Chapter 3

Dirichlet Problems for Homogeneous PDEs

In this chapter, we mainly consider some Dirichlet boundary value problems for polyharmonic functions and poly-analytic-harmonic functions in the unit disc. As a preliminary, we begin with the classic result of the Dirichlet problem for analytic functions.

3.1 Dirichlet Problem for Analytic Functions

In the theory of BVPs for analytic functions, the Dirichlet boundary value problem is one of the classical BVPs. It is expressed as follows.

Dirichlet boundary value problem Find a function $w \in H_1(\mathbb{D})$ such that

$$w = \gamma \text{ on } \partial\mathbb{D},$$

where $\gamma \in C(\mathbb{D})$ is a given complex function.

The following theorem is well-known and can be found in many places [4, 22, 26]. The proof here is due to Begehr [4] with some modification.

Theorem E. *The Dirichlet problem is solvable if and only if for* $|z| < 1$,

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau)\frac{\overline{z}}{\overline{\tau}-\overline{z}}\frac{\mathrm{d}\tau}{\tau} = 0. \tag{3.1}$$

Then the solution is uniquely given by the Cauchy integral

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\gamma(\tau)}{\tau - z}\,\mathrm{d}\tau,\ \ z \in \mathbb{D}. \tag{3.2}$$

Proof. For (3.1) to be necessary, suppose that w is the solution of the Dirichlet problem. Then w can be expressed as (3.2) which is analytic in $\mathbb{D}$ and has continuous boundary values

$$\lim_{z\to\tau,\,|z|<1} w(z) = \gamma(\tau) \tag{3.3}$$

for all $\tau \in \partial\mathbb{D}$.

Define

$$w^*(z) = \overline{w\Big(\frac{1}{\overline{z}}\Big)},\ z \in \overline{\mathbb{C}}\backslash\overline{\partial\mathbb{D}},$$

then $w^* \in H_1(\overline{\mathbb{C}}\backslash\overline{\partial\mathbb{D}})$ follows from (3.2). Since

$$w\Big(\frac{1}{\overline{z}}\Big) = -\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau)\frac{\overline{z}}{\overline{\tau}-\overline{z}}\frac{\mathrm{d}\tau}{\tau},$$

therefore

$$w(z) - w\Big(\frac{1}{\overline{z}}\Big) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau)\Big(\frac{\tau}{\tau - z} + \frac{\overline{\tau}}{\overline{\tau}-\overline{z}} - 1\Big)\frac{\mathrm{d}\tau}{\tau}.$$

So from the above equality and the properties of the Poisson kernel, for $\tau \in \partial\mathbb{D}$,

$$\lim_{z\to\tau, |z|<1} [\, w(z) - \overline{w^*(z)}\,] = \gamma(\tau) \tag{3.4}$$

follows. (3.3) and (3.4) show that $\lim_{z\to\tau,\,|z|<1} w^*(z) = 0$. By the maximum principle for analytic functions, $w^*(z) = 0$ for all $|z| < 1$. Therefore

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau)\frac{\overline{z}}{\overline{\tau}-\overline{z}}\frac{\mathrm{d}\tau}{\tau} = -w\Big(\frac{1}{\overline{z}}\Big) = -\overline{w^*(z)} = 0,\ \ |z| < 1.$$

For the sufficiency, by (3.1) and (3.2),

$$\begin{aligned} w(z) =& \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau)\Big(\frac{\tau}{\tau - z} + \frac{\overline{z}}{\overline{\tau}-\overline{z}}\Big)\frac{\mathrm{d}\tau}{\tau} && (3.5)\\ =& \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau)\Big(\frac{\tau}{\tau - z} + \frac{\overline{\tau}}{\overline{\tau}-\overline{z}} - 1\Big)\frac{\mathrm{d}\tau}{\tau}, && (3.6)\end{aligned}$$

therefore

$$\lim_{z\to\tau,\,|z|<1} w(z) = \gamma(\tau)$$

follows from the properties of the Poisson kernel. □

Remark 9. From the above proof, we find that the existence $\lim_{z\to\tau,\,|z|<1} w(z)$ implies the existence of $\lim_{z\to\tau,\,|z|<1} w^*(z)$. Since $w(z) = [w^*]^*(z) = \overline{w^*\big(\frac{1}{\overline{z}}\big)}$, $|z| > 1$, then the existence $\lim_{z\to\tau,\,|z|<1} w(z)$ implies the existence of $\lim_{z\to\tau,\,|z|>1} w(z)$.

For antianalytic functions, we similarly consider the following Dirichlet boundary value problem.

Associated Dirichlet boundary value problem Find a function $w \in \overline{H}_1(\mathbb{D})$ such that

$$w = \gamma \text{ on } \partial\mathbb{D},$$

where $\gamma \in C(\partial\mathbb{D})$ is a given complex function.

By the above theorem, we have

Corollary 6. *The associated Dirichlet problem is solvable if and only if for $|z| < 1$,*

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau)\frac{z}{\tau - z}\frac{\mathrm{d}\tau}{\tau} = 0.$$

Then the solution is uniquely given by

$$w(z) = -\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\gamma(\tau)}{\overline{\tau} - \overline{z}}\,\mathrm{d}\overline{\tau},\ \ z \in \mathbb{D}.$$

Proof. Since $\overline{\partial_z w} = \partial_{\overline{z}}\overline{w}$, then $w \in \overline{H}_1(\mathbb{D})$ implies $\overline{w} \in H_1(\mathbb{D})$. So it easily follows from Theorem E. □

3.2 Dirichlet Problem for Polyharmonic Functions

In the present section, we considered a Dirichlet problem for polyharmonic functions which is also called polyharmonic Dirichlet problem (PHD problem) as follows.

