FREIE UNIVERSITÄT BERLIN

Doctoral Thesis

Spaces of convex n-partitions

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A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

at the

Fachbereich Mathematik und Informatik
Freie Universität Berlin

Berlin, 2015

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Date of defense:

February 19, 2015

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Chapter 1

Introduction

We look at the set $\mathcal{C}(\mathbb{R}^d, n)$ of all partitions of $\mathbb{R}^d$ into n convex regions, for d and n positive integers. Since we are able to find a metric on this set, we can talk of $\mathcal{C}(\mathbb{R}^d, n)$ as the space of all convex n-partitions of $\mathbb{R}^d$, which is also a topological space. These spaces for different n and d are our main object of study. As far as we know, there is no reference of them in the literature, although some similar spaces and particular cases have been studied. Here we introduce some basic concepts and definitions about them, investigate their general properties and look at some examples and related spaces.

The spaces of n-partitions can be described as unions of semialgebraic sets. Thus, in particular, the spaces have well-defined dimensions. We give two possible ways to decompose spaces of n-partitions as unions of semialgebraic pieces. Since all the regions of a partition are polyhedral, we obtain one parameterization from from the hyperplane description of the regions. We also define a face structure on each partition and use this to distinguish different combinatorial types. The realization spaces are spaces of partitions that share the same combinatorial type. These realization spaces also give us semialgebraic pieces that we glue together to obtain the whole space $\mathcal{C}(\mathbb{R}^d, n)$. We will be interested to find the dimensions of these realization spaces.

Inside the space $\mathcal{C}(\mathbb{R}^d, n)$ there are other spaces that catch our attention, as for example the subspace $\mathcal{C}_{\text{reg}}(\mathbb{R}^d, n)$ of regular partitions, which can be obtained by projecting the facets of a convex polyhedron one dimension higher. Regular partitions appear in different contexts and are much better understood than general partitions, since they are easier to generate and parameterize. We would like to know how the space of regular partitions is embedded in the space of all convex n-partitions.

We find that there is a big difference between the case $d = 2$ and the case when $d \geq 3$. For $d = 2$ and large n, the subspace $\mathcal{C}_{\text{reg}}(\mathbb{R}^2, n)$ of regular n-partitions has much smaller dimension than $\mathcal{C}(\mathbb{R}^2, n)$. On the other hand, for $d \geq 3$, a theorem by Whiteley [35] and Rybnikov [31] shows that simple n-partitions are regular (see Theorem 5.11). For $d = 3$, we conjecture that $\dim \mathcal{C}(\mathbb{R}^3, n) = \dim \mathcal{C}_{\text{reg}}(\mathbb{R}^3, n)$. However $\mathcal{C}_{\text{reg}}(\mathbb{R}^3, n)$ is not a dense subset in $\mathcal{C}(\mathbb{R}^3, n)$ for $n > 3$, and there are also non-simple combinatorial types whose realization spaces have the same dimension as $\mathcal{C}_{\text{reg}}(\mathbb{R}^3, n)$, where almost all partitions are non-regular. We give a heuristic count for the dimensions of the realization spaces that supports the conjecture, and on the way we derive a nice incidence theorem (Theorem 6.11) that is an example of the dependencies among the algebraic relationships that may arise.

In general, realization spaces of partitions of a given combinatorial type are expected to be complicated objects. We relate this to the work by Richter-Gebert [28] on realization spaces of polytopes, where the main result is the Universality Theorem, showing that realization spaces of d-dimensional polytopes for $d \geq 4$ can be "as complicated as possible" as semialgebraic sets. A similar result is established here for realization spaces of regular partitions (Theorem 5.17).

Another family of subspaces of $\mathcal{C}(\mathbb{R}^d, n)$ that is interesting for us is obtained as follows. Given a positive bounded measure μ in $\mathbb{R}^d$ we can look at the space $\mathcal{C}^{\text{equi}}(\mathbb{R}^d, n, \mu)$ of convex equipartitions, for which all regions have the same measure.

One of the motivations of this work come from different questions about the existence of convex partitions with special properties. For example, a question asked by Nandakumar and Ramana Rao [25] about the existence of partitions of a convex region into a prescribed number n of convex pieces with equal area and equal perimeter has led to many interesting results about equipartitions of measures (see for example [8], [22], and [6]). In particular, Blagojević & Ziegler [6] proved the Nandakumar & Ramana Rao conjecture for n equals a prime power. However, all results in the area use only rather restricted types of partitions, such as iterated hyperplane dissections or regular partitions, which are reasonably well understood — e.g. in the case of regular partitions they can be parameterized by configuration spaces of n distinct points, via Optimal Transport, as was first noticed by Karasev [22] [6]. Here we thus try to go one step further and set out to understand the structure of the larger space of *all* convex n-partitions. We also look at the subspace $\mathcal{C}^{\text{equi}}(\mathbb{R}^d, n, \mu)$ of convex equipartitions of a positive continuous measure μ and compare it with $\mathcal{C}^{\text{equi}}_{\text{reg}}(\mathbb{R}^d, n, \mu)$.

1.1 Overview of the thesis

We begin with some basic notions and results of convex geometry that we need, as polyhedra, cones, spherical polyhedra, hyperplane arrangements and CW-complexes. In Chapter 3 we introduce convex n-partitions and we prove that all the regions of a partition must be polyhedral. Then we define some related notions, such as spherical partitions and the face structure, and prove some basic facts about them.

In Chapter 4 we look at the space $\mathcal{C}(\mathbb{R}^d, n)$ of all convex n-partitions of $\mathbb{R}^d$, describing the metric structure there that fixes the topology of the space and also a natural compactification $\mathcal{C}(\mathbb{R}^d, \leq n)$ where empty regions are allowed. Then we prove that spaces of n-partitions are union of semialgebraic pieces in two different ways. We look at hyperplane arrangements carrying an n-partition, and give a description of $\mathcal{C}(\mathbb{R}^d, n)$ where the pieces depend on the hyperplanes used to obtain the partition (Theorem 4.14). For the second description we need to introduce nodes and node systems that are a generalization of the vertices, and define the combinatorial type of a partition. These combinatorial types give the semialgebraic pieces that build the spaces (See Theorem 4.47). At the end of the chapter we describe explicitly particular spaces of n-partitions of $\mathbb{R}^d$ and their compactifications for $n = 2$ and also for $d = 1$.

In Chapter 5 we talk about regular partitions and mention some known results about them. Using these results we compute the dimension of the space of regular partitions $\mathcal{C}_{\text{reg}}(\mathbb{R}^d, n)$. Then we prove a universality theorem that says that realization spaces of regular partitions can be stably equivalent to any primary basic semialgebraic set (Theorem 5.17).

In Chapter 6 we investigate the dimensions of realization spaces. We first study the case $d = 2$ and find that for large n the dimension of $\mathcal{C}(\mathbb{R}^2, n)$ is much bigger than $\dim(\mathcal{C}_{\text{reg}}(\mathbb{R}^2, n))$. Then we focus on the case $d = 3$, where we conjecture that the dimension of $\mathcal{C}(\mathbb{R}^3, n)$ is equal to the dimension of $\mathcal{C}_{\text{reg}}(\mathbb{R}^3, n)$ and try to justify this with a heuristic counting for the dimension of each realization space. From this counting we find an incidence theorem for 3-polytopes (Theorem 6.11) and find many examples of partitions where this counting works.

In Chapter 7 we introduce the spaces of equipartitions $\mathcal{C}^{\text{equi}}(\mathbb{R}^d, n, \mu)$ given a positive bounded measure μ. We explore the topological structure of some small cases of spaces of equipartitions and using this, we describe the spaces of n-partitions for $d = 2$ and $n = 3$. We also discuss the Nandakumar and Ramana Rao problem [25] and different equivariant maps that show that considering regular equipartitions is as good as considering all equipartitions with respect to the approach based on configuration spaces to find fair partitions. We end by listing some further questions that for now remain open.

1.2 Acknowledgements

First of all I wanted to thank Günter for all his patience and support throughout all these years, his always accurate advice and for offering me this nice topic and all the insight to make it become a complete thesis. Thanks to the Berlin Mathematical School and the ERC grant for their financial support and to Birgit Trogisch, Elke Pose and to all the people from the BMS one stop office for their constant and invaluable help with administrative issues. Also thanks to Thorsten Theobald for accepting being the second referee.

Many people helped with ideas and comments that ended up being part of the results here. Special thanks to Karim Adiprasito, Cesar Ceballos, Bernd Gonska, Benjamin Matschke, Alexander Bobenko, Moritz Firsching, Jean Philippe Labbé, Felipe Rincón, Katarina Jochemko, Kaie Kubjas, Federico Ardila and Bernardo Uribe. I also wanted to thank Peter McMullen, Bruno Benedetti, Axel Werner, Michael Joswig, Thilo Rörig, Bernd Schulze, Gil Kalai, Vincent Pilaud, Francisco Santos, Arnaud Padrol, Bernd Sturmfels, Raman Sanyal and Eran Nevo for their help on early stages of my research and to Eran Nevo and Satoshi Murai for solving a problem that got me busy for quite a while.

But the biggest thanks are for all the people who shared with me their time and smiles through all these years. To those who wanted to see this thesis done, being a motivation to keep going on. To my family and their support from the distance. To all friends and nice people I met along the way. Some of them where mentioned above already. I hope to express personally my gratitude to each of them soon.

Chapter 2

Basic concepts

We begin here by reviewing some basic definitions and results from discrete geometry, convex polytopes, cones and polyhedra. As a basic reference for these topics we refer to the books by Gruber [18] and Ziegler [36]. Spherical polyhedra, hyperplane arrangements and CW-complexes are also important for us and will be briefly introduced.

2.1 Convex sets and polyhedra

Given two points $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^d$, the straight line segment joining them is given by the set $[\boldsymbol{x}, \boldsymbol{y}] = \{\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} : \lambda \in \mathbb{R}, 0 \leq \lambda \leq 1\}$. A subset $K \subset \mathbb{R}^d$ is *convex* if for any two points $\boldsymbol{x}, \boldsymbol{y} \in K$, the segment $[\boldsymbol{x}, \boldsymbol{y}]$ is contained in K. Clearly, line segments are convex, as well as $\mathbb{R}^d$ and the empty set $\emptyset \subseteq \mathbb{R}^d$. Also, the intersection of convex sets is convex. Given any set $X \subseteq \mathbb{R}^d$, the *convex hull* of X is the smallest convex set that contains X. It is denoted as $\operatorname{conv}(X)$ and it is well-defined, since it can be obtained as the intersection of all convex sets K with $X \subseteq K$.

A *hyperplane* in $\mathbb{R}^d$ is given by the set of points that satisfy a linear equation of the form $\boldsymbol{a} \cdot \boldsymbol{x} = b$ for some vector $\boldsymbol{a} \in \mathbb{R}^d$ and $b \in \mathbb{R}$, where $\boldsymbol{a} \neq \boldsymbol{0}$ and $\boldsymbol{a} \cdot \boldsymbol{x}$ denotes the standard scalar product in $\mathbb{R}^d$. An open (closed) *halfspace* is determined by a linear inequality of the form $\boldsymbol{a} \cdot \boldsymbol{x} < b$ (respectively $\boldsymbol{a} \cdot \boldsymbol{x} \leq b$). The complement of a hyperplane in $\mathbb{R}^d$ decomposes into two open halfspaces.

A *polyhedron* in $\mathbb{R}^d$ is the solution set of finitely many linear inequalities, i.e. the intersection of finitely many closed halfspaces. Since each halfspace is convex and closed, we can conclude that all polyhedra are convex and closed. A *polytope* is a bounded polyhedron. An important result for polytopes is that they can also be expressed as the convex hull of a finite set of points (see [36, Theorem 1.1]).

An *affine subspace* of $\mathbb{R}^d$ is a translation of a linear subspace. The *affine hull* of a convex polyhedron $P \subseteq \mathbb{R}^d$ is the minimal affine subspace containing P. It is denoted as $\operatorname{aff} P$. The *dimension* $\dim P$ of a polyhedron P is given by the dimension of its affine hull, namely $\dim P = \dim(\operatorname{aff} P) \leq d$. For $P = \emptyset$, we define $\dim P = -1$. The *relative interior* of P are the points in the interior of P considered as a subset of $\operatorname{aff} P$. It is denoted by $\operatorname{relint}(P)$.

A *face* of a polyhedron P is given by the set of points $\boldsymbol{y} \in P$ that satisfy $\boldsymbol{a} \cdot \boldsymbol{y} = b$ for some $\boldsymbol{a} \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that $\boldsymbol{a} \cdot \boldsymbol{x} \leq b$ for all $\boldsymbol{x} \in P$. Since the faces of a polyhedron P are also polyhedra, they also have a well defined dimension. We refer to faces of P of dimension k as *k-faces* of P. In particular, 0-faces are also called *vertices* and 1-faces are called *edges*. The set of all faces of a polytope P ordered by inclusion is known as the *face lattice* of P, and it determines the combinatorial structure of the polytope.

2.2 Cones and pointed cones

The *cone* over a subset $X \subseteq \mathbb{R}^d$, denoted as $\operatorname{cone}(X)$, consists of all non-negative combinations of the vectors in X; more precisely

$$\operatorname{cone}(X) = \{\boldsymbol{x} = \lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_m \boldsymbol{x}_m : \boldsymbol{x}_i \in X, \lambda_i \in \mathbb{R}_{\geq 0} \text{ for all } i \leq m, m \in \mathbb{N}\}.$$

A cone $C \subseteq \mathbb{R}^d$ is generated by positive combinations of a finite set of vectors $C = \operatorname{cone}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)$ if and only if it is a finite intersection of closed linear halfspaces with defining hyperplanes through the origin. Such cones are called *polyhedral cones*. All cones we consider are convex and polyhedral, and therefore they have faces, vertices, edges, etc.

The *linearity space* $\operatorname{lineal}(C)$ of a cone C is the maximal linear subspace contained in it. The *recession cone* of a set $X \subseteq \mathbb{R}^d$ is the set of vectors $\boldsymbol{v} \in \mathbb{R}^d$ with the property that for any point $\boldsymbol{x} \in X$ and any real $\lambda \geq 0$ the point $\boldsymbol{x} + \lambda \boldsymbol{v}$ is in X. It is denoted by $\operatorname{rec}(X)$. If X is bounded, then $\operatorname{rec}(X) = \{\boldsymbol{0}\}$.

A cone C is *pointed* if $\operatorname{lineal}(C) = \{0\}$. Equivalently, a cone is pointed if it doesn't contain a straight line. Any polyhedron P can be written as the Minkowski sum of a polytope Q and a cone C, where C is the recession cone $\operatorname{rec}(P)$. Besides, any polyhedral cone C can be written as the Minkowski sum of its linearity space $\operatorname{lineal}(C)$ and a pointed cone C'.

Lemma 2.1. *Pointed polyhedral cones C can be obtained as* $\operatorname{cone}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)$*, where the $\boldsymbol{x}_i$ are non-zero vectors, one from each of the edges e_i of C.*

Proof. If C is a pointed cone, it is possible to find a hyperplane H such that $C \cap H = \{\mathbf{0}\}$ and all other points of C lie on the same open halfspace determined by H. Let H' be a translation of one such hyperplane H, such that H' intersects the relative interior of C. To see that the intersection $C \cap H'$ is a bounded polytope, suppose it is unbounded and notice that the closure of C will contain points in H. Then $C \cap H'$ is a polytope and can be expressed as the convex hull of its vertices. Since the vertices of $C \cap H'$ are in correspondence with the edges of C, and any point in C is a positive scaling of a point in $C \cap H'$, we conclude that if we take $\boldsymbol{x}_i$ as the vertices of $C \cap H'$ (or any positive scalar multiple of them) then $C = \text{cone}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$. □

2.3 Spherical convexity

A vector $\boldsymbol{v} = \begin{pmatrix} v_0 \\ \vdots \\ v_d \end{pmatrix} \in \mathbb{R}^{d+1}$ is on the sphere S^d if $\sum_{i=0}^{n} v_i^2 = 1$. For subsets of S^d, when we say that they are open or closed, this refers to the induced topology of S^d as a subset of $\mathbb{R}^{d+1}$.

An open (respectively closed) *hemisphere* of S^d is the intersection of S^d with any open (closed) halfspace whose bounding hyperplane goes through the origin in $\mathbb{R}^{d+1}$. A closed hemisphere is obtained by taking the closure of its respective open hemisphere.

Definition 2.2 (Convex and strictly convex subsets of S^d)**.** A *convex subset of* S^d is the intersection of S^d with a convex cone in $\mathbb{R}^{d+1}$. It is *strictly convex* if in addition it is contained in an open hemisphere of S^d.

The first notion of spherical convexity could be weaker than what one might expect. For example, two diametrically opposite points on the sphere form a non-connected set that is convex but not strictly convex. These definitions of convexity are similar to the ones in Horn [20]. Open convex subsets of the sphere are always strictly convex, with the exception of S^d itself. Any convex subset of S^d is an intersection of closed hemispheres and any strictly convex subset can be obtained as an intersections of open hemispheres (maybe infinitely many).

Definition 2.3 (Spherical polyhedron)**.** An open (resp. closed) *spherical polyhedron* is the intersection of finitely many open (closed) hemispheres in S^d. A *pointed spherical polyhedron* is the intersection of a pointed polyhedral cone in $\mathbb{R}^{d+1}$ with the sphere S^d.

Spherical polyhedra are convex subsets of S^d, that can be obtained as the intersection of a polyhedral cone with the sphere S^d. A closed spherical polyhedron is strictly convex if and only if it is pointed.

Definition 2.4 (Faces and boundary of spherical polyhedron)**.** The *faces* of a closed spherical polyhedron Q are obtained as the intersection of the faces of cone(Q) with S^d. The vertices of Q are its zero dimensional faces. The *boundary* of Q is obtained as the union of all faces of Q strictly contained in Q. The *relative interior* relint(Q) is the intersection of the relative interior of cone(Q) with S^d. Thus the boundary of Q is the complement of relint(Q) in Q.

Definition 2.5 (Spherical convex hull)**.** The *spherical convex hull* of a set of points $V \subseteq S^d$ is the intersection of the cone generated by V in $\mathbb{R}^{d+1}$ with S^d.

From Lemma 2.1 we can see that any closed pointed spherical polyhedron Q is the spherical convex hull of its vertices. (These vertices are obtained by intersecting the edges of cone(Q) with S^d.)

If a polyhedral cone is not pointed, it can still be obtained as a combination of a finite set of vectors, but this time there is no unique way to obtain a set of generating vectors (see Avis et al. [2], Schrijver [32] for discussions about canonical representations of polyhedra). Therefore, any spherical polyhedron Q is the spherical convex hull of a finite set of points V in S^d, but there is no canonical choice for the set V.

2.4 Hyperplane arrangements

A *hyperplane arrangement* $\mathcal{A}$ in $\mathbb{R}^d$ is a finite set of hyperplanes of $\mathbb{R}^d$. A hyperplane arrangement is *oriented* in case we fix a normal vector $\boldsymbol{a} \in \mathbb{R}^d$ orthogonal to each hyperplane $H \in \mathcal{A}$, so that H is given by the points $\boldsymbol{x}$ that satisfy an equation of the form $\boldsymbol{a} \cdot \boldsymbol{x} = b$. A hyperplane arrangement is called *central* if all hyperplanes in $\mathcal{A}$ go through the origin, i.e. they are determined by an equation of the form $\boldsymbol{a} \cdot \boldsymbol{x} = 0$.

A central hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^{d+1}$ defines an affine hyperplane arrangement $\mathcal{A}^{\mathit{aff}}$ in $\mathbb{R}^d$, where each of the hyperplanes H' of $\mathcal{A}^{\mathit{aff}}$ is obtained as the intersection of a hyperplane $H \in \mathcal{A}$ with the hyperplane H_1 of points $\boldsymbol{s} \in \mathbb{R}^{d+1}$ with first coordinate $\boldsymbol{x}_0 = 1$ (except if H is parallel H_1). Hyperplanes in the affine arrangement are not necessarily central. The hyperplane $H_1 \subseteq \mathbb{R}^{d+1}$ is often identified with $\mathbb{R}^d$.

2.5 CW complexes

A *regular CW complex* (a.k.a. regular cell complex) is a topological space constructed as the union of a collection of cells homeomorphic to closed balls such that the relative interiors of the cells are disjoint and the boundary of each k-cell is the union of finitely many cells of dimension smaller than k. See for example Munkres [23], Cooke & Finney [9], Björner [5]. All the CW complexes we consider are finite, thus compact.

Each regular CW complex is determined by its *face poset*, which has a minimal element $\hat{0}$ corresponding to the empty cell $\emptyset$. The *order complex* of a poset P is an abstract simplicial complex with vertex set given by the elements of P and with a simplex for each chain of the poset P, i.e. a subset of elements of the poset where every two elements can be compared. The order complex of the face poset of a regular CW complex $\mathcal{C}$ (without including the empty face) is homeomorphic to $\mathcal{C}$. It can be realized by taking the barycentric subdivision of $\mathcal{C}$. Therefore regular CW complexes are triangulable.

Chapter 3

Convex n-partitions

We begin here with the definition of the convex partitions of the space. Throughout this chapter we state some of its basic properties and define many of the related concepts that are useful to understand these objects better.

Definition 3.1 (Convex partitions of $\mathbb{R}^d$, regions, n-partitions)**.** Let n and d be two positive integers. A *convex partition of* $\mathbb{R}^d$ is an ordered list $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ of non-empty open convex subsets $P_i \subseteq \mathbb{R}^d$ that are pairwise disjoint, so that the union $\bigcup_{i=1}^{n} \overline{P_i}$ equals $\mathbb{R}^d$, where $\overline{P_i}$ denotes the closure of P_i. Each of the sets P_i is called a *region* of $\mathcal{P}$. Partitions into n convex regions are also called *n-partitions.*

Since all partitions we are dealing with here are convex, we will often omit this word. The regions of an n-partitions are labeled from 1 to n, where the order is important.

Definition 3.2 (Space of convex n-partitions)**.** The set of all convex n-partitions of $\mathbb{R}^d$ is denoted by $\mathcal{C}(\mathbb{R}^d, n)$.

In Chapter 4 we will analyze the structure of the space $\mathcal{C}(\mathbb{R}^d, n)$ as a whole, but in this chapter we focus only on the structure of a single n-partition.

3.1 Polyhedral structure

As a first observation, we prove that all regions of an n-partition are polyhedral. This means that each region can be described as the set of points satisfying a finite number of linear inequalities.

Proposition 3.3. *Let $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ be an n-partition of $\mathbb{R}^d$. Then each region P_i is the solution set of $n-1$ strict linear inequalities.*

Proof. By the Hahn–Banach Separation Theorem (see Rudin [30, Theorem 3.4]), for two disjoint non-empty open convex sets in $\mathbb{R}^d$ we can always find a hyperplane that separates them. Therefore, for each region P_i and for each $j \neq i$, we can find an affine hyperplane $H_{ij}^{aff} = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{a}_{ij} \cdot \boldsymbol{x} = b_{ij}\}$ for some $\boldsymbol{a}_{ij} \in \mathbb{R}^d$ and $b_{ij} \in \mathbb{R}$, such that if $\boldsymbol{x} \in P_i$ then $\boldsymbol{a}_{ij} \cdot \boldsymbol{x} < b_{ij}$ and if $\boldsymbol{x} \in P_j$ then $\boldsymbol{a}_{ij} \cdot \boldsymbol{x} > b_{ij}$.

All points in P_i satisfy the linear system of inequalities $\boldsymbol{a}_{ij} \cdot \boldsymbol{x} < b_{ij}$ for all $j \neq i$. On the other hand, let R be the set of points $\boldsymbol{x}$ satisfying these inequalities. Any $\boldsymbol{x}' \in \overline{P_j}$ is such that $\boldsymbol{a}_{ij} \cdot \boldsymbol{x}' \geq b_{ij}$ and therefore, if $\boldsymbol{x} \in R$ then $\boldsymbol{x} \notin \bigcup_{j \neq i} \overline{P_j}$. Since $\mathcal{P}$ is an n-partition, then $\boldsymbol{x}$ must belong to $\overline{P_i}$ and $R \subseteq \overline{P_i}$. Moreover, $R \subseteq P_i$ since the set R is open and then it is necessarily contained in the interior of $\overline{P_i}$. We conclude that $P_i = R$ is the set of solutions of $n-1$ strict linear inequalities. □

3.2 Spherical representation and partitions of S^d

We now introduce convex partitions of the unit d-sphere S^d. Even if we are only interested in partitions of $\mathbb{R}^d$, partitions of the sphere appear naturally and they are in a way more fundamental objects, as they generalize partitions of the euclidean space $\mathbb{R}^d$. Therefore, it is convenient to see the n-partitions of $\mathbb{R}^d$ as partitions of $S^d \subseteq \mathbb{R}^{d+1}$, by constructing what we call the spherical representation (see Definition 3.5). This point of view has the following advantages:

- Most of the definitions can be generalized to convex partitions of S^d. For many questions, the treatment is easier in the spherical case and needs only minor adjustments for $\mathbb{R}^d$.
- The faces of partitions of $\mathbb{R}^d$ will be defined in the spherical representation. In this way, the faces at infinity can be treated explicitly, and have mainly the same properties as all other faces. Also, this yields a regular cell complex structure on the set of faces (see Theorem 3.24).

The faces of an n-partition $\mathcal{P}$ ordered by inclusion will form the face poset of $\mathcal{P}$. We will see that it is the poset of a finite regular CW complex homeomorphic to a closed ball. (As usual, *poset* stands for *partially ordered set*, see e.g. Stanley [33, Chapter 3].)

Analogous to Definition 3.1, in the spherical case we have the following.

Definition 3.4 (Convex partitions of the sphere)**.** Let n and d be two positive integers. A *convex partition of* S^d is a list $\mathcal{Q} = (Q_1, Q_2, \ldots, Q_n)$ of non-empty open convex subsets $Q_i \subseteq S^d$ that are pairwise disjoint, so that the union $\bigcup_{i=1}^n \overline{Q_i}$ equals S^d.

A vector $\boldsymbol{v} \in S^d$ lies in the *upper hemisphere* S^d_+ if $\boldsymbol{v} \in S^d$ and its first coordinate is positive, $v_0 > 0$. Respectively, $\boldsymbol{v}$ is in the *lower hemisphere* S^d_- if $\boldsymbol{v} \in S^d$ and $v_0 < 0$. The *equator* S^d_0 of S^d is formed by all $\boldsymbol{v} \in S^d$ with $v_0 = 0$.

For a point $\boldsymbol{x} \in \mathbb{R}^d$ we construct the point

$$\widehat{\boldsymbol{x}} = \frac{1}{\sqrt{1+|x|^2}}\begin{pmatrix}1\\ \boldsymbol{x}\end{pmatrix} \in \mathbb{R}^{d+1},$$

that is, the intersection of the ray $r(\boldsymbol{x}) = \{\lambda\binom{1}{\boldsymbol{x}} \in \mathbb{R}^{d+1} : 0 \leq \lambda \in \mathbb{R}\}$ with S^d. The map $\boldsymbol{x} \mapsto \widehat{\boldsymbol{x}}$ gives a bijection between $\mathbb{R}^d$ and S^d_+.

Definition 3.5 (Spherical representation)**.** The *spherical representation* of an n-partition $\mathcal{P}$ of $\mathbb{R}^d$ is the spherical $(n+1)$-partition $\widehat{\mathcal{P}} = (\widehat{P}_1, \ldots, \widehat{P}_n, \widehat{P}_\infty)$ of S^d, with regions $\widehat{P}_i = \{\widehat{\boldsymbol{x}} : \boldsymbol{x} \in P_i\}$ for $i = 1, \ldots, n$ and an extra region $\widehat{P}_\infty = S^d_-$.

When we talk about a spherical representation $\widehat{\mathcal{P}}$ for $\mathcal{P} \in \mathcal{C}(\mathbb{R}^d, n)$, we denote by ∞ the subindex $n+1$.

Lemma 3.6. *The spherical representation $\widehat{\mathcal{P}}$ of an n-partition $\mathcal{P}$ of $\mathbb{R}^d$ is a convex partition of S^d with $n+1$ regions.*

Proof. Each region $\widehat{P}_i$ is the intersection of S^d with the open convex cone

$$\text{cone}\left(\{\tbinom{1}{\boldsymbol{x}} \in \mathbb{R}^{d+1} : \boldsymbol{x} \in P_i\}\right).$$

All these regions are therefore open and convex in S^d. Also the sets $\widehat{P}_i$ are pairwise disjoint, and their closures completely cover the upper hemisphere, since the P_i form an n-partition of $\mathbb{R}^d$. The face $\widehat{P}_\infty$ is also convex, disjoint from all other $\widehat{P}_i$, and such that the union of the closures of all regions is S^d. □

Example 3.7. Figure 3.1 shows an 4-partition $\mathcal{P}$ of $\mathbb{R}^2$ together with an upper view of its spherical representation $\widehat{\mathcal{P}}$, where we only depict the upper hemisphere S^2_+. The face $\widehat{P}_\infty$ corresponds to the side of the sphere hidden to us. This partition includes two parallel lines as the boundary of P_3 that in the spherical representation meet at two points "at infinity" (on the boundary of S^2_+).

3.3 Faces and the face poset

First we define what are the euclidean faces of a partition. Since we also want to study the behavior of n-partitions at infinity as part of the face structure, it will be

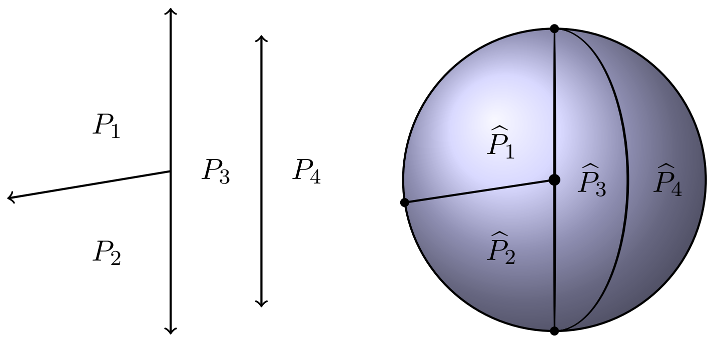

FIGURE 3.1: A 4-partition $\mathcal{P} \in \mathcal{C}(\mathbb{R}^2, 4)$ together with an upper view of its spherical representation.

convenient to look at the faces of their spherical representations. We will see later that euclidean faces of a partition are in correspondence with interior faces of the spherical representation.

Definition 3.8 (Euclidean faces of an n-partition)**.** Let $\mathcal{P} = (P_1, \dots, P_n)$ be an n-partition of $\mathbb{R}^d$. A *euclidean face* of $\mathcal{P}$ is a set of the form

$$E_{\boldsymbol{x}} \;=\; \bigcap_{i:\boldsymbol{x}\in\overline{P}_i} \overline{P_i} \;\subseteq\; \mathbb{R}^d$$

for a point $\boldsymbol{x}$ in $\mathbb{R}^d$.

Now we introduce the faces of spherical partitions.

Definition 3.9 (Index sets and faces of spherical partitions)**.** Let $\mathcal{Q} = (Q_1, \dots, Q_n)$ be a partition of S^d. Let $\overline{Q}_i$ be the closure of Q_i in S^d and $C_i = \operatorname{cone}(\overline{Q}_i)$ for $1 \le i \le n$. For any point $\boldsymbol{x}$ in $\mathbb{R}^{d+1}$, we define the *index set* $I(\boldsymbol{x})$ to be the set of values $i \in \{1, 2, \dots, n\}$ such that $\boldsymbol{x} \in C_i$. We define $\mathcal{I}(\mathcal{Q})$ to be the set of all index sets $I(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{R}^{d+1}$.

The *faces of a spherical partition* $\mathcal{Q}$ are all sets $F_I \subseteq S^d$ that can be obtained as an intersection of the form $F_I = \bigcap_{i\in I} \overline{Q_i}$ for some $I \in \mathcal{I}(\mathcal{Q})$. That is, for each $\boldsymbol{x} \in \mathbb{R}^{d+1}$ we obtain the spherical face

$$F_{I(\boldsymbol{x})} = \bigcap_{i\in I(\boldsymbol{x}))} \overline{Q_i} \subseteq S^d.$$

Lemma 3.10. *If* $I(\boldsymbol{x}) \subsetneq I(\boldsymbol{x}')$ *then* $F_{I(\boldsymbol{x}')} \subsetneq F_{I(\boldsymbol{x})}$.

Proof. The inclusion $F_{I(\boldsymbol{x}')} \subseteq F_{I(\boldsymbol{x})}$ is clear since the intersection $F_{I(\boldsymbol{x}')} = \bigcap_{i\in I(\boldsymbol{x}')} \overline{Q_i}$ includes all terms involved in computing $F_{I(\boldsymbol{x}')}$. Also if $I(\boldsymbol{x}) \ne I(\boldsymbol{x}')$ then $\boldsymbol{x} \notin F_{I(\boldsymbol{x}')}$, since there is at least one $i \in I(\boldsymbol{x}') - I(\boldsymbol{x})$ such that $\boldsymbol{x} \notin \overline{Q_i}$. Since $\boldsymbol{x} \in F_{I(\boldsymbol{x})}$ we get the strict inclusion $F_{I(\boldsymbol{x}')} \subsetneq F_{I(\boldsymbol{x})}$. $\square$

Definition 3.11 (Faces of partitions of $\mathbb{R}^d$, faces at infinity, interior faces, bounded faces)**.** The *faces of an n-partition* $\mathcal{P}$ of $\mathbb{R}^d$ are all the faces of the spherical representation $\widehat{\mathcal{P}}$, with the exception of $F_{\{\infty\}} = \overline{S^d_-}$. Faces $F_{I(\boldsymbol{x})}$ of $\mathcal{P}$ with $\infty \in I(\boldsymbol{x})$ are called *faces at infinity* of $\mathcal{P}$. All other faces are called *interior faces.* A face is *bounded* if it does not contain any face at infinity.

With this definition, faces of an n-partition $\mathcal{P}$ of $\mathbb{R}^d$ are *not* subsets of $\mathbb{R}^d$, but they are contained in the closure of S^d_+. Faces at infinity are precisely the faces of $\mathcal{P}$ contained in the boundary of S^d_+, which is the equator S^d_0, while interior faces are in bijection with the euclidean faces $E_{\boldsymbol{x}}$ introduced before (which are subsets of $\mathbb{R}^d$), where $F_{I(\widehat{\boldsymbol{x}})}$ is the closure of $\widehat{E_{\boldsymbol{x}}}$ for $\boldsymbol{x} \in \mathbb{R}^d$.

For a convex n-partition $\mathcal{P}$ we set $\mathcal{I}(\mathcal{P}) = \mathcal{I}(\widehat{\mathcal{P}}) \setminus \big\{\{\infty\}\big\}$ to be the set of indices of faces of $\mathcal{P}$.

