \neq \emptyset$, the set $\{p \in P\colon p \leq q \text{ for all } q \in M\}$ is either empty or of the form $\{p\colon p \leq q\}$ for a $q \in P$, we consider the map $s^P\colon \Delta P \to P$ arising from the covering of ΔP by the subspaces $\Delta(\{p'\colon p' \leq p\})$, $p \in P$.

2.3.10 Lemma. *For a finite poset P and $R \subset P$, both satisfying the condition regarding minima of the preceding definition, the map $s^P_*\colon H_*(\Delta P, \Delta R) \xrightarrow{\cong} H_*(P, R)$ is an isomorphism.*

Proof. We may assume $R = \emptyset$, because the general case will follow by an application of the five lemma.

We consider the covering of P by the open subsets $X(p) := \{q\colon q \geq p\}$. These together with the inclusion maps form a P-diagram of spaces. If $\mathbb{Z}$ denotes the constant diagram, then $S(X) \otimes_{P^o} \mathbb{Z} = \sum_{p \in P} S(X(p))$ and $H(S(X) \otimes_{P^o} \mathbb{Z}) \cong H(P)$ induced by inclusion. As in Proposition 1.3.9 the diagram $S(X)$ is free. For this note that the existence of maxima follows from the existence of minima. Since s^P maps $\Delta X(p)$ to $X(p)$, it induces a map $\mathbb{Z} \otimes_{P^o} B(P^o) \to S(X)$. Regarding $\mathbb{Z}$ as a chain complex concentrated in dimension 0 this is a ZŽ–map, because $X(p)$ is acyclic for all p. The resulting isomorphism

$$H_*(\Delta P) \cong H(\mathbb{Z} \otimes_{P^o} B(P^o) \otimes_{P^o} \mathbb{Z}) \to H(S(X) \otimes_{P^o} \mathbb{Z}) \cong H(P)$$

is easily identified with s^P_*. □

2.3.11 Proposition. *The composition*

$$H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \xrightarrow{h_k} H_*(PV, \bigcup P\mathcal{A}) \xrightarrow{s^{\mathcal{A}}_*} H_*(Q_{[0,n]}, Q_{[0,n)})$$

is an isomorphism for $k = 0$ and zero for $k > 0$.

Proof. Consider the diagram

$$\begin{array}{ccccc} \mathbb{C}P^k \times (\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) & & \xrightarrow{\quad f^k \quad} & & (PV, \bigcup P\mathcal{A}) \\ \downarrow \pi & & & & \downarrow s^{\mathcal{A}} \\ (\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) & \xrightarrow{s^{Q_{[k,n]}}} & (Q_{[k,n]}, Q_{[k,n)}) & \xrightarrow{i} & (Q_{[0,n]}, Q_{[0,n)}) \end{array}$$

where i is the inclusion map and π the projection onto the second factor. By construction of f^k, $f^k\left[\{x\} \times \langle u_0, \ldots, u_q \rangle\right] \subset Pu_q$, that is $s^{\mathcal{A}}(f^k(x,y)) \leq s^Q(y)$, and by Lemma 2.3.7 this implies the homotopy commutativity of the diagram.

For $k = 0$, π is a homeomorhism, i the identity, and h_0 equals f^0_* up to an isomorphism. Therefore $s^{\mathcal{A}}_* \circ h_0$ is an isomorphism, because $s^{Q_{[0,n]}}_*$ is an isomorphism by Lemma 2.3.10.

For $k > 0$, $s^{\mathcal{A}}_*(h_k(c)) = s^{\mathcal{A}}_*(f^k_*([\mathbb{C}P^k] \times c)) = (i \circ s^{Q_{[k,n]}})_*(\pi_*([\mathbb{C}P^k] \times c)) = (i \circ s^{Q_{[k,n]}})_*(0) = 0$. □

Since we are concerned with the vanishing of certain intersection products, we will use the following immediate corollary.

2.3.12 Corollary. *Let* $0 \leq i \leq n$, $c_i \in H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$, *and* $x = \sum_i h_i(c_i) \in H_*\left(PV, \bigcup P\mathcal{A}\right)$. *If* $s^{\mathcal{A}}_*(x) = 0 \in H_*(Q_{[0,n]}, Q_{[0,n)})$, *then* $c_0 = 0$. □

Vanishing for projective arrangements

We are now ready to prove the last step in the proof of the product formula for complex projective arrangements, namely the following proposition.

2.3.13 Proposition. *Let* $\mathbb{K} = \mathbb{C}$ *and* $k + l < n$, $c \in H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$, $d \in H_*(\Delta Q_{[l,n]}, \Delta Q_{[l,n)})$. *Then*

$$h_k(c) \bullet h_l(d) = 0.$$

We will assume $\mathbb{K} = \mathbb{C}$ from now on. We will prove the proposition in three steps.

We would like to have the classes $h_k(c)$ and $h_l(d)$ represented by chains as much as possible in general position with respect to each other. To this end we consider an arrangement that is the union of two arrangements $\mathcal{A}$ and $\tilde{\mathcal{A}}$ with intersection posets Q and $\tilde{Q}$ such that $\tilde{\mathcal{A}}$ is in general position with respect to $\mathcal{A}$ (see Definition 2.3.4).

