Chapter 10

Weak convergence

In this appendix, we will present a very convenient way to compute the adiabatic limit. It is based on the concept of weak convergence. The structure behind this concept is best understood by some theorems given in [13]. We will follow the very instructive introduction of [13] and cite some of the theorems given therein and of [121]. This empowers us to present the technique in application to our two examples. The omitted proofs can be found in [13] and [121].

Let us consider a sequences $\{x_\epsilon\}$ of functions which are indexed by a sequence $\{\epsilon\}$ of real numbers. We are interested in the case that the latter converges to zero, $\epsilon \to 0$. We assume that all functions x_ϵ are defined on some bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$. Let us furthermore denote any partial derivative by $\partial_j x, j = 1, \ldots, d$.

Definition 10.1 (Definition 1 in [13]) *A sequence* $\{x_\epsilon\}$ *of* $L^\infty(\Omega)$ converges weakly* *to the limit* $x_0 \in L^\infty(\Omega)$*, denoted as* $x_\epsilon \overset{*}{\rightharpoonup} x_0$*, if and only if*

$$\int_\Omega x_\epsilon(t)\phi(t)dt \to \int_\Omega x_0(t)\phi(t)dt \quad as \quad \epsilon \to 0$$

for all test functions $\phi \in L^1(\Omega)$

This definition is based on the fact that $L^\infty(\Omega)$ is isometrically isomorphic to the dual space of $L^1(\Omega)$ (see [96, Thm. 6.16]):

$$L^\infty(\Omega) = \left(L^1(\Omega)\right)^*.$$

Weak* convergence is closely connected to an averaging of the rapid fluctuations of x_ϵ. This is particularly expressed by the following theorem

Theorem 10.2 (Lemma 1 in [13]) *A sequence* $\{x_\epsilon\}$ *of* $L^\infty(\Omega)$ *converges weakly* to a limit* $x_0 \in L^\infty(\Omega)$*, if and only if the following two properties hold:*

1. *the sequence is bounded in* $L^\infty(\Omega)$*,*
2. *for every open rectangle* $I \subset \Omega$ *the corresponding integral mean value converges,*

$$\frac{1}{|I|}\int_I x_\epsilon(t)dt \to \frac{1}{|I|}\int_I x_0(t)dt.$$

The first theorem we state connects the uniform convergence of functions to the weak* convergence of their derivatives.

Theorem 10.3 (Principle 1 in [13]) *Let* $\{x_\epsilon\}$ *be a sequence in* $C^1(\bar{\Omega})$ *such that* $x_\epsilon \to 0$ *in* $C(\bar{\Omega})$*. Then, if and only if the sequence* $\{\partial x_\epsilon\}$ *is bounded in* $L^\infty(\Omega)$*, there holds*

$$\partial x_\epsilon \overset{*}{\rightharpoonup} 0 \quad in \quad L^\infty(\Omega).$$

One of the crucial differences between uniform convergence and weak* convergence lies in the fact that nonlinear functionals are not weakly* sequentially continuous. That means in general

$$x_\epsilon \overset{*}{\rightharpoonup} x_0 \qquad \not\Rightarrow \qquad f(x_\epsilon) \overset{*}{\rightharpoonup} f(x_0)$$

for a continuous nonlinear function f. Nevertheless, one obtains a convergence result for the product of a weak* converging and a uniformly converging function.

Theorem 10.4 (PRINCIPLE 2 IN [13]) *Let there be the convergences $x_\epsilon \overset{*}{\rightharpoonup} x_0$, weakly* in $L^\infty(\Omega)$, and $y_\epsilon \to y_0$, uniformly in $C(\bar{\Omega})$. Then, we obtain*

$$x_\epsilon \cdot y_\epsilon \overset{*}{\rightharpoonup} x_0 \cdot y_0 \quad \text{in} \quad L^\infty(\Omega).$$

Now, *Alaoglu's theorem* which states that a closed ball in $L^\infty(\Omega)$ is compact with respect to the weak*–topology.

Theorem 10.5 (PRINCIPLE 3 IN [13]) *Let $\{x_\epsilon\}$ be a bounded sequence in the space $L^\infty(\Omega)$. Then, there is a subsequence $\{\epsilon'\}$ and a function $x_0 \in L^\infty(\Omega)$, such that*

$$x_{\epsilon'} \overset{*}{\rightharpoonup} x_0 \quad \text{in} \quad L^\infty(\Omega).$$

An upper bound for the weak* limit $x_0 \in L^\infty(\Omega)$ is given in next theorem.

Theorem 10.6 (CF. THM. V.1.9 IN [121] OR THM. 1.1 IN [30]) *Let $x_\epsilon \overset{*}{\rightharpoonup} x_0$ for $\epsilon \to 0$ in $L^\infty(\Omega)$. Then the following assertion holds*

$$\|x_0\| \le \liminf_{\epsilon \to 0} \|x_\epsilon\|.$$

Furthermore, by applying the *extended Arzelà-Ascoli theorem* we derive some convergence properties of x as well as its derivative.

Theorem 10.7 (PRINCIPLE 4 IN [13]) *Let $\{x_\epsilon\}$ be a bounded sequence in the space $C^{0,1}(\bar{\Omega})$ of uniformly Lipschitz continuous functions. Then, there is a subsequence $\{\epsilon'\}$ and a function $x_0 \in C^{0,1}(\bar{\Omega})$, such that*

$$x_{\epsilon'} \to x_0 \quad \text{in} \quad C(\bar{\Omega}), \qquad \partial x_{\epsilon'} \overset{*}{\rightharpoonup} \partial x_0 \quad \text{in} \quad L^\infty(\Omega).$$

The partial derivatives ∂x_ϵ and ∂x_0 are classically defined almost everywhere.

A criterium for the overall convergence of a sequence when all convergence subsequences converge to the same element is given in the next theorem.

Theorem 10.8 (PRINCIPLE 5 IN [13]) *Let $\{x_\epsilon\}$ be a sequence in a sequentially compact Hausdorff space $\mathfrak{H}$. If every convergent subsequence of $\{x_\epsilon\}$ converges to one and the same element $x_0 \in \mathfrak{H}$, then the sequence converges itself,*

$$x_\epsilon \to x_0.$$