Blow-up in complex time

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Hiermit bestätige ich, Hannes Stuke, dass ich die vorliegende Dissertation mit dem Thema

Blow-up in complex time

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Abstract

Scalar reaction-diffusion type partial differential equations (PDE) exhibit a phenomenon called blow-up. A solution blows-up in finite time if it ceases to exist in the solutions space, i.e. the norm grows to infinite. On the other hand, in reaction-diffusion type PDE there exists the notion of global attractor, the maximal compact invariant set, that attracts all bounded solutions. In this thesis we study a hidden kinship between solutions in the global attractor and blow-up solutions in analytic PDEs by allowing for complex time.

In the first chapter we prove that heteroclinic orbits in one-dimensional unstable manifolds are accompanied by blow-up solutions. Furthermore, we study in more detail the quadratic nonlinear heat equation

$$u_t = u_{xx} + u^2,$$

and the heteroclinic orbit starting from the unique positive equilibrium. In this setting we show, that blow-up solution can be continued back to the real axis after the blow-up, but continuations along different time paths do not coincide. The proof relies on analyticity of unstable manifolds. This does not hold for center manifolds. In the second chapter we show that in special cases we can continue one-dimensional center manifolds of PDEs to sectors in the complex plane.

Using the result of the second chapter, we prove the existence of blow-up solutions of PDEs in the presence of one-dimensional non-degenerate center manifolds.

Zusammenfassung

Skalare partielle reaktions-diffusions Differentialgleichungen (PDE) weisen das Phänomen des "blow-ups" auf. Eine Lösung "blows-up" in endlicher Zeit, falls sie aufhört im Lösungsraum der PDE zu existieren, also die Norm gegen unendlich geht. Auf der anderen Seite existieren globale Attraktoren - sie sind die maximale, kompakte und invariante Menge, die alle beschränkten Lösungen anzieht. In der vorliegenden Arbeit untersuchen wir den Zusammenhang des globalen Attraktors und "blow-up" in analytischen PDEs durch die Benutzung von komplexer Zeit.

In dem ersten Kapitel zeigen wir, dass heterokline Lösungen auf eindimensionalen instabilen Mannigfaltigkeiten zusammen mit einem "blow-up" Orbit kommen. Wir studieren weiterhin die quadratische nichtlineare Wärmeleitungsgleichung

$$u_t = u_{xx} + u^2,$$

und den heteroklinen Orbit, der von dem eindeutigen Gleichgewicht startet. In diesem Beispiel sind wir in der Lage zu zeigen, dass der "blow-up" Orbit durch die komplexe Zeit am "blow-up" Zeitpunkt vorbei zurück auf die reelle Zeitachse fortgesetzt werden kann. Allerdings müssen die Fortsetzungen entlang verschiedener komplexer Zeit Pfade nicht übereinstimmen. Der Beweis benutzt die Analytizität der instabilen Mannigfaltigkeit. Zentrumsmannigfaltigkeiten hingegen sind nicht analytisch. Dennoch können wir im zweiten Kapitel zeigen, dass sich eindimensionale PDE Zentrumsmannigfaltigkeiten in speziellen Fällen in Sektoren der komplexen Ebene fortsetzen lassen. In dem dritten Kapitel zeigen wir unter der Verwendung der Resultate des zweiten Kapitels die Existenz von "blow-up" Lösungen auf eindimensionalen Zentrumsmannigfaltigkeiten.

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Contents

Introduction and Overview

Current developments in the analysis of dynamics of partial differential equations (PDEs) seem to pursue two opposite directions.

On the one hand, the concept of global attractors has received ongoing attention, ever since their discovery by Olga Alexandrowna Ladyschenskaja in the 1970s. By definition the global attractor is the maximal compact invariant set that attracts all bounded sets, see e.g. [51], [20].

On the other hand there is the phenomenon of finite time blow-up. A solutions blows up in finite time if it becomes unbounded in finite time.

Assuming analytic nonlinearities this thesis attempts to reconcile both concepts by passing to the complex time domain. The solution flow to a differential equation satisfies the complex flow property as long as the solution is analytic. Here complex flow property is the same as complex time path independence. Consider for example a solution flow $\Phi : D \subset \mathbb{C} \times X \to X$ with appropriate phase space X. Then for $z_1, z_2 \in \mathbb{C}$ and $u_0 \in X$ it holds,

$$\Phi\left(z_1, \Phi\left(z_2, u_0\right)\right) = \Phi\left(z_1 + z_2, u_0\right) = \Phi\left(z_2, \Phi\left(z_1, u_0\right)\right),$$

if there is no singularity in the interior of the two different complex time paths from zero to $z_1 + z_2$, i.e. the solution is analytic in the interior.

The idea to establish a dichotomy between bounded solutions and in particular equilibria and unbounded solutions is not as new as it may sound. Indeed the well-known theorem of Liouville (1809–1882) is a first result in that direction.

Theorem (Liouville's theorem)**.** *An entire complex analytic function that is uniformly bounded must be constant.*

Suppose we have found a nonconstant solution in a global PDE attractor in real time. The complex time continuation can not remain uniformly bounded by Liouville's theorem. In other words the solution must blow up in finite time or tend to infinity as time tends to complex infinity. Our results extend and refine this first superficial observation.

Generic solutions of PDEs however exist neither for negative nor for complex time. Extensions to complex time can be obtained by using analytic semigroups. They allow to solve PDEs in sectors of the complex time domain, but do not allow to solve PDEs in negative real time. This is where invariant manifold theorems from the theory of dynamical systems come into play. Unstable manifolds, for example, consist of all complex-valued solutions which exist and converge to equilibrium in backwards time. In variational settings global attractors consist of the union of all such unstable manifold. We consider parabolic differential equations of the reaction-diffusion type

$$u_t = u_{xx} + f(u),$$

with entire f and in particular $f(u) = u^2$. The latter equation is called nonlinear heat equation. In the parabolic setting the unstable manifolds are finite dimensional, [22]. The PDE reduces to an ordinary differential equation (ODE) on them. The analysis of the ODEs allows us to construct complex time continued solutions and thus in complex time Liouville's theorem applies.

Once we have established a connection between blow-up solutions and solutions in the global attractor, in particular equilibria, heteroclinic and homoclinic orbits, immediately a lot of interesting questions arise, for example

(i) Which blow-up solutions can really be related to bounded solutions on the global attractor via complex time detours?
For example consider the positive equilibrium u_+ of the quadratic nonlinear heat equation

$$u_t = u_{xx} + u^2,$$

with Dirichlet boundary conditions, [48]. The equilibrium u_+ has a one-dimensional unstable manifold W^u, [48]. Then we encounter finite time blow-up above u_+, on W^u. Below u_+ we obtain a uniformly bounded heteroclinic solution from u_+ to $u \equiv 0$. Through complex W^u and complex time extension, both solutions are one and the same solution evaluated at different complex time paths.

(ii) In the presence of blow-up the set of all uniformly bounded eternal solutions is the proper replacement of the global attractor. If this set, itself, is bounded then we call it the eternal core. The analysis in this thesis suggests a hidden kinship between blow-up and eternal core. The question how this kinship actually manifests is a guiding thread of this thesis.

In this thesis we are only able to scratch the surface of these questions. In Chapter 1 we show that there exists indeed a connection between bounded eternal solutions, in particular heteroclinic orbits, equilibria and blow-up solutions. In the example of the quadratic nonlinear heat equation, the blow-up solution on the unstable manifold of u_+ can be analytically continued through complex time back to the real axis. The real time blow up introduces a branched Riemann surface by analytic continuation of the blow-up solution around the singularity along different complex time paths. The Riemann surface is connected to the local behaviour of the heteroclinic orbit at the connected equilibria.

We will elaborate in the next two Chapters 2 and 3 on the idea developed in Chapter 1. We show that under extra conditions the purely local analysis of a one-dimensional center manifold can already guarantee the existence of blow-up solutions. In other words, the existence of blow-up is shown by local analysis around a single equilibrium. More specifically we prove the existence of a real initial condition which has blow-up in a complex time half strip. Furthermore, we show that there exists a family of complex initial conditions with arbitrarily small imaginary part and real finite time blow-up. However we can not exclude the possibility that the blow-up time goes to infinity if the imaginary part goes to zero. Moreover, we obtain a relation between the branch-type of the analytic continuation of the blow-up orbit and the behaviour of the solution close to the equilibrium. As far as we know this is the first time blow-up in parabolic PDEs is proven and described by complex time extensions of bounded solutions.

Each of the chapters contains its own introduction and overview over the existing literature and can be seen as rather independent part. For the sake of readability we included an Appendix that contains important Theorems.

Chapter 1

Blow-up of the nonlinear heat equation

Abstract

In this chapter we analyze the equation $u_t = u_{xx} + u^2$ *with Dirichlet boundary conditions. We show that the complex unstable manifold of the unique positive equilibrium* u_+ *is foliated by heteroclinic orbits, which are complex time continuations of the real time blow-up orbit. We furthermore show that the blow-up solution can be continued back to the real time axis after blow-up, but analytic continuations along different paths do not coincide. Thus blow-up can be detected by regular solutions and even proven rigorously by numerical methods.*

1.1 Introduction

In this chapter we study the nonlinear heat equation with Dirichlet boundary conditions, i.e

$$u_t = \Delta u + u^p, \qquad x \in \Omega \subset \mathbb{R}^N, \qquad u|_{\partial\Omega} = 0. \tag{1.1}$$

It is well-known that solutions may blow-up, see [48], [24]. For example the following questions are addressed for different p and N.

(i) What is the blow-up rate of solutions?
Suppose that the solutions blows-up at time $T > 0$, i.e. $\|u(t,x)\|_\infty \to \infty$ for $t \nearrow T$. The question is at which rate the norm approaches infinity, e.g. is there a function $g : (0,T) \to \mathbb{R}_+$ such that

$$\|u(t,x)\|_\infty \leq g(t), \qquad t \in (0,T).$$

One might expect that blow-up of solutions happens due to the reaction term, since the Laplace operator regularizes the solution. The pure reaction part $u_t = u^p$ has an explicit solution

$$u(t) = \left(\frac{1}{c-t}\right)^{1/(p-1)}.$$

If the sup-norm of u is bounded by the ODE rate, i. e.

$$|u(t,x)| \leq C\left(\frac{1}{T-t}\right)^{1/(p-1)}, \qquad \forall (t,x) \in (0,T) \times \Omega. \tag{1.2}$$

the blow-up is called type-I blow up. There are several theorems which give sufficient conditions on p and N such, that every blow-up is of type-I. But inequality (1.2) does not always hold. Then the blow-up is called type-II blow up.

For solutions in complex time the definition of blow-up rate is not straight forward anymore. In the complex plane, we can take time limits $t_n \to T$ from various directions and it is not clear whether the limit is independent of the sequence. Consider for example the function

$$f(t) := t^{-1}e^{-1/t}, \qquad t \in \mathbb{C} \setminus \{0\}.$$

As long as we take a sequence $t_n \to 0$ with positive real part the limit will be zero, whereas otherwise the limit might become unbounded or does not exist at all. In complex analysis, there is the notion of sectoral limits [10]. A sectorial limit exists at $t = 0$ if the function $f(t)$ converges uniformly to the same value inside a sector attached to zero. In the above example the sectorial limit is defined for subsectors of the right complex half plane.

(ii) What is the blow-up set?
The blow-up set is the following set

$$B := \left\{x \in \bar{\Omega} : \exists (x_k, t_k) \in \Omega \times (0,T), \text{ such that } (x_k,t_k) \to (x,T) \text{ and } |u(t_k,x_k)| \to \infty\right\}.$$

It has been shown that, if Ω is convex, the blow-up set is compact and thus contained in the interior of Ω. Furthermore, there are examples in which the blow-up set consist of discrete points and one can even construct solution (for a different nonlinearity) such that the blow-up set contains an open set, [48]. Moreover one can construct solutions with prescribed sign changing blow-up profiles [14]. But still the definition of the blow-up set has the same difficulty as the blow-up rate, since t_k can be complex.

(iii) How does the blow-up profile look like?
This question is more abstract. One possible answer is to study rescaled versions of the solution u. Here the blow-up rate becomes important, since type-I blow-up solutions converge to equilibrium in rescaled coordinates.

For $T = 0$, the following change of coordinates

$$s = -\log -t, \qquad y = \frac{x}{\sqrt{-t}}, \qquad w = (-t)^{1/(p-1)} u. \tag{1.3}$$

transforms equation the nonlinear heat equation to

$$w_s = \partial_{yy} w + \frac{y}{2} w_y + \frac{1}{p-1} w = |w|^{p-1} w. \tag{1.4}$$

One can show by Pohozaev identity [48] and assuming subexponential growth for $|y| \to \infty$, that equation (1.4) has only constant equilibria, i.e.

$$w = 0, \qquad w = \pm \left(\frac{1}{p-1} \right)^{1/(p-1)}.$$

This yields the following convergence result [63],

Theorem 1.1.1. *If $x = 0$ is a blow-up point and*

$$|u(t,x)| \leq \frac{C}{(T-t)^{1/(p-1)}}, \qquad (x,t) \in I \times (0,T),$$

then for any $K > 1$,

$$(T-t)^{1/(p-1)} u\left(t, y\sqrt{T-t}\right) \to \left(\frac{1}{p-1} \right)^{1/(p-1)},$$

uniformly for $|y| \leq K$ as $t \nearrow 0$.

A better description, e.g. asmyptotic rates of convergence or convergence on sets $|y| \leq K\sqrt{|\log(T-t)|}$ can then be obtained by a very subtle analysis of the "center-stable manifold" of the linearisation around the equilibrium w^+. The main problem is, that due to the weighted spaces in which the solution is considered, there is no known proper functional analytic setting to really obtain a center-stable manifold [54], [53] so that it can not be more than a guideline for the analysis. This makes the analysis of converging solutions much more complicated, since one needs to control the information coming from large $|y|$, see among many others [34], [8].

There have also been attempts to study the complex-valued nonlinear heat equation [27], [41] for real time only, as one can under that circumstances rewrite (1.11) as system of real and imaginary part of $u = v + iw$.

$$\begin{aligned} v_t &= v_{xx} + v^2 - w^2, \\ w_t &= w_{xx} + 2wv. \end{aligned}$$

Especially [41] was able to derive detailed asymptotic expansions of the blow-up profile close to the blow-up.

(iv) A further question asked already by [59] concerns the continuation of solutions after blow-up. One possibility to tackle the problem is monotone approximation from below, which let to the notion of complete blow-up, see e.g. [4], [31], [36], [46], [48].

Consider the following truncations of the nonlinearity f defined by $f_k(u) := \min(u, k)$. Then the equation to the truncated nonlinearity

$$u_t^k = \Delta u^k + f_k\left(u^k\right), \qquad u^k(0) = u_0,$$

possess global solutions. For any k the solution u^k coincides with the solution u of (1.1) as long as u is below k. Suppose now, that u is positive and blows up at time T. Then the solutions u^k monotonically approximate u from below before the blow-up and even exist after the blow-up. So the question is now, if the solutions u^k converge for $k \to \infty$ and $t > T$, that is in which sense the function

$$\bar{u}(t, x) := \lim_{k\to\infty} u^k(t, x),$$

exists.

The time of complete blow-up is defined as follows

$$T_c := \inf\{t \geq T,\ \bar{u}(t, x) = \infty,\ \forall x \in \Omega\}.$$

Note, that after T_c, the solution is unbounded for all $x \in \Omega$. This implies that there are no "weak" solutions that are compatible with point wise approximations from below, e.g. solutions of weaker integrability or measure-valued solutions after blow-up. A solution blows-up completely at T if $T_c = T$. The quadratic one-dimensional nonlinear heat equation has only complete blow-up [48].

In this chapter we take a different point of view to address the questions posed above which we started to develop in [55]. We will especially tackle the problem of continuation, since in the situation of complete blow-up we are lacking a proper notion of continued solutions. We consider the quadratic nonlinear heat equation

$$u_t = u_{xx} + u^2, \qquad x \in (-1, 1), \qquad u(\pm 1) = 0. \tag{1.5}$$

and try to address the following questions

(i) Can we prove the existence of blow-up by studying only bounded solutions?

(ii) Can we extend the solution after blow-up though complex time back to the real axis? And do analytic continuations coincide after real axis along different paths of continuation?

In Section 1.2 we prove that if there exists a one-dimensional fast unstable manifold of an evolution equation of the form

$$u_t = Au + f(u),$$

which contains a heteroclinic orbit connecting two equilibria, there must be a complex-time blow-up orbit starting with real initial data and a real time blow-up orbit starting with complex initial data.

We give a more detailed description for blow-up solutions of equation (1.5). Equation (1.5) possesses a positive equilibrium u_+ with one-dimensional fast unstable manifold W^u. Locally the manifold is a graph Υ over the eigenspace of the fast unstable mode. One side of the fast unstable manifold is a heteroclinic orbit to zero, whereas the other side is a blow up orbit, [48].

We denote the solution flow of equation (1.5) by Φ. The domain to which we can continue solutions are so called spall strips.

Definition 1.1.2 (Spall strip)**.** *For any $\delta > 0$ and $0 < T < T_1$ we denote the upper/lower spall strip as*

$$S_{\pm}(\delta, [T, T_1]) := \{t \in \mathbb{C} \setminus [T, T_1],\ 0 \leq \pm Im(t) \leq \delta\}.$$

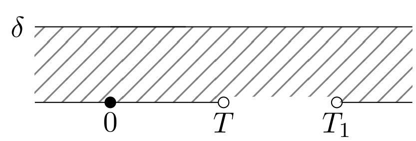


Figure 1.1: Spall strip domain

We will prove the following theorem.

Theorem 1.1.3. *There exists a $\delta > 0$ such that the time analytic continuations of the real blow up orbit on the fast unstable manifold of u_+, i.e. $\Phi(t, (\tau, \Upsilon(\tau)))$, $0 < \tau < \delta$ exists and has the following properties:*

(i) It blows up completely after at time T.

(ii) It can be continued to upper and lower spall strips $S_{\pm}(\delta, [T, T_1])$ for some

$$T_1 < 2C_0 \max\left\{\max_{x \in I} \frac{v_0(x)}{w_0(x)}, \|u_+\|_\infty\right\}.$$

The constant $C_0 > 0$ depends only on the heat semigroup.

(iii) The upper and lower time path continuations do not coincide after $T_1 > 0$

We furthermore show that the branch-type of the complex-time continuation is related to the quotient of the eigenvalues of the eigenfunctions to which the heteroclinic orbit is tangent at u_+ resp at $u \equiv 0$. If the quotient is rational, the heteroclinic orbit lives on the a compact root-type Riemann surface, whereas if the quotient is irrational it will live on the Riemann surface of the logarithm.

There have already been previous attempts to the question of analytic time continuations, mainly [27] and [37], [38].

Masuda [37], [38] has already proven the following results for the quadratic nonlinear heat equation with Neumann boundary conditions

Define the constant a as follows

$$a := \frac{1}{2} \int_{-1}^{1} u_0(x)\, dx.$$

Theorem 1.1.4. *Let u_0 be a non-negative function ($u_0 \neq 0$) in $W_p^2(\Omega)$, $p > n$, and set $a = Pu_0$. If $\left\|\partial_x^2 u_0\right\|_p / |a|^2$ is sufficiently small, then there exists a unique solution u_j ($j = 1, 2$) of the quadratic nonlinear heat equation with Neumann boundary conditions which is analytic in $t \in D_j$ as a $W_p^2(\Omega)$-valued function and converges to u_0 as $t \to 0$, $|\arg t| < \theta$, in the norm of $W_p^2(\Omega)$.*

This result shows the existence of analytic continuations for solutions with almost constant initial data in the regions $D_{1,2}$. Note, that $D_1 = \bar{D}_2$, see Figure 1.2.

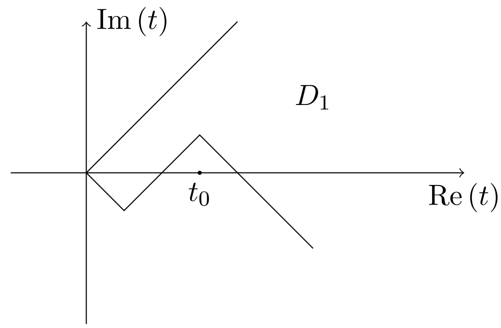


Figure 1.2: Existence of solution in complex time

Theorem 1.1.5. *Let u_j , $j = 1, 2$ be as in Theorem 1.1.4. If $\left\|\partial_x^2 u_0\right\|_p / |a|^2$ is sufficiently small and $u_1(t, \cdot) = u_2(t, \cdot)$ for some $t \in D_1 \cap D_2$ for $Re(t) > t_0 + \delta$, then is u_0 a constant function.*

The proofs of Masuda rely on the implicit function theorem with respect to the explicit solution to spatially constant initial conditions.

Also [27] has tackled the problem of the quadratic nonlinear heat equation. They considered the equation for real and imaginary part of $u = v + iw$ separately, but for real time only

$$\begin{pmatrix} v_t \\ w_t \end{pmatrix} = \begin{pmatrix} v_{xx} \\ w_{xx} \end{pmatrix} + \begin{pmatrix} v^2 - w^2 \\ 2vw \end{pmatrix}, \tag{1.6}$$

and $x \in \mathbb{R}$.

They observed that the image of the real time flow $t \in \mathbb{R} \mapsto \eta(t, v_0, w_0)$ of the reaction term

$$\begin{pmatrix} v_t \\ w_t \end{pmatrix} = \begin{pmatrix} v^2 - w^2 \\ 2vw \end{pmatrix}, \tag{1.7}$$

is a circle in $\mathbb{R}^2$ if $w_0 \neq 0$. If the image of the ODE flow of the reaction term in a system of reaction-diffusion equations is a convex subset of $\mathbb{R}^2$ and the diffusion parts do not couple, then the system possesses a maximum principle [62], [12]:

If the initial and the boundary conditions of system (1.6) is contained in the interior of a solution to (1.7), then is the solution of (1.6) contained in the interior for all positive times.

Note, that the maximum principle may not hold anymore if the diffusion parts start to couple. This happens if one rotates the time axis into the complex plane, e.g. $t \mapsto e^{i\theta}t$, which yields the equation

$$\begin{pmatrix} v_t \\ w_t \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \left[\begin{pmatrix} v_{xx} \\ w_{xx} \end{pmatrix} + \begin{pmatrix} u^2 - v^2 \\ 2vw \end{pmatrix} \right].$$

But if we solve (1.5) along complex time paths that are parallel to the real time axis ($\theta = 0$), we can always rewrite the equation as a system of real and imaginary part of the form (1.6).

Using the maximum principle, [27] also proved convergence to zero if the initial values $v_0(x)$, $w_0(x)$ satisfy

$$v_0(x) - Bw_0(x) < 0, \qquad \forall x \in \mathbb{R},$$

and some $B \in \mathbb{R}$.

We will use a very similar result to show existence and convergence of complex time analytically continued solutions for times with large real part.

Throughout the chapter we need the following definitions.

Definition 1.1.6 (Sector)**.** *We denote by $S_{r,\alpha}$ sectorial subsets of the complex plane, that is*

$$S_{r,\alpha} := \left\{ se^{i\varphi} \in \mathbb{C},\ 0 < s < r,\ |\varphi| < \alpha \right\}.$$

Furthermore the unbounded sector $S_{\infty,\alpha}$ is abbreviated by S_α.

Definition 1.1.7 (Strip)**.** *We denote by* $St_{r_1,r_2,\delta}$ *rectangular subsets of the complex plane, that is*

$$St_{r_1,r_2,\delta} := \{x + iy \in \mathbb{C},\ r_1 < x < r_2,\ |y| < \delta\}.$$

The strip $St_{0,r_2,\delta}$ *is abbreviated by* $St_{r_2,\delta}$ *and the unbounded half-strip* $St_{0,\infty,\delta}$ *by* St_{δ}*.*

Definition 1.1.8 (Time p–path)**.** *We define a path* $\gamma : I \to \mathbb{C}, t \mapsto t + i\tilde{\delta}$ *for some fixed* $\tilde{\delta}$ *and some interval* I *as time parallel-path or time p–path, since the line* γ *is parallel to the real axis.*

1.2 The unstable manifold theorem and its complex consequences

We analyze the equation

$$u_t = Au + f(u), \tag{1.8}$$

with $f(0) = 0$ and $Df(0) = 0$.

(i) Consider the complex separable Banach spaces Z, Y and X such that the embeddings $Z \hookrightarrow Y \hookrightarrow X$ are continuous.

(ii) $A : Z \to X$ is a bounded operator.

(iii) A is a sectorial operator with $|\mathrm{Re}(\sigma(A))| \geq \delta > 0$.

(iv) $f \in C^{\omega}(Z, Y)$ and $f(\mathbb{R}) \subset \mathbb{R}$.

(v) The operator A has a compact resolvent.

As canonical example one should think of the quadratic nonlinear heat equation with $X = L^2(I, \mathbb{C})$, $Y = H_0^1(I, \mathbb{C})$ and $Z = H^2(I, \mathbb{C}) \cap H_0^1(I, \mathbb{C})$, $I = (-1, 1)$.

We also assume that there exists a second real-valued equilibrium u_+ of the equation (1.8) whose linearization $A_+ := A + Df(u_+)$ has similar properties to A.

The above conditions imply the existence of stable and unstable manifolds around $u \equiv 0$ and u_+, see e.g. [33], [22] and [26]. Since the resolvent of A is compact, the spectrum is discrete and we can construct strong unstable manifolds. The operator A introduces spectral projection to each of the discrete eigenvalues, see Appendix. Throughout this chapter, we make the further assumption that the largest eigenvalue of A_+ is algebraically and geometrically simple with eigenvalue $\mu > 0$ and eigenfunction φ_μ. This implies, that the strong unstable manifold associated to μ is one-dimensional. We will denote the associated spectral projection P_+ and $P_- = \mathrm{Id} - P_+$.

