1 Differential forms on $\mathbb{R}^n$

By $\mathbb{R}^n$ we denote the n-dimensional Euclidean space consisting of elements of the form $x = (x^1, \ldots, x^n)$, $x^i \in \mathbb{R}$. The Euclidean space is equipped with the standard inner product $\langle x, y \rangle = \sum_{i=1}^n x^i y^i$ and the norm $|x| = \langle x, x \rangle^{1/2} = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}$.

By $\Lambda_k(\mathbb{R}^n)$ we denote the linear space of all k-vectors, by $\Lambda^k(\mathbb{R}^n)$ the space of all k-covectors or differential forms of degree k. The mutually dual spaces $\Lambda_k(\mathbb{R}^n)$ and $\Lambda^k(\mathbb{R}^n)$ are associated with the Euclidean space $\mathbb{R}^n$. We have $\Lambda^0(\mathbb{R}^n) = \mathbb{R} = \Lambda_0(\mathbb{R}^n)$ and $\Lambda_k(\mathbb{R}^n) = \{0\} = \Lambda^k(\mathbb{R}^n)$ in the case $k > n$ or $k < 0$. Further we have for every k with $1 \leq k \leq n$

$$\dim \Lambda^k = \dim \Lambda^{n-k} = \binom{n}{k} .$$

The direct sums

$$\Lambda_*(\mathbb{R}^n) = \bigoplus_{0 \leq k \leq n} \Lambda_k(\mathbb{R}^n) , \quad \Lambda^*(\mathbb{R}^n) = \bigoplus_{0 \leq k \leq n} \Lambda^k(\mathbb{R}^n) \tag{1.1}$$

generate the contravariant and covariant Grassmann algebras on $\mathbb{R}^n$ with the exterior multiplication operator $\wedge$.

Let $\omega \in \Lambda^k(\mathbb{R}^n)$ be a covector. We denote by $\mathcal{I}(k, n)$ the set of ordered multi-indices $I = (i_1, \ldots, i_k)$ of integers $1 \leq i_1 < \ldots < i_k \leq n$. The differential form ω can be written in a unique way as the linear combination

$$\omega = \sum_{I \in \mathcal{I}(k,n)} \omega_I \, dx^I . \tag{1.2}$$

Here ω_I are the coefficients of ω with respect to the standard basis

$$dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_k}, \quad I = (i_1, \ldots, i_k) \in \mathcal{I}(k, n)$$

of $\Lambda^k(\mathbb{R}^n)$. Let $I = (i_1, \ldots, i_k)$ be a multi-index from $\mathcal{I}(k, n)$. The complement J of the multi-index I is the multi-index $J = (j_1, \ldots, j_{n-k})$ in $\mathcal{I}(n-k, n)$ where the components j_p are in $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$. We have

$$dx^I \wedge dx^J = \sigma(I) \, dx^1 \wedge \ldots \wedge dx^n \tag{1.3}$$

where $\sigma(I)$ is the signature of the permutation $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$ in the set $\{1, \ldots, n\}$. Note that $\sigma(J) = (-1)^{k(n-k)} \sigma(I)$.

With the notions mentioned above we define

$$\star dx^I = \sigma(I) dx^J \ . \tag{1.4}$$

For $\omega \in \Lambda^k(I\!R^n)$ with $\omega = \sum_{I\in\mathcal{I}(k,n)} \omega_I \, dx^I$ we set

$$\star\omega = \sum_{I\in\mathcal{I}(k,n)} \omega_I \star dx^I \ . \tag{1.5}$$

The differential form $\star\omega$ is of degree $n-k$, i.e. it belongs to $\Lambda^{n-k}(I\!R^n)$ and is called the orthogonal complement of the differential form ω. The linear operator $\star : \Lambda^k(I\!R^n) \to \Lambda^{n-k}(I\!R^n)$ is called the Hodge star operator. For $\alpha, \beta \in \Lambda^k(I\!R^n)$ and $a, b \in I\!R$ we have