We will denote the intersection poset of the arrangement $\check{\mathcal{A}} := \mathcal{A} \cup \tilde{\mathcal{A}}$ by $\check{Q}$ and so on. The map

$$\begin{aligned} \sigma\colon (Q \times \tilde{Q})_{[n,2n]} &\to \check{Q}_{[0,n]} \\ (u,v) &\mapsto u \cap v \end{aligned} \tag{2.58}$$

is an isomorphism.

2.3.14 Proposition. *In the above situation let* $c \in H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$ *and* $d \in H_*(\Delta \tilde{Q}_{[l,n]}, \Delta \tilde{Q}_{[l,n)})$ *with* $k + l < n$. *Then*

$$h_k(c) \bullet \tilde{h}_l(d) = \sum_{i>0} \check{h}_i(r_i)$$

for classes $r_i \in H_*\left(\Delta \check{Q}_{[i,n]}, \Delta \check{Q}_{[i,n)}\right)$.

Proof. By Corollary 2.3.12 we have to show $s_*^{\check{\mathcal{A}}}(h_k(c) \bullet \tilde{h}_l(d)) = 0$. It will be in doing so that we employ the ideas laid out in the discussion of the real example.

We set $(X, A) := \mathbb{C}P^k \times \mathbb{C}P^l \times \Big(\Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]}), \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,2n)}\Big)$, $D := \{(x, x) \in PV \times PV\}$, $\mathcal{C}D := (PV \times PV) \setminus D$ and use the map g from Definition 2.3.1. We denote the diagonal map $PV \to PV \times PV$ by Δ and define $\bar{g}\colon (g^{-1}[D], g^{-1}[D] \cap A) \to (PV, \bigcup P\mathcal{A} \cup \bigcup P\tilde{\mathcal{A}})$ by $\Delta \circ \bar{g} = g$. Note that in the real example the projection of $g^{-1}[D]$ to the order complex is the set represented by a dotted line in (2.56). We will first show $h_k(c) \bullet \tilde{h}_l(d) \in \operatorname{im} \bar{g}_*$ and then $s_*^{\check{\mathcal{A}}} \circ \bar{g}_* = 0$.

There is a commutative diagram

$$
\begin{array}{ccc}
H^*(PV \times PV, \mathcal{C}D) \otimes H_*(X, A) & \xrightarrow{\mathrm{id} \otimes g_*} & \begin{array}{c} H^*(PV \times PV, \mathcal{C}D) \otimes \\ H_*\Big((PV, \bigcup P\mathcal{A}) \times (PV, \bigcup P\tilde{\mathcal{A}})\Big) \end{array} \\
\Big\downarrow {\scriptstyle g^* \otimes \mathrm{id}} & & \\
H^*(X, X \setminus g^{-1}[D]) \otimes H_*(X, A) & & \Big\downarrow {\scriptstyle \frown} \\
\Big\downarrow {\scriptstyle \frown} & & \\
H_*(g^{-1}[D], g^{-1}[D] \cap A) & \xrightarrow{g_*} & H_*\big(D, D \cap (\bigcup P\mathcal{A} \times PV \cup PV \times \bigcup P\tilde{\mathcal{A}})\big) \\
 & \searrow^{\bar{g}_*} & \cong \Big\uparrow {\scriptstyle \Delta_*} \\
 & & H_*(PV, \bigcup P\mathcal{A} \cup \bigcup P\tilde{\mathcal{A}}).
\end{array}
$$

Regarding the existence of the cap products in this diagram and commutativity, note that we are entirely dealing with algebraic sets and polynomial maps. Now, if $\vartheta \in H^*(PV \times PV, \mathcal{C}D)$ is the Thom class determined by $\vartheta \frown [PV \times PV] = \Delta_*([PV])$, then

$$
\begin{aligned}
h_k(c) \bullet \tilde{h}_l(d) &= \Delta_!(h_k(c) \times \tilde{h}_l(d)) \\
&= \Delta_*^{-1}\Big(\vartheta \frown (h_k(c) \times \tilde{h}_l(d))\Big) \\
&= \Delta_*^{-1}\Big(\vartheta \frown g_*\big([\mathbb{C}P^k] \times [\mathbb{C}P^l] \times (c \times d)\big)\Big) \\
&= \Delta_*^{-1}\Big(g_*\Big(g^*(\vartheta) \frown \big([\mathbb{C}P^k] \times [\mathbb{C}P^l] \times (c \times d)\big)\Big)\Big) \\
&= \bar{g}_*\Big(g^*(\vartheta) \frown \big([\mathbb{C}P^k] \times [\mathbb{C}P^l] \times (c \times d)\big)\Big).
\end{aligned}
$$

