and

$$|A(m, du_I)| \le \nu_2 \, |du_I|^{p-1} \, , \tag{7.27}$$

for $p > 1$ and for all differential forms $du_I \in L^1_{\mathrm{loc}}(\mathcal{M})$ of degree k, $0 \le k \le n$.

Proof. Because of Lemma 7.9 the norm of du_I on the manifold $\mathcal{M}$ and the norm generated by $H(m)$ are equivalent and thus

$$\nu \, |du_I| \;\le\; |du_I|_H \; .$$

With (7.16), (7.18) and (7.22) we get

$$\begin{aligned} \nu_1 \, |du_I|^p \;&\le\; |du_I|_H^p \;=\; J_f(m) \\ &=\; \langle du_I, d^* v_J \rangle \;=\; \langle du_I, A(m, du_I) \rangle \; . \end{aligned}$$

The second estimation follows directly from the definition of the mapping $A(m, \xi)$

$$\begin{aligned} |A(m, du_I)| \;&=\; |\langle H(m) du_I, du_I \rangle^{\frac{p-2}{2}} H(m) du_I| \;\le\; |H(m)|^{\frac{p}{2}} |du_I|^{p-1} \\ &=\; \nu_2 \, |du_I|^{p-1} \; . \end{aligned}$$

□

8 Quasiregular mappings and $\mathcal{WT}$-classes

In this chapter we want to consider the connection of quasiregular mappings and the $\mathcal{WT}$-classes of differential forms.

8.1. Theorem. *If $f \in W^{1,s}_{\mathrm{loc}}(\mathcal{M})$, $s = \max\{k, n-k\}$, is weakly K-quasiregular, then the differential form du_I (7.4), $\deg du_I = k$, is of the class $\mathcal{WT}_2$.*

Proof. This result follows direct with the Lemmas 7.21 and 7.25 together with Theorem 5.6. □

We want to show now a different approach, based more on the properties of differential forms of the $\mathcal{WT}$-classes. Here we follow [MMV1] and [FMMVW].

Let $\mathcal{A}$ and $\mathcal{B}$ be Riemannian manifolds of dimensions $\dim \mathcal{A} = k$ and $\dim \mathcal{B} = n-k$, $1 \le k < n$, and with scalar products $\langle\, , \rangle_A$, $\langle\, , \rangle_B$, respectively. On the Cartesian product $\mathcal{N} = \mathcal{A} \times \mathcal{B}$ we introduce the natural structure of a Riemannian manifold with the scalar product

$$\langle\, , \rangle = \langle\, , \rangle_{\mathcal{A}} + \langle\, , \rangle_{\mathcal{B}} \,.$$

We denote by $\pi : \mathcal{A} \times \mathcal{B} \to \mathcal{A}$ and $\eta : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ the natural projections of the manifold $\mathcal{N}$ onto submanifolds.

If $w_{\mathcal{A}}$ and $w_{\mathcal{B}}$ are volume forms on $\mathcal{A}$ and $\mathcal{B}$, respectively, then the differential form $w_{\mathcal{N}} = \pi^* w_{\mathcal{A}} \wedge \eta^* w_{\mathcal{B}}$ is a volume form on $\mathcal{N}$ (see (3.10)).

8.2. Theorem. *Let $f : \mathcal{M} \to \mathcal{N}$ be a quasiregular mapping and let $u = \pi \circ f : \mathcal{M} \to \mathcal{A}$. Then the differential form $u^* w_{\mathcal{A}}$ is of the class $\mathcal{WT}_2$ on $\mathcal{M}$ with the structure constants $p = n/k$, $\nu_1 = \nu_1(n,k,K_O)$ and $\nu_2 = \nu_2(n,k,K_O)$.*

