where the infimum is taken over all possible sequences $\{m_k\}$, $m_k \in \mathcal{M}$, not having accumulation points in $\mathcal{M}$. For the domain D we assume that there exists a constant $c_7 > 0$, such that

$$\mathrm{mes}_n(B(a_1, d) \cap B(a_2, d)) \geq c_7\, d^n \tag{9.20}$$

for all points $a_1, a_2 \in D$, satisfying the condition

$$d = d(a_1, a_2) \leq \frac{1}{2}\delta(D)\ . \tag{9.21}$$

Now we deduce the well-known form of Morrey's lemma for differential forms on Riemannian manifolds. For the special case of functions compare with [GT] §12.1 and [Re] §2.1.

9.22. Theorem. *Suppose that the manifold $\mathcal{M}$ satisfies the properties I), II), and III) with the constant $\delta > 0$. Let $D \subset\subset \mathcal{M}$ be a domain such that $\delta \leq \delta(D)/2$ and (9.20) holds. Let $\omega \in W^{1,p}_{\mathrm{loc}}(\mathcal{M})$ be a differential form of degree k, $0 \leq k \leq n$, $p \geq 1$. If for every point $a \in D$ and for every $r \leq \delta(D)/2$ the inequality*

$$\int\limits_{B(a,r)} |d\omega|^p dv_{\mathcal{M}} \leq c_5\, r^{n-p+\alpha} \tag{9.23}$$

holds, then the differential form ω can be redefined on a set of measure zero such that for all $a_1, a_2 \in D$, $d(a_1, a_2) < \delta$, we get

$$\inf_{\gamma \in \Gamma(a_1,a_2)} \int\limits_{\gamma} |d\omega| ds_{\mathcal{M}} \leq \frac{c_6}{c_7}\, d^{\frac{\alpha}{p}}\ , \tag{9.24}$$

where c_6 is the constant from Lemma 9.7.

Proof. If we replace in Lemma 9.7 the function ρ by the value of the differential form $d\omega$, the theorem follows directly with the help of (9.20). □

10 Estimate for the energy integral

Here we present an estimate for the energy integral of the differential form $d\omega \in \mathcal{WT}_2$.

We need a quantity, the fundamental frequency of the free membrane $\Sigma(a,r)$. G. Pólya and G. Szegö [PS] §5 worked out a similar idea in two dimensions.

Let $a \in \mathcal{M}$ be a fixed point and let $\Sigma(a,r) \subset \mathcal{M}$ be a geodesic sphere and a manifold of dimension $n-1$. Let $\omega \in W^{1,p}(\Sigma(a,r))$, $\deg \omega = k$, $1 \le k \le n-1$, be a differential form. We define the quantity

$$\mu(a,r) = \inf_{\omega} \frac{\left(\int\limits_{\Sigma(a,r)} |d_\Sigma \omega|^p dH^{n-1}\right)^{\frac{1}{p}}}{\left(\inf\limits_{\omega_0} \int\limits_{\Sigma(a,r)} |\omega - \omega_0|^p dH^{n-1}\right)^{\frac{1}{p}}} \tag{10.1}$$

where the operator d_Σ denotes the differential operator on $\Sigma(a,r)$ and where ω_0 is a differential form with constant coefficients and $\deg \omega_0 = \deg \omega$. Here dH^{n-1} is the element of the $(n-1)$-dimensional Hausdorff measure on $\Sigma(a,r)$.

10.2. Theorem. *If the differential form $d\omega$ is in the class $\mathcal{WT}_2$, then with some β, for every $a \in \mathcal{M}$ and for every $\delta < r_{\text{inj}}(a)$ the function*

$$\phi_a(r) = \frac{1}{r^{n-p+\beta}} \int\limits_{B(a,r)} |d\omega|^p dv_{\mathcal{M}}$$

is increasing on $(0,\delta)$.

10.3. Remark. From the proof it will be clear that we can choose

$$\beta = \frac{\nu_1}{\nu_2} \inf_{r\in(0,\delta)} r\mu(a,r) - n + p \tag{10.4}$$

where ν_1, ν_2 are the constants of (5.1) and (5.2).

