

Quasiregular mappings and Hölder continuity of differential forms on Riemannian manifolds

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Dirk Christoph Franke

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Betreuer und Gutachter:

Professor Dr. Ilppo Simo Louhivaara,
Freie Universität Berlin

Professor Dr. Vladimir M. Miklyukov,
Brigham Young University, Provo, USA und Volgograd State University, Russia

Professor Dr. Matti Vuorinen,
University of Helsinki, Finland

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Preface

The present thesis has been written during the years 1996-1999 at the Department of Mathematics of the Freie Universität Berlin under the guidance of Professor Ilppo Simo Louhivaara.

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Dirk Ch. Franke

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Zusammenfassung

In dieser Dissertation wird der Versuch unternommen, gewisse bekannte und teilweise auch neue Resultate über quasireguläre Abbildungen auf Riemannschen Mannigfaltigkeiten mit Hilfe der Theorie der Differentialformen, insbesondere mit einigen erst vor Kurzem eingeführten Klassen von Differentialformen, mit den WT -Klassen, zu erzielen.

Quasireguläre, beziehungsweise quasikonforme Abbildungen in höherdimensionalen Räumen sind seit Ende der 50er Jahre Gegenstand intensiver Forschung. Die Theorie der n -dimensionalen quasiregulären Abbildungen ist für $n \geq 3$ weitgehend nichtlinear, sie hat Verbindungen zu bestimmten nichtlinearen partiellen Differentialgleichungen wie auch zur nichtlinearen Potentialtheorie.

Die WT -Klassen von Differentialformen auf Riemannschen Mannigfaltigkeiten werden definiert. Differentialformen, lokal aus den Komponenten einer quasiregulären Abbildungen zusammengesetzt, fungieren als die wichtigsten Beispiele für die WT -Klassen. Es kann gezeigt werden, dass diese speziellen Beispiele einer quasilinearen elliptischen Gleichung genügen. In dieser Arbeit wird auf zwei unterschiedliche Beweisführungen hingewiesen.

Der Kernpunkt dieser Arbeit ist die neue Definition der Hölder-Stetigkeit von Differentialformen auf Riemannschen Mannigfaltigkeiten. Diese Definition ist eine Verallgemeinerung der Hölder-Stetigkeit von Funktionen. Es wird gezeigt, dass ein großer Teil der Differentialformen aus den WT -Klassen Hölder-stetig sind. Der Beweis beruht auf Morrey's Lemma für Differentialformen auf Riemannschen Mannigfaltigkeiten und der Abschätzung des Energieintegrals von Differentialformen aus den WT -Klassen.

Curriculum vitae

On the 25th of April 1966 I was born in Berlin as the son of Rosemarie and Fritz Franke. My parents gave me the name Dirk Christoph. Three years later my parents moved to the village Durmersheim near Karlsruhe. There I absolved the primary (1972-76) and secondary school (1976-85).

Directly after school I started to study electrical engineering at the Technische Hochschule Karlsruhe. After finishing the intermediate diploma I decided to change town and subject. In October 1988 I started to study mathematics at the Freie Universität Berlin. Two years later I completed the intermediate diploma and in April 1995 the diploma in mathematics with physics as subsidiary subject. Already my diploma thesis was about quasiregular mappings in higher dimensions.

From October 1990 to September 1994 I worked as an assistant teacher at the Department of Mathematics at the Technische Universität Berlin. From June 1996 to February 1999 I got a postgraduate fellowship granted by the Land Berlin (NaFöG).

On the 26th of August 1997 my son Julius was born.