Chapter 1

Introduction

In the classical theory of complex analysis, it is well known that harmonic functions are intimately connected with analytic functions. That is, for any real harmonic function, one can find an analytic function such that the harmonic function becomes its real part. In other words, any real harmonic function can be decomposed as a sum of an analytic function and its conjugate function which is an antianalytic function. The idea is simple but subtle and important because it constructs a bridge linking the two kinds of functions so that they can be mutually applied. In fact, the mutual applications are successfully realized in the classical theory of one complex variable. In this dissertation, one can find that the idea is valid for the generalized analogues of harmonic functions which are called polyharmonic functions. Of course, analytic functions should also be generalized. It is fortunate that some generalized analogues for analytic functions have already been introduced by contribution from many mathematicians (see [3, 29] and references there).

Usually, analytic functions are defined by Cauchy-Riemann operator $\partial_{\bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ and harmonic functions are defined by Laplace operator $\Delta = 4\partial_z\partial_{\bar{z}}$, where $\partial_z = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ is the adjoint operator of the Cauchy-Riemann operator [27]. Different generalizations for Cauchy-Riemann operator yielded many generalized analogues such as generalized analytic functions, polyanalytic functions and metaanalytic functions etc. [3, 29]. Especially, polyanalytic functions are defined by operators $\partial_{\bar{z}}^n$ ($n \geq 2$). By iterating the Laplace operator, one can define the so-called polyharmonic functions by operators Δ^n ($n \geq 2$) [1, 3]. The simplest polyharmonic functions are biharmonic functions which are defined as $\Delta^2 u(= \Delta\Delta u) = 0$ in some domain. Historically, many investigations for the extension of harmonic functions are about biharmonic functions. Of course, there

are also a lot of works on n-analytic and on n-harmonic functions. We mainly refer readers to two of them: the preeminent work [23] given by Goursat and Vekua's excellent paper [30] about the Dirichlet problem for biharmonic functions, which only expresses the author's interest. In [23], Goursat obtained his decomposition theorem of biharmonic functions which indicates that the idea stated in the beginning of this chapter is valid for biharmonic functions. Using Goursat's decomposition formula, in [30], Vekua developed one method to construct an approximative solution of the biharmonic Dirichlet problem in a simply connected domain with a simple closed Jordan curve satisfying the Ljapunov condition as its boundary.

Since complex analysis is closely related to mathematical physics, the theory of boundary value problems (simply, BVPs) in complex analysis were abundantly developed. Especially, the theory of boundary value problems for analytic functions is an important branch of function theory. Many mathematicians contributed to this field such as B. Riemann, D. Hilbert, N. I. Muskhelishvili, F. D. Gakhov, I. N. Vekua and their students. The initial investigations are due to B. Riemann and D. Hilbert. Deep developments were given by the BVPs school of the former Soviet Union. Except for analytic functions, the investigations were also devoted to particular partial differential equations, for example, the Bitsadze equation, elliptic partial differential equations with analytic coefficients and so on. There are many different types of BVPs which are called Riemann, Hilbert, Dirichlet, Schwarz, Neumann, Robin boundary value problems. Among them, the Riemann boundary value problem and the Hilbert boundary value problem are in the center of interest. The Dirichlet boundary value problem is connected to the Riemann boundary value problem. In this dissertation, we mainly are concerned with the Dirichlet boundary value problem. The Schwarz problem is the simplest form of the Hilbert problem. The Neumann problem is related the Dirichlet problem. The Robin problem is contacted to the Dirichlet problem and the Neumann problem. In addition, for some special cases, e.g. the unit disc or the half plane, the Hilbert problem can be transformed to the Riemann problem. To extend the classical theory of BVPs, in recent time, a large number of investigations on various boundary value problems for polyanalytic functions, metaanalytic func-