Also by [22] we know that the strong unstable manifold is actually analytic, that is the graph $\Upsilon : (-a, a) \to Z_-$ is an analytic function which can be extended to a complex neighborhood of zero. Here $Z_- := P_- Z$. The flow on the strong unstable manifold is a one-dimensional analytic differential equation

$$\dot{q} = \mu q + P_+ f(q\varphi_\mu + \Upsilon(q)),$$

Instead of $f(q\varphi_\mu + \Upsilon(q))$ we will also write $f(q, \Upsilon(q))$.

The solution semigroup of ordinary differential equations can be extended to a group in complex time. This will be important together with the fact that the ordinary differential equation depends analytically on the initial data on the unstable manifold, see for example [23].

We begin with a very simple Lemma about the connection of unbounded and bounded solutions for analytic systems. It is a direct corollary to Vitali's theorem 4.2.7 and its general form 4.2.11.

Lemma 1.2.1. *Consider a map $\Phi : \mathbb{R}_+ \times U \subset \mathbb{C} \to Z$ with the following properties*

(i) $\Phi(t, 0) = 0$, $t > 0$,

(ii) $\Phi(t, \cdot) \in C^\omega(U, Z)$ *for* $t > 0$.

$U \subset \mathbb{C}$ is an open and connected neighbourhood of zero. Assume that there exists an element $q_0 \in U$ and a sequence of pairwise different $q_m \in U$ with $\lim_{m\to\infty} q_m = q_0 \in U$ such that

$$\lim_{t\to\infty} \Phi(t, q_m) \to p^*, \qquad m \in \mathbb{N} \cup \{0\}, \qquad p^* \in Z.$$

Then for all $q \in U$ holds,

$$\lim_{t\to\infty} \Phi(t, q) = p^*, \qquad q \in U.$$

Proof. Consider any sequence $t_n \nearrow \infty$ and define the family of analytic functions f_n defined as the time t_n – maps of the neighborhood U, i.e.

$$f_n : U \to Z, \qquad f_n(u) := \Phi(t_n, q).$$

By assumption the set U_0 is defined as

$$U_0 := \left\{ z \in U : \lim_{n\to\infty} f_n(z) \text{ exists} \right\},$$

non–empty and has a limit point $q_0 \in U$. Thus by Vitali's theorem 4.2.7 or 4.2.11 we know that the family f_n converges uniformly to an analytic function $f : U \to Z$ for all $\tilde{q}_0 \in U$. Note, that the limit exists for the full sequence and not just for a subsequence. Here it is important, that the claim does not follow by compactness properties, but uses the analyticity of the functions. By assumption we have $f(q_m) = p^*$ and the sequence q_m also has an interior limit point. We can conclude that f is constant by the identity theorem 4.2.6, that is $f \equiv p^*$. Thus, the full sequence converges to p^*

$$\lim_{n\to\infty} \Phi(t_n, q) = p^*.$$

Since the sequence t_n was arbitrary the claim holds for any sequence $t_n \nearrow \infty$. □

Remark 1.2.2. *Note, that this theorem does not follow from Montel compactness. We get the limit on the full sequence and not just a converging subsequence and furthermore, we do not require any compactness of the image of f_n. The existence of the limit follows from assumptions on the regularity and give stronger convergence results.*

Lemma 1.2.3. *Consider a map $\Phi : \mathbb{R}_+ \times U \subset \mathbb{C}^N \to Z$, with the following properties*

(i) $\Phi(t, 0) = 0$, $t > 0$,

(ii) For $t > 0$, $\Phi(t, \cdot) \in C^\omega(U, K)$ where K is a compact subset of Z

U is an open and connected neighborhood of zero. The equilibrium zero is assumed to have at least one unstable direction, that is there exists $0 \neq p_0 \in Z$ and $q_n \in U$ converging to zero with $\Phi(n, q_n) = p_0$. Furthermore assume, that the flow is analytic with respect to the initial data as long as it stays bounded and has values in a compact subset $K \subset Z$. Then $\Phi(t, U)$ can not stay uniformly bounded with $t > 0$.

Proof. The proof argues by contradiction. Assume that the time n–maps stay uniformly bounded, that is consider the (uniformly bounded) family

$$f_n : U \to K \subset Z, \qquad f_n(u) := \Phi(n, u).$$

By Montel compactness theorem 4.2.5 there exists a subsequence $\tilde{f}_n := f_{\tilde{n}(n)}$ of f_n converging uniformly to an analytic function $\tilde{f}$. Then we have for $\tilde{q}_n := q_{\tilde{n}(n)}$

$$\|p_0\| = \left\|\tilde{f}_n(\tilde{q}_n)\right\| \leq \left\|\tilde{f}_n(\tilde{q}_n) - \tilde{f}_m(\tilde{q}_n)\right\| + \left\|\tilde{f}_m(\tilde{q}_n)\right\|, \qquad m, n \in \mathbb{N}.$$

For all $\varepsilon > 0$, since the convergence is uniform, we can choose, m, n large enough such that

$$\left\|\tilde{f}_n(\tilde{q}_n) - \tilde{f}_m(\tilde{q}_n)\right\| < \varepsilon.$$

Now choosing n large enough, we also have

$$\left\|\tilde{f}_m(\tilde{q}_n)\right\| < \varepsilon.$$

This implies $\|p_0\| < 2\varepsilon$. Since the argument holds for any ε we can conclude $p_0 = 0$, which is a contradiction. □

Remark 1.2.4. *The proof even shows that the functions $\Phi(\cdot, U)$ can not stay bounded for any (complex) neighborhood $\tilde{U} \subset U$ arbitrarily close to zero for all positive time $t > 0$.*

Remark 1.2.5. *The proof also works, if the $\Phi(t, \cdot)$ is only locally analytic for any $q \in U$. Analyticity on the full set U follows by analytic continuation.*

The two Lemmata can be connected the theory of stable and unstable manifolds in dynamicals systems.

Lemma 1.2.6. *Consider a semi flow $\Phi : S_Z \subset \mathbb{R}_+ \times Z \to Z$ of the differential equation* (1.8)*. Assume that the real fast unstable manifold of u_+ is one-dimensional and contains a heteroclinic orbit from u_+ to equilibrium $u_- \equiv 0$. Then solutions on a complex neighborhood in the fast unstable manifold of equilibrium u_+ can not stay uniformly bounded for real positive time.*

Proof. The proof proceeds by contradiction.

(i) Denote the analytic graph of the one–dimensional real fast unstable manifold by Υ. The real analytic graph can be extended to an analytic function on some complex neighborhood U of the equilibrium, see Figure 1.3.

(ii) Denote by f_n the time n - map of the neighborhood U, i.e.

$$f_n : U \to Z, \qquad f_n(q_0) = \Phi(n, (q_0, \Upsilon(q_0))) .$$

where $(q_0, \Upsilon(q_0))$ is on the heteroclinic orbit. Assume that the family f_n stays uniformly bounded.
Take any sequence $t_m \in \mathbb{R}$ with $|t_m|$ small enough and $t_m \to \delta$. Define $u_m \in \mathbb{C}$ by

$$u_m := P_+ (\Phi(t_m, (u_0, \Upsilon(u_0)))) ,$$

where P_+ is the projection on the tangent space of the strong unstable manifold $W^{su}(u_+)$ and δ small enough such that the projection is still contained in U, i.e.

$$P_+ (\Phi(t_m, (q_0, \Upsilon(q_0)))) \in U.$$

. The existence of the projection P_+ is due to theorem 4.1.5. The sequence of q_m converges to $q_0^\delta \in U$. By construction each $(q_m, \Upsilon(q_m))$ is on the heteroclinic orbit and it holds

$$\lim_{t\to\infty} \Phi(t, q_m) = u_-.$$

Since we assumed that the flow stays uniformly bounded, we can apply Lemma 1.2.1 to obtain

$$\lim_{t\to\infty} \Phi(t, U) = u_-.$$

This contradicts $u_+ \in U$.

□

Remark 1.2.7. *The Lemma shows that, a flow that depends analytically on the initial condition, can not change the limit point if the flow stays uniformly bounded. This implies, that there might be a kinship between grow-up, blow-up and the global attractor of differential equations in the complex domain.*

The Lemma required the mere existence of a real heteroclinic orbit. But we can similarly also prove that there exists grow-up or blow-up even if we do not have such heteroclinic orbit, but a finite-dimensional analytic unstable manifold by Lemma 1.2.3.

The theorem tells us that the equilibria of analytic systems are related to blow-up or grow-up of dynamical systems in the complexified domain. Next we show that the solution must actually possess blow-up and that the analytic time continuation of the blow-up orbit is also a real time heteroclinic orbit. Indeed, the one-dimensional fast unstable manifold is foliated by heteroclinic orbits and there exists a non-empty boundary to the foliation, which is a blow-up orbit.

Lemma 1.2.8. *Consider the setting of Lemma 1.2.6. Define the set $\tilde{H}$ as follows*

$$\tilde{H} := \left\{ u_0 \in U \subset \mathbb{C}, \ \sup_{t \in \mathbb{R}_+} \|\Phi(t, (u_0, \Upsilon(u_0)))\|_Z < \infty \right\},$$

and the set H as the connected component of $\tilde{H}$ that contains $U \cap \mathbb{R}_-$. Then H is foliated by real time heteroclinic orbits with complex initial data and furthermore all real time heteroclinic orbits are time p - path continuations of the real heteroclinic orbit.

Proof. We proceed by the following steps.

(i) Take any $q_0 \in U$. Then there exists a $r > 0$ such that a small ball $B_r(q_0)$ in the tangent space of the fast unstable manifold is filled by the complex time flow of q_0. The proof argues via the complex one-dimensional equation on the fast unstable manifold

$$\dot{q} = \mu q + \tilde{f}(q, \Upsilon(q)), \qquad q(0) = q_0. \tag{1.9}$$

where $\tilde{f} := P_{su} q$. For small enough q_0 we know that $\mu q_0 + \tilde{f}(q_0, \Upsilon(u_0)) \neq 0$ for $q_0 \neq 0$ and thus the separation of variables formula is well-defined

$$t(q, q_0) = \int_{q_0}^{q} \frac{1}{\mu\tau + f(\tau, \Upsilon(\tau))} d\tau, \tag{1.10}$$

for any $q \in B_r(q_0)$. This also implies an explicit estimate on the time $t(q, q_0)$ needed to go from q_0 to q

$$|t(q, q_0)| \leq 2 \left| \frac{q - q_0}{\mu q_0 + f(q_0, \Upsilon(q_0))} \right|.$$

(ii) The next step is to prove that the set H is open. Take any $q_0 \in H$. Then by assumption there exists an $M > 0$ such that $\|\Phi(t, (q_0, \Upsilon(q_0)))\|_Z < M$. The main problem is that the time t is unbounded. Thus we can not simply take small perturbations of the initial condition and argue by continuous dependence of initial data of time t maps and let t go to infinity. The idea to remedy that problem is that every initial condition $\tilde{q}_0$ in the unstable manifold close to q_0 can be reached by solving the reduced equation (1.19) for a small complex time q_0 given by (1.10). First note, that since the real time flow of q_0 is uniformly bounded, there exists a $\delta > 0$ such that we can extend the solution analytically to complex time strip

$$S := \{t \in \mathbb{C}, |\mathrm{Im}(t)| \leq \delta\}.$$

Take any $\tilde{q}_0 \in B_r(q_0)$ with $r > 0$. By the previous step we can choose an $r > 0$ small enough such that the neighborhood $B_r(q_0)$ is obtained by the complex time flow of q_0 with imaginary part less than δ. Thus there exists a $t_{\tilde{q}_0} := t(\tilde{q}_0, q_0) \in \mathbb{C}$ such that $u = \Phi(t_{\tilde{q}_0}, q_0)$

and $|\mathrm{Im}\,(t\,(\tilde{q}_0, q_0))| \leq \delta$. Due to time analyticity, the complex flow property holds and we obtain

$$\Phi\,(t, (\tilde{q}_0, \Upsilon\,(\tilde{q}_0))) = \Phi\,(t, \Phi\,(t_{\tilde{q}_0}, (q_0, \Upsilon\,(q_0)))) = \Phi\,(t_{\tilde{q}_0}, \Phi\,(t, (q_0, \Upsilon\,(q_0))))\,.$$

This implies $\tilde{q}_0 \in H$.

(iii) Since H is open we can apply Lemma 1.2.1 to obtain that H is foliated by real time heteroclinic orbits. Note that the backwards in time convergence is given by definition since H is a subset of the fast unstable manifold.

(iv) All in all we can conclude a foliation of the fast unstable manifold as indicated in Figure 1.3.

□

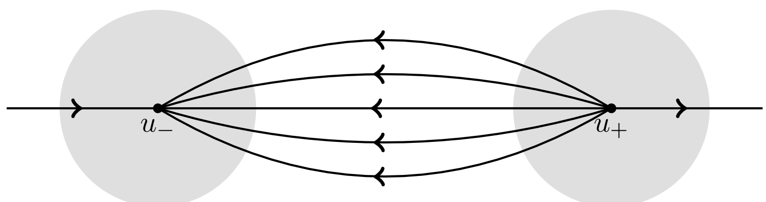


Figure 1.3: Heteroclinic foliation of the fast unstable manifold

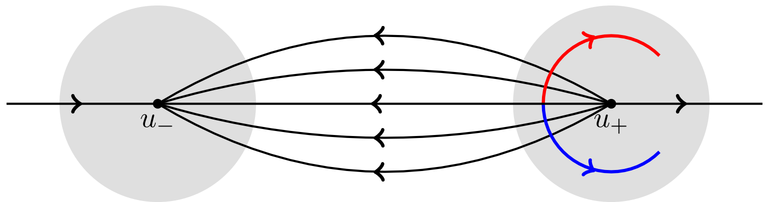
The set of heteroclinic orbits H on the fast unstable manifold is open and we study its topological boundary. We show that the boundary is non-empty and that it is a blow-up orbit. The blow-up orbit has a complex time analytic continuation into the positive or negative complex half-plane and it is heteroclinic along time p - paths.

Lemma 1.2.9. *The set $\partial H \cap U$ is not empty. The boundary orbits consist of a complex conjugate pair of finite time blow-up orbits with time analytic continuation into the positive or negative complex plane. The time p - paths of the analytic continuation are heteroclinic orbits and contained in H.*

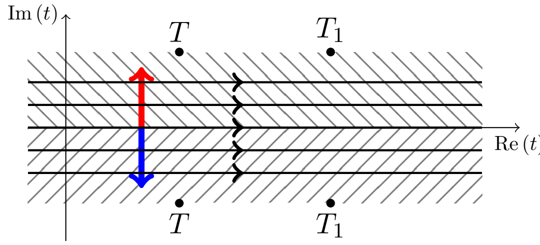
Proof. First we prove that H must have a boundary. As a consequence of Lemma 1.2.6, we know that there must exist a $q_0 \in U$ such that the forward flow is unbounded. This element can not be contained in H. Assume on the contrary that the boundary of the set $\partial H \cap U$ is empty. Then H must be open, connected and closed relative to U and thus $H = U$, which can not be true. We can assume, that there exists $q_0 \in \partial H \cap U$. The proof of the Lemma argues by contradiction. Assume that q_0 was a grow up orbit.

(i) Take $\tilde{q}_0 \in U \cap \mathbb{R}_-$. Arguing as in Lemma 1.2.8 we consider the differential equation on the fast unstable manifold

$$\dot{q} = \mu q + f\,(q, \Upsilon\,(q))\,.$$



(a) Path in the unstable manifold



(b) Associated complex time path of real blow-up orbit

Figure 1.4: Relation between real and imaginary time path in the unstable manifold

Also here we use separation of variables to solve the equation

$$t_{q_0} = \int_{\tilde{q}_0}^{q_0} \frac{1}{\mu\tau + f(\tau, \Upsilon(\tau))} d\tau.$$

Thus taking any closed path $\kappa : [0,1] \to U$ with $\kappa(0) = q_0$ and $\kappa(1) = q_0$ enclosing the origin we get by Cauchy theorem

$$t_{q_0} = \oint_{\tilde{q}_0}^{q_0} \frac{1}{\mu\tau + f(\tau, \Upsilon(\tau))} d\tau = \frac{1}{2i\pi\mu}.$$

This implies that we can take a purely imaginary time path to encircle zero in the fast unstable manifold. Since multiplication with the imaginary unit corresponds to a rotation of $\pi/2$ real and imaginary trajectories are always orthogonal to each other, see Figure 1.4.

(ii) By the above argument we can find a complex time path $\gamma(s) : [0, s_0) \to \mathbb{C}$ for some $s_0 > 0$ such that

$$q(\gamma(s), \tilde{q}_0) \subset H \cap U, \qquad q(\gamma(s_0), \tilde{q}_0) = q_0.$$

By assumption, the real positive time flow for any solution $\Phi(t, (q(\gamma(s), \tilde{q}_0), \Upsilon(q(\gamma(s), \tilde{q}_0)))$ stays bounded for all $0 \leq s < s_0$ and thus the real and complex time flow commute, i.e.

$$\Phi(t, (q_0, \Upsilon(q_0))) = \Phi(\gamma(a) + t, (\tilde{q}_0, \Upsilon(\tilde{q}_0))), \qquad t > 0.$$

For any $\varepsilon > 0$ we can choose a $t_0 > 0$ large enough such that

$$\|\Phi(t_0 + t, (\tilde{q}_0, \Upsilon(\tilde{q}_0)))\| < \varepsilon, \qquad t > 0.$$

This implies for ε small enough

$$\|\Phi(t_0 + t, (q_0, \Upsilon(q_0)))\| = \|\Phi(\gamma(a), \Phi(t_0 + t, (\tilde{q}_0, \Upsilon(\tilde{q}_0))))\| < 2\varepsilon,$$

for all $t > 0$. This contradicts the grow-up assumption.

(iii) Note that this also proves, that the one sided imaginary time flow of q_0 is contained in H.

□

This gives an estimate for the maximal complex time strip where the function can stay regular.

Corollary 1.2.10. *The real heteroclinic orbit has blow-up in a strip of size* $\frac{1}{4\pi\mu}$.

1.3 The non-linear heat equation

In this section, we will focus on the complex nonlinear heat equation with quadratic nonlinearity

$$u_t = u_{xx} + u^2, \qquad x \in (-1, 1), \; u(\pm 1) = 0. \tag{1.11}$$

Consider the space $Z := H^2(I) \cap H_0^1(I)$, $Y := H_0^1(I)$ and $X := L^2(I)$. Here all the functions are complex valued. The Laplace operator $A := \partial_{xx}$ is a bounded operator from Z to X and generates an analytic semigroup [33],

$$A : D(A) := H^2(I) \cap H_0^1(I) \subset L^2(I) \mapsto L^2(I).$$

By Sobolev embedding theorems, the quadratic nonlinearity $f(u) := u^2$ is analytic from $f : Z \to Y$ as well as a function $f : Y \to X$. By [33], [22] the local solution is analytic in time with values in Z.

We give a more advanced description of the time analytic continuation of the blow-up orbit of Lemma 1.2.9. The nonlinear heat equation has an unique positive equilibrium u_+ and it is well-known that solutions starting above u_+ exhibit finite time blow-up. Furthermore, the fast unstable manifold is one-dimensional and solutions that start below u_+ converge to zero.

We follow the idea of [27] to disprove blow-up of solutions on the unstable manifold of u_+ with complex initial conditions. In particular, we show, that the real blow-up orbit stays uniformly bounded on time p-paths with non-vanishing imaginary part. Note, that along time p-paths the equation (1.11) can be written as real system of real and imaginary part for $u = v + iw$

$$
\begin{aligned}
v_t &= v_{xx} + v^2 - w^2, \\
w_t &= v_{xx} + 2vw.
\end{aligned}
\tag{1.12}
$$

Different from [27], we consider Dirichlet boundary conditions, but we use the same maximum principle to show boundedness of solutions. The system (1.12) possesses a maximum principle [62], [49], [12]. One corollary from these maximum principles is that the solutions of systems of the form $w_t = \Delta w + f(w)$ where $w \in \mathbb{R}^2$ is contained in the ODE solution $\dot{w} = f(w)$ if the ODE solution defines a convex set and the initial condition and boundary condition lies in the interior of the convex set.
The ODE solutions of equation (1.12) are circles, see Figure 1.5. We prove that the non-real initial conditions on the fast unstable manifold of u_+ are contained in such circles.

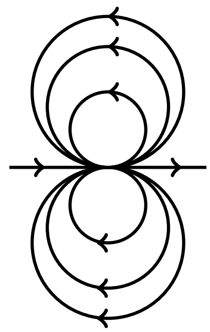


Figure 1.5: Solution to the ODE flow

ODE results

The maximum principle of the parabolic system (1.6) allows to use the ODE flow to the quadratic equation $\dot{U} = U^2$ to show a priori estimates on PDE solutions.

Explicit solution of the ODE Consider the differential equation

$$\dot{U}(t) = U^2(t), \qquad t \in \mathbb{C},\ U(0) = U_0 \in \mathbb{C},$$

The equation has an explicit solution, which can be obtained by separation of variables.

$$U(t) := \frac{1}{1/U_0 - t}. \tag{1.13}$$

The flow $\eta(t, U_0)$ is defined as the solution $U(t)$. The solution maps straight lines in $t = re^{i\theta}$, $r \in \mathbb{R}$ to circles, if it exists as a map to the complex plane $\mathbb{C}$.

But one can also consider the solution $U(t)$ as an analytic function from the complex plane to the Riemann sphere $\hat{\mathbb{C}}$ for all $t \in \mathbb{C}$. The Riemann sphere is the set $\mathbb{C} \cup \{\infty\}$ together with the charts $\tilde{U}_1 := U$ and $\tilde{U}_2 = U^{-1}$. The two charts coincide almost everywhere. The domain of definition of each chart is $\mathbb{C}$. Using the second chart, i.e. introducing $\tilde{U}_2(t) := U_1^{-1}(t)$ yields the differential equation

$$\dot{\tilde{U}}_2(t) = -1, \qquad t \in \mathbb{C}, \qquad (\tilde{U}_2)_0 = U_0^{-1}.$$

This has the trivial solution $\tilde{U}_2(t) = U_0^{-1} - t$, which is the same as (1.13). Note that the solution even exists if $\tilde{U}_2 = 0$, that is $|U| = \infty$. This implies, that the imaginary part of $\operatorname{Im}\left(\tilde{U}_2(t)\right) = \operatorname{Im}\left(U_0^{-1}\right)$ is a conserved quantity for the real time flow. To study the behaviour for unbounded t one has to use the chart $\tilde{U}_1$, in which all orbits converge to zero. The point zero is the only point that can not be described in the chart $\tilde{U}_2$. Since there is a conserved quantity in the $\tilde{U}_2$ chart, there also exists a conserved quantity in the $\tilde{U}_1$ chart.

Writing the function $U(t) := V(t) + iW(t)$, it can also be seen by explicit calculation the quantity

$$H(t) := V^2(t) + \left(W(t) - W_0^{-1}\right)^2,$$

is conserved by the flow. This gives the following definition.

Definition 1.3.1 (Solution circle/disk)**.** *A circle of the form*

$$C_{v_0} := \left\{(u, v) \in \mathbb{R}^2,\ u^2 + (v - v_0^{-1})^2 = R^2\right\},$$

is a solution to the ODE and called solution circle. The interior of the circle is called solution disk.

Some ODE results We use properties of the flow η to prove invariance principles of the PDE. The proofs are based on an idea of [27].

Lemma 1.3.2. *Consider a straight half line $\gamma(s) := se^{i\phi}$, $s \geq 0$, $0 < \phi < \pi$ of initial conditions. Then the curve $z(s) := \eta(t, \gamma(s))$, where η is the ODE flow of $\dot{U} = U^2$, is convex for each fixed $t > 0$.*

Proof. Recall that a plane curve $s \mapsto z(s) \in \mathbb{C}$ is convex if the curvature

$$\kappa(s) := \operatorname{Re}\left(iz'\bar{z}''\right),$$

does not change sign. Explicit calculation of $\kappa(s)$ yields

$$\kappa(s) = \frac{2t\sin(\phi)}{((st - \cos\phi)^2 + \sin(\phi)^2)^3}.$$

Since the straight line starts in the upper half plane, i.e. $0 < \phi < \pi$, we have $\sin\phi > 0$ and the curvature is always positive for $t > 0$, see Figure 1.6a. ☐

(a) Evolution of straight line in invariant region for $0 = t_0 < t_1 < t_2 < t_3$

(b) Initial condition in invariant region

Figure 1.6: Invariant region in complex domain

The straight line is transported by the ODE flow and traces a convex domain, see Figure 1.6b. A solution, which is contained in a solution disk and starting to the left of a straight line, must converge to zero. This idea is due to [27] and allows to prove the following a priori estimate on the analytic continuation.

Lemma 1.3.3. *Assume that the initial condition $u_0(x) := v_0(x) + iw_0(x)$ is contained in the solution disk of $s_0 e^{i\varphi}$, $0 < \varphi < \pi$ and $0 < \max \arg u_0(x) < \pi$, see Figure 1.6b. Then the ODE solution $\eta(t, u_0(x))$ satisfies the following inequality*

$$\|\eta(t, u_0(x))\|_\infty \leq \begin{cases} \left((\alpha/s_0 - t)^2 + \beta^2/s_0^2\right)^{-1/2} & \text{for } \alpha t < 1/s_0, \\ (\alpha\beta t)^{-1} & \text{for } \alpha t \geq 1/s_0. \end{cases}$$

with $\alpha := \cos\phi$ and $\beta := \sin\phi$.

Proof. By the explicit representation of $\eta\left(t, se^{i\varphi}\right)$, we obtain

$$\left|\eta\left(t, se^{i\varphi}\right)\right| = \frac{1}{(r\alpha - t)^2 + r^2\beta^2}.$$

where $r = s^{-1}$, $s_0^{-1} \leq r < \infty$.