Each n-partition has only finitely many faces, since all $I(\boldsymbol{x})$ are contained in the set $I(\mathbf{0}) = \{1, \ldots, n, \infty\}$, where $\mathbf{0}$ represents the origin in $\mathbb{R}^{d+1}$. The union of all faces of $\mathcal{P}$ will be precisely S^d_+, since any point $\boldsymbol{x} \in S^d_+$ is contained in $F_{I(\boldsymbol{x})}$.

Definition 3.12 (Face poset)**.** The *face poset* of an n-partition $\mathcal{P}$ is the set of all faces of $\mathcal{P}$, partially ordered by inclusion. It is denoted as $\mathcal{F}(\mathcal{P})$.

The face poset of a partition $\mathcal{P}$ is isomorphic as a poset to $\mathcal{I}(\mathcal{P})$, ordered by reversed inclusion (see Lemma 3.10). For that reason we say that two partitions $\mathcal{P}$ and $\mathcal{P}'$ have the same face poset if $\mathcal{I}(\mathcal{P}) = \mathcal{I}(\mathcal{P}')$. This have the advantage that a canonical label is given to each face.

Example 3.13. In Figure 3.2 we show the face poset of the partition $\mathcal{P}$ on Example 3.7. Here we denote by F_{123} the face $F_{\{1,2,3\}}$, and similarly for other sets of indices. Notice that $F_{I(\mathbf{0})} = F_{1234\infty} = \emptyset$. To obtain the face poset of $\widehat{\mathcal{P}}$ we have to add the face F_∞ as another maximal face above all faces at infinity (appearing with dotted lines in the figure).

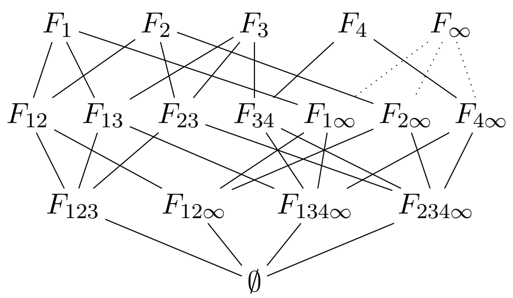

FIGURE 3.2: Face poset of the partition $\mathcal{P}$ on Example 3.7.

Let us provide a little overview of the most relevant notation we have introduced so far related with an n-partition $\mathcal{P}$:

- $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ denotes an n-partition of $\mathbb{R}^d$, $\mathcal{P} \in \mathcal{C}(\mathbb{R}^d, n)$.
- $\widehat{\mathcal{P}} = (\widehat{P}_1, \ldots, \widehat{P}_n, \widehat{P}_\infty)$ is the spherical representation of $\mathcal{P}$, a partition of S^d into $n+1$ regions.
- $\mathcal{I}(\mathcal{P}) = \mathcal{I}(\widehat{\mathcal{P}}) \setminus \big\{\{\infty\}\big\}$ is the set of indices of faces of $\mathcal{P}$.
- $F_I \subset S^d$ are the faces of $\mathcal{P}$, for $I = I(\boldsymbol{x}) \in \mathcal{I}(\mathcal{P})$ and $\boldsymbol{x} \in \mathbb{R}^{d+1}$.
- $C_I = \operatorname{cone}(F_I)$ are the corresponding cones in $\mathbb{R}^{d+1}$.
- $E_{\boldsymbol{x}} \subseteq \mathbb{R}^d$ denote the euclidean faces of $\mathcal{P}$, for $\boldsymbol{x} \in \mathbb{R}^d$. Also we might denote them as E_I where $I = I(\boldsymbol{x}) \in \mathcal{I}(\mathcal{P})$.

Sometimes we denote a face F_I as $F_I(\mathcal{P})$ to specify the partition it belongs to, for $I \in \mathcal{I}(\mathcal{P})$.

Faces of $\mathcal{P}$ of dimension k are also known as k-faces. The 0-faces of $\mathcal{P}$ are called the *vertices* and the 1-faces are called *edges*, but only in case they are contractible. See Example 3.26 for an example of a partition with a 0-face that is not vertex. We introduce now a subset of partitions where those strange things never happen.

Definition 3.14 (Essential partitions)**.** An n-partition $\mathcal{P}$ is *essential* if $F_{I(\mathbf{0})}(\mathcal{P}) = \emptyset$.

Since all $I \in \mathcal{I}(\mathcal{P})$ are contained in $I(\mathbf{0}) = \{1, \ldots, n, \infty\}$, the face $F_{I(\mathbf{0})}$ is the minimal face of the partition. From here it is easy to conclude the following:

Lemma 3.15. *An n-partition $\mathcal{P}$ is essential if and only if it has a bounded face. Also, it is essential if and only if it has an interior vertex.*

Proof. If a partition is not essential, all faces must contain $F_{I(\mathbf{0})}$, which is a non-empty face at infinity. Therefore there are no bounded faces. On the other hand, if there is a bounded face F, this implies that the partition is essential, since $F_{I(\mathbf{0})}$ is the intersection of the closures of all regions $\widehat{P}_i$ of $\widehat{\mathcal{P}}$ and it will be contained in the intersection of F and the closure of $\widehat{P}_\infty$, that is empty for a bounded face F. For the second part, notice that an interior vertex is a bounded face. Also any bounded face will have at least one vertex, since it is a pointed spherical polyhedron (see also Lemma 2.1). □

Definition 3.16 (Subfaces)**.** The *subfaces* of a face F_I of an n-partition $\mathcal{P}$ are the faces of F_I considered as convex spherical polyhedra, i. e. the faces of the cones C_I intersected with S^d for $I \in \mathcal{I}(\mathcal{P})$. Subfaces of a face of $\mathcal{P}$ are also called subfaces of $\mathcal{P}$.

Subfaces of dimension k are denoted as k-subfaces. We will see that each subface is a union of faces (Lemma 3.23). We might also talk about the subfaces of an euclidean face E of $\mathcal{P}$, that are equivalently defined to be the faces of E as a convex polyhedron.

Example 3.17. For the partition in Figure 3.1, we see that the region $\widehat{P}_3$ is bounded by two subfaces of dimension one. One of these subfaces is the face F_{34} of the partition, while the other one is the union of the faces F_{13} and F_{23}.

3.4 Basic lemmas about faces

Here we will prove some basic facts about faces that will be useful later, in particular to prove Theorem 3.24. First we need some definitions that make precise when two regions of an n-partition are adjacent.

Definition 3.18 (Adjacent regions of an n-partition)**.** Let $\mathcal{P}$ be an n-partition of $\mathbb{R}^d$ and $\widehat{\mathcal{P}}$ its spherical representation. Two regions $\widehat{P}_i$ and $\widehat{P}_j$ of $\widehat{\mathcal{P}}$ are *adjacent* if the intersection $F_{ij} = F_i \cap F_j$ is a $(d-1)$-face of $\mathcal{P}$.

In this case, there is a unique hyperplane H_{ij} that separates the regions $\widehat{P}_i$ and $\widehat{P}_j$. The equation for the points $\boldsymbol{x} \in \mathbb{R}^{d+1}$ in H_{ij} can be written as $\boldsymbol{c}_{ij} \cdot \boldsymbol{x} = 0$, where $\boldsymbol{c}_{ij} = (-b_{ij}, a_{ij_1}, \dots, a_{ij_d}) \in \mathbb{R}^{d+1}$, with $\boldsymbol{a}_{ij}$ and b_{ij} as defined in the proof of Proposition 3.3.

Definition 3.19 (Adjacency graph of an n-partition)**.** The *adjacency graph* $A(\mathcal{P})$ of an n-partition $\mathcal{P}$ of $\mathbb{R}^d$ is a simple graph with vertex set $\{1, \dots, n, \infty\}$ and edges $\{i, j\}$ for each pair of adjacent regions $\widehat{P}_i$ and $\widehat{P}_j$. In this way the set of $(d-1)$-faces of $\mathcal{P}$ is in bijection with the edges of $A(\mathcal{P})$, which also includes the pairs of the form $\{i, \infty\}$ corresponding to $(d-1)$-faces at infinity.

Lemma 3.20. *For any face F_I, the cone $C_I = \operatorname{cone}(F_I)$ is described by the intersection of the hyperplanes $H_{ij} = \{\boldsymbol{x} \in \mathbb{R}^{d+1} : \boldsymbol{c}_{ij} \cdot \boldsymbol{x} = 0\}$ for $i, j \in I$ corresponding to adjacent regions and the halfspaces of points $\boldsymbol{x}$ satisfying $\boldsymbol{c}_{ik} \cdot \boldsymbol{x} \leq 0$ for each pair $\{i, k\} \in A(\mathcal{P})$ where $i \in I$ and $k \notin I$. That is,*

$$C_I = \Bigg(\bigcap_{\substack{i,j \in I \\ \{i,j\} \in A(\mathcal{P})}} H_{ij} \Bigg) \cap \Bigg(\bigcap_{\substack{i \in I,\, k \notin I \\ \{i,k\} \in A(\mathcal{P})}} \{\boldsymbol{x} \in \mathbb{R}^{d+1} : \boldsymbol{c}_{ik} \cdot \boldsymbol{x} \leq 0\} \Bigg).$$

Also, the affine span of C_I is given by $\operatorname{aff} C_I = \bigcap_{\{i,j\}} H_{ij}$, where the intersection is taken over all pairs $\{i, j\} \in A(\mathcal{P})$ such that $i, j \in I$.

Proof. Each cone C_i can be described by the inequalities $\boldsymbol{c}_{ij} \cdot \boldsymbol{x} \leq 0$, for i, j adjacent. If $i, j \in I$, then the corresponding inequalities imply that $F_I \subseteq H_{ij}$. All other inequalities involve a $k \notin I$. If $\boldsymbol{x} \in F_I$ then the point $\boldsymbol{x}$ will satisfy all inequalities of the form $\boldsymbol{c}_{ik} \cdot \boldsymbol{x} < 0$ strictly, for $\{i, k\} \in A(\mathcal{P})$ with $i \in I$ and $k \notin I$. Therefore the only linear equations satisfied by all points in C_I are of the form $\boldsymbol{c}_{ij} \cdot \boldsymbol{x} = 0$ for $i, j \in I$ adjacent, and they determine the affine span of C_I. □

Lemma 3.21. *The relative interiors of the faces of $\mathcal{P}$ are pairwise disjoint.*

Proof. Let F_I be a face of $\mathcal{P}$ and $\boldsymbol{x} \in \operatorname{relint} F_I$. Then $\boldsymbol{x} \in \operatorname{relint} C_I$ and it also has to satisfy all inequalities $\boldsymbol{c}_{ik} \cdot \boldsymbol{x} < 0$ involving an index $k \notin I(\boldsymbol{x})$ strictly, otherwise it will belong to the boundary of F_I. We can see then that $I(\boldsymbol{x}') = I$ since $\boldsymbol{x}$ belongs to all F_i for $i \in I$ and it won't be in any F_k for $k \notin I(\boldsymbol{x})$. Therefore every point $x \in \overline{S^d_+}$ can be in at most one relative interior of a face and we conclude that the relative interiors are pairwise disjoint. □

From here we see that for all $\boldsymbol{x} \in S^d$ we have $\operatorname{relint}(F_{I(\boldsymbol{x})}) = \{\boldsymbol{x}' : I(x) = I(x')\}$.

Lemma 3.22. *The face $F_{I(\mathbf{0})}$ is the only face F_I such that its corresponding cone C_I is a linear subspace of $\mathbb{R}^{d+1}$.*

Proof. First, $C_{I(\mathbf{0})}$ is a linear subspace of $\mathbb{R}^{d+1}$, since by Lemma 3.20 the cone $C_{I(\mathbf{0})}$ is given by

$$C_{I(\mathbf{0})} = \bigcap_{\substack{i,j \in I(\mathbf{0}) \\ \{i,j\} \in A(\mathcal{P})}} H_{ij} = \operatorname{aff} F_{I(\mathbf{0})},$$

and no inequality is involved. No other face of $\mathcal{P}$ can be a linear subspace. Otherwise its cone will contain $C_{I(\mathbf{0})}$ in its relative interior. But we can extend the result of Lemma 3.21 to see that also all cones C_I have disjoint relative interiors, and it is not possible for another cone C_I to contain $C_{I(\mathbf{0})}$ if it is a linear subspace, since the relative interior of a linear subspace is the whole linear subspace itself. If $F_{I(\mathbf{0})}$ is empty, then its corresponding cone is only the origin (the vector $\mathbf{0} \in \mathbb{R}^{d+1}$), and it is contained in all other cones. □

Lemma 3.23. *Let $\mathcal{P}$ be an n-partition of $\mathbb{R}^d$ and F_I a face of $\mathcal{P}$.*

(i) *The boundary of F_I is equal to the union of all faces F_J of $\mathcal{P}$ with $I \subsetneq J \subseteq I(\mathbf{0})$.*

(ii) *All subfaces of F_I are unions of some faces of $\mathcal{P}$.*

(iii) *The vertices of a face F_I (as 0-dimensional subfaces) are also vertices of $\mathcal{P}$.*

Proof. If F_I is the minimal face $F_{I(\mathbf{0})}$, then it has no boundary, no subfaces and there is nothing to prove, so we assume that F_I is different from $F_{I(\mathbf{0})}$.

(i) If $I \subsetneq I(\mathbf{0})$, then clearly $F_{I(\mathbf{0})} \subsetneq F_I$. Moreover, if $I \subsetneq J \subseteq I(\mathbf{0})$, then the relative interior of F_J must be on the boundary of F_I since the relative interiors of different faces are disjoint. (J must include at least one index $k \notin I$ that makes F_J be on the boundary.) Since all points $\boldsymbol{x}'$ in the boundary of F_I have $I \subsetneq I(\boldsymbol{x}')$, the result follows.

(ii) We know that the boundary of F_I is the union of faces of $\mathcal{P}$ contained in F_I. Each of the $(k-1)$-faces of that union are contained in a $(k-1)$-subface of F_I, since the affine span of such $(k-1)$-face is a k-dimensional linear subspace defining a facet of C_I. The union of all $(k-1)$-faces contained in a $(k-1)$-subface G will be precisely G.

Now if G is any other subface of F_I of dimension $k' < k-1$, we proceed inductively as follows. Assume that all subfaces of F_I of dimension greater than k' are unions of faces of $\mathcal{P}$. Then G is a subface of a subface G' of F_I of dimension $k'+1$ and G' can be written as a union of faces of $\mathcal{P}$. Each of those faces will have a subface contained in G, and the union of those subfaces will cover G. It is enough to take those faces such that the subface in G has dimension k'. We repeat the argument as before to see that subfaces of dimension k' of a face of dimension $k'+1$ can be expressed as unions of faces of $\mathcal{P}$ to conclude that any subface is a union of faces.

(iii) Since subfaces of a face F_I are unions of faces of $\mathcal{P}$, in the case of a single vertex, the only possibility to obtain it as a union of faces is that the point itself is a vertex of $\mathcal{P}$.

□

3.5 CW complex structure

Theorem 3.24. *If $\mathcal{P}$ is an essential n-partition of $\mathbb{R}^d$, then the faces of $\mathcal{P}$ form a regular CW complex homeomorphic to $\overline{S^d_+}$.*

Proof. From Lemma 3.21 we know that the relative interior of all faces are disjoint. Also the union $\bigcup_{I\in\mathcal{I}(\mathcal{P})} F_I$ is the closure of the upper hemisphere $\overline{S^d_+}$, since $\boldsymbol{x} \in F_{I(\boldsymbol{x})}$ for any point $\boldsymbol{x} \in \overline{S^d_+}$. Also from Lemma 3.23 we know that the boundary of each face is homeomorphic to the union of some other faces of smaller dimension. Therefore the only thing that we need to check is that the faces are k-cells (homeomorphic to closed

k-balls). The problem is that from our definition of convexity we cannot conclude that the faces are balls. Here is where we need to use that the partition is essential.

If a closed convex cone C in $\mathbb{R}^{d+1}$ is not a linear subspace, then there must be a vector $\boldsymbol{x} \in C$ such that $-\boldsymbol{x}$ is not in C (otherwise we will always have all linear combinations in the convex hull). We can use this vector to contract the cone C to the ray cone($\boldsymbol{x}$), by constructing a family of norm preserving maps taking any point $\boldsymbol{y} \in C$ to cone($\boldsymbol{x}$). In this way, the intersection of C with the sphere S^d will be contracted to a point.

We know from Lemma 3.22 that the only face F_I such that its corresponding cone C_I is a linear subspace is $F_{I(\mathbf{0})}$. If $C_{I(\mathbf{0})}$ is a k-dimensional linear subspace, then its intersection $F_{I(\mathbf{0})}$ is a $(k-1)$-sphere in S^d, for $k = 1, \dots, d+1$, and then it is not contractible. The only possibility is that $C_{I(\mathbf{0})} = \mathbf{0}$ is of dimension $k = 0$, and then $F_{I(\mathbf{0})} = \emptyset$ and the partition is essential. In that case all k-faces are contractible and homeomorphic to k-balls for $k = 0, \dots, d$ and form a CW complex homeomorphic to a closed d-ball. □

If the partition is not essential, we have two options to see it as a CW complex. One possibility is to see it as a partition in a lower dimension, and the other is to refine the partition.

Proposition 3.25. *The order complex of the face poset of an n-partition $\mathcal{P}$ is homeomorphic to a ball of dimension $d-k$, where $k = \dim F_{I(\mathbf{0})}$.*

Proof. The face $F_{I(\mathbf{0})}$ is the intersection of a linear subspace L of $\mathbb{R}^{d+1}$ with S^d. (This holds even if the face is empty.) We can intersect all faces of the partition with the orthogonal subspace $L^{\perp}$ to get a new partition $\mathcal{P}^{\text{ess}}$. If $\dim L = k$, then $\mathcal{P}^{\text{ess}}$ is equivalent to an n-partition of $\mathbb{R}^{d-k}$.

The new partition $\mathcal{P}^{\text{ess}}$ has the same face poset as $\mathcal{P}$, because the cones of all faces of $\mathcal{P}$ contain L and will be intersected by $L^{\perp}$. This preserves all containment relationships and each cone of a face in $\mathcal{P}$ can be reconstructed by taking the corresponding cone in $\mathcal{P}^{\text{ess}}$ and the Minkowski sum with L. For the face $F_{I(\mathbf{0})} = L$ the intersection with $L^{\perp}$ will only be the origin $\mathbf{0}$ and $F_{I(\mathbf{0})}(\mathcal{P}^{\text{ess}})$ will be empty, and therefore $\mathcal{P}^{\text{ess}}$ will be essential. From Theorem 3.24 we conclude that the face poset of $\mathcal{P}$ is the poset of a cell complex homeomorphic to a $(d-k)$-ball. □

Example 3.26. In Figure 3.3 we show the spherical representation of a non-essential partition $\mathcal{P}$ of the plane in four regions separated by three parallel lines. Again, the side of the sphere hidden to us corresponds to $\widehat{P}_{\infty}$. In this case $F_{I(\mathbf{0})}$ is non-empty and consists of two antipodal points at the boundary of S^2_+. The cone $L = C_{I(\mathbf{0})}$ is a straight line (a linear subspace of $\mathbb{R}^3$).

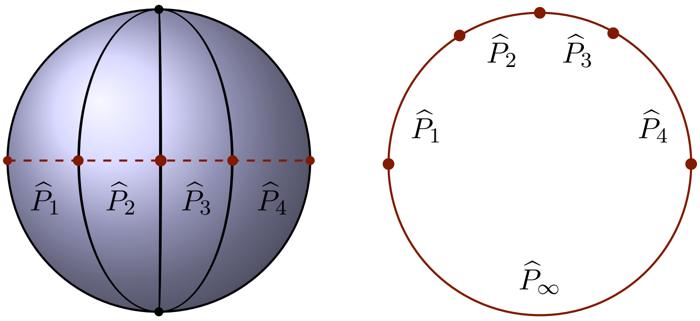

FIGURE 3.3: Non-essential 4-partition of the plane and its essentialization on S_1.

The orthogonal space $L^\perp$ is a plane that intersects the sphere in a great circle, generating in this way the essentialization $\mathcal{P}^{\text{ess}}$. This intersection is the spherical representation of a 4-partition of $\mathbb{R}^1$ (depicted at the right of Figure 3.3).

Instead of reducing the dimension, we can also make a refinement of any non-essential n-partition to get at the end a CW complex homeomorphic to a d-ball. One way to construct such refinement will be presented in Definition 4.31, where node systems are introduced.

Chapter 4

Spaces of n-partitions

Understanding the space $\mathcal{C}(\mathbb{R}^d, n)$ of all convex n-partitions of $\mathbb{R}^d$ is the main goal of this work, and in this chapter we investigate its basic structure. First we define a metric on $\mathcal{C}(\mathbb{R}^d, n)$, which gives rise to a topological structure, and introduce a natural compactification, the space $\mathcal{C}(\mathbb{R}^d, \leq n)$. Both spaces can be obtained by gluing pieces that can be described as semialgebraic subsets in some real vector space. We offer two approaches to obtain this result. The first one is based on the hyperplane description of the partitions, where the pieces are in bijection with the adjacency graph of the partition $A(\mathcal{P})$. The second one is based on the node description of partitions (where nodes generalize the notion of vertices) and there we define the combinatorial types of n-partitions and its corresponding realization spaces.

4.1 Metric structure, topology and compactification

To understand the space $\mathcal{C}(\mathbb{R}^d, n)$ we want to define a metric on it. For non-empty compact convex sets there are two standard ways to measure the distance between them. The *Hausdorff distance* between two compact convex sets A, $B \subset \mathbb{R}^d$ is defined as

$$\delta(A, B) = \max\big(\max_{a\in A}\min_{b\in B}|a-b|, \max_{b\in B}\min_{a\in A}|a-b|\big) \tag{4.1}$$

and the *symmetric difference distance*

$$\theta(A, B) = \mathrm{vol}_d(A \triangle B), \tag{4.2}$$

where $A \triangle B$ denotes the symmetric difference of sets A and B. Both of these metrics induce the same topology (see [19]). Unfortunately we get some troubles when we try to apply this to unbounded regions, since in that case the distances would be typically infinite. To remedy this, instead of the usual volume vol_d over $\mathbb{R}^d$ we need to take a continuous measure μ that is bounded, i.e. $\mu(\mathbb{R}^d) < \infty$. A measure is *positive* if it is supported on the whole space $\mathbb{R}^d$. Throughout our discussion the measures we consider are positive, continuous and bounded.

One possible choice for the measure is the standard d-volume $\mu(P) = \mathrm{vol}_d(\widehat{P})$ of the projection to the sphere for any measurable set $P \subseteq \mathbb{R}^d$; this volume is bounded by $\mathrm{vol}_d(S^d_+) = \frac{1}{2}\,\mathrm{vol}_d(S^d)$. With this measure μ, we can fix a metric on $\mathcal{C}(\mathbb{R}^d, n)$ as follows.

Definition 4.1. Given two n-partitions $\mathcal{P} = (P_1, \dots, P_n)$ and $\mathcal{P}' = (P'_1, \dots, P'_n)$ of $\mathbb{R}^d$, the distance $d_\mu(\mathcal{P}, \mathcal{P}')$ between them is the sum of the measures of the symmetric differences of the corresponding regions, that is,

$$d_\mu(\mathcal{P}, \mathcal{P}') = \sum_{i=1}^{n} \mu(P_i \triangle P'_i).$$

This distance d_μ is a metric and endows $\mathcal{C}(\mathbb{R}^d, n)$ with a topological structure that is relevant for our study.

There is a natural compactification for the space $\mathcal{C}(\mathbb{R}^d, n)$ that is obtained by considering partitions where now it is allowed to have empty regions.

Definition 4.2 (Non-proper and proper n-partitions)**.** Let n and d be two positive integers. A *non-proper n-partition of* $\mathbb{R}^d$ is a list $\mathcal{P} = (P_1, P_2, \dots, P_n)$ of n open convex subsets $P_i \subseteq \mathbb{R}^d$ that are pairwise disjoint, so that the union $\bigcup_{i=1}^n \overline{P_i}$ equals $\mathbb{R}^d$, where now the P_i are allowed to be empty and at least one of the P_i is empty. The convex n-partitions as introduced in Definition 3.1 are called *proper* in this context. We denote by $\mathcal{C}(\mathbb{R}^d, \leq n)$ the set of all proper or non-proper n-partitions.

In other words, the elements of $\mathcal{C}(\mathbb{R}^d, \leq n)$ are n-partitions where empty regions are allowed, so that $\mathcal{C}(\mathbb{R}^d, n)$ is the subset of proper partitions in $\mathcal{C}(\mathbb{R}^d, \leq n)$ and the non-proper n-partitions are all other elements of $\mathcal{C}(\mathbb{R}^d, \leq n)$ that contain at least one empty region. Non-proper partitions can also be seen as a k-partition with $k < n$, whose regions have distinct labels in the range from 1 to n, while labels that are not used correspond to empty regions.

Most of the results and definitions we have introduced up to now can be extended to non-proper partitions. The distance d_μ can be extended to $\mathcal{C}(\mathbb{R}^d, \leq n)$, so that it is also a metric and topological space. Non-proper partitions also have polyhedral regions as claimed by Theorem 3.3 (now possibly empty). We can also talk about non-proper partitions of a d-sphere, spherical representation of non-proper partitions and face structure, where now the labels of the faces $I(\boldsymbol{x})$ are contained in $I(\mathbf{0}) = \{i : C_i \neq \emptyset\}$, the set of labels of all non-empty regions. For a region $P_i = \emptyset$, we define C_i to be empty as well, so that we don't get new faces by adding extra empty regions. As before, a non-proper n-partition is *essential* if $F_{I(\mathbf{0})} = \emptyset$.

Theorem 4.3. *The space $\mathcal{C}(\mathbb{R}^d, \leq n)$ is compact.*

Proof. To prove this, we will introduce couple of spaces that will also be important in the next section when we talk about the semialgebraic structure of the spaces of partitions.

First consider the space $(S^d)^{\binom{n}{2}}$, which is a compact subset of $\mathbb{R}^{(d+1)\times\binom{n}{2}}$. Each of the points $\boldsymbol{c} \in (S^d)^{\binom{n}{2}}$ is represented by $\binom{n}{2}$ unit vectors $\boldsymbol{c}_{ij} \in S^d$ for $1 \leq i < j \leq n$. Each point $\boldsymbol{c}$ can be identified with a central hyperplane arrangement $\mathcal{A}_{\boldsymbol{c}}$ in $\mathbb{R}^{d+1}$ that is oriented, with $\binom{n}{2}$ hyperplanes H_{ij}, one for each pair i, j with $1 \leq i < j \leq n$. Each hyperplane $H_{ij} \in \mathcal{A}_{\boldsymbol{c}}$ is given by the linear equation $\boldsymbol{c}_{ij} \cdot \boldsymbol{x} = 0$ and comes with an orientation given by the vector $\boldsymbol{c}_{ij} \in \mathbb{R}^{d+1}$. To keep the symmetry of the notation, H_{ji} denote the same hyperplane H_{ij} with the opposite orientation, with corresponding normal vector $\boldsymbol{c}_{ji} = -\boldsymbol{c}_{ij}$. The following space will contain the space $\mathcal{C}(\mathbb{R}^d, \leq n)$.

Definition 4.4 (Space of n disjoint open polyhedra)**.** Let $\mathcal{D}(\mathbb{R}^d, \leq n)$ be the set of n labeled, disjoint, possibly-empty, open polyhedral subsets $(Q_1, \dots, Q_n)$ of $\mathbb{R}^d$.

We fix the topological structure of $\mathcal{D}(\mathbb{R}^d, \leq n)$ in the same way as we did for $\mathcal{C}(\mathbb{R}^d, \leq n)$ using the metric structure from Definition 4.1. For this, we take a metric in $\mathcal{D}(\mathbb{R}^d, \leq n)$ where the distance of two lists $(Q_1, \dots, Q_n)$ and $(Q'_1, \dots, Q'_n)$ in $\mathcal{D}(\mathbb{R}^d, \leq n)$ is given by

$$\sum_{i=1}^{n} \mathrm{vol}_d(\widehat{Q}_i \triangle \widehat{Q}'_i),$$

that is, the sum of the measure of the symmetric differences of the projections to S^d of the pairs of corresponding polyhedra in both lists. In this way $\mathcal{C}(\mathbb{R}^d, \leq n)$ is a subspace of $\mathcal{D}(\mathbb{R}^d, \leq n)$, with the corresponding subspace topology.

Equivalently, $\mathcal{D}(\mathbb{R}^d, \leq n)$ can be considered the space of n labeled, disjoint, possibly-empty, open spherical polyhedral subsets of S^d_+ (the upper hemisphere), since we can map each polyhedron $Q_i \subseteq \mathbb{R}^d$ to the spherical polyhedron $\widehat{Q}_i$. The space of partitions $\mathcal{C}(\mathbb{R}^d, \leq n)$ can be considered as the subspace of lists $(Q_1, \dots, Q_n) \in \mathcal{D}(\mathbb{R}^d, \leq n)$ where the union of the closure of the Q_i is the whole $\mathbb{R}^d$.

We can define a map $\pi : (S^d)^{\binom{n}{2}} \to \mathcal{D}(\mathbb{R}^d, \leq n)$ obtained by taking for each $\boldsymbol{c} \in (S^d)^{\binom{n}{2}}$ the polyhedra

$$Q_i = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{c}_{ij} \cdot \binom{1}{\boldsymbol{x}} < 0 \text{ for } 1 \leq j \leq n, j \neq i\},$$

for $1 \leq i \leq n$ where $\binom{1}{\boldsymbol{x}} \in \mathbb{R}^{d+1}$ is the vector obtained by adding to $\boldsymbol{x}$ a first coordinate equal to 1. In other words, each Q_i is determined by intersecting the halfspaces $\boldsymbol{c}_{ij} \cdot \binom{1}{\boldsymbol{x}} < 0$ determined by the affine hyperplanes $H_{ij} \in \mathcal{A}$ in $\mathbb{R}^d$, where the orientation of the $\boldsymbol{c}_{ij}$ indicates the side of H_{ij} that must be taken. We recall the convention that $\boldsymbol{c}_{ji} = -\boldsymbol{c}_{ij}$, which implies that all Q_i are disjoint. The polyhedral sets Q_i might be empty.

Lemma 4.5. *The map* $\pi : (S^d)^{\binom{n}{2}} \to \mathcal{D}(\mathbb{R}^d, \leq n)$ *is continuous.*

Proof. This is because if we move the hyperplanes a small amount, the polyhedra projected to the sphere also change slightly and the sum of the d-volume of the symmetric differences must be small. □

Now we can complete the proof of Theorem 4.3. Since the space $(S^d)^{\binom{n}{2}}$ is compact, the image of the continuous map π is also a compact space (see e.g. [24, Theorem 26.5]). On this image we have a continuous function f to $\mathbb{R}$, given by $f(Q_1, \ldots, Q_n) = \sum_{i=1}^{n} \text{vol}_d(\widehat{Q}_i)$.

This is a continuous function, so the preimage of the maximal value, namely the d-volume of S^d_+, is a closed subset of a compact space, so it is compact as well. This preimage is denoted by $\mathcal{H}(R^d, \leq n)$ (as explained later in Definition 4.12). We conclude that $\mathcal{C}(R^d, \leq n)$ is compact, as it is the image under π of $\mathcal{H}(R^d, \leq n)$ that is compact (again by [24, Theorem 26.5]). □

We cannot claim that $\mathcal{C}(\mathbb{R}^d, n)$ is also compact, since the limit of a sequence of proper partitions might have empty regions. On the other hand, any non-proper partition can be obtained as a limit of proper partitions. To check this, take a non-proper partition and subdivide one of its regions into one big and some small convex pieces, to get a proper n-partition out of it. If the measure of the small pieces goes to zero, in the limit we end up at the non-proper partition we started with.

Therefore we can think of $\mathcal{C}(\mathbb{R}^d, \leq n)$ as a compactification of $\mathcal{C}(\mathbb{R}^d, n)$. Other compactifications are possible for $\mathcal{C}(\mathbb{R}^d, n)$, but $\mathcal{C}(\mathbb{R}^d, \leq n)$ is a natural choice with a concrete interpretation, so we usually think of $\mathcal{C}(\mathbb{R}^d, n)$ as a subset of $\mathcal{C}(\mathbb{R}^d, \leq n)$.

4.2 Hyperplane description and semialgebraic structure

A subset of $\mathbb{R}^m$ is semialgebraic if it can be described as a finite union of solution sets of systems given by finitely many polynomial equations and strict inequalities on the coordinates of $\mathbb{R}^m$. In this section we prove that each of the spaces $\mathcal{C}(\mathbb{R}^d, n)$ and

$\mathcal{C}(\mathbb{R}^d, \leq n)$ is a union of finitely many pieces that can be parameterized by semialgebraic sets.

We refer to Bochnak, Coste & Roy [7] and Basu, Pollack & Roy [3] as general references on semialgebraic sets. For the following we will use some basic results about semialgebraic sets, such as the fact that finite unions and intersections of semialgebraic sets are semialgebraic, and the fact that the complements of semialgebraic sets are again semialgebraic. Most notably, we will use the Tarski–Seidenberg Theorem (see [7, Theorem 2.2.1]), which claims that semialgebraic sets are closed under projections.

Theorem 4.6 (Tarski–Seidenberg)**.** *If $X \subset \mathbb{R}^n \times \mathbb{R}^m$ is a semialgebraic set, and if p is the projection onto the first n coordinates, then $p(X) \subseteq \mathbb{R}^n$ is also semialgebraic.*

We will use here some of the notation introduced in the proof of Theorem 4.3, such as the map $\pi : (S^d)^{\binom{n}{2}} \to \mathcal{D}(\mathbb{R}^d, \leq n)$. Note that the space $(S^d)^{\binom{n}{2}} \subset \mathbb{R}^{(d+1)\binom{n}{2}}$ is semialgebraic.

Definition 4.7 (Hyperplane arrangement carrying a partition)**.** Let $\mathcal{P}$ be an n-partition of $\mathbb{R}^d$. An oriented hyperplane arrangement $\mathcal{A}_{\boldsymbol{c}}$ for $\boldsymbol{c} \in (S^d)^{\binom{n}{2}}$ *carries the partition* $\mathcal{P}$ if $\pi(\boldsymbol{c}) = \mathcal{P}$.

In other words, an oriented hyperplane arrangement $\mathcal{A}_{\boldsymbol{c}}$ for $\boldsymbol{c} \in (S^d)^{\binom{n}{2}}$ carries the partition $\mathcal{P} = (P_1, \dots, P_n)$ if the regions $\widehat{P}_i$ and $\widehat{P}_j$ are separated by the hyperplane H_{ij}, so that $\boldsymbol{c}_{ij} \cdot \boldsymbol{x} < 0$ for $\boldsymbol{x} \in \widehat{P}_i$ and $\boldsymbol{c}_{ij} \cdot \boldsymbol{x} > 0$ for $\boldsymbol{x} \in \widehat{P}_j$.

