Kac–Moody Eisenstein series in string theory

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Zusammenfassung

Seit jeher ist es ein zentrales Problem der Physik, die Natur auf ihrer kleinsten physikalischen Längenskala zu verstehen und zu beschreiben. In den letzten fünfzig Jahren war die theoretische Arbeit an diesem Problem hauptsächlich motiviert durch die Zielsetzung, die Allgemeine Relativitätstheorie, welche die Gravitation und die Dynamik der Raumzeit beschreibt, mit der Theorie der Quantenmechanik, welche die Physik auf kleinen (subatomaren) Längenskalen dominiert, miteinander in Einklang zu bringen in einer einzigen Theorie, deren allgemein gebräuchliche Bezeichnung Quantengravitation ist. Die String Theorie, die auf der grundlegenden Annahme basiert, dass die kleinsten Bausteine der Natur nicht durch Punktteilchen, sondern durch eindimensionale 'Fäden', die Strings, gegeben sind, deren Länge der Größenordnung der Plancklänge entspricht, ist ein mögliches Erklärungsmodell. Weiterhin, so der Gedanke, wird das in der Natur beobachtete Teilchenspektrum durch unterschiedliche Schwingungszustände der Strings erzeugt. Ausgehend von dieser Annahme, stellt die String Theorie nicht nur eine Theorie der Quantengravitation dar, sondern beschreibt darüber hinaus auch alle anderen fundamentalen Wechselwirkungen der Natur.

Seit ihrer ersten Beschreibung, hat sich die String Theorie zu einem komplexen Netz verschiedener Theorien entwickelt, die jedoch alle verschiedene Aspekte einer größeren, übergeordneten Theorie zu sein scheinen. Ein wichtiger Ansatz zum Verständnis dieser komplexen mathematischen Struktur ist die Rolle von Symmetrien. Diese Symmetrien, auch Dualitäten genannt, manifestieren sich zum Beispiel in speziellen mathematischen Funktionen, die in Amplituden von Streuprozessen von Strings auftauchen. Ein relevantes Beispiel für solch eine Funktion sind die Eisensteinreihen, welche eine Invarianz unter diskreten Dualitätsgruppen aufweisen. Das zentrale Ziel der vorliegenden Arbeit ist es, die Eigenschaften von Eisensteinreihen zu untersuchen, die unter sehr großen, insbesondere unendlich-dimensionalen Kac–Moody Gruppen invariant sind. Der Großteil dieser Dissertation ist dem mathematischen Problem der Fourierentwicklung von Eisensteinreihen gewidmet, jedoch werden die erzielten Resultate auch in dem relevanten physikalischen Kontext dargestellt.

Abstract

Understanding nature on its very smallest 'physical-length' scale has always been a central goal of physics. Theoretical investigations into this problem over the last fifty years or so were largely driven by the aim of reconciling the theory of general relativity, the theory which describes the fundamental force of gravity and therefore the dynamics of space-time, with the theory of quantum mechanics, which dominates the physical phenomena on very small (sub-atomic) scales, within one big framework, referred to as the theory of quantum gravity. One candidate for such a theory is *string theory*. The fundamental assumption of this theory is that the smallest constituents of nature are not given by point particles, but rather by one dimensional strings the size of the Planck length. Through their different vibrational modes, strings are thought to produce the different properties of the observed spectrum of particles in nature. With this basic idea, string theory is not only predicted to describe the gravitational force, but also all other known forces of nature, and therefore extends far beyond the concept of only being a theory of quantised gravity.

Since its initial proposal, the theory has developed into a vast and complex mathematical web of different theories, which all seem to be part of a larger, all-encompassing theory. Key to understanding the complicated mathematical structure of this theory is the concept of *symmetries*. Such symmetries, which are also known as duality relations, for instance manifest themselves in special mathematical functions, contained in the amplitudes that capture information about the interaction processes of strings with one another. A particularly relevant example of such a function is given by the so-called *Eisenstein series*, which display invariance under certain discrete duality groups. The central goal of this thesis is to study the properties of Eisenstein series invariant under special, particularly large (in fact infinite-dimensional) symmetry groups, known as Kac–Moody groups. While a large part of this thesis is dedicated to the mathematical problem of calculating the *Fourier expansion* of these series, our results are also explained within the relevant physical context.

Contents

Chapter 1

From string theory to automorphic forms

The objective of this thesis is to provide a discussion of *Eisenstein series* and the role that they play within string theory. Eisenstein series are examples of a special type of mathematical function. More precisely, they fall into the class of automorphic forms, which display invariance under certain symmetry transformations. This makes them interesting objects to study, particularly from a mathematical point of view. In our case, the Eisenstein series which we are going to consider are invariant under a set of symmetries, which appear naturally in string theory. Therefore, by studying these Eisenstein series, we can also hope to learn something new about string theory and in particular gain new insights into the non-perturbative sector of the theory. As we will show, the work presented in this thesis provides an important step towards achieving that goal.

This first chapter provides the reader with a brief introduction to some general concepts and terminology used throughout this thesis. These concepts include: quantum gravity and in particular string theory; symmetries of string theory; as well as some familiarity with the definition of Eisenstein series and automorphic forms. This will give sufficient motivation and set the stage for the concrete questions studied in the remaining chapters of this thesis. We will close the first chapter with section 1.3, where we clearly outline the goals and objectives that we set out to achieve, providing a detailed list of our original work and giving a summary on the structure of this thesis.

1.1 From quantum gravity to string theory

We want to start this thesis by giving a motivation for the physical problem that the work presented here is related to. This is the problem of finding a theory of *quantum gravity*, which combines the theory of quantum mechanics with the theory of general relativity. Our exposition loosely follows parts of the book [1] and the article [2] by C. Kiefer. The book [1] provides a general introduction to the field of quantum gravity with its many different approaches.

In our present understanding a theory of quantum gravity is necessary in situations where extremely large amounts of mass or energy are concentrated on a very short length scale. This is for example the case when a space-time singularity occurs, for instance inside a black hole or at the Big Bang, the presumed initial singularity of the universe. The assumption that such singularities indeed occur is supported by the singularity theorems of S. Hawking and R. Penrose [3], which state that under rather general assumptions, space-time will develop singularities, in particular for the universe as a whole, the Big Bang as the initial singularity and possibly a Big Crunch after a re-collapse of the universe. Assuming the existence of an initial singularity, a natural question to ask is how the initial conditions at this point in space-time are determined. One would expect that a theory of quantum gravity would provide an answer to this question.

There are also a number of reasons why, in the physical regimes discussed above, one should expect that the gravitational field is quantised. The most obvious reason is provided by the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.1.1}$$

themselves. In these equations the left-hand side describes the geometry of space-time, and therefore gravity, while the energy-momentum tensor $T_{\mu\nu}$

on the right-hand side of the equation includes all matter or energy present in the space-time. Since all known forms of matter and their associated fundamental forces (electromagnetism, the weak and the strong force) have been quantised successfully, it seems only reasonable that the left-hand side of the equation, associated with gravity as the fourth fundamental force, should be quantised. Further reasons for the quantisation of gravity are provided by: 1. The breakdown of the semiclassical description of black hole radiation [4] after a certain time; 2. The up-to-now unsuccessful attempts to couple classical gravity to quantum fields; 3. The presumed existence of a fundamental cutoff at small scale, in order to avoid divergences of quantum field theories.

To discuss the possible existence of a fundamental cutoff scale further, it is indeed possible to define from the speed of light c, the Planck energy quantum $\hbar$ and the gravitational constant G a set of three quantities, l_p, t_p and m_p, which are the Planck length, time and mass, respectively:

$$\begin{aligned} l_p &= \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-35}\,\mathrm{m}\,, \\ t_p &= \sqrt{\frac{\hbar G}{c^5}} \approx 5.40 \times 10^{-44}\,\mathrm{s}\,, \\ m_p &= \sqrt{\frac{\hbar c}{G}} \approx 2.17 \times 10^{-8}\,\mathrm{kg} \approx 1.22 \times 10^{19}\,\mathrm{GeV}\,. \end{aligned}$$

Due to their unique character and smallness, compared to the energy scales that have been experimentally tested so far (roughly 20 orders of difference in magnitude), one might expect that they could represent such a fundamental cutoff. Finally one might consider the philosophically desirable feature of complete background independence. This means that in a truly fundamental theory, all structures should be quantised such that they become dynamical entities and do not maintain an absolute character. In particular this should also be true for the concept of time, which in ordinary quantum mechanics, and also special relativity, is an absolute entity (it is not represented by a quantum operator), whereas in general relativity it is dynamical, since it is interwoven in a dynamically changing space-time. This apparent difference in the concept of time between the two theories, should find a resolution in a theory of quantum gravity.

On a more technical level, the central reason behind the problem of quantising gravity is that upon applying the usual quantum field theoretic methods to gravity, one finds that the theory is non-renormalizable, as was shown by t'Hooft and Veltman [5].

Given the reasons listed above, most theorists would agree that it is indeed necessary to find a theory of quantised gravity. However there is disagreement about how to approach the problem and about what a theory of quantum gravity will look like. Many different proposals have been made and we will describe one of them, namely string theory, in some more detail below. Before doing this, it should be remarked however that ultimately the source of all disagreement and the different, largely diverging, ideas is the fact that there is practically no experimental input for our search of a theory of quantum gravity. This is of course due to the fact that the energy regime that one would need to probe in order to get such an input, lies far higher than the energy scales that are currently being probed at particle accelerators such as the Large Hadron Collider (LHC) at CERN and in other experiments. The search for a theory of quantum gravity is thus limited to trying to make reasonable guesses and building mathematical theories, based on analogies inferred from physical concepts observed at much lower energies. The guiding principles here can be certain symmetry principles (which we will discuss more later on), uniqueness of the theory and to some extent how economical the theory is from a mathematical point of view. It should finally be remarked that one thing that pretty most candidate theories of quantum gravity have in common at present is their mathematical complexity, which often makes it hard to identify possible underlying physical principles.

The various approaches to quantum gravity can be put into two categories. In the first category an attempt is made to find the quantised version of general relativity only, without including any of the other fundamental forces in the picture. In contrast, the second category follows the more ambitious approach of complete unification, where all four fundamental forces are treated together in the framework of one quantum field theory. The only serious candidate theory in this second category is string theory, which will be discussed in the following. The number of contenders of theories in the first category is by far greater and one can again distinguish between two different approaches, the first one being the so-called covariant theories, which include a perturbative treatment of gravity, effective field theories, renormalisation-group approaches and path integral methods. In the second type, the canonical theories, gravity is re-formulated using the Hamiltonian formalism, in which canonical momenta and their conjugates are identified. Two prominent examples of this approach are Quantum Geometrodynamics and Loop Quantum Gravity. For a more detailed summary of these approaches see [1].

Let us now enter into a discussion of string theory, giving a compact outline of the features which will be of relevance for the technical part of this thesis.

1.1.1 String theory – a short summary

In the following we will provide a very brief summary of some selected aspects of string theory. This will provide some context for our discussion of automorphic forms and in particular Eisenstein series in string theory.
As general references on the subject of string theory, we recommend the books [6–8]. Furthermore, although by now outdated (in particular on the topic of dualities in string theory, which will feature rather prominently in this thesis), the books [9,10] nevertheless provide useful background material.

The theory now known as 'string theory' grew out of early attempts to try to understand the observed properties of hadrons and their spectrum of excitations. In particular a problem was to understand why the various hadronic resonances seemed to all lie on straight lines (the Regge trajectories) in a plot of the squared mass, M^2, versus the angular momentum, J. An attempt to explain this linear relation between M^2 and J was initially provided in G. Veneziano's and subsequent work [11–13], often simply referred to as the Veneziano amplitude. In this work it was suggested that the interaction of two quarks could be modelled using a string with a certain tension T, 'spanned' between the two particles. In accord with the size of a nucleon of

$\sim 10^{-15}$ m, the length of the string ℓ_s was assumed to be of the same order. For the string's tension the relation $\ell_s^2 \propto T^{-1}$ holds and it was taken to be of the order of $(1\text{GeV})^2$. The string introduced into the QCD picture in this way, is referred to as the 'QCD string'. The slope of the Regge trajectories is denoted by α' and its relation to the string length is $\alpha' \propto \ell_s^2$.

The picture of the QCD string was abandoned as a working physical theory once it was realised (amongst other reasons) that the excitation spectrum of the string also contained a resonance with the associated signature of a massless, spin 2 excitation, which is not present in the hadronic spectrum. Eventually the theory of these strings was taken up again in a completely different context and the seeming disadvantage of having a massless, spin 2 excitation was turned into an advantage. Namely, the theory was reinterpreted as describing a whole spectrum of elementary particles, where the massless, spin 2 excitation would correspond to the exchange-particle responsible for the transmission of the gravitational force, known as the graviton and would therefore incorporate Einstein gravity in a natural way. Interpreting strings in this way, the characteristic length scale l_s of the theory was set to be of the order of the Planck length $l_p \approx 1.62 \times 10^{-35}$ m, such that the theory can also claim to be a candidate for a theory of quantum gravity. Furthermore, since every elementary particle observed in nature is thought to correspond to a particular string excitation, string theory is seen as a theory which in principle unifies all the fundamental forces of nature.

A consistent quantisation of the string, carrying only bosonic degrees of freedom, implies that the string itself should live in a higher-dimensional background space-time (the target space), which is 26 dimensional, where 26 is also known as the critical dimension [14]. The picture one has in mind is that the one-dimensional string sweeps out a two-dimensional worldsheet, embedded in the target space, and the governing quantum field theory on the worldsheet is conformally invariant. Taking into account also fermionic excitations of the string and imposing supersymmetry, the critical dimension is lowered to 10, cf. [15].