8.3. Remark. From the proof of the theorem it will be clear that the structure constants can be chosen to be

$$\nu_1 = (k + \frac{n-k}{\overline{c}^2})^{n/2} n^{-n/2}\, K_O^{-1}, \quad \nu_2 = \underline{c}^{k-n} \,,$$

where $\overline{c} = \overline{c}(k,n,K_O)$ and $\underline{c} = \underline{c}(k,n,K_O)$ are, respectively, the greatest and least positive roots of the equation

$$(k\xi^2 + (n-k))^{n/2} - n^{n/2}\, K_O\, \xi^k = 0 \,. \tag{8.4}$$

Through similar considerations and with the same proof, only swapping the roles of $u^* w_{\mathcal{A}}$ and $v^* w_{\mathcal{B}}$, it can be shown that also $v^* w_{\mathcal{B}} \in \mathcal{WT}_2$.

Proof. Setting $v = \eta \circ f : \mathcal{M} \to \mathcal{B}$ we choose $\theta = v^* w_{\mathcal{B}}$. The volume form $w_{\mathcal{B}}$ is weakly closed.

In fact, if the mapping v is sufficiently regular, then

$$d\theta = dv^* w_{\mathcal{B}} = v^* dw_{\mathcal{B}} = 0 \,.$$

In the general case for the verification of the condition (3.14) we approximate the mapping $v : \mathcal{M} \to \mathcal{B}$ in the norm of $W^{1,n}(\mathcal{M})$ by smooth maps v_l,

$l = 1, 2, \ldots$. Because the condition (3.14) holds for each of the differential forms $v_l^* w_{\mathcal{B}}$, it holds also for the differential form $v^* w_{\mathcal{B}}$.

The weak closedness of the differential form $u^* w_{\mathcal{A}}$ follows similarly.

Fix a point $m \in \mathcal{M}$, at which the relation (6.3) holds. Set $a = u(m)$, $b = v(m)$. Then

$$T_{f(m)}(\mathcal{N}) = T_a(\mathcal{A}) \times T_b(\mathcal{B}).$$

The computations can be conveniently carried out as follows. We first rewrite the condition (6.3) with the help of (3.11) in the form

$$|Df(m)|^n \leq K_O |f^* w_{\mathcal{N}}| \,, \tag{8.5}$$

where $w_{\mathcal{N}}$ is a volume form on $\mathcal{N}$.

For the points $a \in \mathcal{A}$, $b \in \mathcal{B}$ we choose neighborhoods and local systems of coordinates $x^1, \ldots, x^k$, and $x^{k+1}, \ldots, x^n$, orthonormal at a and b, respectively. With (3.10) we have

$$\begin{aligned} u^* w_{\mathcal{A}} &= u^*(dx^1 \wedge \ldots \wedge dx^k) = u^* dx^1 \wedge \ldots \wedge u^* dx^k \\ &= df^1 \wedge \ldots \wedge df^k \,. \end{aligned} \tag{8.6}$$

Because the form $w_{\mathcal{A}}$ is simple, we obtain by the inequality between the geometric and arithmetic means

$$|df^1 \wedge \ldots \wedge df^k|^{1/k} \leq \Big(\prod_{i=1}^k |df^i| \Big)^{1/k} \leq \frac{1}{k} \sum_{i=1}^k |df^i| \leq \Big(\frac{1}{k} \sum_{i=1}^k |df^i|^2 \Big)^{1/2} . \tag{8.7}$$

Similarly we obtain

$$v^* w_{\mathcal{B}} = df^{k+1} \wedge \ldots \wedge df^n \tag{8.8}$$

and

$$|df^{k+1} \wedge \ldots \wedge df^n|^{1/(n-k)} \leq \Big(\frac{1}{n-k} \sum_{i=k+1}^n |df^i|^2 \Big)^{1/2} . \tag{8.9}$$