$$\star(a\alpha + b\beta) = a \star \alpha + b \star \beta. \tag{1.6}$$

It follows that

$$\star 1\!\!1 = dx^1 \wedge \ldots \wedge dx^n$$

and the Hodge star operator twice applied to a differential form ω of degree k yields

$$\star(\star\,\omega) = (-1)^{k(n-k)}\omega \ . \tag{1.7}$$

For $\omega \in \Lambda^k(I\!R^n)$ we also introduce the operator $\star^{-1} := (-1)^{k(n-k)}\star$. The operator $\star^{-1}$ is an inverse to $\star$ in the sense that

$$\star^{-1}(\star\,\omega) = \star(\star^{-1}\omega) = \omega \ . \tag{1.8}$$

For $\alpha, \beta \in \Lambda^k(I\!R^n)$ the inner or scalar product is defined as

$$\langle \alpha, \beta \rangle := \star^{-1}(\alpha \wedge \star\beta) = \star\,(\alpha \wedge \star\beta) \ . \tag{1.9}$$

The scalar product of differential forms has the usual properties of the scalar product. Thus, the norm of a differential form $\omega \in \Lambda^*(I\!R^n)$ is given by the formula

$$|\omega|^2 = \langle \omega, \omega \rangle = \star\,(\omega \wedge \star\,\omega) \ .$$

A differential form ω of degree k is called simple if there are differential forms $\omega_1, \ldots, \omega_k$ of degree 1 such that

$$\omega = \omega_1 \wedge \ldots \wedge \omega_k \, .$$

For $\alpha,\ \beta \in \Lambda^*(I\!R^n)$ we have the following estimation of the Euclidean norm

$$|\alpha \wedge \beta| \leq |\alpha|\,|\beta| \, ,$$

if at least one of the differential forms α, β is simple. If α and β are simple and non-zero, then equality holds if and only if the subspaces associated with α and β are orthogonal. More generally, for $\alpha, \beta \in \Lambda^*(I\!R^n)$ with $\deg \alpha = p$ and $\deg \beta = q$ we get

$$|\alpha \wedge \beta| \leq (C_{p,q})^{1/2} |\alpha|\,|\beta| \, . \tag{1.10}$$

The constant $C_{p,q}$ can be choosen to be $\binom{p+q}{p}$. For details see [Fe] §1.7.

Let $A : I\!R^n \to I\!R^n$ be a linear transformation with the norm $|A| = \sup_{|x|=1} |Ax|$ and let $\omega = \omega_1 \wedge \ldots \wedge \omega_k$ be a simple differential form of degree k, i.e. $\omega_1, \ldots, \omega_k \in \Lambda^1(I\!R^n)$. For every $k = 1, \ldots, n$ the linear operator $A_\# : \Lambda^k(I\!R^n) \to \Lambda^k(I\!R^n)$ is defined by

$$A_\# \omega := A\omega_1 \wedge \ldots \wedge A\omega_k \, . \tag{1.11}$$

The operator $A_\#$ is called the kth exterior power of A. It follows that the matrix $A_\#$ consists of the $k \times k$ matrices of minors. For $A, B \in \mathrm{GL}(n)$, the linear space of $n \times n$ matrices with real entries and non-zero determinant, the properties

$$(AB)_\# = A_\# B_\# \, , \quad (A^{-1})_\# = (A_\#)^{-1} \, , \quad (A^t)_\# = (A_\#)^t \tag{1.12}$$

hold, see [Fl] §2. By $\mathrm{S}(n)$ we denote the subspace of $\mathrm{GL}(n)$ consisting of the positive definite symmetric matrices whose determinant is equal to one. We need the following lemmas in a later proof, see also [IM] §2.