The spin 2 (graviton) excitation of the string appears in the spectrum of the closed string. While there are also theories of open strings, which do not have the spin 2 excitation, the fact that open strings can interact in such a way as to form a closed string, means that the graviton and therefore Einstein gravity, is naturally incorporated in (or one might even say 'predicted by') string theory.

An essential ingredient in modern string theory is the concept of compactification. This aspect of the theory is motivated by the question of how we can obtain a theory with the four space-time dimensions that we observe around us, from a higher-dimensional theory like string theory. The idea, going back to the work of T. Kaluza and O. Klein [16, 17], is to 'compactify' the spatial dimensions of, say, a ten-dimensional space-time, on a d-dimensional, internal manifold, X. The physical scale of the manifold X is such that it is very small compared to the physical scales accessible to our everyday perception, or even the smallest scale that can be probed in current high-energy particle physics experiments, and it is therefore effectively invisible. It was found that there is a very large number of possible types of manifolds X that one can compactify string theory on, e.g. the class of Calabi-Yau manifolds. Since the geometry and topology of the manifold affects the spectrum of the strings moving in target space, one hope is that only a very small number of manifolds exists, that reproduce the characteristics of the observed physical world, e.g. the correct elementary particle spectrum.

One of the simplest and best understood compactification manifolds is that of a circle, S^1, which allows for a compactification of a ten-dimensional string theory to a theory with only nine extended dimensions. One can continue down to lower dimensions by compactifying the ten-dimensional theory on a torus instead, down to eight dimensions, and to even lower dimensions by a compactification on the general d-dimensional analogue of a torus, which we denote by T^d. These toroidal compactifications are the ones that we will consider in the remainder of this thesis. Strings moving on such a manifold have two different modes that contribute towards the energy spectrum of the string. These are the momentum modes, which as the name suggests quantify the momentum of the string along a compactified direction and the winding modes, which roughly speaking counts the number of times a string is wound around such a direction.

In total, there are five different superstring theories. They are the type IIA and IIB theories, the type I and the heterotic $E_8 \times E_8$ and $SO(32)$ superstring theories. The 'super' here refers to the fact that they all display supersymmetry. The five string theories are distinguished by the properties of the strings that they describe. It has, however, been shown, that there exist maps, so-called duality relations, between the different theories, which make them nothing but five different manifestations of some larger theory, termed M theory. This theory, which at this point is not well understood, can be argued to be an eleven-dimensional theory, as opposed to the five string theories, which naturally live in ten dimensions. Furthermore it is known that the low-energy limit of the theory is a theory called eleven-dimensional supergravity. Taking the low-energy limit of any string theory corresponds to taking the limit $\alpha' \to 0$, such that strings loose their 'stringy' nature and are approximated by point particles. Supergravity theories [18] are therefore theories of point particles and display supersymmetry [19]. We will come back to the concept of a low-energy expansion in section 2.2. While, as described above, we used the T^d torus to compactify the ten-dimensional string theories, we use a $\mathcal{T}^{d+1}$ torus in order to compactify the eleven-dimensional M theory or supergravity. We thus distinguish between the string theory and M theory torus respectively. Let us also note that apart from one dimensional strings, string- and M theory also contain higher dimensional objects, known as Dp-branes [20], with p spatial and one time-like dimension. Loosely speaking such branes can be defined as the *locus* of the endpoints of open strings.

As mentioned above, dualities play an important role in string theory. They are symmetries of string theory which for example act on the compactification space X or exchange the strong with the weak coupling sector of theories, where the couplings controls the strength of string interactions. Dualities are expressed in terms of symmetry groups G and are described using group theoretical techniques. A particular type of duality is U-duality [21], which we will discuss extensively in this thesis. This duality manifests itself in the amplitudes of string scattering processes, where it acts as a discrete symmetry leaving the amplitude invariant. We will now take this as a motivation to discuss special types of functions, which are defined on groups and display invariance under discrete subgroups. Such functions are known as *automorphic forms.*

1.2 Automorphic forms – a first glimpse

In this section, which contains excerpts from article **IV**, we introduce the concept of an automorphic form. We base our discussion mostly on explicit examples and hope to give the reader a general, intuitive outlook on some of the topics that will be discussed in detail in subsequent chapters. Some comments on the physical interpretation of automorphic forms (in this case Eisenstein series) in the context of string theory are also provided. Let us start by giving the mathematical definition of an automorphic form.

Automorphic forms are functions $f(g)$ on a Lie group G that

(1) are invariant under the action of a discrete subgroup $\Gamma \subset G$: $f(\gamma \cdot g) = f(g)$ for all $\gamma \in \Gamma$,

(2) satisfy eigenvalue differential equations under the action of the ring of G-invariant differential operators and

(3) have well-behaved growth conditions.

Throughout the thesis we will refer back to this definition at various places. In what follows now in this section, we will provide a qualitative description of automorphic forms, based on examples. We will mainly be interested in automorphic forms $f(g)$ that are invariant under the action of the maximal compact subgroup K of G when acting from the right: $f(gk) = f(g)$ for all $k \in K$; such forms are called *K-spherical.* The automorphic forms are then functions on the coset G/K.

Let us now introduce an example of an automorphic form for $G = SL(2,\mathbb{R})$. Due to the explicitness of this case, it will serve as a kind of canonical example throughout this thesis.

The prime example of an automorphic form is obtained when considering $G = SL(2,\mathbb{R})$ and $\Gamma = SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$. The maximal compact subgroup is $K = SO(2,\mathbb{R})$ and the coset space G/K is a constant negative curvature space isomorphic to the Poincaré upper half plane $\mathcal{H} = \{z = x + iy \,|\, x \in \mathbb{R} \text{ and } y > 0\}$. A function satisfying the three criteria above is then given by the non-holomorphic function

$$f_s(z) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{y^s}{|mz+n|^{2s}}. \tag{1.2.1}$$

The parameter s is in general a complex number and the sum converges absolutely for $\mathrm{Re}(s) > 1$. The action of an element $\gamma \in SL(2,\mathbb{Z})$ on $z \in \mathcal{H}$ is given by the standard fractional linear form

$$\gamma \cdot z = \frac{az+b}{cz+d} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \tag{1.2.2}$$

Property (1) is then verified by noting that the integral lattice $(m,n) \in \mathbb{Z}^2$ is preserved by the action of $SL(2,\mathbb{Z})$ and acting with $\gamma \in SL(2,\mathbb{Z})$ in (1.2.1) merely reorders the terms in the absolutely convergent sum. Property (2) in this case reduces to a single equation since there is only a single primitive G-invariant differential operator for the real rank one group $SL(2,\mathbb{R})$. This operator is given by

$$\Delta = y^2 \left(\partial_x^2 + \partial_y^2\right) \tag{1.2.3}$$

and corresponds to the Laplace–Beltrami operator on the upper half plane $\mathcal{H}$. In group theoretical terms it is the quadratic Casimir operator. Acting with it on the function (1.2.1) one finds

$$\Delta f_s(z) = s(s-1) f_s(z) \tag{1.2.4}$$

and hence $f_s(z)$ is an eigenfunction of Δ, and therefore also of the full ring of differential operators generated by Δ. Condition (3) relating to the growth of the function here corresponds to the behavior of $f_s(z)$ near the boundary of the upper half plane, more particularly near the so-called *cusp at infinity* when $y \to \infty$. The growth condition requires $f_s(z)$ to grow at most as a power law as $y \to \infty$. For this we refer the reader to a brief first discussion of the Fourier expansion below and to chapters 4, 5, 6 for a detailed discussion.

As we explain in detail in chapter 3, the type of automorphic functions appearing in the context of string theory that we will consider are the so-called *Eisenstein series*. The simplest example of an Eisenstein series can be obtained from the form of the $G = SL(2,\mathbb{R})$ series given in (1.2.1) above. Namely, the Eisenstein series on $SL(2,\mathbb{R})$, which we denote by $E^{SL(2,\mathbb{R})}(s,z)$, is related to $f_s(z)$ through the relation

$$E^{SL(2,\mathbb{R})}(s,z) := \frac{1}{2\zeta(2s)} f_s(z)\,. \tag{1.2.5}$$

This relation is derived in equation (3.1.1) in the chapter 3.

Since we are ultimately interested in Eisenstein series and their significance in the context of string theory, we will have to ask the question of how one can extract physical information (e.g. perturbative and non-perturbative effects) from such series. The key to this question is the realisation that Eisenstein series are periodic functions with respect to certain discrete subgroups of $G(\mathbb{R})$. Again this is most easily seen in the $G = SL(2,\mathbb{R})$ series example.

The discrete Borel subgroup $B(\mathbb{Z}) \subset SL(2,\mathbb{R})$ acts on the variable $z = x + iy$, through (1.2.2), as translations by

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot z = z + m \quad \text{for } m \in \mathbb{Z} \tag{1.2.6}$$

and therefore any automorphic function (that is by definition invariant under any discrete transformation) is periodic in the x direction with period equal to 1 corresponding to the smallest non-trivial $m = 1$. This means that we can Fourier expand it in modes $e^{2\pi inx}$. Applying this to (3.1.7) leads to

$$E^{SL(2,\mathbb{R})}(s,z) = \underbrace{C(y)}_{\substack{\text{constant term}\\ \text{zero mode}}} + \underbrace{\sum_{n\neq 0} F_n(y) e^{2\pi inx}}_{\text{non-zero mode}} . \tag{1.2.7}$$

As we indicated, it is natural to divide the Fourier expansion into two parts depending on whether one deals with the zero Fourier mode, also known as the *constant term*, or with a non-zero mode. Since the Fourier expansion is developed only along the x direction, the Fourier coefficients still depend on the second variable y.

Determining the explicit form of the Fourier coefficients is one of the key problems in the study of Eisenstein series. In the example of $SL(2,\mathbb{R})$ this can be done by making recourse to the formulation in terms of a lattice sum given in (1.2.1) and using the technique of Poisson resummation, see for example [6, 7, 22, 23]. Such an analysis leads to the following explicit expression

$$E^{SL(2,\mathbb{R})}(s,z) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} + \frac{2y^{1/2}}{\xi(2s)} \sum_{n\neq 0} |n|^{s-1/2} \mu_{1-2s}(n) K_{s-1/2}(2\pi|n|y) e^{2\pi inx} , \tag{1.2.8}$$

where $K_s(q)$ is the modified Bessel function of the second kind (that decreases exponentially for $q \to \infty$ in accordance with the growth condition) and

$$\mu_{1-2s}(n) = \sum_{d|n} d^{1-2s} \tag{1.2.9}$$

is called a divisor sum (or the instanton measure in physics) and given by a sum over the positive divisors of $n \neq 0$. The function $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function (1.2.10)), with the definition of the *Riemann zeta function* [24] given by

$$\zeta(s) = \sum_{k>0} k^{-s} . \tag{1.2.10}$$

The two types of terms introduced above are the *constant term* (zeroth Fourier mode) and the *Fourier coefficients* (non-zero Fourier modes). The physical interpretation of the constant term is that it contains those contributions from string- or M theory, which can be calculated using perturbative methods. The second type of term, the Fourier coefficients, appearing in the expansion, is generally associated with non-perturbative effects, or more precisely instanton corrections, see for example [25–29]. See also 2.1.2 for a description of the non-perturbative effects due to the (euclidean) $D(-1)$-instanton.

As is evident from (1.2.8), the explicit form of the Fourier expansion can appear quite complicated and involves special functions as well as number theoretic objects. For the case of more general $G(\mathbb{R})$ the method of Poisson resummation is not necessarily available as there is not always a form of the Eisenstein series as a lattice sum. It is therefore desirable to develop alternative techniques for obtaining (parts of) the Fourier expansion under more general assumptions. This is achieved by lifting the Eisenstein series into an adelic context which is explained in some detail in section 3.6.

The books [30, 31] are recommended as further reading on the subject of automorphic forms.

1.3 Outline and goals

The bulk of the candidate's original work presented in this thesis is based on the following publications, that were made by the candidate, in collaboration with the respective co-authors indicated. We also provide a brief summary of each publication.

I. P. Fleig and A. Kleinschmidt, *Eisenstein series for infinite-dimensional U-duality groups*, *Journal of High Energy Physics* **6** (June, 2012) 54, `arXiv:1204.3043 [hep-th]`.
In this article the zero modes in the Fourier expansion of certain Eisenstein series defined on the Kac–Moody groups E_9, E_{10} and E_{11} are shown to contain a finite (and only a very small) number of terms. Explicit expressions of the zero mode of the Fourier expansion for some

examples are given and an extensive discussion of physical consistency checks is provided.

II. P. Fleig, A. Kleinschmidt, and D. Persson, *Fourier expansions of Kac–Moody Eisenstein series and degenerate Whittaker vectors*, ArXiv e-prints (Dec., 2013), `arXiv:1312.3643 [hep-th]`.

In this article the results of article **I** above are extended to the higher Fourier modes in the expansion of certain Kac–Moody Eisenstein series and specific examples are calculated explicitly.

Furthermore, the candidate was and is also involved in the following list of publications, some of which have been published and some of which are still in preparation. They contain original work, cf. **III**, as well as proceedings, cf. **V** and **VI**, and review type articles, cf. **IV**. We have chosen not to include the original work of article **III** as a separate chapter in this thesis, since, even though related, its topic stands somehow apart from the main theme of this thesis. The other articles should be seen as providing a summary as well as additional context for the work discussed in this thesis and we will also refer to them where appropriate.

III. P. Fleig, M. Koehn, and H. Nicolai, *On Fundamental Domains and Volumes of Hyperbolic Coxeter-Weyl Groups*, Letters in Mathematical Physics **100** (June, 2012) 261, `arXiv:1103.3175 [math.RT]`.

In this article the fundamental domains of hyperbolic Coxeter-Weyl groups are discussed. In particular, geometric information, such as their finite volume and shape, is extracted from the algebraic structure of the various hyperbolic Weyl groups. This work is related to the articles **I** and **II**, through the fact that the hyperbolic Weyl group $\mathcal{W}(E_{10})$ is the Weyl group of the Kac–Moody group E_{10} on which we define particular Eisenstein series.