As explained in the proof of Proposition 3.3, for each n-partition $\mathcal{P} \in \mathcal{C}(\mathbb{R}^d, n)$ it is always possible to find a hyperplane arrangement $\mathcal{A}$ such that it carries the partition $\mathcal{P}$. Such hyperplane arrangement is usually not unique. For a non-proper partition $\mathcal{P} \in \mathcal{C}(\mathbb{R}^d, \leq n)$ it is also possible to find a hyperplane arrangement carrying $\mathcal{P}$. If a region P_i is empty, any hyperplane that doesn't intersect P_j is good enough to separate this two regions. If $P_i = \mathbb{R}^d$, we can still take $\boldsymbol{c}_{ij} = (1, 0, \dots, 0)$.

Example 4.8. In Figure 4.1, to the left we show a partition of $\mathbb{R}^2$ into four regions, but to make the example more interesting (so that it is in fact two examples at once) we will consider it as a non-proper partition in $\mathcal{C}(\mathbb{R}^2, \leq 5)$, by taking an extra empty region $P_5 = \emptyset$.

To the right we show an affine picture of a hyperplane arrangement carrying $\mathcal{P}$. For adjacent regions P_i and P_j, with $\{i, j\} \in A(\mathcal{P})$, there is only one possible hyperplane H_{ij}^{aff} that separates them, namely the affine span of the points on the intersection of the boundaries. The extension of these hyperplanes appears on the figure as dashed lines. For all other hyperplanes there is some freedom to choose them. In the figure, there is

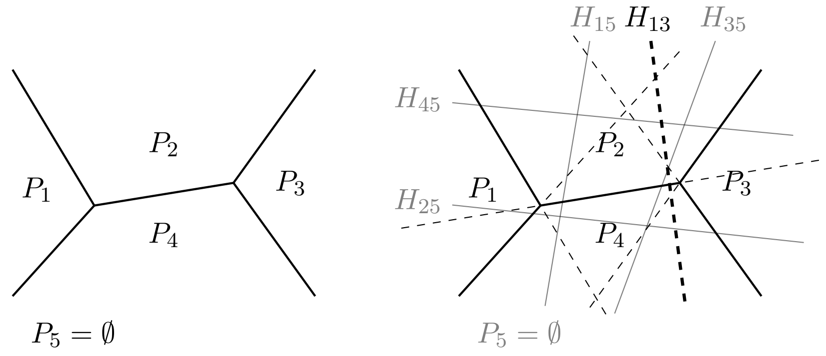

FIGURE 4.1: Non-proper partition $\mathcal{P}$ in $\mathcal{C}(\mathbb{R}^2, \leq 5)$ together with a possible hyperplane arrangement carrying it.

a label that appears next to each of them. For the hyperplanes involving the region P_5, it is only necessary that the other region lies entirely on one side of the hyperplane.

The unit vector $\boldsymbol{c}_{ij}$ is uniquely determined by the hyperplane H_{ij} and the requirement that P_i and P_j lie on the correct sides of H_{ij}, unless $P_i = P_j = \emptyset$. We remind the reader that an affine hyperplane H^{aff} given by the points $\boldsymbol{x} \in \mathbb{R}^d$ that satisfy an equation of the form $\boldsymbol{a} \cdot \boldsymbol{x} = b$ for $\boldsymbol{a} \in \mathbb{R}^d$ and $b \in \mathbb{R}$ is represented projectively by its corresponding vector $\boldsymbol{c} = (-b, a_1, \ldots, a_d) \in \mathbb{R}^{d+1}$ or by the vector $(b, -a_1, \ldots, -a_d)$ in case that the opposite orientation is required. This vector can be normalized later.

Definition 4.9 (Regions of a hyperplane arrangement)**.** Let $\mathcal{A}_{\boldsymbol{c}}$ for $\boldsymbol{c} \in (S^d)^{\binom{n}{2}}$ be a hyperplane arrangement with hyperplanes $H_{ij} = \{\boldsymbol{x} \in \mathbb{R}^{d+1} : \boldsymbol{c}_{ij} \cdot \boldsymbol{x} = 0\}$. A *region* of the affine hyperplane arrangement $\mathcal{A}^{aff}_{\boldsymbol{c}}$ is a subset of the form

$$R_{\boldsymbol{s}} = \{\boldsymbol{x} \in \mathbb{R}^d : s_{ij}\boldsymbol{c}_{ij} \cdot \begin{pmatrix} 1 \\ \boldsymbol{x} \end{pmatrix} < 0 \text{ for all } i < j\},$$

where $\boldsymbol{s} \in \{+1, -1\}^{\binom{n}{2}}$ is a sign vector with coordinates s_{ij} for $i < j$ and $\binom{1}{\boldsymbol{x}} \in \mathbb{R}^{d+1}$ is the vector obtained by adding a first coordinate equals one to $\boldsymbol{x}$.

That means that $R_{\boldsymbol{s}}$ is determined by intersecting the halfspaces $s_{ij}\boldsymbol{c}_{ij} \cdot \binom{1}{\boldsymbol{x}} < 0$ with the affine hyperplanes of $\mathcal{A}^{aff}_{\boldsymbol{c}}$ in $\mathbb{R}^d$, where each coordinate s_{ij} indicates the side of H_{ij} that contains $R_{\boldsymbol{s}}$, for $1 \leq i < j \leq n$. Regions $\mathbb{R}_{\boldsymbol{s}}$ of a hyperplane arrangement may be empty.

We symmetrize the notation by setting $s_{ji} = -s_{ij}$ for $i > j$. To avoid confusion with the regions of partitions, we always talk about regions $R_{\boldsymbol{s}}$ when we refer to a region of hyperplane arrangements. Also, regions $R_{\boldsymbol{s}}$ of $\mathcal{A}_{\boldsymbol{c}}$ simply denote regions of $\mathcal{A}^{aff}_{\boldsymbol{c}}$.

The *complete graph* K_n is the graph with vertex set $\{1, \ldots, n\}$ and with an edge between each pair of vertices. (It has $\binom{n}{2}$ edges.) An *orientation* of K_n is obtained by taking the graph K_n and fixing a direction to each edge e of K_n, by choosing which of the vertices of e is the *tail* and which is the *head*. (You can consider this directed edge e as an arrow that goes from the tail to the head.) A graph with all its edges oriented is also known as a *directed graph*. Each sign vector $\boldsymbol{s} \in \{+1,-1\}^{\binom{n}{2}}$ generates an orientation $G_{\boldsymbol{s}}$ of the complete graph K_n, where the edge ij is directed from i to j if $s_{ij} = +1$, and from j to i otherwise, for $1 \leq i < j \leq n$. A *source* of $G_{\boldsymbol{s}}$ is a vertex v of K_n that is not the head of any of the edges involving v in $G_{\boldsymbol{s}}$. Since the graph K_n is complete, there can be at most one source in the directed graph $G_{\boldsymbol{s}}$.

Lemma 4.10. *A hyperplane arrangement $\mathcal{A}_{\boldsymbol{c}}$ for $\boldsymbol{c} \in (S^d)^{\binom{n}{2}}$ carries a (possibly non-proper) n-partition $\mathcal{P}$ if and only if for all non-empty regions $R_{\boldsymbol{s}}$ of $\mathcal{A}_{\boldsymbol{c}}$ the oriented complete graph $G_{\boldsymbol{s}}$ has a source. The partition is proper if for each $i \in \{1,\ldots,n\}$, there is at least one non-empty region whose source is the vertex i.*

Proof. If $\mathcal{P}$ is an n-partition carried by $\mathcal{A}_{\boldsymbol{c}}$, all non-empty regions $R_{\boldsymbol{s}}$ of $\mathcal{A}_{\boldsymbol{c}}$ must be contained in some fixed region P_i of $\mathcal{P}$. If $R_{\boldsymbol{s}} \subseteq P_i$, then we have that $s_{ij} = +1$ for all $j \neq i$, so i is a source in $G_{\boldsymbol{s}}$.

On the other hand, if the directed graphs of all non-empty regions $\mathbb{R}_{\boldsymbol{s}}$ have a source, then we obtain an n-partition by taking

$$P_i = \bigcap_{j \neq i} \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{c}_{ij} \cdot \begin{pmatrix} 1 \\ \boldsymbol{x} \end{pmatrix} \leq 0\}.$$

The regions P_i are clearly disjoint, and their union cover all regions $R_{\boldsymbol{s}}$ of the hyperplane arrangement, since each $R_{\boldsymbol{s}} \subseteq P_i$ whenever i is the unique source of $G_{\boldsymbol{s}}$. Therefore the union of the closure of the regions must be the whole $H_1 \cong \mathbb{R}^d$. The regions P_i as defined might still be empty, but if there is a non-empty region $R_{\boldsymbol{s}}$ in $\mathcal{A}_{\boldsymbol{c}}$ with $G_{\boldsymbol{s}}$ having as source the vertex i for each i, then $R_{\boldsymbol{s}} \cap H \subseteq P_i$ is non-empty and the partition is proper. □

Lemma 4.11. *For any $\boldsymbol{s} \in \{+1,-1\}^{\binom{n}{2}}$, the set of hyperplane arrangements $\mathcal{A}_{\boldsymbol{c}}$ for $\boldsymbol{c} \in (S^d)^{\binom{n}{2}}$ such that the region $R_{\boldsymbol{s}}$ is empty is semialgebraic. Also the set of hyperplane arrangements $\mathcal{A}_{\boldsymbol{c}}$ such that the region $R_{\boldsymbol{s}}$ is non-empty is semialgebraic.*

Proof. A region $\mathbb{R}_{\boldsymbol{s}}$ is non-empty if exists $\boldsymbol{x} \in \mathbb{R}^{d+1}$ such that $s_{ij}\boldsymbol{c}_{ij} \cdot \boldsymbol{x} < 0$ for each pair $i < j$. We can add the coordinates of $\boldsymbol{x}$ as slack variables and construct a semialgebraic set X on the coordinates of $\boldsymbol{c}_{ij}$ for $1 \leq i < j \leq n$ and of $\boldsymbol{x}$, so that all inequalities $s_{ij}\boldsymbol{c}_{ij} \cdot \boldsymbol{x} < 0$ are satisfied. The parameterization of the set of all hyperplane arrangements $\mathcal{A}_{\boldsymbol{c}}$ with $R_{\boldsymbol{s}} \neq \emptyset$ can be obtained as a projection of X to the coordinates $\boldsymbol{c} \in (S^d)^{\binom{n}{2}}$ and by Theorem 4.6, we conclude that the set of arrangements with $R_{\boldsymbol{s}} \neq \emptyset$ is semialgebraic.

Since the complement of a semialgebraic set is semialgebraic, the set of arrangements such that $R_{\boldsymbol{s}} = \emptyset$ is semialgebraic. Alternatively, we can use a suitable version of the Farkas Lemma (see [36, Section 1.4]) to get another semialgebraic description of this set. □

Definition 4.12 (The spaces $\mathcal{H}(\mathbb{R}^d, \leq n)$ and $\mathcal{H}(\mathbb{R}^d, n)$)**.** We denote by $\mathcal{H}(\mathbb{R}^d, \leq n)$ the space of all $\boldsymbol{c} \in (S^d)^{\binom{n}{2}}$ such that the hyperplane arrangement $\mathcal{A}_{\boldsymbol{c}}$ carries a possibly non-proper n-partition of $\mathbb{R}^d$. The subspace of $\mathcal{H}(\mathbb{R}^d, \leq n)$ corresponding to hyperplane arrangements carrying a proper n-partition is denoted as $\mathcal{H}(\mathbb{R}^d, n)$.

We have the following chain of inclusions.

$$\mathcal{H}(\mathbb{R}^d, n) \;\subseteq\; \mathcal{H}(\mathbb{R}^d, \leq n) \;\subseteq\; (S^d)^{\binom{n}{2}} \subseteq \;\mathbb{R}^{(d+1)\times\binom{n}{2}}.$$

Theorem 4.13. *The spaces $\mathcal{H}(\mathbb{R}^d, \leq n)$ and $\mathcal{H}(\mathbb{R}^d, n)$ are semialgebraic sets.*

Proof. By Lemma 4.10, a hyperplane arrangement $\mathcal{A}_{\boldsymbol{c}}$ for $\boldsymbol{c} \in (S^d)^{\binom{n}{2}}$ carries an n-partition $\mathcal{P}$ if and only if for all regions $R_{\boldsymbol{s}}$ in $\mathcal{A}_{\boldsymbol{c}}$ the oriented graph $G_{\boldsymbol{s}}$ have a source. Therefore, we need to characterize all hyperplane arrangements $\mathcal{A}_{\boldsymbol{c}}$ such that all regions $R_{\boldsymbol{s}}$ of $\mathcal{A}_{\boldsymbol{c}}$ are empty for all sign vector $\boldsymbol{s}$ in $S = \{\boldsymbol{s} \in \{+1,-1\}^{\binom{d}{2}} : G_{\boldsymbol{s}} \text{ has no source}\}$. By Lemma 4.11 and the (obvious) fact that finite intersections of semialgebraic sets are semialgebraic, we find that $\mathcal{H}(\mathbb{R}^d, \leq n)$ is a semialgebraic set over the coordinates of $\boldsymbol{c}_{ij}$ as variables.

Also the set $\mathcal{H}(\mathbb{R}^d, n)$ of hyperplane arrangements carrying a proper n-partition, where at least one region $\mathbb{R}_{\boldsymbol{s}}$ has source i for each $i \leq n$, is semialgebraic, again by Lemma 4.11 and the fact that finite unions and intersections of semialgebraic sets are again semialgebraic. □

From Theorem 4.13 we can see that $\mathcal{H}(\mathbb{R}^d, \leq n)$ is the union of all sets of arrangements with an adjacency graph that satisfies the source conditions specified on Lemma 4.10.

Theorem 4.14. *The space $\mathcal{C}(\mathbb{R}^d, \leq n)$ is the union of finitely many subspaces indexed by adjacency graphs $A(\mathcal{P})$, which can be parameterized as semialgebraic sets. The same statement is true for the space $\mathcal{C}(\mathbb{R}^d, n)$.*

Proof. The map $\pi : \mathcal{H}(\mathbb{R}^d, \leq n) \to \mathcal{C}(\mathbb{R}^d, \leq n)$ is a surjective continuous map taking each oriented hyperplane arrangement $\mathcal{A}$ in $\mathcal{H}(\mathbb{R}^d, \leq n)$ to its corresponding partition. The pieces of $\mathcal{C}(\mathbb{R}^d, \leq n)$ are given by the partitions in $\mathcal{C}(\mathbb{R}^d, \leq n)$ that share the same adjacency graph $A(\mathcal{P})$, for any given n-partition $\mathcal{P}$. Each of these pieces is denoted as

$\mathcal{C}_{A(\mathcal{P})}(\mathbb{R}^d, n)$ for each partition $\mathcal{P}$, and the inverse image $\pi^{-1}(\mathcal{C}_{A(\mathcal{P})}(\mathbb{R}^d, n))$ is denoted as $\mathcal{H}_{A(\mathcal{P})}(\mathbb{R}^d, n)$.

To see that $\mathcal{H}_{A(\mathcal{P})}(\mathbb{R}^d, n)$ is a semialgebraic set, we take the description of $\mathcal{H}(\mathbb{R}^d, n)$ and add extra restrictions to express the fact that certain hyperplanes do not determine any $(d-1)$-face of the partition. These extra restrictions are described in what follows.

A pair $\{i, j\}$ is in $A(\mathcal{P})$ for $\mathcal{P} = \pi(\mathcal{A})$ if and only if there are $\boldsymbol{s}$, $\boldsymbol{s}' \in \{+1, -1\}^{\binom{n}{2}}$ with exactly the same entries, except only by the entry $s_{ij} = -s'_{ij}$, with oriented graphs $G_{\boldsymbol{s}}$, $G_{\boldsymbol{s}'}$ having sources i and j respectively, so that the regions $R_{\boldsymbol{s}}$, $R_{\boldsymbol{s}'}$ are non-empty.

Using Lemma 4.11 we find that the subset of arrangements $\mathcal{A}' \in \mathcal{H}(\mathbb{R}^d, n)$ with $\{i, j\} \in A(\pi(\mathcal{A}))$ for a given $\mathcal{A} \in \mathcal{H}(\mathbb{R}^d, n)$ is semialgebraic, since it is the union over all pairs $\boldsymbol{s}$, $\boldsymbol{s}'$ that differ only in the ij-coordinate and with respective graphs sources i and j of the subsets of $\mathcal{H}(\mathbb{R}^d, n)$ where $R_{\boldsymbol{s}}$ and $R_{\boldsymbol{s}'}$ are non-empty. The complement of those subsets, that represent hyperplane arrangements with $\{i, j\} \notin A(\pi(\mathcal{A}))$ are also semialgebraic.

Finally $\mathcal{H}_{A(\mathcal{P})}(\mathbb{R}^d, n)$ is the intersection of subsets of $\mathcal{H}(\mathbb{R}^d, n)$ where $\{i, j\} \in A(\pi(\mathcal{A}))$ for $\{i, j\} \in A(\mathcal{P})$ and $\{k, \ell\} \notin A(\pi(\mathcal{A}))$ for $\{k, \ell\} \notin A(\mathcal{P})$ and thus it is also a semialgebraic set. Since the map $\pi : \mathcal{H}(\mathbb{R}^d, \leq n) \to \mathcal{C}(\mathbb{R}^d, \leq n)$ is a projection obtained by deleting the coordinates $\boldsymbol{c}_{ij}$ of the hyperplanes H_{ij} for $\{i, j\} \notin A(\mathcal{P})$, by Theorem 4.6 we conclude that $\mathcal{C}_{A(\mathcal{P})}(\mathbb{R}^d, n)$ is a semialgebraic set on the coordinates of the vectors $\boldsymbol{c}_{ij}$ for $\{i, j\} \in A(\mathcal{P})$ and $\mathcal{C}(\mathbb{R}^d, \leq n)$ is a union of semialgebraic pieces.

If there are two ore more non-empty regions in $\mathcal{P}$, the vertices of $A(\mathcal{P})$ contained in at least one edge correspond to the non-empty regions of $\mathcal{P}$. Therefore, we can obtain $\mathcal{C}(\mathbb{R}^d, n)$ as the union of the semialgebraic pieces of the form $\mathcal{C}_{A(\mathcal{P})}(\mathbb{R}^d, n)$ where $A(\mathcal{P})$ is a connected graph on the vertices from 1 to n. □

The pieces $\mathcal{C}_{A(\mathcal{P})}(\mathbb{R}^d, n)$ are not necessarily closed. Their closure in $\mathcal{C}(\mathbb{R}^d, \leq n)$ might include partitions with graphs that are subsets of $A(\mathcal{P})$. Only knowing these semialgebraic pieces it is not enough to reconstruct the spaces $\mathcal{C}(\mathbb{R}^d, n)$ and $\mathcal{C}(\mathbb{R}^d, \leq n)$. We also need the topological structure induced by the metric given in Section 4.1 to know how to glue the different semialgebraic pieces of the form $\mathcal{C}_{A(\mathcal{P})}(\mathbb{R}^d, n)$ in order to obtain the spaces of n-partitions $\mathcal{C}(\mathbb{R}^d, n)$ and $\mathcal{C}(\mathbb{R}^d, \leq n)$.

On each semialgebraic piece $\mathcal{C}_{A(\mathcal{P})}(\mathbb{R}^d, n)$ we have a topological structure by seeing it as a subset of $\mathbb{R}^{(d+1)\times E}$ given by the parameterization through the $\boldsymbol{c}_{ij}$, where E is the number of edges in $A(\mathcal{P})$. We claim that this topological structure is equivalent as the one obtained as a subset of $\mathcal{C}(\mathbb{R}^d, \leq n)$. To see this, notice that a sequence of partitions $(\mathcal{P}^k)_{k \in \mathbb{N}}$ in $\mathcal{C}_{A(\mathcal{P})}(\mathbb{R}^d, n)$ converges to a partition $\mathcal{P}$ in the δ_μ-topology if and only if each sequence of coordinates c_{ij}^k of the parameterizations of $\mathcal{P}^k$ for $\{i,j\} \in A(\mathcal{P})$ converge to the coordinates of $\boldsymbol{c}_{ij}$.

Lemma 4.15. *The set $Q_{ij} \subseteq S^d$ of all vectors $\boldsymbol{c}_{ij} \in S^d$ corresponding to separating hyperplanes H_{ij} for two disjoint open convex spherical polyhedra $\widehat{P}_i$ and $\widehat{P}_j$ in S^d is an strictly convex spherical polyhedron.*

Proof. The set $Q_{ij} \subseteq S^d$ of possible values $\boldsymbol{c}_{ij}$ is the solution set of a system of inequalities of the form $\boldsymbol{c}_{ij} \cdot \boldsymbol{x} < 0$ for all points $\boldsymbol{x} \in \widehat{P}_i$ and $\boldsymbol{c}_{ij} \cdot \boldsymbol{x} > 0$ for all points $\boldsymbol{x} \in \widehat{P}_j$. This is an intersection of open hemispheres and therefore strictly convex.

If the sets $\widehat{P}_i$ and $\widehat{P}_j$ are polyhedral, it is possible to find finite sets of points N_i and N_j in S^d such that $\widehat{P}_i$ is the interior of the spherical convex hull of the points in N_i and similarly for $\widehat{P}_j$. In that case, the possible values for $\boldsymbol{c}_{ij}$ are the solution set of a finite system of inequalities of the form $\boldsymbol{c}_{ij} \cdot \boldsymbol{x} \leq 0$ for all points $\boldsymbol{x} \in N_i$ and $\boldsymbol{c}_{ij} \cdot \boldsymbol{x} \geq 0$ for all points $\boldsymbol{v} \in N_j$. We conclude that Q_{ij} is also polyhedral. □

For the proof of the last lemma we need to use the fact that any spherical polyhedron can be described as the spherical convex hull of a finite set of points. In case that the sets $\widehat{P}_i$ and $\widehat{P}_j$ come from an n-partition, this is a consequence of Lemma 4.30, that will be explained in the next section when we talk about node systems. If $\widehat{P}_i$ and $\widehat{P}_j$ are pointed spherical polyhedra, it is enough to take their vertex set.

Proposition 4.16. *The fibers of the projection $\pi : \mathcal{H}(\mathbb{R}^d, n) \to \mathcal{C}(\mathbb{R}^d, n)$ are contractible.*

Proof. For a given partition $\mathcal{P} \in \mathcal{C}(\mathbb{R}^d, n)$, all hyperplane arrangements $\mathcal{A}$ in the preimage $\pi^{-1}(\mathcal{P})$ have the same coordinates for the vectors $\boldsymbol{c}_{ij}$ corresponding to pairs $\{i,j\} \in A(\mathcal{P})$. If a pair $\{i,j\} \notin A(\mathcal{P})$, due to Lemma 4.15 we find that the sets Q_{ij} of all points $\boldsymbol{c}_{ij}$ corresponding to separating hyperplanes are strictly convex spherical polyhedra, and therefore contractible. We know that they are non-empty by the Hahn–Banach Separation Theorem. Since the choices of the different $\boldsymbol{c}_{ij}$ are independent of each other to obtain a point in the preimage, we conclude that the fiber $\pi^{-1}(\mathcal{P})$ is the product of contractible sets Q_{ij} for $\{i,j\} \notin A(\mathcal{P})$, and therefore it is contractible. □

If there are two empty regions $P_i = P_j = \emptyset$, then the set $Q_{ij} \cong S^d$ is not contractible. Therefore the result in Proposition 4.16 does not hold for the space $\mathcal{C}(\mathbb{R}^d, \leq n)$.

We would like to construct a map τ that chooses a preimage of π in such a way that $\tau(\mathcal{C}(\mathbb{R}^d, n))$ is a semialgebraic subset of $\mathcal{H}(\mathbb{R}^d, n)$ homeomorphic to $\mathcal{C}(\mathbb{R}^d, n)$. Due to Lemma 4.15 one option would be to choose one point in the interior of a spherically convex polyhedron Q_{ij} for each $\{i,j\} \notin A(\mathcal{P})$. One first attempt to do this is looking at the barycenters. Unfortunately this idea doesn't work properly since the polyhedra Q_{ij} don't need to be full dimensional, and the barycenter is not continuous when we jump to polyhedra in smaller dimensions. Instead of barycenters, another option is to look at the center of the smallest sphere that contains Q_{ij}. This points are well defined and move continuously in terms of the vertices, even when we go to smaller dimensions. We will discuss a bit about them in the intermezzo, but unfortunately they are also not sufficient for the construction we wanted to do. Moreover, we found out that no such map τ could possibly exist.

Proposition 4.17. *There is no continuous map* $\tau : \mathcal{C}(\mathbb{R}^d, n) \to \mathcal{H}(\mathbb{R}^d, n)$ *such that* $\pi \circ \tau = \mathrm{id}_{\mathcal{C}(\mathbb{R}^d,n)}$, *for* $n \geq 3$ *and* $d \geq 2$.

Proof. First we will show this result for $n = 3$ and $d = 2$. Consider a non-essential 3-partition $\mathcal{P}$ with three regions separated by two parallel lines ℓ_{12} and ℓ_{23} (as depicted on the left of Figure 4.2). The possible preimages under π of the partition $\mathcal{P}$ consist of three vectors representing the oriented hyperplanes ℓ_{12}, ℓ_{23} and a extra hyperplane ℓ_{13} that has to be parallel the two former ones and between them. For each of such possible hyperplane ℓ_{13} we want to construct a sequence of essential 3-partitions $\mathcal{P}^i$ whose limit is $\mathcal{P}$ and such that the hyperplane ℓ_{13} is the only hyperplane that separates regions P_1^i and P_3^i for all partitions in the sequence. Therefore if such continuous function τ exists, the value of $\tau(\mathcal{P})$ must be given by the hyperplane ℓ_{13}, as the limit of the images of the sequence under τ. Since it is not possible that $\tau(\mathcal{P})$ take all those different values, we conclude that no such function τ might exist.

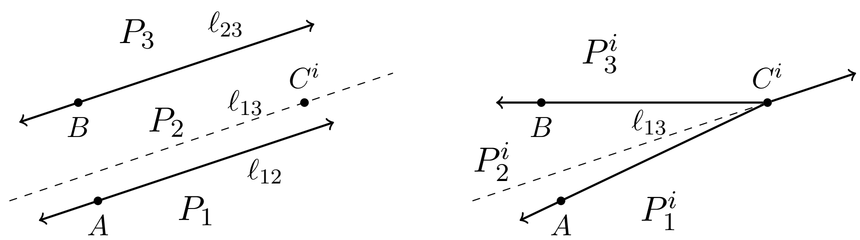

FIGURE 4.2: A 3-partition $\mathcal{P} \in \mathcal{C}(\mathbb{R}^2, 3)$ as the limit of partitions $\mathcal{P}^i$ that use ℓ_{13}.

To construct the sequence of partitions for a given line ℓ_{13}, we fix two points A and B in ℓ_{12} and ℓ_{23} respectively, and for any point C^i in ℓ_{13} we construct a partition $\mathcal{P}^i$ by taking three rays centered at C^i, one in direction of the line ℓ_{13} and the other two passing through the points A and B respectively, as it is shown on the right of Figure 4.2. Now take a sequence of points C^i that goes far away, every time moving in the same direction over the line ℓ_{13}. The corresponding sequence of partitions $\mathcal{P}^i$ in the limit will go to $\mathcal{P}$. All partitions in the sequence have ℓ_{13} as a defining hyperplane, as desired.

We conclude that no such function τ might exist for $n = 3$ and $d = 2$. For bigger values of n and d we can construct similar sequences that prove that the function τ cannot be well defined, by taking parallel hyperplanes in higher dimensions and adding extra regions if needed. Therefore no such τ function might exist on those other cases either. □

Intermezzo: Smallest bounding spheres of spherical polyhedra

Here we will look at the center of the smallest sphere that contains a set of points, in the spherical case. In the euclidean case, this center is uniquely defined, lies always in the convex hull of the set (but not necessarily on its interior), and besides it is continuous with respect to Hausdorff distance. (See [12] and [15] for known results and generalizations).

For subsets of the sphere we will prove some of those properties and see that many structural properties of minimal bounding spheres are geometrically clearer in this setup. The following definition captures the essence of being the center of the smallest bounding sphere.

Definition 4.18 (Bounding sphere center)**.** Let Q be a closed subset of S^d contained in an open hemisphere. The *bounding sphere center* of Q is the vector $\boldsymbol{q}(Q)$ in S^d that maximize the value of $\min\limits_{\boldsymbol{x}\in Q} \boldsymbol{q}\cdot\boldsymbol{x}$ among all points $\boldsymbol{q} \in S^d$.

Proposition 4.19. *The bounding sphere center is uniquely defined. If Q is a closed strictly convex polyhedral subset of S^d then $\boldsymbol{q}(Q)$ belongs to Q. Moreover, it is contained in the spherical convex hull of the set W of vertices $\boldsymbol{v}$ of Q such that $\boldsymbol{v}\cdot\boldsymbol{q}(Q) = \min_{\boldsymbol{x}\in Q} \boldsymbol{q}(Q)\cdot\boldsymbol{x}$.*

Proof. For any point $\boldsymbol{q} \in S^d$ the value $\min\limits_{\boldsymbol{x}\in Q} \boldsymbol{q}\cdot\boldsymbol{x}$ always exists and it must be maximal at some closed subset of S^d, since S^d is compact. This maximal d must be positive since Q is contained in a hemisphere.

If there are two different points $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ such that they attain the maximal value d, then $\boldsymbol{q}_1\cdot\boldsymbol{x} \geq d$ and $\boldsymbol{q}_2\cdot\boldsymbol{x} \geq d$ for any $\boldsymbol{x} \in Q$. If $\boldsymbol{q} = (\boldsymbol{q}_1+\boldsymbol{q}_2)/|\boldsymbol{q}_1+\boldsymbol{q}_2|$ then

$$\boldsymbol{x}\cdot\boldsymbol{q} = \frac{1}{|\boldsymbol{q}_1+\boldsymbol{q}_2|}(\boldsymbol{x}\cdot\boldsymbol{q}_1+\boldsymbol{x}\cdot\boldsymbol{q}_2) \geq \frac{2d}{|\boldsymbol{q}_1+\boldsymbol{q}_2|} > d$$

for any $\boldsymbol{x} \in Q$, since $|\boldsymbol{q}_1+\boldsymbol{q}_2| < 2$ for $\boldsymbol{q}_1 \neq \boldsymbol{q}_2$. Therefore d is not maximal, creating a contradiction. This implies that $\boldsymbol{q}(Q)$ is uniquely defined.

Now let Q be strictly convex and $\boldsymbol{q} = \boldsymbol{q}(Q)$, so that $\boldsymbol{q}$ maximizes $d = \min\limits_{\boldsymbol{x}\in Q} \boldsymbol{q}\cdot\boldsymbol{x}$, and suppose that $\boldsymbol{q}$ is not in the spherical convex hull of the set W of vertices of Q such that

$\boldsymbol{v} \cdot \boldsymbol{q} = d$. Then by the Hahn–Banach Separation Theorem we can find a vector $\boldsymbol{p} \in S^d$ such that $\boldsymbol{p} \cdot \boldsymbol{v} > 0$ for all $\boldsymbol{v} \in W$ and $\boldsymbol{p} \cdot \boldsymbol{q} < 0$. Then we can find a small $\varepsilon > 0$ such that $|\boldsymbol{q} + \varepsilon \boldsymbol{p}| < |\boldsymbol{q}| = 1$, and then for $\boldsymbol{q}' = \dfrac{\boldsymbol{q} + \varepsilon \boldsymbol{p}}{|\boldsymbol{q} + \varepsilon \boldsymbol{p}|}$ and any $\boldsymbol{v} \in W$ it holds that

$$\boldsymbol{v} \cdot \boldsymbol{q}' = \frac{\boldsymbol{v} \cdot \boldsymbol{q} + \boldsymbol{v} \cdot \varepsilon \boldsymbol{p}}{|\boldsymbol{q} + \varepsilon \boldsymbol{p}|} \geq \frac{d}{|\boldsymbol{q} + \varepsilon \boldsymbol{p}|} > d.$$

If ε is small enough we can also ensure that $\boldsymbol{q}' \cdot v > d$ for all other vertices $\boldsymbol{v} \notin W$ since $\boldsymbol{q} \cdot \boldsymbol{v} > d$. Therefore d is not maximal and we conclude that if Q is a strictly convex polyhedron, then $\boldsymbol{q}(Q)$ is in the spherical convex hull of W. This also implies that $\boldsymbol{q}(Q) \in Q$. □

Proposition 4.20. *A point $\boldsymbol{q}$ is the bounding sphere center $\boldsymbol{q}(Q)$ of a strictly convex polyhedron Q if and only if there is a subset of vertices W of Q such that $\boldsymbol{q} \cdot \boldsymbol{v} = d$ for $\boldsymbol{v} \in W$ and some $d > 0$, $\boldsymbol{q} \cdot \boldsymbol{v} > d$ for all other vertices $\boldsymbol{v} \notin W$, and $\boldsymbol{q}$ lies on the spherical convex hull of W.*

Proof. If $\boldsymbol{q} = \boldsymbol{q}(Q)$, we can take W as in Proposition 4.19, and from there we know that $\boldsymbol{q}$ is in the convex hull of W. Also, for $d = \min_{\boldsymbol{x} \in Q} \boldsymbol{q} \cdot \boldsymbol{x}$ it holds that $\boldsymbol{q} \cdot \boldsymbol{v} = d$ for $\boldsymbol{v} \in W$ and $\boldsymbol{q} \cdot \boldsymbol{v} > d$ for all other vertices $\boldsymbol{v} \notin W$, as desired.

By the other hand, suppose that $\boldsymbol{q}$ is such that there is a subset of vertices W of Q such that $\boldsymbol{q} \cdot \boldsymbol{v} = d$ for $\boldsymbol{v} \in W$ and some $d > 0$, $\boldsymbol{q}(Q) \cdot \boldsymbol{v} > d$ for all other vertices $\boldsymbol{v} \notin W$ and such that $\boldsymbol{q}$ is in the spherical convex hull of W.

Let Q' be the polyhedron in $\mathbb{R}^{d+1}$ that is the affine convex hull of the vertices of Q. Then W must be the set of vertices of a face F of Q'. If F is a facet of Q', call $n(F)$ to the unit outer normal vector of Q' at F. Then, the only option is that $\boldsymbol{q} = -n(F)$. If F is not a facet, it is the intersection of some facets of Q'. Let $\sigma(F)$, the outer normal cone of F, to be the cone spaned by the vectors $n(G)$ corresponding to the facets G of Q' containing F. With this notation, $\boldsymbol{q}$ must belong to $-\sigma(F)$, so that $\boldsymbol{q} \cdot \boldsymbol{x} = d$ is minimal in F, for $\boldsymbol{x} \in Q'$.