The derivative with respect to r is

$$\partial_r \eta\left(t, se^{i\varphi}\right) = \frac{2(r\alpha - t)\alpha + 2r\beta^2}{\left((r\alpha - t)^2 + (r\beta)^2\right)^2} r^2.$$

Setting the derivative to zero we obtain

$$2(r\alpha - t)\alpha + 2r\beta^2 = 0 \Rightarrow r = t\alpha.$$

Thus as long as $\alpha t < 1/s_0$ we have $r = r_0$, i.e.

$$\|\eta(t, u_0(x))\|_\infty \leq \left((r_0\alpha - t)^2 + r_0^2\beta^2\right)^{-1/2},$$

and for $\alpha t > r_0$ we have

$$\left\|\eta\left(t, u_0\left(x\right)\right)\right\|_\infty \leq \frac{1}{\alpha\beta t}.$$

In particular does the solution converge to zero as expected. □

Lemma 1.3.4. *Assume, that the smooth function* $u_0\left(x\right) := v_0\left(x\right) + iw_0\left(x\right) : \left(-1,1\right) \to \mathbb{C}$*,* $u_0\left(\pm 1\right) = 0$ *has positive imaginary part and positive boundary derivative. Then an angle* $0 < \varphi < \pi$ *exists, such that* $u_0\left(x\right)$ *is to the left of the half-line* $z\left(s\right) := se^{i\varphi}$*. The angle* φ *satisfies*

$$\varphi = \min_{x \in I} \arctan\left(\frac{w_0\left(x\right)}{v_0\left(x\right)}\right).$$

Proof. On any compact subset $K \subset I$, $w_0\left(x\right)$ is uniformly bounded from below and there exists a $\delta > 0$ such that

$$\delta < \arctan\left(\frac{w_0\left(x\right)}{v_0\left(x\right)}\right) < \pi - \delta, \qquad x \in K.$$

The only problem may occur at the boundary. Expansion of x close to the boundary gives for $x = \pm 1 + y$.

$$\frac{\mp(w_0)_x\left(\pm 1\right)y + Cy^2}{\mp(v_0)_x\left(\pm 1\right)y + Cy^2} > c_1 > 0. \tag{1.14}$$

for y small enough. This implies, that φ is positive and less than π. □

Lemma 1.3.5. *Again, let the smooth function* $u_0\left(x\right) := v_0\left(x\right) + iw_0\left(x\right) : \left(-1,1\right) \to \mathbb{C}$*,* $u_0\left(\pm 1\right) = 0$ *have positive imaginary part and positive boundary derivative. Then the solution is contained in an invariant ODE disk.*

Proof. The proof is similar to the proof above. Geomtrically it is already clear, that since the initial condition is contained in a cone in the upper half-plane, we can find in invariant ODE disk.

(i) We need that the curve $\gamma\left(x\right) := \left(v_0\left(x\right), w_0\left(x\right)\right)$ is contained in a circle around iR of radius R for some $R > 0$. Thus we have to calculate

$$R^* := \sup_{x \in I} R\left(v_0\left(x\right), w_0\left(x\right)\right),$$

where $R\left(y, z\right)$ is defined as follows

$$R\left(y, z\right) := \frac{y^2 + z^2}{2z}.$$

The expression for $R\left(y, z\right)$ comes from the equation $y^2 + \left(R - z\right)^2 = R^2$ solved for R and describes the circle around iR containing the point $\left(y, z\right)$ and the origin. We need to show that R^* is bounded.

(ii) Consider any compact subset $K \subset I$. Define $\alpha := \min_K \varphi(x) > 0$. This implies

$$\sup_{x \in I} R(v_0(x), w_0(x)) = \sup_{x \in I} \frac{v_0(x)^2 + w_0(x)^2}{2v_0(x)} \le \frac{4\|u_0\|_\infty^2}{\min_{x \in K} w_0(x)} < \infty. \tag{1.15}$$

(iii) The only problem can occur at the boundary of I. Here we obtain again by expansion close to the boundary with $x = \pm 1 + y$, $y > 0$ small enough

$$R(v_0(x), w_0(x)) \le \frac{(v_0)_x^2(\pm 1)\, y^2 + (w_0)_x^2(\pm 1)\, y^2 + Cy^3}{\mp (w_0)_x(\pm 1)\, y + Cy^2} < c_1 y. \tag{1.16}$$

□

Analysis of the PDE

Denote by u_+ the unique positive, symmetric equilibrium of stationary problem

$$u_{xx}(x) + u^2(x) = 0, \qquad u(\pm 1) = 0.$$

The linearisation is hyperbolic and has simple eigenvalues and a one-dimensional analytic fast unstable manifold W^u. The graph of the unstable manifold $\Upsilon(\tau)$ is analytic, see [22].

Lemma 1.3.6. *There exists a $r > 0$ such that the positive time flow $\Phi(t, (\tau, \Upsilon(\tau)))$, $t > 0$ is bounded for $\tau \in \mathbb{C} \setminus \mathbb{R}_+$ and $|\tau| < r$.*

Proof. The proof is based on the ODE results.

(i) The L^2-normalized eigenfunction $\varphi(x)$ to the fast unstable manifold is positive [48]. It satisfies the equation

$$\varphi_{xx} + 2u_+\varphi = \mu\varphi,\ \varphi(\pm 1) = 0,\ \mu > 0.$$

for some $\mu > 0$. Thus by Hopf lemma we obtain

$$\min\{\mp\varphi_x(\pm 1)\} = \delta > 0.$$

(ii) Assume first that $\tau < 0$. Then $u_+(x) + \tau\varphi(x) + \Upsilon(\tau) \le u_+(x)$ for all $x \in I$ and $|\tau|$ small enough. This implies, that the solution converges to zero for $\tau < 0$.

(iii) Consider the initial data $u_0(\tau) := u_+ + \tau\varphi + \Upsilon(\tau)$ with $\tau \in \mathbb{C} \setminus \mathbb{R}$ and $|\tau| < r$. Since we consider solutions on the fast unstable manifold we know, that the graph Υ is of quadratic order at zero. This implies, that for small enough τ the imaginary part of $u_+ + \tau\varphi + \Upsilon(\tau)$ is positive and has positive boundary derivative.

(iv) The claim follows by Lemma 1.3.5.

□

Corollary 1.3.7. *The real time flow on the complex fast unstable manifold stays bounded in the slit disk of the fast unstable manifold.*

Remark 1.3.8. *From the perspective of Lemma 1.2.9 we have shown, that the boundaries of the two complex conjugated heteroclinic nests (see Figure 1.3) coincide. More precisely it is the positive real blow-up orbit. Furthermore note, that we also have proven, that the every heteroclinic obtained in Lemma 1.3.6 is actually a time p – path of the single real time heteroclinic orbit.*

Continuation back to the real axis For the nonlinear heat equation one can extend the heteroclinic orbit up to the real axis, but not back onto the real axis as indicated in Figure 1.3 directly after blow-up.
Naturally the question arises what happens to the analytic continuation of the real heteroclinic orbit along time p – paths. We can show, that even if we are not able to continue the solution back to the real axis immediately after the blow-up, we can continue back to the real axis after some time $T_1 > 0$.

In this paragraph we prove that the solution can be continued to the regions $S_\pm$, see Figure 1.7.

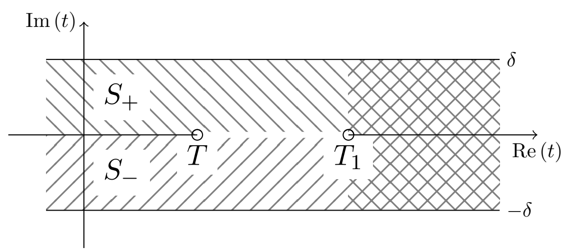


Figure 1.7: Domain of existence of continued solutions

This is already true, due to Lemma 1.2.9 since it implies, that solutions along time p-pathes are heteroclinic orbits. Once the solution is close to zero, which is completely stable, we can tilt the time path to obtain a solution on the real time axis. But in this paragraph, we derive an upper estimate for the time T_1.

The next to Lemma give a lower bound on the blow-up rate. The proofs are very similar to the real-valued case in [48].

Lemma 1.3.9. *If* $\nu_0 := \|u_0\|_\infty < r$*. Then the flow* $\Phi(t, u_0)$ *is regular along* $t = re^{i\theta}$ *for* $|\theta| < \frac{\pi}{2}$ *for at least time* $r = \frac{1}{C^2 \nu_0}$*, where* $C > 0$ *is independent of* ν_0*.*

Proof. Since the Laplace operator generates an analytic semigroup in the space of continuous functions [33], we solve the variation of constants formula

$$u(t) = T(t) u_0 + \int_0^z T(z - s) u(s)^2 \, ds.$$

Taking the sup-norms we obtain

$$\nu(t) := \|u(t)\|_\infty \leq C\nu_0 + C\int_0^r \nu^2(s)\,ds.$$

since t is contained in the sector of the left half-plane. We can compare this inequality to the solution $\eta\left(t, C^2\nu_0\right)/C$. The difference of the solutions satisfies the inequality

$$h(r) := \eta\left(r, C^2\nu_0\right)/C - q(r) \geq C\int_0^r \left(q(s) + \eta\left(s, C^2 q_0\right)/C\right) h(s)\,ds,$$

Since $h(0) > 0$ we have $h(s) > 0$ for all $0 < s < r$. By Gronwall inequality $h(s) < \infty$ as long as $q(s) + \eta\left(s, C^2\nu_0\right)/C < 2\eta\left(s, C^2\nu_0\right)/C < \infty$. But $\eta\left(s, C^2\nu_0\right)/C$ exists up to time $r = \frac{1}{C^2\nu_0}$. □

The lower blow-up rate estimate gives an estimate on the blow-up time.

Corollary 1.3.10. *Suppose that* $|t-T|^p \|u(t-T)\|_\infty \leq M < \infty$ *for some* $0 < p < 1$. *Then does the solution* $u(t,x)$ *stay regular until* T.

Proof. Argue by contradiction. By Lemma 1.3.9 and assumptions, it holds,

$$|T-t| \geq \frac{1}{C^2\|u(T-t)\|_\infty} \geq \frac{|T-t|^p}{MC^2}.$$

Since $p < 1$, this gives a contradiction for t close enough to T. □

The two Lemmata above give a lower bound on the existence time depending on the initial condition. This allows to prove, that the analytic continuations of the blow-up orbit can be continued back to the real axis once they are small enough. We show first that this is not true without the Laplace operator.

Proposition 1.3.11. *Consider the ODE flow of*

$$\begin{aligned} u_t(t,x) &= u^2(t,x), \qquad x \in I := (-1,1) \\ u(0,x) &:= u_0(x) \in C^0(I,\mathbb{R}_+),\ u_0 \not\equiv 0, \end{aligned} \tag{1.17}$$

and Dirichlet boundary conditions. Then the solution to equation (1.17) *exists for all* $t \in \mathbb{C}\setminus[T,\infty)$ *and is unbounded on* $[T,\infty)$*, where* $T := \frac{1}{\max_{x\in I} u_0(x)}$.

Proof. We can solve equation (1.17) explicitly and calculate the blow-up time for each point $x \in I$,

$$u(t,x) = \eta(t, u_0(x)) = \frac{1}{\frac{1}{u_0(x)} - t}.$$

Thus the blow-up time $T(x)$ is $T(x) = \frac{1}{u_0(x)}$.

Note, that $x \to \partial I$ implies $T(x) \to \infty$ and also $T = \min_{x\in I} t_b(x) = \frac{1}{\max_{x\in I} u_0(x)}$. Now the claim follows by the intermediate value theorem. □

Let us compare the ODE flow (1.17) with the PDE semiflow $\phi(t, u_0)$, for real initial conditions $0 \leq u_0 \neq 0$, but allowing for, both, real and complex times t. We keep Dirichlet conditions in either case. We have seen how the ODE solution $u(t,x)$ must blow up whenever t returns to the real axis after $T = \frac{1}{\max_{x\in I} u_0(x)}$, no matter which complex detour our time path t might take. Due to the presence of the Laplacian u_{xx}, however, the PDE solution $u(t,x)$ may behave quite different. We have already seen how the Laplacian generates complete blow-up in the real domain, due to infinite propagation speed outwards from the singularity. In the complex time domain, in contrast, the Laplacian prevents quadratic tangencies at the boundary, via the Hopf Lemma. Moreover the initially real spatial profile $u_0(x)$ is pushed into the complex upper half plane, for small imaginary times $t = i\delta$, provided that u_0 satisfies

$$(u_0)_{xx} + f(u_0) \geq 0.$$

Immediately afterwards, comparison with a straight line solution $\eta(t, s\exp(i\phi))$ strikes, and traps the solution for all later real times,

$$t = i\delta + \tau, \qquad \tau \geq 0.$$

We formulate this simple argument in the following theorem. The uniform boundedness, and decay, established here will allow us to even return to the real axis itself, boundedly.

Also note, the proof is completely different from Masuda [38]. He used the explicit spatial constant solution to show boundedness on the real axis again. For initial conditions u_0 that are close to a spatial constant profile, the ODE flow $\eta(t, u_0(x))$ is also uniformly bounded for $t \in \mathbb{C} \setminus \left(\frac{1}{\max_{x\in I} u_0(x)}, \frac{1}{\min_{x\in I} u_0(x)}\right)$. In particular, for spatially constant solutions $\frac{1}{\max_{x\in I} u_0(x)} = \frac{1}{\min_{x\in I} u_0(x)}$ and the solution just blows up at a single time point. Masuda showed that solutions, which are sufficiently close to a spatially constant solution the Laplacian can not desynchronize the blow-up much for different x. But he has also shown, that time analyticity is immediately destroyed by the Laplacian operator.

Theorem 1.3.12. *Let $0 \leq u_0(x) = u_+(x) + w(x)$ with $w_{xx}(x) + w^2(x) \geq 0$ and $w \not\equiv 0$. Then the flow blows up at some finite time $0 < T < \infty$ completely. However, there exists a upper and lower spall strip $S(\delta, [T, T_1])$ with $T_1 < 2C_0 \max\left\{\max_{x\in I} \frac{v_0(x)}{w_0(x)}, \|u_+\|_\infty\right\}$ to which the solution can be extended. The constant $C_0 > 0$ depends only on the heat semigroup. In particular, it can be continued after time T_1 back to the real axis.*

Proof. By the maximum principle we can assume that there exists a time $0 < t_0 < T$ such that $u_t(t_0, x) \geq 0$. By the Hopf Lemma $u_t(t_0, x)$ has a positive boundary derivative, i.e.

$$\pm u_{tx}(t_0, \pm 1) > c_0 > 0.$$

We can expand the function around $t = t_0 + \tau$ with respect to $\tau \in \mathbb{R}$, $|\tau| < \varepsilon$,

$$u(t_0 + i\tau, x) = u(t_0, x) + i\tau u_t(t_0, x) - \tau^2 g(\tau, x).$$

where $g(\tau, x)$ is bounded and differentiable with respect to x. The imaginary part of the boundary derivative is

$$\pm \mathrm{Im}\left(u_x\left(t_0 + \tau, \pm 1\right)\right) = \pm \tau u_{tx}\left(t_0, \pm 1\right) \mp \tau^2 \mathrm{Im}\left(g_x\left(\tau, \pm 1\right)\right) > c_0 \tau + M\tau^2.$$

Choosing $\varepsilon > 0$ small enough, implies that the imaginary part of the boundary derivative is positive. Thus, we can apply Lemma 1.3.5 to show that the solution exists for any positive $\tilde{t} \in \mathbb{R}_+$, i.e.

$$\left\| u\left(t_0 + i\tau + \tilde{t}, \cdot\right) \right\|_\infty \leq M < \infty.$$

From Lemma 1.3.4 we know, that the solution function $u(t_0 + i\tau)$ is to the left of a straight line of angle $\varphi > C\tau$ for $C = \min_{x \in I} \frac{w_0(x)}{2v_0(x)}$. Furthermore it is contained in a solution disk to $s_0 e^{i\varphi}$ with $s_0 < \|u(t_0 + i\tau, \cdot)\|_\infty < 2\|u_0\|_\infty$ for τ and t_0 small enough.

From Lemma 1.3.3 the solution satisfies the following bound

$$\left\| u\left(t_0 + i\tau + \tilde{t}, \cdot\right) \right\|_\infty \leq \frac{1}{\tilde{t} \cos\varphi \sin\varphi} \leq \frac{1+\varepsilon}{C\tilde{t}\tau}.$$

for $\tilde{t} > 2\|u_0\|_\infty$ and any $\varepsilon > 0$, if one chooses τ small enough.

Lemma 1.3.3 guarantees the existence of solutions, when solving along a slanted time line $t = (1-i)r/\sqrt{2}$, if the sup-norm of the initial condition is less than, i.e. if $\|u_0\|_\infty < \frac{1}{\sqrt{2}C_0 r}$. Thus, we can solve back to the real axis, boundedly, if

$$\frac{\sqrt{2}\tau}{1+\varepsilon} < C\tilde{t}\tau \Rightarrow \frac{\sqrt{2}}{1+\varepsilon} < C\tilde{t}.$$

This implies that we need to wait for time

$$T_1 = 2C_0 \max\left\{ \max_{x \in I} \frac{v_0(x)}{w_0(x)}, \|u_+\|_\infty \right\} < \infty,$$

until we can return back to the real axis. The solution exists on an upper and lower spall strip

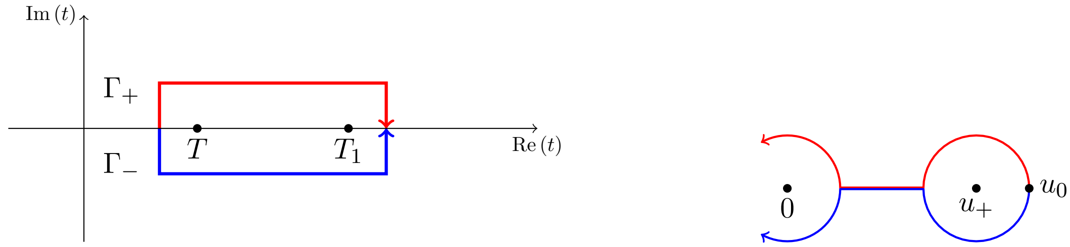
$$S_\pm\left(\delta, [T, T_1]\right).$$

□

Remark 1.3.13. *We can choose in particular $w = \varphi$, where φ is the first eigenfunction at u_+.*

This time estimate is unfortunately far from optimal. One might expect continuation to spall strip where $T = T_1$, i.e. the real blow up singularity becomes just a singleton from the complex time point of view. At the present we are not able to resolve the question.

Description of continued solutions Using the Cauchy formula we can prove that blow-up solutions on the fast unstable manifold can not coincide after blow up again.



(a) Time path (b) Path in the fast unstable manifold

Figure 1.8: Time path and solution in the local complex tangent space

The main idea is depicted in Figure 1.8. Choosing the time paths $\Gamma_{\pm}$, the path the solution $u\left(\Gamma_{\pm}, x\right)$ traces in the unstable manifold is shown on the right. The argument, why the solution does not close up again, is that the eigenvalues at zero and u_+ are not the same. Suppose, that the local flow around zero and u_+ is one-dimensional and purely linear with eigenvalues μ_0 and μ_1. To pass from u_0 to the other side of the equilibrium u_+ would then take time $t_0 = \frac{i\pi}{\mu_1}$, whereas it would take time $t_1 = \frac{i\pi}{\mu_0}$ at zero. Thus the solutions along $\Gamma_{\pm}$ can only coincide if $t_0 = t_1 + 2nt_1$ for some $n \in \mathbb{N}$.

Lemma 1.3.14. *The eigenvalue μ of the linearization at $u_+(x)$ satisfies $\frac{\pi^2}{4} < \mu$.*

Proof. We want to show that the largest eigenvalue of the operator

$$L := \partial_{xx} + 2u_+(x)$$

is not equal to the absolute value of the first eigenvalue at zero. For that reason, we follow closely the discussion on Sturm-Liouville problems in [57]. For Sturm - Liouville problems, we can express the largest eigenvalue by the Rayley quotient.

$$R(Q) := \frac{\int_{-1}^{1} -Q_x^2(x) + 2u_+(x) Q^2(x)\, dx}{\|Q\|_{L^2}^2}.$$

By [57] the eigenvalue μ satisfies the following variational principle

$$\mu := \max_{Q \in H^1} R(Q).$$

Thus, we can test with any function to obtain a lower bound of μ. Taking $Q = u_+$ we obtain

$$R(u_+) = \frac{\int_{-1}^{1} -((u_+)_x)^2(x) + 2(u_+)^3(x)\, dx}{\|u_+\|_{L^2}^2} = \frac{\int_{-1}^{1} -((u_+)_x)^2 - 2(u_+)_{xx}(x)(u_+)(x)\, dx}{\|u_+\|_{L^2}^2} = \frac{\|(u_+)_x\|_{L^2}}{\|u_+\|_{L^2}^2}$$

$\leq \mu.$

This yields the inequality

$$\|(u_+)_x\|_{L^2}^2 \leq \mu \|u_+\|_{L^2}^2 . \tag{1.18}$$

We can expand $(u_+)_x(x) := \sum_{n=0}^{\infty} a_n e_n(x)$ where $e_n(x) := \sin\left(\frac{n\pi}{2}(x+1)\right)$ is the eigenbasis at zero. By Plancherel we can rewrite inequality (1.18) as

$$\sum_{n=1}^{\infty} \frac{n^2\pi^2}{4} a_n^2 \leq \mu \sum_{n=1}^{\infty} a_n^2,$$

for

$$u_+ = \sum_{n=1}^{\infty} a_n e_n,$$

If, it holds that $u_+(x) \neq \sin\left(\frac{\pi}{2}(x+1)\right)$, also $\frac{\pi^2}{4} < \mu$ holds. As simple calculation yields

$$-\frac{\pi^2}{4}\sin\left(\frac{\pi}{2}(x+1)\right) + \sin\left(\frac{\pi}{2}(x+1)\right)^2 \neq 0,$$

for $x = 0$. □

The previous lemma allows to show that the blow-up singularity is indeed a branch point of the analytic continued solutions. We summarize the results of the chapter in the following theorem, which was already quoted in the introduction.

Theorem 1.3.15. *There exists a $\delta > 0$ such that the time analytic continuations of the real blow up orbit on the fast unstable manifold of u_+, i.e. $\Phi(t, (\tau, \Upsilon(\tau)), 0 < \tau < \delta$ exists and has the following properties:*

(i) It blows-up completely at time T.

(ii) It can be continued to upper and lower spall strips $S_\pm(\delta, [T, T_1])$ for

$$T_1 < 2C_0 \max\left\{\max_{x\in I}\frac{v_0(x)}{\varphi(x)}, \|u_+\|_\infty\right\}.$$

The constant $C_0 > 0$ depends only on the heat semigroup.

(iii) The upper and lower time path continuations do not coincide after $T_1 > 0$

Proof. The first two claims are already known or have already been shown. The idea to prove the third claim is to choose a complex time path as indicated in the Figure 1.8 and to show that the solution after continuation along path Γ_+ and Γ_- does not coincide.

Since the fast unstable manifold is analytic, we have can consider the reduced equation

$$\dot{q} = \mu q + \tilde{f}(q), \qquad q(0) = q_0, \tag{1.19}$$

where $\tilde{f}(u) := P_{su}u(q, \Upsilon(q))$ is an analytic function vanishing of quadratic order. Since $\tilde{f}$ respects the real axis, real q_0 implies that the solution $q(t)$ is real for real time t. Take now $q_0 > 0$. The solution to the nonlinear heat equation with initial condition

$$u_0 = u_+ + q_0\varphi + \Upsilon(q_0).$$

blows-up.

The time needed to pass from the right side of u_+ to the left side of u_+ (see Figure 1.8) is $t = \frac{i\pi}{\mu}$.

This is due to the Cauchy formula by separation of variables of the reduced equation (1.19). By Cauchy residue theorem, the total time needed to go around the stationary solution u_+ is

$$2t = \oint \frac{1}{\mu q + \tilde{f}(q)} dq = \frac{2\pi i}{\mu}.$$

The factor two is due to conjugation symmetry. Furthermore the heteroclinic orbit converges to the first eigenfunction at zero, since the heteroclinic orbit does not change sign. This implies, that the heteroclinic converges to zero on the slow stable manifold and is tangent to the first eigenfunction.

Similarly as in the fast unstable manifold we prove that the time needed to pass from right to left around u_- in the first eigenfuction is close to

$$t = \frac{4i}{\pi^2}.$$

But from Lemma 1.3.14 we know that $\frac{\pi^2}{4} < \mu$ which implies, that the solution is not real. In particular the solution is not real when going first for time $\frac{i\pi}{\mu}$ and then along the real heteroclinic orbit for time $t > 0$, such that the reduced equation on the slow stable manifold holds and then for time $\frac{-i\pi}{\mu}$. □

Remark 1.3.16. *Alternatively we could also argue by analytic linearization [3]. The reduced one-dimensional analytic differential equation on the fast unstable resp. slow stable manifold can be linearized analytically and we have the pure flow of the linear part. This gives the same result as the Cauchy formula.*

Remark 1.3.17. *The proof indicates that the Riemann surface introduced by the analytic continuations of the blow-up orbit is induced by the quotient of the eigenvalues μ and $\tilde{\mu}$ at the connected equilibria of the heteroclinic orbit. If the quotient is rational, then is the Riemann surface compact otherwise non-compact. Suppose the quotient satisfies $\frac{\tilde{\mu}}{\mu} = \frac{m}{n}$ for some $n \in \mathbb{N}$ and $m \in \mathbb{Z} < 0$. This implies that going for time $i\frac{n}{\mu}$ into the complex plane and then back for time $-i\frac{m}{\tilde{\mu}}$ gives a closed orbit. Note, that the time path encircles many singularities and we can just make a statement about the net branching.*

This allows to study how the branch point of the blow-up orbit changes depending on a parameter as for example, in the following equation

$$u_t = u_{xx} + \lambda u + u^2, \qquad \lambda \in \mathbb{R},\ x \in (-1,1)\,,\ u\,(\pm 1) = 0.$$

Upper blow-up rate estimate We prove a geometric characterization of blow-up points: If there is blow-up at $t = 0$ and $x = 0$, the image of the function $\mathbb{D} \mapsto u\,(t+i,0)$ must cover a half-plane in the complex plane, see Figure 1.9. In particular, the image can not be contained in any sector of opening less than π. This surprising result follows from analytic functions theory. Since we already now, that the solution on the fast unstable manifold is contained in the upper half-plane by Lemma 1.3.6, we also have an upper estimate on the blow-up rate for complex time. Note, that in contrast to lower estimates upper blow-up rate estimates are more difficult to obtain, and is not clear how to transfer the proofs e.g. in [48] from the real to the complex case. As already mentioned in the introduction, upper blow-up rate estimates allow for rescaled coordinates. In rescaled coordinates, the blow-up point becomes an equilibrium.