IV. P. Fleig, H. A. P. Gustafsson, A. Kleinschmidt, D. Persson, *A physicists' invitation to: Adelic Eisenstein series and automorphic represen-*

tations, in preparation.

This review article offers an extensive introduction to Eisenstein series formulated in the adelic language. Since a detailed treatment of adelic methods as applied to Eisenstein series would require a separate and long discussion in itself, we have chosen to only include some of the relevant parts in this thesis. Particularly chapter 3 contains excerpts from this article.

V. P. Fleig and A. Kleinschmidt, *Perturbative terms of Kac-Moody–Eisenstein series*, submitted to the proceedings of the 'String–Math 2012' conference, ArXiv e-prints (Nov., 2012), `arXiv:1211.5296 [hep-th]`.

These proceedings contain a concise summary of the article **I** listed above and may serve as an introduction to the subject.

VI. P. Fleig and H. Nicolai, *Hidden Symmetries: from BKL to Kac-Moody*, to appear in the proceedings of the 'Marcel Grossmann 13' meeting.

In these proceedings of a talk given by H. Nicolai at the MG13 meeting, the idea of the BKL conjecture, cf. [32], is introduced and its relation to Kac–Moody algebras and a possible approach quantising gravity, cf. [33, 34], is briefly discussed. In the conclusion chapter 7 we provide some comments about a possible relation to the automorphic forms discussed in this thesis.

Let us now summarise and give an outline of this thesis. The primary goal of the thesis is to discuss particular Eisenstein series, which are defined on the infinite-dimensional Kac–Moody groups E_9, E_{10} and E_{11}. In chapter 2 we will explain in some detail how these Kac–Moody groups, together with some other finite-dimensional groups, collectively known as U-duality groups, appear naturally as symmetries of string theory. For the remainder of that chapter, we discuss how the U-duality groups manifest themselves as symmetries in certain string scattering amplitudes, whose general structure we are going to present. In chapter 3, we will provide the general definition of an Eisenstein series, following the work [35] of R. P. Langlands, and discuss Eisenstein series as examples of automorphic forms. For this we resort to the specific case of forms defined on the upper-half complex plain and show how a *lift* can be used to generalise their definition to an automorphic form on $SL(2, \mathbb{R})$. We will then focus on a certain type of Eisenstein series, which is of the form that also appears in the context of string theory. From there we will extend the classical definition of an Eisenstein series to the case of a Kac–Moody group. In the following chapter 4 we will discuss the general structure of the Fourier expansion of Eisenstein series. For this we will write out and manipulate the Fourier integrals that one needs to solve in order to obtain the constant term (zeroth Fourier modes) and the higher order Fourier modes. The two subsequent chapters in a sense form the heart of this thesis, in as much as they contain the central insights of our work. In chapter 5, we will introduce Langlands' formula, which provides a method to compute the constant term. A central point of the discussion will be how one can evaluate this formula in the case of Eisenstein series defined on infinite-dimensional Kac–Moody groups. For this we will derive a special *collapse property* (or mechanism) of the constant term, which holds for the particular Eisenstein series relevant to string theory. In chapter 6, we will show how the collapse mechanism extends to the higher order Fourier modes. In chapter 6 and in the appendix, we will provide explicit expressions for the constant terms and Fourier modes of some Kac–Moody Eisenstein series. The final chapter 7 concludes this thesis. In particular we will make some speculations about other possible applications of Eisenstein series, in the context of the quantum cosmological billiard approach.

Finally, let us summarise again the scientific goal of the work contained in this thesis:

By unravelling some of the clearly very rich and complex mathematical structure encoded in Eisenstein series, we hope to contribute towards an understanding of the non-perturbative structure of string- and M theory. In particular we hope to convince the reader that the Eisenstein series and their Fourier expansions discussed here, provide a very concrete example, where the infinite-dimensional duality groups E_9, E_{10} and E_{11} manifest themselves in terms of quantities. At least in principle these quantities, also afford a physical interpretation.

Chapter 2

Dualities in string theory

In this chapter we discuss dualities of maximally supersymmetric, compactified, string theories. The discussion will reveal a set of discrete symmetry groups, generally referred to as the U-duality groups, which, together with the Eisenstein series, will take centre stage in the discussion of our work in subsequent chapters. More precisely, we will define particular Eisenstein series, which are defined as functions on the continuous version of the U-duality groups, but which display invariance under the discrete U-duality groups themselves. This is done much in the spirit of the brief introduction to automorphic forms given in the previous chapter.

However for now, let us focus on explaining some of the dualities and symmetries which are present in string theory and supergravity theories. References to original, as well as review articles are provided for further reading.

2.1 String dualities

As mentioned in section 1.1.1, there are a present five different string theories that have been discovered. Amongst these are the type IIA and IIB string theory. Over the last 15 years a lot of research work has been devoted to understanding the structural relations between the various string theories, also known as duality relations. We will now discuss three types of dualities, with particular emphasis on dualities related to the type II string theories.

While we are mainly interested in dualities of string theories, we shall also discuss the continuous symmetries of the supergravity theories (low-energy limit theories) of string theories.

The following sections on T-, S- and U-duality follow loosely the presentations of [21] and [8]. The review article [21] is particularly recommended for further reading on the topic of U-duality.

2.1.1 T-duality

We start with a discussion of T-duality. This duality, where the 'T' stands for target space, is associated with discrete transformations performed on the compactification space, in our case the string theory torus, T^d. More precisely, performing the transformation $r \leftrightarrow \alpha'/r$, where r is the radius of one of the compactified directions on the torus, implies an exchange of the momentum and winding modes of the string, for its spectrum to remain unchanged under the transformation. The discrete group of these transformations is generated by a set of Weyl generators. The generators can be obtained from the explicit form of the action of a theory after compactification. It can be shown [21] that the Weyl generators (supplemented by Borel generators) generate the discrete T-duality group $SO(d, d, \mathbb{Z})$, with its associated Dynkin diagram shown in figure 2.1.

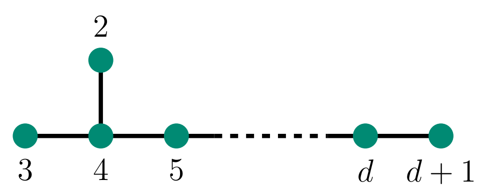

Figure 2.1: *The $SO(d, d, \mathbb{Z})$ Dynkin diagram of T-duality*

The Dynkin digram is a sub-diagram of the $E_{d+1}(\mathbb{Z})$ Dynkin diagram shown in figure 2.3, associated with the discrete exceptional group $E_{d+1}(\mathbb{Z})$. This fact is emphasised by the particular labelling of nodes we have chosen. Note also that the above diagram contains the discrete group $SL(d, \mathbb{Z})$, with the long horizontal line as its Dynkin diagram. It is the modular group of the string theory torus, and its Weyl generators S_{ij} imply an *exchange* of two radii $r_i \leftrightarrow r_j$ of the torus T^d. The full T-duality group also includes the Weyl generators T_{ij}, which yield the simultaneous *inversion* of two radii, $(r_i, r_j) \leftrightarrow (r_j^{-1}, r_i^{-1})$. A possible choice for a minimal set of generators of the full group is given by

$$\begin{aligned} S_i &: r_i \leftrightarrow r_{i+1} \quad \text{with} \quad i = 1, 2, ...d-1 \\ T_{12} &: (g_s, r_1, r_2) \leftrightarrow \left(\frac{g_s}{r_1 r_2}, \frac{1}{r_2}, \frac{1}{r_1} \right) . \end{aligned} \tag{2.1.1}$$

Let us provide some more details on the way in which the Weyl generators act. Namely, the Weyl generators can be interpreted as discrete actions on the $d+1$-dimensional weight space (a vector space) of the T-duality group. Let us define the vector H_S, where the subscript label refers to the string theory torus, as

$$H_S := (\log(g_s), \log(r_1), \log(r_2)..., \log(r_d)) . \tag{2.1.2}$$

This function is a function on the moduli space of the theory and assigns a vector to each point in this space. It contains the string coupling g_s and the radii r_i of the string theory torus. Upon choosing basis elements e_0, e_1,..., e_d of the vector space, we can define an object τ, which we refer to as the 'tension', given by

$$\tau = e^{\langle \lambda | H_S \rangle} = g_s^{x_0} r_1^{x_1} r_2^{x_2} ... r_d^{x_d} , \tag{2.1.3}$$

where $\lambda = x_0 e_0 + x_1 e_1 + x_2 e_2 ... + x_d e_d$ and $\langle \cdot | \cdot \rangle$ is the pairing between the weight space and its dual space. A Weyl generator of the T-duality group will then act on λ, according to the transformations (2.1.1). Let us also point out the role of the Weyl group as a reflection group, which is explained in the appendix A.1.1. In particular, acting with a Weyl generator on the weight λ, is interpreted as the reflection of λ on the hyperplane of a root α, as is clear from formula (A.1.12) in the appendix. Group theoretic objects, very similar to the tension τ, will appear frequently throughout this thesis and in particular in the definition of Eisenstein series (3.2.20).

Before we conclude our discussion of T-duality, let us make two more remarks. Firstly, the duality is defined order by order in α' [36–38]. This is for instance the case in low-energy effective actions, cf. section 2.2.1, where string theory is defined as an expansion in terms of the Regge slope α'. Secondly, T-duality as a symmetry does not appear in quantum field theories which describe ordinary point particles, since the concept of exchanging momentum and winding modes is not applicable.

2.1.2 S-duality

The 'S' in the name stands for 'strong-weak' duality. It states that a strongly coupled quantum field theory, $\mathcal{A}$, with coupling constant $g > 1$ can be mapped to the weak coupling regime of another theory, $\mathcal{B}$, with coupling constant g^{-1} and vice versa. The figure 2.2, which is inspired by a similar diagram in [8], illustrates this duality relation diagrammatically. The first example of such a strong-weak duality was provided in [39] for supersymmetric Yang-Mills theory.

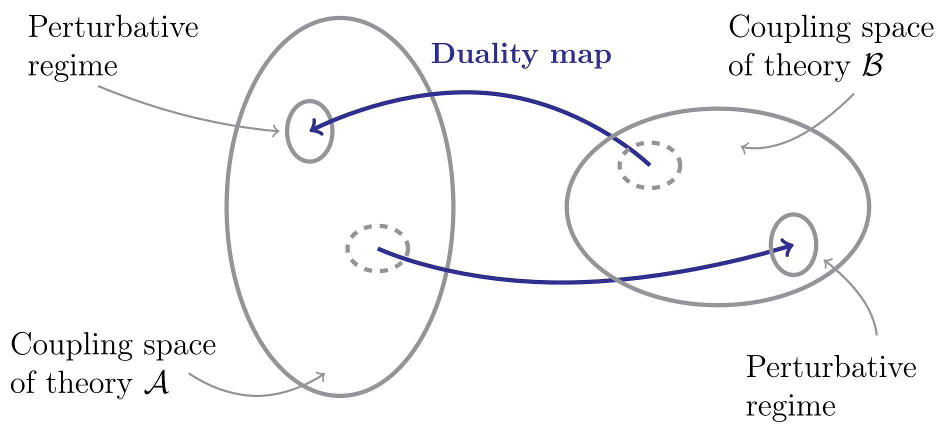

Figure 2.2: The schematics of strong-weak coupling between two quantum field theories $\mathcal{A}$ and $\mathcal{B}$.

In the following we illustrate this duality with a prominent example, namely type IIB string theory. The following example follows a similar exposition provided in [8]. We start by discussing a hidden symmetry in the corresponding supergravity theory of type IIB.

Namely, the type IIB supergravity theory displays an hidden invariance under the $SL(2,\mathbb{R})$, see [40–42]. In Einstein frame, the action for the bosonic sector of this theory can be written in a manifestly $SL(2,\mathbb{R})$ invariant way, see for example [7, 8]. In particular, the axion χ and the dilaton ϕ, as the scalar fields of IIB supergravity, are then gathered in a complex parameter τ which is defined as

$$\tau = \chi + ie^{-\phi}\,. \tag{2.1.4}$$

The complex field τ transforms under an $SL(2,\mathbb{R})$ transformation in the standard way

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})\,. \tag{2.1.5}$$

The term for the scalar fields in the action of type IIB supergravity then takes the form $-\partial\bar{\tau}\partial\tau/2|\operatorname{Im}\tau|^2$, which under the above transformation is manifestly $SL(2,\mathbb{R})$ invariant. Let us consider a transformation under the particular $SL(2,\mathbb{R})$ element

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \tag{2.1.6}$$

for the case when $\chi = 0$. The results of such a transformation will be that $\tau = ie^{-\phi}$ will be mapped to $-1/\tau = ie^{\phi}$. As already discussed in the introduction, in string theory, the expectation value of the dilaton determines the string coupling constant through the relation, $g_s^{IIB} = e^{\phi}$. Hence the transformation just shown leads to an inversion of the coupling constant, $(g_s^{IIB})^{-1} \to g_s^{IIB}$, of the theory, which is an instance of the strong-weak duality mentioned above. There is a subtlety in string theory, however, which plays an important role:

While the supergravity theory displays invariance under the continuous $SL_B(2,\mathbb{R})$ group, this symmetry does not hold any longer for the full type IIB string theory, but is broken to the discrete duality group $SL_B(2,\mathbb{Z})$, [43, 44], the S-duality group. More precisely, type IIB string theory is self-dual under

S-duality meaning that the regime of strong string coupling is mapped to its weakly coupled regime.