Polyharmonic Dirichlet Problem Find a function $w \in Har_n^{\mathbb{C}}(\mathbb{D})$ satisfying the Dirichlet type boundary conditions

$$[(\partial_z\partial_{\overline{z}})^j w]^+(t) = \gamma_j(t),\ \ t \in \partial\mathbb{D},\ \ 0 \le j < n, \tag{3.7}$$

where $\gamma_j \in C(\partial\mathbb{D})$ which denotes the set of all complex continuous functions on $\partial\mathbb{D}$ for $0 \le j < n$.

With the higher order Poisson kernels, the above PHD problem is uniquely solvable. To do so, we need the following lemmas, one of which is about another property of the higher order Poisson kernels.

Lemma 7. *Let Ω_1 be a domain and Ω_2 be a compact set in the complex plane, $\Omega_1 \cap \Omega_2 = \emptyset$, $g(z, \xi)$ is a continuous function defined in $\Omega_1 \times \Omega_2$ such that $g(z, \xi) \in H_1(\Omega_1)$ as a function of z with fixed $\xi \in \Omega_2$. For any fixed $z_0 \in \Omega_1$, take $D_{z_0,R} = \{z : 0 < |z - z_0| < R\} \subset \Omega_1$ and define*

$$F_z(z_0, \xi) = \frac{g(z, \xi) - g(z_0, \xi)}{z - z_0}, \ \ \xi \in \Omega_2 \tag{3.8}$$

and

$$G_z(z_0, \xi) = \frac{g(z, \xi) - g(z_0, \xi)}{\overline{z} - \overline{z}_0}, \ \ \xi \in \Omega_2 \tag{3.9}$$

with fixed $z \in D_{z_0,R/2}$. Then $F_z(z_0, \cdot)$, $G_z(z_0, \cdot) \in L(\Omega_2)$.

Proof. Since $g(z, \xi) \in H_1(\Omega_1)$ with respect to z for fixed $\xi \in \Omega_2$, by Cauchy integral formula, for fixed $\xi \in \Omega_2$,

$$g(z_0, \xi) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = R} \frac{g(\zeta, \xi)}{\zeta - z_0} \, \mathrm{d}\zeta$$

and

$$g(z, \xi) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = R} \frac{g(\zeta, \xi)}{\zeta - z} \, \mathrm{d}\zeta.$$

Thus

$$F_z(z_0, \xi) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = R} \frac{g(\zeta, \xi)}{(\zeta - z)(\zeta - z_0)} \, \mathrm{d}\zeta.$$

So

$$\begin{aligned} \int_{\Omega_2} |F_z(z_0, \xi)| \mathrm{d}\nu(\xi) &= \int_{\Omega_2} \left| \frac{1}{2\pi i} \int_{|\zeta - z_0| = R} \frac{g(\zeta, \xi)}{(\zeta - z)(\zeta - z_0)} \, \mathrm{d}\zeta \right| \mathrm{d}\nu(\xi) \\ &\leq \frac{1}{2\pi i} \int_{|\zeta - z_0| = R} \frac{|\tilde{g}(\zeta)|}{|(\zeta - z)|} \frac{\mathrm{d}\zeta}{\zeta - z_0} \\ &\leq \frac{2 \sup |\tilde{g}(\zeta)|}{R}, \end{aligned}$$

where ν is the Lebegue measure on Ω_2, $\tilde{g}(\zeta) = \int_{\Omega_2} |g(\zeta, \xi)| \mathrm{d}\nu(\xi)$ is bounded on $\{\zeta : |\zeta - z_0| = R\}$ since $g(z, \xi) \in C(\Omega_1 \times \Omega_2)$ and Ω_2 is compact. That is to say $F_z(z_0, \cdot) \in L(\Omega_2)$. Note that

$$G_z(z_0, \xi) = \frac{z - z_0}{\overline{z} - \overline{z}_0} F_z(z_0, \xi),$$

therefore $G_z(z_0, \cdot) \in L(\Omega_2)$. $\square$

Lemma 8 (Differentiability of Integral)**.** *Let* $\{ g_n(z,\tau) \}_{n=1}^{\infty}$ *be the sequence of higher order Poisson kernels, then for any* $\gamma \in C(\partial\mathbb{D})$,

$$(\partial_z \partial_{\overline{z}})\Big[\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau) g_n(z,\tau)\frac{\mathrm{d}\tau}{\tau}\Big] = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau) g_{n-1}(z,\tau)\frac{\mathrm{d}\tau}{\tau},\ n = 2, 3, \ldots. \tag{3.10}$$

Proof. For any fixed $z \in \mathbb{D}$, arbitrarily choose a sequence $\{z_l\}$ such that $z_l \neq z$ for any l and $z_l \to z$ as $l \to \infty$. Define

$$Z_l(z,\tau) = \frac{g_n(z_l,\tau) - g_n(z,\tau)}{z_l - z}$$

for fixed l. Obviously, $Z_l(z,\tau) \in C(\partial\mathbb{D}) \subset L(\mathbb{D})$ with respect to τ and

$$\lim_{l\to\infty} Z_l(z,\tau) = \partial_z g_n(z,\tau).$$

In addition, by the decomposition (2.18) of $g_n(z,\tau)$ and the last lemma with $\Omega_1 = \mathbb{D}$ and $\Omega_2 = \partial\mathbb{D}$, it is easy to see that $Z_l(z,\cdot) \in L(\partial\mathbb{D})$. Note the continuity of $\partial_z g_n(z,\tau)$, by the dominated convergence theorem,