Besides, for all faces F of Q' it holds that $\dim F + \dim \sigma(F) = d + 1$, and moreover the affine span of $\sigma(F)$ intersects the affine span of $V(F)$ in a unique point $\boldsymbol{p}(F)$, since $\sigma(F)$ is spanned by a set of normal unit vectors to F that generate an orthogonal space. If F is the face corresponding to W, the only option for $\boldsymbol{q}$ is to be the unit vector in the direction of $\boldsymbol{p}(F)$ corresponding to this face F. Since it is also required that $\boldsymbol{q}$ belong to the spherical convex hull of W, then it is needed that $\boldsymbol{p}(F) \in F \cap -\sigma(F)$.

Now we want to see that there is only one possible choice of $\boldsymbol{q}$. For this, we construct a partition of $\mathbb{R}^{d+1}$ where there is a region P_F for each face F of Q' obtained by taking the interior the Minkowski sum of F and the outer normal cone $\sigma(F)$. Then, $\boldsymbol{p}(F) \in F \cap -\sigma(F)$ if and only if the origin belongs to the closure of the region $P_F = F + \sigma(F)$. But the origin can only belong to the interior of one of the regions $P_{F'}$, corresponding to a face F'. In case the origin is on the boundary of more than one of those regions, take F' to be the face that have maximal dimension. (In that case, all those faces F are contained in F', and the corresponding vector $\boldsymbol{p}(F)$ is always the same.) Since Q is strictly convex, then the origin does not belong to $P_{Q'} = Q'$ and $\boldsymbol{p}(F') \neq 0$.

Then the only option is that W is the set of vertices of F' and $\boldsymbol{q}$ is the unit normal vector in the direction of $\boldsymbol{p}(F')$. Since the bounding sphere center $\boldsymbol{q}(Q)$ also satisfies all those properties, then necessarily $\boldsymbol{q} = \boldsymbol{q}(Q)$. □

We observe that the coordinates of the bounding sphere center can be described by a system of equations and inequalities on the coordinates of the vertices of the polyhedron Q using an extra slack variable d.

4.3 Pointed partitions and node systems

Pointed partitions are an important class of partitions, where every face is completely determined by its set of vertices (see Definition 4.21, Proposition 4.22). For general partitions the same doesn't hold. To get similar properties for any n-partition, we need to define node systems (Definition 4.27).

Definition 4.21 (Pointed partitions)**.** An n-partition $\mathcal{P} = (P_1, \ldots, P_n)$ of $\mathbb{R}^d$ is *pointed* if for each region P_i the cone C_i is pointed.

We remind that we exceptionally defined $C_i = \emptyset$ in case a region $P_i = \emptyset$ (see comments after Definition 4.2). Therefore pointed partitions must be proper. We establish now some simple lemmas about pointed partitions.

Proposition 4.22. *If $\mathcal{P}$ is a pointed n-partition, every face F_I of $\mathcal{P}$ can be obtained as the spherical convex hull of all vertices in F_I.*

Proof. If $\mathcal{P}$ is a pointed n-partition, every d-face F_i is the intersection of the pointed cone C_i with S^d. Also any intersection of such pointed cones is pointed, and so we can conclude that all faces F_I of $\mathcal{P}$ can be obtained as the intersection of a pointed cone C_I with S^d. By Lemma 2.1, a pointed cone C_I is the cone of the set of points obtained by intersecting each edge of C with S^d. These are the vertices of F_I (see Lemma 3.23) and therefore each F_I is the spherical convex hull of its vertices. □

Lemma 4.23. *Pointed n-partitions are essential.*

Proof. Let $\mathcal{P}$ be a pointed partition in $\mathcal{C}(\mathbb{R}^d, n)$. For any region P_i of a pointed n-partition $\mathcal{P}$, the face F_i is a strictly convex spherical polyhedron, and by Lemma 2.1 it is the spherical convex hull of its vertices. Not all of these vertices can be at infinity, otherwise its spherical convex hull won't have dimension d. Therefore there is at least one interior vertex and by Lemma 3.15 this implies that the partition is essential. □

The converse of Lemma 4.23 is not true. Example 3.7 shows a counterexample.

Lemma 4.24. *A partition $\mathcal{P}$ is pointed if and only if the recession cones* $\mathrm{rec}(P_i)$ *of all regions of $\mathcal{P}$ are pointed.*

Proof. The recession cone of a face P_i can be identified with the cone over the points at infinity of the face F_i. This can be obtained as a face of C_i that lies over the hyperplane H bounding the upper hemisphere S^d_+. If the partition $\mathcal{P}$ is pointed, all cones C_i are pointed and therefore also all recession cones $\mathrm{rec}(P_i)$ are pointed as well.

On the other hand, if $\mathcal{P}$ is not pointed, some of the cones C_i contain a straight line. Since the interior of C_i is contained in the upper hemisphere, this line must be contained in the boundary face of C_i over the hyperplane H, and therefore it will create a line on the recession cone $\mathrm{rec}(P_i)$, and not all recession cones can be pointed. □

Now we define the node systems of an n-partition, in order to get that every face is the spherical convex hull of its corresponding nodes and so that for a pointed partition $\mathcal{P}$ the nodes coincide with the vertices of $\mathcal{P}$. First we need to introduce the half-linear faces of a partition.

Definition 4.25 (Half-linear faces)**.** A face F of a partition $\mathcal{P}$ is *half-linear* if it is the intersection of S^d with a linear subspace of $\mathbb{R}^{d+1}$ and a unique closed halfspace given by a linear inequality. The set of half-linear faces of a partition is denoted as $\mathcal{F}^H(\mathcal{P})$.

If a face F is half-linear, then it has a unique linear subface F' in its relative boundary. The subface F' is the union of some faces of $\mathcal{P}$, and is the intersection of a linear subspace with S^d. Then F' cannot have any boundary since it is topologically a sphere (of dimension $\dim F' = \dim F - 1$) and is the union of some faces at infinity of $\mathcal{P}$. Since $\hat{P}_\infty$ is not a face of $\mathcal{P}$, then in particular it is not a half-linear face of $\mathcal{P}$ (but it is a half-linear face of the spherical partition $\hat{\mathcal{P}}$).

By Lemma 3.22, the only face F_I of $\mathcal{P}$ such that its corresponding cone C_I is a linear subspace is the minimal face $F_{I(\mathbf{0})}$. This face has no boundary and no subfaces. All faces covering $F_{I(\mathbf{0})}$ in the face poset are half-linear. If a partition is essential, all vertices are half-linear faces.

Example 4.26. Let us describe the half-linear faces of the partitions $\mathcal{P}$ and $\mathcal{P}'$ presented in Example 3.7 and Example 3.26 respectively. For the first partition, every vertex of $\mathcal{P}$ is half-linear (there are four of them). Besides, there are two more half-linear faces in the figure, namely the faces F_{34} and $F_{4\infty}$. For these two 1-faces, there is a unique linear subface that covers the relative boundary and is the union of two vertices of $\mathcal{P}$.

The second partition $\mathcal{P}'$ is non-essential, where its minimal face $F_{I(\mathbf{0})}(\mathcal{P}')$ consists of two antipodal points. Then all 1-faces are half-linear, since they cover $F_{I(\mathbf{0})}(\mathcal{P}')$. There are no other half-linear faces on this partition.

Definition 4.27 (Node systems, nodes)**.** Let $\mathcal{P}$ be a partition in $\mathcal{C}(\mathbb{R}^d, \leq n)$. In case that $\mathcal{P}$ is essential, a *node system* N of $\mathcal{P}$ is a set of points $\boldsymbol{v}_F$, one in the relative interior of each half-linear face F of $\mathcal{P}$. If the partition $\mathcal{P}$ is non-essential, with $\dim F_{I(\mathbf{0})} = k \geq 0$, then a node system again contains one point $\boldsymbol{v}_F$ in the relative interior of each half-linear face F of $\mathcal{P}$, and additionally an ordered sequence of $k+2$ extra points $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{k+2}$ on the face $F_{I(\mathbf{0})}$ such that they positively span the linear subspace $C_{I(\mathbf{0})}$.

The points in a node system are referred as *nodes*. We denote as $N(\mathcal{P})$ the set of all node systems of $\mathcal{P}$. Note that all vertices of $\mathcal{P}$ are also nodes in any node system of $\mathcal{P}$.

We sometimes add the node system to the notation of the node by setting $\boldsymbol{v}_F(N) = \boldsymbol{v}_F \in N$, in case that it is not clear the node system we are using. If $\mathcal{P}$ is non-essential, the same applies to the nodes $\boldsymbol{v}_i$ in the minimal face.

Example 4.28. Now we construct node systems for both partitions of Example 4.26. In the first case, every vertex of $\mathcal{P}$ must be a node. We need to include two more nodes $\boldsymbol{v}_{F_{34}}$ and $\boldsymbol{v}_{F_{4\infty}}$ in the relative interior of the faces F_{34} and $F_{4\infty}$ respectively. We have one degree of freedom to choose each of these two nodes. On the left of Figure 4.3 we depict one possible choice for a node system N of $\mathcal{P}$.

For the second partition (on the right of Figure 4.3), we need to have two nodes $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ on the linear face $F_{I(\mathbf{0})}(\mathcal{P}')$, so that they positively span $C_{I(\mathbf{0})}$. There are two possibilities to choose $\boldsymbol{v}_1$, and $\boldsymbol{v}_2$ must be the antipodal point $-\boldsymbol{v}_1$. Besides these two nodes, we need five more nodes, one on the relative interior of each half-linear face, in order to get a node system N' of $\mathcal{P}'$.

Proposition 4.29. *If $\mathcal{P}$ is an essential n-partition, then the space $N(\mathcal{P})$ of all node systems can be seen as a semialgebraic set obtained as a product $N(\mathcal{P}) = \prod_{\operatorname{relint} F \in \mathcal{F}^H(\mathcal{P})} F$. In that case* $\dim N(\mathcal{P}) = \sum_{F \in \mathcal{F}^H} \dim(F)$.

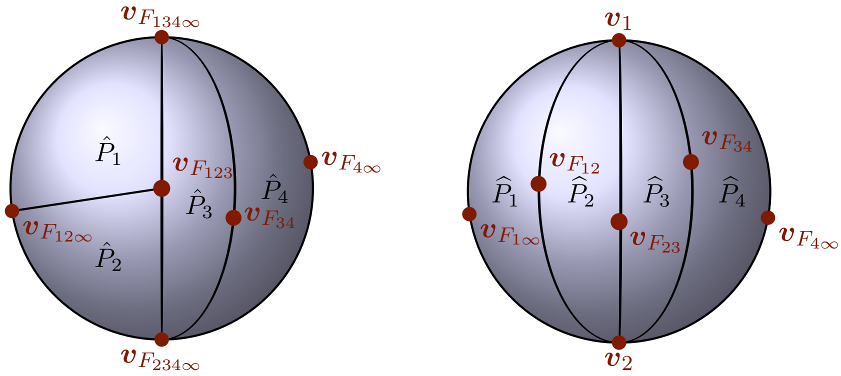

FIGURE 4.3: Node systems for two different 4-partitions. There is a node in the relative interior of each half-linear face.

If $\mathcal{P}$ is non-essential and $k = \dim(F_{I(\mathbf{0})})$, the space $N(\mathcal{P})$ can be obtained as the product

$$N(\mathcal{P}) = (GL(\mathbb{R}^{k+1})/\mathbb{R}_+) \times \prod_{F \in \mathcal{F}^H(\mathcal{P})} F.$$

The dimension of the space of node systems is $\dim N(\mathcal{P}) = k(k+2) + \sum_{F\in\mathcal{F}^H} \dim(F)$ for non-essential partitions, since there are k degrees of freedom for each node in the minimal face.

Proof. For essential partitions, a node system is obtained by taking one node in each half-linear face, and the result is clear. A precise description of $N(\mathcal{P})$ in the non-essential case can be obtained as follows. The choice of nodes on the minimal face can be parameterized by the general linear group $GL(\mathbb{R}^{k+1})$ modulo $\mathbb{R}_+$. To see this, notice that if $X \in GL(\mathbb{R}^{k+1})$ is a matrix, we can obtain nodes $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}$ given by the direction of the columns $X_1, \ldots, X_{k+1}$ of X after fixing a basis for the $(k+1)$-dimensional subspace $C_{I_\mathbf{0}}$. The direction of $\boldsymbol{v}_{k+2}$ can be fixed by the direction of the vector $-\sum_{i=1}^{k+1} X_i$ (all nodes have to be normalized).

In this way, every possible choice of nodes in $F_{I(\mathbf{0})}$ is represented by a unique matrix in $GL(\mathbb{R}^{k+1})$, up to multiplication by a scalar, since the direction of the vectors $X_1, \ldots, X_{k+1}$ is fixed by the nodes $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}$, and there is a unique way to scale them in such a way that $-\sum_{i=1}^{k+1} X_i = \boldsymbol{v}_{k+2}$. If we scale the whole matrix X obtained this way by a positive scalar, we get all the matrices $X' \in GL(\mathbb{R}^{k+1})$ that produce the nodes $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+2}$. Then the space $N(\mathcal{P})$ can be described as the product

$$N(\mathcal{P}) = (GL(\mathbb{R}^{k+1})/\mathbb{R}_+) \times \prod_{F \in \mathcal{F}^H(\mathcal{P})} F,$$

and since $\dim GL(\mathbb{R}^{k+1}) = (k+1)^2$, we obtain that $\dim N(\mathcal{P}) = k(k+2) + \sum_{F\in\mathcal{F}^H} \dim(F)$.

□

Lemma 4.30. *If N is a node system of a partition $\mathcal{P}$, then any face F of $\mathcal{P}$ can be obtained as the spherical convex hull of the set of nodes in N contained in F.*

Proof. By induction on the dimension of F, first take $F = F_{I(\mathbf{0})}$ to be the minimal face of $\mathcal{P}$, with $\dim(F_{I(\mathbf{0})}) = k$. Then we have $k+2$ nodes in $F_{I(\mathbf{0})}$ that positively span the $(k+1)$-dimensional linear subspace $C_{I(\mathbf{0})}$, and therefore its spherical convex hull is equal to $F_{I(\mathbf{0})}$. If the partition is essential, $F_{I(\mathbf{0})} = \emptyset$ doesn't contain any node, and its convex hull is also empty.

Now suppose that $\dim(F) = m$ and every face F' of $\mathcal{P}$ with $\dim F' < m$ is equal to the convex hull of the nodes contained F'. If F is half-linear, we have an extra node $\boldsymbol{v}_F$ in the interior of F, and any other point $\boldsymbol{x}$ in F is in an interval between $\boldsymbol{v}_F$ and a point $\boldsymbol{x}'$ in the boundary of F. Since $\boldsymbol{v}_F$ cannot be antipodal to $\boldsymbol{x}'$, then $\boldsymbol{x}$ can be written as a positive combination of $\boldsymbol{v}_F$ and $\boldsymbol{x}'$. By the induction hypothesis, $\boldsymbol{v}$ is a positive combination of the nodes in the face where it belongs (we use here Lemma 3.23) that are also contained in F, and therefore $\boldsymbol{p}$ is in the spherical convex hull of the nodes in F.

If F is not half-linear, we can find a point v in the boundary such that its antipodal point is not in F. Now we can repeat the argument given before, and the result follows. □

Definition 4.31 (Cell complex from a node system)**.** Given a node system N of a partition $\mathcal{P}$, we construct a CW complex $\mathcal{P}_N$ in such a way that the vertices of this complex are precisely the nodes in N, and such that each face of $\mathcal{P}$ is union of faces of $\mathcal{P}_N$. The *complex* $\mathcal{P}_N$ is constructed recursively as follows:

- Include a face F_S in $\mathcal{P}_N$ for every subset S of nodes contained in $F_{I(\mathbf{0})}$, with $|S| \leq k+1$, where F_S is the spherical convex hull of S and $k = \dim F_{I(\mathbf{0})}$. For essential partitions, only the empty set is included in this step.
- For every half-linear face F of $\mathcal{P}$ such that the boundary is already covered by faces of $\mathcal{P}_N$, the spherical convex hull of every face G of $\mathcal{P}_N$ contained on the boundary of F together with the node $\boldsymbol{v}_F$ is also a face of $\mathcal{P}_N$. (These faces of $\mathcal{P}_N$ are pyramids over the faces on the boundary of F.)
- All other faces of $\mathcal{P}$ that are not linear or half-linear are also faces of $\mathcal{P}_N$.

Example 4.32. For the two partitions given in Figure 4.3, the cell complex obtained from this construction coincides precisely with what is shown in the picture, where every half-linear 1-face is subdivided in two segments and every non-pointed region forms a 2-cell with four nodes and four 1-faces on the boundary. For a more interesting example, consider the 1-partition of $\mathbb{R}^2$ into one region. This "partition" is non-essential, with minimal face $F_{I(\mathbf{0})} = F_{1\infty}$ of dimension one, equals to the boundary of $\overline{S}^d_+$ (this face is homeomorphic to S^1 and cannot be a cell). There is also one half-linear face F_1, that coincides with $\overline{S}^d_+$. Therefore a node system here would have four nodes, three on the boundary face $F_{I(\mathbf{0})}$ that positively span the plane containing that face, and one more node n in the interior of $\overline{S}^d_+$. The cell complex in this case is obtained by first taking the spherical convex hull of every subset of nodes in the boundary with two or less elements, that form a subdivision of $F_{I(\mathbf{0})}$ in three edges and three vertices, and then taking the pyramid over all those faces, with apex on the interior node n, to obtain a cell decomposition as shown in Figure 4.4.

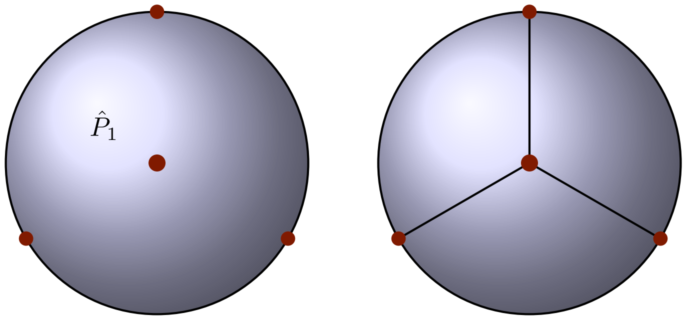

FIGURE 4.4: Node system and cell complex $\mathcal{P}_N$ corresponding to the partition of $\mathbb{R}^2$ with only one region

Lemma 4.33. *The complex $\mathcal{P}_N$ is a regular CW complex homeomorphic to a d-ball.*

Proof. If $k = \dim F_{I(\mathbf{0})}$, the $k+2$ nodes placed on $F_{I(\mathbf{0})}$ together with all faces of $\mathcal{P}_N$ that are the spherical convex hull of some of these nodes will cover $F_{I(\mathbf{0})}$, since the $k+2$ nodes in $F_{I(\mathbf{0})}$ where chosen to be positively spanning. Each face F of $\mathcal{P}_N$ contained in $F_{I(\mathbf{0})}$ is homeomorphic to a ball and its boundary will be covered by the faces generated by subsets of the nodes in F.

Also the pyramidal faces of $\mathcal{P}_N$ contained in any half-linear face F will be strictly convex spherical polyhedra, homeomorphic to balls and with its boundary covered by the faces of $\mathcal{P}_N$ contained in the basis of the pyramid, and the pyramid over the faces on the boundary of the basis. The union of all these faces cover F.

Since faces of $\mathcal{P}$ are homeomorphic to balls, with the unique exception of $F_{I(\mathbf{0})}$ (as discussed in the proof of Theorem 3.24), all faces of $\mathcal{P}_N$ that come from faces of $\mathcal{P}$ are also balls, with boundary covered by faces of $\mathcal{P}$. Since all faces of $\mathcal{P}$ are covered by faces of $\mathcal{P}_N$, we conclude that the union of faces in $\mathcal{P}_N$ is $\overline{S}^d_+$ and $\mathcal{P}_N$ is a CW complex homeomorphic to a d-ball, which is regular since all cells are strictly convex spherical polyhedra and there are no identifications on the boundaries of the faces. □

Proposition 4.34. *For a pointed partition $\mathcal{P}$, the complex $\mathcal{P}_N$ coincides with the cell complex $\mathcal{P}$ described in Theorem 3.24. The vertices of $\mathcal{P}_N$ are precisely the vertices of $\mathcal{P}$.*

Proof. For essential partitions all vertices are half-linear faces, and the corresponding node must be precisely at the vertex. If the partition $\mathcal{P}$ is pointed, there are no other half-linear faces, since the cone of a half-linear face F of dimension $\dim F \geq 1$ contains antipodal points on its boundary and therefore is not pointed. Then no other nodes are included, and all faces of $\mathcal{P}_N$ are precisely the faces of $\mathcal{P}$, so that we end up with the same complex. □

Lemma 4.35. *Let $\mathcal{P}$ be a fixed n-partition. Then combinatorial structure of the complex $\mathcal{P}_N$ doesn't depend on the choice of the nodes in N, i. e. the face poset of two complexes $\mathcal{P}_N$ and $\mathcal{P}_{N'}$ is always isomorphic, for any pair of node systems $N, N' \in N(\mathcal{P})$.*

Proof. The face poset of $\mathcal{P}_N$ can be obtained from the face poset of $\mathcal{P}$, once we know which are the linear and half-linear faces, independently of the choice of the node system N. Following the construction in Definition 4.31, we can see how to obtain recursively the face poset of $\mathcal{P}_N$ from the face poset of $\mathcal{P}$.

First instead of the minimal element of the face poset of $\mathcal{P}$ we include a Boolean poset on $k+2$ vertices without a top element (where $k = \dim F_{I(\mathbf{0})}$), that is represented by all subsets of the set $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_{k+2}\}$ with size smaller than $k+1$. Then replace in order each half-linear face F by a copy of the elements below F in the current poset, starting from the faces of smaller dimensions. Each copied element covers its original element in the poset. This represents the new faces that are pyramids over faces $G \subset F$ of the current poset (and therefore faces of $\mathcal{P}_N$). At the end we get the face poset of the complex $\mathcal{P}_N$, regardless of the choice of the nodes in N. □

Definition 4.36 (Node frame, node basis and flats)**.** Let $\mathcal{P}$ be an n-partition, together with a node system N. A *node frame* of N is a list $(\boldsymbol{v}_0, \dots, \boldsymbol{v}_d)$ of $d+1$ different nodes in N such that the nodes $\boldsymbol{v}_0, \dots, \boldsymbol{v}_k$ are contained on a k-face G_k of $\mathcal{P}_N$ and the faces $G_0 \subset \cdots \subset G_d$ form a flag. A *node basis* is a node frame whose vectors are linearly independent and a *flat* is a node frame whose vectors are linearly dependent.

Since the vertices of $\mathcal{P}_N$ are precisely the nodes in N and the face poset of $\mathcal{P}_N$ is always the same for any node system N, then for any node frame $(\boldsymbol{v}_0, \dots, \boldsymbol{v}_d)$ and any other node system N' of $\mathcal{P}$, the corresponding list of nodes $(v_0(N'), \dots, v_d(N'))$ is a node frame of N'. Also two partitions $\mathcal{P}$ and $\mathcal{P}'$ with the same face poset and the same corresponding half-linear faces will have a bijection between node frames, for any pair of node systems on them. This is clear since node frames can be read completely from the combinatorial structure of $\mathcal{P}_N$.

Lemma 4.37. *Let $G_0 \subset \cdots \subset G_d$ be a complete flag of faces in the complex $\mathcal{P}_N$. Then for any list $\boldsymbol{x}_0, \dots, \boldsymbol{x}_k$ of linearly independent vectors in S^d such that $\boldsymbol{x}_i \in G_i$ for all*

$0 \leq i \leq d$, the sign of the determinant $\det(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_d)$ *depends uniquely on the flag* $G_0 \subset \cdots \subset G_d$.

Proof. Let $\boldsymbol{b}_0, \ldots, \boldsymbol{b}_d$ be the basis of $\mathbb{R}^{d+1}$ where $\boldsymbol{b}_0 \in G_0$ and every $\boldsymbol{b}_i$ for $0 < i \leq d$ is the vector in the linear space spanned by the face G_i orthogonal to the subspace spanned by G_{i-1}, such that any point $\boldsymbol{x} \in G_i$ satisfy the inequality $\boldsymbol{b}_i \cdot \boldsymbol{x} \geq 0$. This basis is uniquely defined by the flag $G_0 \subset \cdots \subset G_d$.

Then the vectors $(\boldsymbol{b}_0, \ldots, \boldsymbol{b}_i)$ span the same linear subspace as the face G_i. In terms of this basis, the list of vectors $(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_d)$ is represented by an upper triangular matrix

$$\begin{pmatrix} 1 & a_{01} & \cdots & a_{0d} \\ 0 & a_{11} & \cdots & a_{1d} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{dd} \end{pmatrix}$$

where all diagonal entries a_{ii} are greater than zero. Then we conclude that

$$\det(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_d) = (\prod_{i=1}^{d} a_{ii}) \det(\boldsymbol{b}_0, \ldots, \boldsymbol{b}_d).$$

will always have the same sign, independently of the choice of the points $\boldsymbol{x}_i$. (This determinant cannot be zero since we require that the vectors $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_d$ are linearly independent.) □

4.4 Combinatorial types and realization spaces

Now we will explain a second approach to prove that $\mathcal{C}(\mathbb{R}^d, n)$ is a union of semialgebraic pieces. With the different tools we have now, we can now define when two partitions are combinatorially equivalent, and use this to construct the realization space of any partition $\mathcal{P}$ (made by all partitions that are combinatorially equivalent to $\mathcal{P}$). This will be useful in the discussion about the dimension of the spaces of convex n-partitions.

Given an n-partition $\mathcal{P}$, we want to describe all n-partitions that are combinatorially equivalent to $\mathcal{P}$. Two partitions $\mathcal{P}$ and $\mathcal{P}'$ have the same face poset if $\mathcal{I}(\mathcal{P}) = \mathcal{I}(\mathcal{P}')$. They have the same corresponding half-linear faces if the indices $I \in \mathcal{I}(\mathcal{P})$ such that $F_I(\mathcal{P})$ is half-linear are the same indices for which $F_I(\mathcal{P}')$ is half-linear.

Definition 4.38 (Orientation of a partition)**.** The *orientation* of a partition $\mathcal{P}$ of $\mathbb{R}^d$ is given by the signs of the determinants $\det(\boldsymbol{v}_0, \ldots, \boldsymbol{v}_d)$ of all node frames of a node system N of $\mathcal{P}$.

Orientations of partitions are closely related with orientations of cell complexes. If we consider the barycentric subdivision sd $\mathcal{P}_N$ obtained by taking a point $\boldsymbol{y}_G$ in the relative interior of each face G of $\mathcal{P}_N$, and with maximal simplices that are the spherical convex hull of sets $\boldsymbol{y}_{G_0}, \ldots, \boldsymbol{y}_{G_d}$ for each complete flag $G_0 \subset \cdots \subset G_d$ in $\mathcal{P}_N$, then by Lemma 4.37, we can read an orientation of the simplicial complex sd $\mathcal{P}$ from the orientation of $\mathcal{P}$.

Since orientations of oriented simplicial complexes are determined after fixing the orientation of one simplex, then it is enough to know the sign of one node basis to determine the sign of all other node bases of $\mathcal{P}_N$. In particular, if $\mathcal{P}$ is an essential partition, then any node system on $\mathcal{P}$ will give rise to the same orientation. If $\mathcal{P}$ is non-essential, there are two possible orientations, depending on the choice of the nodes on the minimal face $F_{I(\mathbf{0})}$.

Orientations also keep track of which node frames are node basis and which are flats. Two partitions $\mathcal{P}$ and $\mathcal{P}'$ with the same face poset and corresponding half-linear faces have the same orientation if there are node systems N and N' on each of them, so that the sign of the determinants of corresponding node basis are always the same, and they have the same corresponding flats.

Definition 4.39 (Combinatorial type of a partition)**.** The *combinatorial type* of an n-partition $\mathcal{P}$ consists in the following information: the set $\mathcal{I}(\mathcal{P})$ of labels of the face poset, the set of half-linear faces of $\mathcal{P}$ and the orientation given by a node system of $\mathcal{P}$.

The orientation allows us to differentiate the combinatorial type of an essential partition and of its reflection on a hyperplane. If a partition has some reflection symmetry, it implies that it is non-essential. Orientations also make sure that combinatorially equivalent partitions have the same π-angles, as defined next.

Definition 4.40 (π-angles)**.** Two $(d-1)$-faces F_{ij} and F_{ik} make a π*-angle* if they belong to the same $(d-1)$-subface of a d-face F_i of $\mathcal{P}$ and their intersection is $(d-2)$-dimensional. It means that the dihedral angle between these two $(d-1)$-faces is equal to π.

Two partitions have the same *shape* if they are combinatorially equivalent up to a permutation of the indices from 1 to n. If a partition contains empty regions, it will have the same shape than other partitions with smaller number of regions.

For our next result (Theorem 4.42) we need a characterization for cone partitions from Firla and Ziegler [14, Theorem 4]. A *cone partition* (or simply a partition on that paper) of a cone C is a collection of cones $C_1, \cdots, C_r$ contained in C such that every point of C is contained in one of the subcones C_i, where also the intersection of any two subcones $C_i \cap C_k$ is a face of both cones.

Theorem 4.41 (Firla–Ziegler, [14])**.** *A set of cones $C_1, \cdots, C_r$ contained in a bigger cone $C \subset \mathbb{R}^{d+1}$ form a cone partition of C if and only if there is a generic point g contained in exactly one of the cones C_k, and for any d-face F of a $(d+1)$-cone C_i that is not contained in the boundary of C there is a second cone C_j with $C_i \cap C_j = F$ such that F is a face of C_j.*

Theorem 4.42. *Let $\mathcal{P}$ be a partition of $\mathbb{R}^d$ together with a node system N. Consider a list of vectors $\boldsymbol{x_v} \in \mathbb{R}^{d+1}$ for every node $\boldsymbol{v} \in N$ that satisfy the following algebraic relationships and inequalities:*

(i) $|\boldsymbol{x_v}| = 1$ *for every $\boldsymbol{v}$ vertex of $\mathcal{P}$.*

(ii) $\det(\boldsymbol{x}_{\boldsymbol{v}_0}, \ldots, \boldsymbol{x}_{\boldsymbol{v}_d}) > 0$*, for every node basis $(\boldsymbol{v}_0, \ldots, \boldsymbol{v}_d)$ with $\det(\boldsymbol{v}_0, \ldots, \boldsymbol{v}_d) > 0$.*

(iii) $\det(\boldsymbol{x}_{\boldsymbol{v}_0}, \ldots, \boldsymbol{x}_{\boldsymbol{v}_d}) = 0$*, for every node flat $(\boldsymbol{v}_0, \ldots, \boldsymbol{v}_d)$.*

(iv) $\boldsymbol{e}_0 \cdot \boldsymbol{x_v} = 0$*, for any node $\boldsymbol{v} \in N$ at infinity (i.e. in the boundary of S^d_+).*

(v) $\boldsymbol{e}_0 \cdot \boldsymbol{x_v} > 0$*, for any other node $\boldsymbol{v} \in N$, not at infinity.*

Assume also that there is a point $g \in \mathbb{R}^{d+1}$ that is generic (i. e. not contained in a hyperplane spanned by d vectors $\boldsymbol{x}_{\boldsymbol{v}_i}$) that belongs to the interior of exactly one of the cones spanned by all vectors $\boldsymbol{x_v}$ corresponding to the nodes $\boldsymbol{v}$ that belong to a d-face of $\mathcal{P}_N$.

Then there is a partition $\mathcal{P}'$ that is combinatorially equivalent to $\mathcal{P}$ with a node system given by the points $x_{\boldsymbol{v}}$ for $\boldsymbol{v} \in N$.

Proof. We want to see first that we can construct a regular CW complex $\mathcal{P}_X$ by taking a face G' for each face G in $\mathcal{P}_N$, where G' is the spherical convex hull of the points $\boldsymbol{x_v}$ for all nodes $\boldsymbol{v} \in G$. Then we will obtain the partition $\mathcal{P}'$ out of the complex $\mathcal{P}_X$.

Consider the barycentric subdivision $\operatorname{sd}\mathcal{P}_N$ of the complex $\mathcal{P}_N$ obtained by taking points $\boldsymbol{y}_G$ in the relative interior of each face G of $\mathcal{P}_N$. The maximal simplices of $\operatorname{sd}\mathcal{P}_N$ correspond to complete flags $G_0 \subset \cdots \subset G_d$ in $\mathcal{P}_N$ and have $\boldsymbol{y}_{G_0}, \ldots, \boldsymbol{y}_{G_d}$ as vertices. Then take a point $\boldsymbol{y}'_G$ in the relative interior of each spherical polyhedral set G' in $\mathcal{P}_X$. We want to see that if we construct the family S_X of simplicial cones over the sets $\boldsymbol{y}'_{G_0}, \ldots, \boldsymbol{y}'_{G_d}$ for each complete flag $G_0 \subset \cdots \subset G_d$ in $\mathcal{P}_N$, then we obtain a cone partition of the upper halfspace of $\mathbb{R}^{d+1}$ (with first coordinate $x_0 \geq 0$), by making use of Theorem 4.41.

By assumption, there is a generic vector g contained in exactly one of the cones spanned by the vectors $\boldsymbol{x_v}$ for all nodes v that belong to a d-face G of the complex $\mathcal{P}_N$. This is precisely the cone over the spherical polyhedron G'.

Lemma 4.43. *Let G be a d-face of $\mathcal{P}_N$. Then the algebraic relationships and inequalities for node frames (of type (ii) and (iii)) imply that G' is combinatorially equivalent to G as a polyhedral cone.*

Proof. The relationships of type *(iii)* coming from flats tell us that the points $\boldsymbol{x_v}$ corresponding to nodes $\boldsymbol{v}$ on the same d-subface of G are all on the same hyperplane and the inequalities of type *(ii)* for node bases tell us that this hyperplane defines a facet of G'. Moreover, for each node $\boldsymbol{v}$, the set of facets on G where it belongs must be similar to the set of facets of G' where the point $\boldsymbol{x_v}$ is contained.