By construction of g, $\bar{g}\Big(x, y, \sum_{j=0}^{r} \lambda_j(u_j, v_j)\Big) \in u_r \cap v_r$. Firstly this implies that $g^{-1}[D]$ misses $\mathbb{C}P^k \times \mathbb{C}P^l \times \Delta(Q \times \tilde{Q})_{[0,n)}$, and secondly from the reformulation $s^{\check{\mathcal{A}}}(\bar{g}(x, y, z)) \leq \sigma\Big(s^{Q \times \tilde{Q}}(z)\Big)$, where σ is the isomorphism from (2.58), and

Lemma 2.3.7 it can be seen that the diagram

$$\begin{array}{ccc}
(g^{-1}[D], g^{-1}[D]\cap A) & \xrightarrow{\ s^{\check{A}}\circ\bar{g}\ } & (\check{Q}_{[0,n]}, \check{Q}_{[0,n)}) \\
 & \searrow^{\pi} & \uparrow{\scriptstyle\sigma} \\
 & & \big((Q\times\tilde{Q})_{[n,2n]}, (Q\times\tilde{Q})_{[n,2n)}\big) \\
 & & \nearrow_{s^{Q\times\tilde{Q}}} \\
\multicolumn{3}{c}{\big(\Delta(Q_{[k,n]}\times\tilde{Q}_{[l,n]})\setminus\Delta(Q\times\tilde{Q})_{[0,n)}, \Delta(Q_{[k,n]}\times\tilde{Q}_{[l,n]})_{[0,2n)}\setminus\Delta(Q\times\tilde{Q})_{[0,n)}\big),}
\end{array}$$

where π denotes projection onto the third factor, is homotopy commutative. The two arrows on the right hand side of the diagram should be compared to (2.57).

By Proposition 2.3.5 and because the subcomplex $\Delta\big((Q_{[k,n]}\times\tilde{Q}_{[l,n]})\cap(Q\times\tilde{Q})_{[0,n)}\big)$ has dimension at most $n-1-k-l$, we may assume that $\pi[g^{-1}[D]]$ will not only miss this subcomplex, but any cone over it. Therefore π factorizes over the pair

$$\begin{aligned}\big(\Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]}) \setminus \Delta((Q \times \tilde{Q})_{[0,n)} \cup \{(V,V)\}),& \\ \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,2n)} \setminus \Delta(Q \times \tilde{Q})_{[0,n)}\big).&\end{aligned}$$

This pair is homeomorphic to $(\Delta(Q_{[k,n]}\times\tilde{Q}_{[l,n]})_{[0,2n)}\setminus\Delta(Q\times\tilde{Q})_{[0,n)})\times([0,1),\{0\})$ and has trivial homology. So $(s^{\check{A}}\circ\bar{g})_* = s_*^{Q\times\tilde{Q}}\circ\pi_* = s_*\circ 0 = 0$. □

2.3.15 Proposition. *In the above situation let* $c \in H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$ *and* $d \in H_*(\Delta\tilde{Q}_{[l,n]}, \Delta\tilde{Q}_{[l,n)})$ *with* $k+l<n$*. Then*

$$h_k(c)\bullet\tilde{h}_l(d) = 0.$$

Proof. We choose a hyperplane H in V in general position with respect to the arrangement $\check{A} = \mathcal{A}\cup\tilde{\mathcal{A}}$ and use notation as in Section 2.1. By Proposition 2.1.35 and induction on the dimension of V

$$i_!(h_k(c)\bullet\tilde{h}_l(d)) = i_!(h_k(c))\bullet i_!(\tilde{h}_l(d)) = h^H_{k-1}(\eta_*(c))\bullet\tilde{h}^H_{l-1}(\eta_*(d)) = 0,$$

since the arrangement A^H is again in general position with respect to the arrangement $\tilde{A}^H$ and $(k-1)+(l-1)<n-1$. This implies $h_k(c)\bullet\tilde{h}_l(d)\in\ker i_! = \operatorname{im}\check{h}_0$ by Proposition 2.1.35. But $h_k(c)\bullet\tilde{h}_l(d)\in\bigoplus_{j>0}\operatorname{im}\check{h}_j$ by Proposition 2.3.14 and hence $h_k(c)\bullet\tilde{h}_l(d) = 0$. □

Proof of Proposition 2.3.13. We choose a neighbourhood U of $\bigcup P\mathcal{A}$ such that the inclusion $(PV,\bigcup P\mathcal{A})\to(PV,U)$ is a homotopy equivalence. We then choose a copy $\tilde{\mathcal{A}}$ of $\mathcal{A}$ also contained in U, in general position with respect to $\mathcal{A}$ and such that the diagram

$$\begin{array}{ccc}
\mathbb{C}P^i\times(\Delta Q_{[i,n]},\Delta Q_{[i,n)}) & \xrightarrow{\ \tilde{f}^i\ } & (PV,\bigcup P\tilde{\mathcal{A}}) \\
\downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle \text{incl.}} \\
(PV,\bigcup P\mathcal{A}) & \xrightarrow{\ \text{incl.}\ } & (PV,U)
\end{array}$$