$$\begin{aligned}
&\lim_{l\to\infty} \frac{1}{z_l - z}\Big[\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau) g_n(z,\tau)\frac{\mathrm{d}\tau}{\tau} - \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau) g_n(z,\tau)\frac{\mathrm{d}\tau}{\tau}\Big] \\
=&\lim_{l\to\infty} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau)\frac{g_n(z_l,\tau) - g_n(z,\tau)}{z_l - z}\frac{\mathrm{d}\tau}{\tau} \\
=&\lim_{l\to\infty} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau) Z_l(z,\tau)\frac{\mathrm{d}\tau}{\tau} \\
=&\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau)\partial_z g_n(z,\tau)\frac{\mathrm{d}\tau}{\tau}.
\end{aligned}$$

Because of the arbitrariness of $\{z_l\}$, therefore in view of the Heine principle

$$\partial_z\Big[\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau) g_n(z,\tau)\frac{\mathrm{d}\tau}{\tau}\Big] = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau)\partial_z g_n(z,\tau)\frac{\mathrm{d}\tau}{\tau}.$$

Further, similarly define

$$H_l(z,\tau) = \frac{\partial_z g_n(z_l,\tau) - \partial_z g_n(z,\tau)}{\overline{z}_l - \overline{z}},$$

again by (2.18), Lemma 7, the dominated convergence theorem and the Heine principle,

$$\partial_{\overline{z}}\Big[\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau)\partial_z g_n(z,\tau)\frac{\mathrm{d}\tau}{\tau}\Big] = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau)\partial_{\overline{z}}[\partial_z g_n(z,\tau)]\frac{\mathrm{d}\tau}{\tau}.$$

So (3.10) follows from the last two equalities and the induction property of the higher order Poisson kernels. □

Lemma 9. *If* $\varphi \in H_1(\mathbb{D})$ *and* $\frac{\partial \varphi}{\partial z} \in C(\overline{\mathbb{D}})$*, then* $\varphi \in C(\overline{\mathbb{D}})$*.*

Proof. It immediately follows from

$$\varphi(z) = \int_0^z \frac{\partial \varphi}{\partial z}(\zeta)\mathrm{d}\zeta - \varphi(0), \ z \in \mathbb{D}.$$ □

Theorem 10. *The PHD problem (3.7) is solvable and its unique solution is*

$$w(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma_{k-1}(\tau) g_k(z,\tau) \frac{\mathrm{d}\tau}{\tau}, \ z \in \mathbb{D}, \tag{3.11}$$

where $g_k(z,\tau)\,(1 \le k \le n)$ *is the kth order Poisson kernel given by* (2.39)*.*

Proof. At first, we show that (3.11) is a solution. By Lemma 8 and the induction property of the higher order Poisson kernels, using the operators $(\partial_z \partial_{\overline{z}})^j$, $j = 1, 2, \ldots, n-1$ to act on two sides of (3.11), we get

$$(\partial_z \partial_{\overline{z}})^j w(z) = \sum_{k=j+1}^{n} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma_{k-1}(\tau) g_{k-j}(z,\tau) \frac{\mathrm{d}\tau}{\tau}. \tag{3.12}$$

Thus

$$[\partial_z \partial_{\overline{z}})^j w]^+(t) = \gamma_j(t), \ t \in \mathbb{D}, \, 0 \le j < n$$

follows from (3.12) and the other properties of the higher order Poisson kernels, i.e., (3.11) is a solution.

Next, we turn to the uniqueness of (3.11). To do so, we must show that (3.7) only has zero as its solution when all $\gamma_j = 0$ on $\partial\mathbb{D}$. It is enough to consider $w \in Har_n(\mathbb{D})$ for this case. Since $w \in Har_n(\mathbb{D})$, by Theorem B, there exist some functions $w_j \in H^j_{1,0}(\mathbb{D})$, $j = 0, 1, \ldots, n-1$ such that

$$w(z) = 2\Re\Big\{ \sum_{j=0}^{n-1} \overline{z}^j w_j(z) \Big\}, \quad z \in \mathbb{D}. \tag{3.13}$$

Applying the operators $(\partial_z\partial_{\overline{z}})^j,\ j=1,2,\dots,n-1$ to both sides of (3.13), we have

$$(\partial_z\partial_{\overline{z}})^j w(z) = 2\Re\Big\{\sum_{k=j}^{n-1}\frac{k!}{(k-j)!}\overline{z}^{k-j}\partial_z^j w_k(z)\Big\},\quad z\in\mathbb{D}. \tag{3.14}$$

By (3.14), Lemma 9 and the boundary value conditions of (3.7) with $\gamma_j=0$,

$$\Re[\partial_z^j w_j(t)] = 0,\ \ t\in\partial\mathbb{D},\ 0\le j\le n-1.$$

So it is easy to get $w_j\in\Pi_{1,0}^j(\mathbb{D})$ from the last equality and then $w=0$. □

Remark 10. In [9], Begehr, Du and Wang only considered the PHD problem (3.7) with Hölder continuous boundary conditions not continuous boundary conditions. So it happens since they solve the problem by reflection method which transfers the problem to the classical Riemann jump problems for analytic functions. However, the Hölder continuity is necessary for the latter considering the singular integrals on the unit circle. In [14], to solve the same problem when $n=3$, Begehr and Wang used a new approach which transfers the problem to the classical Schwarz problem for analytic functions in the unit disc. So the Hölder continuity is weaken to the condition of continuity. In fact, in view of the above proof, with continuous boundary conditions discussed in the last theorem, the unique solvability of PHD problem (3.7) obviously follows from the properties of the higher order Poisson kernels $g_n(z,\tau)$ by induction.