We can tell which nodes are vertices of G from the set of facets where each node belong. Vertices are in the maximal sets under inclusion, because if a node $\boldsymbol{v}$ is not a vertex, the set $A_{\boldsymbol{v}}$ of facets of G containing $\boldsymbol{v}$ is determined by the subface of G that contains it, and this is a subset of the set $A_{\boldsymbol{v}'}$ of facets of G containing a vertex $\boldsymbol{v}'$ of that subface. Therefore G and G' have the same vertex-facet incidences, and this imply that they are combinatorially equivalent (this is a direct consequence of the analogous result for convex polytopes, see [36, Chapter 2]). □

This together with Lemma 4.43 implies that the cone over G' is subdivided by all cones of the form $\text{cone}(\boldsymbol{y}'_{G_0}, \ldots, \boldsymbol{y}'_{G_d})$ where $G_0 \subset \cdots \subset G_d$ is a complete flag on $\mathcal{P}_n$ with $G = G_d$, and we conclude that if the vector g is generic enough, it will belong to the interior of exactly one of the subcones $\text{cone}(\boldsymbol{y}'_{G_0}, \ldots, \boldsymbol{y}'_{G_d})$ corresponding to a complete flag with $G = G_d$. By a similar argument, if $G_d \neq G$, it is not possible that g belong to any other cone corresponding to a flag ending in G_d and g is in the interior of a unique cone from S_X. This implies that the cone over G' is subdivided by all cones of the form $\text{cone}(\boldsymbol{y}'_{G_0}, \ldots, \boldsymbol{y}'_{G_d})$ where $G_0 \subset \cdots \subset G_d$ is a complete flag on $\mathcal{P}_n$ with $G = G_d$, and we conclude that if the vector g is generic enough, it will belong to the interior of exactly one of the subcones $\text{cone}(\boldsymbol{y}'_{G_0}, \ldots, \boldsymbol{y}'_{G_d})$ corresponding to a complete flag with $G = G_d$. By a similar argument, if $G_d \neq G$, it is not possible that g belong to any other cone corresponding to a flag ending in G_d and g is in the interior of a unique cone from S_X.

Now we want to see that for any d-face F of a $(d+1)$-cone C_i in S_X that is not contained in the boundary of the upper halfplane in $\mathbb{R}^{d+1}$ there is a second cone C_j with $C_i \cap C_j = F$ such that F is a face of C_j. Notice that the cones spanned by $\boldsymbol{y}_{G_0}, \ldots, \boldsymbol{y}_{G_d}$ form a simplicial cone partition S_N of the upper halfspace of $\mathbb{R}^{d+1}$, since they arise from a barycentric subdivision of $\mathcal{P}_N$.

Lemma 4.44. *The determinant* $\det(\boldsymbol{y}'_{G_0}, \ldots, \boldsymbol{y}'_{G_d})$ *have the same sign as the determinant* $\det(\boldsymbol{y}_{G_0}, \ldots, \boldsymbol{y}_{G_d})$ *for any node flag* $G_0 \subset \ldots \subset G_d$.

Proof. By Lemma 4.37 we know that the sign of the determinant $\det(\boldsymbol{y}_{G_0}, \ldots, \boldsymbol{y}_{G_d})$ is the same than the sign of $\det(\boldsymbol{v}_0, \ldots, \boldsymbol{v}_d)$ for any node basis $(\boldsymbol{v}_0, \ldots, \boldsymbol{v}_d)$ in N with $\boldsymbol{v}_i \in G_i$.

By the algebraic conditions on the $\boldsymbol{x}_{\boldsymbol{v}}$, this sign is also the same as the determinant $\det(\boldsymbol{x}_{v_0}, \ldots, \boldsymbol{x}_{v_d})$ for any node basis $(\boldsymbol{v}_0, \ldots, \boldsymbol{v}_d)$ in N with $\boldsymbol{v}_i \in G_i$. We want to see that this is also the same sign of the determinant $\det(\boldsymbol{y}'_{G_0}, \ldots, \boldsymbol{y}'_{G_d})$.

The fact that $\boldsymbol{y}'_G \in \operatorname{relint} G'$ can be expressed by a linear combination

$$\boldsymbol{y}'_G = \sum_{\boldsymbol{v} \in N \cap G} \alpha_{\boldsymbol{v}} \boldsymbol{x}_{\boldsymbol{v}},$$

where all $\alpha_{\boldsymbol{v}} > 0$. Since determinants are multilinear, we can expand it as follows.

$$\det(\boldsymbol{y}'_{G_0}, \ldots, \boldsymbol{y}'_{G_d}) = \sum \Big(\big(\prod_{i=0}^{d} \alpha_{\boldsymbol{v}_i} \big) \det(\boldsymbol{x}_{v_0}, \ldots, \boldsymbol{x}_{v_d}) \Big)$$

where the sum goes over all lists $(\boldsymbol{v}_0, \ldots, \boldsymbol{v}_d)$ such that $\boldsymbol{v}_i \in G_i$, namely the node systems on the flag $G_0 \subset \cdots \subset G_d$. We can see that all summands on the right have the same sign as $\det(\boldsymbol{y}_{G_0}, \ldots, \boldsymbol{y}_{G_d})$ or are zero. □

Lemma 4.44 imply that two adjacent cones in S_X don't overlap on their interiors, since the corresponding cones in S_N don't overlap. All d-faces of S_X corresponding to faces on the boundary of S_N are also in the boundary of the upper halfspace (due to relationships of type *(iv)*) while a d-face of a cone $C_i \in S_X$ corresponding to an interior d-face of S_N are always interior (due to the inequalities of type *(v)*), and by the lemma we can find that there is a second cone in S_X such that its intersection with C_i is the corresponding d-face, by looking at the cone with analogous property in S_N.

Now we are in conditions to use Theorem 4.41 to conclude that the cones in S_X don't overlap and make a cone partition of the upper hemisphere.

Each of the faces G' of $\mathcal{P}_X$ can be obtained as the intersection with S^d of the union of the cones over sets $\boldsymbol{y}'_{G_0}, \ldots, \boldsymbol{y}'_{G_k}$ where $G_0 \subset \ldots \subset G_k = G$ are partial flags on $\mathcal{P}_N$, and then the faces of $\mathcal{P}_X$ are obtained by gluing some of the simplices from the spherical simplicial complex $S_X \cap S^d$ (the interior of each face of the simplicial complex $S_X \cap S^d$ belongs only to the relative interior of the maximal face G_k from the labels of the generators $\boldsymbol{y}'_{G_k}$ of the cone). Then $\mathcal{P}_X$ is a CW complex since the relative interiors of its faces are pairwise disjoint and that the boundary of each face G' is covered by the faces of $\mathcal{P}'_X$ contained in G'.

By construction, we have inclusion between faces $G'_1 \subset G'_2$ if and only if the corresponding faces in $\mathcal{P}_N$ satisfy that $G_1 \subset G_2$. Therefore the resulting complex $\mathcal{P}_X$ will have the same face poset as $\mathcal{P}_N$. Half-linear faces F of $\mathcal{P}$ can be obtained as union of faces of $\mathcal{P}_N$, and the union F' of the corresponding faces of $\mathcal{P}_N$ have to be in a linear subspace of the right dimension, due to equations of type *(iii)* that tell that points $\boldsymbol{x_v}$ for $\boldsymbol{v} \in F$ have to be coplanar for all facets of all regions of $\mathcal{P}$ containing F, and besides, F' have in the boundary the same faces at infinity as F (due to equations of type *(iv)*), so they will be the half-linear faces for a new partition $\mathcal{P}'$ that have as faces in its spherical representation the same faces as $\mathcal{P}_X$, but gluing together those faces corresponding to the same half-linear face.

The fact that $\mathcal{P}'$ is a partition of $\mathbb{R}^d$ is a consequence that S_X is a cone partition of the upper halfspace. By Lemma 4.44 we can find that the $\mathcal{P}$ and $\mathcal{P}'$ have the same orientations, and we conclude that they are combinatorially equivalent as we wanted. □

The condition of the existence of a point g in the interior of only one of the d-faces G is important and cannot be omitted. To see this, consider a 5-partition of $\mathbb{R}^2$ as in the left of Figure 4.5, and the choice of points $x_{\boldsymbol{v}_i}$ depicted on the right. For simplicity we called the vertices $\boldsymbol{v}_i$ and all nodes are vertices since the partition is pointed. In that example, all conditions from Theorem 4.42 are satisfied, except the existence of the point g. In this case we get that the expected spherical regions form a double covering of the upper hemisphere.

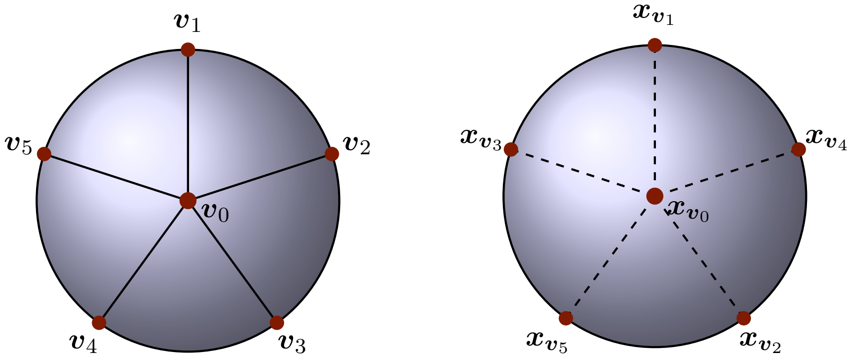

FIGURE 4.5: Nodes of a 5-partition together with points $\boldsymbol{x_v}$ that satisfy all algebraic relationships and inequalities in Theorem 4.42 but don't make a new 5-partition.

Proposition 4.45. *Let $\mathcal{P}$ be an n-partition of $\mathbb{R}^d$. The space of pairs $(\mathcal{P}', N')$ of partitions $\mathcal{P}'$ combinatorially equivalent to together with a node system N' on $\mathcal{P}'$ is a semi-algebraic set.*

Proof. On Theorem 4.42 it is given the algebraic description by equations and inequalities of a set that parameterize all these pairs, under the condition of the existence of the point g. Notice that if a partition $\mathcal{P}'$ is combinatorially equivalent to $\mathcal{P}$, then any node system give rise to an equivalent system of equations and inequalities, and therefore it satisfies the system given by $\mathcal{P}$. The condition about the point g can be also given as a system of algebraic conditions after introducing new slack variables for g. We recall that unions and intersections of semialgebraic sets are semialgebraic. Then by Theorem 4.6 we find that the set we are interested in is semialgebraic. □

Definition 4.46 (Realization spaces)**.** The *realization space* of an n-partition $\mathcal{P}$ is the subspace of $\mathcal{C}(\mathbb{R}^d, n)$ of all partitions $\mathcal{P}'$ with the same combinatorial type as $\mathcal{P}$. It is denoted as $\mathcal{C}_{\mathcal{P}}(\mathbb{R}^d, n)$.

Theorem 4.47. *Let $\mathcal{P}$ be an n-partition of $\mathbb{R}^d$. Then the realization space $\mathcal{C}_{\mathcal{P}}(\mathbb{R}^d, n)$ is a semialgebraic set.*

Proof. Proposition 4.45 shows that for pointed partitions $\mathcal{P}$ the space $\mathcal{C}_{\mathcal{P}}(\mathbb{R}^d, n)$ is semialgebraic, since all vertices are nodes, and there is a unique node system on each partition in the realization space. In general, the realization space of $\mathcal{P}$ can be obtained as the image of the space of pairs $(\mathcal{P}', N')$ described in Proposition 4.45 to the space $\mathbb{R}^{h(d+1)}$ describing by the equations of the h hyperplanes that define $(d-1)$-faces of the partition, where each partition corresponds a unique point. We make use of an equivalent formulation of Theorem 4.6 that claims that the image under a polynomial mapping $f : \mathbb{R}^m \to \mathbb{R}^{m'}$ of a semi-algebraic set is semi-algebraic. □

This result gives us an alternative proof of the fact that spaces of n-partitions $\mathcal{C}(\mathbb{R}^d, n)$ are union of semialgebraic pieces since the union of all realization spaces of n-partitions of $\mathbb{R}^d$ is equal to $\mathcal{C}(\mathbb{R}^d, n)$.

4.5 Examples

We will analyze here the easiest examples of spaces of n-partitions, namely what happens for small values of n and d. The easiest case is when $n = 1$. In that case our space of partitions $\mathcal{C}(\mathbb{R}^d, 1)$ will simply consists of one point. A more interesting example is what happens for $n = 2$.

Proposition 4.48. *The space $\mathcal{C}(\mathbb{R}^d, \leq 2)$ is homeomorphic to the sphere S^d. The space of partitions $\mathcal{C}(\mathbb{R}^d, 2)$ is homotopy equivalent to S^{d-1} and is obtained from $\mathcal{C}(\mathbb{R}^d, \leq 2)$ by removing two points.*

Proof. To parameterize our space of 2-partitions for fixed d we only need to choose the coordinates $\boldsymbol{c}_{1,2}$, that describe the normal to the hyperplane H_{ij} by a point in S^n.

Two special cases have to be taken into account that characterize the cases when the combinatorial type of the 2-partition is not the generic one. These are precisely when $\boldsymbol{c}_{ij} = \pm(1,0,\ldots,0)$. In those cases, there is no hyperplane in $\mathbb{R}^d$, representing the partitions with only one (labeled) non-empty region. These extreme partitions can be obtained as a limit of proper 2-partitions, and S^d will handle the topological structure of $\mathcal{C}(\mathbb{R}^d, \leq 2)$ in the right way. □

For $n \geq 3$, things begin to be more complicated, even in the case of $d = 1$.

Proposition 4.49. *The space $\mathcal{C}(\mathbb{R}^1, \leq n)$ is homeomorphic to a CW complex with n vertices and $k!\binom{n}{k}$ simplicial $(k-1)$-cells for $0 \leq k \leq n$. It is made out of $n!$ simplices of dimension $(n-1)$ glued appropriately on the boundaries. The space $\mathcal{C}(\mathbb{R}^1, n)$ is homeomorphic to $n!$ open $(n-1)$-balls.*

Proof. For a combinatorial type with k non-empty regions, its realization space is contractible and can be realized as a $(k-1)$-simplex. To do this, take an order preserving homeomorphism from $\mathbb{R}$ to the open interval $(0,1)$. Then the coordinates of the $k-1$ interior vertices (hyperplanes!) $\boldsymbol{v}_{i,j} = F_{i,j} \in \mathbb{R}$ need to be in a prescribed order, and via the homeomorphism we can map any partition to a point inside a $(k-1)$ -simplex contained in the unit cube $(0,1)^{k-1}$.

For example, if the n-partition have the region i at the left of region $i+1$ for all $i < n$ (and no empty region) then we only need to specify the coordinates of the vertices $\boldsymbol{v}_{i,i+1}$ such that $\boldsymbol{v}_{1,2} \leq \ldots \leq \boldsymbol{v}_{n-1,n}$. Mapping these $n-1$ values to the unit cube $(0,1)^{n-1}$ via the homeomorphism, we identify the realization space of this particular n-partition with the interior of an $(n-1)$-simplex.

The boundary of each of those simplices will represent the case when some of the points coincide, and can be naturally identified with the realization spaces of other combinatorial types with some extra empty regions. In this way we give to $\mathcal{C}(\mathbb{R}^1, \leq n)$ the structure of a regular cell complex (start with n vertices corresponding to the realization spaces of partitions with only one non-empty region, and then for higher dimensions, identify the boundary with a subspace of the union of the cells of smaller dimension).

There will be $n!$ combinatorial types without empty regions. The space $\mathcal{C}(\mathbb{R}^1, n)$ is the union of the interior of all those simplices. All other combinatorial types can be obtained in the limit (in the boundary) of those proper combinatorial types and therefore $\mathcal{C}(\mathbb{R}^1, \leq n)$ will have $n!$ top-dimensional simplicial $(n-1)$-cells, and $\binom{n}{k}k!$ cells of dimension $k-1$. □

Example 4.50. The space $\mathcal{C}(\mathbb{R}^1, \leq 3)$ is homeomorphic to a two dimensional space made out topologically by gluing six simplices along the boundaries in a special way, since there are two different edges joining each pair of vertices. The vertices represent the partitions with one non-empty region, and the edges represent the partitions with two non-empty regions.

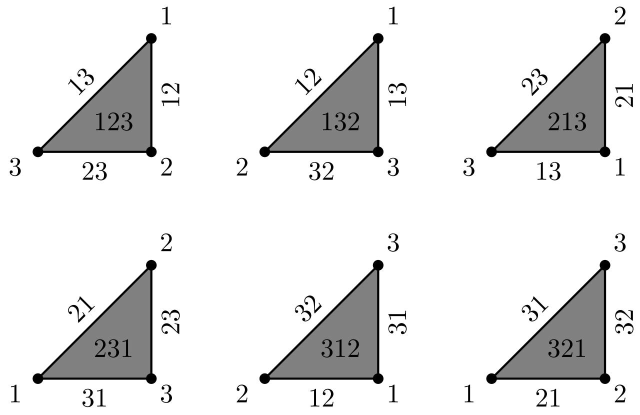

FIGURE 4.6: Simplices to build a cell complex homeomorphic to $\mathcal{C}(\mathbb{R}^1, 3)$.

On Figure 4.6 we can see the six simplices of this CW complex with the corresponding labels on the different cells. These simplices have to be glued along the edges corresponding to the same partitions, in such way that the corresponding vertices coincide. Each edge appears in three of the simplices

The next examples of spaces of n-partitions we completely describe are $\mathcal{C}(\mathbb{R}^2, 3)$ and $\mathcal{C}(\mathbb{R}^2, \leq 3)$; they will be discussed in detail in Section 7.3.

Chapter 5

Regular n-partitions

Regular partitions are very important and much better understood than general partitions. They are related with many other interesting objects and constructions in discrete geometry, as for example *weighted Voronoi partitions*, *k-stresses* and *reciprocals* (see [22], [31]). These different descriptions help to parameterize the subspace of regular partitions, making easy to generate many partitions that can be controlled nicely in order to obtain partitions with special properties.

Definition 5.1 (Regular partitions)**.** A partition $\mathcal{P} \in \mathcal{C}(\mathbb{R}^d, \leq n)$ is *regular* if it can be obtained as the projection of a $(d+1)$-dimensional convex polyhedron Q to $\mathbb{R}^d$, where the regions of $\mathcal{P}$ are the image of the interior of the facets of Q under the projection. Such a polyhedron Q is called a *convex lifting* of $\mathcal{P}$.

Definition 5.2. We denote by $\mathcal{C}_{\text{reg}}(\mathbb{R}^d, n)$ the subspace of all regular partitions in $\mathcal{C}(\mathbb{R}^d, n)$. Similarly $\mathcal{C}_{\text{reg}}(\mathbb{R}^d, \leq n)$ is the subspace of all regular partitions in $\mathcal{C}(\mathbb{R}^d, \leq n)$.

A main question we would like to understand is if there are any differences between the spaces $\mathcal{C}_{\text{reg}}(\mathbb{R}^d, \leq n)$ and $\mathcal{C}(\mathbb{R}^d, \leq n)$ at the level of homology or homotopy, for fixed n and d.

5.1 Dimension of the subspace of regular partitions

We can introduce coordinates to $\mathcal{C}_{\text{reg}}(\mathbb{R}^d, n)$ by parameterizing the hyperplanes that describe a polyhedral lifting of a partition. Suppose that $\mathcal{P}$ is a regular partition on $\mathcal{C}_{\text{reg}}(\mathbb{R}^d, n)$ that can be obtained by projecting the lower convex hull of a convex polyhedron $Q \in \mathbb{R}^d$. We can parameterize the equation for each facet Q_i of Q (that projects to the region P_i) as a linear inequality of the form $x_0 \geq d_i \cdot \boldsymbol{x} + c_i$, where $d_i \in \mathbb{R}^d$,

$c_i \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ denote the extra height coordinate of a point $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, so that $(x_0, \dots, x_d) \in \mathbb{R}^{d+1}$ and the projection of Q to $\mathbb{R}^d$ is made by deleting the first coordinate. Suppose also that the facet of Q bounded by the hyperplane $x_0 = d_i \cdot \boldsymbol{x} + c_i$ is projected down to the region P_i.

Definition 5.3 (Simple partitions)**.** An essential n-partition $\mathcal{P}$ of $\mathbb{R}^d$ is *simple* if each k-face F_I satisfies that $|I| = d - k + 1$, for all $k \geq 0$. In particular, each vertex belong to $d+1$ maximal faces of $\widehat{\mathcal{P}}$.

This definition of simple n-partitions also takes into account all faces at infinity. For example, the partition in the Example 3.7 is not simple, since there are vertices at infinity contained in too many regions.

Proposition 5.4. *For $d \geq 2$ and $n \geq 2$, the space $\mathcal{C}_{\text{reg}}(\mathbb{R}^d, n)$ of regular partitions is a semialgebraic set of dimension*

$$\dim \mathcal{C}_{\text{reg}}(\mathbb{R}^d, n) = (d+1)(n-1) - 1.$$

Proof. For each of these d-faces of $\mathcal{P}$ we have $d+1$ parameters to choose the equation of an affine hyperplane for the facet Q_i (namely d_i and c_i) of the lifted polyhedron. However, if we add the same linear function to each equation, we will get at the end the same partition. We can therefore assume that the first region is generated by a fixed hyperplane. Also we can scale all equations by the same value, and get at the end the same partition after the projection. In that way we get the dimension count of $\dim \mathcal{C}_{\text{reg}}(\mathbb{R}^d, n) \leq (d+1)(n-1) - 1$. In case that the partition is simple and all facets of Q appear in the lower convex hull, no other degrees of freedom leave the partition invariant. This is true, because there is a unique possible lifting of a partition once we fix the equation of a one facet of Q and a dihedral angle with a facet corresponding to a neighboring region (see Rybnikov [31, Theorem 8.3]). □

The cases when $n = 1$ or $d = 1$ don't satisfy this formula. If $n = 1$, then $\dim \mathcal{C}_{\text{reg}}(\mathbb{R}^d, 1) = 0$. If $d = 1$, all partitions are regular, and by Proposition 4.49 we get that $\dim \mathcal{C}_{\text{reg}}(\mathbb{R}^1, n) = n - 1$. In this case the liftings are not unique after fixing one face and a dihedral angle.

Another way to compute the dimension of $\mathcal{C}_{reg}(\mathbb{R}^d, n)$ is via weighted Voronoi partitions. Given n different points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in \mathbb{R}^d$ and real weights $w_1, \dots, w_n \in \mathbb{R}$, the weighted Voronoi partition $\mathcal{P}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n; w_1, \dots, w_n)$ is given by the regions

$$P_i = \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x} - \boldsymbol{x}_i\|^2 - w_i \leq \|\boldsymbol{x} - \boldsymbol{x}_j\|^2 - w_j \text{ for } 1 \leq j \leq n\}.$$

This $\mathcal{P}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n; w_1, \dots, w_n)$ is a regular partition, since the minimum of the linear equations $\|\boldsymbol{x} - \boldsymbol{x}_i\|^2 - \|\boldsymbol{x}\|^2 - w_i$ can be seen as the boundary of a convex polyhedron in one dimension higher, whose facets project to the regions of the generalized Voronoi partition.

Theorem 5.5 (Kantorovich [21]). *Let μ a continuous bounded measure on $\mathbb{R}^d$. For any $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in \mathbb{R}^d$ with $\boldsymbol{x}_i \neq \boldsymbol{x}_j$ for $i \neq j$ and any $m_1, \dots, m_n \in \mathbb{R}^+$ such that $\sum_{i=1}^n m_i = \mu(\mathbb{R}^d)$ there are unique weights $w_1, \dots, w_n \in \mathbb{R}$ such that the regions of the partition*

$$\mathcal{P}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n; w_1, \dots, w_n)$$

have measures $\mu(P_i) = m_i$.

The proof of Theorem 5.5 can be found in [34], [13], but two more proofs are now available: a geometric argument by Geiß et al. [16] and a topological proof by Moritz Firsching (personal communication). Since any regular partition can be obtained as a weighted Voronoi diagram, then we can also parameterize the space of regular n-partitions using point configurations and weights $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n; w_1, \dots, w_n$. This give us a $(d+1)n$ family of parameters, but since we can obtain the same partition after translating and scaling the point configuration (and fixing the weights accordingly) and after adding a constant to all weights, we find once again that $\dim \mathcal{C}_{\text{reg}}(\mathbb{R}^d, n) = (d+1)(n-1) - 1$, as shown in Proposition 5.4.

Already in dimension two, not all partitions are regular. One example of such partition is shown on Figure 5.1. If we imagine that this partition can be obtained as the lower projection of a three dimensional polyhedron, the three unbounded regions would be the projections of facets contained in three planes in $\mathbb{R}^3$ that intersect. These planes intersect pairwise in lines that project to the unbounded edges and these lines intersect where the three planes meet. But then also their projections should intersect. In the example we draw the prolongation of these edges and they don't intersect, an then this cannot be a regular partition.

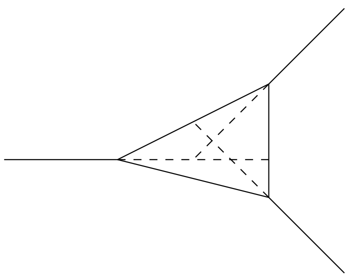

FIGURE 5.1: Non-regular partition on the plane.

For $d = 2$, we will see in Proposition 6.3 that $\dim \mathcal{C}(\mathbb{R}^2, n) = 4n - 7$ for $n > 3$. It means that the dimension of the space of all n-partitions of the plane in that case is much higher than the dimension of the regular ones, since $\dim \mathcal{C}_{\text{reg}}(\mathbb{R}^2, n) = 3n - 4$.

To recognize if a partition of $\mathbb{R}^2$ is regular, there is a projectively invariant criterion in terms of suitable intersections of some lines (generalizing the intersection needed in Figure 5.1) known as the callote condition (Crapo [11]), that can be expressed as an algebraic equation. The following proposition gives a similar criterion, using parallel lines.

Proposition 5.6. *If $\mathcal{P}$ is a simple regular partition of $\mathbb{R}^2$ with a bounded euclidean 2-face E with vertices $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_m$, going around E in that order. Let e_i be the edge at the vertex $\boldsymbol{v}_i$ that is not a face of E, and ℓ_i the straight line in $\mathbb{R}^2$ containing the edge e_i. If we start from a point $\boldsymbol{x}_1$ on the line ℓ_1 and draw a line through $\boldsymbol{x}_1$ parallel to $\boldsymbol{v}_1\boldsymbol{v}_2$ that intersects the line ℓ_2 in a point $\boldsymbol{x}_2$, and continue drawing parallels $\boldsymbol{x}_i\boldsymbol{x}_{i+1}$ to the sides $\boldsymbol{v}_i\boldsymbol{v}_{i+1}$ of E with $\boldsymbol{x}_i \in \ell_i$, then at the end $\boldsymbol{x}_n\boldsymbol{x}_1$ will also be parallel to $\boldsymbol{v}_n\boldsymbol{v}_1$.*

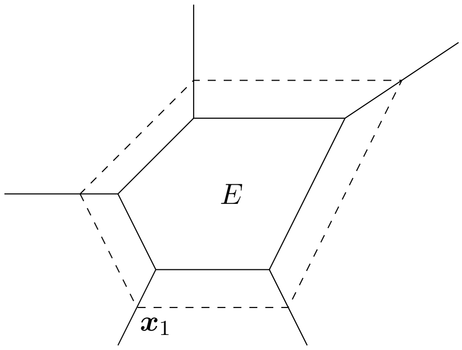

FIGURE 5.2: Parallel lines around a bounded face in a simple regular partition of $\mathbb{R}^2$.

Proof. Since $\mathcal{P}$ is regular, consider a convex lifting Q, that is a 3-dimensional polyhedron. One of its facets F projects to the region E. If we translate the plane H containing the facet F to a parallel plane H' going through the lifting of the point $\boldsymbol{x}_1$, we see inductively that the intersection of H' with the lines ℓ'_i containing the edges of Q (that project to the edges e_i) project to the points $\boldsymbol{x}_i$, because the intersection of the parallel planes H and H' with the facets adjacent to F project to parallel lines in $\mathbb{R}^2$. At the end, we also get that $\boldsymbol{x}_n\boldsymbol{x}_1$ is parallel to $\boldsymbol{v}_n\boldsymbol{v}_1$ for the same reason. □

For three dimensions and more, the situation is very different than for $d = 2$. We will see that for these values of d, simple partitions are regular and they form a dense subset of $\mathcal{C}_{\text{reg}}(\mathbb{R}^d, n)$ (see Theorem 5.11 and Corollary 5.14).

5.2 Generic and simple partitions

We will start by fixing the meaning of generic partitions and discuss some properties of them. One of the main uses of them is Theorem 5.11, that claims that generic simple partitions of $\mathbb{R}^d$ are regular, for $d \geq 3$.

Definition 5.7 (Generic Partitions)**.** A partition $\mathcal{P}$ is *generic* if it is essential and it doesn't have any π-angle (as introduced in Definition 4.40).

A partition $\mathcal{P}$ can be essential, and hence generic, only if $n > d$. In case $n \leq d$ we might require the existence of one $(d-n+1)$-cell to obtain an analogous definition of generic partitions. This is anyway a bit different and not so interesting, since it implies that the partition is the product of a simple n-partition of $\mathbb{R}^{n-1}$ with a linear $(d-n+1)$-space. Therefore we restrict to the case of essential partitions.

Lemma 5.8. *Every generic partition $\mathcal{P}$ is pointed.*

Proof. By Lemma 4.24 it is enough to prove that the recession cone of each region of $\mathcal{P}$ is pointed. First observe that if $\mathcal{P}$ is generic, a region of $\mathcal{P}$ is pointed if and only if it has a vertex on its boundary. If the region contains a vertex, it will be pointed, since the recession cone will be a subset of the cone spanned locally at that vertex. Since no π-angles are allowed, then this local cone must be pointed. On the other hand, if the recession cone is pointed, its apex optimize a linear functional over the cone. There must also be a vertex optimizing that functional over the region.

Assume that there is an unbounded region P_i without any vertex on the boundary. If $\mathcal{P}$ is generic, it is essential and it has at least one vertex $\boldsymbol{v}$ (Lemma 3.15). Choose a path from the interior of a region R_1 with $\boldsymbol{v}$ at the boundary to the interior of P_i, that is generic in the sense that only cross $(d-1)$-faces on the interior, and call $R_1, \ldots, R_k = P_i$ to the regions crossed by that path. Then R_1 is pointed, as well as the $(d-1)$-face $R_1 \cap R_2$, and therefore there should be a vertex on that intersection face. Then R_2 is also pointed, and we can repeat the argument to find that also $R_3, \ldots, R_k = P_i$ are pointed regions. □

Together with Lemma 4.23 we get the following chain of implications on partitions:

$$\text{Generic} \implies \text{Pointed} \implies \text{Essential}.$$

From the proof of Lemma 5.8 it is easy to conclude the following corollary.

Corollary 5.9. *If $\mathcal{P}$ is a generic partition, then the graph induced by interior vertices is connected.*

Proof. For each pointed region, the graph of vertices on the boundary together with the bounded edges is connected, since we can move optimizing that linear functional, and end up over a bounded face with maximal value, that has connected graph.

If we repeat the argument from the proof of Lemma 5.8, then walking from one region to the next we can always find a path on the surface of the first region to a vertex of the next and repeating this we can connect any pair of points. □

We will prove something stronger in Proposition 6.9, after defining the bounded complex of a partition.

Lemma 5.10. *Let $\mathcal{P}$ be a regular partition that is essential. Then $\mathcal{P}$ is generic.*

Proof. We need to see that a regular partition doesn't have π-angles. Suppose that $\mathcal{P}$ have a π-angle between two $(d-1)$-faces F_{ij} and F_{ik} that belong to the same $(d-1)$-subface of a d-face F_i of $\mathcal{P}$, that meet at a $(d-2)$ face F, and let Q be a convex lifting of $\mathcal{P}$, i. e. a convex polyhedron in $\mathbb{R}^{d+1}$ whose facets project to the regions of $\mathcal{P}$. If we compare the dihedral angles in Q at the lifting of the faces F_{ij} and F_{ik} (between the lifting of F_i and the lifting of regions F_j and F_k respectively), we see that if one is bigger than the other then Q cannot be convex, and both must be equal. Therefore the liftings of the d-faces F_j and F_k belong to the same hyperplane in $\mathbb{R}^{d+1}$. Moreover, the only way to get convexity in the lifting is placing all other facets of Q that meet in F different than the lifting of F_i on the same hyperplane where the liftings of F_j and F_k belong. But this cannot be, since it implies that all these regions will be contained in the same facet of Q and this is not a proper lifting. In case that $\mathcal{P}$ is also essential, we can conclude that it is generic. □

The following theorem appeared in Whiteley [35] and was generalized later by Rybnikov [31].

Theorem 5.11 (Whiteley [35])**.** *Let $\mathcal{P}$ be a generic partition of $\mathbb{R}^d$ that is simple, for $d \geq 3$. Then $\mathcal{P}$ is a regular partition.*

In Rybnikov [31] something stronger is proven. He uses a more general setup of d-manifolds PL-realized in $\mathbb{R}^d$, possibly with boundary, and the statement of theorem still holds for 3-simple partitions.

Definition 5.12 (k-simple partitions)**.** Let $0 \leq k \leq d$ be a fixed integer. An n-partition of $\mathbb{R}^d$ is *k-simple* if each $(d-k)$-face F_I satisfies that $|I| = d-k+1$.

A partition $\mathcal{P}$ of $\mathbb{R}^d$ is simple if and only if it is d-simple. Therefore, to check that $\mathcal{P}$ is simple it is enough to see that each interior vertex is contained in $d+1$ maximal faces and each vertex at infinity is in d maximal faces and in F_∞. Also k-simple partitions are always k'-simple for any $k' < k$.

Theorem 5.13 (Rybnikov [31])**.** *Let $\mathcal{P}$ be a generic partition of $\mathbb{R}^d$ that is 3-simple (i.e. each $(d-3)$-cell is contained in 4 of the d-cells). Then $\mathcal{P}$ is a regular partition. If $\mathcal{P}$ is 2-simple, there is a convex lifting if and only if the star of each interior $(d-3)$-cell is liftable.*

In [31] they use a slightly different notion of genericity than here. There it is required that no pair of adjacent $(d-1)$-cells (containing a common $(d-2)$-cell) lie on the same hyperplane. This condition is stronger than having no π-angle, but in case the partition is 2-simple, both conditions are equivalent. This includes the most interesting cases for us, where Theorem 5.13 applies.

There is also a technical condition needed to conclude Theorem 5.11 from the work of Rybnikov [31, Condition 9.1]. He claims that partitions of $\mathbb{R}^d$ into convex regions and convex tilings of convex regions in $\mathbb{R}^d$ satisfy that condition. However it is needed that the partition is essential to be able to conclude that. For that reason we include the essential requirement in Definition 5.7 and we don't need to worry about that condition any longer.

One easy corollary of Theorem 5.11 is the following.