It is not difficult to see that

$$f^* w_{\mathcal{N}} = f^*(\pi^* w_{\mathcal{A}} \wedge \eta^* w_{\mathcal{B}}) = u^* w_{\mathcal{A}} \wedge v^* w_{\mathcal{B}} = u^* w_{\mathcal{A}} \wedge \theta$$

and further that

$$|f^*w_{\mathcal{N}}| = |u^*w_{\mathcal{A}} \wedge v^*w_{\mathcal{B}}| \leq |df^1 \wedge \ldots \wedge df^k||df^{k+1} \wedge \ldots \wedge df^n| \ .$$

We have

$$|df|^2 = \sum_{i=1}^{k} |df^i|^2 + \sum_{i=k+1}^{n} |df^i|^2 \ \leq n\,|Df(m)|^2,$$

and therefore from (8.5), (8.7) and (8.9) we get

$$\begin{aligned}
(8.10)\Big(k|u^*w_{\mathcal{A}}|^{2/k} \ &+ \ (n-k)|v^*w_{\mathcal{B}}|^{2/(n-k)}\Big)^{n/2} \\
&\leq \ \Big(\sum_{i=1}^{k} |df^i|^2 + \sum_{i=k+1}^{n} |df^i|^2\Big)^{n/2} \ \leq \ \Big(n|Df(m)|^2\Big)^{n/2} \\
&\leq \ n^{n/2}\, K_O \langle u^*w_{\mathcal{A}}, \star\,\theta\rangle \\
&\leq \ n^{n/2}\, K_O |u^*w_{\mathcal{A}}|\,|v^*w_{\mathcal{B}}| \ .
\end{aligned}$$

With

$$\xi = \frac{|u^*w_{\mathcal{A}}|^{1/k}}{|v^*w_{\mathcal{B}}|^{1/(n-k)}}$$

the preceding relation takes the form

$$(k\xi^2 + (n-k))^{n/2} \leq n^{n/2}\, K_O \xi^k \ .$$

Using the notations $\underline{c}$ and $\overline{c}$ for the least and greatest positive roots of the equation (8.4) we have $\underline{c} \leq \xi \leq \overline{c}$ and

$$(8.11) \qquad \underline{c}|v^*w_{\mathcal{B}}|^{1/(n-k)} \leq |u^*w_{\mathcal{A}}|^{1/k} \leq \overline{c}|v^*w_{\mathcal{B}}|^{1/(n-k)} \ .$$

As above, from (8.11) it follows that

$$|u^*w_{\mathcal{A}}|^{n/k} \leq \Big(k + \frac{n-k}{\overline{c}^2}\Big)^{-n/2} n^{n/2}\, K_O \langle u^*w_{\mathcal{A}}, \star\,\theta\rangle \ .$$

Thus the condition (4.6) for the membership of the differential form $u^*w_{\mathcal{A}}$ of degree k in the class $\mathcal{WT}_2$ is indeed satisfied.

To verify the condition (4.7), it is enough to observe that from (8.11) it follows that

$$\underline{c}^{n-k}|\theta| \leq |f^* w_{\mathcal{A}}|^{(n-k)/k} .$$

$\square$

Let $x^1, \ldots, x^k$ be an orthonormal system of coordinates in R^k, $1 \leq k < n$. Let $\mathcal{A}$ be a domain in R^k and let $\mathcal{B}$ be an $(n-k)$-dimensional Riemannian manifold. We consider the manifold $\mathcal{N} = \mathcal{A} \times \mathcal{B}$.

Let $f : \mathcal{M} \to \mathcal{N}$ be a mapping of the class $W^{1,n}_{\mathrm{loc}}(\mathcal{M})$. Locally we write the mapping f in the components $f^1, \ldots, f^n$. Let $u = \pi \circ f$ and $v = \eta \circ f$ be as defined above. We have $u^* w_{\mathcal{A}} = df^1 \wedge \ldots \wedge df^k$.