1.13. Lemma. *For every matrix $A \in \mathrm{GL}(n)$ we have*

$$A^t_\# \star A_\# = (\det A) \star \quad : \quad \Lambda^k(I\!R^n) \to \Lambda^{n-k}(I\!R^n) \, . \tag{1.14}$$

Proof. For simple differential forms $\alpha \in \Lambda^{n-k}(I\!R^n)$ and $\beta \in \Lambda^k(I\!R^n)$ and with (1.9) we compute

$$\begin{aligned} \langle \alpha, A^t_\# \star A_\# \beta \rangle \star 1\!\!1 &= \langle A_\# \alpha, \star A_\# \beta \rangle \star 1\!\!1 \\ &= A_\# \alpha \wedge \star \star A_\# \beta \\ &= A_\# (\alpha \wedge \star \star \beta) \\ &= A_\# \langle \alpha, \star \beta \rangle \star 1\!\!1 \\ &= (\det A) \langle \alpha, \star \beta \rangle \star 1\!\!1 \ . \end{aligned}$$

Hence $(A^t_\# \star A_\#)\, \beta = (\det A) \star \beta$ and the lemma is proved. □

1.15. Lemma. *Let $G \in \mathrm{S}(n)$ be a matrix with the representation $G = |\det A|^{-2/n} A A^t$ for a matrix $A \in \mathrm{GL}(n)$. Then on $\Lambda^k(I\!R^n)$ we have*

$$G_\# \star A_\# = |\det A|^{\frac{2(k-n)}{n}} (\det A) A_\# \star \ . \tag{1.16}$$

Proof. Let $\omega = \omega_1 \wedge \ldots \wedge \omega_k$ be a simple differential form of degree k, then $A_\# \omega \in \Lambda^k(I\!R^n)$ and $\star A_\# \omega \in \Lambda^{n-k}(I\!R^n)$. For $\lambda \in I\!R$ we have

$$(\lambda G)_\# \omega = \lambda G \omega_1 \wedge \ldots \wedge \lambda G \omega_k = \lambda^k G_\# \omega \ .$$

This together with (1.12) and (1.14) yields

$$\begin{aligned} |\det A|^{\frac{-2(k-n)}{n}} G_\# \star A_\# \omega &= (|\det A|^{\frac{2}{n}} G)_\# \star A_\# \omega \\ &= (A A^t)_\# \star A_\# \omega \\ &= A_\# A^t_\# \star A_\# \omega \\ &= (\det A) A_\# \star \omega \ . \end{aligned}$$

□

2 Riemannian manifolds

A manifold $\mathcal{M}$ of dimension n is a connected paracompact Hausdorff space for which every point has a neighborhood U that is homeomorphic to an open subset Ω of $I\!R^n$. Such a homeomorphism

$$x : U \to \Omega$$

is called a local chart. A collection $(U_i, x_i)_{i\in I}$ of local charts such that $\bigcup_{i\in I} U_i = \mathcal{M}$ is called an atlas. The (local) coordinates of $m \in U$, related to x, are the coordinates of the point $x(m)$ of $\mathbb{R}^n$. An atlas of class C^k, $k \geq 2$, on $\mathcal{M}$ is an atlas for which all changes of coordinates are C^k. That is to say, if (U_1, x_1) and (U_2, x_2) are two local charts with $U_1 \cap U_2 \neq \emptyset$, then the mapping $x_1 \circ x_2^{-1}$ of $x_2(U_1 \cap U_2)$ onto $x_1(U_1 \cap U_2)$ is a diffeomorphism of class C^k. Two atlases of class C^k are said to be equivalent if their union is an atlas of class C^k.

A differentiable manifold $\mathcal{M}$ of class C^k, $k \geq 2$, is a manifold together with an equivalence class of C^k atlases.

A mapping $f : \mathcal{M} \to \mathcal{N}$ between differentiable manifolds $\mathcal{M}$ and $\mathcal{N}$ of the same dimension with charts $(U_i, x_i)_{i\in I}$ and $(U_j, x_j)_{j\in J}$ is called differentiable if all mappings $x_j \circ f \circ x_i^{-1}$ are differentiable.

An atlas for a differentiable manifold is called oriented if all changes of coordinates have positive functional determinant. A differentiable manifold is called orientable if it possesses an oriented atlas.