Corollary 5.14. *For $d \geq 3$, the set of generic simple n-partitions of $\mathbb{R}^d$ form an open dense subset of $\mathcal{C}_{\text{reg}}(\mathbb{R}^d, n)$.*

Proof. Theorem 5.11 gives the inclusion. Also, any regular partition has a small perturbation that makes it simple. From Proposition 5.4 we can see how to make this perturbation, using a parameterization of regular partitions by choosing the lifted hyperplanes and modding out scaling and adding a linear functional. There we can perturb the hyperplanes generically to obtain a simple polyhedron, and its projection will be a new partition close to the original one. On the other hand, if we perturb a simple partition an small amount, it will still have the same combinatorial type, and therefore simple partitions form an open subset inside $\mathcal{C}_{\text{reg}}(\mathbb{R}^d, n)$. □

5.3 Universality

The realization spaces of regular partitions are closely related to realizations of polytopes. This fact allow us to extend known results for polytopes to the realization spaces of n-partitions, in particular, Richter-Gebert's Universality Theorem for 4-polytopes [28]. It states that realization spaces of 4-polytopes can be "as complicated as possible" as a semialgebraic set.

The *realization space* of a polytope P is the space of all polytopes combinatorially equivalent to P, modulo affine equivalence. The realization spaces of polytopes are semialgebraic sets. Two semialgebraic sets are *stably equivalent* if they are in the same equivalence class generated by rational homeomorphisms and stable projections (see [28]).

Theorem 5.15 (Richter-Gebert [28], Universality for 4-polytopes)**.** *For every primary basic semialgebraic set X defined over $\mathbb{Z}$, there is a 4-polytope P whose realization space is stably equivalent to X.*

Let $\mathcal{P}$ be a regular n-partition of $\mathbb{R}^d$. Let Q be an unbounded polyhedron that is a convex lifting for $\mathcal{P}$. We can construct a convex polytope Q' projectively equivalent to Q. The polytope Q' will have the same face structure as Q, with an extra facet F_∞ that corresponds to the points at infinity on Q. In fact, the polytope Q' will have the same face poset as the spherical representation $\widehat{\mathcal{P}}$, so we know its combinatorial structure in advance. (We will sometimes also denote F_∞ by $F_\infty(Q')$ to specify the polytope where it belongs.) This correspondence via projective transformations will help us to relate realization spaces of polytopes with realizations of a partition $\mathcal{P}$. We will see how to obtain such polytope Q' in the proof of Proposition 5.16 (see Figure 5.3).

We say that Q is a *positive convex lifting* of $\mathcal{P}$ if all points of Q have last coordinate greater than zero. A *lifting pair* $(\mathcal{P}, Q)$ consists of a regular partition $\mathcal{P}$ together with a positive convex lifting. Two lifting pairs $(\mathcal{P}_1, Q_1)$ and $(\mathcal{P}_2, Q_2)$ are affinely equivalent if there is an affine transformation in $\mathbb{R}^{d+1}$ sending each region of $\mathcal{P}_1$ to the corresponding region in $\mathcal{P}_2$, and sending Q_1 to Q_2.

Proposition 5.16. *Let $\mathcal{P}$ be a regular n-partition. There is a correspondence between the space of lifting pairs $(\mathcal{P}_1, Q_1)$ up to affine equivalence, where $\mathcal{P}_1$ is a regular partition combinatorially equivalent to $\mathcal{P}$, with pairs $(Q_1', \boldsymbol{x})$ of realizations of a polytope Q' projectively equivalent to Q_1 together with a point $\boldsymbol{x}$ on the relative interior of the facet $F_\infty(Q')$, up to affine transformations.*

Proof. We identify the space $\mathbb{R}^d$ where the partition $\mathcal{P}_1$ lives with the affine hyperplane V_0 with last coordinate equals 0 in a vector space $V \cong \mathbb{R}^{d+1}$. The projection from Q_1 is made "vertically" to $\mathbb{R}^d$, by changing the last coordinate to 0.

Let φ be a fixed projective map from V to $W \cong \mathbb{R}^{d+1}$ such that sends the point $\boldsymbol{v}_\infty$ "high above" in V to the origin $\underline{\mathbf{0}}$ in W, so that it interchanges the hyperplane V_0 in V with the hyperplane at infinity in W and vice versa.

On Figure 5.3 we sketch how to construct such map φ in case our partition $\mathcal{P}$ lives in $\mathbb{R}^1$. On that figure, both spaces V and W are embedded in $\mathbb{R}^3$ in such a way that the projective transformation φ is simply given by a projectivity from the origin $\mathbf{0}$ of $\mathbb{R}^3$ (i. e. by mapping each point $\boldsymbol{y} \in V$ to the point $\boldsymbol{z} \in W$ such that $\boldsymbol{y}$, $\boldsymbol{z}$ and $\mathbf{0}$ are collinear). In general, we can assume that V and W are embedded in $\mathbb{R}^{d+2}$, where W has last coordinate $x_{d+2} = 1$, V is given by $x_{d+1} = 1$, and the projectivity from V to W is made from the origin $\mathbf{0} \in \mathbb{R}^{d+2}$, by sending any point $\boldsymbol{x} \in V$ with $x_{d+2} \neq 0$ to the point $\varphi(\boldsymbol{x}) = \boldsymbol{x}/x_{d+2} \in W$. Other points in V are sent to the plane at infinity in W.

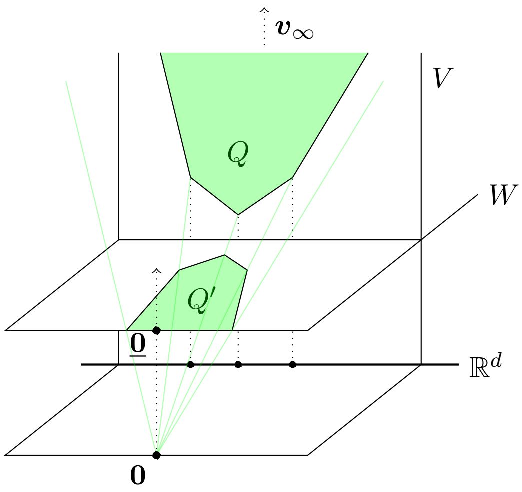

FIGURE 5.3: Projective map φ taking a positive convex lifting Q over a regular partition $\mathcal{P}$ of $\mathbb{R}^d$ to a convex polytope Q'.

For every partition $\mathcal{P}_1$ of $\mathbb{R}^d$ and lifting $Q_1 \subseteq V$ construct $Q'_1 = \varphi(Q_1)$ and set $\boldsymbol{x}$ to be the origin $\underline{\mathbf{0}}$ in W. Then $\boldsymbol{x} \in F_\infty(Q'_1)$ since it is the image of the point v_∞ that always belong to Q, at the hyperplane at infinity of V. We want to see that the map φ induces the desired bijection. To prove this, we need to show that affinely equivalent pairs will be mapped to affinely equivalent polytopes where the point $\boldsymbol{x}$ is also preserved, and that for any realization of Q' together with a point $\boldsymbol{x} \in F_\infty(Q')$ we can obtain a unique lifting pair $(\mathcal{P}_1, Q_1)$ up to affine equivalence.

Let $(\mathcal{P}_1, Q_1)$ and $(\mathcal{P}_2, Q_2)$ two affinely equivalent lifting pairs. Then there is an affine map ψ such that it preserves V_0 and sends the partition $\mathcal{P}_1$ to $\mathcal{P}_2$ and the polyhedron Q_1 to Q_2. Such affine map ψ can be seen as a projective map that preserves the hyperplane at infinity of V. After applying φ we will get affinely equivalent bounded polytopes Q'_1 and Q'_2, since they are related by a projective function $\varphi \circ \psi \circ \varphi^{-1}$ that preserves the hyperplane at infinity in W, because ψ preserves the hyperplane V_0 (the polytopes are bounded since Q_1 and Q_2 are positive convex liftings). Also the point $\boldsymbol{x}$ will be preserved, because ψ keeps invariant the point $\boldsymbol{v}_\infty$, so that the lifted polyhedra Q_1 and Q_2 keep being sent to the corresponding partition by projecting down.

Now consider a realization Q'_1 of the polytope Q' together with a point $\boldsymbol{x} \in F_\infty(Q'_1)$. We can place Q'_1 in the space W in such a way that the point $\boldsymbol{x}$ is at the origin of W and the facet F_∞ is on the hyperplane W_0 at height 0, with Q'_1 at the right side of W_0, given by the image under φ of the points of V with positive height. (We will see that we don't care about possible rotations of Q'_1 and actually we also don't care of Q'_1 up to affine equivalence. We only need to care that the point $\boldsymbol{x}$ is also preserved by the affine transformation.)

Once we place Q'_1 in W, we can get $Q_1 = \varphi^{-1}(Q'_1)$ and the partition $\mathcal{P}_1$ is obtained by projecting down the facets of Q_1. Since the point $\boldsymbol{x}$ in the interior of $F_\infty(Q)$ is mapped "high above" to $\boldsymbol{v}_\infty$, then there are directions at infinity around $\boldsymbol{v}_\infty$ in such a way that after projecting down we will completely cover V_0 and obtain a partition. Also $\mathcal{P}_1$ will have the right combinatorial structure, due to the combinatorial structure of Q'.

Now suppose we have an affinely equivalent pair $(Q'_2, \boldsymbol{x}_2)$. After positioning Q'_2, we can assume that $\boldsymbol{x}_2 = \boldsymbol{x}$ is also at the origin of W and the face $F_\infty(Q'_2)$ is properly positioned. Then the affine transformation between Q'_1 and Q'_2 is such that it preserves the origin of W and the hyperplane W_0. Again, such affine transformation can be seen as a projective transformation ψ' that besides maps the hyperplane at infinity in W to itself. By a similar analysis as we did before, we can conclude that the pairs $(\mathcal{P}_1, Q_1)$ and $(\mathcal{P}_2, Q_2)$ obtained by applying φ^{-1} are affinely equivalent, since they will by related by the map $\varphi^{-1} \circ \psi' \circ \varphi$, that preserves V_0, $\boldsymbol{v}_\infty$ and the hyperplane at infinity of V. □

Theorem 5.17. *For any primary basic semialgebraic set X and $d \geq 3$, there is an n-partition $\mathcal{P}$ of $\mathbb{R}^d$ such that the set of regular partitions combinatorially equivalent to $\mathcal{P}$, up to affine equivalence, form a semialgebraic set stably equivalent to X.*

Proof. By Theorem 5.15 there is a 4-polytope P whose realization space is stably equivalent to X. Take F_∞ to be any facet of P. Then the set of pairs $(Q', \boldsymbol{x})$ with Q' realization of P and $\boldsymbol{x}$ in the interior of $F_\infty(Q')$, up to affine equivalence, is also a semialgebraic set stably equivalent to the realization space of P, since the extra information given by the position of $\boldsymbol{x}$ is bounded by algebraic (linear) equations, and the projection $(Q', \boldsymbol{x}) \mapsto Q'$ is an stable projection with open convex polytopes as fibers.

Now, by Proposition 5.16, we get that the space of lifting pairs for a corresponding partition $\mathcal{P}$ obtained from projecting down a polyhedron $Q = \varphi^{-1}(Q')$, up to affine equivalence, also have a semialgebraic structure stably equivalent to X. It remains to check that the projection $(\mathcal{P}, Q) \mapsto \mathcal{P}$ from lifting pairs up to affine equivalence to its corresponding partition is a stable projection. Notice that for a generic partition $\mathcal{P}$, two lifting pairs $(\mathcal{P}, Q_1)$ and $(\mathcal{P}, Q_2)$ are affinely equivalent only in case that Q_1 and Q_2 only differ by multiplying by a scalar, since these are the only affine transformations that preserve $\mathcal{P}$ Partitions of $\mathbb{R}^d$ coming from $(d+1)$-polytopes are always generic.

From [31, Theorem 5.1], we know that for a fixed partition $\mathcal{P}$, the convex liftings of $\mathcal{P}$ are given by a polyhedral cone, that they denote $\mathrm{CLift}(\mathcal{P})$. Requiring that a convex lifting is positive only adds an extra linear inequality to the cone $\mathrm{CLift}(\mathcal{P})$. The lifting pairs $(\mathcal{P}', Q')$ that project to a partition $\mathcal{P}'$ affinely equivalent to $\mathcal{P}$ can be always be represented by a pair $(\mathcal{P}, Q)$ up to affine equivalence, and a positive convex liftings Q are given by a polyhedral set. Therefore the map $(\mathcal{P}, Q) \mapsto \mathcal{P}$ is a stable projection. We conclude that the space $\mathcal{C}_{\mathcal{P},\mathrm{reg}}(\mathbb{R}^d, n)$ modulo affine transformations is stably equivalent to X. □

On the other hand, realization spaces of 3-polytopes are always contractible. We expect similar behavior for regular n-partitions of the plane (see Conjecture 8.2).

Chapter 6

Dimension of realization spaces

We know that the space $\mathcal{C}(\mathbb{R}^d, n)$ is a topological space glued together from different semialgebraic pieces corresponding to the different combinatorial types. These pieces might have different dimensions. The dimension $\dim \mathcal{C}(\mathbb{R}^d, n)$ of the space of n-partitions is the maximal dimension $\dim \mathcal{C}_{\mathcal{P}}(\mathbb{R}^d, n)$ of the realization spaces of partitions $\mathcal{P} \in \mathcal{C}(\mathbb{R}^d, n)$. We would like to understand what are these dimensions for different values of n and d, and which combinatorial types attain the maximal dimension. For $d = 1$, we can see from Proposition 4.49 that $\dim \mathcal{C}(\mathbb{R}^1, n) = n - 1$. In what follows, we find this maximal dimension in for $d = 2$, characterizing the combinatorial types that attain it. Then we try to understand the case $d = 3$ and offer some conjectures about its maximal dimension.

6.1 Partitions of the plane

We consider here the spaces $\mathcal{C}(\mathbb{R}^2, n)$ and compute their dimensions. For the planar case, a partition is simple if each interior vertex is contained in three regions, and vertices at infinity are contained only in two regions.

Lemma 6.1. *Generic partitions are dense in $\mathcal{C}(\mathbb{R}^2, n)$ for $n \geq 3$.*

Proof. Let $\mathcal{P}$ be a partition that is not generic. If $\mathcal{P}$ is essential, then it has some interior vertices contained in a pair of edges that form a π-angle. Let S be the set of all those vertices. For each $v \in S$ let n_v the unit normal vector orthogonal to the 1-subface containing the edges that make a π-angle at $\boldsymbol{v}$, exterior to region where the 1-subface belongs (as depicted in Figure 6.1). We want to perturb every vertex $v \in S$ a small amount in direction n_v, to get a new partition without π-angles.

Let $\varepsilon > 0$ be a small enough real number. If we move every $v \in S$ by εn_v (using always the same value of ε) then we will end up with a convex configuration, where the π-angles will get smaller, unless there are many π-angles on the same subface.

To see this, notice that for ε small enough we can neglect the motions in direction orthogonal to n_v to find the convexity condition that have to be satisfied for each v. We get at the end the following inequality

$$|\varepsilon n_v| \geq \frac{\ell_2 |n_v \cdot \varepsilon n_{v_1}| + \ell_1 |n_v \cdot \varepsilon n_{v_2}|}{\ell_1 + \ell_2} \tag{6.1}$$

where v_i are the vertices adjacent to v on the edges of angle π (for $i = 1, 2$), $\ell_i = |v - v_i|$, are the length of those edges, and $n_{v_i} = 0$ in case that $v_i \notin S$. (The case where some of the ℓ_i are not finite will be discussed in more detail later.)

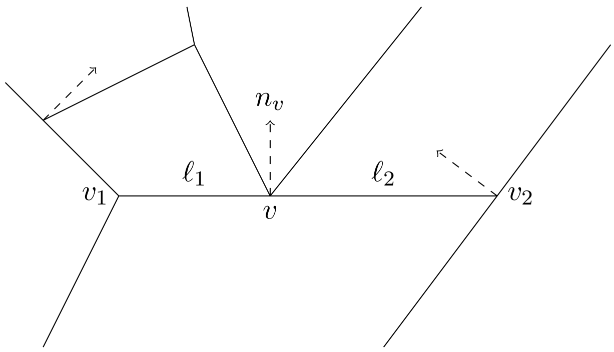

FIGURE 6.1: Normal directions n_v for vertices in S

These inequalities are always true since $n_v \cdot n_{v_i} \leq 1$, and we get strict inequality unless $n_v = n_{v_1} = n_{v_2}$. Since we want to get always strict inequalities to ensure that the final partition is generic, we modify the amount of perturbation in case that there are more than three vertices on the interior of an edge (1-subface) of a region. Call the vertices in that subface $v_0, v_1, \ldots, v_n$ in the respective order on the segment, where v_0 and v_n are on the boundary (see Figure 6.2).

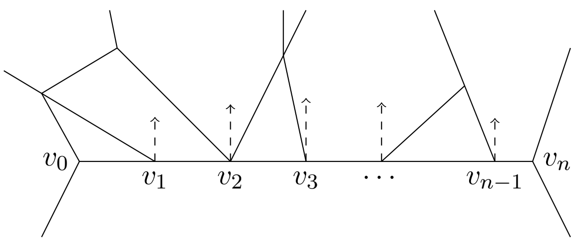

FIGURE 6.2: Many vertices v_i on the same subface.

Now find real values λ_i for $i = 0, \ldots, n$ so that they satisfy

$$\lambda_i > \frac{\ell_{i+1} \lambda_{i-1} + \ell_i \lambda_{i+1}}{\ell_i + \ell_{i+1}} \tag{6.2}$$

for $0 < i < n$, where $\ell_i = |v_i - v_{i-1}|$ and $\lambda_0 = \lambda_n = 0$. These values can be easily obtained from a convex function on the interval. We need special care in case that we have vertices at infinity. In case that ℓ_1 is infinite, the Equation (6.2) reduces to $\lambda_1 < \lambda_2$ that in principle leads to no complication, but implies that the λ_i will be in decreasing order. The problem appears in case that both v_0 and v_n are at infinity. In that case, no λ_i values can be found and we need to perturb at least one of them before applying our procedure. If one of them is a simple vertex, we can easily move it to another vertex at infinity that is close enough, by rotating the ray at infinity a small amount, so that region with the π-angle is still convex. If both vertices at infinity are not simple, we can find an interior vertex v "close" to v_0 such that it is between some of the unbounded edges at v_0. Then replace all edges to v_0 by edges to v and add a new edge from v to v_0. (This kind of perturbation is analogous to the one we describe in the proof of Theorem 6.2.) After doing this, we can always find values λ_i for the new partition.

Once we find these values of λ_i, we can perturb each of the v_i by $(\varepsilon + \lambda_i \varepsilon^2) n_{v_i}$ for a small ε. We do the same for all sets of vertices on a common 1-subface. For ε small enough, these new values of perturbation should not interfere with the convexity inequalities obtained before, and now we will get at the end a strictly convex partition, since now all inequalities will hold strictly.

If $\mathcal{P}$ is non-essential, it consists of regions separated by parallel lines and we can perturb one of those lines slightly to obtain essential partitions close to $\mathcal{P}$. Here we need that $n \geq 3$. Since those partitions can be approximated by generic partitions, we conclude that generic partitions are dense in $\mathcal{C}(\mathbb{R}^2, n)$. □

Theorem 6.2. *Simple partitions are dense in* $\mathcal{C}(\mathbb{R}^2, n)$.

Proof. Due to Lemma 6.1 we only need to prove that any generic partition of $\mathbb{R}^2$ can be slightly perturbed to a simple partition. If a partition is not simple, take a vertex V with degree higher than three and find a generic line ℓ through V, so that it leaves at least two edges at each side. It means that two small intervals in ℓ at different sides of V (with V in the boundary) are initially contained in two different convex regions of the partition, only intersecting at V.

Now add a new vertex V' very close to V in the direction of ℓ and join it with V by an edge. Also erase the two edges at V of the region containing V' and change them for edges to V'. This will give rise to a new partition with smaller average degree at interior vertices.

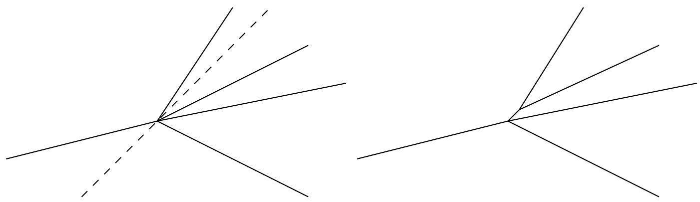

Figure 6.3: Duplicating a vertex

Since the partition we started with is taken to be generic, then we can make these perturbations without losing the convexity of the regions. We can repeat this procedure until we end up with a simple partition. □

If we consider the space of partitions for some given combinatorial type, we can parameterize it by giving two coordinates for each interior vertex and one coordinate to parameterize the direction of the vertices at infinity (corresponding to unbounded edges). It is easy to see now that we will get bigger dimensions for simple combinatorial types (since we are duplicating vertices several times to get simple types). Based on that, we compute the dimension of $\mathcal{C}(\mathbb{R}^2, n)$ by finding the combinatorial types with configuration space of maximal dimension.

Proposition 6.3. *The space of partitions of* $\mathbb{R}^2$ *into* n *convex pieces has dimension* $\dim \mathcal{C}(\mathbb{R}^2, n) = 4n - 7$*, for* $n \geq 3$*. The partitions whose combinatorial types that attain the top dimension are simple partitions with exactly three unbounded regions.*

Proof. The dimension of the space of partitions with a simple combinatorial type with v interior vertices and m unbounded regions is $2v + m$. Consider the graph of vertices and edges of this partition, including vertices and edges at infinity as well. Call e the number of edges of this graph. Then, by the Euler formula, we know that

$$(n + 1) - e + (v + m) = 2,$$

$$(n - 1) + (v + m) = e.$$

Since all vertices have three edges, $2e = 3(v + m)$, and therefore $2n - 2 = v + m$. The dimension $2v + m$ is then maximal whenever m is as small as possible, namely 3, for $n \geq 3$. In that case, $v = 2n - 5$ and the dimension of the space is $4n - 7$. □

These partitions exist for any $n \geq 3$. Figure 6.4 is an example of such a partition, for $n = 6$.

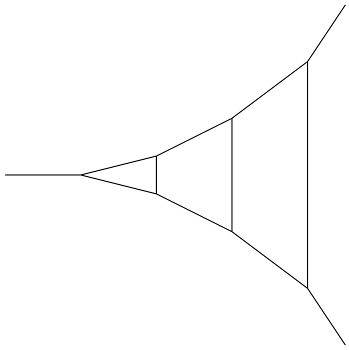

FIGURE 6.4: Shape of a partition $\mathcal{P}$ with maximal $\dim \mathcal{C}_{\mathcal{P}}(\mathbb{R}^2, 6)$.

6.2 Dual and bounded complex

We define here two cell complexes related with any pointed partition $\mathcal{P}$. A cell complex $\mathcal{M}$ is a *d-manifold* (with or without boundary) if its corresponding topological space is a manifold of dimension d. It means that every interior point has a neighborhood homeomorphic to an open d-ball. All other points are on the boundary, and have a neighborhood homeomorphic to the halfspace of $\mathbb{R}^d$ with last coordinate $x_d \geq 0$. (See [23], [5]). A manifold is *PL* if it is homeomorphic to a piecewise linear realization where all cells are polyhedra. We denote by $\partial\mathcal{M}$ to the boundary of a d-manifold with boundary.

For any PL d-manifold $\mathcal{M}$ with boundary there is a dual cell complex $\mathcal{M}^*$ that has the same underlying space and has one interior $(d-k)$-face for each k-face of $\mathcal{M}$. The faces on the boundary of $\mathcal{M}^*$ make the complex $\partial(\mathcal{M}^*) = (\partial\mathcal{M})^*$ that has no boundary. (See [23], [4]).

We now define a complex that has its faces in bijection with the interior faces of $\mathcal{P}$. The complex $\mathcal{P}_N$ given in Definition 4.31 is a d-ball, and therefore a PL d-manifold with boundary. From Proposition 4.34 we know that for pointed partitions the complex whose cells are the faces of $\mathcal{P}$ coincide with the cell complex $\mathcal{P}_N$.

Definition 6.4 (Dual complex $\mathcal{P}_D$)**.** Let $\mathcal{P}$ be a pointed n-partition. The *dual complex* $\mathcal{P}_D$ is a cell complex obtained from $\mathcal{P}_N^*$ by removing all faces in the boundary $\partial(\mathcal{P}_N^*)$ together with all faces containing one of those boundary faces, namely

$$\mathcal{P}_D = \mathcal{P}_N^* - \partial(\mathcal{P}_N^*).$$

Example 6.5. On Figure 6.5 we show how to obtain the dual complex $\mathcal{P}_D$ for the partition $\mathcal{P}$ from 3.7. First we construct the complex $\mathcal{P}_N^*$, depicted on the left, that is a covering of the upper hemisphere using six 2-cells, one for each node of $\mathcal{P}$. After deleting the boundary faces we obtain the complex $\mathcal{P}_D$, as depicted on the right. It has four vertices, five 1-faces and two 2-faces, one for each interior face of $\mathcal{P}_N$.

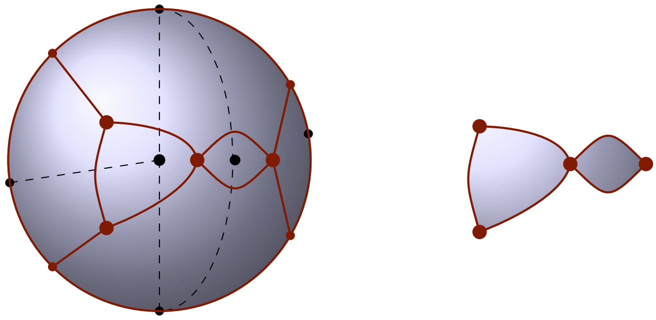

FIGURE 6.5: Complexes $\mathcal{P}_N^*$ and $\mathcal{P}_D$ for the partition $\mathcal{P}$ given in Example 3.7.

The complex $\mathcal{P}_D$ has a $(d-k)$-face for each interior k-face of $\mathcal{P}$. Also, the boundary faces of $\mathcal{P}_D$ are in correspondence with the unbounded faces of $\mathcal{P}$ (i.e. the interior faces of $\mathcal{P}$ that contain a face at infinity). The following lemma tell us about the topology of $\mathcal{P}_D$.

Lemma 6.6. *If $\mathcal{M}$ is a PL manifold with boundary, then $\mathcal{M}^* - \partial(\mathcal{M}^*)$ is a deformation retract of $\mathcal{M}^*$.*

The proof of this lemma can be found on [23, Lemma 70.1] and also in [4, Lemma 3.2] where it is shown that M^* collapses to $M^* - \partial(\mathcal{M}^*)$. In particular, Lemma 6.6 says that $\mathcal{P}_N$ and $\mathcal{P}_D$ are homotopy equivalent.

Lemma 6.7. *If $\mathcal{P}$ is a generic partition, then the dual complex $\mathcal{P}_D$ is homeomorphic to a d-ball.*

Proof. Let Q be a big d-ball strictly contained on the interior of the upper hemisphere S^d_+, such that it is spherically convex and all bounded faces of $\mathcal{P}_N$ are contained in Q. We can construct a cell complex $\mathcal{Q}$ with interior (and respectively boundary) cells given by intersecting the interior (respectively boundary) of Q with the faces of $\mathcal{P}$. Since Q is convex, the intersection of the interior of Q with any face of $\mathcal{P}$ will be also strictly convex, and therefore homeomorphic to a ball. Since the partition is pointed, each unbounded face of $\mathcal{P}$ only intersects the boundary of Q in a contractible cell. The boundary of each cell $C = F \cap Q$ of $\mathcal{Q}$ is covered by other cells of $\mathcal{Q}$ given by the intersection of faces in the boundary of F with the interior of Q, and the intersection of unbounded faces of F and the boundary of Q.

Each interior face of $\mathcal{Q}$ will correspond to a unique face in the dual complex $\mathcal{P}_D$, so that $\mathcal{P}_D = (\mathcal{Q})^*$. We conclude that $\mathcal{P}_D$ is a d-ball. □

Using the same notation as before, boundary faces of $\mathcal{Q}$ correspond to unbounded faces of $\mathcal{P}$, and it holds that for generic partitions $\partial\mathcal{P}_D = (\partial Q)^*$ is homeomorphic to a $(d-1)$-ball, since its union is the boundary of Q. Notice that for the complex $\mathcal{P}_D$ in Figure 6.5, Theorem 6.7 doesn't apply, since its corresponding partition $\mathcal{P}$ is not generic.

There is one more cell complex that will be important for us, and we define it now.

Definition 6.8. Let $\mathcal{P}$ be a pointed n-partition. The *bounded complex* $\mathcal{P}_B$ is the subcomplex of all bounded faces of $\mathcal{P}_N$.

Proposition 6.9. *Let $\mathcal{P}$ be a generic n-partition of $\mathbb{R}^d$. Then the bounded complex $\mathcal{P}_B$ is contractible.*

Proof. By Lemma 6.7 we know that $\mathcal{P}_D$ is a d-ball, in particular a d-manifold with boundary. The bounded complex of $\mathcal{P}$ can be obtained as $\mathcal{P}_B = \mathcal{P}_D^* - \partial(\mathcal{P}_D^*)$ if we identify the faces of $\mathcal{P}_B \subset \mathcal{P}_D^* \subset (\mathcal{P}_N^*)^*$ with the faces of $\mathcal{P}_N$, since only the cells corresponding to interior faces in $\mathcal{P}_N$ are left. By Lemma 6.6, we can conclude that $\mathcal{P}_B$ is contractible. □

6.3 Partitions of $\mathbb{R}^3$

We would like to identify the partitions $\mathcal{P}$ such that $\mathcal{C}_\mathcal{P}(\mathbb{R}^d, n)$ has maximal dimension for a given n. For $d = 2$, Proposition 6.3 gives us the characterization of such partitions. Now we focus on the case $d = 3$. We know already that if $\mathcal{P}$ is simple and generic, then $\dim \mathcal{C}_\mathcal{P}(\mathbb{R}^d, n) = (d + 1)(n - 1) - 1$ since it is regular. For $d = 3$ we conjecture that this formula gives the maximal dimension $4n-5$ although we prove that there are other combinatorial types that also attain this dimension. To justify this conjecture we offer an argument based on a naive counting of the dimension.

The naive way to try to compute the dimension of a semialgebraic set proceeds by counting the number of variables and then subtracting the number of "independent" algebraic equations. If the equations are not independent, this gives us a lower bound on the dimension of the space, and we have to add a quantity from the algebraic relationships between the equations. Intuitively, this value correspond to "hidden incidence theorems" between the points of the configuration. (Other instances of this naive counting appear in [10] and [29].)

We will apply this naive count for the dimension the realization space $\mathcal{C}_\mathcal{P}(\mathbb{R}^3, n)$ for a generic n-partition $\mathcal{P}$ of $\mathbb{R}^3$, based on the description given in Theorem 4.42. This naive count will be denoted as z and it will be expressed in terms of the f-vectors of

$\mathcal{P}_N$, $\mathcal{P}_D$ and of their boundary complexes $\partial\mathcal{P}_N$ and $\partial\mathcal{P}_D$. The entries of the f-vector of a complex $\mathcal{C}$ are the values $f_i(\mathcal{C})$ that denote the number of i-cells in that complex. We will assume here an extra genericity assumption, namely that there are no parallel unbounded edges in $\mathcal{P}$, so that there is a vertex at infinity for each of them.

The f-vector of $\mathcal{P}_D$ counts the number of interior faces of $\mathcal{P}$ in reversed order, where $f_3(\mathcal{P}_D)$ equals the number of interior vertices of $\mathcal{P}$, $f_2(\mathcal{P}_D)$ counts the interior edges, $f_1(\mathcal{P}_D)$ counts the interior 2-faces and $f_0(\mathcal{P}_D) = f_3(\mathcal{P}_N) = n$ since all the regions are interior. Also, the faces on $\partial\mathcal{P}_D$ are in correspondence to the unbounded faces of $\mathcal{P}$ in reversed order, where $f_0(\partial\mathcal{P}_D)$ equals the number of unbounded 3-faces of $\mathcal{P}$, $f_1(\partial\mathcal{P}_D)$ counts the unbounded 2-faces and $f_2(\partial\mathcal{P}_D)$ the unbounded edges. The complex $\mathcal{P}_D$ is a 3-ball (by Lemma 6.7) and its boundary $\partial\mathcal{P}_D$ is a 2-sphere.

We count initially $3f_0(\mathcal{P}_N) - f_0(\partial\mathcal{P}_N)$ corresponding to the three dimensions of freedom for each vertex of $\mathcal{P}$, where we need to subtract one for each of the equations of the form $\boldsymbol{e}_0 \cdot \boldsymbol{x}_{\boldsymbol{v}} = 0$ corresponding to the vertices at infinity. We have to subtract from that a value coming from equations of the form $\det(\boldsymbol{x}_{\boldsymbol{v}_0}, \ldots, \boldsymbol{x}_{\boldsymbol{v}_d}) = 0$ telling that our 2-faces have to be planar. Since most of these equations are related, so we have to find this value more carefully.