8.12. Theorem. *If the mapping f is quasiregular, then the differential form $u^* w_{\mathcal{A}}$ is of the class $\mathcal{WT}_3$ on $\mathcal{M}$ with the structure constants $p = n/k$, $\nu_2 = \nu_2(k, n, K_O)$, $\nu_3 = \nu_3(k, n, K_O)$.*

8.13. Theorem. *If the mapping $f : \mathcal{M} \to R^n$ is quasiregular, then the differential form $u^* w_{\mathcal{A}}$ is of the class $\mathcal{WT}_4$ on $\mathcal{M}$ with the structure constants $p = n/k$, $\nu_3 = \nu_3(k, n, K_O)$, $\nu_4 = \nu_4(k, n, K_O)$.*

8.14. Remark. We can choose the constants ν_2, ν_3 and ν_4 to be

$$\nu_2 = \underline{c}^{k-n}, \qquad \nu_3 = \Big(1 + \frac{1}{\overline{c}_1^2}\Big)^{n/2} n^{-n/2} k^{n/2} K_O^{-1} ,$$
$$\nu_4 = \Big((n-k)^{-n/2} (1 + \underline{c}_1^2)^{-n/2} n^{n/2} K_O\Big)^{(n-k)/k} ,$$

where $\underline{c}$ is the least positive root of (8.4), where $\underline{c}_1$ is the least and $\overline{c}_1$ the greatest positive root of the equation

$$(\xi^2 + 1)^{n/2} - n^{n/2} k^{-k/2} (n-k)^{-(n-k)/2} K_O \, \xi^k = 0 . \tag{8.15}$$

In contrast to the proof of Theorem 8.2 the differential form $u^* w_{\mathcal{A}}$ of degree k has now a global coordinate representation. The results of the previous proof stay applicable in the proofs of Theorem 8.12 and 8.13. The main difference is that we use now the inequality between geometric and arithmetic means (8.7) and (8.9) and with the help of (8.10) we get

$$\Big(\sum_{i=1}^{k}|df^i|^2 \;+\; \sum_{i=k+1}^{n}|df^i|^2\Big)^{n/2}$$
$$\leq\; k^{-n/2}(n-k)^{-(n-k)/2}n^{n/2}K_O\Big(\sum_{i=1}^{k}|df^i|^2\Big)^{k/2}\Big(\sum_{i=k+1}^{n}|df^i|^2\Big)^{(n-k)/2}.$$

From this point the proofs follow the concept of the proof of Theorem 8.2. For details of the proofs of Theorem 8.12 and 8.13 see [FMMVW] §6. Our slightly better constants ν_3 and ν_4 follow directly from the definitions of the classes $\mathcal{WT}_3$ and $\mathcal{WT}_4$.

There exist some differences between the Theorems 8.1 and 8.2. In the first theorem the mapping f is only weakly quasiregular. This could be weakened by a theorem from T. Iwaniec ([Iw1] §11) which says that a weakly K-quasiregular mapping $f \in W^{1,p}_{\mathrm{loc}}$, $p < n$, is also K-quasiregular, if p is close enough to n, here p depends only on n and K, see also [FW] §9. The theorem depends on a Caccioppoli-type estimate, which recently was refined in [Iw2].

The differential form du_I (7.4) depends on a multi-index, we have more possibilities for a differential form of the class $\mathcal{WT}_2$. The differential form $u^*w_{\mathcal{A}}$ in Theorem 8.2 is fixed, but we gave concreter constants ν_1 and ν_2.

9 Morrey's Lemma on manifolds

In this chapter we follow mostly the considerations of [MMV3]. Let $\mathcal{M}$ be a Riemannian manifold of dimension n and without boundary. We assume that $\mathcal{M}$ is orientable and of the class C^3. Let $d(m_1, m_2)$ be the geodesic distance between the points $m_1, m_2 \in \mathcal{M}$. We denote by

$$B(a,t) = \{m \in \mathcal{M} : d(a,m) < t\}$$
$$\Sigma(a,t) = \{m \in \mathcal{M} : d(a,m) = t\}$$

the geodesic ball and the geodesic sphere, respectively, with center $a \in \mathcal{M}$ and radius $t > 0$.