Proof. Let $a \in \mathcal{M}$ be a fixed point. We prove that for some $\beta > 0$ the derivative $\phi_a'(r) \ge 0$ almost everywhere in $(0,\delta)$. For almost every $r \in (0,\delta)$ we have with (9.2)

$$\frac{d}{dr} \int\limits_{B(a,r)} |d\omega|^p \, dv_{\mathcal{M}} = \int\limits_{\Sigma(a,r)} |d\omega|^p \, dH^{n-1} \, .$$

The condition $\phi_a'(r) \geq 0$ is equivalent to the inequality

$$(10.5) \qquad (n-p+\beta) \int\limits_{B(a,r)} |d\omega|^p \, dv_{\mathcal{M}} \leq r \int\limits_{\Sigma(a,r)} |d\omega|^p \, dH^{n-1} \,.$$

We take $r_0 \in (0, \delta)$, choose $\varepsilon > 0$ such that $r_0 + \varepsilon < \delta$ and define the function

$$\phi(t) = \begin{cases} 1 & \text{for} \quad t < r_0 \,, \\ 1 + r_0/\varepsilon - t/\varepsilon & \text{for} \quad t \in [r_0, r_0 + \varepsilon] \,, \\ 0 & \text{for} \quad t > r_0 + \varepsilon \,. \end{cases}$$

For every differential form ω_0 with constant coefficients, $\deg \omega_0 = \deg \omega$, the function $\phi(d(a,m))\,(\omega(m) - \omega_0)$ belongs to the class $W^{1,p}_{\text{loc}}(\mathcal{M})$ and is equal to 0 for $m \in \mathcal{M} \setminus B(a, r_0 + \varepsilon)$. Because of Theorem 5.6 the differential form ω is A-harmonic, therefore (5.5) yields

$$\int\limits_{\mathcal{M}} \langle A(m, d\omega), d(\phi(d(a,m))\,(\omega - \omega_0))\rangle \, dv_{\mathcal{M}} = 0$$

and we get

$$\int\limits_{\mathcal{M}} \phi(d(a,m)) \langle A(m, d\omega), d\omega\rangle \, dv_{\mathcal{M}}$$
$$= - \int\limits_{\mathcal{M}} \langle A(m, d\omega), d\phi \wedge (\omega - \omega_0)\rangle \, dv_{\mathcal{M}}.$$

Because $\phi(d(a,m)) = 0$ on $\mathcal{M} \setminus B(a, r_0 + \varepsilon)$ we obtain

$$\int\limits_{B(a, r_0+\varepsilon)} \phi \, \langle A(m, d\omega), d\omega\rangle \, dv_{\mathcal{M}}$$
$$\leq \int\limits_{r_0 < d(a,m) < r_0 + \varepsilon} |d\phi| \, |\omega - \omega_0| \, |A(m, d\omega)| \, dv_{\mathcal{M}}$$
$$\leq \frac{1}{\varepsilon} \int\limits_{r_0 < d(a,m) < r_0 + \varepsilon} |\nabla d(a,m)| \, |\omega - \omega_0| \, |A(m, d\omega)| \, dv_{\mathcal{M}} \,.$$

Observing that

$$|\nabla d(a,m)| = 1 \quad \text{in} \quad B(a, \delta)$$

with (5.1) and (5.2) and with (9.2) we get

$$\begin{aligned}\nu_1 \int\limits_{B(a,r_0)} |d\omega|^p dv_{\mathcal{M}} &\leq \frac{1}{\varepsilon}\nu_2 \int\limits_{r_0<d(a,m)<r_0+\varepsilon} |\omega-\omega_0|\,|d\omega|^{p-1}\,dv_{\mathcal{M}} \\ &\leq \frac{1}{\varepsilon}\nu_2 \int\limits_{r_0}^{r_0+\varepsilon} dt \int\limits_{\Sigma(a,t)} |\omega-\omega_0|\,|d\omega|^{p-1}\,dH^{n-1}\,.\end{aligned}$$

Passing to the limit $\varepsilon \to 0$, one gets

$$\begin{aligned}(10.6)\qquad \nu_1 \int\limits_{B(a,r_0)} |d\omega|^p dv_{\mathcal{M}} &\leq \nu_2 \int\limits_{\Sigma(a,r_0)} |\omega-\omega_0|\,|d\omega|^{p-1}\,dH^{n-1} \\ &= \nu_2 I\,.\end{aligned}$$