A possible line of reasoning for this breaking of symmetry is related to the existence of non-perturbative effects in type IIB theory. One such non-perturbative effect is for example implied by the presence of D-instantons, which were already mentioned in section 1.2. The D-instantons are euclidean $D(-1)$-branes and their contribution is of the form $\exp(2\pi i\tau)$, since their (classical) action takes the form $S_{\text{inst.}} = 2\pi\tau$. It is then clear from the exponential form of the functional that shifting, $\tau \to \tau + b$, according to (2.1.5), can only hold as a symmetry provided b is an integer, i.e. $b \in \mathbb{Z}$. See for example [25] for a discussion of the effects of D-instantons in type IIB supergravity and string theory. The review article [45] provides a general introduction to instanton solutions.

2.1.3 U-duality

In this section we discuss U-duality which contains both the T- and the S-duality as subgroups. We begin the discussion by introducing M theory as an eleven-dimensional theory, providing a larger framework from which the various other types of string theories derive as particular limits.

Eleven-dimensional 'M theory' perspective

In order to obtain a broader perspective on dualities in string theory, we will now discuss the role played by the eleven-dimensional M theory and its low-energy limit $11D$ supergravity.

Let us focus on the case of type IIA string theory and its string coupling constant g_s. In [46, 47] it was shown that type IIA supergravity and $11D$ supergravity are related by dimensional reduction of the eleventh dimension on a circle. It was, however, not clear how type IIA string theory relates to the eleven-dimensional theory. This was clarified in [48], where it was shown that a similar relation as for the supergravity theory also holds for the string theory. The crucial point here is to interpret the string coupling $g_s = e^{\phi}$ of the IIA theory, as the radius r_M of the eleventh dimension. Distinguishing between the strong and weak coupling limits of the theory one obtains different regimes. In the first, for weak coupling, i.e. $g_s \to 0$, the theory is in its perturbative regime, which is under control. In the strong coupling regime for $g_s \to \infty$, the theory can be interpreted as developing an eleventh dimension, and turning into $11D$ supergravity. More precisely, one can then express type IIA quantities in terms of M theory quantities:

$$r_M/l_p = g_s^{2/3}\,, \quad l_p^3 = g_s l_s^3\,. \tag{2.1.7}$$

The duality

Since their introduction in [48] and [49], a lot of work has been devoted to understanding the above mentioned examples of dualities in a bigger framework. We would now like to summarise the picture that has emerged in some detail, taking into consideration our discussion above.

A few years before the emergence of the discrete U-dualities in string theory, it had already been established that toroidally compactified (maximal) $11D$ supergravity theory displays invariance under a set of continuous symmetry groups, known as the $E_n(\mathbb{R})$ groups. To be more precise, after compactification from eleven down to D dimensions on torus $\mathcal{T}^{d+1}$, the symmetry group of the compactified theory is given by the split real group $E_{d+1}(\mathbb{R})$, cf. [50–53]. It is important to note that this result was restricted to the case where $D \geq 2$ (or $d \leq 8$). The list of $E_{d+1}(\mathbb{R})$ groups is provided in the top nine rows of the first column of table 2.1. We note in particular the appearance of the exceptional groups E_6, E_7 and E_8 in $D = 5, 4$ and 3 dimensions, respectively, with E_8 being the largest, finite-dimensional Lie group. Furthermore, in $D = 2$ dimensions, the duality group is E_9, which is an affine (infinite-dimensional) Kac–Moody group and was introduced in this context in [52].

The moduli fields (scalars) that are present in the compactified theory, after having performed all possible dualisations of higher-rank form fields, parameterise the moduli space

$$\mathcal{M}_{d+1} = E_{d+1}(\mathbb{R})/K(E_{d+1}(\mathbb{R}))\,, \tag{2.1.8}$$

D	$E_{d+1}(\mathbb{R})$	$K(E_{d+1})$	$E_{d+1}(\mathbb{Z})$
10_B	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$\mathbb{R}^+ \times SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
8	$SL(2,\mathbb{R}) \times SL(3,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(2,\mathbb{Z}) \times SL(3,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$SO(5,5,\mathbb{R})$	$SO(5) \times SO(5)$	$SO(5,5,\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$
2	$E_9(\mathbb{R})$	$K(E_9(\mathbb{R}))$	$E_9(\mathbb{Z})$
1	$E_{10}(\mathbb{R})$	$K(E_{10}(\mathbb{R}))$	$E_{10}(\mathbb{Z})$
0	$E_{11}(\mathbb{R})$	$K(E_{11}(\mathbb{R}))$	$E_{11}(\mathbb{Z})$

Table 2.1: *List of the split real forms of the duality groups of compactified type IIB theory in $D \leq 10$ dimensions. We also list the corresponding maximal compact subgroups, and the last column contains the discrete versions, which appear in string theory. The last two rows are conjectural as are the corresponding discrete groups for $D \leq 3$. As explained in the main text, the relation, $D = 10 - d$, holds.*

where K is the maximal compact subgroup of $E_{d+1}(\mathbb{R})$. The list of maximal compact subgroups is also shown in table 2.1.

A central question now is: what happens to the continuous symmetry of compactified maximal supergravity when it is embedded in string or M theory? The widely accepted conjecture [43, 49] is that the continuous symmetry is broken, and one is left with discrete duality groups which are the corresponding Chevalley groups of the continuous symmetry groups. We list these conjectured, so-called U-duality groups, $E_{d+1}(\mathbb{Z})$, also, in the first nine

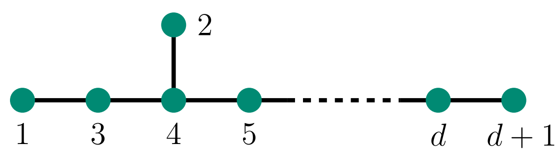

Figure 2.3: *Dynkin diagram for $E_{d+1}(\mathbb{Z})$ with Bourbaki labelling of nodes, cf. [54].*

rows of the last column of table 2.1. The figure 2.3 shows the Dynkin diagram of $E_{d+1}(\mathbb{Z})$ in the particular Bourbaki labelling [54] of nodes, which is used throughout this thesis. It is clear from the diagram that $E_{d+1}(\mathbb{Z})$ contains $SO(d, d, \mathbb{Z})$ and $SL(d + 1, \mathbb{Z})$ (horizontal line) as subgroups.

Quantisation and Weyl generators

Let us now provide the standard reasoning for the breaking of the continuous group $E_{d+1}(\mathbb{R})$ to the discrete version $E_{d+1}(\mathbb{Z})$. For this one resorts to the particular case of a reduction to $D = 4$ dimensions, where the corresponding continuous symmetry of the compactified maximal supergravity theory is $E_7(\mathbb{R})$. Maximal supergravity in four dimensions also contains, in its bosonic sector, along with the metric and scalar fields, 28 Maxwell one-form fields. Each of the Maxwell fields has an electric and a magnetic charge associated with it, and together these charges, which we denote by (q^n, p_n) with $n = 1, ..., 28$, transform in a 56-dimensional representation of E_7. In the quantum theory, the set of charges is subject to a Dirac-Schwinger-Zwanziger type quantisation condition [55–57]. Consider two electrically and magnetically charged particles with charges, (q^n, p_n) and $(\bar{q}^n, \bar{p}_n)$. These charges have to obey the constraint

$$q^n \bar{p}_n - \bar{q}^n p_n \in \mathbb{Z}\,. \tag{2.1.9}$$

This constraint, which we recognise to be the symplectic product of the charge vectors, forces the charges to lie on a 56-dimensional integral lattice. The lattice is invariant under the symplectic group $Sp(56, \mathbb{Z})$ and one may refer to it as the duality group of electric and magnetic charges. Therefore, the conjecture made in [49] is that the duality group $E_7(\mathbb{Z})$ that remains a symmetry of the quantum theory is defined as the intersection $E_7(\mathbb{R}) \cap Sp(56, \mathbb{Z})$.

Just as in the case of the T-duality group discussed in section 2.1.1, we can similarly identify a generating set of Weyl transformations for the (discrete) U-duality group $E_{d+1}(\mathbb{Z})$. The set contains the Weyl generators S_I for the exchange of directions of the M theory torus, as well as a T-duality transformation on three of the directions. Hence we have the following possible choice for a minimal generating set:

$$\begin{aligned} S_I \;&:\; \tilde{r}_I \leftrightarrow \tilde{r}_{I+1}\,, \\ T_{123} \;&:\; (l_p^3, \tilde{r}_1, \tilde{r}_2, \tilde{r}_3) \leftrightarrow \left(\frac{l_p^6}{\tilde{r}_1 \tilde{r}_2 \tilde{r}_3}, \frac{l_p^3}{\tilde{r}_2 \tilde{r}_3}, \frac{l_p^3}{\tilde{r}_1 \tilde{r}_3}, \frac{l_p^3}{\tilde{r}_1 \tilde{r}_2} \right)\,, \end{aligned} \tag{2.1.10}$$

where $I = 1, 2, ...d$. In the definition of the T_{123} generator, we have used the second one of the relations (2.1.7) to express the type IIA string coupling in terms of the eleven-dimensional Planck length. The string length l_s is set to unity. This set generates the U-duality group with the T-duality group $SO(d, d, \mathbb{Z})$ and the modular group $SL(d + 1, \mathbb{Z})$ of the torus as subgroups. One can write $E_{d+1}(\mathbb{Z})$ as the semi-direct product of the two non-commuting subgroups as

$$E_{d+1}(\mathbb{Z}) = SL(d + 1, \mathbb{Z}) \ltimes SO(d, d, \mathbb{Z}) \tag{2.1.11}$$

In order to see the explicit action of the U-duality group, let us define the vector

$$H_M := (\log(\tilde{r}_1), \log(\tilde{r}_2)..., \log(\tilde{r}_{d+1}))\ , \tag{2.1.12}$$

where the subscript label is now chosen with reference to the M theory torus. Upon choosing basis elements $\tilde{e}_1$,..., $\tilde{e}_{d+1}$ of the $d + 1$-dimensional vector space, we define the tension τ similarly as in the case of the T-duality group,

$$\tau = e^{\langle \lambda | H_M \rangle} = \tilde{r}_1^{x_1} \tilde{r}_2^{x_2} ... \tilde{r}_{d+1}^{x_{d+1}}\ , \tag{2.1.13}$$

where now $\lambda = x_1\tilde{e}_1 + x_2\tilde{e}_2 ... + x_{d+1}\tilde{e}_{d+1}$. A Weyl generator of the U-duality group will then act on λ, according to the transformations (2.1.10).

Duality groups in $D < 2$

In [33, 58–61] it was conjectured, that the above discussed pattern of duality groups in $D \geq 2$ could be extended to lower dimensions $D = 1, 0$ respectively. The corresponding duality groups are the over-extended, hyperbolic Kac–Moody group E_{10} and the very extended Kac–Moody group E_{11}. Both group are of course infinite-dimensional. While the precise role of these groups is still not clear, we think of them here as symmetries of the toroidally compactified theory. This way of thinking about these groups is in contrast to farther-reaching conjectures of [33] and [60], which will also be mentioned briefly in the concluding chapter 7.

2.2 Schematics of scattering amplitudes

In this section we provide a discussion of superstring scattering amplitudes. Our interest in scattering amplitudes is that the amplitudes we will consider have the property of being automorphic under the U-duality groups that we have just discussed. The automorphic property is given by the appearance of automorphic forms in the amplitude. More precisely, these are Eisenstein series such as the series (3.1.7), and more complicated generalisations thereof, which will be intoduced in the next chapter 3. The following exposition will be rather concise and we refer the reader to the appropriate literature for a detailed introduction to the rich field of string scattering amplitudes.

In general, the amplitude $\mathcal{A}$ of a string scattering process organises most naturally as a perturbative double expansion in two string theory parameters. One is the Regge slope, $\alpha' = \ell_s^2$, and the other is the string coupling constant g_s. We have already seen earlier that the value of the coupling g_s is determined by the expectation value of the dilation field ϕ, as $g_s = e^{\phi}$. In the context of our work, we are interested in the scattering of closed strings in D dimensions, e.g. in compactified type IIB theory, and in this case the amplitude then takes the form:

$$\mathcal{A}^D = \sum_{n=0}^{\infty}\sum_{g=0}^{\infty}(\alpha')^{n-4} g_s^{2(g-1)} \mathcal{A}^D_{(n,g)} \,. \tag{2.2.1}$$

In the sum over orders of α', each term in the series introduces additional string theory contributions, which are not captured by the low-energy theory in the limit of $\alpha' \to 0$.

On the other hand, the sum over orders of the string coupling g_s is quite analogous to a perturbative series in standard quantum field theory of point particles, where one is summing over the Feynman diagrams of the process associated with all possible orders in the coupling constant. In string theory, however, the analogues of the Feynman diagrams are now two-dimensional worldsheets, which capture the interaction process of the closed strings. The different worldsheets are characterised by their topology. The topology is captured by the integer number χ, known as the Euler number, which is a topological invariant, and is obtained by integrating the two-dimensional Ricci scalar over the worldsheet. In particular, one finds that the Euler number, in our case of interest, is given by $\chi = -2(g-1)$, where g is the genus of the surface. The genus counts the number of 'handles' of the two-dimensional surface. In the case of a sphere, doughnut and a double-doughnut (doughnut with two holes/handles), we have that $g = 0, 1$ and 2, respectively. In this picture, the sphere corresponds to a tree-level, the doughnut to a one-loop and the double-doughnut to two-loop string scattering process. The contribution to a scattering process at a particular loop order, then receives a weighting of the form $(g_s^{-1})^{2(g-1)}$, which is also visible in the general form of the amplitude (2.2.1).

The U-duality symmetry discussed in the previous section 2.1.3, becomes manifest for instance in superstring scattering amplitudes, where it acts as a discrete symmetry. The amplitude $\mathcal{A}$ of a scattering process is in general a function of all the scalar fields (after dualisation of higher rank form fields) present in the compactified superstring theory. This means it is a function of the moduli fields $\mathcal{M}_{d+1}$. For instance, as discussed above, in type IIB string theory, compactified on a d-dimensional torus T^d, the moduli space $\mathcal{M}_{d+1}$ is just the one that was introduced in equation (2.1.8).

