Appendix A: How to Define $W_{n,j}(z,\tau)$

As expressed in Remark 8, the vertical sum $W_{n,j}(z,\tau)$ has some very nice circulatory structures as (2.40) in it. Its summands are of the form (2.41) with (2.42) and their multiplicities take on some extraordinary sequential properties (1)-(3) there.

It must be noted that, for $W_{n,j}(z,\tau)$, there are $n-j-1$ vertical sums as its summands in the outmost vertical sum. From the top down, these vertical sums respectively have 2^{n-j-3}, 2^{n-j-4}, …, 2, 1, 1 summands of the form as (2.41), it is the same as the above property (3) for the variance of the multiplicites about the first summand of the form as (2.41) between two adjacent vertical sums and the coefficients appearing before the vertical sum symbols are in turn 1, $-\frac{1}{2!}$, …, $\frac{(-1)^{n-j-3}}{(n-j-2)!}$, $\frac{(-1)^{n-j-2}}{(n-j-1)!}$. Interestingly, any one of these vertical sums has a similar structure and properties as the outmost vertical sum.

Just because of the above sequential properties of the multiplicities and the nice circulatory structure, we can sequentially define $W_{n,j}(z,\tau)$ as the vertical sum (2.34) only from the first summand $\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{n-j}(k+1)\cdots(k+j-1)} + \frac{1}{j!}$.

For example, in what follows, we give the definition of $W_{8,1}(z,\tau)$.

At first, we look at the outmost vertical sum of $W_{8,1}(z,\tau)$. As above, there are $6\,(=8-1-1)$ vertical sums as its summands. From the top down, these vertical sums respectively have 2^4, 2^3, 2^2, 2, 1, 1 summands of the form as (2.41) and the coefficients appearing before the vertical sum symbols are in turn 1, $-\frac{1}{2!}$, $\frac{1}{3!}$, $-\frac{1}{4!}$, $\frac{1}{5!}$, $-\frac{1}{6!}$. Thus the outmost vertical sum is of the form

$$\sum \begin{cases} V_1 \\ -\frac{1}{2!}V_2 \\ \frac{1}{3!}V_3 \\ -\frac{1}{4!}V_4 \\ \frac{1}{5!}V_5 \\ -\frac{1}{6!}V_6, \end{cases}$$

where $V_1, V_2, V_3, V_4, V_5, V_6$ are respectively some vertical sums with $16, 8, 4, 2, 1, 1$ summands of the form as (2.41).

Since every V_j has a similar structure and properties as the outmost vertical sum. e.g., for V_1, from the top down, these vertical sums respectively have 2^3, 2^2, 2, 1, 1 summands of the form as (2.41) and the coefficients appearing before the vertical sum symbols are in turn 1, $-\frac{1}{2!}$, $\frac{1}{3!}$, $-\frac{1}{4!}$, $\frac{1}{5!}$. Thus the vertical sum V_1 is of the form

$$\sum \begin{cases} V_{1,1} \\ -\frac{1}{2!}V_{1,2} \\ \frac{1}{3!}V_{1,3} \\ -\frac{1}{4!}V_{1,4} \\ \frac{1}{5!}V_{1,5}, \end{cases}$$

where $V_{1,1}, V_{1,2}, V_{1,3}, V_{1,4}, V_{1,5}$ are respectively some vertical sums with $8, 4, 2, 1, 1$ summands of the form as (2.41).

All and the same, every vertical $V_{1,j}$ has a similar structure and properties as the outmost vertical sum. Again, e.g., for $V_{1,1}$, from the top down, these vertical sums respectively have 2^2, 2, 1, 1 summands of the form as (2.41) and the coefficients appearing before the vertical sum symbols are in turn 1, $-\frac{1}{2!}$, $\frac{1}{3!}$, $-\frac{1}{4!}$. Thus the vertical sum $V_{1,1}$ is of the form

$$\sum \begin{cases} V_{1,1,1} \\ -\frac{1}{2!}V_{1,1,2} \\ \frac{1}{3!}V_{1,1,3} \\ -\frac{1}{4!}V_{1,1,4}, \end{cases}$$