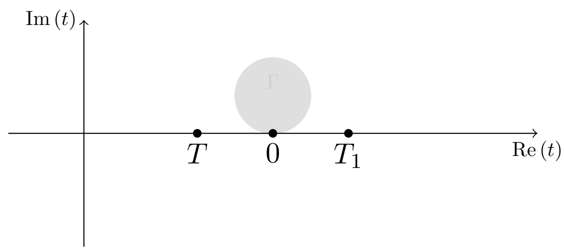


Figure 1.9: Complex time disk attached to the blow-up point

Lemma 1.3.18. *Consider the solution constructed in Theorem 1.3.12. Then it holds* $|u\,(t,x)| \leq \frac{M}{Im(t)}$ *for some* $M > 0$ *independent of* x*. The constant* $M > 0$ *is related to the height* δ *of the time strip in which the solution exists.*

Proof. The image of $u\,(S_+ \setminus \mathbb{R}, I)$ is contained in the upper half-plane of the complex plane by Lemma 1.3.6.. The point evaluation $\delta_x : C^0\,(I) \to \mathbb{C}$

$$\delta_x\,(u) := u\,(x)\,,$$

is a bounded linear functional. The function $u^x\,(t) := \delta_x\,(u\,(t))$ is a holomorphic function. Take now a disk in the upper half-plane tangent to t^* of radius $\delta/2$. First assume $\delta = 2$. Then the function $u^x\,(t)$ is a holomorphic function from the unit disk to $\mathbb{C}$. The proof follows from the classical subordination [10] principle as follows

(i) Set $z = t - t^* - i$ and define the function

$$\tilde{u}^x\,(z) := u^x\,(t^* + i + z) - u^x\,(t^* + i)\,.$$

The function $\tilde{u}^x$ is a holomorphic function on the unit disk at zero with $\tilde{u}^x(0) = 0$. Furthermore, the image $\tilde{u}^x(\mathbb{D})$ is contained in a shifted upper half-plane.

(ii) Next we look for a Möbius transformation $\Gamma(z) := \frac{z+b}{cz+d}$, which satisfies

$$\Gamma(0) = 0, \qquad \Gamma(\infty) = 1, \qquad \Gamma(i\alpha) = -1.$$

for some $\alpha \in \mathbb{R}$. The first two conditions yield $b = 0$ and $c = 1$. The last condition implies $d = -2i\alpha$, thus

$$\Gamma(z) = \frac{z}{z - 2i\alpha}.$$

The line $t \mapsto i\alpha + t$, $t \in \mathbb{R}$ is mapped to the unit circle since

$$|\Gamma(t + i\alpha)| = \left| \frac{t + i\alpha}{t - i\alpha} \right| = 1, \qquad t \in \mathbb{R}.$$

Now, we choose α such that $\tilde{u}^x(\mathbb{D}) \subset \Gamma^{-1}(\mathbb{D})$. The inverse transformation is given by

$$\Gamma^{-1}(z) = \frac{-2i\alpha z}{1 - z}.$$

(iii) Consider the function $\omega(z) = \Gamma \circ \tilde{u}^x$. The function is a holomorphic self-map of the unit disk and satisfies $\omega(0) = 0$. Thus by Schwarz lemma $|\omega(z)| \leq |z|$. This implies $\tilde{u}^x(z) = \Gamma^{-1}(\omega(z))$ and thus

$$\sup_{|z| \leq r} |\tilde{u}^x(z)| \leq \sup_{|z| \leq r} |\Gamma_\alpha(\omega(z))| \leq \sup_{|z| \leq r} |\Gamma_\alpha(z)| \leq \frac{2\alpha}{1 - r},$$

for any $r < 1$.

(iv) Rescaling time, we can always rescale a disk of radius $\delta/2$ to the unit disk. This gives the estimate

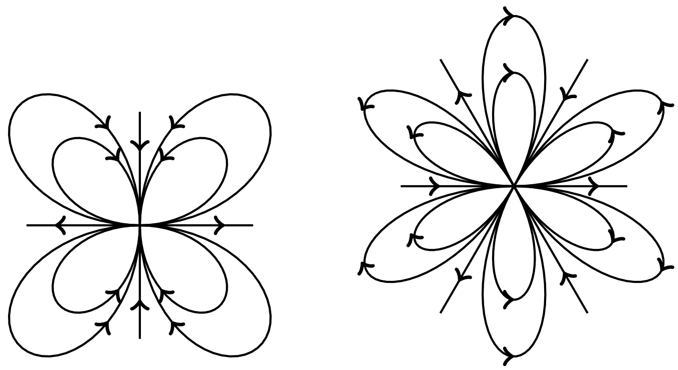
$$\sup_{|z| \leq r} |\tilde{u}^x(z)| \leq \frac{2\alpha}{1 - 2r\delta^{-1}}.$$

for $r < \delta/2$.

□

Remark 1.3.19. *This implies that the nontagential limit of $u^x(t)(t - t^*)$ to t^* exists.*

Remark 1.3.20. *The result is quite astonishing, since we do not just get an a priori estimate on the blow-up rate very easily, but we also get a geometric condition on the range of the image of points close to blow-up from the lower blow-up rate estimate. Suppose, that there exists an interval $\tilde{I} \subset I$ and a disk Ω in the upper complex plane touching the real axis such, that im $\left(u\left(\Omega, \tilde{I}\right)\right)$ is contained in a sector of the upper half-plane, then the solution is bounded on $\tilde{I}$ up to the real axis. Then we get subordination of the solution by some $p < 1$ of the Cayley transform, which contradicts blow-up by Corollary 1.3.10.*



(a) Solution for $p = 3$ (b) Solution for $p = 4$

Figure 1.10: Real time ODE invariant solutions for different $\dot{x} = x^p$

Remark 1.3.21. *For $p > 1$ all of the proofs in this Chapter also hold. The main reason is, that also in the case $p > 1$ there exists invariant regions of the ODE flow for positive solutions, see Figure 1.10. All the arguments of section 1.3 work analogously.*

1.4 Outlook

In the first part of this chapter we have shown that the global attractor introduces unbounded solutions in the complex domain. In particular in the presence of heteroclinic orbits, there are also blow-up solutions.
In the second part, we studied the concrete example of the quadratic nonlinear heat equation. Here, we were able to derive finer results. For example there exists an analytic continuation of the blow-up orbit to a complex time strip cut out a finite time interval. Furthermore we showed that the analytic continuation along different paths around the blow-up point does not close.

Quite recently a paper [9] with numerical simulations of the quadratic nonlinear heat equation with periodic boundary conditions in complex time appeared. The numerical simulations suggest, similar to our rigorous proven results in the Dirichlet case, that complex time continuation after the blow-up introduces a Riemann surface and in particular continuations along different paths around the singularity do not close up. Even though the analysis is purely numerical it indicates how continued solutions on the Riemann surfaces look like and give a starting point of a more detailed study of continued solutions.

In the next chapter, we pursue the approach from the first part. We try to deepen the connection between blow-up solutions and the eternal core. In particular, we will show that a one-dimensional center manifold can already be enough to prove the existence of finite time blow-up. In contrast to the heteroclinic orbit, which is a non-local object, the center manifold allows to conclude blow-up by local analysis around a single equilibrium without considering any global properties of the flow.

Chapter 2

Continuation of center manifolds to complex sectors

Abstract

In this chapter, we analyze one-dimensional center manifolds of analytic reaction-diffusion equation

$$u_t = Au + f(u), \qquad f(0) = 0, \qquad Df(0) = 0,$$

with sectorial operator A. We show that if the unstable spectrum of A is empty and the equilibrium is weakly unstable of quadratic order, there exists an analytic continuation of the real center manifold into complex sectors.

2.1 Introduction

The results of the previous chapter crucially depended on the analyticity of stable and unstable manifolds. By analyticity we were able to describe the flow on the unstable manifold by an analytic ordinary differential equation (ODE). By the ODE we proved the existence of blow-up solutions by local analysis close to an equilibrium.

Center manifolds however need not to be analytic even for analytic systems, e.g. [58], [33].

In this chapter we show that it is still possible to analytically continue the graph of one-dimensional center manifolds of PDEs to complex sectors. This is already known for ODEs [25], but to the best of our knowledge not for PDEs. We will use the continued center manifold in the next chapter to analyze the connection between local center manifolds and blow-up solutions.

Consider the following abstract evolution equation

$$u_t = Au + f(u), \qquad f(0) = 0, \qquad Df(0) = 0. \tag{2.1}$$

The standard center manifold theory, see Appendix 4.1, guarantees the existence of a center manifold under certain assumptions on A and f. One important assumption is that A is a sectorial operator, such that the spectrum on the imaginary axis is finite and the rest of the spectrum is bounded away from the axis. The spectral splitting allows for projections, which decompose the space $X = X_0 \oplus X_h$ into the center space resp. hyperbolic part of A.
The center manifold is an invariant manifold that is close to the equilibrium a graph $\psi : X_0 \to X_h$. It contains in particular all solutions, which stay small enough for all positive and negative real times.
We want to study whether we can continue the graph of the center-manifold to complex values. Here one needs to clarify what extension means. We can, on the one hand, try to continue the center manifold as geometric object, i.e. extend the graph function ψ to a larger, complexed valued domain by analytic continuation. On the other hand we can also try to extend the center manifold by its dynamical properties, i.e. to characterize the set of small complex-valued initial conditions, such that solutions exist for all real times after suitable cut-off of the nonlinearity.
It is not clear a priori that both approaches will give the same extensions of the real center manifold. We will address this question in the next chapter, where we will prove, that among the non-unique real center manifolds there exists an unique center manifold, which contains small complex-valued solutions and is also analytic. Thus the two extensions coincide. Furthermore, this special center manifold is linked to blow-up solution.
In this chapter we will take the first approach. As already mentioned, the graph is not necessarily an analytic function, which manifests for example in divergent power series of the power series expansion ψ. Or the power series expansion can be zero when the graph contains exponentially flat terms, that can not be represented by power series. From a technical point of view the failure of analyticity stems from the multiplication of the nonlinearity f with a cut-off function in the proof of the existence of center manifolds. It is clear that cut-off functions can not be analytic. That is why we use the invariance equation of the center manifold as in [25] to obtain an analytic extension. Plugging $u(t) = x(t) + \psi(x(t))$ into equation (3.1) gives the following invariance equation for ψ

$$\psi_x(x)\, P_0 f(x, \psi(x)) = A_h \psi(x) + P_h f(x, \psi(x)). \tag{2.2}$$

The linear operator $A_h = P_h A$ is still a sectorial operator which is why we attempt to solve the invariance equation with “time” x in sectorial regions of the complex plane. Again, this has the advantage that we do not have to consider any cut-off function at all. But the disadvantage is that it is not clear whether solutions on the continued graph share the dynamical properties of solutions on center manifolds, e.g. that they contain all globally bounded solutions of the system after cut-off.

Throughout the chapter we use the following definition.

Definition 2.1.1 (Sector)**.** *We denote by $S_{r,\alpha}$ sectorial subsets of the complex plane, that is*

$$S_{r,\alpha} := \left\{ se^{i\varphi} \in \mathbb{C},\ 0 < s < r,\ |\varphi| < \alpha \right\}.$$

Furthermore, the unbounded sector $S_{\infty,\alpha}$ is abbreviated by S_α.

2.2 A simple example

In this section we consider a very simple example, which already conveys the main idea. Consider the following system of ordinary differential equations.

$$\begin{aligned} \dot{x} &= x^2, \qquad x \in \mathbb{R}, \\ \dot{y} &= -y + x^2, \qquad y \in \mathbb{R}. \end{aligned} \tag{2.3}$$

The equation has a one-dimensional center manifold at zero with center direction x. The standard center manifold theory guarantees the existence of an invariant center manifold over x. This implies that we may look for solutions of the form $y = \psi(x)$. Plugging this ansatz for y into the system (2.3) gives the following invariance equation

$$\psi_x x^2 = -\psi + x^2. \tag{2.4}$$

Usually one tries to expand ψ as power series $\psi(x) := \sum_{n=2}^{\infty} \psi_n x^n$. The power series ansatz in equation (2.4) yields the following recursion of the coefficients ψ_n.

$$-n\psi_{n-1} = \psi_n, \ n \geq 3, \qquad \psi_2 = -1.$$

The recursion can be solved explicitly and gives $\psi_n = (-1)^{n+1}(n-1)!$. The power series for ψ does not converge for any small x.

Throughout this work, we will take a different approach to equation (2.4). Since we assume, that the center space is one-dimensional, we can view the center direction x as new "time" of equation (2.4) and ask for which "times" there is a solution of equation (2.4). Setting $z = -\frac{1}{x}$ transforms equation (2.4) to

$$\psi_z = -\psi + z^{-2}. \tag{2.5}$$

Equation (2.5) is a non autonomous linear differential equation, which we attempt to solve by variation of constants

$$\psi(z) = e^{-(z-z_0)}\psi_0 + \int_{z_0}^{z} e^{-(z-s)} s^{-2} ds.$$

Since we look for solutions with $\psi(z_0) \to 0$ for $z_0 \to -\infty$ we have

$$\psi(z) = \int_{-\infty}^{z} e^{s-z} s^{-2} ds = \int_0^{\infty} e^{-\tau} (z-\tau)^{-2} d\tau. \tag{2.6}$$

The integral converges as long as $z < 0$. We could also look for analytic continuation of ψ to complex z. For $z \notin \mathbb{R}_+$ the integral in equation (2.6) is well-defined. This immediately implies that we can analytically continue the integral to $\mathbb{C} \setminus \{\mathbb{R}_+\}$ or, in the original coordinates, we can continue $\psi(x)$ to $\mathbb{C} \setminus \{\mathbb{R}_-\}$.

Furthermore, we recover the alternating factorial sum of the power series expansion by partial integration of equation (2.6)

$$\psi(z) = -z^{-2} - 2\int_0^\infty e^{-\tau}(z-\tau)^{-3}\,d\tau = -z^{-2} + 2z^{-3} + 6\int_0^\infty e^{-\tau}(z-\tau)^{-4}\,d\tau.$$

Repeated partial integration yields the alternating factorial sum. Note, that since the error term is multiplied by a factorial it does not become small.

Remark 2.2.1. *For the infinite dimensional case sectorial regions become important. Note, that the change from x to z maps a sector at zero to a sector at negative infinity, that is*

$$x = re^{i\phi} \Leftrightarrow z = -r^{-1}e^{-i\phi}.$$

This simple observation is why we can obtain similar results also in the infinite dimensional case.

2.3 Extension of center manifolds to complex sectors

We restrict to the case in which the unstable spectrum of A is empty.

2.3.1 Setting

In this section we study the following setting.

$$u_t = Au + f(u). \tag{2.7}$$

Let the assumptions for the center manifold theorem hold, see Appendix 4.1 with $f(u) \in C^\omega(V, Y)$ and $Z = D(A)$ and $Y = D(A^\alpha)$ for some $0 < \alpha < 1$. $V \subset Z$ is a neighborhood of zero. Furthermore, let the following conditions hold

$$\sigma(A) \subset \mathbb{R}_- \cup \{0\}, \tag{2.8}$$

$$\dim\ker(A) = 1, \tag{2.9}$$

$$f(\mathbb{R}) \subset \mathbb{R}. \tag{2.10}$$

Note, that the first condition can actually be replaced by the usual sectorial assumption, i. e. that the spectrum is contained in a sector in the complex plane. The sector to which we can extend the semigroup associated to A determines the sector to which we can extend the graph of the center manifold. The third condition is natural, if the differential equation leaves the real axis invariant. The actual limitation is the second condition.

By Theorem 4.1.6 we obtain a local center-manifold of the real equation. The center manifold theorem guarantees the existence of solutions of the form

$$u(t) = x(t) + \psi(x(t)),$$

where $\psi : C^k(X_0, Z_h)$ and $x(t) \in X_0$.

This implies that we can rewrite the system (2.7) for solutions as mentioned in the beginning of this chapter

$$\dot{x} = P_0 f(x, \psi(x)), \tag{2.11}$$
$$\psi_x(x) P_0 f(x, \psi(x)) = A_h \psi(x) + P_h f(x, \psi(x)). \tag{2.12}$$

The operator A_h and its corresponding spaces are well-defined by Appendix.

For simplicity we make the following non-genericity assumption

$$P_0 f_{xx}(0,0) = 1, \tag{2.13}$$

such that the projected equation vector field starts quadratically in x. The quadratic term is solely determined by the vector field itself and does not depend on the graph ψ.

Remark 2.3.1. *Note, that we can rescale time, such that whenever $P_0 f_{xx}(0,0) > 0$ equation (2.13) holds.*

2.3.2 Generation of evolutions in complex time domains

In this preliminary section, we show that the differential equation

$$\psi_x = \gamma(x) A_h \psi, \tag{2.14}$$

generates an evolution that can be extended to sectors for functions γ that are also analytic on sectors in the complex plane. Since we assumed that the spectrum of A_h is on the real line, we can actually extend the evolution to the right half-plane.

The basic idea of this chapter, that a change of time $z_x = \gamma(x)$ transforms equation (2.14) into standard linear form. Instead of following that approach, we take explicit care of the time change. For real time an evolution satisfies the following properties [33].

Definition 2.3.2 (Evolution)**.** *A family of operators $G(t,s) \subset L(X)$ is said to be an evolution operator for problem* (2.14) *if*

(i) $G(t,s)G(s,r) = G(t,r)$, $G(s,s) = Id$, $0 \le r \le s \le t \le T$,

(ii) $G(t,s) \in L(X,D)$, $0 \le s < t \le T$,

(iii) $t \to G(t,s)$ *is differentiable in* $(s,T]$ *with values in* $L(X)$ *and*

$$\partial_t G(t,s) = \gamma(t) AG(t,s)$$

We extend the definition of evolutions analogously to semigroups to sectorial regions.

Lemma 2.3.3. *Assume that there exists* $\theta < \frac{\pi}{2}$*, such that* $\Gamma(t,s) := \int_s^t \gamma(\tau)\,d\tau$ *satisfies* $\Gamma(t,s) \in S_\theta$ *for all* $t - s \in S_\theta$*.*

Define the following function

$$G(t,s) := T(\Gamma(t,s)). \tag{2.15}$$

Then $G(t,s)$ *is an evolution of the differential equation* (2.14)*.*

Proof. Note, that the function $G(t,s)$ is well-defined by definition since $\Gamma(t,s)$ is always contained in the domain of definition of the semigroup $T(z)$.

(i) By definition $T(0) = \mathrm{Id}$ and thus $G(s,s) = 0$.

(ii) Furthermore, due to additivity of the integral we have the semigroup property of $T(z)$

$$G(t,s)\,G(s,r) = T(\Gamma(t,s) + \Gamma(s,r)) = T(\Gamma(t,r)) = G(t,r).$$

(iii) Obviously $G(t,s) \in L(X, D(A))$ holds for $t - s \in S_\alpha$ by the properties of the analytic semigroup.

(iv) And also

$$\partial_t G(t,s) = \partial_t T(\Gamma(t,s)) = A\gamma(t)\,T(\Gamma(t,s)) = \gamma(t)\,A\Gamma(t,s).$$

□

We will only consider very particular functions $\gamma(x) := \frac{1}{x^2+h(x)}$ with $h(x) \in O\left(|x|^3\right)$. The essential idea is that the change of coordinates $z = \frac{-1}{x}$ maps sectorial regions into each other, see Figure 2.1.

Note, that the left sector is bounded whereas the region on the right is unbounded and contained in the left half-plane. The analytic semigroup generated by A_h can be extended to the whole left half-plane.

Lemma 2.3.4. *Assume, that* $h(x) = \frac{1}{x^2+\gamma(x)}$ *with* $\gamma(x) \in O\left(|x|^3\right)$*. Then the integral*

$$\Gamma(x,s) := \int_s^x h(s)\,ds,$$

can be approximated by

$$\Gamma(x,s) = -x^{-1} + s^{-1} + O\left(\log\left(\frac{|x|}{|s|}\right)\right),$$

for small enough $|s|, |x|$*. Furthermore, for each* $0 < \phi < \frac{pi}{2}$ *there exists a constant* $r > 0$*, such that the real part of* $\Gamma(x,s)$ *positive if* $Re(s) < Re(x)$*,* $s, x \in S_{\theta,r}$*.*

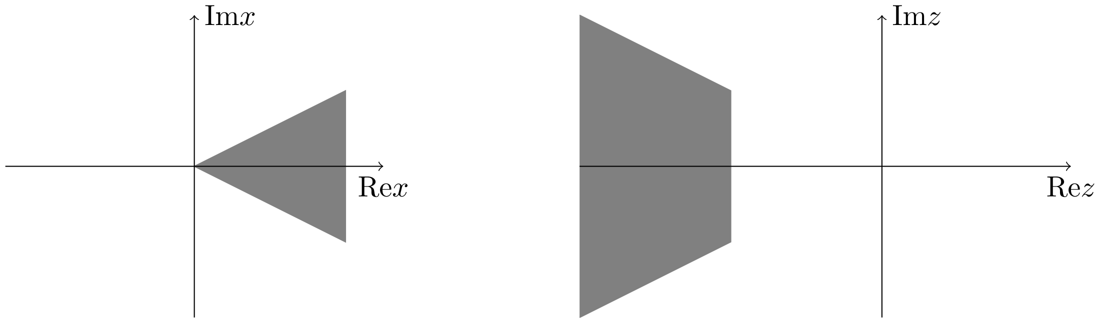


Figure 2.1: Time regions

Proof. For small enough $|x|$ and $|s|$ we can expand $\gamma(x)$ as geometric series, which yields the following integral:

$$\int_{x_0}^{x} h(s)\,ds = \int_{s}^{x} \frac{1}{s^2}\sum_{n=0}^{\infty}\left(\frac{-\gamma(s)}{s^2}\right)^n ds. \tag{2.16}$$

Since $\gamma(x) \in O\left(|x|^3\right)$, we know that the n-term is bounded by $C^n s^n$. This implies

$$\Gamma(x,s) = -x^{-1} + s^{-1} + O\left(\log\left(\frac{|x|}{|s|}\right)\right).$$

We set $x = re^{i\phi}$ and $x_0 = r_0 e^{i\phi}$ for $|\phi| < \frac{\pi}{2}$ to obtain

$$\int_{s}^{x} h(s)\,ds = -\frac{e^{-i\phi}}{r} + \frac{e^{-i\phi}}{s} + O\left(\log\left(\frac{r}{s}\right)\right).$$

This implies that the real part of $\Gamma(x,x_0)$ is

$$\operatorname{Re}\left(\Gamma(x,x_0)\right) \geq \cos(\phi)\left[\frac{1}{s} - \frac{1}{r}\right] - C\log\left(\frac{r}{s}\right).$$

Setting $\frac{1}{s} = \frac{\tau}{r}$ for some $\tau > 1$, we obtain by concavity of the logarithm, that the following inequality holds for small enough $r > 0$,

$$\frac{\cos(\phi)}{r}\left[\tau - 1\right] - C\log\tau > \frac{\cos(\phi)}{r} - C > 0,$$

Note, that the estimate is uniform in sectors where $\cos(\phi)$ is bounded from below. □

Corollary 2.3.5. *For every $0 < \phi < \frac{\pi}{2}$ exists an $r > 0$ such, that evolution $G(x,s)$ is well-defined on $S_{\phi,r}$. In particular for $x, s \in S_{\phi,r}$ with $Re(s) < Re(x)$ the operator $G(x,s) : X \to Z$ is bounded.*

The next lemma shows, that the evolution $G(x,s)$ inherits regularization properties from the semigroup $T(z)$.

Lemma 2.3.6. *For $x - s \in S_\theta$ the evolution $G(x,s)$ satisfies the following properties* *(i)* $\lim_{s\in S_\theta\to 0} A^n G(x,s) = 0.$

(ii) $\|A^\alpha G(t,s)\| \leq M\left(\frac{1}{\frac{-1}{x}+\frac{1}{s}}\right)^\alpha \exp\left(-\beta\left(\frac{-1}{x}+\frac{1}{s}\right)\right).$

for some $\beta > 0$ *for all* s, x, n.

Proof. From equation (2.15) we have for some $\beta > 0$,

$$\lim_{s\in S_\theta\to 0} \|A^n G(t,s)\|_{L(X,X)} = \lim_{s\in S_\theta\to 0} C\left|-\frac{1}{t}+\frac{1}{s}\right|^n \exp\left(-\beta\left(-\frac{1}{t}+\frac{1}{s}\right)\right) = 0.$$

Secondly we have,

$$\|A^\alpha G(t,s)\| \leq \frac{C}{|\Gamma(t,s)|^\alpha}\exp\left(\beta\left(\frac{1}{t}-\frac{1}{s}\right)\right).$$

But by Lemma 2.3.4 we obtain that $|\Gamma(t,s)| \geq \frac{C}{\frac{1}{t}+\frac{1}{s}}$ for small enough t and s. □

Corollary 2.3.7. *Consider the variational formulation*

$$u(t,s) = G(t,s)\,u(s) + \int_s^t G(t,\sigma)\,f(s)\,ds, \tag{2.17}$$

where f *is complex-differentiable with values in* $D(A^\alpha)$ *for some* $0 < \alpha < 1$. *Then* $u(t,s)$ *is complex differentiable with values in* $D(A)$.