2.2.1 Four-graviton scattering

Let us now discuss one of the most prominently studied examples of superstring scattering, namely the four-graviton superstring scattering process. Our presentation follows [62], as well as [63]. The expansion at low energies of the four-graviton superstring scattering amplitude $\mathcal{A}^D(s,t,u;\Phi)$ in $D = 10 - d$ space-time dimensions is a function of the Mandelstam variables s, t and u (see below) and of moduli $\Phi \in \mathcal{M}_{d+1}$. The amplitude can be written as a sum

$$\mathcal{A}^D(s,t,u;\Phi) = \mathcal{A}^D_{\text{analytic}} + \mathcal{A}^D_{\text{non-analytic}} \,. \tag{2.2.2}$$

Here the first term is an analytic function of the Mandelstam variables and the second term is non-analytic in these variables [26]. In our work, we are mainly interested in the analytic part $\mathcal{A}^D_{\text{analytic}}$. The non-analytic contribution also plays a role in the analysis and we will provide some more comments on this term later on. Written in its most general form, the amplitude takes the form

$$\mathcal{A}^D_{\text{analytic}}(s,t,u;\Phi) = \ell_D^6 \, T^D(s,t,u;\Phi)\mathcal{R}^4 \,, \tag{2.2.3}$$

where ℓ_D is the Planck length in D space-time dimensions (see section 5.4.1 for further explanation on this). The symbol $\mathcal{R}^4$ denotes a contraction of four Riemann tensors with a standard 16-index tensor. The 16-index tensor is the $t_8 t_8$ tensor, which can, for example, be found in [64]. The Mandelstam variables s, t and u are defined in terms of the four (null) particle momenta k_i, with $i = 1,2,3,4$ of the external legs, and satisfy momentum conservation $\sum_{i=1}^4 k_i = 0$. The (invariant) Mandelstam variables are defined as

$$s = -(k_1 + k_2)^2 \,, \quad t = -(k_1 + k_4)^2 \,, \quad \text{and} \quad u = -(k_1 + k_3)^2 \,, \tag{2.2.4}$$

and satisfy the relation $s+t+u=0$. The (scalar) function $T_D(s,t,u;\Phi)$ is given by the following sum

$$T_D(s,t,u;\Phi) = \sum_{p,q} \mathcal{E}^D_{(p,q)}(\Phi)\sigma_2^p\sigma_3^q\,. \tag{2.2.5}$$

Here the momentum insertions σ_n are defined as a dimensionless combinations of the Mandelstam variables

$$\sigma_n = (s^n + t^n + u^n)\frac{\ell_D^{2n}}{4^n}\,. \tag{2.2.6}$$

The coefficients $\mathcal{E}^D_{(p,q)}(\Phi)$ are automorphic forms of the moduli $\Phi \in \mathcal{M}_{d+1}$. The superscript D indicates that $\mathcal{E}$ is an automorphic from under the duality group in $D = 10-d$ space-time dimensions, i.e. E_{d+1}. The orders $2p+3q \leq 3$, with positive integers p and q, have been studied extensively in the literature and a considerable amount of evidence for their precise form has been accumulated in $D \geq 3$. Then, writing out the first few orders in the expansion, it takes the form

$$T_D(s,t,u;\Phi) = \mathcal{E}^D_{(0,-1)}\sigma_3^{-1} + \mathcal{E}^D_{(0,0)} + \mathcal{E}^D_{(1,0)}\sigma_2 + \mathcal{E}^D_{(0,1)}\sigma_3 + \dots\,. \tag{2.2.7}$$

Note that on the right-hand side of this equation, we have suppressed the dependence on the moduli. The first term on the right-hand side of this expansion is the classical supergravity tree-level term, determined by the Einstein-Hilbert action. The function $\mathcal{E}^D_{(0,-1)} = 3$. Starting with the work of [25] which was subsequently developed in many further publications [26–29, 65–74], it was demonstrated that precise statements can be made about the form of the three lowest orders beyond the Einstein-Hilbert term in the low-energy expansion of the four-graviton scattering amplitude.

As we will see shortly, the coupling functions $\mathcal{E}^D_{(0,0)}$, $\mathcal{E}^D_{(1,0)}$ and $\mathcal{E}^D_{(0,1)}$ are automorphic forms (namely Eisenstein series), with invariance under the duality group $E_{11-D}(\mathbb{Z})$. Let us emphasise at this point that the function $\mathcal{E}^D_{(0,-1)}$, associated with the supergravity theory, is constant and does not have a dependence on the moduli. This is a reflection of the fact that the automorphic property of the scattering amplitude only appears when taking into account string theory corrections to the low-energy, supergravity, limit.

Let us make an aside remark. In string theory there are a two different frames (determined by the metric), namely the string- and the Einstein frame, which are relevant in the context of our discussion. The two frames are related by a dilaton dependent re-scaling of the metric. The string frame is the one associated with the Polyakov action for the string, whereas the Einstein frame is characterised by the fact that the pre-factor of the Einstein-Hilbert term does not depend on the dilaton. Furthermore, in the Einstein frame quantities are measured in terms of Planck units and not string units. For our discussion of four-graviton scattering we are working in Einstein frame and hence the Planck length ℓ_D in D dimensions appears in the equations above. For more details on frames see [6–8].

Before turning to a discussion of the explicit expression of the automorphic couplings let us briefly develop an alternative view point of the low-energy expansion.

The string effective action

Instead of considering the amplitude of the scattering process directly, one can translate the information of the amplitude into the form of an effective action. Let us provide a recipe for translating the amplitude of four-graviton scattering into the corresponding effective action. Focusing on the analytic part of the amplitude $\mathcal{A}^D$, then each term in (2.2.3) has a corresponding term in the effective action. Due to the momentum insertions σ_2 and σ_3 in (2.2.3), the effective action will contain higher-derivative contributions, beyond the Einstein-Hilbert term. The translation rule looks as follows

$$\mathcal{E}^D_{(p,q)}(\Phi)\sigma_2^p\sigma_3^q\mathcal{R}^4 \longrightarrow \mathcal{E}^D_{(p,q)}(\Phi)\partial^{2(2p+3q)}\mathcal{R}^4\,. \tag{2.2.8}$$

Each such term in the effective action then constitutes a string theory correction term at a particular order in α' (of dimension $(\text{length})^2$) to the usual Einstein-Hilbert term of the supergravity theory. The infinite series of correction terms, obtained by translation from the scattering amplitude, is of the form

$$\ell_D^{8-D} \sum_k \int \mathrm{d}^D x \, \ell_D^{2k} \, \mathcal{E}^D_{(p,q)}(\Phi) \partial^{2k} R^4 \,, \tag{2.2.9}$$

where we have defined $k := 2p+3q$. The first few terms in this effective action expansion, beyond the Einstein-Hilbert term, then occur for $k = 0, 2, 3, \ldots$, i.e. yield the curvature corrections (including automorphic couplings) of the form

$$\mathcal{E}^D_{(0,0)} \mathcal{R}^4 \,, \quad \mathcal{E}^D_{(1,0)} \partial^4 \mathcal{R}^4 \quad \text{and} \quad \mathcal{E}^D_{(0,1)} \partial^6 \mathcal{R}^4 \,, \tag{2.2.10}$$

where the dependence on the moduli has been suppressed.

Properties of automorphic couplings

As mentioned, we will mostly consider the two lowest orders of string theory curvature corrections in the effective action of four-graviton scattering. They are the $\mathcal{R}^4$ and $\partial^4 \mathcal{R}^4$ terms, with coefficients $\mathcal{E}^D_{(0,0)}$ and $\mathcal{E}^D_{(1,0)}$, respectively. For statements about the precise form of these couplings (in terms of Eisenstein series), we refer the reader to section (3.5). Here we would like to collect some more statements about the general properties of the low order $\mathcal{E}^D_{(p,q)}$ couplings.

First, we note that the curvature couplings are functions on the moduli, invariant under the the discrete U-duality group, $E_{11-D}(\mathbb{Z})$.

Secondly, let us discuss the Laplace eigenvalue equations satisfied by the coefficient functions that we have introduced. We will also provide the explicit form of the Laplace eigenvalue equation satisfied by the automorphic coefficient $\mathcal{E}^D_{(0,1)}$ of the curvature correction term $\partial^6 \mathcal{R}^4$ in the effective action.

As proven in [69] for $D \geq 3$, the coefficient functions $\mathcal{E}^D_{(0,0)}$, $\mathcal{E}^D_{(1,0)}$ and $\mathcal{E}^D_{(0,1)}$ of the lowest three orders of curvature corrections each satisfy a Laplace eigenvalue equation defined by the E_{d+1} invariant Laplace operator Δ^D on the moduli space $\mathcal{M}_{d+1}$, cf. (2.1.8), in $D = 10 - d$ dimensions. In the first two cases this Laplace eigenvalue equation is homogeneous (with source terms only in dimensions when there is a known divergence). For the third case

$\mathcal{E}^D_{(0,1)}$, the coefficient of the $\partial^6\mathcal{R}^4$ correction, the equation is always inhomogeneous, where the inhomogeneous term is given by $(\mathcal{E}^D_{(0,0)})^2$. The explicit expressions of these Laplace equations are

$$\left(\Delta^D - \frac{3(11-D)(D-8)}{D-2}\right)\mathcal{E}^D_{(0,0)} = 6\pi\delta_{D,8}\,, \qquad (2.2.11)$$

$$\left(\Delta^D - \frac{5(12-D)(D-7)}{D-2}\right)\mathcal{E}^D_{(1,0)} = 40\zeta(2)\delta_{D,7}\,. \qquad (2.2.12)$$

The inhomogeneous Laplace equation for the $\mathcal{E}^D_{(0,1)}$ coefficient takes the form

$$\left(\Delta^D - \frac{6(14-D)(D-6)}{D-2}\right)\mathcal{E}^D_{(0,1)} = -\left(\mathcal{E}^D_{(0,0)}\right)^2 + 120\zeta(3)\delta_{D,6}\,. \qquad (2.2.13)$$

Here the $\delta_{i,j}$ are discrete Kronecker deltas and Δ^D is the Laplace operator defined on $\mathcal{M}_{d+1}$ where $d = 10 - D$. These were derived in [72] using the decompactification limit of the Laplace operator from D to $D+1$ dimensions. This decompactification limit is discussed in detail in section 5.4. We note that for $D = 2$ all three equations appear to break down, for the respective eigenvalues are singular in this case. This is, however an artefact of the method of derivation which needs to be refined for $D = 2$. For the details of the corrected derivation of the $D = 2$ case, we refer the reader to article **I**.

For a discussion of modular- and automorphic forms in the context of non-supersymmetric theories of gravity see [75].

Chapter 3

Eisenstein series

As was shown in chapter 2, the type of automorphic functions appearing in the context of string theory that we will consider are the so-called *Eisenstein series*. In the present chapter we therefore outline the theory of these Eisenstein series. We start by providing further details on the $SL(2,\mathbb{R})$ mentioned in the introduction and discuss its relation to (non-)holomorphic forms defined on the upper-half complex plane. After introducing the definition of an Eisenstein series on a general group $G(\mathbb{R})$ according to R. P. Langlands [35], we discuss the types of Eisenstein series which appear in the context of string theory. The next step is to extend the definition of Eisenstein series to the case when they are defined on infinite-dimensional Kac–Moody groups. In particular the case for affine Kac–Moody groups is introduced in some detail, following the work of H. Garland [76].

The chapter contains excerpts from the unpublished article **IV**, although with considerable changes in the presentation.

We point out to the reader that the algebraic and group theoretical concepts and notions that are freely used in this chapter are introduced in the appendix A.1.

3.1 Eisenstein series on $SL(2,\mathbb{R})$ in context

The simplest example of an Eisenstein series can be obtained from the form of the $G = SL(2,\mathbb{R})$ series given in (1.2.1) above. Therefore to give an idea of the more general definition of an Eisenstein series and in order to reproduce the relation (1.2.5), we rewrite the expression for $f_s(z)$ in (1.2.1). The first thing we do is to take out the greatest common divisor of the coordinates of the lattice point $(c,d) \in \mathbb{Z}^2$:

$$f_s(z) = \left(\sum_{k\neq 0} k^{-2s}\right) \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1}} \frac{y^s}{|mz+n|^{2s}} = 2\zeta(2s) \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1}} \frac{y^s}{|mz+n|^{2s}}\,, \tag{3.1.1}$$

where we have evaluated the sum over the common divisor k using the Riemann zeta function (1.2.10). The series after the second equality sign above (excluding the factor $2\zeta(2s)$) is what we call the (non-holomorphic) $SL(2,\mathbb{R})$ Eisenstein series

$$E^{SL(2,\mathbb{R})}(s,z) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1}} \frac{y^s}{|mz+n|^{2s}}\,. \tag{3.1.2}$$

A central goal of this thesis is to present an extension of the definition of an Eisenstein series to groups G of higher rank than the $SL(2,\mathbb{R})$ case. In particular we aim to define Eisenstein series on the $E_n(\mathbb{R})$ duality groups and in particular for the infinite-dimensional Kac–Moody duality groups $E_9(\mathbb{R})$, $E_{10}(\mathbb{R})$ and $E_{11}(\mathbb{R})$. In order to achieve this, we have to employ a formalism for defining Eisenstein series which is more powerful than the 'sum over a lattice' definition given in (1.2.1) and (3.1.2), which is available for the simple case of $G = SL(2,\mathbb{R})$. In order to pave the way for such a general formalism, which we present in the subsequent section 3.2, let us again resort to our canonical example.