commutes up to homotopy. Because of the commutativity of

$$\begin{array}{ccc}
H_*(PV, \bigcup P\mathcal{A}) \otimes H_*(PV, \bigcup P\mathcal{A}) & \xrightarrow{\bullet} & H_*(PV, \bigcup P\mathcal{A}) \\
\uparrow{\scriptstyle h_k \otimes h_l} & & \downarrow{\scriptstyle \mathrm{incl}_*\ \cong} \\
H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \otimes H_*(\Delta Q_{[l,n]}, \Delta Q_{[l,n)}) & & H_*(PV, U) \\
\downarrow{\scriptstyle h_k \otimes \tilde{h}_l} & & \uparrow{\scriptstyle \mathrm{incl}_*} \\
H_*(PV, \bigcup P\mathcal{A}) \otimes H_*(PV, \bigcup P\tilde{\mathcal{A}}) & \xrightarrow{\bullet} & H_*(PV, \bigcup P\mathcal{A} \cup \bigcup P\tilde{\mathcal{A}})
\end{array}$$

the result follows from Proposition 2.3.15 □

Proof of Theorem 2.2.1. The theorem follows directly from Proposition 2.3.13 and Proposition 2.2.10. □

2.4 Projective c-arrangements

Descriptions of the cohomology ring of the complement of a linear arrangement in terms of generators and relations go back to Arnol'd [Arn69] who gave such a description for the classifying space of the coloured braid group, which is the complement of a complex hyperplane arrangement. He also conjectured a similar formula for general complex hyperplane arrangements, later to be proved by Orlik and Solomon [OS80] (see Remark 2.4.4). Since then several such results on other classes of linear arrangements have been obtained. The approach most useful to us is that of Yuzvinsky, who derived from the complex case of Theorem 2.2.3 (with rational coefficients) a description in terms of generators and relations of the cohomology ring of the complement of a complex linear arrangement with geometric intersection lattice [Yuz99]. Generalizations of his results to real ≥ 2-arrangements and integral coefficients have been stated in [dLS01].

A presentation of the cohomology ring

We will now use a route similar to Yuzvinsky's to obtain from Theorem 2.2.1 a simple description of the cohomology of the complement of a complex projective c-arrangement. These probably form the simplest class of arrangements that still yield a proper generalization of the classical result on complex hyperplane arrangements in this way. The result presented in this section complements results of Feichtner and Ziegler in [FZ00].

2.4.1 Definition. For a positive integer c, we call $\mathcal{A}$ a c-arrangement, if every $A \in \mathcal{A}$ is a subspace of codimension c and $d(q)$ is an integral multiple of c for every $q \in Q$.

2.4.2 Definition. We call a subset M of $\mathcal{A}$ independent, if $n - d\left(\bigcap M\right) = \sum_{A \in M}(n - d(A))$, dependent, if it is not independent, and minimally dependent, if it is dependent but all of its proper subsets are independent.

We will assume $\mathbb{K} = \mathbb{C}$ in this section. Our goal is the following.

2.4.3 Theorem. *Let $\mathcal{A}$ be a complex c-arrangement, $|\mathcal{A}| - 1 =: t \geq 0$, $\mathcal{A} = \{A_0, \dots, A_t\}$. Let R be the free graded commutative (in the graded sense) ring over the set of generators $\{x\} \cup \{y_i : 1 \leq i \leq t\}$ with $|x| = 2$, $|y_i| = 2c - 1$. Let I be the ideal generated by*

$$\left\{ \sum_{j=0}^{r} (-1)^j y_{i_0} \cdots \hat{y}_{i_j} \cdots y_{i_r} : i_0 < \cdots < i_r,\ \left\{A_{i_j}\right\} \text{ is minimally dependent.} \right\}$$
$$\cup \left\{ y_{i_1} \cdots y_{i_r} : i_1 < \cdots < i_r,\ \{A_0\} \cup \left\{A_{i_j}\right\} \text{ is minimally dependent. } \right\}$$
$$\cup \{x^c\}\,.$$

The map

$$\begin{aligned} \pi\colon R &\to H^*\left(PV \setminus \bigcup P\mathcal{A}\right), \\ x &\mapsto P(h_{n-1}([\langle V \rangle])), \\ y_i &\mapsto P(h_{n-c}([\langle A_i, V\rangle - \langle A_0, V\rangle])), \end{aligned}$$

where $P\colon H_*(PV, \bigcup P\mathcal{A}) \xrightarrow{\cong} H^*\left(PV \setminus \bigcup P\mathcal{A}\right)$ denotes Poincaré duality, is an epimorphism and $\ker \pi = I$.

We now fix the arrangement $\mathcal{A} = \{A_0, \dots, A_t\}$.

2.4.4 Remark. For $c = 1$ the complement $PV \setminus \bigcup P\mathcal{A}$ can be regarded as the complement in the affine space $PV \setminus PA_0$ of the linear hyperplane arrangement $\mathcal{A}' := \{PA_i \setminus PA_0 : 1 \leq i \leq t\}$. In this case, the generator x and the corresponding relation can be omitted.