3.3 Dirichlet Problems for Poly-analytic-harmonic Functions

In this section, three kinds of Dirichlet type boundary value problems for poly-analytic-harmonic functions in $M_{m,n}(\mathbb{D})$ are given.

One of which is of the form: find a function $L(z)\in M_{m,n}(\mathbb{D})\,(m>n)$ satisfying the boundary conditions

$$[(\partial_z\partial_{\overline{z}})^j L]^+(t)=\gamma_j(t),\ 0\le j<n\ \text{ and }\ [\partial_z^{n+k}\partial_{\overline{z}}^n L]^+(t)=\sigma_k(t),\ 0\le k<m-n, \tag{3.15}$$

where $t\in\partial\mathbb{D}$, $\gamma_j,\sigma_k\in C(\partial\mathbb{D})$ for $0\le j<n,\ 0\le k<m-n$.

By the harmonic decomposition theorem, we have

Theorem 11. *Set*

$$A(t)=\begin{pmatrix} n! & (n+1)!t & \cdots & \frac{(m-2)!}{(m-n-2)!}t^{m-n-2} & \frac{(m-1)!}{(m-n-1)!}t^{m-n-1} \\ 0 & (n+1)! & \cdots & \frac{(m-2)!}{(m-n-3)!}t^{m-n-3} & \frac{(m-1)!}{(m-n-2)!}t^{m-n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (m-2)! & (m-1)!t \\ 0 & 0 & \cdots & 0 & (m-1)! \end{pmatrix}, \tag{3.16}$$

$$a(t)=\begin{pmatrix} \sigma_0(t) \\ \sigma_1(t) \\ \vdots \\ \sigma_{m-n-2}(t) \\ \sigma_{m-n-1}(t) \end{pmatrix}, \tag{3.17}$$

$$\Xi_l(z)=\frac{1}{n!(n+1)!\cdots(m-1)!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{\overline{\det(A_l(\tau))}}{\tau-z}\mathrm{d}\tau, \tag{3.18}$$

and

$$\widetilde{\varphi}_l(z)=\int_0^z\int_0^{\zeta_{n-1}}\cdots\int_0^{\zeta_1}\Xi_l(\zeta)\mathrm{d}\zeta\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{n-1}+\pi_l(z), \tag{3.19}$$

where $t\in\partial\mathbb{D}$, $\pi_l\in\Pi_{n-1}$, the matrix $A_l(t)$ is given by replacing the lth column of $A(t)$ by $a(t)$, $0\le l\le m-n-1$. Then

$$\begin{aligned} L(z)=&\sum_{k=1}^{n}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}g_k(z,\tau)\Big[\gamma_{k-1}(\tau) \\ &-\sum_{l=0}^{m-n-1}\frac{(n+l)!}{(n+l-k+1)!}\tau^{n+l-k+1}\overline{\partial_z^{k-1}\widetilde{\varphi}_l(\tau)}\Big]\frac{\mathrm{d}\tau}{\tau} \\ &+z^n\sum_{l=0}^{m-n-1}z^l\overline{\widetilde{\varphi}_l(z)} \end{aligned} \tag{3.20}$$

are all solutions of the problem (3.15) if and only if

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{z\det A_l(\tau)}{\tau-z}\frac{\mathrm{d}\tau}{\tau}=0,\ z\in\mathbb{D},\ 0\le l\le m-n-1, \tag{3.21}$$

where $g_k(z,\tau)\,(1\le k\le n)$ are the former n higher order Poisson kernels.

Proof. By Theorem C, since $L \in M_{m,n}(\mathbb{D})\,(m > n)$, then

$$L(z) = 2\Re\Big\{\sum_{k=0}^{n-1} \overline{z}^k \varphi_k(z)\Big\} + 2i\Re\Big\{\sum_{k=0}^{n-1} \overline{z}^k \widehat{\varphi}_k(z)\Big\} + z^n \sum_{l=0}^{m-n-1} z^l \overline{\widetilde{\varphi}_l(z)}, \ \ z \in \mathbb{D},$$

where $\varphi_k, \widehat{\varphi}_k \in H^k_{1,0}(\mathbb{D})$ and $\widetilde{\varphi}_l \in H_1(\mathbb{D})$. So

$$(\partial_z^{n+k}\partial_{\overline{z}}^n)L(z) = \sum_{l=k}^{m-n-1} \frac{(n+l)!}{(l-k)!} z^{l-k}\overline{\partial_z^n \widetilde{\varphi}_l(z)}, \ \ z \in \mathbb{D}, \ 0 \le k \le m-n-1.$$

Note that from (3.15), by Lemma 9, it follows that $[\partial_z^j \widetilde{\varphi}_l]^+(t)$ exists for all $0 \le j \le n, 0 \le l \le m-n-1$, $t \in \partial\mathbb{D}$. Therefore,

$$\sum_{l=k}^{m-n-1} \frac{(n+l)!}{(l-k)!} t^{l-k}\overline{[\partial_z^n \widetilde{\varphi}_l]^+(t)} = \sigma_k(t), \ \ t \in \partial\mathbb{D}, \ 0 \le k \le m-n-1. \tag{3.22}$$