Referring back to (1.2.1) and using (1.2.2), we can rewrite the summand of $f_s(z)$ using an element of the group $SL(2,\mathbb{Z})$ as:

$$\frac{y^s}{|cz+d|^{2s}} = [\mathrm{Im}\,(\gamma \cdot z)]^s \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.1.3}$$

For this to be possible, two things have to occur: (i) For any co-prime pair (c, d) such a matrix $\gamma \in SL(2, \mathbb{Z})$ must exist, and (ii) if several matrices exist we must form equivalence classes such that the sum over co-prime pairs (c, d) corresponds exactly to the sum over equivalence classes. For (i), we note that the condition that c and d be co-prime is necessary since it would otherwise be impossible to satisfy the determinant condition $ad - bc = 1$ over $\mathbb{Z}$. At the same time, co-primality is sufficient to guarantee existence of integers a_0 and b_0 that complete c and d to a matrix $\gamma \in SL(2, \mathbb{Z})$. In fact, there is a one-parameter family of solutions for γ that can be written as

$$\begin{pmatrix} a_0 + mc & b_0 + md \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 & b_0 \\ c & d \end{pmatrix} \tag{3.1.4}$$

for any integer $m \in \mathbb{Z}$. That these are all solutions to the determinant condition over $\mathbb{Z}$ is an elementary lemma of number theory, sometimes called *Bézout's lemma* [77]. The form (3.1.4) tells us also how to resolve point (ii): We identify matrices that are obtained from each other by left multiplication by a matrix belonging to the *Borel subgroup*

$$B(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \middle| \, m \in \mathbb{Z} \right\} \subset SL(2, \mathbb{Z}). \tag{3.1.5}$$

The interpretation of this group is that it is the stabiliser of the y-axis.

Summarising the steps we have performed, we find that we can write the function (1.2.1) as

$$f_s(z) = 2\zeta(2s) \sum_{\gamma \in B(\mathbb{Z}) \backslash SL(2,\mathbb{Z})} \left[\mathrm{Im} \left(\gamma \cdot z \right) \right]^s . \tag{3.1.6}$$

Dropping the multiplicative ζ-factor, we obtain the function

$$E^{SL(2,\mathbb{R})}(\chi_s, z) = \sum_{\gamma \in B(\mathbb{Z}) \backslash SL(2,\mathbb{Z})} \chi_s(\gamma \cdot z) \, , \tag{3.1.7}$$

where we have also introduced the notation $\chi_s(z) = [\mathrm{Im}(z)]^s$. The reason for this notation is that χ_s is actually induced from a *character* on the real

Borel subgroup. We will explain this in more detail below in section 3.2.1. Note that this way of writing the automorphic form makes the invariance under $SL(2,\mathbb{Z})$ completely manifest because it is a sum over images. Let us also note again, that the form (3.1.7), which we call an Eisenstein series on $SL(2,\mathbb{R})$, generalises straight-forwardly to Lie groups $G(\mathbb{R})$ other than $SL(2,\mathbb{R})$, cf. (3.2.19), whereas the form with the sum over a lattice in (1.2.1) does not.

3.1.1 Forms on the upper-half complex plane

This section is intended to provide the reader with some perspective on the place that Eisenstein series take in the theory of automorphic forms. In particular we want to give a qualitative description of how automorphic forms, defined on the group $SL(2,\mathbb{R})$, are related to the classical theory of forms defined on the upper-half complex plane, $\mathcal{H}$.

The key here, is that the $SL(2,\mathbb{R})$ automorphic forms are obtained by *lifting* of the forms on the upper-half complex plane to the full group $SL(2,\mathbb{R})$. For forms on the upper-half complex plane, one draws the distinction between holomorphic and non-holomorphic forms, both of which we will give examples for in the following. On the level of the automorphic forms (after lifting) one finds that there exists an *interpolating series*, which provides a connection between the holomorphic vs. non-holomorphic classification. Although we will not present a detailed derivation of the interpolating series here, we will state its form and give a brief discussion of it. The scheme just described here is summarised in a diagrammatic way in figure 3.1.

Finally we also comment on the possibility of generalising this picture to the case of automorphic forms defined on groups of higher rank.

Holomorphic modular forms

Let $\mathcal{H}$ be the upper-half complex plane of points $\{z = x+iy \in \mathbb{C} | \operatorname{Im}(z) > 0\}$. This carries an action of $SL(2,\mathbb{R})$ given by the Möbius transformation

$$z \to g \cdot z = \frac{az+b}{cz+d}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}). \tag{3.1.8}$$

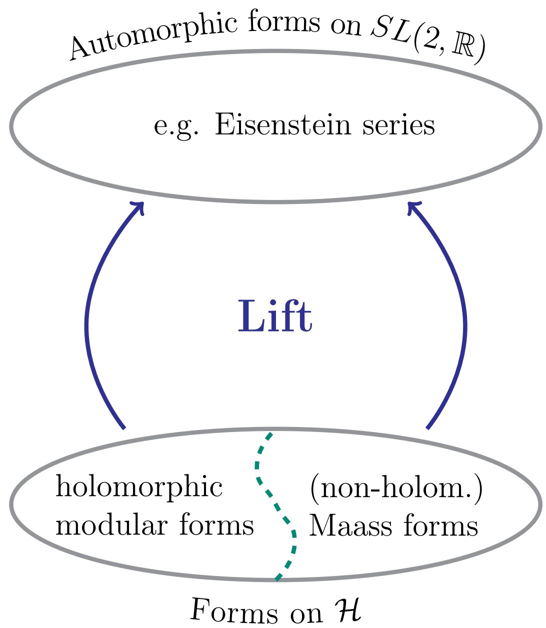

Figure 3.1: *Schematics of the lift from the classical theory of forms on upper-half complex plane $\mathcal{H}$ to the theory of automorphic forms on $SL(2,\mathbb{R})$.*

The appearance of the $SL(2,\mathbb{R})$-action is very natural since we have in fact an isomorphism $\mathcal{H} \cong SL(2,\mathbb{R})/SO(2,\mathbb{R})$, where $SO(2,\mathbb{R}) \subset SL(2,\mathbb{R})$ is the stabiliser of the point $i \in \mathcal{H}$.

A *modular form* of weight $w \geq 0$ is a *holomorphic* function $f : \mathcal{H} \to \mathbb{C}$ which transforms according to

$$f(\gamma \cdot g) = f\left(\frac{az+d}{cz+d}\right) = (cz+d)^w f(z), \tag{3.1.9}$$

under the discrete action of

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \tag{3.1.10}$$

If $f(z)$ has zero weight, $w = 0$, we call it a *modular function* [78]. The standard example of a modular form, is then given by the classical holomorphic Eisenstein series of weight w:

$$E_w(z) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ gcd(m,n)=1}} \frac{1}{(mz+n)^w}\,. \tag{3.1.11}$$

For further reading on the theory of modular forms, see for example [79]. We shall now see how to adapt the theory of holomorphic modular forms to make them amenable to lifting to the more general theory of automorphic forms. There is indeed a standard way of passing from a holomorphic modular form on $\mathcal{H}$ to an automorphic form on $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ [80]. The treatment followed here is based on [81].

Given a weight w holomorphic modular form f of weight w on $\mathcal{H}$ we define a new function on $SL(2,\mathbb{R})$ through the assignment

$$f \longmapsto \varphi_f(g) = (ci+d)^{-w} f(g\cdot i)\,, \tag{3.1.12}$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$. The function φ_f obtained in this way, satisfies the relation

$$\varphi_f(\gamma g) = \varphi_f(g)\,, \tag{3.1.13}$$

with $\gamma \in SL(2,\mathbb{Z})$, and is therefore automorphic according to our definition in section 1.2. Moreover, under the right-action of an $SO(2,\mathbb{R})$ element

$$k = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\,, \tag{3.1.14}$$

it transforms with a phase pre-factor:

$$\varphi_f(gk) = e^{-iw\theta}\varphi_f(g)\,. \tag{3.1.15}$$

This implies that the original transformation property (3.1.9) of f under $SL(2,\mathbb{Z})$ has been traded for the above phase transformation of $\varphi_f(g)$ under $K = SO(2,\mathbb{R})$. While f itself was invariant under $SO(2,\mathbb{R})$ one instead says that φ_f is K-*finite*, implying that the action of K on f generates a finite-dimensional vector space; in the present example this is represented by the one-dimensional space of characters $\sigma : k \mapsto e^{-iw\theta}$.

We should also understand how the automorphic form φ_f incorporates the holomorphicity of the form f on $\mathcal{H}$. The fact that f is a holomorphic function implies

$$\frac{\partial}{\partial \bar{z}} f = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = 0\,, \tag{3.1.16}$$

where $z = x + iy \in \mathcal{H}$. The condition (3.1.16) translates into the property that φ_f is annihilated by a differential operator F on $SL(2,\mathbb{R})$:

$$F\varphi_f = -2ie^{-2i\theta}\left(y\frac{\partial}{\partial \bar{z}} - \frac{1}{4}\frac{\partial}{\partial \theta}\right)\varphi_f = 0\,, \tag{3.1.17}$$

To see this we use the Iwasawa decomposition of an element $g \in SL(2,\mathbb{R})$:

$$g = nak = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\,, \tag{3.1.18}$$

with $n \in N(\mathbb{R}), a \in A(\mathbb{R}), k \in SO(2,\mathbb{R})$, see also appendix A.3. Thus, we can view φ_f as a function of the three variables (x, y, θ):

$$\varphi_f(g) = \varphi_f(x, y, \theta) = e^{iw\theta} y^{w/2} f(x + iy)\,. \tag{3.1.19}$$

Applying the operator F in (3.1.17) on this expression then immediately shows that it annihilates φ_f whenever f satisfies the holomorphicity condition (3.1.16).

Before we proceed we shall mention one final important property of φ_f, namely that it is an eigenfunction of the Laplacian on $SL(2,\mathbb{R})$:

$$\Delta\varphi_f = -\frac{w}{2}\left(\frac{w}{2} - 1\right)\varphi_f\,, \tag{3.1.20}$$

where

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - y\frac{\partial^2}{\partial x\partial\theta}\,. \tag{3.1.21}$$

As we will see, all the properties of φ_f discussed above will have counterparts in the general theory of automorphic forms.

Maass wave forms (non-holomorphic)

In addition to the holomorphic modular forms, the classical theory on the upper-half plane also contains an interesting class of *non-holomorphic* functions $f : SL(2,\mathbb{Z})\backslash\mathcal{H} \to \mathbb{R}$. These non-holomorphic functions are eigenfunctions of the hyperbolic Laplacian $\Delta_{\mathcal{H}}$ on $\mathcal{H} = SL(2,\mathbb{R})/SO(2,\mathbb{R})$:

$$\Delta_{\mathcal{H}} = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) = -(z-\overline{z})^2\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}\,, \tag{3.1.22}$$

where $\partial/\partial z = (\partial/\partial x - i\partial/\partial y)/2$ and $\partial/\partial z = (\partial/\partial x + i\partial/\partial y)/2$. Such functions are called *Maass (wave) forms* and an important example of a Maass form is provided by the non-holomorphic Eisenstein series already stated in equation (3.1.2) in the introduction:

$$E^{SL(2,\mathbb{R})}(s,z) = \sum_{\substack{(m,n)\in\mathbb{Z}^2\\ gcd(m,n)=1}} \frac{y^s}{|mz+n|^{2s}}. \tag{3.1.23}$$

This converges absolutely for $\text{Re}(s) > 1$, but, as already mentioned in section 3.2.3, according to Langlands' general theory [82], it can be analytically continued to a meromorphic function of $s \in \mathbb{C}$. One can verify that this indeed defines an eigenfunction of the Laplacian $\Delta_{\mathcal{H}}$ that is invariant under $SL(2,\mathbb{Z})$, with eigenvalue $s(s-1)$.

Let us now see how Maass forms fit into the general framework of the automorphic forms on $SL(2,\mathbb{R})$. We will see that this requires even less effort than for the holomorphic modular forms. Given a Maass form f on $\mathcal{H}$

we lift it to a function φ_f on $SL(2,\mathbb{R})$ according to the simple assignment

$$f \longmapsto \varphi_f(g) = \varphi_f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} k\right) = f(x+iy)\,, \qquad (3.1.24)$$

where we have used the Iwasawa decomposition $g = nak \in SL(2,\mathbb{R})$ of equation (A.3.2). The associated function $\varphi_f(g)$ then satisfies the relation

$$\varphi_f(\gamma g k) = \varphi_f(g)\,, \qquad (3.1.25)$$

with $\gamma \in SL(2,\mathbb{Z})$ and $k \in SO(2,\mathbb{R})$ and so is indeed an automorphic form on $SL(2,\mathbb{R})$.

3.1.2 Interpolating series

In this section, we provide a brief discussion of a non-holomorphic series, which represents a kind of interpolating step between the classical holomorphic modular forms and the non-holomorphic Maass wave forms. We have chosen not to present the slight lengthy and technical proof here, which is however outlined in detail in article **IV**.

The mentioned interpolating series is of the form:

$$E^{SL(2,\mathbb{R})}(s,w,g) = e^{iw\theta} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ gcd(m,n)=1}} \frac{y^s}{(mz+n)^w |mz+n|^{2s-w}}\,. \qquad (3.1.26)$$

It is a holomorphic form on $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ with weight $e^{iw\theta}$ under the right action of $k \in SO(2,\mathbb{R})$, cf. (3.1.15). Let us now demonstrate the 'interpolating' role of this series. Namely, by $s = w/2$ the series becomes

$$E^{SL(2,\mathbb{R})}(w/2,w,g) = e^{iw\theta} y^{w/2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ gcd(m,n)=1}} \frac{1}{(mz+n)^w} = e^{iw\theta} y^{w/2} E_w(z)\,, \qquad (3.1.27)$$

which we recognise immediately as the function $\varphi_f(g)$, see equation (3.1.19), in the terminology of section 3.1.1. Here $f = E_w(z)$ is the classical holomorphic Eisenstein series (3.1.11) on the upper-half complex plane $\mathcal{H}$ with weight w.