If A_0 is in general position with respect to $\mathcal{A} \setminus \{A_0\}$, the second kind of generators does not occur. This is in particular the case if the arrangement $\mathcal{A}'$ is central, i.e. if $\bigcap \mathcal{A}' \neq \emptyset$. In this case the theorem reduces to the description of the cohomology ring of the complement of $\mathcal{A}'$ given by Orlik and Solomon.

The atomic complex

We now turn to the proof of the theorem. When using simplicial chain complexes, we will always use the complex of non-degenerate simplices and view it as the complex of all simplices modulo degenerate simplices if necessary.

2.4.5 Definition. For an integer k with $0 \leq k \leq n$, we define S_k to be the simplicial complex which has the vertex set $\{0, \dots, t\}$ and as simplices the sets $I \subset \{0, \dots, t\}$ with $d\left(\bigcap_{i \in I} A_i\right) \geq k$. This is the *atomic complex* of $Q_{[k,n]}$. We also define D^k to be the reduced ordered (using the natural order of $\{0, \dots, t\}$) simplicial chain complex of S_k shifted by one, i.e. $D^k_r = \tilde{C}_{r-1}(S_k)$ and in particular $D^k_0 \cong \mathbb{Z}$ generated by the empty simplex.

As is well known, the atomic complex and the order complex, of $Q_{[k,n)}$ in this case, are homotopy equivalent. We describe a homotopy equivalence to fix a concrete isomorphism between their homology groups. Before doing this, we state a useful lemma.

2.4.6 Lemma. *Let P_0, P_1 be posets, $P_i' \subset P_i$. If $f, g\colon (P_0, P_0') \to (P_1, P_1')$ are order preserving functions such that $f(p) \le g(p)$ for all $p \in P_0$, then the maps $f, g\colon (\Delta P_0, \Delta P_0') \to (\Delta P_1, \Delta P_1')$ are homotopic.*

Proof. The map $H\colon \{0,1\} \times P_0 \to P_1$ defined by $H(0,x) := f(x)$, $H(1,x) := g(x)$ is order preserving and hence yields the desired homotopy

$$I \times (\Delta P_0, \Delta P_0') \approx \big(\Delta(\{0,1\} \times P_0), \Delta(\{0,1\} \times P_0')\big) \xrightarrow{H} (\Delta P_1, \Delta P_1'),$$

where we view $\{0,1\}$ as a poset. □

2.4.7 Remark. This lemma is a special case of [Seg68, Prop. 2.1] which is proved in the same way.

2.4.8 Definition and Proposition. *We denote the face poset of S_k by FS_k, but order it by $M \le M'$ if M' is a face of M, that is if $M' \subset M$. We also set $\tilde{F}S_k := FS_k \cup \{\emptyset\}$. The map*

$$\begin{aligned} s\colon (\tilde{F}S_k, FS_k) &\to (Q_{[k,n]}, Q_{[k,n)}) \\ M &\mapsto \bigcap \{A_i \colon i \in M\} \end{aligned}$$

is then order preserving and moreover satisfies $s(M \wedge M') = s(M \cup M') = s(M) \cap s(M') = s(M) \wedge s(M')$, if one side, and therefore the other, exists.

With these definitions, the map $s\colon (\Delta\tilde{F}S_k, \Delta FS_k) \to (\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$ is a homotopy equivalence.

2.4.9 Remark. ΔFS_k is the barycentric subdivision of S_k, and $\Delta\tilde{F}S_k$ is a cone over ΔFS_k.

Proof. We define an order preserving map

$$\begin{aligned} r\colon (Q_{[k,n]}, Q_{[k,n)}) &\to (\tilde{F}S_k, FS_k), \\ q &\mapsto \{i \colon A_i \supset q\}. \end{aligned}$$

We have $s(r(q)) \ge q$ for $q \in Q_{[k,n]}$ and $r(s(i)) \le i$ for $i \in \tilde{F}S_k$. Hence, by the preceding lemma r is a homotopy inverse to s, when both maps are regarded as simplicial maps between order complexes. □

2.4.10 Definition and Proposition. *We define chain maps*

$$\begin{aligned} f^k\colon D_r^k &\to C_r(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \\ \langle i_1, \ldots, i_r\rangle &\mapsto \langle A_{i_1}, V\rangle \hat{\times} \cdots \hat{\times} \langle A_{i_r}, V\rangle. \end{aligned}$$

For $r = 0$ this is to be understood as $f^k(\langle\rangle) = \langle V\rangle$.

2.4.11 Notation. To simplify the following calculations, we set set

$$\alpha_i := \langle A_i, V\rangle \in C_1(\Delta Q_{[n-c,n]})$$

and sometimes write the multiplication $\hat{\times}$ as juxtaposition.

Proof. To see that f^k is well-defined, we have to check that the right hand side is in $C_*(\Delta Q_{[k,n]})$. But $d\,(A_{i_1} \wedge \cdots \wedge A_{i_r}) \geq k$ by definition of S_k and hence D^k_r.