The tangent space $T_m(\mathcal{M})$ at $m \in \mathcal{M}$ is the set of tangent vectors at m. It has a natural vector space structure. We denote by $T(\mathcal{M})$ the disjoint union of the tangent spaces $T_m(\mathcal{M})$, $m \in \mathcal{M}$. Let $\pi : T(\mathcal{M}) \to \mathcal{M}$ with $\pi(w) = m$ for $w \in T_m(\mathcal{M})$ be the projection onto the “base point”. The triple $(T(\mathcal{M}), \pi, \mathcal{M})$ is called tangent bundle of $\mathcal{M}$, and $T(\mathcal{M})$ is called total space of the tangent bundle. Often the tangent bundle is simply denoted by its total space. The total space $T(\mathcal{M})$ is also a differentiable manifold.

2.1. Definition. *A Riemannian metric on a differentiable manifold $\mathcal{M}$ is given by a scalar product on each tangent space $T_m(\mathcal{M})$ which depends smoothly on the base point m. A Riemannian manifold is a differentiable manifold equipped with a Riemannian metric.*

Let $x = (x^1, \ldots, x^n)$ be local coordinates. In these coordinates, a metric is represented by a positive definite symmetric matrix $(g_{ij}(x))_{i,j=1,\ldots,n}$ where the coefficients depend smoothly on x. The scalar product of two tangent vectors $v, w \in T_m(\mathcal{M})$ with coordinate representations $(v^1 \frac{\partial}{\partial x^1}, \ldots, v^n \frac{\partial}{\partial x^n})$ and $(w^1 \frac{\partial}{\partial x^1}, \ldots, w^n \frac{\partial}{\partial x^n})$ is

$$\langle v, w \rangle := \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}(x(m))\, v^i \frac{\partial}{\partial x^i}\, w^j \frac{\partial}{\partial x^j} \ . \tag{2.2}$$

In particular, one has $\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle = g_{ij}$. The length of v is given by

$$|v| := \langle v, v \rangle^{\frac{1}{2}} \ .$$

A well-known theorem says that each differentiable manifold $\mathcal{M}$ may be equipped with a Riemannian metric. For details see [Jo] or [AMR] §5.5.

Let now $[a, b]$ be a closed interval in $I\!R$ and $\gamma : [a, b] \to \mathcal{M}$ a curve of class C^k, $k \geq 2$. The length of γ is defined as

$$L(\gamma) := \int_a^b |\frac{d\gamma}{dt}(t)| \, dt$$

and the energy of γ as

$$E(\gamma) := \frac{1}{2} \int_a^b |\frac{d\gamma}{dt}(t)|^2 dt \ .$$

On a Riemannian manifold $\mathcal{M}$, the geodesic distance between two points m, p can be defined by

$$\begin{aligned} (2.3)\quad d(m,p) \ &:= \ \inf \{ L(\gamma) : \gamma : [a,b] \to \mathcal{M} \text{ a curve piecewise of class } C^k, \\ &\qquad \text{with } \gamma(a) = m,\ \gamma(b) = p \}, \quad k \geq 2\,. \end{aligned}$$

Any two points m, p can be connected by a curve like this, and $d(m, p)$ therefore is always defined. Clearly d is a metric.

Working with the coordinates $(x^1(\gamma(t)), \ldots, x^n(\gamma(t)))$ of a curve γ we use the abreviation $\dot{x}^i(t) := \frac{d}{dt}(x^i(\gamma(t)))$. The Euler-Lagrange equations for the energy functional E are

$$(2.4) \qquad \ddot{x}^i(t) + \sum_{j=1}^{n} \sum_{i=1}^{n} \Gamma^i_{jk}(x(t))\, \dot{x}^j(t)\, \dot{x}^k(t) = 0 \ , \quad i = 1, \ldots, n \ ,$$

with

$$\Gamma^i_{jk} = \sum_{l=1}^{n} \frac{1}{2} g^{il} \left(g_{jl,k} + g_{kl,j} - g_{jk,l} \right) ,$$

where

$$(g^{ij})_{i,j=1,\ldots,n} = (g_{ij})^{-1} \quad \text{and} \quad g_{jl,k} = \frac{\partial}{\partial x^k} g_{jl} \ .$$

The expressions Γ^i_{jk} are called Christoffel symbols.