2.3.3 Sector in the positive real plane

We want to solve the invariance equation

$$\psi_x(x) = \frac{A_h}{P_0 f(x,\psi(x))}\psi(x) + \frac{P_h f(x,\psi(x))}{P_0 f(x,\psi(x))} = \gamma(x,s,\psi)\,A_h\psi + g(s,\psi(s)).$$

Assume that the operator $\gamma(x,s,\psi)\,A_h$ generates the evolution $G(x,s,\psi)$. Then the variation of constants formula is

$$\psi(x) = \int_0^x G(x,s,\gamma)\,g(s,\psi(s))\,ds.$$

The difficulty is that both, the evolution $G(x,s,\gamma)$ and the function $g(s,\psi(s))$ depend on ψ. We split the dependence into three steps.

(i) Show that the following differential equation yields a well-defined evolution

$$\psi_x = (x^2+\gamma(x))^{-1}A_h\psi,$$

where γ is an analytic function from a sector to complex plane $\mathbb{C}$ that decays at least like $|z|^3$ close to zero.

(ii) Show that the equation

$$\psi_x(x)(x^2+\gamma(x)) = A_h\psi(x) + P_h f(x,\psi(x)), \tag{2.18}$$

has a solution $\psi(\gamma)$ that depends continuously on γ.

(iii) Show, that the map $F : \tilde{\psi} \to \psi\left(P_0 f\left(x,\tilde{\psi}\right) - x^2\right)$ has a fixed point ψ. The function ψ is then a solution of the equation (2.2) in a sector in the complex plane.

Lemma 2.3.8. *Consider the evolution generated by*

$$\psi_x = (x^2+\gamma(x))^{-1}A_h\psi,$$

for $x \in S_{\theta,r_0}$ *with* $0<\theta<\frac{\pi}{2}$.
Define the set

$$U := \left\{\gamma \in C^\omega\left(S_{\theta,r_0},\mathbb{C}\right)\cap C^0\left(\bar{S}_{\theta,r_0},\mathbb{C}\right),\ \|\gamma\|_U \le M\right\}, \qquad \|\gamma\|_U = \sup_{x\in S_{\theta,r_0}}\left|\frac{\gamma(x)}{|x|^3}\right|,$$

for some fixed constant $M>0$ *and* $r_0>0$ *to be chosen during the proof. Then it holds*

$$\lim_{\tilde{\gamma}\to\gamma}\sup_{x\in S_{\theta,r_0}} x^{-2}\int_0^x \|G(x,s,\tilde{\gamma}) - G(x,s,\gamma)\|_{L(Y_h,Z_h)}\,ds = 0.$$

Proof. Throughout the proof we suppress constants that do not depend on δ, s and x into a generic constant C. Using Lemma 2.3.3 we obtain

$$\|G(x,s,\gamma) - G(x,s,\tilde{\gamma})\|_{L(Y_h,Z_h)} = \left\|T(\Gamma(x,s)) - T\left(\tilde{\Gamma}(x,s)\right)\right\|_{L(Y_h,Z_h)}. \tag{2.19}$$

But since T is generated by the sectorial operator A_h we can represent the semigroup $T(z)$ by the Dunford integral, see for example [33]

$$T(\Gamma(x,s)) = \frac{1}{2\pi i}\int_\omega e^{\lambda\Gamma(x,s)}R(\lambda,A_h)\,d\lambda.$$

where ω is a the boundary curve of a sector in the negative real half-plane containing $\sigma(A_h)$. Due to the Dunford integral representation we can calculate the difference (2.19) as follows

$$G(x,s,\gamma) - G(x,s,\tilde{\gamma}) = C\int_\omega\left(e^{\lambda\Gamma(x,s)} - e^{\lambda\tilde{\Gamma}(x,s)}\right)R(\lambda,A_h)\,d\lambda = C(I_1+I_2). \tag{2.20}$$

Writing down the path integral in (2.20) we obtain

$$I_1 = \int_0^\infty e^{i\phi}\left(e^{\lambda e^{i\phi}\Gamma(x,s)} - e^{\lambda e^{i\phi}\tilde{\Gamma}(x,s)}\right)R\left(\lambda e^{i\phi},A_h\right)d\lambda,$$
$$I_2 = \int_0^\infty e^{-i\phi}\left(e^{\lambda e^{-i\theta}\Gamma(x,s)} - e^{\lambda e^{-i\phi}\tilde{\Gamma}(x,s)}\right)R\left(\lambda e^{-i\phi},A_h\right)d\lambda.$$

We prove everything for I_1 since the result follows completely analogously for I_2. Pulling out $e^{\lambda e^{-i\phi}\Gamma(x,s)}$ gives

$$I_1 = \int_0^\infty e^{i\phi} e^{\lambda e^{i\phi}\Gamma(x,s)} \left(1 - e^{\lambda e^{i\phi}\left(\tilde{\Gamma}(x,s)-\Gamma(x,s)\right)}\right) R\left(\lambda e^{i\phi}, A_h\right) d\lambda. \tag{2.21}$$

We get the following inequality

$$\begin{aligned}\left|\Gamma(x,s) - \tilde{\Gamma}(x,s)\right| &= \int_s^x \left|\frac{\gamma(s) - \tilde{\gamma}(s)}{\left(s^2 + \gamma(s)\right)\left(s^2 + \tilde{\gamma}(s)\right)}\right| ds \\ &\leq 2\left\|\tilde{\gamma} - \gamma\right\|_U \log\frac{|x|}{|s|}.\end{aligned}$$

Furthermore we have

$$\left|e^{\lambda e^{i\phi}\Gamma(x,s)}\right| \leq \left|\exp\left[\lambda\cos(\phi - \theta)\left(\frac{1}{\operatorname{Re}(x)} - \frac{1}{\operatorname{Re}(s)}\right) + C_2\lambda\left(\log\left(\frac{\operatorname{Re}(x)}{\operatorname{Re}(s)}\right)\right)\right]\right|.$$

But since the spectrum is on the real axis, we can make ϕ such large, that $\frac{\pi}{2} < \phi - \theta < \frac{3\pi}{2}$. This implies $\cos(\phi - \theta) < 0$. Now choosing $\operatorname{Re}(x)$ small enough we get for some $\tilde{x} \geq \frac{1}{C_3 \operatorname{Re}(x)}$ for some $C_3 > 0$

$$\left|e^{\lambda e^{i\phi}\Gamma(x,s)}\right| \leq \exp\left(\lambda\tilde{x}\left(1 - \frac{1}{\tau}\right)\right),$$

where $s = x\tau$ for some $1 < \tau$.

Furthermore, from [33] we know that $\left\|R(\lambda, A_h)\right\|_{L(Y_h,X_h)}$ decays for large $|\lambda|$,

$$(1+\lambda)^\alpha \left\|R\left(\lambda e^{i\phi}, A_h\right)\right\|_{L(Y_h,X_h)} \leq M < \infty.$$

Thus changing $\tau x = s$ in the integral 2.21 yields

$$\begin{aligned}&\int_0^x \left\|\int_0^\infty e^{i\phi} e^{\lambda e^{i\phi}\Gamma(x,s)}\left(1 - e^{\lambda e^{i\phi}\left(\tilde{\Gamma}(x,s)-\Gamma(x,s)\right)}\right) R\left(\lambda e^{i\phi}, A_h\right) d\lambda\right\| ds \\ &\leq Cx\int_0^1\int_0^\infty \exp\left(\lambda\tilde{x}\left(1 - 1/\tau + C\log(1/\tau)\right)\right)\left|1 - \exp\left(\lambda\log(1/\tau)\,\delta\right)\right|(1+\lambda)^{-\alpha}\, d\lambda d\tau.\end{aligned}$$

Set $z = \frac{1}{\tau}$ which implies $-z^{-2}dz = d\tau$ and thus with $\tilde{x} = Cx^{-1}$, the integral becomes,

$$\begin{aligned}&x\int_0^1\int_0^\infty \exp\left(\lambda\left(\tilde{x}\left(1 - 1/s\right) + C\log(1/\tau)\right)\right)\left|1 - \exp\left(\lambda\log(1/\tau)\,\delta\right)\right|(1+\lambda)^{-\alpha}\, d\lambda dz \\ &\leq Cx\int_1^\infty\int_0^\infty \exp\left(\lambda\left(\tilde{x}(1-z) + C\log(z)\right)\right)\left|1 - \exp\left(\lambda\log(z)\,\delta\right)\right| z^{-2}(1+\lambda)^{-\alpha}\, d\lambda dz.\end{aligned}$$

After shifting and reversing z, we obtain

$$
\begin{aligned}
&x\int_1^\infty\int_0^\infty \exp\left(\lambda\left(\tilde{x}\left(1-z\right)+C\log\left(z\right)\right)\right)\left|1-\exp\left(\lambda\log\left(z\right)\delta\right)\right|z^{-2}\left(1+\lambda\right)^{-\alpha}d\lambda dz\\
&=x\int_0^\infty\int_0^\infty \exp\left(\lambda\left(-\tilde{x}z+C\log\left(1+z\right)\right)\right)\left|1-\exp\left(\lambda\log\left(1+z\right)\delta\right)\right|\left(z+1\right)^{-2}\left(1+\lambda\right)^{-\alpha}d\lambda dz
\end{aligned}
$$

Using the concavity of logarithm we get $\log(1+z)\le z$. This implies for $\tilde{x}:=\tilde{x}-C$ the following bound,

$$
\begin{aligned}
&x\int_0^\infty\int_0^\infty \exp\left(\lambda\left(-\tilde{x}z+C\log\left(1+z\right)\right)\right)\left|1-\exp\left(\lambda\log\left(1+z\right)\delta\right)\right|\left(z+1\right)^{-2}\left(1+\lambda\right)^{-\alpha}d\lambda dz\\
&\le Cx\int_0^\infty\int_0^\infty \exp\left(-\lambda\tilde{x}z\right)\left|1-\exp\left(\lambda\log\left(1+z\right)\delta\right)\right|\left(z+1\right)^{-2}\left(1+\lambda\right)^{-\alpha}d\lambda dz.
\end{aligned}
$$

Our goal is to show, that the integral converges to zero with $\delta\to 0$ uniformly in x^2.

Suppose we have shown, that the integral is uniformly bounded after dividing by x^2, i.e.

$$
x^{-1}\int_0^\infty\int_0^\infty \exp\left(-\lambda\tilde{x}z\right)\left|1-\exp\left(\lambda\log\left(1+z\right)\delta\right)\right|\left(z+1\right)^{-2}\left(1+\lambda\right)^{-\alpha}d\lambda dz<C_0<\infty,\qquad \forall x\in S_{\theta,r}.
$$

Then we can argue by contradiction. Assume, that exists an $\varepsilon>0$ and sequences $\gamma_n,\tilde{\gamma}_n\in U$ with $\|\tilde{\gamma}_n-\gamma_n\|_U=\delta_n\le\frac{1}{n}$, $x_n\in S_{\theta,r}$ such, that the following holds

$$
\begin{aligned}
\varepsilon&\le x_n^{-2}\int_0^x\left\|G\left(x_n,s,\gamma\right)-G\left(x_n,s,\tilde{\gamma}\right)\right\|_{L(Y_h,Z_h)}ds\\
&\le\frac{C}{x_n}\int_0^\infty\int_0^\infty \exp\left(-\lambda z x_n^{-1}\right)\left|1-\exp\left(\lambda\log\left(1+z\right)\delta_n\right)\right|\left(z+1\right)^{-2}\left(1+\lambda\right)^{-\alpha}d\lambda dz.
\end{aligned}
$$

Define the following sequence of functions

$$
f_n\left(\lambda,z\right):=x_n^{-1}\exp\left(-\lambda z x_n^{-1}\right)\left|1-\exp\left(\lambda\log\left(1+z\right)\delta_n\right)\right|\left(z+1\right)^{-2}\left(1+\lambda\right)^{-\alpha}.
$$

Consider any fixed $\lambda>0$ and $z>0$. Then the pointwise limit of $f_n(\lambda,z)$ is zero, i.e.

$$
\lim_{n\to\infty}f_n\left(\lambda,z\right)\le\limsup_{n\to\infty}\exp\left(-\lambda z x_n^{-1}\right)x_n^{-1}\limsup_{n\to\infty}\left[\exp\left(\lambda\log\left(1+z\right)\delta_n\right)-1\right]=0.
$$

This implies by dominated convergence theorem

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{1}{x_n}\int_0^\infty\int_0^\infty \exp\left(-\lambda z x_n^{-1}\right)\left|1-\exp\left(\lambda\log\left(1+z\right)\delta_n\right)\right|\left(z+1\right)^{-2}\left(1+\lambda\right)^{-\alpha}d\lambda dz\\
&=\lim_{n\to\infty}\int_0^\infty\int_0^\infty f_n\left(\lambda,z\right)d\lambda dz\\
&=0.
\end{aligned}
$$

This is a contradiction.

Now we prove that the integral

$$x^{-1}\int_0^\infty\int_0^\infty \exp\left(-\lambda\tilde{x}z\right)\left|1-\exp\left(\lambda\log\left(1+z\right)\delta\right)\right|\left(z+1\right)^{-2}\left(1+\lambda\right)^{-\alpha}d\lambda dz,$$

is uniformly bounded for all $0<|x|<r$.

We bound the integral as before by getting rid of the logarithm (and modification of the constant $\tilde{x}$),

$$\begin{aligned}&x^{-1}\int_0^\infty\int_0^\infty \exp\left(-\lambda\tilde{x}z\right)\left|1-\exp\left(\lambda\log\left(1+z\right)\delta\right)\right|\left(z+1\right)^{-2}\left(1+\lambda\right)^{-\alpha}d\lambda dz\\&\leq Cx^{-1}\int_0^\infty\int_0^\infty \exp\left(-\lambda\tilde{x}z\right)\left(z+1\right)^{-2}\left(1+\lambda\right)^{-\alpha}d\lambda dz.\end{aligned}$$

We change to polar coordinates $z=r\cos\left(\varphi\right)$, $\lambda=r\sin\left(\varphi\right)$ to obtain

$$\begin{aligned}&x^{-1}\int_0^\infty\int_0^\infty \exp\left(-\lambda\tilde{x}z\right)\left(z+1\right)^{-2}\left(1+\lambda\right)^{-\alpha}d\lambda dz\\&=x^{-1}\int_0^\infty dr\int_0^{\pi/2}\exp\left(-r^2\sin\varphi\cos\varphi\tilde{x}\right)r\left(r\cos\varphi+1\right)^{-2}\left(1+r\sin\varphi\right)^{-\alpha}drd\varphi.\end{aligned}$$

For each r we now rewrite the inner integral as the sum of three integrals. First consider the integral from 0 to $\varphi_0<<1$. That implies by setting $\tilde{x}:=\frac{\tilde{x}}{2}$

$$\begin{aligned}&\int_0^{\varphi_0}\exp\left(-r^2\sin\varphi\cos\varphi\tilde{x}\right)r\left(z+1\right)^{-2}\left(1+\lambda\right)^{-\alpha}d\varphi\\&\leq\int_0^{\varphi_0}\exp\left(-r^2\varphi\tilde{x}\right)r\left(r+1\right)^{-2}d\varphi=\frac{1-\exp\left(-r^2\tilde{x}\varphi_0\right)}{r\tilde{x}\left(1+r\right)^{-\alpha}}.\end{aligned}$$

Thus, the following estimate holds

$$\int_0^\infty\int_0^{\varphi_0} r\exp\left(-r^2\varphi\tilde{x}\right)\left(r/2+1\right)^{-\alpha}=\int_0^\infty\frac{1-\exp\left(-r^2\tilde{x}\varepsilon\right)}{r\tilde{x}\left(1+r\right)^{-\alpha}}\leq\frac{C}{\tilde{x}}.$$

A similar estimate holds for $\pi/2-\varphi_0<\varphi<\pi/2$.

Furthermore we have

$$\begin{aligned}&\int_0^\infty\int_{\varphi_0}^{\pi/2-\varphi_0}\exp\left(-r^2\sin\varphi\cos\varphi\tilde{x}\right)r\left(1+r\sin\varphi\right)^{-\alpha}\left(r\cos\varphi+1\right)^{-2}d\varphi dr\\&\leq\int_0^\infty dr\int_\varepsilon^{\pi/2-\varepsilon}\exp\left(-r^2\tilde{x}\varphi_0\right)r\left(1+r\sin\varphi\right)^{-\alpha}\left(r\cos\varphi+1\right)^{-2}d\varphi dr\\&\leq\frac{C}{\tilde{x}}.\end{aligned}$$

□

Lemma 2.3.9. *There exists a sector $S_{\theta,r}$ for some $r > 0, \theta > 0$ to be chosen during the proof, there exists a unique analytic function $\psi : S_{\theta,r} \to Z_h$ with*

$$\|\psi(x)\|_{Z_h} \leq \nu |x|^2 \text{ for } x \in S_{\theta,r},$$

which solves equation (2.18).

Proof. Consider the sets

$$M := \{\psi \in C^\omega(S_{\theta,r}, Z_h),\ \|\psi(x)\|_M \leq \nu\}, \qquad \|\psi(x)\|_M := \sup_{x \in S_{\theta,r_0}} \left\| \frac{\psi(x)}{x^2} \right\|_{Z_h},$$

$$U := \left\{\gamma \in C^\omega(S_{\theta,r_0}, \mathbb{C}),\ \|\gamma\|_U \in \tilde{M}\right\}, \qquad \|\gamma(x)\|_U := \sup_{x \in S_{\theta,r_0}} \left| \frac{\gamma(x)}{x^3} \right|.$$

By Lemma 2.3.4 the evolution of $G(x,s,\gamma)$ is analytic for $x - s \in S_{\theta,r_0}$. For functions in the set M and $\gamma \in U$ consider the mapping $F : M \times U \to M$:

$$F(\psi,\gamma)(x) := \int_0^x G(x,s,\gamma)\, g(s,\psi(s))\, ds, \tag{2.22}$$

with $g(s,\psi(s)) := \frac{P_h(x,\psi(x))}{(x^2+\gamma(x))}$. We want to show that the mapping $F(\psi,\gamma)(x) : M \to M$ is continuous and a uniform contraction in ψ for appropriate $r, \nu > 0$. Then we can apply contraction mapping principle with parameters to conclude.

(i) The set M is complete.

$$\|\psi_n - \psi_m\|_\infty \leq \|\psi_n - \psi_m\|_M < \varepsilon.$$

This implies that ψ_n converges to an analytic function ψ. Furthermore holds $\|\psi\|_M \leq \nu$.

(ii) The function $F(\psi,\gamma)(x)$ is analytic with values in Z_h by Lemma 2.3.7 for all $\theta < \pi/2$.

(iii) There exists a $r > 0$ such that for all $x \in S_{\theta,r}$ and $\psi_1, \psi_2 \in M_C$ we get

$$\|F(\psi_1,\gamma)(x) - F(\psi_2,\gamma)(x)\|_M \leq \sup_{x \in S_{\theta,r_0}} |x|^{-2} \int_0^x \|G(x,s,\gamma)\|_{L(Y_h,Z_h)} \|g(s,\psi_1(s)) - g(s,\psi_2(s))\|_{Y_h}\, ds.$$

By Lemma 2.3.6 we have the following estimate

$$\|G(x,s,\gamma)\|_{L(Y_h,Z_h)} \leq C \left(\frac{1}{\frac{-1}{\mathrm{Re}(x)} + \frac{1}{\mathrm{Re}(s)}} \right)^{\alpha-1} \exp\left(-\beta \left(\frac{-1}{\mathrm{Re}(x)} + \frac{1}{\mathrm{Re}(s)} \right)\right).$$

Furthermore, we have

$$\|g(s,\psi_2(s)) - g(s,\psi_1(s))\|_{Y_h} \leq C_2 \left(|s| + |s|^2\right) \|\psi_2 - \psi_1\|_M.$$

To obtain a contraction in M we prove that the following integral becomes arbitrarily small

$$\int_0^x \left| \left(\frac{-1}{\mathrm{Re}\,(x)} + \frac{1}{\mathrm{Re}\,(s)} \right)^{\alpha-1} |s|^{-1} \exp\left(-\delta \left(\frac{-1}{\mathrm{Re}\,(x)} + \frac{1}{\mathrm{Re}\,(s)} \right) \right) \right| ds.$$

for small enough x. Set $x = re^{i\phi}$ and $s = \tau e^{i\phi}$ to obtain

$$\begin{aligned} &\int_0^r \left| \left(\frac{-1}{\mathrm{Re}\,(r)} + \frac{1}{\mathrm{Re}\,(s)} \right)^{\alpha-1} \tau^{-1} \exp\left(-\beta \left(\frac{-1}{\mathrm{Re}\,(r)} + \frac{1}{\mathrm{Re}\,(s)} \right) \right) \right| d\tau \\ &\leq C \int_0^r \left(\frac{-1}{r} + \frac{1}{s} \right)^{\alpha-1} \tau^{-1} \exp\left(-\tilde{\beta} \left(\frac{-1}{r} + \frac{1}{s} \right) \right) d\tau, \end{aligned}$$

with $0 < \tilde{\beta} < \beta \cos\phi$. Furthermore, $\cos(\phi)$ is uniformly bounded from below. We set $z := \frac{-1}{s} + \frac{1}{r}$, which changes the integral to

$$\begin{aligned} &\int_0^r \left(\frac{-1}{r} + \frac{1}{s} \right)^{1-\alpha} s^{-1} \exp\left(-\tilde{\beta} \left(\frac{-1}{r} + \frac{1}{s} \right) \right) ds \\ &\leq \int_0^\infty z^{\alpha-1} \exp\left(-\tilde{\beta} z \right) \left| r^{-1} - z \right|^{-1} d\tau < \infty. \end{aligned}$$

Choosing r_0 small enough yields by dominated convergence theorem

$$\| F(\psi_1, \gamma)(x) - F(\psi_2, \gamma)(x) \|_{Z_h} \leq q \| \psi_2 - \psi_1 \|_M ,$$

for some $q < 1$ indepenent of γ.

(iv) To show that F is a self-map, we need to establish the bound $\|\psi(x)\|_M \leq \eta |x|^2$ for $x \in S_{\theta,r}$. By definition of g we have

$$\begin{aligned} &\lim_{x \to 0} |x|^{-2} \int_0^x \| G(x,s,\gamma) \|_{L(Y_h, Z_h)} \| g(s, \psi(s)) \|_{Y_h} \\ &\leq |x|^{-2} \int_0^z \| G(x,s,\gamma) \|_{L(Y_h, Z_h)} \| g_{xx}(0) + q(s) \|_{Y_h} \end{aligned}$$

with $q(s) \in O(|s|)$. The existence of an appropriate constant C follows by the previous calculation. Doing essentially the same as above we obtain the estimate

$$\begin{aligned} &\int_0^x \left| \left(\frac{-1}{\mathrm{Re}\,(x)} + \frac{1}{\mathrm{Re}\,(s)} \right)^{\alpha-1} |x|^{-2} \exp\left(-\delta \left(\frac{-1}{\mathrm{Re}\,(x)} + \frac{1}{\mathrm{Re}\,(s)} \right) \right) \right| ds \\ &\leq \int_0^\infty z^{\alpha-1} \exp\left(-\tilde{\beta} z \right) r^{-2} \left| r^{-1} - z \right|^{-2} d\tau \leq C \int_0^\infty z^{\alpha-1} \exp\left(-\tilde{\beta} z \right) =: \tilde{\eta}. < \infty \end{aligned}$$

Set now $\eta := \tilde{\eta} \|g_{xx}\|$ to obtain the claim.

(v) If $F : M \times U \to M$ is continuous the contraction mapping principle would imply the continuous dependence of the fixed point on γ. To prove continuity we use Lemma 2.3.8. We have to show that the following holds,

$$\lim_{(\tilde{\psi},\tilde{\gamma})\to(\psi,\gamma)} \left\| F\left(\tilde{\psi},\tilde{\gamma}\right) - F(\psi,\gamma)\right\|_M = 0.$$

From equation (2.22), we obtain

$$F\left(\tilde{\psi},\tilde{\gamma}\right)(x) - F(\psi,\gamma)(x) = \int_0^x G(x,s,\tilde{\gamma})\, g\left(s,\tilde{\psi}(s)\right) - G(x,s,\gamma)\, g(s,\psi(s))\, ds.$$

This implies

$$\begin{aligned}\left\| F\left(\tilde{\psi},\tilde{\gamma}\right) - F(\psi,\gamma)\right\|_C \leq & \sup_{x\in S_{\theta,r}} |x|^{-2} \int_0^x \left\| \left(G(x,s,\tilde{\gamma}) - G(x,s,\gamma)\right) g\left(s,\tilde{\psi}(s)\right)\right\| ds \\ & + \sup_{x\in S_{\theta,r}} |x|^{-2} \int_0^x \left\| G(x,s,\gamma)\left(g\left(s,\tilde{\psi}(s)\right) - g(s,\psi(s))\right)\right\| ds.\end{aligned}$$

We have already shown that the second term converges to zero for $\tilde{\psi} \to \psi$ and the first term converges to zero by Lemma 2.3.8.

(vi) Applying now the contraction mapping principle depending on a parameter we obtain that there exists for each $\gamma \in U$ a unique fixed point $\psi(\gamma)$ of equation (2.18) fixing γ and this fixed point depends continuously on γ.

□

The function γ still depends on ψ. But since we have shown continuous dependence on γ of the fixed point ψ, we can use an additional fixed point argument to prove the existence of a solution to the full equation. Even though we will employ Schauder's fixed point theorem in the last step, uniqueness is not a problem, since the real center-unstable manifold is unique and has to coincide with ψ on the real axis. Since ψ is analytic a the sector around the real axis, we know that ψ is the unique analytic continuation of the real center-unstable manifold. For analytic continuation in Banach spaces see Section 4.2.1.

Theorem 2.3.10. *The function $G : M \mapsto C^\omega\left(S_{\theta,r_0}, X_h\right)$ defined by*

$$G\left(\tilde{\psi}\right) := \psi\left(\gamma\left(\tilde{\psi}\right)\right), \qquad \gamma\left(\tilde{\psi}\right) := P_0 f\left(x,\tilde{\psi}(x)\right) - x^2,$$

has a fixed point for small enough r_0 and $0 < \theta < \frac{\pi}{2}$.