To see that f^k is a chain map, we calculate

$$\begin{aligned}
\mathfrak{d}(f^k(\langle i_1,\dots,i_r\rangle)) &= \sum_{j=1}^{r}(-1)^{j+1}\langle A_{i_1}, V\rangle \,\hat{\times}\cdots\hat{\times}\, \mathfrak{d}\langle A_{i_j}, V\rangle \,\hat{\times}\cdots\hat{\times}\langle A_{i_r}, V\rangle \\
&= \sum_{j=1}^{r}(-1)^{j+1}\alpha_{i_1}\cdots\alpha_{i_{j-1}}\langle V\rangle\alpha_{i_{j+1}}\cdots\alpha_{i_r} \\
&\quad - \sum_{j=1}^{r}(-1)^{j+1}\alpha_{i_1}\cdots\alpha_{i_{j-1}}\langle A_{i_j}\rangle\alpha_{i_{j+1}}\cdots\alpha_{i_r} \\
&= \sum_{j=1}^{r}(-1)^{j+1}\alpha_{i_1}\cdots\alpha_{i_{j-1}}\hat{\alpha}_{i_j}\alpha_{i_{j+1}}\cdots\alpha_{i_r} \\
&\quad - \sum_{j=1}^{r}(-1)^{j+1}\alpha_{i_1}\cdots\alpha_{i_{j-1}}\langle A_{i_j}\rangle\alpha_{i_{j+1}}\cdots\alpha_{i_r}.
\end{aligned}$$

The first summand equals $f^k(\mathfrak{d}\langle i_1,\dots,i_r\rangle)$. Since $\langle A_i\rangle \in \mathbb{C}_0(Q_{[n-c,n)})$, the second summand is in $C_*(\Delta Q_{[k,n-c]}) \subset C_*(\Delta Q_{[k,n)})$. □

2.4.12 Proposition. *The induced maps* $f^k_*\colon H(D^k) \to H(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$ *are isomorphisms.*

Proof. Defining

$$\begin{aligned}
\bar{f}^k\colon D^k_r &\to C_r(\Delta\tilde{F}S_k, \Delta FS_k) \\
\langle i_1,\dots,i_r\rangle &\mapsto \langle\{i_1\},\emptyset\rangle \,\hat{\times}\cdots\hat{\times}\langle\{i_r\},\emptyset\rangle
\end{aligned}$$

the diagram

$$\begin{array}{ccccc}
 & & H_r(D^k) & & \\
 & \swarrow{\scriptstyle f^k_*} & \downarrow{\scriptstyle \bar{f}^k_*} & \searrow{\scriptstyle sd_*,\ \cong} & \\
H_r(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) & \xleftarrow[\cong]{s_*} & H_r(\Delta\tilde{F}S_k, \Delta FS_k) & \xrightarrow[\cong]{\mathfrak{d}} & \tilde{H}_{r-1}(\Delta FS_k)
\end{array}$$

commutes, where sd is the barycentric subdivision map $\tilde{C}_*(S_k) \to \tilde{C}_*(\Delta FS_k)$. The connecting homomorphism is an isomorphism, because $\tilde{F}S_k$ has the maximum $\emptyset$. The map s_* is an isomorphism because of Proposition 2.4.8. It follows that f^k_* is an isomorphism. □

Proof of the presentation

The chain maps f^k would be more useful in a situation in which the chains $\langle A_i, V\rangle$ are cycles. For example, think of affine arrangements, where $(\Delta Q_{[k,n]}, \Delta Q_{[k,n)} \cup \Delta Q_{(k,n]})$ takes the place of $(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$. In our situation they are not. The chains $\langle A_i, V\rangle - \langle A_j, V\rangle$ however are cycles, we will therefore replace the maps f^k by the following maps.

2.4.13 Definition and Proposition. *For a c-arrangement $\mathcal{A}$, we define chain maps*

$$g^k\colon D^k_r \to C_r(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$$
$$\langle i_1, \dots, i_r\rangle \mapsto \begin{cases} (\langle A_{i_1}, V\rangle - \langle A_0, V\rangle) \hat{\times} \cdots \hat{\times} (\langle A_{i_r}, V\rangle - \langle A_0, V\rangle), & r = a, \\ 0, & r \neq a, \end{cases}$$

where a is defined by $n - (a+1)c < k \leq n - ac$.

Proof. We check that g^k is a well-defined chain map. For $r > a$ we have $C_r(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \cong 0$, since $n - k < (a+1)c \leq rc$. So we just have to show that $g^k(\langle i_1, \dots, i_a\rangle)$ is a cycle in $C_a(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$. This is true, because each $\langle A_i, V\rangle - \langle A_0, V\rangle$ is a cycle in $C_1(\Delta Q_{[n-c,n]}, \Delta Q_{[n-c,n)})$ and $n - ac \geq k$. □

2.4.14 Proposition. *The maps f^k and g^k are chain homotopic.*

Proof. We define

$$K\colon D^k_r \to C_{r+1}(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$$
$$\langle i_1, \dots, i_r\rangle \mapsto \begin{cases} f^k(\langle 0, i_1, \dots, i_r\rangle), & r < a, \\ 0, & r \geq a. \end{cases}$$

The right hand side is well defined, because for $r < a$ we have

$$d(A_0 \cap A_{i_1} \cap \cdots \cap A_{i_r}) \geq n - (r+1)c \geq n - ac \geq k.$$

We calculate $K\mathfrak{d} + \mathfrak{d}K$.