On the other hand, restricting to $w = 0$ in (3.1.26) we obtain the classical non-holomorphic Eisenstein series (Maass form)

$$E^{SL(2,\mathbb{R})}(s,0,g) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ gcd(m,n)=1}} \frac{y^s}{|mz+n|^{2s}} = E^{SL(2,\mathbb{R})}(s,z)\,, \tag{3.1.28}$$

cf. (3.1.23). Thus we see how for particular choices of the parameters s and w, we recover either the classical holomorphic or non-holomorphic theory.

3.1.3 Generalisation to higher rank groups

Since we can view modular forms as holomorphic functions on the coset $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ with simple transformation properties under $SL(2,\mathbb{Z})$, it now of course seems natural to consider generalisations of this to higher rank real Lie groups $G(\mathbb{R})$. To this end we note that $SO(2,\mathbb{R}) \cong U(1)$ is the maximal compact subgroup of $SL(2,\mathbb{R})$. Hence, one might suspect a generalization to holomorphic functions $f : G(\mathbb{R})/K \to \mathbb{C}$, where K is the maximal compact subgroup of $G(\mathbb{R})$, transforming with some weight under the action of a discrete subgroup $G(\mathbb{Z}) \subset G(\mathbb{R})$. However, this only works whenever the coset $G(\mathbb{R})/K$ carries a complex structure. A standard example is provided by $G = Sp(2n,\mathbb{R})$, $K = U(n)$, in which case $Sp(2n;\mathbb{R})/U(n)$ is a hermitian symmetric domain known as the Siegel upper half space. This leads to the theory of holomorphic Siegel modular forms, see for example [83] for a review.

However, in general, symmetric spaces G/K do not carry a complex structure, and therefore we can not expect to have a general theory of holomorphic modular forms on G/K. Nonetheless, one can look for a theory of (non-holomorphic) functions $f : G(\mathbb{R}) \to \mathbb{C}$ which transform nicely under the action of some discrete subgroup $G(\mathbb{Z}) \subset G(\mathbb{R})$. This leads to the notion of an automorphic form.

3.2 Defining Eisenstein series

In this section we will state Langlands' general definition of an Eisenstein series in [35]. We will do this by induction from a parabolic subgroup, which we denote by $P(\mathbb{R})$. For the purpose of this section we will take $P(\mathbb{R}) = B(\mathbb{R})$, the Borel subgroup, which yields to the particular definition of what we will refer to as the *minimal* parabolic Eisenstein series. In section 3.3, we will extend the definition by defining Eisenstein series induced from parabolic subgroups other than the Borel subgroup. In the following, Eisenstein series will be induced through a character χ, which will be defined now, together with a list of some of its properties.

3.2.1 Multiplicative characters on the Borel subgroup

Let us begin by fixing a Borel subgroup, $B(\mathbb{R}) \subset G(\mathbb{R})$. In a *Levi decomposition* of the Borel subgroup, we obtain

$$B(\mathbb{R}) = A(\mathbb{R})N(\mathbb{R}) = N(\mathbb{R})A(\mathbb{R})\,. \tag{3.2.1}$$

Here $A(\mathbb{R})$ is the Cartan torus and $N(\mathbb{R})$ is the unipotent radical. The Lie algebra of the Borel therefore include the Cartan subalgebra generators, together with the raising generators associated with the positive roots of the algebra. For a more careful definition of the Borel subgroup in terms of its Lie algebra see appendix A.2. Of central importance in the definition of Eisenstein series is the multiplicative character χ, which is defined on the Borel subgroup in the following way:

$$\chi\,:\,B(\mathbb{Z})\backslash B(\mathbb{R}) \to \mathbb{C}^{\times}. \tag{3.2.2}$$

The defining property of the character χ is that it is invariant under the action of elements of the unipotent subgroup N on its arguement, from the left:

$$\chi(na) = \chi(a)\,, \tag{3.2.3}$$

where $a \in A(\mathbb{R})$ and $n \in N(\mathbb{R})$.

Employing the Iwasawa decomposition, $G = NAK$, see also appendix A.3, to a group element $g \in G(\mathbb{R})$, we can then extend the definition of χ to the full group $G(\mathbb{R})$, by demanding the character to be trivial on the compact subgroup $K(\mathbb{R}) \subset G(\mathbb{R})$:

$$\chi(g) = \chi(nak) = \chi(na) = \chi(an) = \chi(a)\,, \tag{3.2.4}$$

with $k \in K(\mathbb{R})$. Although we have extended the character to the full group $G(\mathbb{R})$ it is nevertheless only multiplicative on the Borel subgroup $B(\mathbb{R})$:

$$\chi(bb') = \chi(b)\chi(b') = \chi(a)\chi(a')\,, \tag{3.2.5}$$

where $b, b' \in B(\mathbb{R})$. On the other hand, to evaluate it on a product of two elements $g, g' \in G(\mathbb{R})$ we have

$$\chi(gg') = \chi(bkb'k') = \chi(bkb') = \chi(b\tilde{b}\tilde{k}) = \chi(b\tilde{b}) = \chi(b)\chi(\tilde{b})\,. \tag{3.2.6}$$

Here we have made use of the fact that the character is trivial on $K(\mathbb{R})$ and we have defined $\tilde{b}\tilde{k}$ as the Iwasawa decomposition of the kb' factor. From this we also see that

$$\chi(bg) = \chi(b)\chi(g)\,, \tag{3.2.7}$$

where $b \in B(\mathbb{R})$ and $g \in G(\mathbb{R})$.

3.2.2 Characters and weights

We will now introduce a one-to-one correspondence between the multiplicative character χ defined above and weights of the Lie algebra $\mathfrak{g}(\mathbb{R})$. More precisely, there is a correspondence between the character χ and complex linear functionals

$$\lambda \in \mathfrak{h}^\star_{\mathbb{C}} = \mathfrak{h}(\mathbb{R})^\star \otimes_{\mathbb{R}} \mathbb{C}\,, \tag{3.2.8}$$

where $\mathfrak{h}(\mathbb{R})$ is the Cartan subalgebra of $\mathfrak{g}(\mathbb{R})$, see appendix A.1 for a precsie definition. In order to develop an explicit parameterisation of the character χ, we define a logarithmic map $H(\cdot)$ in the following way:

$$H : G(\mathbb{R}) \to \mathfrak{h}(\mathbb{R}), \tag{3.2.9}$$

such that

$$H(g) = H(nak) = \log |a| \,, \tag{3.2.10}$$

where log denotes the natural logarithm. The map therefore essentially yields the abelian part in an Iwasawa decomposition of the group element g. The absolute value here is defined as follows.

We parametrize the group element $a \in A(\mathbb{R})$ by

$$a = \exp\left(\sum_{\alpha\in\Pi} u_\alpha H_\alpha\right), \qquad H_\alpha \in \mathfrak{h}(\mathbb{R}), \quad u_\alpha \in \mathbb{R}, \tag{3.2.11}$$

where Π denotes the set of simple roots of $\mathfrak{g}(\mathbb{R})$. Then the absolute value simply means

$$\log |a| := \log\exp\left(\sum_{\alpha\in\Pi} |u_\alpha| H_\alpha\right) = \sum_{\alpha\in\Pi} |u_\alpha| H_\alpha \,. \tag{3.2.12}$$

The character χ can now be parametrized by the choice of linear functional λ according to the form:

$$\chi(g) = e^{\langle\lambda+\rho|H(g)\rangle} = |a^{\lambda+\rho}| \,, \tag{3.2.13}$$

where in the last equality, we have introduced a convenient short-hand notation. The 'inner product' $\langle\cdot|\cdot\rangle$ is defined as the standard pairing between the space $\mathfrak{h}$ and its dual space $\mathfrak{h}^\star$. Furthermore the translation by the Weyl vector ρ in the above formula, constitutes a convenient choice of normalisation. The Weyl vector is define as half the sum of positive roots, or alternatively, as the sum of fundamental weights of the algebra $\mathfrak{g}$.

For subsequent calculations, it will also be useful to have the following definitions at hand. Namely, we define

$$b \mapsto e^{\langle 2\rho | H(b) \rangle} \equiv \delta_B(b)\,, \tag{3.2.14}$$

where $b \in B(\mathbb{R})$. This is often called the modular function (or "modulus character") of $B(\mathbb{R})$. It is defined by

$$\delta_B(b) = \Big| \det \mathrm{ad}(b)\big|_{\mathfrak{n}} \Big|. \tag{3.2.15}$$

In words, it is the modulus of the determinant of the adjoint representation of $b \in B(\mathbb{R})$, restricted to the Lie algebra $\mathfrak{n}$ of the unipotent radical N. The modulus character corresponds to the Jacobian that relates the left- and right-invariant Haar measures on B. This implies in particular that under conjugation by $a \in A(\mathbb{R})$, i.e.

$$n \mapsto ana^{-1}, \tag{3.2.16}$$

the Haar measure dn on $N(\mathbb{R})$ transforms by

$$dn \mapsto \delta_B(b) dn. \tag{3.2.17}$$

This fact will be used in the calculations of chapter 4. Finally, using the modulus character we can write χ of (3.2.13) in the alternative form

$$\chi(g) = e^{\langle \lambda | H(g) \rangle} \delta_B^{1/2}(g). \tag{3.2.18}$$

3.2.3 Minimal parabolic Eisenstein series

We now define the following (Langlands-)Eisenstein series [35] on the group $G(\mathbb{R})$, by using the multiplicative character χ which we introduced above. The series is defined as

$$E^{G(\mathbb{R})}(\chi, g) \equiv \sum_{\gamma \in B(\mathbb{Z}) \backslash G(\mathbb{Z})} \chi(\gamma g)\,. \tag{3.2.19}$$

By employing the explicit parameterisation (3.2.13) of the character χ, we can also write this series as

$$E^{G(\mathbb{R})}(\lambda, g) \equiv \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda+\rho | H(\gamma g)\rangle} . \tag{3.2.20}$$

Here $G(\mathbb{Z})$ is the Chevalley group of $G(\mathbb{R})$ and $B(\mathbb{Z}) = B(\mathbb{R}) \cap G(\mathbb{Z})$ the corresponding discrete version of the Borel subgroup $B(\mathbb{R})$. The parameter λ is a general weight vector of G (which does not have to lie on the weight lattice) and relates directly to our discussion of the character χ and its parameterisation in section 3.2.2 above. As mentioned earlier, the Weyl vector ρ, is defined as half the sum of all positive roots or alternatively as the sum over all fundamental weights which we denote by Λ_i, with $i = 1, \ldots, \mathrm{rk}(G)$. The angled brackets in the definition are the standard pairing between the space of weights $\mathfrak{h}^*$ and the Cartan subalgebra $\mathfrak{h}$.

Due to the invariance of the character under the compact subgroup $K(\mathbb{R})$, the Eisenstein series defined in (3.2.20), as a function, depends on the continuous variables that parameterise the coset $G(\mathbb{R})/K(\mathbb{R})$. Overall, the series displays invariance under the discrete group $G(\mathbb{Z})$.

We refer to the function defined in (3.2.20) as a *minimal parabolic* Eisenstein series, since it is associated with the minimal parabolic subgroup B through the inducing character χ. The sum (3.2.20) converges when the real parts of the inner products $\langle \lambda | \alpha_i \rangle$ for all simple roots α_i are sufficiently large and can be analytically continued to the complexified space of weights except for certain hyperplanes [35]. In slightly more technical terms, Langlands proved that the sum converges absolutely whenever λ lies in the open subset

$$\{\lambda \in \mathfrak{h}^\star_{\mathbb{C}} \mid \mathrm{Re}(\lambda) \in \rho + (\mathfrak{h}^\star)^+\}, \tag{3.2.21}$$

known as *Godement's domain.* Here the positive chamber $(\mathfrak{h}^\star)^+$ is defined by

$$(\mathfrak{h}^\star)^+ = \{\Lambda \in \mathfrak{h}^\star \mid \langle \Lambda, H_\alpha \rangle > 0, \, \forall \, \alpha \in \Pi\}, \tag{3.2.22}$$

so that we require $\langle\lambda, H_\alpha\rangle > 1$ for all simple roots α. Remarkably Langlands showed that the Eisenstein series $E(\lambda, g)$ can in fact be analytically continued outside of the domain (3.2.21) to a meromorphic function on all of $\mathfrak{h}^\star_{\mathbb{C}}$. To establish the analytic continuation a crucial property of $E(\lambda, g)$ is the *functional relation*

$$E(\lambda, g) = M(w, \lambda)E(w\lambda, g), \tag{3.2.23}$$

which it satisfies. The relation relates the value of the series at λ to its value at the Weyl transform (by a Weyl word w) of λ. The factor $M(w, \lambda)$, whose properties we will discuss extensively in subsequent chapters, is defined as

$$M(w, \lambda) = \prod_{\alpha>0\,|\,w\alpha<0} \frac{\xi(\langle\lambda|\alpha\rangle)}{\xi(1 + \langle\lambda|\alpha\rangle)}\,. \tag{3.2.24}$$

What this means, is that the product runs over all *positive* roots α of the algebra $\mathfrak{g}$, that are mapped to negative roots by w. The function $\xi(k) = \pi^{-k/2}\Gamma(k/2)\zeta(k)$ is the completed Riemann ζ-function and the angled bracket denotes the canonical inner product on the space of weights. The factor $M(w, \lambda)$ furthermore enjoys the important multiplicative property

$$M(w\tilde{w}, \lambda) = M(w, \tilde{w}\lambda)M(\tilde{w}, \lambda)\,, \tag{3.2.25}$$

where $w, \tilde{w} \in \mathcal{W}$. Here $\mathcal{W}$ is the Weyl group of G and is discussed in detail in section A.1.1 of the appendix.