Proof. By Theorem 4.2.5 the set M is compact, convex subset of $C^{\omega}\left(S_{\theta,\pi}, X_h\right)$. G is continuous, since

$$\begin{aligned}\left\|\gamma\left(\tilde{\psi}_1\right)-\gamma\left(\tilde{\psi}_2\right)\right\|_U &\leq C \sup_{x\in S_{\theta,r}} |x|^{-3}\left(|x|\left\|\tilde{\psi}_1(x)-\tilde{\psi}_2(x)\right\|_Z+\left\|\tilde{\psi}_1(x)-\tilde{\psi}_2(x)\right\|_Z^2\right)\\ &\leq C\left\|\tilde{\psi}_1-\tilde{\psi}_2\right\|_U.\end{aligned}$$

By Lemma 2.3.9 is G a self map of M.

Schauder's fixed point theorem gives the existence of a fixed point $\tilde{\psi}\in M$ which satisfies

$$\tilde{\psi}(x)=\psi\left(\gamma\left(\tilde{\psi}\right)\right)(x)=\int_0^x G\left(x,s,P_0 f\left(x,\tilde{\psi}(x)\right)-x^2\right)g\left(s,\tilde{\psi}(s)\right)ds.$$

□

This is the variational formulation of solutions to equation (2.2).

Remark 2.3.11. *A similar arguments holds for any center manifold for which the lowest order term of the center manifold dynamics is obtained by the direct projection onto the center-manifold and not induced by the graph function* ψ.

Remark 2.3.12. *Similarly, one can extend graphs of center-unstable manifolds to complex sectors.*

2.4 Complexification of the real center manifold

In the previous section, we have proved the existence of an analytic continuation of the graph ψ of the center-unstable manifold to complex x to sectors of opening less than $\frac{\pi}{2}$.
In this section, we want to analyze complex valued center manifolds constructed by dynamical properties. In particular, we want to construct center manifolds for complex values that are in sectors with larger opening angle.

$$u_t=\left(Au+f(u)\right). \tag{2.23}$$

The results of this section would also follow using the standard center manifold theory for equation (2.23) and cut-offs in the complex plane, which are smooth as functions from $\mathbb{R}^2$ in . However in this section we take a different approach, which is closer to real-valued partial differential equations.

Equation (2.23) can be written as a real two-dimensional system for $u=v+iw$,

$$\begin{pmatrix} v_r\\ w_r\end{pmatrix}=\begin{pmatrix} Av+h(v,w)\\ Aw+g(v,w)\end{pmatrix}, \tag{2.24}$$

with $h(v,w) := \mathrm{Re}(f(v+iw))$ and $g(v,w) := \mathrm{Im}(f(v+iw))$.

We can apply the usual real center manifold theorem to the system (3.6). The center manifold theorem guarantees the existence of a neighborhood $U \subset \mathbb{R} \times \mathbb{R}$ and a (smooth) graph $\psi_\theta : U \to Z_h \times Z_h$ such that all small solutions have to be contained in the graph of ψ_θ. If we consider $(x_1, x_2) \in U$ and set $x = x_1 + ix_2$, it is clear that the graph ψ_θ does not just exist in sectors of opening less then $\frac{\pi}{2}$, but surprisingly in a full neighbourhood of the origin. Moreover, it is also not clear, whether it is an analytic function of the complex variable x. We can not use the invariance equation anymore to obtain analyticity. Nevertheless, small solutions are time analytic since they do not feel the cut-off. We use the center manifold ψ to prove via the reduced dynamics of solutions on the center manifold of (3.6) that solutions on the center manifold stay small for all real r. Then the solution must be time analytic. So time analyticity follows a posteriori and not because the center manifold ψ is analytic from the very beginning.

2.4.1 The real two dimensional center manifold

In this subsection we consider the equation $u = v + iw$ on the space $X \times X$.

The real two dimensional system We consider the following system of equations (3.6),

$$\begin{pmatrix} v_t \\ w_t \end{pmatrix} = \begin{pmatrix} Av \\ Aw \end{pmatrix} + \begin{pmatrix} g(v,w) \\ h(v,w) \end{pmatrix}. \tag{2.25}$$

Note, that by definition of the complexification of A we have

$$A_0 \begin{pmatrix} v \\ w \end{pmatrix} := \begin{pmatrix} Av \\ Aw \end{pmatrix} = A(v+iw).$$

We consider the differential equation (2.25) in the space $\tilde{Z} := Z \times Z$, $\tilde{Y} := Y \times Y$ and $\tilde{X} := X \times X$. We prove that the operator A_θ is a sectorial operator if A is a sectorial operator and $|\theta| \leq \delta$ for some $\delta > 0$.

We prove that the rotated operator is still a sectorial operator on $X \times X$.

Lemma 2.4.1. *The operator A_0 is a sectorial operator, if A is sectorial.*

Proof. By definition the operator A_0 is written in components $A_0(v,w) = (Av, Aw)$.

Consider the equation eigenvalue equation of A_θ,

$$(\lambda - A_0)(v,w) = (\tilde{v}, \tilde{w}).$$

which has a unique solution as long as $\lambda \notin \sigma(A)$. This implies that for $\theta < \frac{\pi}{2}$ the operator A_0 is sectorial. Also the resolvent estimates hold

$$\|(v,w)\|_{X\times X} \leq C \left\| \left((\lambda - A)^{-1} \tilde{v}, (\lambda - A)^{-1} \tilde{w} \right) \right\|$$
$$\leq \|\lambda - A\|^{-1} \left(\|\tilde{v}\|_X + \|\tilde{w}\|_X \right).$$

□

It is also true that the eigenspaces are direct products of the eigenspaces of the real-valued operator.

Corollary 2.4.2. *Suppose that* $\left\{u_n^k, k \in 1, ...k_n\right\}$ *is a eigenbasis to the eigenspace of eigenvalue* λ_n *and operator* A*. Then the eigenspace to* λ_n *and operator* A_0 *has the following basis*

$$\begin{pmatrix} u_n^k \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ u_n^k \end{pmatrix} \qquad k \in 1, ...k_n.$$

We can prove a further corollary for simple eigenvalues.

Corollary 2.4.3. *Suppose, that there is a simple eigenvalue* $\lambda = 0$ *of the real equation* $Au = 0$*. Then the kernel of* A_0 *is two-dimensional and spanned to the direct product of the kernel functions.*

Since the operator A_0 is closed it satisfies the assumption of the center manifold theorem in $X \times X$.

We denote the center manifold of equation (2.25) by $\psi(x_1, x_2) := (\psi^1(x_1, x_2), \psi^2(x_1, x_2))$.

Corollary 2.4.4. *Assume that* (v, w) *solves equation* (2.25) *for small enough* v, w*. Then*

$$u(t) := v(t) + iw(t),$$

solves equation (2.23)*.*

Taylor expansion of the center manifold Instead of writing equation (2.23) as explicitly as system for real and imaginary part, we could have considered the center manifold ψ as a map from $\mathbb{R}^2$ to Z_h from the very beginning. The cut-off function is not complex differentiable, but smooth as a function of $\mathbb{R}^2$. The standard center would then guarantee the existence of a smooth center manifold depending in $\mathbb{R}^2$ with values in the complex Banach space Z_h, see Appendix.

In the next Lemma, we show that the Taylor expansion of the real two-dimensional center manifold is induced by the Taylor expansion of the real one-dimensional center manifold.

Lemma 2.4.5. *The Taylor coefficients of the center manifolds* $\psi(x_1, x_2)$ *are induced by the real one-dimensional center manifold* $\Upsilon(x)$*, i.e. for any finite* $N \in \mathbb{N}$ *we have*

$$T_N(\psi(x_1, x_2)) = T_N(\Upsilon(x_1 + ix_2))$$

where T_N *is the Taylor expansion up to order* N*.*

Proof. For any $N \in \mathbb{N}$ consider the Taylor expansion of the real center manifold $\Upsilon(x) := \sum_{n=2}^{N} a_n x^n$, where $x \in \mathbb{R}$. We know that the Taylor series solves the truncated polynomial equation

$$T_N \left(D\Upsilon(x) P_0 f(x, \Upsilon(x)) \right) = T_N A \Upsilon(x) + T_N P_h f(x, \Upsilon(x)), \tag{2.26}$$

and is unique.

We need to show that the following equation,

$$\begin{aligned} &T_N \left(D\psi P_0 \left(g(x_1, x_2, \psi(x_1, x_2)), h(x_1, x_2, \psi(x_1, x_2)) \right) \right) \\ &= T_N \left(A_0 \psi(x_1, x_2) \right) + T_N \left(P_h \left(g(x_1, x_2, \psi(x_1, x_2)), h(x_1, x_2, \psi(x_1, x_2)) \right) \right), \end{aligned} \tag{2.27}$$

is solved by Υ. Then we could conclude by uniqueness of the power series expansion. Uniqueness is here implied by the equation (2.27) in the space of power series which start with second order, [58].

On the left hand side of the equation we obtain,

$$D\psi = \begin{pmatrix} \partial_{x_1} \psi_1(x_1, x_2) & \partial_{x_2} \psi_1(x_1, x_2) \\ \partial_{x_1} \psi_2(x_1, x_2) & \partial_{x_2} \psi_2(x_1, x_2) \end{pmatrix} \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$$

with $(f_1(x_1, x_2), f_2(x_1, x_2)) = P_0 \left(g(x_1, x_2, \psi(x_1, x_2)), h(x_1, x_2, \psi(x_1, x_2)) \right)$.

The Taylor expansion $\Upsilon(x)$ to any finite order is by definition analytic and unique by the center manifold theorem.

Since we can consider not just $x \in \mathbb{R}$, but also $x \in \mathbb{C}$ equation (2.26) does hold for complex values of x. Thus also the following equation holds

$$T_N \left(D\Upsilon(x) P_0 f(x, \Upsilon(x)) \right) = T_N A \Upsilon(x) + T_N P_h f(x, \Upsilon(x)), \qquad x \in \mathbb{C}.$$

Rewriting this equation as system of real and imaginary part we obtain exactly the system (2.27). This follows from the Cauchy-Riemann equations.

Set $\Upsilon(x_1 + ix_2) := \Upsilon^1(x_1 + ix_2) + i\Upsilon^2(x_1 + ix_2)$,

$$\begin{aligned} &D_x \Upsilon(x) P_0 f(x, \Upsilon(x)) = \left(\partial_{x_1} \Upsilon^1(x_1 + ix_2) + i\partial_{x_1} \Upsilon^2(x_1 + ix_2) \right) \left(f_1(x_1, x_2) + if_2(x_1, x_2) \right) \\ &= \partial_{x_1} \Upsilon^1(x_1 + ix_2) f_1(x_1, x_2) - \partial_{x_1} \Upsilon^2(x_1 + ix_2) f_2(x_1, x_2) \\ &+ i\partial_{x_1} \Upsilon^1(x_1 + ix_2) f_2(x_1, x_2) + i\partial_{x_1} \Upsilon^2(x_1 + ix_2) f_1(x_1, x_2). \end{aligned}$$

Since we are only interested in truncations, the Cauchy - Riemann equations hold and we have to any finite order

$$\begin{aligned} &D_x \Upsilon(x) P_0 f(x, \Upsilon(x)) \\ &= \partial_{x_1} \Upsilon^1(x_1 + ix_2) f_1(x_1, x_2) + \partial_{x_2} \Upsilon^1(x_1 + ix_2) f_2(x_1, x_2) \\ &+ i\partial_{x_2} \Upsilon^2(x_1 + ix_2) f_2(x_1, x_2) + i\partial_{x_1} \Upsilon^2(x_1 + ix_2) f_1(x_1, x_2). \end{aligned}$$

But this can be written as matrix vector multiplication of the form

$$\begin{pmatrix} \partial_{x_1}\Upsilon^1(x_1,x_2) & \partial_{x_2}\Upsilon^1(x_1,x_2) \\ \partial_{x_1}\Upsilon^2(x_1,x_2) & \partial_{x_2}\Upsilon^2(x_1,x_2) \end{pmatrix} \begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix},$$

which gives the left hand side of invariance equation (2.27).

By definition of A_0 is the linear part

$$T_N A\Upsilon(x) = T_N A_0 \left(\Upsilon^1(x_1+ix_2), \Upsilon^2(x_1+ix_2)\right).$$

which is the linear part of equation (2.27).

Also since f is analytic, it can be represented as power series, which implies, that the power series of the real and imaginary parts, g, h are the real and imaginary parts of power series of f.

This implies that the complex polynomial also solves the truncated invariance equation of the real system and the claim follows by uniqueness of the Taylor expansions of the center manifold. □

Chapter 3

Complex center manifold and blow-up

Abstract

In this chapter we consider solution of the equation

$$u_t = Au + f(u), \qquad f(0), \qquad Df(0) = 0,$$

with equilibrium $u = 0$ *and sectorial operator* A. *We show that if* f *is analytic and* A *generates an analytic semigroup whose spectrum has a single eigenvalue zero, then there exists an orbit that has finite real time blow-up.*

3.1 Introduction

In this chapter, we study the following abstract evolution equation

$$u_t = Au + f(u), \qquad f(0) = 0, \; Df(0) = 0, \tag{3.1}$$

with analytic nonlinearity f. The equation (3.1) can also be a system of differential equations. The solution flow is denoted by $\Phi(t, u_0)$. One should think for example of equations of the following type

Example 3.1.1.

$$\begin{aligned}
&u_t = \Delta u + \lambda u^p + u^2, \qquad \lambda \in \mathbb{R},\\
&u_t = \Delta u + e^u + \lambda u, \qquad \lambda \in \mathbb{R},\\
&u_t = \Delta u + f(u), \qquad u \in \mathbb{R}^N,\\
&u_t - \Delta u + (u \cdot \nabla) u + \nabla p = f(x), \; \nabla \cdot u = 0, \qquad x \in \mathbb{R}^3, u \in \mathbb{R}^3.
\end{aligned}$$

The goal of this chapter is to show that the two opposed worlds of blow-up and bounded solutions are actually closely related. More precisely we show the following two theorems.

Theorem 3.1.2. *Assume, that the assumptions of Section 3.3 hold. In particular, assume, that the origin is an weakly unstable equilibrium with one-dimensional center manifold and empty unstable spectrum. Then there exists $\delta > 0$ and a real initial condition u_0, such that the solution $\Phi(t, u_0)$ must have finite time blow-up in the half strip*

$$S_\delta = \{t \in \mathbb{C},\ Re(z) > 0,\ |Im(z)| < \delta\}.$$

The constant δ does only depend on the local expansion of the center manifold around the equilibrium at the origin.

The next theorem is essentially a reformulation of the first theorem.

Theorem 3.1.3. *Under the conditions of Theorem 3.1.2 there exists for any $\delta > 0$ an initial condition u_0 with $|Im(u_0)| \leq \delta$ such that the solution $\Phi(t, u_0)$ (3.1) has blow-up at a real time $0 < T(u_0) \in \mathbb{R}_+ < \infty$.*

The theorems connect stationary solutions and the local analysis of small solutions to its counterpart blow-up. Even though this results seems to be very surprising at first sight, at the heart of the argument is the ordinary Liouville's theorem:

Theorem 3.1.4 (Liouville's theorem)**.** *An analytic function can not stay uniformly bounded on the whole complex plane.*

The idea is to use the reduced flow close on the center manifold to construct a time analytic orbit, which would contradict Liouville's theorem if it was analytic (but not necessarily uniformly bounded) on the whole complex plane.

The analysis has a similar flavour to the proof of blow-up by heteroclinic orbits in Chapter 1. The main argument was that a real homoclinic orbit induces a foliation of the complex unstable manifold with heteroclinic orbits and that this foliation is actually due to a single complex time orbit. On the boundary of the heteroclinic foliation, there must be a blow-up orbit.
In this chapter, we do not study heteroclinic connections, but homoclinic orbits. We will prove that there exists a foliation with homoclinic orbits of the analytic continuation of one-dimensional center manifolds. On the boundary of the homoclinic foliation, there again must exist a blow-up homoclinic orbit.

The main tool to prove the result is center manifold theory. The dynamics of solutions on center manifolds is close to the equilibrium governed by an ODE. In our case small solutions, which live in an infinite dimensional space, follow a one-dimensional dynamics.

From the standard center manifold theory of ODEs [58], [35] and PDEs [26], [21], [22] we know that there exists a center manifold for the real equation under suitable conditions on equation (3.1).

Unfortunately, as addressed in Chapter 2 the center manifold is not necessarily analytic even if the system is analytic. Detailed analysis of analytic continuations of center manifolds for certain ODEs have been addressed in literature, e.g [25], but to the best of our knowledge not for PDEs. The solution to the problem is not to use the algebraic properties of the center manifold (e.g. Taylor expansions, invariance equation) but to take also into account the dynamical properties. It is the manifold that contains, after suitable cut-off, all bounded solutions. Therefore, we will also consider the equation (3.1) as a real system of equations by writing real and imaginary parts separately, see Chapter 2. In this situation the standard center manifold theory guarantees the existence of a center manifold of real dimension two with corresponding graph function Υ. The center manifold is not unique. We show that there are nests of small homoclinic orbits that foliate the regions $H_{\pm}$ of the center manifold, see Figure 3.4. This implies that the center manifold is unique in the regions $H_{\pm}$. Furthermore, we show, that the center manifold is even analytic in $H_{\pm}$, i.e. that the function Υ is actually an analytic function Ψ of one complex variable. We can study the analytic continuations of Ψ. We will show that if the unstable spectrum of A is empty, there exists indeed a unique center manifold that is analytic in the bean region $B := H_{\pm} \cup H_0$. This implies that the unique real orbit on the unstable part of the center manifold has complex finite time blow-up.

Throughout this chapter we fix the following notation.

Definition 3.1.5. *We denote as time p - path a line in the complex plane that is parallel to the real axis, which is a set γ of the form*

$$\gamma := \{t + i\delta, t \in \mathbb{R}\} .$$

We will also denote by Z, Y, X (complex) Banach spaces.

3.2 Homoclinic orbits and blow-up - the main idea

In this section we outline the main idea of the proof of Theorems 3.1.2 and 3.1.3. At the heart of the argument is Liouville's theorem for complex analytic functions - the only globally bounded analytic functions are constant. The blow-up solutions of Theorems 3.1.2 and 3.1.3 will be on the center manifold of the equilibrium at zero. The local analysis of the center manifold allows to show that there exists a solution of equation (3.1) which is bounded in the following region of complex time, see Figure 3.1 (c).

One can also show that if the orbit was also analytic in the left out half-strip, it would stay globally bounded and thus constant by Liouville's theorem, which contradicts the assumption that the initial condition is not an equilibrium.

To be more specific, we prove the Theorems in the following steps:

(i) Show that in the real two dimensional center manifold, the regions $H_\pm$ are foliated by real time homoclinic orbits with complex initial conditions. The proof relies on the analysis of the reduced equation on the center manifold

$$\dot{x} = x^p, \qquad x \in \mathbb{C},\ t \in \mathbb{R},\ p \in \mathbb{N}. \tag{3.2}$$

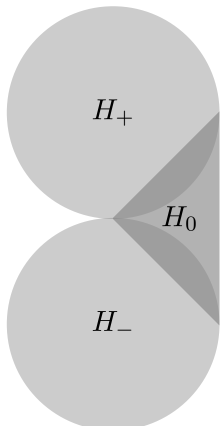
The homoclinic orbits remain small and are not affected by the cut-off employed in the construction of the center manifold. They are thus solutions of equation (3.1), see Section 3.4. This also implies that the center manifold Υ is unique on $H_\pm$.

(ii) Next, we show that the real time homoclinic orbits stem from one and the same complex time solution. They are evaluations along complex time paths, which are parallel to the real axis, i. e. along time p-paths. The center manifold Υ is then complex analytic on $H_\pm$, i.e. there exists a complex analytic function Ψ such that

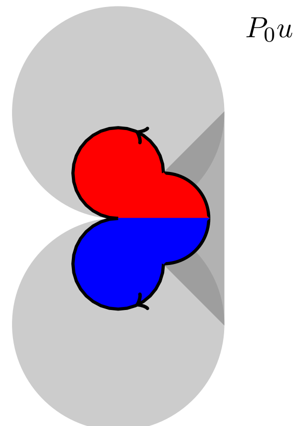
$$\Upsilon(x_1, x_2) = (\mathrm{Re}\Psi(x_1 + ix_2), \mathrm{Im}\Psi(x_1 + ix_2)).$$

We show next that Ψ can be analytically continued to the bean region B, if the unstable spectrum is empty. This is rather delicate, since if one was able to define an analytic continuation from H_- and from H_+ it is still not guaranteed that they coincide on the common domain of definition, e.g. the real axis. However, if the unstable spectrum is empty, the two different analytic continuations coincide because of uniqueness of the center-unstable manifold. See Section 3.5.

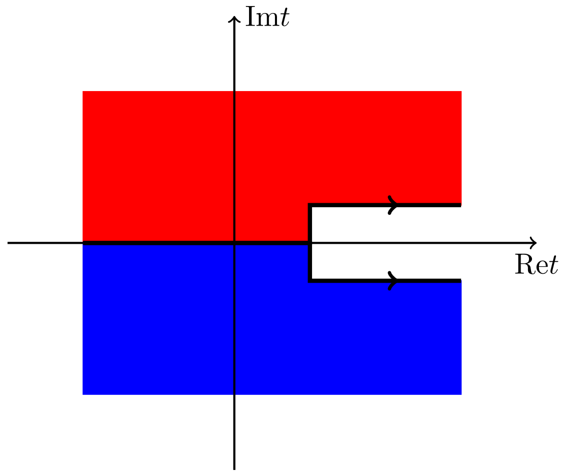
(iii) The center manifold is analytic in the bean region B, and it is actually filled by the complex time flow of a single orbit, see Figure 3.1.



(a) Bean region of analyticity of the center manifold

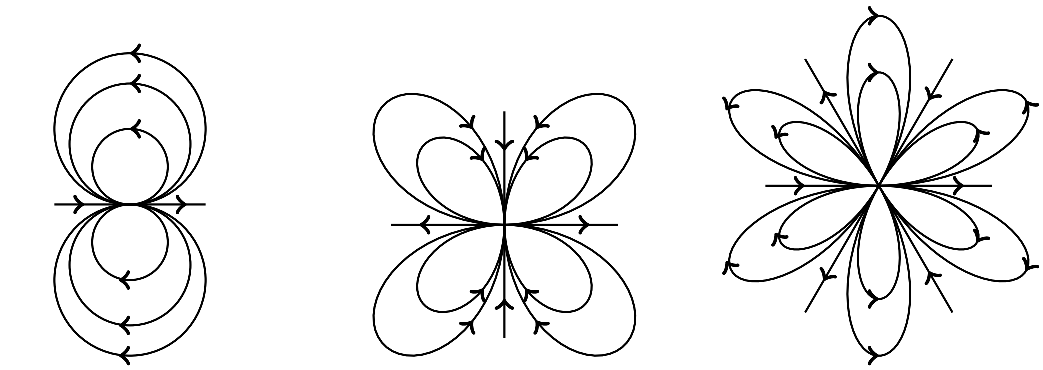


(b) Part of the center manifold traced by the complex time flow



(c) Region in complex time, where the constructed orbit stays bounded by center manifold analysis

Figure 3.1: Relationsship between center manifold and complex time path



(a) Solution for $p = 2$ (b) Solution for $p = 3$ (c) Solution for $p = 4$

Figure 3.2: Solutions of the reduced equation for real time

(iv) By Liouville's theorem we know that the orbit must either have grow-up or blow-up. Assume that there does not blow-up in finite time. Then, since the function is holomorphic in $\mathbb{C}$, instead of solving along the real time axis we solve the equation along an time orbit parallel to the real axis, that is in the red or blue region. We know that the solution is homoclinic there. The imaginary and the real time flow commute and we end up close to the origin for times with large real part. This contradicts blow-up.

3.3 Setting

The assumptions we make are basically those which allow to conclude the existence of a center manifold. The center manifold theorem for partial differential equations has been studied, e.g. in [21], [35], [58], [26] and [22]. See also the Appendix.

As already said in the introduction, we will consider equations of the form,

$$u_t = Au + f(u), \qquad u(0) = u_0,$$

with solution flow to u by Φ, i.e.

$$\Phi(t, u_0) := u(t, u_0).$$

For the existence of a center manifold we need the following assumptions.

Assumption 3.3.1. *Let the following assumptions hold throughout this chapter.*

(i) *The (complex) Banach spaces* X, Y *and* Z *are continuously embedded, that is* $X \hookrightarrow Y \hookrightarrow Z$.

(ii) $A \in L(Z, X)$

(iii) *For some* $k \geq 2$ *there exists a neighborhood* $U \subset Z$ *of zero, such that* $f \in C^k(U, Y)$ *and*

$$f(0) = 0, \; Df(0) = 0.$$

(iv) *The spectrum of* $\sigma(A)$ *has a finite number of eigenvalues with zero real part and the other eigenvalues are bounded away from the imaginary axis.*

Moreover the hyperbolic part should generate unique eternal solutions, i.e.

Assumption 3.3.2. *There exists* $\gamma > 0$ *such that for every* $0 \leq \eta \leq \gamma$ *and any* $f \in C_\eta(\mathbb{R}, Y_h)$ *the linear problem*

$$\dot{u}_h = A_h u_h + f.$$

has a unique solution $u = K_h f \in C_\eta(\mathbb{R}, Z_h)$. *Furthermore the linear map* $K_h f$ *belongs to* $L(C_\eta(\mathbb{R}, Y_h), C_\eta(\mathbb{R}, Z_h))$ *and there exists a continuous map* $C : [0, \gamma] \to \mathbb{R}$ *such that*

$$\|K_h\| \leq C(\eta).$$

Assumption 3.3.3 (Resolvent estimates)**.** *Assume that there exist postive constants* $\omega_0 > 0$, $c > 0$ *and* $\alpha \in [0, 1)$ *such that for all* $\omega \in \mathbb{R}$, *with* $|\omega| \geq \omega_0$ *we have that* $i\omega$ *belongs to the resolvent of* A, *and*

$$\|R(i\omega, A)\|_{L(X)} \leq \frac{c}{|\omega|}, \tag{3.3}$$

$$\|R(i\omega, A)\|_{L(Y,Z)} \leq \frac{c}{|\omega|^{1-\alpha}}. \tag{3.4}$$

Here does $R(\lambda, A)$ *denote the resolvent operator, that is*

$$R(\lambda, A) := (\lambda - A)^{-1}.$$

We end this subsection with a useful Remark [21].