Set

$$X(t) = \begin{pmatrix} \overline{[\partial_z^n \widetilde{\varphi}_0]^+(t)} \\ \overline{[\partial_z^n \widetilde{\varphi}_1]^+(t)} \\ \vdots \\ \overline{[\partial_z^n \widetilde{\varphi}_{m-n-2}]^+(t)} \\ \overline{[\partial_z^n \widetilde{\varphi}_{m-n-1}]^+(t)} \end{pmatrix}, \quad a(t) = \begin{pmatrix} \sigma_0(t) \\ \sigma_1(t) \\ \vdots \\ \sigma_{m-n-2}(t) \\ \sigma_{m-n-1}(t) \end{pmatrix}$$

and

$$A(t) = \begin{pmatrix} n! & (n+1)!t & \cdots & \frac{(m-2)!}{(m-n-2)!}t^{m-n-2} & \frac{(m-1)!}{(m-n-1)!}t^{m-n-1} \\ 0 & (n+1)! & \cdots & \frac{(m-2)!}{(m-n-3)!}t^{m-n-3} & \frac{(m-1)!}{(m-n-2)!}t^{m-n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (m-2)! & (m-1)!t \\ 0 & 0 & \cdots & 0 & (m-1)! \end{pmatrix},$$

then (3.22) becomes

$$A(t)X(t) = a(t).$$

By Cramer rule, we get

$$[\partial_z^n \widetilde{\varphi}_l]^+(t) = \frac{\overline{\det(A_l(t))}}{n!(n+1)!\cdots(m-1)!}, \tag{3.23}$$

where the matrix $A_l(t)$ is given by replacing the lth column of $A(t)$ by $a(t)$, $0 \le l \le m-n-1$. Let

$$\Xi_l(z) = \frac{1}{n!(n+1)!\cdots(m-1)!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\overline{\det(A_l(\tau))}}{\tau - z}\mathrm{d}\tau,$$

then

$$\widetilde{\varphi}_l(z) = \int_0^z \int_0^{\zeta_{n-1}} \cdots \int_0^{\zeta_1} \Xi_l(\zeta)\mathrm{d}\zeta\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{n-1} + \pi_l(z),$$

where $\pi_l \in \Pi_{n-1}$, $0 \le l \le m-n-1$.

Let

$$\widetilde{L}(z) = z^n \sum_{l=0}^{m-n-1} z^l \overline{\widetilde{\varphi}_l(z)},$$

then $L - \widetilde{L} \in Har_n^{\mathbb{C}}(\mathbb{D})$ and

$$[(\partial_z\partial_{\overline{z}})^j(L-\widetilde{L})]^+(t) = \gamma_j(t) - \sum_{l=0}^{m-n-1} \frac{(n+l)!}{(n+l-j)!} t^{n+l-j}\overline{\partial_z^j\widetilde{\varphi}_l(t)},\ t \in \partial\mathbb{D},\ 0 \le j < n.$$

So, from the last section,

$$\begin{aligned} L(z) - \widetilde{L}(z) =& \sum_{k=1}^{n} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} g_k(z,\tau)\Big[\gamma_{k-1}(\tau) \\ &- \sum_{l=0}^{m-n-1} \frac{(n+l)!}{(n+l-k+1)!}\tau^{n+l-k+1}\overline{\partial_z^{k-1}\widetilde{\varphi}_l(\tau)}\Big]\frac{\mathrm{d}\tau}{\tau}, \end{aligned}$$

where $g_k(z,\tau)\,(1 \le k \le n)$ are the higher order Poisson kernels. Therefore,

$$\begin{aligned} L(z) =& \sum_{k=1}^{n} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} g_k(z,\tau)\Big[\gamma_{k-1}(\tau) \\ &- \sum_{l=0}^{m-n-1} \frac{(n+l)!}{(n+l-k+1)!}\tau^{n+l-k+1}\overline{\partial_z^{k-1}\widetilde{\varphi}_l(\tau)}\Big]\frac{\mathrm{d}\tau}{\tau} + \widetilde{L}(z). \end{aligned} \tag{3.24}$$

Note that by (3.23) and Theorem E, we know that (3.24) are all solutions of (3.15) if and only if

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{z\det A_l(\tau)}{\tau - z}\frac{\mathrm{d}\tau}{\tau} = 0,\ z \in \mathbb{D},\ 0 \le l \le m-n-1. \qquad \square$$

The second Dirichlet problem is to find a function $R(z) \in M_{m,n}(\mathbb{D})$ $(m < n)$ satisfying the boundary conditions

$$[(\partial_z \partial_{\overline{z}})^j R]^+(t) = \rho_j(t),\ 0 \le j < m \quad \text{and} \quad [\partial_z^m \partial_{\overline{z}}^{m+k} R]^+(t) = \varrho_k(t),\ 0 \le k < n-m, \tag{3.25}$$

where $t \in \partial\mathbb{D}$, $\rho_j, \varrho_k \in C(\partial\mathbb{D})$ for $0 \le j < m,\ 0 \le k < n-m$.