For every interior 2-face F in $\mathcal{P}$, we need to fix three vertices to fix the plane where it belongs. Each interior 2-face in $\mathcal{P}$ correspond to a 1-face in $\mathcal{P}_D$, so there are $f_1(\mathcal{P}_D)$ of those 2-faces. For each extra vertex $\boldsymbol{v}$ in F, there is one dimension less of freedom telling that it belongs to that plane. It corresponds to the equation $\det(\boldsymbol{x}_{\boldsymbol{v}_0}, \boldsymbol{x}_{\boldsymbol{v}_1}, \boldsymbol{x}_{\boldsymbol{v}_2}, \boldsymbol{x}_{\boldsymbol{v}}) = 0$, where we assume that the three fixed vertices are $\boldsymbol{v}_0, \boldsymbol{v}_1$ and $\boldsymbol{v}_2$ where v_0 and v_1 are chosen so that they belong to the same edge, in order to have $(\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v})$ to be a node flat. Equations corresponding to other node flats of vertices contained in F will be dependent on the equations mentioned before. If $\mathcal{P}$ is generic, no other algebraic equations take place (otherwise $\mathcal{P}$ would have some π-angles). Therefore we have to subtract

$$f_{12}(\mathcal{P}_D) + f_1(\partial\mathcal{P}_D) - 3f_1(\mathcal{P}_D) \tag{6.3}$$

where $f_{12}(\mathcal{P}_D)$ counts the number of inclusions of a 1-face in an interior 2-face in $\mathcal{P}_D$, and also the number of inclusions of an interior 1-face of $\mathcal{P}_N$ in an interior 2-face of $\mathcal{P}_N$. The expression $f_{12}(\mathcal{P}_D) + f_1(\partial\mathcal{P}_D)$ counts the number of inclusions of a 1-face in an interior 2-face in $\mathcal{P}_N$ in case that every unbounded 2-face of $\mathcal{P}_N$ has one edge at infinity. This is always true under our extra genericity assumption of having no parallel unbounded edges. Since every 2-face have the same number of 0-faces and 1-faces, the expression (6.3) counts at each interior 2-face of $\mathcal{P}_N$ the number of vertices minus three. We get the desired naive dimension z by subtracting equation (6.3) from $3f_0(\mathcal{P}_N) - f_0(\partial\mathcal{P}_N)$, namely

$$z = 3f_0(\mathcal{P}_N) - f_0(\partial\mathcal{P}_N) - f_{12}(\mathcal{P}_D) + 3f_1(\mathcal{P}_D) - f_1(\partial\mathcal{P}_D). \tag{6.4}$$

Proposition 6.10. *Let $\mathcal{P}$ be a generic partition in $\mathcal{C}(\mathbb{R}^3, n)$. Then the naive count z for the dimension of $\mathcal{C}_\mathcal{P}(\mathbb{R}^3, n)$ satisfies $z \leq 4n - 5$. Equality is attained if and only if $\mathcal{P}$ is 2-simple (as in Definition 5.12) and doesn't have bounded regions.*

Proof. First, since $f_3(\mathcal{P}_D)$ is equal to the number of interior vertices of $\mathcal{P}$, then $f_0(\mathcal{P}_N) = f_3(\mathcal{P}_D) + f_0(\partial\mathcal{P}_N)$. Replacing this in equation (6.4) we get

$$z = 3f_3(\mathcal{P}_D) + 2f_0(\partial\mathcal{P}_N) - f_{12}(\mathcal{P}_D) + 3f_1(\mathcal{P}_D) - f_1(\partial\mathcal{P}_D). \tag{6.5}$$

We know that $f_0(\partial\mathcal{P}_N) \leq f_2(\partial\mathcal{P}_D)$, since the number of vertices at infinity is smaller than the number of unbounded edges in the partition (every vertex at infinity is contained in at least one unbounded edge). Also, notice that $f_{12}(\mathcal{P}_D) \geq 3f_2(\mathcal{P}_D)$, with equality in case that all 2-faces in $\mathcal{P}_D$ are triangles. It means that in $\mathcal{P}_D$ all 3-faces are simplicial and therefore the link of each interior vertex in $\mathcal{P}$ is simple. This is equivalent to say that $\mathcal{P}$ is 2-simple. Replacing these inequalities in (6.3), we get

$$z \leq 3f_3(\mathcal{P}_D) + 2f_2(\partial\mathcal{P}_D) - 3f_2(\mathcal{P}_D) + 3f_1(\mathcal{P}_D) - f_1(\partial\mathcal{P}_D). \tag{6.6}$$

Using the Euler characteristics for $\mathcal{P}_D$ and $\partial\mathcal{P}_D$, we know that

$$\begin{aligned} f_3(\mathcal{P}_D) - f_2(\mathcal{P}_D) + f_1(\mathcal{P}_D) &= f_0(\mathcal{P}_D) - 1, \\ -f_2(\partial\mathcal{P}_D) + f_1(\partial\mathcal{P}_D) &= f_0(\partial\mathcal{P}_D) - 2. \end{aligned}$$

Together with the inequality that tells that every 2-face has at least three edges

$$3f_2(\partial\mathcal{P}_D) \leq 2f_1(\partial\mathcal{P}_D) \tag{6.7}$$

and the fact that $f_0(\partial\mathcal{P}_D) \leq f_0(\mathcal{P}_D) = n$ we get

$$\begin{aligned} z &\leq 3n - 3 + 2f_2(\partial\mathcal{P}_D) - f_1(\partial\mathcal{P}_D) \\ &= 3n - 3 + (3f_2(\partial\mathcal{P}_D) - 2f_1(\partial\mathcal{P}_D)) - f_2(\partial\mathcal{P}_D) + f_1(\partial\mathcal{P}_D) \\ &\leq 3n - 3 + f_0(\partial\mathcal{P}_D) - 2 \\ &\leq 4n - 5. \end{aligned}$$

Inequality (6.7) is sharp in case the boundary is a simplicial complex (this is true if $\mathcal{P}$ is 2-simple) and to get equality is also needed that there are no interior vertices in $\mathcal{P}$ so that $f_0(\partial\mathcal{P}_D) = f_0(\mathcal{P}_D) = n$. ☐

Unfortunately this naive count doesn't always provide the right dimension for all realization spaces of partitions in $\mathbb{R}^3$. For example, if $\mathcal{P}$ is a simple partition we know that $\dim \mathcal{C}_{\mathcal{P}}(\mathbb{R}^3, n) = 4n - 5$ and the equality should be satisfied. Therefore there are relationships among the planarity restrictions, in fact one for each bounded face. This is one instance of the "hidden incidence theorems" mentioned before.

Theorem 6.11. *Let $Q \subset \mathbb{R}^3$ be a 3-dimensional polytope with vertices $A_1, \ldots, A_m$ (where $m = f_0(Q)$). Let $B_1, \ldots, B_m$ be other points in $\mathbb{R}^3$ different from the vertices of Q, for $i \leq m$. If the points A_i, A_j, B_i and B_j are coplanar for all edges A_iA_j of Q except one, then for the remaining edge the corresponding four points are also coplanar.*

Proof. Let A_p and A_q be the vertices on the remaining edge where the coplanarity of A_p, A_q, B_p and B_q has to be checked. The precise location of the points B_i is not important as long as they generate the same straight line A_iB_i. We want to prove that there are points C_i on the lines A_iB_i different from the A_i, so that for every edge A_iA_j of Q, the segment C_iC_j is parallel to the edge A_iA_j. In this way, we obtain at the end that C_pC_q is parallel to A_pA_q, and therefore we conclude the desired coplanarity, since B_p is on the line A_pC_p and B_q is on the line A_qC_q.

First fix $C_1 = B_1$. For each edge A_iA_j different from A_pA_q such that C_i is already fixed on the line A_iB_i, we can find the point C_j by taking a parallel line to the edges A_iA_j going through the point C_i and intersecting it wity the line A_jB_j. Since the points A_i, A_j, B_i and B_j are coplanar, this line will always meet.

In this way we can find all the vertices C_i, but we need to check that they are well defined, independently on the chosen path from C_1 to C_i. To see this, it is enough to check that going around the edges of a 2-face F of Q not containing the edge A_pA_q we end up always in the same point. But this is clear since all points $C_{i'}$ for $A_{i'}$ in F must be on the same plane P_F parallel to F, since all the segments $C_{i'}C_{j'}$ are parallel to the edges $A_{i'}A_{j'}$ of F. This plane is fixed once we know at least one of the points $C_{i'}$ and then the point C_i is at the intersection of the plane P_F parallel to F with the line through A_i and B_i. Since A_i, A_j, B_i and B_j are coplanar, as well as the points A_i, A_k, B_i and B_k, the intersection of those planes with the plane P_F contain the segments C_iC_j and C_iC_k respectively, and this segments must be parallel to the corresponding edges A_iA_j and A_iA_k.

Let G and H be the 2-faces of Q containing the edge A_pA_q. The points C_p and C_q belong to two planes P_G and P_H parallel to the facets G and H since we can walk from one of them to the other through segments parallel to edges of those faces. Then C_pC_q is parallel to A_pA_q since this edge lies on the intersection of P_G and P_H. We conclude that A_p, A_q, C_p and C_q lie on the same plane, as well as B_p and B_q, as desired. □

The idea of the proof of Theorem 6.11 is based on the assumption that Q is a bounded region of an n-partition of $\mathbb{R}^3$ that is simple and generic. From Theorem 5.11 we know that such partitions must always be regular, and then a similar property to the one mentioned in Proposition 5.6 for partitions of the plane must always hold, for $d \geq 3$. It is not necesary that the polytope is simple to have this parallel property. This is related with the fact that if we construct a partition $\mathcal{P}(Q)$ of $\mathbb{R}^3$ by taking a 3-polytope Q and constructing one unbounded edge from each vertex of Q and unbounded regions for each facet, then $\mathcal{P}(Q)$ is always regular (as a consequence of Theorem 5.13).

We can improve our naive counting z by adding the number of bounded faces to it. They represent the relationships among the coplanarity conditions. Then we get a new value $z' = z + f_3(\mathcal{P}_B) = z + f_0(\mathcal{P}_D) - f_0(\partial\mathcal{P}_D)$. From the proof of Proposition 6.10 it is easy to see that $z' \leq 4n - 5$ also holds for any generic partition of $\mathbb{R}^3$.

Conjecture 6.12. *For $d = 3$, $n \geq 4$, the dimension of the realization space of any n-partition of $\mathbb{R}^3$ is at most $4n - 5$, with equality if (but not only if) the partition is simple.*

It is possible that other restrictions other than the coplanarities can arise while describing the vertices of a combinatorial type of partition. However in some interesting cases the naive count gives the right answer, as we will see.

Let Q be a d-polytope with n facets (i. e. $(d-1)$-faces) labeled from 1 to n. The face fan of Q is an n-partition $\mathcal{P}(Q)$ of $\mathbb{R}^d$ in convex regions obtained by choosing a point $\boldsymbol{x}$ in the interior of Q and with a region P_i that is the interior of the cone with apex $\boldsymbol{x}$ over each of the facets Q_i of Q. The combinatorial type of $\mathcal{P}(Q)$ is independent of the choice of $\boldsymbol{x}$.

Proposition 6.13. *Let Q be a simple 3-polytope, with $f_2(Q) = n$ labeled facets. Then the face fan $\mathcal{P}(Q)$ is an n-partition of $\mathbb{R}^3$ into convex pieces, such that $\mathcal{C}_{\mathcal{P}(Q)}(\mathbb{R}^3, n)$ has dimension $4n - 5$.*

Proof. There are three degrees of freedom to choose the unique vertex $\boldsymbol{x}$ of the partition. To select the directions of each of the rays from $\boldsymbol{x}$ we have always two dimensions, and they are independent of each other, since there are no coplanarity restrictions to be satisfied, but only inequalities saying that the regions are convex and disjoint and for generic choices they do not affect the dimension. Then we get that the dimension is $3+2f_0(Q)$. Since Q is simple, we know that $3f_0(Q) = 2f_1(Q)$. Using this, and the Euler relationship $f_0(Q) - f_1(Q) + n = 2$ we find that $f_0 = 2f_1(Q) - 2f_0(Q) = 2n - 4$ and then $\dim \mathcal{C}_{\mathcal{P}(Q)}(\mathbb{R}^3, n) = 4n - 5$. □

Example 6.14. On Figure 6.6 is the sketch of a non-simple 5-partition of $\mathbb{R}^3$ that attain the conjectured upper bound in the dimension of the realization space, coming from the construction described in Proposition 6.13, in the case that Q is a triangular prism.

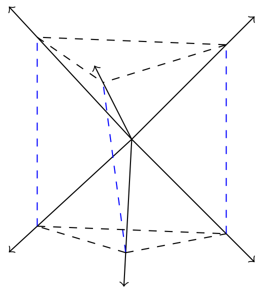

FIGURE 6.6: Non-simple 5-partition $\mathcal{P}$ with $\dim \mathcal{C}_{\mathcal{P}}(\mathbb{R}^3, 5) = 4 \cdot 5 - 5 = 15$.

There is a unique interior vertex $\boldsymbol{x}$, six unbounded edges and nine interior 2-faces represented by the cone from $\boldsymbol{x}$ over each of the dashed segments lines. The dimension of the realization space is $3 + 6 \times 2 = 15$, where we have three degrees of freedom to choose $\boldsymbol{x}$ and two for each other vertex at infinity contained on each unbounded edge, independent of each other. This is equal to the dimension of the realization space of simple 5-partitions $4n - 5 = 15$, but in this case $\mathcal{P} = \mathcal{P}(Q)$ is not simple.

Not all partitions in this space $\mathcal{C}_{\mathcal{P}}(\mathbb{R}^3, n)$ are regular. The subspace of regular partitions inside this realization space has codimension one, since an extra restriction is needed for a partition in order to be regular, namely that the three "vertical" 2-faces (corresponding to the blue edges) intersect in a line. This restriction can be visualized for example in case we perturb such regular partition to get a simple partition with a small "vertical" edge surrounded by three regions. This edge is the intersection of three planes containing 2-faces. Such perturbations can only be obtained for regular partitions, and since such planes intersect in any small perturbation, they should also intersect for the original regular partition as well.

If Q is not a simplex, then $\mathcal{P}(Q)$ is not a simple partition. Also if we allow Q to be any 3-polytope, then we have the inequality $3f_0(Q) \leq 2f_1(Q)$ that implies that $\dim \mathcal{C}_{\mathcal{P}(Q)}(\mathbb{R}^3, n) \leq 4n - 5$.

Although the face fans of polytopes are always regular partitions, most of the partition with the combinatorial type of such face fans are non-regular. We know already that inside the space of regular partitions, the simple ones have maximal dimension since they are generic. On the other hand, for the non-simple combinatorial types of fans, the regular partitions among them have dimension smaller than $4n - 5$, and it means that to reach dimension $4n - 5$, as stated in Proposition 6.13, most of the partitions have to be non-regular.

All n-partitions obtained as a face fan of a simple 3-polytope are 2-simple and don't have bounded regions. We offer the following conjecture, based on the equality case of the naive count inequality given in Proposition 6.10.

Conjecture 6.15. *Let $\mathcal{P}$ be a generic n-partition in $\mathcal{C}(\mathbb{R}^3, n)$ that is 2-simple, without parallel unbounded edges. Then the realization space of $\mathcal{P}$ has dimension $4n-5$.*

Other examples of non-simple combinatorial types $\mathcal{P}$ with $\dim \mathcal{C}_{\mathcal{P}}(\mathbb{R}^3, n) = 4n-5$ can be obtained by gluing some of the fans described before. Here is a partial result supporting Conjecture 6.15.

Proposition 6.16. *Let $\mathcal{P}$ be a generic n-partition of $\mathbb{R}^3$ that is 2-simple. If the bounded complex $\mathcal{P}_B$ is one-dimensional, then $\dim \mathcal{C}_{\mathcal{P}}(\mathbb{R}^3, n) = 4n-5$.*

Proof. By Proposition 6.9, $\mathcal{P}_B$ is contractible. If $\mathcal{P}_B$ is a point, then $\mathcal{P} = \mathcal{P}(Q)$ for some simple 3-polytope Q, and by Proposition 6.13 the result holds, since all combinatorial 2-spheres are polytopal. If not, then $\mathcal{P}_B$ is a contractible graph, namely a tree. We can find a vertex $\boldsymbol{v}$ of $\mathcal{P}_B$ of degree one (a leaf in $\mathcal{P}_B$) that belongs to a unique edge $e \in \mathcal{P}_B$. Locally, $\text{star}(\boldsymbol{v}, \mathcal{P})$ is also combinatorially equivalent to the face fan over a simple 3-polytope. Call $\mathcal{P}_{\boldsymbol{v}}$ to partition of $\mathbb{R}^3$ with a unique interior vertex, such that the star at this vertex is as $\text{star}(\boldsymbol{v}, \mathcal{P})$, and $\mathcal{P}_{\setminus \boldsymbol{v}}$ to the partition obtained from $\mathcal{P}$ when $\boldsymbol{v}$ goes infinitely far away (in the direction of e) and becomes a vertex at infinity.

By induction, assume that $\dim \mathcal{C}_{\mathcal{P}_{\boldsymbol{v}}}(\mathbb{R}^3, n) = 4n_1 - 5$ and $\dim \mathcal{C}_{\mathcal{P}_{\setminus \boldsymbol{v}}}(\mathbb{R}^3, n) = 4n_2 - 5$, where n_1 and n_2 count the number of regions of $\mathcal{P}_{\boldsymbol{v}}$ and $\mathcal{P}_{\setminus \boldsymbol{v}}$ respectively. Notice that $n = n_1 + n_2 - 3$. Since $\mathcal{P}$ is 2-simple, there are only three regions adjacent to the edge containing $\boldsymbol{v}$. These are the only regions counted in both complexes.

Now, to choose any partition in $\mathcal{C}_{\mathcal{P}}(\mathbb{R}^3, n)$, we can choose a partition in $\mathcal{C}_{\mathcal{P}_{\setminus \boldsymbol{v}}}(\mathbb{R}^3, n)$ by fixing $4n_2 - 5$ parameters. To fix the point $\boldsymbol{v}$ along e is needed only one parameter. Now to choose all other edges through $\boldsymbol{v}$, there are two degrees of freedom for each, except for e that was already chosen, and for the three edges contained in the planes through e, since those planes were fixed by the choice of $\mathcal{C}_{\mathcal{P}_{\setminus \boldsymbol{v}}}(\mathbb{R}^3, n)$, and there is only one degree of freedom left. If we compare with the degree count made in Proposition 6.13, we find that from the $4n_1 - 5$ we are losing two degrees of freedom on choosing $\boldsymbol{v}$, two in the choice of e and three in the choice ofthe edges in the planes through e. We conclude that $\dim \mathcal{C}_{\mathcal{P}}(\mathbb{R}^3, n) = 4n_2 - 5 + 4n_1 - 5 - 7 = 4(n_1 + n_2 - 3) - 5$. □

Chapter 7

Spaces of equipartitions

Let μ be a positive continuous bounded measure on $\mathbb{R}^d$, as in Section 4.1. All measures we consider are going to be that way. Here we introduce spaces of equipartitions where all regions have the same measure. Spaces of equipartitions are of great interest for partition problems such as the Nandakumar & Ramana Rao problem [25–27]. With a better understanding of the spaces of equipartitions for small cases we explore further examples of spaces of partitions, more precisely the cases $d = 2$ and $n = 3$.

Definition 7.1. Let μ be a positive continuous bounded measure on $\mathbb{R}^d$. The space of convex equipartitions $\mathcal{C}^{\text{equi}}(\mathbb{R}^d, n, \mu)$ is the subspace of $\mathcal{C}(\mathbb{R}^d, n)$ of partitions $\mathcal{P}$ such that all convex regions P_i in $\mathcal{P}$ have the same measure $\mu(P_i) = \frac{1}{n}\mu(\mathbb{R}^d)$. We also denote by $\mathcal{C}^{\text{equi}}_{\text{reg}}(\mathbb{R}^d, n, \mu)$ the subspace of $\mathcal{C}^{\text{equi}}(\mathbb{R}^d, n, \mu)$ of regular equipartitions.

We are interested to know whether $\mathcal{C}^{\text{equi}}(\mathbb{R}^d, n, \mu)$ is homotopy equivalent to $\mathcal{C}^{\text{equi}}_{\text{reg}}(\mathbb{R}^d, n, \mu)$.

7.1 Looking for fair partitions

The Nandakumar and Ramana Rao problem [25–27] ask for the existence of partitions of a convex region into n convex pieces with equal area and equal perimeter. This has led to many interesting results about equipartitions of measures (see for example [8], [22], and [6]).

The approach of Karasev [22] for the Nandakumar & Ramana Rao problem relies on the observation that if there was a counterexample to the conjecture for some n, then this would imply the existence of equivariant maps

$$\mathcal{F}(\mathbb{R}^d, n) \underset{\mathfrak{S}_n}{\longrightarrow} \mathcal{C}^{\text{equi}}_{\text{reg}}(\mathbb{R}^d, n) \underset{\mathfrak{S}_n}{\longrightarrow} S(W_n),$$

where $\mathfrak{S}_n$ is the symmetric group of size n, $\mathcal{F}(\mathbb{R}^d, n)$ denotes the space of n different points in $\mathbb{R}^d$, and $S(W_n)$ is the set of points $(y_1, \ldots, y_n) \in \mathbb{R}^n$ such that $y_1 + \cdots + y_n = 0$ and $y_1^2 + \cdots + y_n^2 = 1$.

The first map is a consequence of the theory of optimal transport and is obtained from Theorem 5.5, by setting all $m_i = \frac{1}{n}\mu(\mathbb{R}^d)$. The second map is given by the perimeter of each region minus the average perimeter. If no fair partition exists, then such function never goes through the origin, and we can normalize it to get a point in $S(W_n) \cong S^{n-2}$.

To see if such a map can exist, different topological tools can be used. In particular, Blagojević & Ziegler [6] proved the Nandakumar & Ramana Rao conjecture for prime powers n. There it is described a cell complex structure for an equivariant strong deformation retract of $\mathcal{F}(\mathbb{R}^d, n)$. Since the resulting partitions of the map from $\mathcal{F}(\mathbb{R}^d, n)$ are always regular, we wanted to know if considering the whole space $\mathcal{C}^{\text{equi}}(\mathbb{R}^d, n)$ and not only the regular equipartitions could improve the results by this approach. The following proposition explains why just considering regular equipartitions is as good as considering all equipartitions for the existence of such equivariant maps.

Theorem 7.2. *There are $\mathfrak{S}_n$-equivariant maps*

$$\mathcal{F}(\mathbb{R}^d, n) \underset{\mathfrak{S}_n}{\longrightarrow} \mathcal{C}^{\text{equi}}_{\text{reg}}(\mathbb{R}^d, n) \underset{\mathfrak{S}_n}{\hookrightarrow} \mathcal{C}^{\text{equi}}(\mathbb{R}^d, n) \underset{\mathfrak{S}_n}{\longrightarrow} \mathcal{F}(\mathbb{R}^d, n).$$

Proof. Here the first map is the one obtained from Theorem 5.5 we just described above. The second is inclusion. The third one is obtained by considering the centers of mass of the convex regions with respect to a positive continuous bounded measure. Since the regions are non-empty, these points are on the interior of each region and must be all different. All these maps are equivariant, where the symmetric group acts by permuting the regions or the corresponding points in the same way. □

Thus the existence of a $\mathfrak{S}_n$-equivariant map from any of those spaces implies the existence of such map from all three spaces. Therefore the question about the existence of a $\mathfrak{S}_n$-equivariant map

$$\mathcal{C}^{\text{equi}}(\mathbb{R}^d, n) \underset{\mathfrak{S}_n}{\longrightarrow} S(W_n),$$

is equivalent to the existence of a map on the restricted domain $\mathcal{C}^{\text{equi}}_{\text{reg}}(\mathbb{R}^d, n)$. We still don't know if the maps given by Theorem 7.2 are homotopy equivalences. One partial result in this direction is the following theorem, due to Bernardo Uribe (personal communication).

Theorem 7.3 (B. Uribe)**.** *The composition of the three maps described in Theorem 7.2 is homotopy equivalent to the identity map on $\mathcal{F}(\mathbb{R}^d, n)$.*

Proof. Let $f : \mathcal{F}(\mathbb{R}^d, n) \to \mathcal{F}(\mathbb{R}^d, n)$ be the composition of the functions described in Theorem 7.2.We want to see that the map

$$\psi : \mathcal{F}(\mathbb{R}^d, n) \times [0, 1] \to \mathcal{F}(\mathbb{R}^d, n)$$

given by $\psi(\boldsymbol{x}, t) = t\boldsymbol{x} + (1 - t)f(\boldsymbol{x})$ that linearly interpolate between $\boldsymbol{x}$ and $f(\boldsymbol{x})$ is a homotopy between the identity in $\mathcal{F}(\mathbb{R}^d, n)$ and the function f.

We only need to check that $t\boldsymbol{x} + (1 - t)f(\boldsymbol{x})$ is also a point in $\mathcal{F}(\mathbb{R}^d, n)$. For $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots \boldsymbol{x}_n) \in \mathcal{F}(\mathbb{R}^d, n)$ denote by $\boldsymbol{y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n) = f(\boldsymbol{x})$. Then the coordinates of $t\boldsymbol{x} + (1 - t)f(\boldsymbol{x})$ are of the form $t\boldsymbol{x}_i + (1 - t)\boldsymbol{y}_i$ for $1 \leq i \leq n$ and we have to check that all those coordinates are different. For this we define $\boldsymbol{v}_{ij} = \boldsymbol{x}_i - \boldsymbol{x}_j$ for $1 \leq i < j \leq n$. Clearly $\boldsymbol{v}_{ij} \cdot (\boldsymbol{x}_i - \boldsymbol{x}_j) = \|\boldsymbol{v}_{ij}\|^2 > 0$, since $\boldsymbol{x}_i \neq \boldsymbol{x}_j$, and then $\boldsymbol{v}_{ij} \cdot \boldsymbol{x}_i > \boldsymbol{v}_{ij} \cdot \boldsymbol{x}_j$.

Also $\boldsymbol{v}_{ij} \cdot \boldsymbol{y}_i > \boldsymbol{v}_{ij} \cdot \boldsymbol{y}_j$, since $\boldsymbol{y}_i$ and $\boldsymbol{y}_j$ are the barycenters of the corresponding regions in the equipartition $\mathcal{P}$ obtained from the Voronoi map with the nodes given by $\boldsymbol{x} \in \mathcal{F}(\mathbb{R}^d, n)$. The regions P_i and P_j of this equipartition can be separated by a hyperplane orthogonal to the vector $\boldsymbol{v}_{ij}$, so that its corresponding barycenters are on different sides of that hyperplane, and such that $\boldsymbol{v}_{ij} \cdot \boldsymbol{y}_i > \boldsymbol{v}_{ij} \cdot \boldsymbol{y}_j$.

Then, for any value of $t \in [0, 1]$ and any pair $1 \leq i < j \leq n$, it holds that $t\boldsymbol{x}_i + (1-t)\boldsymbol{y}_i \neq t\boldsymbol{x}_j + (1 - t)\boldsymbol{y}_j$ since

$$\boldsymbol{v}_{ij} \cdot (t\boldsymbol{x}_i + (1 - t)\boldsymbol{y}_i) > \boldsymbol{v}_{ij} \cdot (t\boldsymbol{x}_j + (1 - t)\boldsymbol{y}_j).$$

Since all coordinates of $t\boldsymbol{x}+(1-t)f(\boldsymbol{x})$ are different, then $t\boldsymbol{x}+(1-t)f(\boldsymbol{x}) \in \mathcal{F}(\mathbb{R}^d, n)$ and ψ is a homotopy, showing that f is homotopy equivalent to the identity in $\mathcal{F}(\mathbb{R}^d, n)$. □

7.2 3-equipartitions of $\mathbb{R}^2$

We will present explicitly the example of the space of equipartition $\mathcal{C}^{\text{equi}}_{\text{reg}}(\mathbb{R}^2, 3, \mu)$. This will be useful later to analyze the spaces $\mathcal{C}(\mathbb{R}^2, 3)$ and $\mathcal{C}(\mathbb{R}^2, {\leq}3)$ (see Propositions 7.6 and 7.7). The space $\mathcal{C}^{\text{equi}}(S^2, 3, \mu)$ of 3-equipartitions of the sphere S^2 was already studied by Bárány et al. in [8]. Since they looked at equipartitions of a 2-sphere by a fan, the degeneracies in case that for example the partition is made by parallel lines are excluded. However, there the topology of $\mathcal{C}^{\text{equi}}(S^2, 3, \mu)$ is not always the same and depends on the measure μ. An example of this situation in a smaller dimension is presented next.

Proposition 7.4. *The topology of the space $\mathcal{C}^{\text{equi}}(S^1, 3, \mu)$ depends on the choice of the measure μ in S^1,* *Proof.* If the measure μ is equally distributed on S^1, we get that $\mathcal{C}^{\text{equi}}(S^1, 3, \mu)$ is homeomorphic to two copies S^1 (one for each orientation of the labels of the regions), but if the measure is concentrated in a small interval, the convexity requirement makes that not any first choice of the starting point of the first region gives a convex equipartition giving at the end two copies of the union of some intervals on the circle S^1. □

It is not clear yet if the topology of the space of equipartitions of $\mathbb{R}^d$ also depends on the choice of the measure μ, for $d \geq 2$. For $d = 1$ it always consists on $n!$ points and doesn't depend on the measure. The next proposition shows another example where the topology is independent of the measure.

Proposition 7.5. *For any positive continuous bounded measure μ on $\mathbb{R}^2$, the space $\mathcal{C}^{\text{equi}}(\mathbb{R}^2, 3, \mu)$ is homeomorphic to $S^1 \times T$, where T is the 2-dimensional simplicial cell complex obtained from two disjoint triangles by identifying corresponding vertices.*

Proof. The labeled partitions of the plane into three convex regions are generically given by three cones around a point. Therefore $\mathcal{C}^{\text{equi}}(\mathbb{R}^2, 3, \mu)$ has two main components depending the orientation. We will analyze first the space of equipartitions with a fixed orientation (take P_1, P_2, P_3 appearing clockwise in that order) and such that the ray between the regions P_1 and P_3 has a constant direction given by a unit vector w. In that case, we only need to fix the vertex $v = F_{123} \in \mathbb{R}^2$ to completely determine the equipartition, since the other rays through v are given by the measure.

We only need to determine where can this point be, since not for every v in $\mathbb{R}^2$ we get a convex equipartition. The first observation is that the point v has to be between two parallel lines ℓ_1 and ℓ_2 in the direction of w that splits the measure in three equal parts. Otherwise, if both of the regions P_1 or P_3 are convex, one of them would have measure less than one third. On the other hand, if v is between ℓ_1 and ℓ_2, then P_1 or P_3 must be convex, and in case v lie over one of the two lines, one of the regions will be a half space, and the corresponding equipartition won't be regular.

To see in which cases the region P_2 is convex, we analyze the extreme case when it is bounded by a straight line. Such line should split two thirds of the measure in the direction of w. All those lines create a continuous family of lines going from ℓ_1 to ℓ_2. On each of those lines ℓ there will be a unique point v such that the ray in direction w through v makes an equipartition together with ℓ. As the line ℓ moves, v goes in a continuous path, as depicted in Figure 7.1. The main observation is that it will be asymptotic to both ℓ_1 and ℓ_2, and the space of partitions with a fixed orientation and direction w is homeomorphic to a triangle (without the vertices).

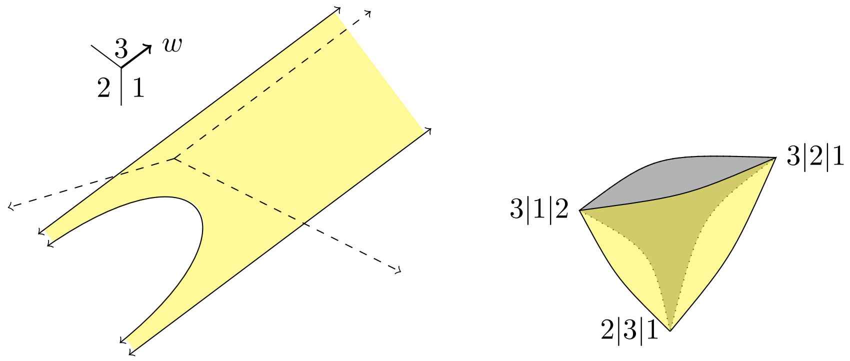

FIGURE 7.1: Equipartitions of $\mathcal{C}^{\mathrm{equi}}(\mathbb{R}^2, 3, \mu)$ given a fixed direction w and an orientation, together with the complex T.

One way to see that this is really a triangle is looking at the possible angles at v that have to be positive, smaller than π and add up to 2π. For each selection of angles there will be a unique equipartition (this is a consequence of Theorem 5.5 by fixing the measures to one third of $\mu(\mathbb{R}^2)$). The set of possible angles has two dimensions of freedom and is a triangle (2-simplex) Δ_2 bounded by three inequalities. The interior of the boundary edges of these triangles correspond to non-regular partitions, since there we get a π-angle.

There are three extreme equipartitions when the point v goes to infinity (i. e. to the vertices of the triangle). The regions of these partitions are bounded precisely by the lines ℓ_1 and ℓ_2. These partitions by parallel lines don't have an orientation. If we consider the opposite orientation and take $-w$ as the basic direction instead of w, we end up with exactly the same three equipartitions in the limit, and therefore both main generic pieces corresponding to the two possible orientations are connected through partitions by parallel lines. Notice that these partitions are still regular, despite the fact that they are on the boundary of the generic types.

If we consider the two triangles corresponding to the equipartitions with opposite orientation and opposite direction (w and $-w$), we see that the vertices correspond to the same equipartitions by parallel lines. Gluing the corresponding vertices we obtain a graph homotopy equivalent to the complex T, depicted at the right of Figure 7.1. Now, as we rotate w around each possible direction in S^1 we get all possible equipartitions by this construction (both regular and non-regular as the boundary case), and we conclude that both spaces $\mathcal{C}^{\mathrm{equi}}(\mathbb{R}^2, 3, \mu)$ and $\mathcal{C}^{\mathrm{equi}}_{\mathrm{reg}}(\mathbb{R}^2, 3, \mu)$ are homotopy equivalent to $S^1 \times T$, independently of the choice of μ. $\square$

7.3 More examples

The space $\mathcal{C}(\mathbb{R}^2, 3)$ can be parameterized in a similar way as we did with the space $\mathcal{C}^{\text{equi}}(\mathbb{R}^2, 3, \mu)$ in Proposition 7.5, after fixing a measure μ. First consider $\mathcal{C}(\mathbb{R}^2, 3)$ up to rotations, by fixing the direction of the ray w between regions P_3 and P_1. Then parameterize both simple combinatorial types (corresponding to the clockwise and anticlockwise orientations), this time by choosing the angles at the interior vertex and the measures of each of the regions. The set of possible angles is parameterized by a 2-simplex, where the edges represent the angles of non-regular partitions and the vertices represent partitions by two parallel lines. The possible measures $(\mu(P_1), \mu(P_2), \mu(P_3))$ are also parameterized by a 2-simplex, where this time the boundary is not allowed, since it corresponds to partitions with regions of measure zero. (We have to include this boundary to get as well non-proper partitions. This will be done later when we study $\mathcal{C}(\mathbb{R}^d, \leq 3)$.)

Then the set of generic 3-partitions with a fixed orientation, up to rotations, can be parameterized by the interior of the product of two triangles $\Delta_2 \times \Delta_2$. This is a 4-dimensional polytope and in Figure 7.2 we depict its boundary as a Schlegel diagram. For more information about Schlegel diagrams, see [36].

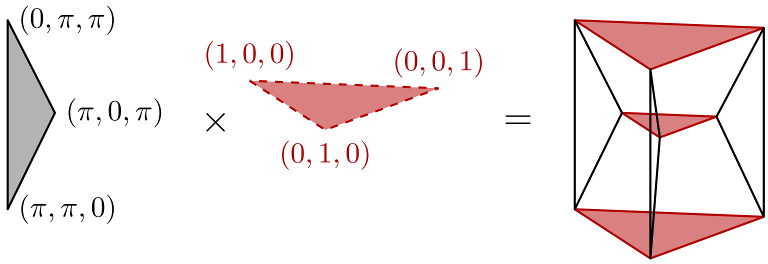

FIGURE 7.2: Schlegel diagram of the boundary of partitions in $\mathcal{C}(\mathbb{R}^2, 3)$ with a fixed orientation, up to rotations.