where $V_{1,1,1}, V_{1,1,2}, V_{1,1,3}, V_{1,1,4}$ are respectively some vertical sums with $4, 2, 1, 1$ summands of the form as (2.41). Obviously, $V_{1,1}$ just has the basic circulatory structure as (2.40) and the coefficients appearing before the vertical sum symbols are exactly 1, $-\frac{1}{2!}$, $\frac{1}{3!}$, $-\frac{1}{4!}$. Once such case appears, we should stop. This principle is applicable to any vertical sum with 4 vertical sums or more as its summands. But for the vertical sum with four or two summands of the form as (2.41), the coefficients are always 1, $-\frac{1}{2!}$, $-\frac{1}{2!}$, $\frac{1}{3!}$ or 1, $-\frac{1}{2!}$ from the top down. The vertical sum with one summand of the form as (2.41) only has 1 as its coefficient.

$$
\sum \left\{ \begin{array}{l}
\sum \left\{ \begin{array}{l}
\sum \left\{ \begin{array}{l}
\sum \left\{ \begin{array}{l} A \\ -\frac{1}{2!}B \\ -\frac{1}{2!}B \\ \frac{1}{3!}C \end{array} \right. \\
-\frac{1}{2!} \sum \left\{ \begin{array}{l} B \\ -\frac{1}{2!}D \end{array} \right. \\
\frac{1}{3!}C \\
-\frac{1}{4!}E
\end{array} \right. \\
-\frac{1}{2!} \sum \left\{ \begin{array}{l} B \\ -\frac{1}{2!}D \\ -\frac{1}{2!}D \\ \frac{1}{3!}F \end{array} \right. \\
\frac{1}{3!} \sum \left\{ \begin{array}{l} C \\ -\frac{1}{2!}F \end{array} \right. \\
-\frac{1}{4!}E \\
\frac{1}{5!}G
\end{array} \right. \\
-\frac{1}{2!} \sum \left\{ \begin{array}{l}
\sum \left\{ \begin{array}{l} B \\ -\frac{1}{2!}D \\ -\frac{1}{2!}D \\ \frac{1}{3!}F \end{array} \right. \\
-\frac{1}{2!} \sum \left\{ \begin{array}{l} D \\ -\frac{1}{2!}H \end{array} \right. \\
\frac{1}{3!}F \\
-\frac{1}{4!}I
\end{array} \right. \\
\frac{1}{3!} \sum \left\{ \begin{array}{l} C \\ -\frac{1}{2!}F \\ -\frac{1}{2!}F \\ \frac{1}{3!}J \end{array} \right. \\
-\frac{1}{4!} \sum \left\{ \begin{array}{l} E \\ -\frac{1}{2!}I \end{array} \right. \\
\frac{1}{5!}G \\
-\frac{1}{6!}K
\end{array} \right.
$$

Repeating the above process for all the vertical sums appeared or appearing, by the properties of the summands of the form as (2.40) and their multiplicities, we will find that $W_{8,1}(z,\tau)$ is a vertical sum of the above form, in which, the summands A-K have the expressions as follows.

$$A = \sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^7} + \frac{1}{1!},$$

$$B = \sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^6(k+1)} + \frac{1}{1! \cdot 2!},$$

$$C = \sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^5(k+1)(k+2)} + \frac{1}{1! \cdot 3!},$$

$$D = \sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^5(k+1)^2} + \frac{1}{1! \cdot 2! \cdot 2!},$$

$$E = \sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^4(k+1)(k+2)(k+3)} + \frac{1}{1! \cdot 4!},$$

$$F = \sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^4(k+1)^2(k+2)} + \frac{1}{1! \cdot 2! \cdot 3!},$$

$$G = \sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^3(k+1)(k+2)(k+3)(k+4)} + \frac{1}{1! \cdot 5!},$$

$$H = \sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^4(k+1)^3} + \frac{1}{1! \cdot 2! \cdot 2! \cdot 2!},$$

$$I = \sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^3(k+1)^2(k+2)(k+3)} + \frac{1}{1! \cdot 2! \cdot 4!},$$

$$J = \sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^3(k+1)^2(k+2)^2} + \frac{1}{1! \cdot 3! \cdot 3!},$$

$$K = \sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^2(k+1)(k+2)(k+3)(k+4)(k+5)} + \frac{1}{1! \cdot 6!}.$$