Similarly, by the harmonic decomposition theorem, we have

Theorem 12. *Set*

$$A'(t) = \begin{pmatrix} m! & (m+1)!\overline{t} & \cdots & \frac{(n-2)!}{(n-m-2)!}\overline{t}^{n-m-2} & \frac{(n-1)!}{(n-m-1)!}\overline{t}^{n-m-1} \\ 0 & (m+1)! & \cdots & \frac{(n-2)!}{(n-m-3)!}\overline{t}^{n-m-3} & \frac{(n-1)!}{(n-m-2)!}\overline{t}^{n-m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (n-2)! & (n-1)!\overline{t} \\ 0 & 0 & \cdots & 0 & (n-1)! \end{pmatrix}, \tag{3.26}$$

$$a'(t) = \begin{pmatrix} \varrho_0(t) \\ \varrho_1(t) \\ \vdots \\ \varrho_{m-n-2}(t) \\ \varrho_{m-n-1}(t) \end{pmatrix}, \tag{3.27}$$

$$\Xi'_l(z) = \frac{1}{m!(m+1)!\cdots(n-1)!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\det(A'_l(\tau))}{\tau - z}\mathrm{d}\tau, \tag{3.28}$$

and

$$\widetilde{\psi}_l(z) = \int_0^z \int_0^{\zeta_{m-1}} \cdots \int_0^{\zeta_1} \Xi'_l(\zeta)\mathrm{d}\zeta\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{m-1} + \pi'_l(z), \tag{3.29}$$

where $t \in \partial\mathbb{D}$, $\pi'_l \in \Pi_{m-1}$, the matrix $A'_l(t)$ is given by replacing the lth column of $A'(t)$ by $a'(t)$, $0 \le l \le n-m-1$. Then

$$\begin{aligned} R(z) =& \sum_{k=1}^{m} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} g_k(z,\tau)\Big[\rho_{k-1}(\tau) \\ & - \sum_{l=0}^{n-m-1} \frac{(m+l)!}{(m+l-k+1)!}\overline{\tau}^{m+l-k+1}\partial_z^{k-1}\widetilde{\psi}_l(\tau)\Big]\frac{\mathrm{d}\tau}{\tau} \\ & + \overline{z}^m \sum_{l=0}^{n-m-1} \overline{z}^l \widetilde{\psi}_l(z) \end{aligned} \tag{3.30}$$

are all solutions of the problem (3.25) if and only if

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{z\,\overline{\det A_l'(\tau)}}{\tau - z}\,\frac{\mathrm{d}\tau}{\tau} = 0,\ \ z\in\mathbb{D},\ 0\le l\le n-m-1, \tag{3.31}$$

where $g_k(z,\tau)\,(1\le k\le m)$ *are the former* m *higher order Poisson kernels.*

Proof. By Theorem C, since $R\in M_{m,n}(\mathbb{D})\,(m<n)$, then

$$R(z) = 2\Re\Big\{\sum_{k=0}^{m-1}\overline{z}^k\psi_k(z)\Big\} + 2i\Re\Big\{\sum_{k=0}^{m-1}\overline{z}^k\widehat{\psi}_k(z)\Big\} + \overline{z}^m\sum_{l=0}^{n-m-1}\overline{z}^l\widetilde{\psi}_l(z),\ \ z\in\mathbb{D},$$

where $\psi_k,\widehat{\psi}_k\in H^k_{1,0}(\mathbb{D})$ and $\widetilde{\psi}_l\in H_1(\mathbb{D})$. So

$$(\partial_z^m\partial_{\overline{z}}^{m+k})R(z) = \sum_{l=k}^{n-m-1}\frac{(m+l)!}{(l-k)!}\overline{z}^{l-k}\partial_z^m\widetilde{\psi}_l(z),\ \ z\in\mathbb{D},\ 0\le k\le n-m-1.$$

Note that from (3.25), by Lemma 9, it follows that $[\partial_z^j\widetilde{\psi}_l]^+(t)$ exists for all $0\le j\le m, 0\le l\le n-m-1$, $t\in\partial\mathbb{D}$. Therefore,

$$\sum_{l=k}^{n-m-1}\frac{(m+l)!}{(l-k)!}\overline{t}^{l-k}[\partial_z^m\widetilde{\psi}_l]^+(t) = \varrho_k(t),\ \ t\in\partial\mathbb{D},\ 0\le k\le n-m-1. \tag{3.32}$$

Set

$$X'(t) = \begin{pmatrix} [\partial_z^m\widetilde{\psi}_0]^+(t)\\ [\partial_z^m\widetilde{\psi}_1]^+(t)\\ \vdots\\ [\partial_z^m\widetilde{\psi}_{n-m-2}]^+(t)\\ [\partial_z^m\widetilde{\psi}_{n-m-1}]^+(t)\end{pmatrix},\quad a'(t) = \begin{pmatrix}\varrho_0(t)\\ \varrho_1(t)\\ \vdots\\ \varrho_{n-m-2}(t)\\ \varrho_{n-m-1}(t)\end{pmatrix}$$

and

$$A'(t) = \begin{pmatrix} m! & (m+1)!\overline{t} & \cdots & \frac{(n-2)!}{(n-m-2)!}\overline{t}^{n-m-2} & \frac{(n-1)!}{(n-m-1)!}\overline{t}^{n-m-1}\\ 0 & (m+1)! & \cdots & \frac{(n-2)!}{(n-m-3)!}\overline{t}^{n-m-3} & \frac{(n-1)!}{(n-m-2)!}\overline{t}^{n-m-2}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & (n-2)! & (n-1)!\overline{t}\\ 0 & 0 & \cdots & 0 & (n-1)!\end{pmatrix},$$

then (3.32) becomes

$$A'(t)X'(t) = a'(t).$$

By Cramer rule, we get

$$[\partial_z^m \widetilde{\psi}_l]^+(t) = \frac{\det(A_l'(t))}{m!(m+1)!\cdots(n-1)!}, \tag{3.33}$$

where the matrix $A_l'(t)$ is given by replacing the lth column of $A'(t)$ by $a'(t)$, $0 \le l \le n-m-1$. Let

$$\Xi_l'(z) = \frac{1}{m!(m+1)!\cdots(n-1)!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\det(A_l'(\tau))}{\tau - z}\mathrm{d}\tau,$$

then

$$\widetilde{\psi}_l(z) = \int_0^z \int_0^{\zeta_{m-1}} \cdots \int_0^{\zeta_1} \Xi_l'(\zeta)\mathrm{d}\zeta\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{m-1} + \pi_l'(z),$$

where $\pi_l' \in \Pi_{m-1}$, $0 \le l \le n-m-1$.