For $r < a$:

$$\begin{aligned}
&(K\mathfrak{d} + \mathfrak{d}K)\langle i_1, \dots, i_r\rangle \\
&\quad = f^k\left(\sum_{j=1}^{r} (-1)^{j+1} \langle 0, i_1, \dots, \hat{i}_j, \dots, i_r\rangle\right) + \mathfrak{d} f^k(\langle 0, i_1, \dots, i_r\rangle) \\
&\quad = f^k\left(\sum_{j=1}^{r} (-1)^{j+1} \langle 0, i_1, \dots, \hat{i}_j, \dots, i_r\rangle + \mathfrak{d}\langle 0, i_1, \dots, i_r\rangle\right) \\
&\quad = f^k(\langle i_1, \dots, i_r\rangle) = (f^k - g^k)\langle i_1, \dots, i_r\rangle.
\end{aligned}$$

For $r = a$: We first calculate

$$\begin{aligned} g^k(\langle i_1,\dots,i_a\rangle) &= (\alpha_{i_i}-\alpha_0)\cdots(\alpha_{i_a}-\alpha_0) \\ &= \alpha_{i_1}\cdots\alpha_{i_a} - \sum_{j=1}^{a}\alpha_{i_1}\cdots\alpha_{i_{j-1}}\alpha_0\alpha_{i_{j+1}}\cdots\alpha_{i_a} \\ &= \alpha_{i_1}\cdots\alpha_{i_a} + \sum_{j=1}^{a}(-1)^j\alpha_0\alpha_{i_1}\cdots\hat{\alpha}_{i_j}\cdots\alpha_{i_a} \end{aligned}$$

and with this

$$\begin{aligned} (K\mathfrak{d}+\mathfrak{d}K)\langle i_1,\dots,i_a\rangle &= f^k\left(\sum_{j=1}^{a}(-1)^{j+1}\langle 0,i_1,\dots,\hat{i}_j,\dots,i_a\rangle\right) \\ &= \sum_{j=1}^{a}(-1)^{j+1}\alpha_0\alpha_{i_1}\cdots\hat{\alpha}_{i_j}\cdots\alpha_{i_a} \\ &= (f^k-g^k)\langle i_1,\dots,i_a\rangle. \end{aligned}$$

For $r > a$ we have $(K\mathfrak{d}+\mathfrak{d}K)\langle i_1,\dots,i_r\rangle = 0 = (f^k-g^k)\langle i_1,\dots,i_r\rangle$, since $C_r(\Delta Q_{[k,n]},\Delta Q_{[k,n)})\cong 0$ as noted before. □

2.4.15 Proposition. *The map π is surjective.*

Proof. By Proposition 2.4.12, Proposition 2.4.14, and of course Proposition 2.1.25, $H^*(PV\setminus\bigcup P\mathcal{A})$ is additively generated by the elements $P\left(h_k\left([g^k\langle i_i,\dots,i_r\rangle]\right)\right)$ with $k\le n-rc$. By Theorem 2.2.1

$$\begin{aligned} &P\left(h_k\left([g^k\langle i_i,\dots,i_r\rangle]\right)\right) \\ &\quad = P(h_k([(\alpha_{i_1}-\alpha_0)\,\hat{\times}\cdots\hat{\times}(\alpha_{i_r}-\alpha_0)])) \\ &\quad = P(h_{n-1}([\langle V\rangle]))^{n-k-rc}P(h_{n-c}([\alpha_{i_1}-\alpha_0]))\cdots P(h_{n-c}([\alpha_{i_r}-\alpha_0])) \\ &\quad = \pi(x)^{n-k-rc}\pi(y_{i_1})\cdots\pi(y_{i_r}). \end{aligned}$$

This shows that π is surjective. □

2.4.16 Proposition. $I\subset\ker\pi$.

Proof. First of all

$$\begin{aligned} \pi(x^c) &= P(h_{n-1}([\langle V\rangle]))^c \\ &= P(h_{n-c}([\langle V\rangle])) \\ &= P(h_{n-c}([\mathfrak{d}\langle A_0,V\rangle])) = P(h_{n-c}(0)) = 0. \end{aligned}$$