2.5. Definition. *A curve $\gamma : [a, b] \to \mathcal{M}$ of class C^2 which satisfies (2.4) is called a geodesic curve.*

Thus, geodesic curves are critical points of the energy functional. A minimizing curve γ from m to p is a geodesic curve.

Let $\mathcal{M}$ be a Riemannian manifold with $m \in \mathcal{M}$ and $v \in T_m(\mathcal{M})$. It can be shown that there exists an $\varepsilon > 0$ and precisely one geodesic curve

$$c : [0, \varepsilon] \to \mathcal{M}$$

with $c(0) = m$ and $\dot{c}(0) = v$. In addition, c depends smoothly on m and v. We denote this geodesic curve by c_v.

2.6. Definition. *Let $\mathcal{M}$ be a Riemannian manifold with $m \in \mathcal{M}$ and*

$$V_m := \{v \in T_m(\mathcal{M}) : c_v \text{ is defined on } [0, 1]\}$$

then the function

$$\exp_m : V_m \to \mathcal{M}$$

with $v \mapsto c_v(1)$ is called the exponential mapping of $\mathcal{M}$ at m.

The domain of definition of the exponential mapping always at least contains a small neighborhood of $0 \in T_m(\mathcal{M})$. The exponential mapping $\exp_m$ maps a neighborhood of $0 \in T_m(\mathcal{M})$ diffeomorphically onto a neighborhood of $m \in \mathcal{M}$.

Let now $e_1, \ldots, e_n$ be a basis of $T_m(\mathcal{M})$ which is orthonormal with reference to the scalar product on $T_m(\mathcal{M})$ defined by the Riemannian metric. Writing for each vector $v \in T_m(\mathcal{M})$ its components with reference to this basis, we obtain a map $\Phi : T_m(\mathcal{M}) \to I\!R^n$ with $v = \sum_{i=1}^n v^i e_i \mapsto (v^1, \ldots, v^n)$. Thus we can identify $T_m(\mathcal{M})$ with $I\!R^n$. An isomorphism $\Phi : T_m(\mathcal{M}) \to I\!R^n$ is called a (n-dimensional) frame at $m \in \mathcal{M}$, often also v is called a frame.

The local coordinates defined by the chart $(U, \exp_m^{-1})$ are called Riemannian normal coordinates with center m. For Riemannian polar coordinates on $\mathcal{M}$, obtained by transforming the Euclidean coordinates of $I\!R^n$, on which the normal coordinates with center m are based, we have the same situation as for Euclidean polar coordinates. It follows that for each $m \in \mathcal{M}$ there exists a $\delta > 0$ such that Riemannian polar coordinates may be introduced on $B(m, \delta) := \{p \in \mathcal{M} : d(m, p) \leq \delta\}$ with $d(m, p)$ given in (2.3).

We denote by $B_\delta(0) := \{y \in I\!R^n : |y| \leq \delta\} \subset T_m(\mathcal{M})$.

2.7. Definition. *Let $\mathcal{M}$ be a Riemannian manifold and $m \in \mathcal{M}$. The radius of injectivity of m is defined by*

$$r_{\text{inj}}(m) := \sup\{\delta > 0 : \exp_m \textit{ is defined and injective on } B_\delta(0)\} .$$

The radius of injectivity of $\mathcal{M}$ is

$$r_{\text{inj}}(\mathcal{M}) := \inf_{m \in \mathcal{M}} r_{\text{inj}}(m) .$$

We call a Riemannian manifold geodesically complete if for all $m \in \mathcal{M}$, the exponential mapping $\exp_m$ is defined on all of $T_m(\mathcal{M})$. The Theorem of Hopf-Rinow (see for example [Jo] §1.4 or [Au] §4) shows that if a Riemannian manifold $\mathcal{M}$ is geodesically complete, then every two points $m, p \in \mathcal{M}$ can be joined by a geodesic curve of length $d(m, p)$, i.e. by a geodesic curve of shortest length.

For a geodesically complete Riemannian manifold $\mathcal{M}$, $m \in \mathcal{M}$, it can be shown, that the injectivity radius $r_{\text{inj}}(m)$ at m is defined as the largest $r > 0$ for which every geodesic curve γ of length less than r and having m as an endpoint is minimizing. One has $r_{\text{inj}}(m) > 0$ for every m. The radius of injectivity of $\mathcal{M}$ may be zero.