Let

$$\widetilde{R}(z) = \overline{z}^m \sum_{l=0}^{n-m-1} \overline{z}^l \widetilde{\psi}_l(z),$$

then $R - \widetilde{R} \in Har_m^{\mathbb{C}}(\mathbb{D})$ and

$$[(\partial_z\partial_{\overline{z}})^j(R-\widetilde{R})]^+(t) = \rho_j(t) - \sum_{l=0}^{n-m-1} \frac{(m+l)!}{(m+l-j)!}\overline{t}^{m+l-j}\partial_z^j\widetilde{\psi}_l(t),\ t \in \partial\mathbb{D},\ 0 \le j < m.$$

So, from the last section,

$$\begin{aligned} R(z) - \widetilde{R}(z) = &\sum_{k=1}^m \frac{1}{2\pi i}\int_{\partial\mathbb{D}} g_k(z,\tau)\Big[\gamma_{k-1}(\tau) \\ &- \sum_{l=0}^{n-m-1} \frac{(m+l)!}{(m+l-k+1)!}\overline{\tau}^{m+l-k+1}\partial_z^{k-1}\widetilde{\psi}_l(\tau)\Big]\frac{\mathrm{d}\tau}{\tau}, \end{aligned}$$

where $g_k(z,\tau)\,(1 \le k \le m)$ are the higher order Poisson kernels. Therefore,

$$\begin{aligned} R(z) = &\sum_{k=1}^m \frac{1}{2\pi i}\int_{\partial\mathbb{D}} g_k(z,\tau)\Big[\rho_{k-1}(\tau) \\ &- \sum_{l=0}^{n-m-1} \frac{(m+l)!}{(m+l-k+1)!}\overline{\tau}^{m+l-k+1}\partial_z^{k-1}\widetilde{\psi}_l(\tau)\Big]\frac{\mathrm{d}\tau}{\tau} + \widetilde{R}(z). \end{aligned} \tag{3.34}$$

Note that by (3.33) and Theorem E, we know that (3.34) are all solutions of (3.25) if and only if

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{z\,\overline{\det A_l'(\tau)}}{\tau - z}\frac{\mathrm{d}\tau}{\tau} = 0,\ z \in \mathbb{D},\ 0 \le l \le n-m-1. \qquad \square$$

The third Dirichlet problem is to find a function $N(z) \in M_{m,n}(\mathbb{D})$ which fulfills the boundary conditions

$$[(\partial_z^m \partial_{\bar{z}}^j)N]^+(t) = \chi_j(t),\ 0 \le j < n \ \text{ and } \ [\partial_z^k \partial_{\bar{z}}^n N]^+(t) = \lambda_k(t),\ 0 \le k < m, \tag{3.35}$$

where $t \in \partial\mathbb{D}$, $\chi_j, \lambda_k \in C(\partial\mathbb{D})$ for $0 \le j < n,\ 0 \le k < m$.

By the canonical decomposition theorem, we have

Theorem 13. *Set*

$$B(t) = \begin{pmatrix} 1 & \bar{t} & \bar{t}^2 & \cdots & \bar{t}^{n-1} \\ 0 & 1 & 2\bar{t} & \cdots & (n-1)\bar{t}^{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (n-2)! & (n-1)!\bar{t} \\ 0 & 0 & \cdots & 0 & (n-1)! \end{pmatrix}, \tag{3.36}$$

$$C(t) = \begin{pmatrix} 1 & t & t^2 & \cdots & t^{m-1} \\ 0 & 1 & 2t & \cdots & (m-1)t^{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (m-2)! & (m-1)!t \\ 0 & 0 & \cdots & 0 & (m-1)! \end{pmatrix}, \tag{3.37}$$

$$b(t) = \begin{pmatrix} \chi_0(t) \\ \chi_1(t) \\ \vdots \\ \chi_{n-1}(t) \end{pmatrix}, \quad c(t) = \begin{pmatrix} \lambda_0(t) \\ \lambda_1(t) \\ \vdots \\ \lambda_{m-1}(t) \end{pmatrix} \tag{3.38}$$

and

$$\Theta_p(z) = \frac{1}{1!2!\cdots(n-1)!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\det B_p(\tau)}{\tau - z}\,\mathrm{d}\tau, \tag{3.39}$$

$$\Lambda_q(z) = \frac{1}{1!2!\cdots(m-1)!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\overline{\det C_q(\tau)}}{\tau - z}\,\mathrm{d}\tau, \tag{3.40}$$

as well as

$$\mu_p(z) = \int_0^z \int_0^{\zeta_{m-1}} \cdots \int_0^{\zeta_1} \Theta_p(\zeta)\mathrm{d}\zeta\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{m-1} + \kappa_p(z), \tag{3.41}$$

$$\nu_q(z) = \int_0^z \int_0^{\zeta_{n-1}} \cdots \int_0^{\zeta_1} \Lambda_q(\zeta)\mathrm{d}\zeta\mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_{n-1} + \xi_q(z), \tag{3.42}$$