If $\{A_{i_0},\dots,A_{i_r}\}$ is minimally dependent, then $d\left(\bigcap_j A_{i_j}\right) = n - rc$ and

$$\begin{aligned} 0 &= P(h_{n-rc}(g_*^{n-rc}([\mathfrak{d}\langle i_0,\dots,i_r\rangle]))) \\ &= (P\circ h_{n-rc}\circ g_*^{n-rc})\left(\left[\sum_{j=0}^{r}(-1)^j\langle i_0,\dots,\hat{i}_j,\dots,i_r\rangle\right]\right) \\ &= \pi\left(\sum_{j=0}^{r}(-1)^j y_{i_0}\cdots\hat{y}_{i_j}\cdots y_{i_r}\right) \end{aligned}$$

and similarly if $\{A_0, A_{i_1},\dots,A_{i_r}\}$ is minimally dependent, then

$$\begin{aligned} 0 &= P(h_{n-rc}(g_*^{n-rc}([\mathfrak{d}\langle 0,i_1,\dots,i_r\rangle]))) \\ &= (P\circ h_{n-rc}\circ g_*^{n-rc})\left(\left[\langle i_1,\dots,i_r\rangle + \sum_{j=1}^{r}(-1)^j\langle 0,i_1,\dots,\hat{i}_j,\dots,i_r\rangle\right]\right) \\ &= \pi\left(y_{i_1}\cdots y_{i_r}\right) \end{aligned}$$

as claimed. □

2.4.17 Lemma. *If $i_0 < \cdots < i_r$ and $\left\{A_{i_j}\right\}$ is dependent, then $y_{i_0}\cdots y_{i_r} \in I$ and $\sum_j(-1)^j y_{i_0}\cdots\hat{y}_{i_j}\cdots y_{i_r} \in I$.*

Proof. Let $\{A_{i_0},\dots,A_{i_r}\}$ be dependent. To show $y_{i_0}\cdots y_{i_r}\in I$ we may assume that the set is minimally dependent. Then $\sum_j(-1)^j y_{i_0}\cdots\hat{y}_{i_j}\cdots y_{i_r} \in I$ and $y_{i_0}\left(\sum_j(-1)^j y_{i_0}\cdots\hat{y}_{i_j}\cdots y_{i_r}\right) = y_{i_0}\cdots y_{i_r}$ since $y_0^2 = 0$.

For the second part of the lemma we may assume that $\left\{A_{i_j} : j \le s\right\}$ is minimally dependent. Then

$$\begin{aligned} \sum_j(-1)^j y_{i_0}\cdots\hat{y}_{i_j}\cdots y_{i_r} = \\ \underbrace{\left(\sum_{j=0}^{s}(-1)^j y_{i_0}\cdots\hat{y}_{i_j}\cdots y_{i_s}\right)}_{\in I} y_{i_{s+1}}\cdots y_{i_r} \\ + \underbrace{y_{i_0}\cdots y_{i_s}}_{\in I}\sum_{j=s+1}^{r}(-1)^j y_{i_{s+1}}\cdots\hat{y}_{i_j}\cdots y_{i_r} \in I \end{aligned}$$

as claimed. □

2.4.18 Proposition. $\ker\pi \subset I$.

Proof. Let $z \in \ker\pi$. We want to show $z \in I$. We may assume that z is a linear combination of elements $x^s y_{i_1}\cdots y_{i_r}$ with $0 \le s < c$, $i_1 < \cdots < i_r$ and $\{A_{i_1},\dots,A_{i_r}\}$ independent. Since $\pi(x^s y_{i_1}\cdots y_{i_r}) \in \operatorname{im}(P\circ h_{n-cr-s})$ and r and s

are determined by $cr+s$, we may assume z to be homogenous in r and s, i.e. $z = x^s \sum_{i_1<\dots<i_r} \lambda_i y_{i_1} \cdots y_{i_r}$. We set $k := n - cr - s$. The chain

$$z' := \sum_{i_1<\dots<i_r} \lambda_i \langle i_1, \dots, i_r \rangle + \sum_{i_1<\dots<i_r} \lambda_i \sum_{j=1}^{r} (-1)^j \langle 0, i_1, \dots, \hat{i}_j, \dots, i_r \rangle$$

is a cycle in D_r^k (the second summand is a cone over the boundary of the first summand), $0 = \pi(z) = (P \circ h_k)\left(g_*^k([z'])\right)$, and therefore $[z'] = 0$ by Proposition 2.4.12 and Proposition 2.4.14, i.e. z' is a boundary in D^k, which means that there exist μ_i, ν_i such that

$$z' = \mathfrak{d} \left(\sum_{i_1<\dots<i_r} \mu_i \langle 0, i_1, \dots, i_r \rangle + \sum_{i_0<\dots<i_r} \nu_i \langle i_0, \dots, i_r \rangle \right)$$

and with $d\left(\bigcap_j A_{i_j} \cap A_0\right) \geq k$ and therefore $\{A_0\} \cup \left\{A_{i_j}\right\}$ dependent for $\mu_i \neq 0$ and $\left\{A_{i_j}\right\}$ dependent for $\nu_i \neq 0$. Comparing coefficients and sorting the simplices by whether the first vertex is 0 yields

$$z = \sum_i \mu_i y_{i_1} \cdots y_{i_r} + \sum_i \nu_i \sum_j (-1)^j y_{i_0} \cdots \hat{y}_{i_j} \cdots y_{i_r} \in I$$

as claimed. □

This completes the proof of Theorem 2.4.3 □