tions have widely been published, refer to papers [12, 17, 18, 20, 31, 32] and references there. However, the investigations on Dirichlet problems for polyharmonic functions just appeared in recent two years [9, 13]. All of these works are based on two kinds of methods: one is called iterating method by making use of the so-called poly-Cauchy operator [12, 17], the other is called reflection method in terms of Schwarz symmetric extension principle and decomposition theorems for polyanalytic functions and polyharmonic functions due to Begehr, Du and Wang [9, 17]. In [17], Du and Wang established a beautiful decomposition theorem for polyanalytic functions such that BVPs for polyanalytic functions can be easily transformed to BVPs for analytic functions while the theory of the latter is completely developed [22, 25, 26]. Further, in [9], Begehr, Du and Wang also obtained a decomposition theorem for polyharmonic functions by the decomposition theorem for polyanalytic functions. In deed, these decomposition theorems have appeared in the book [3] of Balk in some implicit forms. Just using the decomposition theorem, in [9], Begehr, Du and Wang studied the Dirichlet problem for polyharmonic functions in the unit disc by the reflection method. They found that the problem is uniquely solvable and the solution is closely connected with a sequence of kernel functions with some elegant properties. However, explicit expressions for all kernel functions are not yet attained although the kernel functions exist and satisfy certain inductive relations.

In Chapter 2, we develop a new decomposition theorem for polyharmonic functions which is an extension of Goursat decomposition theorem for biharmonic functions. With a view to the usual decomposition for harmonic functions, our decomposition theorem for polyharmonic functions is more natural than the one established by Begehr, Du and Wang. By our decomposition theorem, we give a unified expression for the kernel functions appearing in [9] which are expressed in terms of some vertical sums with nice structure. Since they are polyharmonic analogues of the classical Poisson kernel, we call them higher order Poisson kernels. The higher order Poisson kernels play an important role to solve the Dirichlet problem for polyharmonic functions (simply, PHD problem).

The main subject of this dissertation is to study some Dirichlet boundary value problems for higher order complex partial differential equations in the unit

disc. It is contained in Chapter 3 and Chapter 4.

For homogeneous equations, we begin with review the Dirichlet problem for analytic functions. Then we consider the PHD problem in the unit disc and finally discuss three kinds of Dirichlet problems for homogeneous mixed-partial differential equations. The key tools are the decompositions for polyanalytic, polyharmonic as well as poly-analytic-harmonic functions which are given in Chapter 2. These are the themes of Chapter 3. For inhomogeneous equations, the corresponding problems have only been little investigated [14, 24], and it is the purpose of Chapter 4 to obtain some results in this direction. In [14], since the explicit expressions of the higher order Poisson kernels are unknown, in fact, Begehr and Wang only solved a Dirichlet problem for inhomogeneous triharmonic equations in the unit disc although the general solution for the polyharmonic case is indicated by a final remark. In [24], Kumar and Prakash consider the same equations appearing in this dissertation with different boundary value conditions using another method. In Chapter 4, we first apply the differentiability of the higher order Pompeiu operators introduced by Begehr and Hile [12] to get special solutions for the inhomogeneous equations. Further, we use the known results of Dirichlet problems for the homogeneous equations [9, 19] due to the higher order Poisson kernels [19] which are obtained by the decompositions of polyanalytic functions and polyharmonic functions, and the continuity of the higher order Pompeiu operators. Combining the special solutions and the homogeneous solutions, we obtain the solutions of Dirichlet problems for the inhomogeneous equations under some suitable conditions of solvability. It is a new view to solve Dirichlet problems for inhomogeneous equations which is different from the usual method depending on the higher order Green functions [7] whose explicit expressions are unknown except for some lower orders up to now. It is more interesting that the view appearing here to solve a Dirichlet problem for inhomogeneous higher order complex partial differential equations is similar to the one usually used in linear algebra to solve an inhomogeneous system of linear equations.

In what follows, the main analytic branch of $\log z$ is always chosen in the complex plane cut along the negative real axis with $\log 1 = 0$.