For example, the injectivity radius of the sphere S^n is π, since the exponential mapping of every point m maps the open ball of radius π in $T_m(\mathcal{M})$ injectively onto the complement of the antipodal point of m.

Before we go on with Riemannian manifolds, we are now able to clarify the connection between polyvectors and differential forms. The linear isomorphism $\text{Hom}\,(\Lambda_k(I\!R^n), I\!R) \simeq \Lambda^k(I\!R^n)$, $1 < k < n$, that defines the duality of the spaces $\Lambda_k(I\!R^n)$ and $\Lambda^k(I\!R^n)$, associates a k-vector with a differential form.

For example a vector $a = (a_1, \ldots, a_n) \in I\!R^n$ defines a differential form of degree 1

(2.8) $$\omega = a_1 dx^1 + a_2 dx^2 + \ldots + a_n dx^n.$$

We denote it by Ω_a. Let $u = (u_1, \dots, u_k)$, $u_i \in \Lambda_1(I\!R^n)$, be a non-degenerated frame. The set of all k-dimensional frames is identified with the set of simple k-vectors. One can prove that the differential form

$$\Omega_u = \Omega_{u_1} \wedge \ldots \wedge \Omega_{u_k}$$

does not depend on the choice of the particular frame from the class of frames equivalent with u. This fact produces a one-to-one correspondence $u \mapsto \Omega_u$ of the set of simple polyvectors onto the set of simple differential forms.

Let E be the lower half-space of $I\!R^n$, $x^1 < 0$, x^1 the first coordinate of $I\!R^n$. Consider $\overline{E} \subset I\!R^n$ with the induced topology.

2.9. Definition. *We say that a manifold $\mathcal{M}$ has a boundary if each point of $\mathcal{M}$ has a neighborhood homeomorphic to an open set of $\overline{E}$.*

A vector bundle consists of a total space E, a base $\mathcal{M}$, and a projection $\pi : E \to \mathcal{M}$, where E and $\mathcal{M}$ are differentiable manifolds and π is differentiable. A fiber is an inverse of the projection π and denoted by $E_m := \pi^{-1}(m)$ for $m \in \mathcal{M}$.

2.10. Definition. *Let $(E, \pi, \mathcal{M})$ be a vector bundle. A section of E is a differentiable mapping $s : \mathcal{M} \to E$ with $\pi \circ s = \mathrm{id}_{\mathcal{M}}$. The space of sections of E is denoted by $\Gamma(E)$.*

An example for a vector bundle is the tangent bundle $T(\mathcal{M})$ of a differentiable manifold $\mathcal{M}$. A section of the tangent bundle $T(\mathcal{M})$ of $\mathcal{M}$ is called a vector field on $\mathcal{M}$.

Let $\mathcal{M}$ be a differentiable manifold and $m \in \mathcal{M}$. The vector space dual to the tangent space $T_m(\mathcal{M})$ is called cotangent space of $\mathcal{M}$ at the point m and denoted by $T^*_m(\mathcal{M})$. The vector bundle over $\mathcal{M}$ whose fibers are the cotangent spaces of $\mathcal{M}$ is called cotangent bundle of $\mathcal{M}$ and denoted by $T^*(\mathcal{M})$. Elements of $T^*(\mathcal{M})$ are called cotangent vectors. It follows that a section of $T^*(\mathcal{M})$ is a differential form of degree 1.

The space $\Lambda^*(T^*_m(\mathcal{M}))$ is the Grassmann algebra generated over the cotangent space of $\mathcal{M}$ at the point m. The vector bundle over $\mathcal{M}$ with fiber $\Lambda^k(T^*_m(\mathcal{M}))$ over m is then denoted by $\Lambda^k(T(\mathcal{M}))$ and called the k-vector tangent bundle. If $\mathcal{M}$ is a Riemannian manifold then $\Lambda^k(T(\mathcal{M}))$ is a Riemannian vector bundle.