Next we employ the following modified form of the Young inequality

$$ab \leq \frac{\tau^p}{p}a^p + \frac{p-1}{p}\tau^{-\frac{p}{p-1}}\,b^{\frac{p}{p-1}}$$

for $a, b > 0$ and some $\tau > 0$. We reach to

$$\begin{aligned}(10.7)\quad I &\leq \frac{\tau^p}{p}\int\limits_{\Sigma(a,r_0)} |\omega-\omega_0|^p\,dH^{n-1} + \frac{p-1}{p}\tau^{-\frac{p}{p-1}}\int\limits_{\Sigma(a,r_0)} |d\omega|^p\,dH^{n-1} \\ &= \frac{\tau^p}{p}I_1 + \frac{p-1}{p}\tau^{-\frac{p}{p-1}}\,I_2\,.\end{aligned}$$

The differential form ω belongs to $W^{1,p}(\Sigma(a,r_0))$ for almost every $r_0 \in (0,\delta)$. Choosing the optimal constant differential form ω_0 in I_1 we obtain from (10.1)

$$(10.8)\quad I_1 = \int\limits_{\Sigma(a,r_0)} |\omega-\omega_0|^p\,dH^{n-1} \leq \frac{1}{\mu(a,r_0)^p}\int\limits_{\Sigma(a,r_0)} |d_\Sigma\omega|^p\,dH^{n-1}\,.$$

If we think of $|d\omega|$ as a composition of $|d_\Sigma\omega|$ and the projection to the orthogonal direction of $d_\Sigma\omega$, we see that

$$|d_\Sigma\omega| \leq |d\omega|\,.$$

Combining (10.6), (10.7) and (10.8) yields

$$\nu_1 \int\limits_{B(a,r_0)} |d\omega|^p dv_{\mathcal{M}} \leq \Big(\nu_2\frac{\tau^p}{p\,\mu(a,r_0)^p} + \nu_2\frac{p-1}{p}\tau^{\frac{-p}{p-1}}\Big)\int\limits_{\Sigma(a,r_0)} |d\omega|^p dH^{n-1}\,.$$

Setting

$$\tau = \mu(a, r_0)^{\frac{p-1}{p}}$$

we get

$$\begin{aligned}
\frac{\nu_1}{\nu_2} \int\limits_{B(a,r_0)} |d\omega|^p dv_{\mathcal{M}} &\leq \Big(\frac{1}{p}\mu(a, r_0)^{-1} + \frac{p-1}{p}\mu(a, r_0)^{-1}\Big) \int\limits_{\Sigma(a,r_0)} |d\omega|^p dH^{n-1} \\
&\leq \mu(a, r_0)^{-1} \int\limits_{\Sigma(a,r_0)} |d\omega|^p dH^{n-1} \\
&\leq \frac{r_0}{c} \int\limits_{\Sigma(a,r_0)} |d\omega|^p dH^{n-1}
\end{aligned}$$

with $c = \inf_{r\in(0,\delta)} r\mu(a, r)$. The theorem follows with $\beta = \frac{\nu_1}{\nu_2}c - n + p$. □

Now we can state an estimate for the energy integral of a differential form of the class $\mathcal{WT}_2$. For the subdomain $D \subset\subset \mathcal{M}$ we set $\delta(D)$ as in (9.19).

10.9. Theorem. *If the differential form $d\omega$ is in the class $\mathcal{WT}_2$, then for every $a \in D$ and for every $\delta \leq \delta(D)/2$ and $\delta < r_{\text{inj}}(a)$ the estimate*

$$\int\limits_{B(a,r)} |d\omega|^p dv_{\mathcal{M}} \leq c_5\, r^{n-p+\beta} \tag{10.10}$$

holds for $r \in (0, \delta]$, with β from (10.4) and

$$c_5 = \frac{1}{\delta^{n-p+\beta}} \int\limits_{D'} |d\omega|^p dv_{\mathcal{M}}\ , \tag{10.11}$$

where $D' = \{m \in \mathcal{M} : \operatorname{dist}(m, D) \leq \delta(D)/2\}$.

Proof. By Theorem 10.2 we have at every point $a \in D$

$$\int\limits_{B(a,r)} |d\omega|^p dv_{\mathcal{M}} \leq \frac{r^{n-p+\beta}}{\delta^{n-p+\beta}} \int\limits_{B(a,\delta)} |d\omega|^p dv_{\mathcal{M}}$$

for all $r \leq \delta$. Therefore we get (10.10) with the constants above. □

We want to say something about the constant β (10.4), especially about the fundamental frequency $\mu(a,r)$ in (10.1). Let $a \in \mathcal{M}$ be a fixed point and let $\Sigma(a,r) \subset \mathcal{M}$ be a geodesic sphere. With a differential form $\omega \in W^{1,p}(\Sigma(a,r))$, $\deg \omega = k$, $1 \le k \le n-1$, we define another quantity