where $t \in \partial\mathbb{D}$, $\kappa_p \in \Pi_{m-1}$, $\xi_q \in \Pi_{n-1}$, *the matrices* $B_p(t), C_q(t)$ *are respectively given by replacing the pth, qth column by* $b(t)$, $c(t)$, $0 \le p \le n-1$, $0 \le q \le m-1$. *Then*

$$N(z) = \sum_{p=0}^{n-1} \overline{z}^p \mu_p(z) + \sum_{q=0}^{m-1} z^q \overline{\nu_q(z)} \tag{3.43}$$

are all solutions of the problem (3.35) if and only if

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{z\overline{\det B_p(\tau)}}{\tau - z} \frac{\mathrm{d}\tau}{\tau} = 0, \ 0 \le p \le n-1 \tag{3.44}$$

and

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{z \det C_q(\tau)}{\tau - z} \frac{\mathrm{d}\tau}{\tau} = 0, \ 0 \le q \le m-1, \tag{3.45}$$

in which $z \in \mathbb{D}$.

Proof. By Theorem D, we have the canonical decomposition

$$N(z) = \sum_{p=0}^{n-1} \overline{z}^p \mu_p(z) + \sum_{q=0}^{m-1} z^q \overline{\nu_q(z)},$$

where $\mu_p, \nu_q \in H_1(\mathbb{D})$, $0 \le p < n, \ 0 \le q < m$. Note that by (3.35), we have

$$\sum_{p=j}^{n-1} \frac{p!}{(p-j)!} \overline{t}^{p-j} [\partial_z^m \mu_p]^+(t) = \chi_j(t), \ 0 \le j < n \tag{3.46}$$

and

$$\sum_{q=k}^{m-1} \frac{q!}{(q-k)!} t^{q-k} \overline{[\partial_z^n \nu_q]^+(t)} = \lambda_k(t), \ 0 \le k < m. \tag{3.47}$$

Set

$$b(t) = \begin{pmatrix} \chi_0(t) \\ \chi_1(t) \\ \vdots \\ \chi_{n-1}(t) \end{pmatrix}, \quad c(t) = \begin{pmatrix} \lambda_0(t) \\ \lambda_1(t) \\ \vdots \\ \lambda_{m-1}(t) \end{pmatrix}$$

and

$$Y(t) = \begin{pmatrix} [\partial_z^m \mu_0]^+(t) \\ [\partial_z^m \mu_1]^+(t) \\ \vdots \\ [\partial_z^m \mu_{n-1}]^+(t) \end{pmatrix}, \quad Z(t) = \begin{pmatrix} \overline{[\partial_z^n \nu_0]^+(t)} \\ \overline{[\partial_z^n \nu_1]^+(t)} \\ \vdots \\ \overline{[\partial_z^n \nu_{m-1}]^+(t)} \end{pmatrix}$$

as well as

$$B(t) = \begin{pmatrix} 1 & \bar{t} & \bar{t}^2 & \cdots & \bar{t}^{n-1} \\ 0 & 1 & 2\bar{t} & \cdots & (n-1)\bar{t}^{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (n-2)! & (n-1)!\bar{t} \\ 0 & 0 & \cdots & 0 & (n-1)! \end{pmatrix}$$

and

$$C(t) = \begin{pmatrix} 1 & t & t^2 & \cdots & t^{m-1} \\ 0 & 1 & 2t & \cdots & (m-1)t^{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (m-2)! & (m-1)!t \\ 0 & 0 & \cdots & 0 & (m-1)! \end{pmatrix},$$

then (3.46) and (3.47) become

$$B(t)Y(t) = b(t), \quad C(t)Z(t) = c(t).$$

So

$$[\partial_z^m \mu_p]^+(t) = \frac{\det B_p(t)}{1!2!\cdots(n-1)!}, \quad [\partial_z^n \nu_q]^+(t) = \frac{\overline{\det C_q(t)}}{1!2!\cdots(m-1)!}, \tag{3.48}$$

where $B_p(t), C_q(t)$ have the same meanings as $A_l(t)$ in (3.23).

Let

$$\Theta_p(z) = \frac{1}{1!2!\cdots(n-1)!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\det B_p(\tau)}{\tau - z}\frac{\mathrm{d}\tau}{\tau}$$

and

$$\Lambda_q(z) = \frac{1}{1!2!\cdots(m-1)!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\overline{\det C_q(\tau)}}{\tau - z}\frac{\mathrm{d}\tau}{\tau},$$

then

$$\mu_p(z) = \int_0^z \int_0^{\zeta_{m-1}} \cdots \int_0^{\zeta_1} \Theta_p(\zeta)\mathrm{d}\zeta\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{m-1} + \kappa_p(z), \tag{3.49}$$

$$\nu_q(z) = \int_0^z \int_0^{\zeta_{n-1}} \cdots \int_0^{\zeta_1} \Lambda_q(\zeta)\mathrm{d}\zeta\mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_{n-1} + \xi_q(z), \tag{3.50}$$

where $\kappa_p \in \Pi_{m-1}$, $\xi_q \in \Pi_{n-1}$. Note that by (3.48) and Theorem E, substituting (3.49) and (3.50) into (3.43), we get all solutions (3.43) of the boundary value problem (3.35) if and only if

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{z\overline{\det B_p(\tau)}}{\tau - z}\frac{\mathrm{d}\tau}{\tau} = 0,\ 0 \le p \le n-1$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{z\det C_q(\tau)}{\tau - z}\frac{\mathrm{d}\tau}{\tau} = 0,\ 0 \le q \le m-1,$$

in which $z \in \mathbb{D}$. $\square$