Remark 3.3.4. *In Hilbert spaces we do not need equation* (3.4) *for the center manifold theorem to hold.*

In addition to this standard assumptions we will assume throughout this section the following

(i) The nonlinearity is analytic from Z to Y, $f(u) \in C^{\omega}(Z, Y)$.

(ii) The nonlinearity f maps real values to real values.

(iii) The spectrum of A is real, $\sigma(A) \subset \mathbb{R}$.

(iv) The kernel of A is one-dimensional, $\dim \ker(A) = 1$.

(v) The quadratic term of the reduced flow on the center manifold is positive, $P_0 f_{xx}(0,0) > 0$. We will set without loss of generality $P_0 f_{xx}(0,0) = 1$.

(vi) The unstable spectrum of A is empty.

Remark 3.3.5. *Assumption (iii) is mainly technical nature. Throughout the proofs, it will become clear, that we only need a sectorial operator.*

This setting allows to define the following system of differential equations by spectral projections such that $u = x + y$ with $x := P_0 u$ and $y := P_h u$,

$$\begin{aligned} \dot{x} &= P_0 f(x+y), \\ \dot{y} &= P_h A y + P_h f(x+y), \end{aligned} \tag{3.5}$$

where P_0 is the projection on the center space and P_h on the hyperbolic part.

Consider now the real and imaginary part of $u := v + iw$ separately on $\tilde{Z} := Z \times Z$, $\tilde{Y} := Y \times Y$ and $\tilde{X} := X \times X$, i.e. for $t = re^{i\theta}$.

$$\begin{pmatrix} v_r \\ w_r \end{pmatrix} = \begin{pmatrix} Av + g(v,w) \\ Aw + h(v,w). \end{pmatrix} \tag{3.6}$$

with $g(v,w) := \operatorname{Re}(f(v+iw))$ and $h(v,w) := \operatorname{Im}(f(v+iw))$.

In Chapter 2, we have shown that the system possesses a real two dimensional center manifold which is locally a graph $\Upsilon(x_1, x_2)$.

By Lemma 2.4.5 the reduced equation on the center manifold is of the form

$$\begin{pmatrix} (x_1)_r \\ (x_2)_r \end{pmatrix} = \begin{pmatrix} x_1^2 - x_2^2 \\ 2x_1x_2 \end{pmatrix} + o(x_1^2 + x_2^2). \tag{3.7}$$

3.4 The reduced equation

In this subsection we analyse the reduced equation on the center manifold. The reduced system is complex one-dimensional or real two-dimensional.

Purely polynomial ordinary differential equation Consider complex-valued solutions of the differential equation

$$\dot{x} = x^p, \qquad x \in \mathbb{C},\ t \in \mathbb{C},\ p \in \mathbb{N}. \tag{3.8}$$

We show that solutions are nests of homoclinic orbits for $t \in \mathbb{R}$, see Figure 3.2. This is immediately clear from the explicit solution

$$x(t) = \left(\frac{1}{x_0^{-p+1} - t}\right)^{1/(p-1)}, \qquad \operatorname{Im}\left(x_0^{-p+1}\right) \neq 0.$$

The perturbed polynomial equation We use solutions of the purely polynomial equation to show that solutions of higher order perturbations of equation (3.8) possess similar invariant regions. Here we, will just consider the quadratic case $n = 2$, since the other cases follows analogously. We consider a perturbed system for $x = x_1 + ix_2$, that is

$$\begin{aligned} \dot{x}_1 &= x_1^2 - x_2^2 + f(x_1, x_2), \\ \dot{x}_2 &= 2x_1x_2 + g(x_1, x_2), \end{aligned} \tag{3.9}$$

with $f, g \in o\left(\left|x_1^2 + x_2^2\right|\right)$.

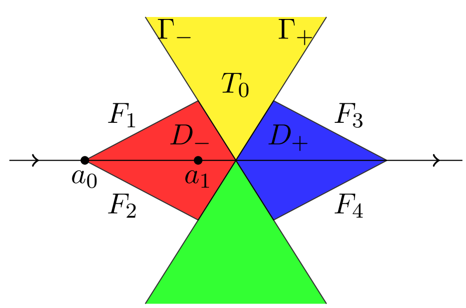


Figure 3.3: Invariant regions of perturbed quadratic flow

We want to show that orbits starting in the upper part of the blue diamond D_+ will end up in the red diamond D_- and then converge to zero, see Figure 3.3.

Let us start with the invariance of the diamond-like regions $D_\pm$. It is enough to show the invariance on each face of the diamonds. Consider the face F_1 first.

Lemma 3.4.1. *For any $a_0 < 0$ and $\beta > 0$ we can choose $a_0 < a_1 < 0$ such that*

$$-\beta\left(x_1^2 - x_2^2\right) + 2x_1x_2 \leq 0, \tag{3.10}$$

for $x_2 = \beta\left(x_1 - a_0\right)$ with $x_1 \in I := [a_0, a_1]$.

Proof. Setting $x_2 = \beta\left(x_1 - a_0\right)$ in inequality (3.10) gives

$$-\beta\left(x_1^2 - \beta^2\left(x_1 - a_0\right)^2\right) + 2x_1\beta\left(x_1 - a_0\right) \leq 0.$$

Since we want to exclude sign changes in I we have to solve

$$-\beta\left(x_1^2 - \beta^2\left(x_1 - a_0\right)^2\right) + 2x_1\beta\left(x_1 - a_0\right) = 0,$$

for x_1. The solutions are

$$x_\pm = a_0 \pm \frac{|a_0|}{\sqrt{1+\beta^2}}.$$

Thus for

$$a_1 \leq a_0\left(1 - \frac{1}{\sqrt{1+\beta^2}}\right). \tag{3.11}$$

holds $x_\pm \notin [a_0, a_1]$. For any $a_0 < 0$, $\beta > 0$, we can choose a_1 small enough such that the inequality (3.11) holds. Furthermore for any $x_1 = a_0$, it holds $-\beta x_1^2 \leq 0$. □

Remark 3.4.2. *Since we can choose a_0 as small as we like and get strict inequalities, the invariance holds for small perturbations of the quadratic vector field.*

Remark 3.4.3. *The argument for the face F_2 is completly similar by conjugation symmetry ($x_2 \mapsto -x_2$). Also the case $0 < a_0 < a_1$ is proven analogously by the symmetry $x_1 \mapsto -x_1$ and $x_2 \mapsto -x_2$.*

We can use the results of the real system to show, that the complexified solution is actually homoclinic in closed regions as indicated in Figure 3.3.

Lemma 3.4.4. *The complex equation*

$$\dot{x} = x^2 + o\left(|x|^3\right), \qquad x \in \mathbb{C},\ t \in \mathbb{R}, \tag{3.12}$$

possesses two regions in the complex plane with homoclinic orbits, see Figure 3.3. Furthermore, the counter clockwise resp. the clockwise rotated vector field points to the inside of the homoclinic orbit in the upper resp. lower half plane.

Proof. We rewrite the complex equation as a real two dimensional system. Set $x = x_1 + ix_2$ in equation (3.12) to obtain the system

$$\dot{x}_1 = x_1^2 - x_2^2 + o\left(\left(x_1^2 + x_2\right)^2\right),$$
$$\dot{x}_2 = 2x_1x_2 + o\left(\left(x_1^2 + x_2\right)^2\right).$$

which is of the form (3.9).

By the considerations of equation (3.8) we can concentrate on the upper half-plane. The proof consists in following two steps:

(i) The diamonds D_+ and D_- are invariant under the forward resp. the backwards flow. From Lemma 3.4.1 we already know the invariance of the faces F_1, F_2, F_3 and F_4. Note, that the axis $x_2 = 0$ is also invariant, since the equation (3.12) is invariant under complex conjugation. Furthermore we need to prove that the flow on the lines Γ_-, Γ_+ in Figure 3.3 points to the inside of D_- resp. to the outside of D_+.

(ii) The maximum drift in the direction x_2 in the triangle T_0 is bounded, such that if the solution starts on the boundary Γ_+ at radius r it hits Γ_- on a radius arbitrary close to r.

For the proof we change to polar coordinates by setting $(x_1, x_2) = re^{i\phi}$. The differential equation for (r, ϕ) is

$$\dot{r} = r^2 \cos(\phi) + o(r^2),$$
$$\dot{\phi} = r \sin\phi + o(r).$$

The boundaries $\Gamma_\pm$ of the invariant triangle T_0 are characterized by the angles $0 < \phi_+ < \phi_- < \pi$.

(i) The conditions on ϕ_-, ϕ_+ imply $\sin(\phi_\pm) > 0$. Thus, for small enough r the flow points outside resp. inside the triangle T_0.

(ii) For small enough $r > 0$ we can consider r as a graph over ϕ, i. e.,

$$\frac{dr}{d\phi} = \frac{r}{\tan\phi} + O(r^2). \tag{3.13}$$

Integrating equation (3.13) from ϕ_- to ϕ_+ gives (setting $\phi_\pm = \pi/2 \pm \alpha$ for some $\alpha > 0$)

$$\left|\log\left(\frac{r(\phi_+)}{r(\phi_-)}\right)\right| \leq Co(r).$$

By taking r small, the drift due to the higher order terms in T_0 becomes arbitrary small.

(iii) Consider now the orbit to $\left(x_1^0, x_2^0\right) \in \Gamma_+$ for $\left(x_1^0, x_2^0\right)$ small enough. Since the orbit can not leave the invariant regions D_+ in backwards resp. D_- in forward time, see Figure 3.3, the α and ω limit set is not empty and has to be the unique equilibrium $x_1 = x_2 = 0$. The real time flow is thus a homoclinic orbit.

(iv) The last step is to show that the orbit $\gamma(t) := (x_1(t), x_2(t))$, $t \in \mathbb{R}$ is an orientable curve. Since the orbit is a non intersecting closed curve the Jordan curve theorem implies orientability.

For $p := (p_1, p_2) \in \gamma$ the inner product between the inwards normal $n(p) := (n_1(p), n_2(p))$ and the rotated vector field $(-2p_1p_2, p_1^2 - p_2^2) + o(p_1^2 + p_2^2)$ is

$$\kappa(p) := -2n_1(p)\, p_1p_2 + n_2(p)\left(p_1^2 - p_2^2\right) + o\left(p_1^2 + p_2^2\right).$$

Since $\kappa(p)$ depends continuously on p and the vector field never vanishes, the sign of κ can not change. This implies that the vector field either points always to the outside or always to the inside of the curve.

(v) We need to find a single point $p \in \gamma$ such that the vector field points inside. Take $p = (0, p_2) \in \gamma$ which gives

$$\kappa(p) = p_2^2 + o\left(p_1^2 + p_2^2\right) > 0.$$

□

Remark 3.4.5. *By the proof it is clear that we can choose* $\phi_\pm = \frac{\pi}{2} \mp \alpha$ *with* α *as close to* $\frac{\pi}{2}$ *as we like. But only if we also choose* $|x_1|, |x_2|$ *small enough.*

Remark 3.4.6. *Although the above argument is due to complex time rotation, we avoid any explicit usage of time analyticity. We just assume that we have an unique projected real two dimensional vector field in the region* $H_\pm$.

3.5 Blow-up on the center manifold

In this subparagraph, we prove analyticity of the one-dimensional center manifold if no unstable spectrum present.

The analyticity originates from two different sources. The first source is the invariance equation as analyzed in Chaper 2. The second source are homoclinic orbits, whose existence is guaranteed in Lemma 3.4.4. The main idea is first to show the existence of homoclinic orbits in the center manifold the two-dimensional real system and then secondly argue that the homoclinic orbits, since they are small solutions, are actually time analytic. Thus, we can continue the solutions into complex time. The continued solution along complex time paths parallel to the real time axis is also homoclinic and have to be contained in the center manifold of the real system. In that region the real two dimensional real center manifold is actually a single complex time orbit.

A second important concept is the center unstable manifold. It is necessary in the construction to remedy the following difficulty:
We need to have a single orbit, whose complex time flow as indicated in Figure 3.6 sweeps the regions on the left. By the above argument the center manifold is analytic in the regions $H_{\pm}$. We can of course study the analytic continuation of the center manifold from H_+ or H_-. Assume we could actually continue both manifolds back to the real axis. Then it is not clear whether they coincide.
But the unstable spectrum is empty. Then we can argue via the center-unstable manifold. The center-unstable manifold is always unique. In this case it is also of real dimension two. By the previous Chapter, we can extend the unique real center-unstable manifold to the complex sector H_0 by extension of the graph function. Solutions on the graph satisfy a reduced equation. The reduced equation allows to show that solutions on the extension of the center manifold converge to zero for negative real time. This implies that the extension has to be contained in the unique two dimension real center-unstable manifold. But since the homoclinic orbits enter the region H_0, they also must be contained in the center-unstable manifold and thus also in the analytic continuation of the real center-unstable manifold.

Lemma 3.5.1. *Under the assumptions of Section 3.3 the real center manifold of the two dimensional system is complex analytic in the regions $H_{\pm}$.*

Proof. We take the following steps:

(i) Show that in the regions $H_{\pm}$, each real time homoclinic orbit on the center manifold can be extended to a complex time strip,

$$S_\delta = \{z \in \mathbb{C}, |\mathrm{Im} z| \leq \delta\}$$

for some $\delta > 0$. Then show that the solution stays small with δ small enough and converges to zero along time in S_δ along time paths parallel to the real axis (time p-paths).

(ii) Solutions of (3.1) along time p - paths can be viewed as a solution of the two dimensional real system by writing real and imaginary part separately. Thus each time p - path in S_δ gives a homoclinic orbit and must be contained in the 2d real center manifold. This implies that the region $H_\pm$ are foliated by homoclinic orbits, which are actually complex time p-pathes of one and the same homoclinic orbit. The center manifold is actually analytic.

By Lemma 3.4.4 the flow of the line segment Γ_+ is homoclinic.

(i) Consider a solution of (3.6) with initial condition $x_0 := \left(x_1^0, x_2^0\right) \in \Gamma_+$ on the center manifold. Since the solution is a small homoclinic orbit by Section 3.4, it is actually a solution to the complex system (3.1). Define $u_0 := \left(x_1^0 + ix_2^0, \Upsilon_0^1(x_0) + i\Upsilon_0^2(x_0)\right)$. There exists for any $\varepsilon > 0$ a T_ε such that $\|\Phi(t, u_0)\|_Z \leq \varepsilon$ for all times $t \in \mathbb{R}$, $T_\varepsilon < |t|$.
As long as the flow is analytic in time, the real time flow and the imaginary time flow commute. Thus take some $\delta > 0$ small enough such, that the time the $\Phi(t, u_0)$ has an analytic continuation onto a strip from $[-T_\varepsilon, T_\varepsilon] \times [-\delta, \delta]$, [33], [22]. Furthermore there exist constants $c_2, c_1 > 0$ as small as we like such

$$\|f(u)\|_Y \leq M\varepsilon^2,$$

for $\|u\|_Z \leq c_1$. This implies for $|t| \geq T_\varepsilon$ with $2\varepsilon < c_1$,

$$\begin{aligned}
&\|\Phi(\delta + i\delta, \Phi(t-\delta, u_0)) - \Phi(t, u_0)\|_Z \\
&\leq \|\Phi(\delta + i\delta, \Phi(t-\delta, u_0)) - \Phi(t-\delta, u_0) + \Phi(t-\delta, u_0) - \Phi(t, u_0)\|_Z \\
&\leq CM \int_0^\delta \left\|T\left(e^{i\pi/4}(t-s)\right)\right\|_{L(Y,Z)} \varepsilon^2 ds + 2\varepsilon \leq \delta M\tilde{C}\varepsilon^2 + 2\varepsilon.
\end{aligned}$$

The above equation holds as long as

$$\delta M\tilde{C}\varepsilon^2 + 2\varepsilon < c_1.$$

Choosing ε small enough this equation holds along the complete path.

(ii) Along each time path of the form $t = i\delta + \tau$, $\tau \in \mathbb{R}$, the complex flow $\Phi(\tau, \Phi(i\delta, u_0))$ solves the equation for the real system (3.6). Since all small solutions have to be contained in the center manifold, we know that all time p-pathes for imaginary time less than δ are homoclinic orbits in the real two dimensional center manifold Υ_0. Because the real center manifold is of dimension two, it is locally already foliated by the time p - homoclinic orbits, which are also of dimension two.

(iii) The homoclinic solutions satisfy an reduced equation in the real time direction, i.e. system (3.9). But since the homoclinic orbit is time analytic, we can also solve into complex time.

By complex analyticity we have for $x(t) := P_0 u(t)$, $t = t_0 + it_1$ and $x(t) := x_1(t) + ix_2(t)$ the following equation

$$\partial_{t_1} x_1(t) = -\partial_{t_0} x_2(t) = -2x_1(t)\, x_2(t) + o\left(x_1^2 + x_2^2\right),$$
$$\partial_{t_1} x_2(t) = \partial_{t_0} x_1(t) = x_1^2(t) - x_2^2(t) + o\left(x_1^2 + x_2^2\right).$$

This vector points to the interior of the projected orbit to $\Phi(t, u_0)$, $t \in \mathbb{R}$. Thus the flow stays uniformly bounded for all positive imaginary times.

(iv) Similar arguments hold for H_-.

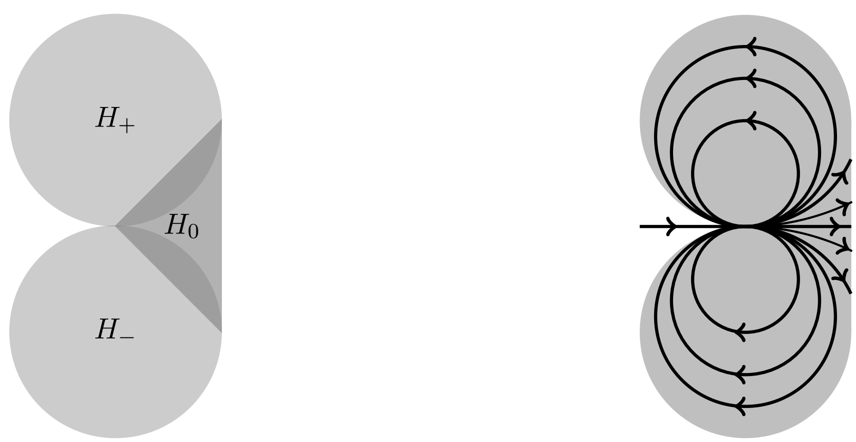
(v) Now, we define the graph function Ψ in $H_\pm$ as follows

$$\Psi_\pm(x) := P_0 u(t(x)), \qquad x \in H_\pm.$$

The reduced solution $x(t)$ is invertible by inverse functions theorem in the analytic category, because the derivative $\dot{x}$ never vanishes in $H_\pm$.

□

The previous argument did not depend on the fact that the unstable spectrum is empty, but to prove the next Lemma we need that the unstable spectrum is empty.



(a) Bean region of analyticity of the center manifold (b) Region of analyticity of the center manifold and the reduced flow

Figure 3.4: Region of analyticity of the center manifold

Lemma 3.5.2. *Let the assumption of Section 3.3 hold. Then the there exists an analytic center manifold Ψ in the region bean region B, see Figure 3.4.*

Proof. We have already defined the center manifolds $\Psi_\pm : H_\pm \to Z$. We need to show that there exists a unique analytic function $\Psi : B \to Z$, which coincides with $\Psi_\pm$ in $H_\pm$.

(i) The real center manifold is unique for $x > 0$. We know by Theorem 2.3.10 that there exists an analytic continuation Ψ_0 of the unique one-dimensional real center manifold $\Upsilon_0(x, 0)$ onto the sector H_0. Furthermore, we have shown, that the function Ψ_0 is of small quadratic order in H_0. On the sector H_0, we can solve the reduced equation for $r > 0$ small enough

$$\dot{x} = x^2 + f(x, \Psi_0(x)), \qquad x \in S_{\theta,r},\ t \in D_x \subset \mathbb{C}.$$

Due to the ODE analysis in Section 3.4, solutions in H_0 converges to zero for negative real times.

(ii) Chapter 2 has proven that there exists a unique center-unstable manifold $\tilde{\Upsilon}$ of the real system of real dimension two. It contains by definition all orbits that converge to zero in real negative time. This implies in particular that $\Upsilon|_{H_0} = \Psi_0$ and also $\Upsilon|_{H_\pm} = \Psi_\pm$.

(iii) The analysis in Section 3.4 showed that the homoclinic orbits in $H_\pm$ enter the sector H_0. This implies that the analytic continuation of the unique real center manifold coincides on a connected open set with the analytic center manifold of $H_\pm$. Thus we can conclude that there exists an analytic continuation Ψ of the unique one-dimensional real center manifold to the bean region, see Figure 3.4

□

Proof of theorem 3.1.2.

(i) As long as the flow stays bounded in the region H_0, we can use the one-dimensional analytic center manifold from flow 3.5.2 to reduce the flow to a complex-valued scalar differential equation, that is

$$\dot{x} = x^2 + P_0 f(x, \Psi(x)) = x^2 + O\left(|x|^3\right). \tag{3.14}$$

(ii) We use separation of variables to obtain

$$t = \int_{x_0}^{x_1} \frac{1}{x^2 + f(x, \Psi(x))} dx. \tag{3.15}$$

as time needed to go from x_0 to x_1. Take any $x_\pm \in \partial H_0 \cap \partial H_\pm$. Then for $x_0 \in \mathbb{R}_+$ small enough there exists a complex valued time path $\gamma_\pm(s)$, such that

$$x(\gamma_\pm(s), x_0) \subset H_0$$

and $x(\gamma_\pm(s_1), x_0) = x_\pm$. This can be seen by calculating the integral explicitly using the geometric series

$$\frac{1}{x^2 + P_0 f(x, \Psi(x))} = \frac{1}{x^2} \sum_{n=0}^{\infty} h^n(x),$$

with $h(x) := \frac{P_0 f(x,\Psi(x))}{x^2}$ and $h \in O(|x|)$.
Integral (3.15) is

$$\int_{x_0}^{x_1} \frac{1}{x^2} h^n(x)\, dx = \sum_{n=0}^{\infty} \int_{x_0}^{x_1} \frac{1}{x^2} + a_0 \frac{1}{x} + g(x)\, dx,$$

with $g(x) = x^{-2} \sum_{n=2}^{\infty} h^n(x) = O(1)$.

Thus

$$\sum_{n=0}^{\infty} \int_{x_0}^{x_1} \frac{1}{x^2} + a_0 \frac{1}{x} + g(x)\, dx = \frac{1}{x_0} - \frac{1}{x_1} + a_0 \log\left(\frac{x_1}{x_0}\right) + o(|x_1 - x_0|).$$

Choose $x_\pm = \frac{x_0}{1 \mp i\alpha}$ to obtain

$$t = \frac{\pm i\alpha}{x_0} + a_0 \log(1 \pm i\alpha) + O(\alpha x_0).$$

Note, that formula (3.15) is valid along the whole time path. In particular we can choose $\alpha = \tau_0 x_0$, which gives

$$t = i\alpha x_0^{-1} + O(1)$$

Since we can choose the angle ϕ_+ of the lines $\Gamma_\pm$ as small as we like, we can first take α small such, that $x_+ \in \Gamma_+$ and then choose x_0 small enough, such that the above calculation holds. This implies, that the solution to the reduced equation enters the invariant triangle T_0, see Figure 3.3.

(iii) By construction the real time solution of $x(t, x_\pm) = x\left(t, x\left(\pm i\alpha x_0^{-1}, x_0\right)\right)$, $t \in \mathbb{R}$ is a homoclinic orbit. Thus the solution $u(t) := (x(t), \Psi(x(t)))$ is bounded in the complex time region of the form

$$S := \{t \in \mathbb{C},\ |\mathrm{Im} z| \geq \tau_0 \wedge \mathrm{Re} z \leq 0\}.$$

as shown in Figure 3.1.

(iv) By Liouville's theorem we know that the orbit can not stay bounded in the region $\mathbb{C} \setminus S$.

(v) To exclude grow-up in S, we argue by contradiction. Assume that there exists grow-up, i. e. the solution was entire. Then again by analyticity the imaginary and real time flow commute. Furthermore for any $\delta > 0$, there exists a $t > 0$ large enough so that $\left|x\left(t + i\alpha x_0^{-1}, x_0\right)\right| \leq \delta$.

This is possible by the results of Section 3.4. Now taking $\delta > 0$ small enough we can know, that the solution must exist at least for time $i\alpha x_0^{-1}$, so that we can continue the solution uniformly bounded back to the real axis. This is a contradiction to Liouville's theorem.

(vi) There must be finite time blow-up in S.

□

Theorem 3.1.3 is a corollary to 3.1.2.