$$\varepsilon(a,r) = \sup_{\omega_0} \frac{\left(\int\limits_{\Sigma(a,r)} |d_\Sigma \omega|^p dH^{n-1} \right)^{\frac{1}{p}}}{\left(\int\limits_{\Sigma(a,r)} |\omega - \omega_0|^p dH^{n-1} \right)^{\frac{1}{p}}} \tag{10.12}$$

where d_Σ denotes again the differential operator on $\Sigma(a,r)$ and where ω_0 is again a differential form with constant coefficients, $\deg \omega_0 = \deg \omega$. We have

$$\mu(a,r) \le \varepsilon(a,r)$$

and Theorem 10.2 and Theorem 10.9 remain valid with the quantity $\varepsilon(a,r)$ instead of $\mu(a,r)$, i.e. we can choose β to be

$$\beta = \frac{\nu_1}{\nu_2} \inf_{r \in (0,\delta)} r\varepsilon(a,r) - n + p \ .$$

For example in [He] §3.3 we find the following Poincaré inequality with proof.

10.13. Lemma. *Let $\mathcal{M}$ be a compact Riemannian manifold of dimension n and let $1 \le p < n$ be a real number. There exists a positive constant $A = A(\mathcal{M},p)$ such that for every $\omega \in W^{1,p}(\mathcal{M})$ we have*

$$\left(\int\limits_{\mathcal{M}} |\omega - \overline{\omega}|^p dv_{\mathcal{M}} \right)^{\frac{1}{p}} \le A \left(\int\limits_{\mathcal{M}} |d\omega|^p dv_{\mathcal{M}} \right)^{\frac{1}{p}} \tag{10.14}$$

where $\overline{\omega} = \frac{1}{\mathrm{mes}_n(\mathcal{M})} \int\limits_{\mathcal{M}} \omega \, dv_{\mathcal{M}}$.

Because we know that $\Sigma(a,r)$ is a compact Riemannian manifold of dimension $n-1$ and $1 \le p < n-1$ we get with Lemma 10.13 and with

$\omega_0 = \frac{1}{\text{mes}_{n-1}(\Sigma(a,r))} \int\limits_{\Sigma(a,r)} \omega \, dH^{n-1}$ the inequalities

$$\varepsilon(a,r) \geq \frac{\left(\int\limits_{\Sigma(a,r)} |d_\Sigma \omega|^p dH^{n-1} \right)^{\frac{1}{p}}}{\left(\int\limits_{\Sigma(a,r)} |\omega - \overline{\omega}|^p dH^{n-1} \right)^{\frac{1}{p}}} \geq \frac{1}{A} \, .$$

Now the problem of finding a lower bound for β or $\varepsilon(a,r)$ is reduced to the problem of finding the best constant for the Poincaré inequality.

In the euclidean case $I\!R^n$ (see for example [BI] §1) we get for the ball $B = B(x,r)$ and for $f \in W^{1,p}(B)$, $1 \leq p < \infty$, the Poincaré inequality

$$\|f - f_B\|_{p,B} \leq 2^{\frac{n}{p}+1} \, r \, \|\nabla f\|_{p,B}$$

with $f_B = \frac{1}{\text{mes}_\text{n}(\text{B})} \int\limits_B f(x) \, dx$.

If we have a geodesically complete Riemannian manifold the situation becomes more difficult, because the Sobolev embedding Theorem 3.4 might be false. For example in [Au] §2.7 or in [He] §3.5 we find the following theorem.

10.15. Theorem. *The Sobolev embedding theorem holds for a complete manifold $\mathcal{M}$ with Ricci curvature bounded from below and positive radius of injectivity.*

For the definition of Ricci curvature see for example [Jo] §3.3. When we have a complete manifold $\mathcal{M}$ it follows that it is only possible to find an A in (10.14) if for example the Ricci curvature is bounded from below and the radius of injectivity is positive.

The search for Poincaré inequalities in various situations has been intensive in recent years. For example in [Se] is shown that every regular, n-dimensional complete metric space, that is also an oriented manifold of dimension n and satisfies a linear local contractibility condition, admits a Poincaré inequality.

We should also mention that in [Kl] geometric estimates for a similar quantity to (10.12) are shown, they also bring us to estimates of the constant in the Poincaré inequality on Riemannian manifolds.