6 Quasiregular mappings

Let $\mathcal{M}$ and $\mathcal{N}$ be orientable Riemannian manifolds of dimension n and let $x^1, \ldots, x^n$ be local coordinates in the neighborhood of a point $m \in \mathcal{M}$. For a mapping $f : \mathcal{M} \to \mathcal{N}$ we define the formal derivative $Df(m)$ in terms of the partial derivatives $D_i f_j$. Through the identification $T_m(\mathcal{M}) \simeq I\!R^n$ the differentiation operator

$$Df(m) : T_m(\mathcal{M}) \to T_{f(m)}(\mathcal{N}) \tag{6.1}$$

is the linear mapping for which $Df(m)e^i = \sum_{j=1}^n D_i f_j(m) e^j$. We denote by $J_f(m)$ the Jacobian of f at the point $m \in \mathcal{M}$, i.e. the determinant of $Df(m)$. For the norm of $Df(m)$ we take the operator norm

$$|Df(m)| = \max_{|\xi|=1} |Df(m)\,\xi| \, .$$

Sometimes $Df(m)$ may be replaced by $f'(m)$. With respect to the standard basis of $I\!R^n$ we will denote by $Df(m)$ also the corresponding matrix.

6.2. Definition. *A mapping $f : \mathcal{M} \to \mathcal{N}$ of the class $W^{1,p}_{\text{loc}}$, $1 \le p \le n$, is said to be weakly quasiregular if the estimation*

$$|Df(m)|^n \le K J_f(m) \tag{6.3}$$

holds for almost every $m \in \mathcal{M}$ with $1 \le K < \infty$. The mapping f is called quasiregular if p is equal to the dimension of $\mathcal{M}$.

The smallest constant $K \ge 1$ in (6.3) is called the outer dilatation of f and denoted dy $K_O(f)$. If f is quasiregular then one has also

$$J_f(m) \le K' \, l(Df(m))^n \tag{6.4}$$

almost everywhere on $\mathcal{M}$ for some $K' \ge 1$. Here we have

$$l(Df(m)) = \min_{|\xi|=1} |Df(m)\xi| \, .$$

The smallest $K' \ge 1$ in (6.4) is called the inner dilatation of f and denoted by $K_I(f)$. The quantity

$$K(f) = \max\{K_O(f), K_I(f)\}$$

is the (maximal) dilatation of f and a quasiregular mapping is called K-quasiregular if $K(f) \le K$. The relationships $K_O(f) \le K_I(f)^{n-1}$ and $K_I(f) \le K_O(f)^{n-1}$ hold. Thus $K_O(f) = K_I(f)$ for $n = 2$. It follows that we also can define a K-quasiregular mapping, $1 \le K < \infty$, to be a mapping $f \in W^{1,n}_{\text{loc}}(\mathcal{M})$ with $J_f(m) \ge 0$ a.e. and that the estimation

$$\max_{|\xi|=1} |Df(m)\xi| \le K \min_{|\xi|=1} |Df(m)\xi| \tag{6.5}$$

holds for almost every $m \in \mathcal{M}$.

If $f : \mathcal{M} \to \mathcal{N}$ is a quasiregular homeomorphism then the mapping f is called quasiconformal. In this case the inverse mapping f^{-1} is also quasiconformal in the domain $f(\mathcal{M}) \subset \mathcal{N}$ and $K(f^{-1}) = K(f)$.

A broad view of quasiregular mappings in higher dimensions is given by S. Rickman in his monograph [Ri].

A fundamental property of quasiregular mappings is that they are almost everywhere differentiable and, if they are non-constant, they are also sense-preserving discrete and open. These results are presented in [Re] §2.

6.6. Example. An important class of examples of quasiregular mappings is provided by mappings that distort lengths of curves by a bounded factor. A continuous mapping $f : \mathcal{M} \to \mathcal{N}$ is for some $L \ge 1$ of L-bounded length distortion, or L-BLD, if $f \in W^{1,1}_{\text{loc}}(\mathcal{M})$, if $J_f(m) \ge 0$ almost everywhere on $\mathcal{M}$, and if for some L the inequality

$$|\xi|/L \le |Df(m)\xi| \le L|\xi| \tag{6.7}$$

holds for all $\xi \in T_m(\mathcal{M})$ and for almost every $m \in \mathcal{M}$. We say that f is a BLD mapping if it is L-BLD for some L. In [HKM] §14 it is shown that every L-BLD mapping is K-quasiregular with $K = L^{2(n-1)}$.

Many properties of quasiregular mappings can be clarified with terms of the Riemannian geometry. First let $G : \mathcal{M} \to \mathrm{GL}(n)$ be a measurable function with values in symmetric positive matrices of determinant one, such that

$$\lambda^{-1}|\xi| \le \langle G(m)\xi, \xi\rangle^{1/2} \le \lambda|\xi| \; , \tag{6.8}$$

for $(m, \xi) \in \mathcal{M} \times T_m(\mathcal{M})$ and $\lambda \ge 1$. The inner product in the tangent space $T_m(\mathcal{M})$ gives rise to a measurable Riemannian metric tensor on $\mathcal{M}$.

The norm of the tangent vector $\xi \in T_m(\mathcal{M})$ at $m \in \mathcal{M}$ with respect to this metric is defined by $|\xi|_G := \langle G(m)\xi, \xi \rangle^{1/2}$. Every K-quasiconformal mapping f induces a metric tensor on $\mathcal{M}$, namely

$$G(m) := J_f(m)^{-2/n} D^t f(m) Df(m) \tag{6.9}$$

if $J_f(m) \neq 0$, and $G(m) = \mathrm{Id}$ if $J_f(m) = 0$. It is clear that f is conformal with respect to this metric. We refer to $G(m)$ as the matrix dilatation of f at $m \in \mathcal{M}$. The following lemma ensures the inequalities in (6.8). For the proof see Lemma 7.9 in the case $k = 1$.

6.10. Lemma. *Let $f \in W^{1,p}(\mathcal{M})$, $1 \leq p \leq n$, be weakly K-quasiregular, then the equation*

$$K^{\frac{1}{n}-1}|\xi| \leq \langle G(m)\xi, \xi \rangle^{\frac{1}{2}} \leq K^{1-\frac{1}{n}}|\xi| \tag{6.11}$$

holds for almost every $m \in \mathcal{M}$ and for all $\xi \in T_m(\mathcal{M})$.

Quasiregular mappings are weak solutions of the differential system

$$D^t f(m) Df(m) = J_f(m)^{2/n} G(m) \;, \tag{6.12}$$

commonly called the n-dimensional Beltrami equation.

7 A-harmonic differential forms and quasiregular mappings

This chapter connects quasilinear elliptic equations with quasiregular mappings. Similar results in Euclidean spaces are shown in [Iw1], [IM] and [FW].

Let $\mathcal{M}$ and $\mathcal{N}$ be orientable Riemannian manifolds of dimension n and $f : \mathcal{M} \to \mathcal{N}$ a mapping of Sobolev class $W^{1,s}_{\mathrm{loc}}(\mathcal{M})$, $1 \leq s \leq n$. We fix an ordered multi-index $I = (i_1, \ldots, i_k) \in \mathcal{I}(k, n)$ and its complementary multi-index $J = (j_1, \ldots, j_{n-k}) \in \mathcal{I}(n-k, n)$ (see also (1.3)), ordered in such a way that

$$dx^I = \star\, dx^J \;. \tag{7.1}$$