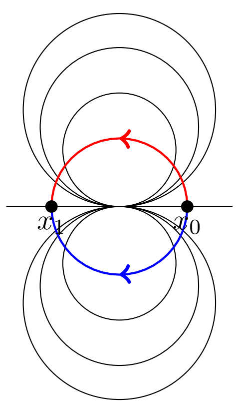
Proof of theorem 3.1.3. By Theorem 3.1.2 we have a real initial condition u_0 such, that there exists complex time blow-up at time $T := T_1 + iT_2$ with $T_1 > 0$. Since the solution converges in backwards time to zero, we can choose for any $\delta > 0$ a $\tilde{T}_1 < 0$ such, that

$$\left\| \mathrm{Im}\Phi\left(\tilde{T}_1 + iT_2, u_0\right)\right\| < \delta.$$

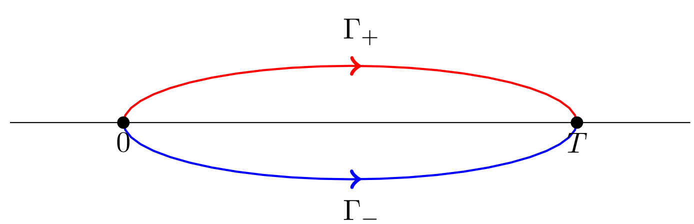
Defining $\tilde{u}_0 := \Phi\left(\tilde{T}_1 + iT_2, u_0\right)$ yields a complex initial condition with real time blow-up at time $T(\tilde{u}_0) := T_1 - \tilde{T}_1$. □

3.6 Analytic center manifold and branching

Analogously to the heteroclinic case, there exists a relation between the branch type of the center manifold and the branch type at the finite time blow-up. It is unclear how the solution at infinity connects back to the origin. We will prove a simple first Lemma towards that direction.



(a) Path in the center manifold



(b) Corresponding time path

Figure 3.5: Time and center manifold path

Suppose for a moment that the center manifold analytic Ψ was analytic in a neighborhood of the equilibrium. Then there exists time paths $\Gamma_\pm$ and a time T such that by following the time paths, such that the difference of the analytic continuation of the solution to $u_0 := (x_0, \Psi(x_0))$ is related to the power series expansion of Ψ, see Figure 3.5.

Theorem 3.6.1. *Assume, that the center manifold $\Psi : U \subset \mathbb{C} \to Z_h$ is analytic on some neighborhood U of zero. If $P_0 f(x, \Psi(x)) := x^2 + \sum_{n=4}^{\infty} \psi_n x^n$, then there exists a time $T > 0$ such that the continuations along $\Gamma_\pm$ coincide.*

Proof. Analyticity of the center manifold in a neighborhood of the equilibrium implies that the reduced equation

$$\dot{x} = h(x) := P_0 f(x, \Psi(x)), \tag{3.16}$$

is analytic and valid in a neighborhood of the equilibrium.

By the preceding section we know that the red and the blue paths in the center manifold are generated by complex time paths $\Gamma_\pm$. In advance, we do not know that the time paths will coincide at the point x_1. The equation (3.16) is solvable by separation of variables

$$\int_{x_0}^{x(t)} \frac{1}{h(x)} dx = t.$$

This implies for $x(t_r) = x(t_b) = x_1$, where $x(t_r)$ is the continuation along the red path and $x(t_b)$ is the continuation along the blue path,

$$\begin{aligned} t_r - t_b &= \int_{x_0}^{x(t_r)} \frac{1}{f(x, \Psi(x))} dx - \int_{x_0}^{x(t_b)} \frac{1}{f(x, \Psi(x))} dx = \oint_{x(t_b)}^{x(t_r)} \frac{1}{f(x, \Psi(x))} dx \\ &= \oint_{x(t_b)}^{x(t_r)} \frac{g(x)}{x^2} dx = g'(0) = 0, \end{aligned}$$

by the Cauchy formula since $g(x) := \left(1 + \sum_{n=2}^{\infty} \psi_{n+2} x^n\right)^{-1}$. □

Remark 3.6.2. *If the center manifold was analytic, the third order term of the reduced vector field induces the branch type.*

Analyticity of the center manifold gives a restriction on the branch-type of the blow-up orbit. But also the converse is true. If the blow-up orbit is not branched, then the center manifold is analytic and the third order term of the vector field must vanish.

Theorem 3.6.3. *Consider the blow-up orbit of Theorem 3.1.2 and empty unstable spectrum. If the time analytic continuations in the negative and positive complex plane conincide for some big enough $T_1 > 0$, and the solution does not have blow-up, but in a domain*

$$S := \{z \in \mathbb{C},\ |Im(z)| < c_1,\ T_0 < Re(z) < T_1\}, \qquad c_1 > 0.$$

Then exsists a neighborhood $U \subset \mathbb{C}$ of zero and an analytic function $\Psi : U \to Z_h$, such, that Ψ is a center manifold.

Proof. The idea is to show that the projection of the blow-up orbit on the center manifold maps $\mathbb{C} \setminus S$ to a neighborhood of the origin, see Figure 3.6.

(i) We again consider the reduced flow on the center manifold. Since conjugation symmetry along $\Gamma_\pm$ holds and the solutions coincide after $t > T_1$, we know that the solution $u(t)$ is real for $t > T_1$. Furthermore, we know that the orbit converges subexponentially to zero with $t > T_1$. It follows the real one-dimensional center manifold and the reduced equation holds for times $\mathrm{Re}(t) > T_1$. This implies that

$$x(t) := P_0 u(t), \qquad \dot{x} = P_0 u_t(t), \qquad t \in \mathbb{C} \setminus S,$$

never vanishes. Thus we can analytically invert the function $x(t)$ on $\mathbb{C} \setminus S$. The function $t(x)$ is analytic in a neighborhood of zero take out zero.

Now, we define the function

$$\Psi(x) := P_h u(t(x)).$$

The function Ψ exists as an analytic function. Since analytic maps are open it maps the domain $\mathbb{C} \setminus S$ to an open neighborhood U of the origin take out the origin.

(ii) But the function Ψ is bounded, such that the origin is a removable singularity and Ψ can be extended to a whole neighborhood by Riemann singularity theorem.

□

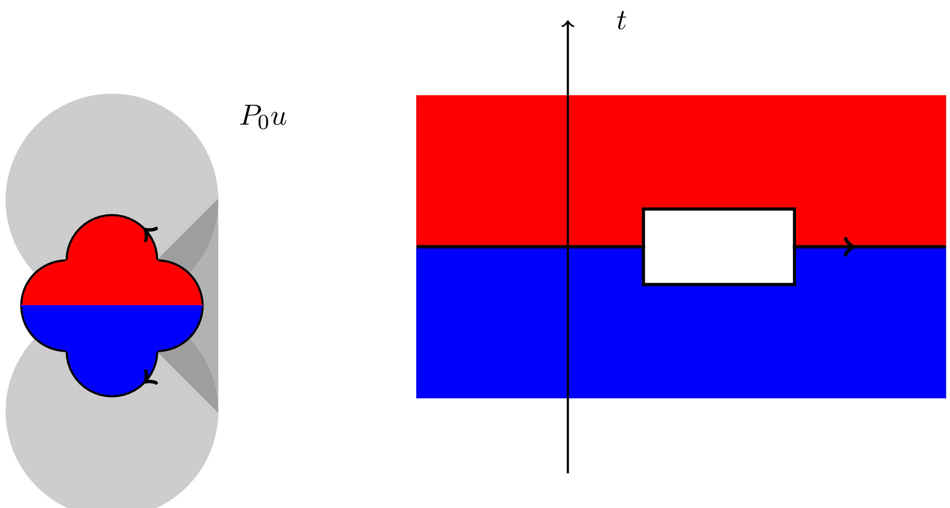


Figure 3.6: Center manifold and complex time orbit

3.7 Outlook

In the one-dimensional setting, there exist a lot of literature with a detailed study of global attractors their topology and heteroclinic connections [15], [32] for dissipative systems. There have been approaches to extend the results to grow-up solutions [45].

Our result show that there is a relation between blow-up of solutions and the global attractor of reaction diffusion equations with entire nonlinearities if one allows for complex time.

We can prove for example, that non-degenerate saddle-node bifurcations come with a (possibly complex time) blow-up orbit. For an introduction to bifurcation theory for partial differential equations, see for example [29].

Theorem 3.7.1. *Consider the equation*

$$u_t = Au + f(\lambda, u),$$

where $f \in C^\omega(\mathbb{R} \times Z, Y)$ and let A satisfy the conditions of Section 3.3. Assume furthermore that there exists a λ_0 such, that

$$Au_0 + f(\lambda_0, u_0) = 0,$$

for some u_0. Furthermore assume, that $D_\lambda f(\lambda_0, u_0) \notin im\ (A + D_u f(\lambda_0, u_0))$.
Let $\ker(A + D_u f(\lambda_0, u_0)) = span(v_0)$ and

$$D_{uu} f(\lambda_0, u_0)(v_0, v_0) \notin im\ (A + D_u f(\lambda_0, u_0)). \tag{3.17}$$

then

(i) There exists a continuously differentiable curve $(u(s), \lambda(s))$ through (λ_0, u_0) such that

$$Au(s) + f(\lambda(s), u(s)) = 0,$$

(ii) At (λ_0, u_0) there exists a blow-up homoclinic orbit, if

$$\langle D_{uu} f(\lambda_0, u_0)(v_0, v_0), v_0\rangle > 0.$$

and $A + D_u f(\lambda_0, u_0)$ has empty unstable spectrum.

Proof. The first statement follows from Theorem I.4.1. in [29]. The second statement follows from Theorem 3.1.2, since condition (3.17) implies

$$P_0 D_{uu} f(\lambda_0, u_0)(v_0, v_0) = \langle D_{uu} f(\lambda_0, u_0)(v_0, v_0), v_0\rangle > 0.$$

□

Remark 3.7.2. *Similar results can also be obtained for pitchfork bifurcation.*

But of course there remain many open questions about the connection between global attractor, eternal core, blow-up and analytic continuation of blow-up solutions.

(i) By the analysis of the ODE invariant regions for $p > 2$ we obtain different flow invariant regions in the center manifold with possible blow-up. How are these regions related to each other?

(ii) If the unstable part is not empty the situation becomes more complicated. In the proof of Theorem 3.1.2 we used the uniqueness of the center unstable manifold to construct a solution that is bounded everywhere in the complex time plane, but on a half-strip. In the half-strip we concluded the existence of blow-up. In the presence of unstable spectrum the center manifold is still unique in the homoclinic regions $H_{\pm}$, but the analytic continuations from $H_{\pm}$ do not necessarily coincide on the common domain of definition. This implies, that one needs different methods and perhaps also extra assumptions to prove of similar result as in Theorem 3.1.2.

(iii) Another question is what happens if the center-manifold of the real equation is not one-dimensional, but of higher dimension? One can try to prove blow-up of solutions similarly to the one-dimensional case, but the complexified reduced equation is now of real dimension four or higher and much harder to study. But as in the one-dimensional case, the blow-up can be induced by a low-dimensional ordinary differential equation close to equilibria and prove the existence of large enough regions in complex time in which the solutions remains bounded to conclude blow-up in the remaining part.

(iv) Moreover, the relation of the Riemann surface induced by analytic continuations of the blow-up orbit around the blow-up time T and the complex analytic continuation of the center manifold function Ψ around the origin is unclear. We have already seen in Chapter 1 and also in Section 3.6 that the branch types are related, but at the current moment the description is very coarse.

(v) Another important example is the three-dimensional Navier-Stokes equation. The Navier-Stokes equation also allow for a center manifold [26] and solutions are analytic in time, see for example [56] as long as they are bounded. If we consider the non-linearity f or the viscosity ν as a parameter, that is if we solve

$$\nu\Delta u + (u \cdot \nabla)\, u + \nabla p = f\,(x)\,,$$

with periodic boundary conditions, see [56] we can look for equilibria with one-dimensional center manifold and empty unstable spectrum. The main problem then to apply theorem 3.1.2 directly is that the nonlinearity vanishes along any possible center direction. Suppose, that the kernel of the linearization around u_0 is one-dimensional, and that v_0 is the corresponding eigenfunction. The nonlinear part projected on the kernel is

$$P_0\left(\left(v_0 \cdot \nabla\right) v_0\right) = \left\langle \left(v_0 \cdot \nabla\right) v_0, v_0 \right\rangle_{L_2} = 0,$$

for any $v_0 \in H^1$ and $\nabla \cdot v_0 = 0$. But since we almost proved the Theorem 3.1.2 when the projected equation has no quadratic term, but starts with a higher order term, a detailed analysis of the center manifold may still yield the existence of a real time blow-up solution for a complex initial condition with arbitrary small imaginary part.

It is however not clear, how this type of blow-up is related to the question of blow-up of real initial data in real time. Especially in the case of the Navier-Stokes equation, the possible real blow-up might be of a completely different type. So if solutions on the global attractor induce blow-up, in what sense it is true that blow-up solutions induce solutions on the global attractor?

(vi) In general one also might attempt to obtain a more detailed picture of the blow-up and attractor structure at bifurcations. For example, consider the saddle-node bifurcation. In the non-degenerate case blow-up homoclinics occur at the bifurcation point. Changing the parameter even further yields a heteroclinic connection of two non hyperbolic equilibria. According to the proofs in Chapter 1 the heteroclinic connection induces a blow-up heteroclinic. It would be worthwhile to have a more detailed picture at the bifurcation points, e.g. how the blow-up homoclinic relates to the blow-up heteroclinic and also whether the heteroclinic orbit can persist if one changes the parameter even further.

Chapter 4

Appendix

4.1 The center manifold theorem

We briefly review the center manifold theorem for partial differential equations. The material is mainly taken from [21] and supplemented by [35], [58], [26] and [22].

Consider an equation of the form

$$u_t = Au + f(u).$$

To conclude the existence of a center manifold assume the following.

Assumption 4.1.1. *Let the following assumptions hold throughout this chapter.*

(i) The Banach spaces (complex) X, Y and Z are continuously embedded, that is $X \hookrightarrow Y \hookrightarrow Z$.

(ii) $A \in L(Z, X)$

(iii) For some $k \geq 2$ there exists a neighborhood $U \subset Z$ of zero, such that $f \in C^k(U, Y)$ and

$$f(0) = 0, \; Df(0) = 0.$$

(iv) The spectrum of $\sigma(A)$ has a finite number of eigenvalues with zero real part and the other eigenvalues are bounded away from the imaginary axis.

The last assumption guarantees the existence of spectral projections, see next Section, which allow for a decomposition of Z, Y, X.

$$X_0 = \text{im } P_0 X \subset Z, \qquad P_h = \text{Id} - P_0, \qquad X_h := P_h X, \qquad Y_h := P_h Y, \qquad Z_h := P_h Z.$$

Moreover the hyperbolic part should generate unique eternal solutions, i.e.

Assumption 4.1.2. *There exists $\gamma > 0$ such that for every $0 \leq \eta \leq \gamma$ and any $f \in C_\eta(\mathbb{R}, Y_h)$ the linear problem*

$$\dot{u}_h = A_h u_h + f,$$

has a unique solution $u = K_h f \in C_\eta(\mathbb{R}, Z_h)$. Furthermore the linear map $K_h f$ belongs to $L(C_\eta(\mathbb{R}, Y_h), C_\eta(\mathbb{R}, Z_h))$ and there exists a continuous map $C : [0, \gamma] \to \mathbb{R}$ such that

$$\|K_h\| \leq C(\eta)$$

Assumption 4.1.3 (Resolvent estimates)**.** *Assume that there exist positive constants $\omega_0 > 0$, $c > 0$ and $\alpha \in [0, 1)$ such that for all $\omega \in \mathbb{R}$ with $|\omega| \geq \omega_0$, we have that $i\omega$ belongs to the resolvent of A, and*

$$\|R(i\omega, A)\|_{L(X)} \leq \frac{c}{|\omega|}, \tag{4.1}$$

$$\|R(i\omega, A)\|_{L(Y,Z)} \leq \frac{c}{|\omega|^{1-\alpha}}. \tag{4.2}$$

Here does $R(\lambda, A)$ denote the resolvent operator, that is

$$R(\lambda, A) := (\lambda - A)^{-1}.$$

We end this subsection with a useful Remark [21][Remark 2.16]

Remark 4.1.4. *In Hilbert spaces we do not need equation (4.2) for the center manifold theorem.*

Spectral projections

One important tool to prove the center manifold theorem are spectral projections, which project the space X onto a stable, unstable and center part, [21][Theorem A.7]

Theorem 4.1.5. *Consider a closed operator $A : D(A) \subset X \to X$. Assume, that the spectrum $\sigma(A)$ can be separated by a closed curve Γ into two parts σ_+ and σ_-, where σ_- contains only finitely many of points. Then the following holds,*

(i) *There exists a decomposition of $X = X_+ \oplus X_-$, where $X_\pm$ is invariant under A.*

(ii) $A|_{X_-} \in L(P_- D(A), X_-)$.

(iii) *The spectra of the restrictions coincide with $\sigma_\pm(A)$, that is $\sigma\left(A|_{X_\pm}\right) = \sigma_\pm$.*

(iv) *The projection P_- of X to X_- is given by the following formula*

$$P_- = \frac{1}{2\pi i} \int_\Gamma (\lambda Id - A)^{-1} \, d\lambda$$

(v) $P_- A = A P_-$.

By Theorem 4.1.5 the following projections are well-defined.

(i) The projections P_0 and P_h define a decomposition of $X = X_0 \oplus X_h$, where $X_0 := \text{im } P_0$ and $X_h := \text{im } P_h$.

(ii) $P_0 A = A P_0$.

(iii) The operator $A_0 := A|_{X_0}$ is well defined with spectrum $\sigma(A_0) := \sigma_0 := \sigma(A) \cap i\mathbb{R}$.

(iv) The operator $A_h := A|_{X_h}$ is well defined with spectrum $\sigma(A_h) := \sigma_+ \cup \sigma_-$ with $\sigma_\pm := \{\lambda \in \sigma(A), \pm \text{Re}\lambda \geq 0\}$.

Furthermore, similar method can be used to prove the existence of semigroup associated to subsets of the spectrum by appropriate resolvent integrals whenever Assumption 3.3.3 holds, see [22].

The center manifold theorem

In this section, we state the center manifold theorem, see [21]. The center manifold contains all global solutions close to an equilibrium and is in general non-unique. But the Taylor expansion at the origin of center manifolds is unique.

Theorem 4.1.6. *Let the assumptions 3.3.1 and 3.3.2 hold, then there exists a map $\psi(E_0, Z_h)$ with*

$$\psi(0) = 0, \ D\psi(0) = 0,$$

and a neighborhood of the origin $O \subset Z$ such that the manifold

$$M_0 = \{u_0 + \psi(u_0); \ u_0 \in E_0\} \subset Z,$$

has the following properties:

(i) M_0 is locally invariant, that is if $u(0) \in M_0 \cap O$ and $u(t) \in O$ for all $t \in [0,T]$, then $u(t) \in M_0$ for all $t \in [0,T]$.

(ii) M_0 constains the set of bounded solutions staying in O for all $t \in \mathbb{R}$, i.e if u is a solution of (3.1) *satisfying $u(t) \in O$ for all $t \in \mathbb{R}$ then $u(0) \in M_0$.*

Remark 4.1.7. *The local center manifold is in general non unique, but all graphs ψ have the same Taylor expansion at the origin. This is due to the cut-off functions used in the proof of the local version of the center manifold.*

Corollary 4.1.8. *Solutions on the center manifold satisfy a reduced, finite dimensional differential equation*

$$\dot{u}_0 = A_0 u_0 + P_0 f(u_0, \psi(u_0)),$$

and furthermore the graph of the center manifold itself obeys the following differential equation

$$D\psi(u_0)\dot{u}_0 = A_h \psi(u_0) + P_h f(u_0, \psi(u_0)). \tag{4.3}$$

4.2 Holomorphic functions

Here, we recap some theorems about holomorphic functions, mainly based on [7], [6].

Definition 4.2.1 (Entire functions)**.** *An analytic function on $C^{\omega}(\mathbb{C})$ is called entire. Denote by $M(r)$ the supremum of the entire function on circles around the origin, that is*

$$M(r) := \sup_{|z|=r} |f(z)| .$$

Then the order ρ of f is defined as

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

Equivalently we know, that of f is of finite order ρ if and only if

$$M(r) = O\left(\exp\left(r^{\rho+\epsilon}\right)\right)$$

for any $\epsilon > 0$.

For example is the exponential function of order $\rho = 1$, since $\frac{\log(cr)}{\log r} \to 1$. The order of an analytic function relates to the coefficients of its power series representation [6][Theorem 2.2.2].

Theorem 4.2.2. *An entire function f is of finite order ρ if and only if*

$$\rho = \limsup_{n\to\infty} \left(-\frac{n\log n}{\log a_n}\right). \tag{4.4}$$

4.2.1 Holomorphic functions in Banach spaces

We introduce the concept of analytic function $f : \Omega \subset X \to Y$ where X, Y are Banch spaces and Ω is an open and connected subset of X. The material is based on [2], [39], [52].

For $X = \mathbb{C}$ the concept of analytic functions is straight forward.

Definition 4.2.3 (Weak and strong holomorphic functions)**.** *A function $f : \Omega \to Y$ is called strongly holomorphic if*

$$\lim_{\tilde{z}\to z} \frac{f(\tilde{z}) - f(z)}{\tilde{z} - z}$$

exists in the topology of Y. The function f is called weakly holomorphic if $\langle y^*, f(z)\rangle$ is holomorphic for all $y^* \in Y^*$.

The main theorem is that weakly holomorphic function are strongly holomorphic and that the Cauchy formula holds.

Theorem 4.2.4. *Let $f : \Omega \to Y$ be weakly holomorphic. Then does following holds:*

(i) The function f is strongly holomorphic.

(ii) For a closed path $\Gamma \subset \Omega$

$$\int_{\Gamma} f(t)\, dt = 0.$$

Furthermore, all properties which can be described in the weak topology inherit the properties of holomorphic function, e.g. if the function $f : \mathbb{C} \to Y$ is weakly bounded, then it is constant. The notion of weak compactness also yields a generalization of Montel compactness.

Theorem 4.2.5. *Assume that* Y *is separable and* $K \subset Y$ *is compact. Let the family of functions* $f_n : \Omega \subset \mathbb{C} \to K \subset Y$ *be uniformly bounded and weakly holomorphic. Then there exists a subsequence* f_k *,converging to a holomorphic function* f *uniformly on compact subsets of* Ω*.*

Furthermore, the following version of the identity theorem holds:

Theorem 4.2.6. *Suppose that* $\tilde{Y} \subset Y$ *is a closed subspace and* $f : \Omega \to Y$ *is holomorphic. Assume, that there exists a convergent sequence* $(z_n) \in \Omega$ *such, that the limit* $z \in \Omega$ *and* $f(z_n) \in \tilde{Y}$ *for all* $n \in \mathbb{N}$*. Then* $f(z) \in Y$ *for all* $z \in \Omega$*.*

A second theorem with a lot of interesting consequences is Vitali's convergence theorem.

Theorem 4.2.7. *Let* $f_n : \Omega \to Y$ *be holomorphic and bounded on compact subsets. Assume that the set*

$$\Omega_0 := \left\{ z \in \Omega : \lim_{n\to\infty} f_n(z) \textit{ exists} \right\},$$

has a limit point in Ω*. Then there exists a holomorphic function* $f : \Omega \to X$ *such that*

$$f^{(k)}(z) = \lim_{n\to\infty} f_n^{(k)}(z)$$

uniformly on compact subsets.

Now, let X be any Banach space. A function $f : \Omega \to Y$ is called analytic or holomorphic if it is locally representable by a converging power series, i.e. for all $x \in \Omega$ there exists a $r > 0$ such that for $\|\tilde{x} - x\| < r$ we have

$$f(\tilde{x}) = \sum_{n=0}^{\infty} P_n(\tilde{x} - x),$$

where P_n are Banach space valued polynomials of degree n. The problem is how to extend the concept of weak holomorphicity if the domain is already a Banach space. We introduce the concept of G - holomorphicity.

Definition 4.2.8 (G – holomorphicity)**.** *A mapping* $f : \Omega \to Y$ *is called* G *– holomorphic if for all* $x \in \Omega$ *and* $\tilde{x} \in X$ *the mapping*

$$f_{\tilde{x}}(\lambda) : \Omega_x \subset \mathbb{C} \to Y$$

is holomorphic on $\Omega_{x_0} := \{\lambda \in C,\ x + \lambda\tilde{x} \in \Omega\}$*.*

The notion of G – holomorphicity yields the following theorem.

Theorem 4.2.9. *For a mapping $f : X \to Y$ the following is equivalent*

(i) f is holomorphic.

(ii) f is continuous and G – holomorphic.

(iii) f is continuous and $f|\Omega \cap M$ is holomoprhic for every finite dimensional subspace $M \subset E$.

Remark 4.2.10. *Actually, weak G - holomorphicity is already sufficient, since weakly G - holomorphic functions are G-holomorphic.*

Theorem 4.2.11. *Let X be separable. Each bounded subset of analytic functions $f_n : \Omega \to Y$ is relatively compact with respect to uniform convergence on compact subsets.*

4.3 Incomplete Gamma function and exponential integral

In this section we briefly summarize the properties of the incomplete Gamma function. This material is taken from [1], [42] and [43]. The incomplete Gamma function $\Gamma_\alpha(z)$ is defined by the following integral

$$\Gamma_\alpha(z) := \int_z^\infty t^{\alpha-1} e^{-t} dt$$

For $z \neq 0$ is $\Gamma(z, \alpha)$ is entire in α.
For $\alpha = 1$ we obtain the principal branch of the exponential integral function Ei.
The incomplete Gamma function is the solution of the following differential equation

$$\Gamma_\alpha'' + \left(1 + \frac{1-\alpha}{z}\right) \Gamma_\alpha = 0$$

For $\alpha < 1$ and $|\mathrm{ph}(z)| < \pi$ we have the following integral relation

$$\Gamma_\alpha(z) = \frac{z^\alpha e^{-z}}{\Gamma(1-\alpha)} \int_0^\infty \frac{t^{-\alpha} e^{-t}}{z+t} dt.$$

Pulling out z, we obtain

$$\Gamma_\alpha(z) = \frac{z^{\alpha-1} e^{-z}}{\Gamma(1-\alpha)} \int_0^\infty \frac{t^{-\alpha} e^{-t}}{1+t/z} dt,$$

which yields the following asymptotic expansion for large $|t|$,

$$\Gamma_\alpha(z) \approx z^{\alpha-1} e^{-z}. \tag{4.5}$$

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