The 2-simplex parameterizing the measures appears there in red color, while the black triangle parameterizes the possible angles. Notice that some of the faces on the boundary of this polytope correspond to other combinatorial types of 3-partitions. Those faces correspond to the product of the interior of the red simplex with the boundary of the black simplex. In the Schlegel diagram, these are namely the interior of the three red triangles (corresponding to partitions by two parallel lines in the direction of w) and the interior of the three prisms with the red triangles as bases (corresponding to non-regular partitions with one π-angle). All other faces are not included here (but will have a meaning later; see Figure 7.3 for more information).

For the anticlockwise orientation we get a similar description. Again we can glue both pieces, this time on three open triangles (shaded in red on Figure 7.2) and take the product with S_1 to finally obtain the space $\mathcal{C}(\mathbb{R}^2, 3)$.

The space $\mathcal{C}(\mathbb{R}^2, 3)$ can be obtained by gluing both cells by the subset in its boundary corresponding to partitions by two parallel lines (that is made out of the interior of three triangles, depicted in red in Figure 7.2).

Proposition 7.6. *The spaces $\mathcal{C}(\mathbb{R}^2, 3)$ and $\mathcal{C}_{\text{reg}}(\mathbb{R}^2, 3)$ are homotopy equivalent. Also the spaces of equipartitions $\mathcal{C}^{\text{equi}}(\mathbb{R}^2, 3, \mu)$ and $\mathcal{C}^{\text{equi}}_{\text{reg}}(\mathbb{R}^2, 3, \mu)$ are homotopy equivalent to the $\mathcal{C}(\mathbb{R}^2, 3)$ for any continuous bounded measure μ.*

Proof. Fix a continuous bounded measure μ. To prove that $\mathcal{C}(\mathbb{R}^2, 3)$ is homotopy equivalent to $\mathcal{C}^{\text{equi}}(\mathbb{R}^2, 3, \mu)$, we can see that the only difference between these two spaces is given by the measures, and $\mathcal{C}(\mathbb{R}^2, 3)$ is homeomorphic to the product of $\mathcal{C}^{\text{equi}}(\mathbb{R}^2, 3, \mu)$ with the interior of a 2-simplex Δ_2 that parameterizes the measures of the three different regions. The homotopy is given by contracting this simplex to a point. Since $\mathcal{C}^{\text{equi}}(\mathbb{R}^2, 3, \mu)$ is a subset of $\mathcal{C}(\mathbb{R}^2, 3)$, then this contraction will be extended to a contraction of $\mathcal{C}(\mathbb{R}^2, 3)$ to its subset of equipartitions. Similarly, the space $\mathcal{C}_{\text{reg}}(\mathbb{R}^2, 3)$ can also be contracted to $\mathcal{C}^{\text{equi}}_{\text{reg}}(\mathbb{R}^2, 3, \mu)$.

To see that the spaces $\mathcal{C}^{\text{equi}}(\mathbb{R}^2, 3, \mu)$ and $\mathcal{C}^{\text{equi}}_{\text{reg}}(\mathbb{R}^2, 3, \mu)$ are homotopy equivalent, notice that the space T described in 7.5 can be contracted to a complex T' with two vertices and three edges between them and this contraction brings also the regular part (that is made out of all points of T except those on the interior of the edges) to T'. Therefore both spaces $\mathcal{C}^{\text{equi}}(\mathbb{R}^2, 3, \mu)$ and $\mathcal{C}^{\text{equi}}_{\text{reg}}(\mathbb{R}^2, 3, \mu)$ can be contracted to $T' \times S_1$. □

Using these ideas we will now describe $\mathcal{C}(\mathbb{R}^2, \leq 3)$ as a cell complex. Here we need to analyze further all other faces on the boundary of the simple cells described in Figure 7.2. When we approach the boundary at different points, we might end up on the same partition, and therefore some faces of the product of simplices $\Delta_2 \times \Delta_2$ have to be contracted in order to get the right cell complex around the simple combinatorial types. In Figure 7.3 we show this situation, still based on the Schlegel diagram from Figure 7.2.

The three black triangles represent the partitions with only one non-empty region, and therefore have to be contracted to a point. Also the prism on the back, depicted in yellow (with black triangular bases corresponding to the non-proper partitions with regions one and three equals $\mathbb{R}^d$ respectively) need to be contracted to the edge labeled as "1|3" on the face poset.

In general, labels in the face poset are direct drawings of how the partition looks like. The only convention is that vertical lines cannot be rotated (representing the choice of w going "down") while lines in other directions can be rotated freely, until reaching a vertical direction or the direction of another edge. Rounded cells are those that belong simultaneously to the boundary of both simple combinatorial types.

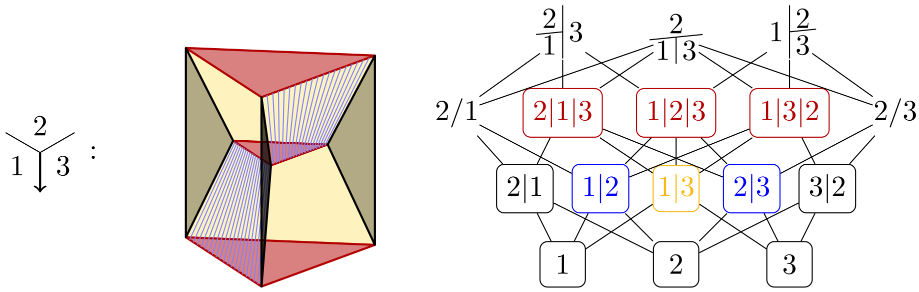

FIGURE 7.3: Sketch of the boundary of the clockwise cell of $\mathcal{C}(\mathbb{R}^2, \leq 3)$, based on the Schlegel diagram on Figure 7.2. Its face poset is shown at the right. Rounded faces on the poset have to be glued with the anticlockwise cell.

Other faces that need to be contracted are the 2-faces colored in blue. For example, the one at the lower left will represent the edge labeled as "1|2" (where each vertical stripe have to be contracted to a point). Similarly, the prism on the left (containing this blue 2-face in the Schlegel diagram) have to be contracted to a 2-face bounded by the new edge "1|2" (the blue 2-face) and the other edge joining the two black triangles, labeled as "2|1" (this edge doesn't have to be confused with "1|2"). The new two face in between is labeled as "2/1". An analogous description can be made for the other blue face, that contracts to the edge "2|3".

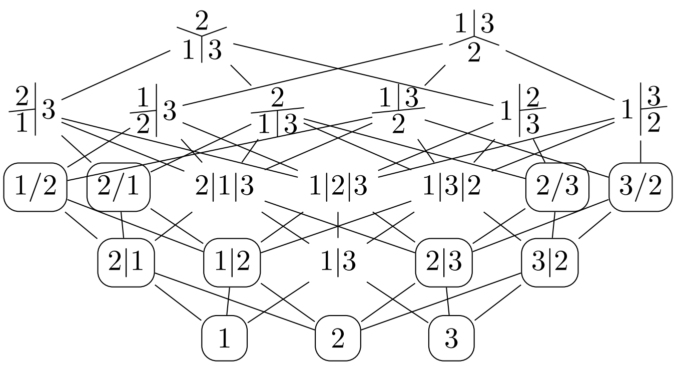

FIGURE 7.4: Face poset of $\mathcal{C}(\mathbb{R}^2, \leq 3)$ up to rotations of H_{13}, obtained by glueing the two cells, whose boundary is described on Figure 7.3. Rounded cells are fixed under the rotations.

For the cell with anticlockwise orientation, we can obtain a similar poset. We can get the labels by "reflecting" vertically those on Figure 7.3. If we glue the two cells through the faces that represent partitions that belong to the boundary of both (rounded faces on the face poset of Figure 7.3), we obtain the face poset of Figure 7.4. This represent all 3-partitions, up to rotations of the direction of the face F_{13} (given by the vector w in the clockwise cell).

To get a complete cell decomposition for $\mathcal{C}(\mathbb{R}^2, \leq 3)$ we only need to take the rotation of the complex described by 7.4, where the rounded cells are fixed under the rotation, while all other cells are replaced by four new cells, as we take the product with the cell decomposition of S_1 by to 1-cells and two 0-cells to get the face poset of Figure 7.5.

Proposition 7.7. *The space $\mathcal{C}(\mathbb{R}^2, \leq 3)$ is a homeomorphic to a 5-dimensional regular cell complex, with face poset as in Figure 7.5.*

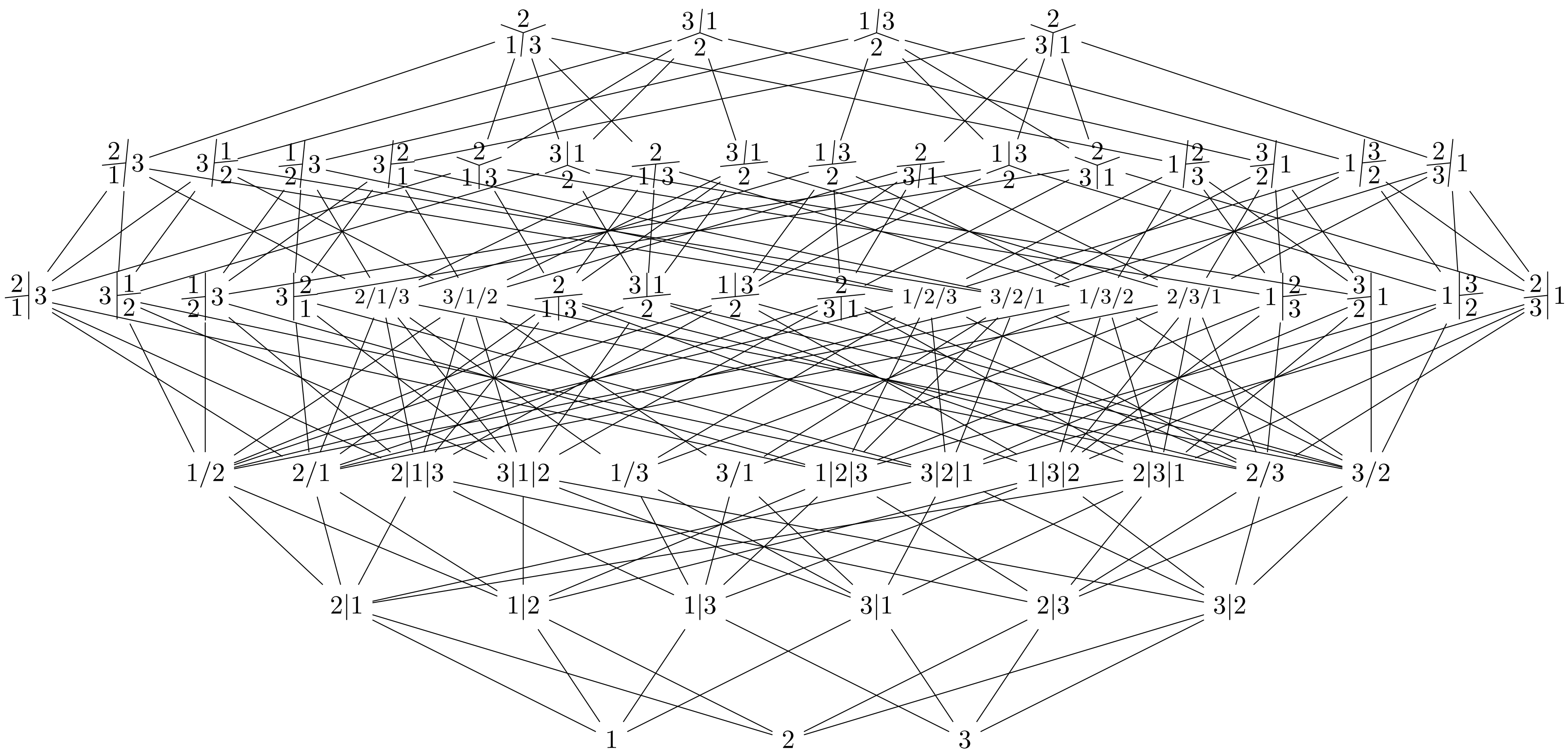

FIGURE 7.5: Face poset of $\mathcal{C}(\mathbb{R}^2, \leq 3)$.

Chapter 8

Further questions

Question 8.1. *How can we give $\mathcal{C}(\mathbb{R}^d, \leq n)$ a nice cell complex structure? (At least for $d = 2$?)*

For $d = 1$ we already described such cell complex structure (Proposition 4.49) and for $n \leq 2$ we know that the space $\mathcal{C}(\mathbb{R}^d, \leq 2)$ is a d-sphere (Proposition 4.48), but already in this case it is not clear how to get a cell decomposition in a canonical way. The combinatorial types give a decomposition into semialgebraic pieces, but they are not homeomorphic to balls in general. Nevertheless, it is required for a cell complex decomposition to respect and refine the combinatorial types.

For $d = 3$, the universality result given in Theorem 5.17 suggests that the combinatorial types might be still far to provide a cell decomposition directly. It is still open if there is also a universality result for the general realization spaces, since Theorem 5.17 only talks about the regular partitions of a given combinatorial type, up to affine equivalence.

For $d = 2$, the following conjecture might help to find a nice cell complex for $\mathcal{C}(\mathbb{R}^2, \leq n)$.

Conjecture 8.2. *The realization space $\mathcal{C}_{\mathcal{P}}(\mathbb{R}^2, n)$ of a generic partition $\mathcal{P} \in \mathcal{C}(\mathbb{R}^2, \leq n)$, up to rotations, is contractible.*

See Definition 5.7 for the meaning of generic partitions. To prove Conjecture 8.2, we suggest to use a similar argument to the one in [28, Theorem 12.2.2] that is used to prove that realization spaces of 3-polytopes are contractible, based in a physical principle of "rubber bands". This is also used in one of the proofs of Steinitz Theorem (see [36, Chapter 4] for more details). If an analogous result of that theorem also holds in a spherical setup, then from any planar graph on the upper hemisphere $\overline{S}^d_+$ such that it is 2-connected and the boundary vertices are fixed on the boundary of $\overline{S}^d_+$, we can obtain a unique planar embedding by minimizing an energy functional on the length of the edges. It is needed that this functional is convex and have a unique minimum on the spherical setting (see [17] where similar considerations on spherical graphs are studied). If this minimization can be done continuously following a gradient of the energy, we can conclude that the possible embeddings of the graph can be contracted. This can also be seen as an application of Morse theory. Also, embeddings where the regions are convex are likely to preserve convexity under the process, as they do in the euclidean case. For higher dimensions we have the universality results (Theorem 5.17) that tell us that this is not likely to happen for n-partitions of $\mathbb{R}^d$ for $d \geq 3$.

Question 8.3. *How can we realize $\mathcal{C}(\mathbb{R}^d, n)$ as a semialgebraic set?*

In theorems 4.14 and 4.47 it is established that the spaces of n-partitions are unions of pieces that can be described as semialgebraic sets. But those pieces are still living in different spaces and to glue them it is necessary to use the topological structure on the space. It is desired to have a global semialgebraic realization of the whole space $\mathcal{C}(\mathbb{R}^d, n)$, or even better of the space $\mathcal{C}(\mathbb{R}^d, \leq n)$. We wanted to obtain such realization from the hyperplane description, but as we found Proposition 4.17, it is not possible to get that immediately inside the space of hyperplane arrangements $\mathcal{H}(\mathbb{R}^d, n)$ from preimages under the map π.

Question 8.4. *What is the dimension of $\mathcal{C}(\mathbb{R}^d, n)$ for $d \geq 3$ and $n \geq 3$? Which combinatorial types attain this maximal dimension?*

In Conjecture 6.12 we suggest that simple partitions have the maximal dimension, while in Conjecture 6.15 we describe the combinatorial types of partitions that possibly attain this upper bound. Proposition 6.16 shows a big family of partitions where this upper bound is attained, together with simple partitions. It is not hard to construct some more examples.We would like to know if there are other combinatorial types with realization space of dimension $4n - 5$ or even higher. Also we would like to know for which combinatorial types the improved naive count give the right dimension for its realization space.

Question 8.5. *Are 2-simple partitions dense in $\mathcal{C}(\mathbb{R}^3, n)$?*

As we already mentioned, there are other combinatorial types in $\mathcal{C}(\mathbb{R}^3, n)$ that attain the maximal conjectured dimension, other than the regular case of simple and generic partitions. It is enough that the partitions are 2-simple and generic in order to attain this upper bound?

Question 8.6. *Is the subspace $\mathcal{C}_{\text{reg}}(\mathbb{R}^d, n)$ homotopy equivalent to $\mathcal{C}(\mathbb{R}^d, n)$?*

For $n \le 2$, these spaces coincide, since in those cases all partitions are regular. For $n = 3$, there are some non-regular partitions, namely those that have a π-angle. These are at the boundary of $\mathcal{C}(\mathbb{R}^d, 3)$, and can be retracted to the interior, at least in the case of $d = 2$ (as discussed in Proposition 7.6).

Question 8.7. *Is $\mathcal{C}^{\text{equi}}_{\text{reg}}(\mathbb{R}^d, n, \mu)$ homotopy equivalent to $\mathcal{C}^{\text{equi}}(\mathbb{R}^d, n, \mu)$?*

Indeed, it would be very interesting to know whether the maps

$$\mathcal{F}(\mathbb{R}^d, n) \underset{\mathfrak{S}_n}{\longrightarrow} \mathcal{C}^{\text{equi}}_{\text{reg}}(\mathbb{R}^d, n) \underset{\mathfrak{S}_n}{\hookrightarrow} \mathcal{C}^{\text{equi}}(\mathbb{R}^d, n) \underset{\mathfrak{S}_n}{\longrightarrow} \mathcal{F}(\mathbb{R}^d, n).$$

indicated in Theorem 7.2 are homotopy equivalences.

Since the Voronoi map is a map with contractible fibers, the first map should be a homotopy equivalence.

Question 8.8. *Is the topology of the space $\mathcal{C}^{\text{equi}}(\mathbb{R}^d, n, \mu)$ independent of μ?*

As suggested by proposition 7.4, it is likely that for $\mathbb{R}^d$ the topology of spaces of equipartitions do depend on the measure. But in the smaller dimensional examples we can check up to now, the topology is independent on the choice of μ. See for example Proposition 7.5 where the topology of the space of equipartitions doesn't depend on μ. See also Conjecture 8.10. Maybe there is something essentially different between equipartitions of S^d and equipartitions of $\mathbb{R}^d$.

Question 8.9. *Is the subspace of equipartitions $\mathcal{C}^{\text{equi}}(\mathbb{R}^d, n, \mu)$ homotopy equivalent to $\mathcal{C}(\mathbb{R}^d, n)$ (at least for some measure μ)?*

We offer the following conjecture in relation with this question.

Conjecture 8.10. *The spaces $\mathcal{C}^{\text{equi}}(\mathbb{R}^2, 4, \mu)$ and $\mathcal{C}^{\text{equi}}_{\text{reg}}(\mathbb{R}^2, 4, \mu)$ are homotopy equivalent, independently of the measure μ.*

The space $\mathcal{C}^{\text{equi}}(\mathbb{R}^2, 4, \mu)$ has two simple combinatorial types (up to permuting the labels): one having a bounded cell as in Figure 5.1 and one without bounded cells. For the case of combinatorial types with one bounded cell, the dimension is six, but the subset of regular partitions there has codimension one. The realization space for the combinatorial types without bounded cells has dimension 5 and there all partitions are regular. First we will construct a homotopy equivalence between the space $\mathcal{C}^{\text{equi}}(\mathbb{R}^2, 4, \mu)$ to the closure of $\mathcal{C}^{\text{equi}}_{\text{reg}}(\mathbb{R}^2, 4, \mu)$ in $\mathcal{C}^{\text{equi}}(\mathbb{R}^2, 4, \mu)$.

Let Δ be a triangle in $\mathbb{R}^2$ with $\mu(\Delta) = \frac{1}{4}\mu(\mathbb{R}^2)$. The set $\mathcal{C}^{\text{equi}}_{\Delta}(\mathbb{R}^2, 4, \mu)$ consists of all equipartitions in $\mathcal{C}^{\text{equi}}(\mathbb{R}^2, 4, \mu)$ that have Δ as bounded cell. They make a one-parameter family of equipartitions, since choosing the direction of one unbounded edge fixes all the rest. The convexity conditions give six boundary inequalities to this set, but only two of them will be relevant, and therefore $\mathcal{C}_\Delta^{\text{equi}}(\mathbb{R}^2, 4, \mu)$ will be an interval, a point or empty. In case $\mathcal{C}_\Delta^{\text{equi}}(\mathbb{R}^2, 4, \mu)$ is an interval, there is a unique equipartitions there that is regular, that is precisely when the three unbounded edges intersect. It is easy to imagine how to contract this interval to this unique regular partition, since the direction of rotation of the unbounded edges is unique in order to make them intersect. If $\mathcal{C}_\Delta^{\text{equi}}(\mathbb{R}^2, 4, \mu)$ is a point, this partition will be typically non-regular, but it can be obtained as a limit of regular partitions (notice that the three unbounded edges intersect, see Figure 8.1). We don't care about the empty case.

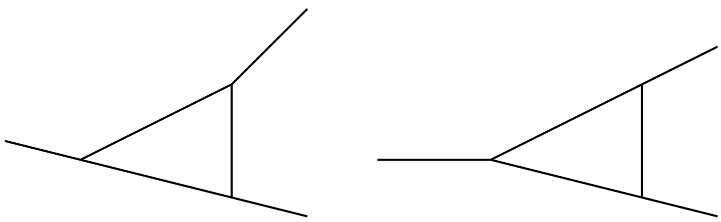

FIGURE 8.1: Non-regular partitions on the boundary of $\mathcal{C}_\Delta^{\text{equi}}(\mathbb{R}^2, 4, \mu)$.

As Δ varies, we can make this retraction into a homotopy equivalence between $\mathcal{C}^{\text{equi}}(\mathbb{R}^2, 4, \mu)$ and the closure of $\mathcal{C}_{\text{reg}}^{\text{equi}}(\mathbb{R}^2, 4, \mu)$. Now it only remain to be checked that the homotopy type of $\mathcal{C}_{\text{reg}}^{\text{equi}}(\mathbb{R}^2, 4, \mu)$ doesn't change by taking the closure, but only adds a boundary that can be retracted to the interior. We don't know if we can retract $\mathcal{C}(\mathbb{R}^d, n)$ to the subspace of partitions without π-angles, at least for $d = 2$.

Question 8.11. *What is the topological structure of the space of "unlabeled" n-partitions (where we don't take into account the order of the regions)?*

This is also a natural object to study. We decided to work with labeled partitions because it is easier to compare and parameterize them, and once understood the labeled partitions, it seems reasonable to get the unlabeled space by modding out the symmetry. Since many of the constructions we considered here depend on the labels, as for example the metric and topological structure, we won't say much here about this question, but the investigation of the *unlabeled configuration space* $\mathcal{F}(\mathbb{R}^d, n)/\mathfrak{S}_n$ is also very interesting, and has been studied extensively since the famous 1969 work of Arnol'd [1], who used it to compute the cohomology of the *coloured braid group*. The spaces $\mathcal{F}(\mathbb{R}^d, n)$ and $\mathcal{F}(\mathbb{R}^d, n)/\mathfrak{S}_n$ are of interest for cohomology computations as they have contractible covers, and thus the cohomology of the spaces is the cohomology of their fundamental groups.

Appendix A

Summaries

A.1 English Summary

We look at the space $\mathcal{C}(\mathbb{R}^d, n)$ of all partitions of $\mathbb{R}^d$ into n convex regions for d and n positive integers. Here we introduce some basic concepts and definitions about them, investigate their general properties and look at some examples and related spaces.

We begin with some basic notions and results of convex geometry that we need, as polyhedra, cones, spherical polyhedra, hyperplane arrangements and CW-complexes.

In Chapter 3 we introduce convex n-partitions and we prove that all the regions of a partition must be polyhedral. Then we define some related notions, such as spherical partitions and the face structure, and prove some basic facts about them.

In Chapter 4 we look at the space $\mathcal{C}(\mathbb{R}^d, n)$ of all convex n-partitions of $\mathbb{R}^d$, describing the metric structure there that fixes the topology of the space and also a natural compactification $\mathcal{C}(\mathbb{R}^d, \leq n)$ where empty regions are allowed. Then we prove that spaces of n-partitions are union of semialgebraic pieces in two different ways. We look at hyperplane arrangements carrying an n-partition, and give a description of $\mathcal{C}(\mathbb{R}^d, n)$ where the pieces depend on the hyperplanes used to obtain the partition (Theorem 4.14). For the second description we need to introduce nodes and node systems that are a generalization of the vertices, and define the combinatorial type of a partition. These combinatorial types give the semialgebraic pieces that build the spaces (See Theorem 4.47). At the end of the chapter we describe explicitly particular spaces of n-partitions of $\mathbb{R}^d$ and their compactifications for $n = 2$ and also for $d = 1$.

In Chapter 5 we talk about regular partitions and mention some known results about them. Using these results we compute the dimension of the space of regular partitions $\mathcal{C}_{\text{reg}}(\mathbb{R}^d, n)$. Then we prove a universality theorem that says that realization spaces of regular partitions can be stably equivalent to any primary basic semialgebraic set (Theorem 5.17).

In Chapter 6 we investigate the dimensions of realization spaces. We first study the case $d = 2$ and find that for large n the dimension of $\mathcal{C}(\mathbb{R}^2, n)$ is much bigger than $\dim(\mathcal{C}_{\text{reg}}(\mathbb{R}^2, n))$. Then we focus on the case $d = 3$, where we conjecture that the dimension of $\mathcal{C}(\mathbb{R}^3, n)$ is equal to the dimension of $\mathcal{C}_{\text{reg}}(\mathbb{R}^3, n)$ and try to justify this with a heuristic counting for the dimension of each realization space. From this counting we find an incidence theorem for 3-polytopes (Theorem 6.11) and find many examples of partitions where this counting works.

In Chapter 7 we introduce the spaces of equipartitions $\mathcal{C}^{\text{equi}}(\mathbb{R}^d, n, \mu)$ given a positive bounded measure μ. We explore the topological structure of some small cases of spaces of equipartitions and using this, we describe the spaces of n-partitions for $d = 2$ and $n = 3$. We also discuss the Nandakumar and Ramana Rao problem [25] and different equivariant maps that show that considering regular equipartitions is as good as considering all equipartitions with respect to the approach based on configuration spaces to find fair partitions.

A.2 Zusammenfassung auf Deutsch

Wir betrachten den Raum $\mathcal{C}(\mathbb{R}^d, n)$ aller Aufteilungen von $\mathbb{R}^d$ in n konvexe Gebiete für positive d und n. Dafür entwickeln wir grundlegende Konzepte und Definitionen, untersuchen allgemeine Eigenschaften und betrachten verwandte Räume sowie Beispiele. Zunächst entwickeln wir dafür die benötigten Konzepte der Konvexgeometrie. In Kapitel 3 definieren wir konvexe n-Aufteilungen und zeigen, dass die Teile immer Polyeder sind. Dann definieren wir sphärische Aufteilungen und Seitenhalbordnungen und leiten grundlegende Strukturergebnisse ab.

Kapitel 4 beschäftigt sich mit dem Raum $\mathcal{C}(\mathbb{R}^d, n)$ aller konvexen n-Aufteilungen des $\mathbb{R}^d$. Wir beschreiben eine Metrik und damit eine Topologie auf diesem Raum, sowie eine natürliche Kompaktifizierung $\mathcal{C}(\mathbb{R}^d, \leq n)$, für die auch leere Teile erlaubt sind. Wir stellen den Raum der n-Aufteilungen dann auf zwei Weisen als eine Vereinigung von semialgebraischen Teilmengen dar: Wir betrachten Hyperebenenarrangements, die Aufteilungen induzieren, und beschreiben $\mathcal{C}(\mathbb{R}^d, n)$ so in Abhängikeit von den Hyperebenen, die die Aufteilung erzeugen (Theorem 4.14). Für die zweite Beschreibung führen wir Knoten und Knotensysteme ein, die Eckenmengen verallgemeinern, und definieren den kombinatorischen Typ einer Aufteilung. Diese kombinatorischen Typen ergeben semialgebraische Teile, aus denen die Räume aufgebaut sind (Theorem 4.47). Am Ende des Kapitels beschreiben wir wir explizit die Räume der n-Aufteilungen von $\mathbb{R}^d$ und ihre Kompaktifizierungen für $n = 2$ und für $d = 1$.

In Kapitel 5 diskutieren wir reguläre Aufteilungen. Wir berechnen die Dimension des Raums der regulären Aufteilungen $\mathcal{C}_{\text{reg}}(\mathbb{R}^d, n)$. Dann beweisen wir einen Universalitätssatz, wonach die Realiserungsräume regulärer Partitionen zu beliebigen primären basischen semialgebraischen Mengen stabil äquivalent sein können (Theorem 5.17).

In Kapitel 6 untersuchen wir die Dimension von Realisierungsräumen. Im Fall $d = 2$ ist die Dimension von $\mathcal{C}(\mathbb{R}^2, n)$ für große n viel größer als $\dim(\mathcal{C}_{\text{reg}}(\mathbb{R}^2, n))$. Dann konzentrieren wir uns auf den Fall $d = 3$, wo wir vermuten, dass die Dimension von $\mathcal{C}(\mathbb{R}^3, n)$ mit der Dimension von $\mathcal{C}_{\text{reg}}(\mathbb{R}^3, n)$ übereinstimmt, und versuchen das mit einer Heuristik für die Zahl der Freiheitsgrade und damit der Dimensionen der Realisierungsräume zu untermauern.

In Kapitel 7 führen wir die Räume von Äquipartitionen $\mathcal{C}^{\text{equi}}(\mathbb{R}^d, n, \mu)$ für beschränkte positive Maße μ ein. Wir untersuchen die topologische Struktur für einige kleine Fälle und beschreiben, darauf aufbauend, die Räume der n-Äquipartitionen für $d = 2$ und $n = 3$. Wir diskutieren auch das Problem von Nandakumar und Ramana Rao über "faire Aufteilungen von Polygonen" [25] und verschiedene äquivariante Abbildungen, die zeigen, dass es für dieses Problem ausreicht, reguläre Äquipartitionen zu betrachten.

A.3 Resumen en español

En este trabajo estudiamos el espacio $\mathcal{C}(\mathbb{R}^d, n)$ de todas las particiones de $\mathbb{R}^d$ en n regiones convexas para d y n enteros positivos. Aquí presentamos algunos conceptos básicos y definiciones sobre estos espacios, investigamos sus propiedades generales y damos un vistazo a algunos ejemplos y espacios relacionados. Comenzamos con algunas nociones básicas y resultados en geometría convexa. Luego en el capítulo 3 se introducen las n-particiones convexas y se demuestra que todas las regiones de una partición deben ser poliedros. Luego definimos algunos conceptos relacionados como particiones esféricas y caras y demostramos algunos resultados básicos sobre ellos.

En el capítulo 4 nos fijamos en el espacio $\mathcal{C}(\mathbb{R}^d, n)$ de todas las n-particiones convexas de $\mathbb{R}^d$. Describimos una estructura métrica que fija la topología del espacio y también una compactificación natural $\mathcal{C}(\mathbb{R}^d, \leq n)$ donde es posible tener regiones vacías. Luego probamos que los espacios de n-particiones son la unión de piezas semialgebraicas de dos maneras diferentes. Nos fijamos en los arreglos de hiperplanos que cargan una n-partición, y damos una descripción de $\mathcal{C}(\mathbb{R}^d, n)$ donde las piezas dependen de los hiperplanos utilizados (Teorema 4.14). Para la segunda descripción se introducen nodos y sistemas de nodos, que son una generalización de los vértices y son útiles para definir el tipo combinatorio de una partición. Estos tipos combinatorios generan los espacios de realizaciones, que son las piezas semialgebraicas que se usan para construir los espacios de particiones (Ver Teorema 4.47). Al final del capítulo se describen como ejemplo los espacios de n-particiones de $\mathbb{R}^d$ y sus compactificaciones para $n = 2$ y para $d = 1$.

En el capítulo 5 se introducen las particiones regulares y algunos resultados conocidos sobre ellas. Utilizando estos resultados se calcula la dimensión del espacio de particiones regulares $\mathcal{C}_{\text{reg}}(\mathbb{R}^d, n)$. Luego se demuestra un teorema de universalidad que dice que los espacios de realización de particiones regulares pueden ser establemente equivalente a cualquier conjunto semialgebraico básico primario (Teorema 5.17).

En el capítulo 6 se investigan las dimensiones de los espacios de realización. En primer lugar, se estudia el caso $d = 2$ en donde para n suficientemente grande la dimensión de $\mathcal{C}(\mathbb{R}^2, n)$ es mucho más grande que $\dim(\mathcal{C}_{\text{reg}}(\mathbb{R}^2, n))$. Luego nos centramos en el caso $d = 3$, donde conjeturamos que la dimensión de $\mathcal{C}(\mathbb{R}^3, n)$ es igual a la dimensión de $\mathcal{C}_{\text{reg}}(\mathbb{R}^3, n)$. Se intenta justificar esto con un conteo heurístico para la dimensión de cada espacio de realización, el cual nos genera un Teorema de incidencia en 3-politopos (Teorema 6.11). También encontramos varios ejemplos de particiones donde el conteo funciona.

En el capítulo 7 se introducen los espacios de equiparticiones $\mathcal{C}^{\text{equi}}(\mathbb{R}^d, n, \mu)$ dada una medida positiva y acotada μ. También exploramos la estructura topológica de algunos pequeños casos de espacios de equiparticiones y esto se usa para describir los espacios de n-particiones para $d = 2$ y $n = 3$. Discutimos el problema Nandakumar y Ramana Rao [25] y diferentes mapas equivariantes que muestran que considerar equiparticiones regulares es equivalente a considerar todas las equiparticiones para encontrar "particiones justas" con respecto al enfoque basado en los espacios de configuración.

Declaration of Authorship

Under Article 7 (4) of the PhD regulations of the Department of Mathematics and computer science at the Free University of Berlin, I hereby certify that I all the tools and results from others are always properly attributed and with the exception of such quotations, this thesis is entirely my own work. Furthermore, I certify that this work has not been submitted at an earlier doctoral procedure before.

Berlin, January 5, 2015

Eidesstattliche Erklärung

Gemäß §7 (4) der Promotionsordnung des Fachbereichs Mathematik und Informatik der Freien Universität Berlin versichere ich hiermit, dass ich alle Hilfsmittel und Hilfen angegeben und auf dieser Grundlage die Arbeit selbstständig verfasst habe. Des Weiteren versichere ich, dass ich diese Arbeit nicht schon einmal zu einem früheren Promotionsverfahren eingereicht habe.

Berlin, den 5. Januar 2015

Emerson León

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