3.2.4 Physical perspective on Langlands' definition

In this section we provide the reader with a different, more physically oriented way of looking at Eisenstein series, which we hope will elucidate the definition (3.2.20) further. The exposition directly relates to the discussion of the previous chapter and in particular to sections 2.1.3, 2.2.1.

As mentioned earlier, Eisenstein series defined on the $E_n(\mathbb{R})$ groups, are of particular interest to us. In order to make the subsequent discussion more concrete, let us therefore restrict ourselves in this section to Eisenstein series defined on these groups. Then, the Eisenstein series are functions on the moduli space $G/K = E_n(\mathbb{R})/K(E_n(\mathbb{R}))$ and their invariance under the discrete $G(\mathbb{Z}) = E_n(\mathbb{Z})$ group, can be constructed from a 'perturbative term' in the following way.

Let us explain what we mean by a perturbative term. For this we first note that the 'coupling constants' of the compactified string theory sit in the maximal split torus $A(\mathbb{R}) \subset G(\mathbb{R})$. We suppose that a particular perturbative term is known and that it is of the form $r_1^{2s_1} r_2^{2s_2} \dots r_n^{2s_n}$, where the r_i label the coupling constants, and the constants s_i parameterise the dependence of the perturbative term on the coupling constants. This is similar to the tension defined in equation (2.1.13) in section 2.1.3 describing U-duality.

Generally, there will also be an overall numerical coefficient of the term that we can absorb in the normalisation of the automorphic function to be defined presently. An example would be the perturbative string tree level correction to four-graviton scattering associated with a term $\mathcal{R}^4$: This term is of the form $2\zeta(3)g_{\mathrm{s}}^{3/2}$ (in Einstein frame) and the string coupling g_{s} is related to the r_i in a polynomial way and the overall coefficient $2\zeta(3)$ is the overall normalisation that we will no longer discuss. Given such a perturbative term, we can construct its $G(\mathbb{Z})$-completion by summing over all its images.

To develop this picture further, let $\chi : G \to \mathbb{C}^\times$ be a function that projects a group element onto the perturbative term, i.e.

$$\chi(g) = r_1^{2s_1} \dots r_n^{2s_n}, \tag{3.2.26}$$

where we parameterise the Cartan torus $A \subset G$ by

$$a = \exp\left[\log(r_1)h_1 + \ldots + \log(r_n)h_n\right] = r_1^{h_1} \dots r_n^{h_n}, \tag{3.2.27}$$

where the h_i are the standard Chevalley generators of the Cartan subalgebra (their Killing inner product matrix is the Cartan matrix). The function χ

satisfies

$$\chi(nak) = \chi(a) \tag{3.2.28}$$

where $g = nak$ is the Iwasawa decomposition and we will, loosely speaking, refer to it as a quasi-character. The function χ is invariant under discrete Borel elements $B(\mathbb{Z})$: $\chi(\gamma g) = \chi(g)$ for $\gamma \in B(\mathbb{Z})$. Physically, these transformations correspond to discrete large gauge transformations of the axions. We also allow the parameters s_i to take complex values. Starting from χ we define the Eisenstein series

$$E(\chi, g) = \sum_{\gamma \in B(\mathbb{Z}) \backslash G(\mathbb{Z})} \chi(\gamma g). \tag{3.2.29}$$

This is the sum over all $G(\mathbb{Z})$-images of the perturbative term where the χ-stabilising $B(\mathbb{Z})$ transformations have been quotiented out in order not to overcount the sum. This discussion then relates to the more mathematical description outlined in the previous section above and which was developed in [35, 84].

In order to mention one other property of the Eisenstein series (3.2.20), we note that it is made up of a simple 'plane-wave type' function $e^{\langle \lambda + \rho | H(g) \rangle} = e^{2 \log(r_i) s_i}$, where a sum over the index i is implied. The terminology 'plane wave' is used here with an application to quantum gravity in mind, where the Eisenstein series should describe wavefunctions [34, 85].
The plane-wave function is trivially an eigenfunction of the quadratic Laplace operator and all higher-order invariant differential operators. Since all these operators are invariant under the group $G(\mathbb{Z})$ (even $G(\mathbb{R})$), the Eisenstein series $E^G(\lambda, g)$ of (3.2.20), which is obtained by summing over all the (in-equivalent) $G(\mathbb{Z})$ images of the plane-wave solution, is an eigenfunction of all these operators. In particular, its eigenvalue under the G-invariant Laplacian $\Delta^{G/K}$ (changing the normalisation of [74]) is

$$\Delta^{G/K} E^G(\lambda, g) = \frac{1}{2} \left(\langle \lambda | \lambda \rangle - \langle \rho | \rho \rangle \right) E^G(\lambda, g) \,. \tag{3.2.30}$$

This is the same eigenvalue as that of the quadratic Casimir on a representation with highest weight $\Lambda = -(\lambda + \rho)$ up to normalisation. The standard pairing $\langle \cdot | \cdot \rangle$ is normalised such that $\langle \alpha_i | \alpha_i \rangle = 2$ for simple roots α_i.

3.3 Eisenstein series on general parabolics

In this section we would like to extend the definition of the minimal parabolic Eisenstein series (3.2.20) to the case when the series is defined with respect to other parabolic subgroups and in particular to *maximal* parabolics. The maximal parabolic Eisenstein series are also the series which are relevant to our discussion in the context of string theory. Let us start by giving some mathematical background on non-minimal parabolic subgroups.

3.3.1 Non-minimal parabolics

To start, we restrict ourselves to standard parabolic subgroups, which by their definition contain the Borel subgroup $B(\mathbb{R}) = A(\mathbb{R})N(\mathbb{R})$ of $G(\mathbb{R})$ as a subgroup. Then, let us consider a particular parabolic subgroup $P(\mathbb{R}) \subset G(\mathbb{R})$. As in the case of the Borel subgroup, there is a canonical *Levi decomposition* of this group

$$P(\mathbb{R}) = L_P(\mathbb{R})U_P(\mathbb{R}) , \tag{3.3.1}$$

where $L_P(\mathbb{R})$ is referred to as the (unique) Levi subgroup that contains $A(\mathbb{R})$ and $U_P(\mathbb{R})$ is the unipotent radical and is contained in $N(\mathbb{R})$. For a Lie algebra definition of these subgroups, see appendix A.2. Let us also note that in this section, we will explicitly put a P label as subscript on all subgroups and objects defined with respect to the parabolic. The Levi factor further decomposes as $L_P(\mathbb{R}) = M_P(\mathbb{R})A_P(\mathbb{R})$, where $A_P(\mathbb{R})$ is the maximal torus in the centre of $L_P(\mathbb{R})$; loosely speaking, we think of $A_P(\mathbb{R})$ as consisting of diagonal matrices commuting with $M_P(\mathbb{R})$.

With the further decomposition of the Levi subgroup L_P from above, we obtain the so-called *Langlands decomposition* of the parabolic subgroup

$$P(\mathbb{R}) = M_P(\mathbb{R})A_P(\mathbb{R})U_P(\mathbb{R}) = U_P(\mathbb{R})A_P(\mathbb{R})M_P(\mathbb{R})\,, \tag{3.3.2}$$

and the full group $G(\mathbb{R})$ factorizes (non-uniquely) to

$$G(\mathbb{R}) = U_P(\mathbb{R})L_P(\mathbb{R})K_{\mathbb{R}} = U_P(\mathbb{R})A_P(\mathbb{R})M_P(\mathbb{R})\,, \tag{3.3.3}$$

with K being the compact subgroup of $G(\mathbb{R})$. For an arbitrary element of $G(\mathbb{R})$ we thus have the decomposition $g = ulk = umak$, where $u \in U_P(\mathbb{R})$, $m \in M_P(\mathbb{R})$, $a \in A_P(\mathbb{R})$ and $k \in K_{\mathbb{R}}$. In the following, we will drop the subscript label P on the various subgroups, whenever it is unambiguously clear which parabolic they are subgroups of.

3.3.2 Multiplicative characters

We now want to define multiplicative characters on $P(\mathbb{R})$ analogously to what was done for the Borel subgroup in section 3.2.1. These will be homomorphisms

$$\chi_P \,:\, P(\mathbb{Z})\backslash P(\mathbb{R}) \,\to \mathbb{C}^\times\,, \tag{3.3.4}$$

determined by their restriction to the Levi subgroup

$$\chi_P(ul) = \chi_P(l)\,, \tag{3.3.5}$$

with $u \in U_P(\mathbb{R})$ and $l \in L_P(\mathbb{R})$. In contrast to the case of the Borel subgroup discussed above, where $L_B(\mathbb{R}) = A(\mathbb{R})$, the Levi part $L_P(\mathbb{R})$ for a general standard parabolic is now non-abelian, and it is therefore not directly clear how to define its value $\chi_P(l)$. Without going into too much detail here, we will do this via a generalisation of the logarithmic map $H(\cdot)$ from (3.2.9) and (3.2.10), which then defines the character χ_P. The map, associated with the parabolic subgroup $P(\mathbb{R})$, is then defined as

$$H_P \,:\, L_P(\mathbb{R}) \,\to\, \mathfrak{h}_P(\mathbb{R})\,. \tag{3.3.6}$$

This is in fact nothing but the restriction of the Cartan subalgebra $\mathfrak{h}(\mathbb{R}) \subset \mathfrak{g}(\mathbb{R})$ to the Lie algebra $\mathfrak{h}_P$ of $A_P(\mathbb{R})$.

Analogously to the case of the Borel subgroup, we define the multiplicative character χ_P in terms of H_P and a choice of complex linear functional $\lambda \in \mathfrak{h}_P^\star(\mathbb{C})$ as follows

$$\chi_P = e^{\langle \lambda + \rho | H_P \rangle} \,. \tag{3.3.7}$$

3.3.3 Maximal parabolic Eisenstein series

We will now consider a particular type of parabolic subgroup, namely the *maximal* parabolic. This parabolic is characterised by the choice of a simple root α_{i_*} from the set of simple roots Π of the group G. The maximal parabolic subgroup is correspondingly denoted by P_{i_*} and the definition via the associated Lie algebra $\mathfrak{p}_{i_*}$ follows appendix A.2.1. As for the minimal parabolic case, the maximal parabolic subgroup is used to induce the Eisenstein series.

The weight λ defining this maximal parabolic Eisenstein series takes the form

$$\lambda = 2s\Lambda_{i_*} - \rho \,. \tag{3.3.8}$$

Here Λ_{i_*} is the fundamental weight associated with our choice of simple root α_{i_*}. This form of the weight λ implies that the combination $\lambda + \rho$ will be orthogonal to all simple roots α_i with $i \neq i_*$. The parameter s which appears here, and which already appeared in (3.1.7), is generically a complex number. However, as we will see in the next section, the cases which are relevant in the context of superstring four-graviton scattering, s will be purely real and take half-integer values.

The particular form (3.3.8) of λ should be viewed as a condition imposed on the general form of an Eisenstein series. Upon making this particular choice for the defining weight, the definition (3.2.20) of the minimal parabolic Eisenstein series is extended to that of a maximal parabolic series. This is demonstrated by a short calculation, see [74]. The sum in (3.2.20) then becomes a sum over the coset $P_{i_*}(\mathbb{Z})\backslash G(\mathbb{Z})$ and taking the form

$$E^G(2s\Lambda_{i_*} - \rho, g) = \sum_{\gamma \in P_{i_*}(\mathbb{Z})\backslash G(\mathbb{Z})} e^{2s\langle \Lambda_{i_*} | H(\gamma g)\rangle} . \tag{3.3.9}$$

We also introduce the following short-hand notation for the maximal parabolic Eisenstein series, $E^G_{i_*;s}(g) := E^G(2s\Lambda_{i_*} - \rho, g)$, at this point.

3.4 Kac–Moody Eisenstein series

So far we have only considered Eisenstein series defined on classical finite-dimensional Lie groups. In the following, we will extend the definition of an Eisenstein series to the more general Kac–Moody groups. In particular, we are interested in defining Eisenstein series which are invariant under $E_9(\mathbb{Z})$, $E_{10}(\mathbb{Z})$ and $E_{11}(\mathbb{Z})$, due to their role as infinite-dimensional extensions of the U-duality groups, cf. 2.1.3. We will begin with a careful treatment of the affine Kac–Moody group case, for which we rely on mathematical results obtained by H. Garland in [76, 86]. This treatment will cover the case of E_9, with all the subtleties of an affine group. The definitions of the E_{10} and E_{11} Eisenstein series can be obtained by direct analogy with the finite-dimensional definition (3.2.20).

3.4.1 Affine Eisenstein series

The theory of Eisenstein series defined on affine (loop) groups was first developed by Garland and is comprehensively described in [76] (see also [86] an [87]). Indeed, the definition of Eisenstein series over affine groups proceeds in much the same way as the one for the finite-dimensional groups. There are, however, some subtleties which we shall explain in the following.

From here on, a hat is used to denote objects of affine type. Starting from a finite-dimensional, simple and $\mathbb{R}$-split Lie algebra $\mathfrak{g}$ one constructs the non-twisted affine extensions as

$$\hat{\mathfrak{g}} = \mathfrak{g}[[t, t^{-1}]] \oplus c\,\mathbb{R} \oplus d\,\mathbb{R} . \tag{3.4.1}$$

The first summand is the algebra of formal Laurent series over $\mathfrak{g}$ (the loop algebra) and the other summands are the central extension and derivation, respectively. The algebra $\hat{\mathfrak{g}}$ has a Cartan subalgebra of dimension $\dim(\mathfrak{h})+2$ and its roots decompose into real roots and imaginary roots, see e.g. [88,89].

The real affine group $\hat{G}$ (in a given representation over $\mathbb{R}$) is defined by taking the closure of exponentials of the *real* root generators of the non-twisted affine algebra. Due to the structure of the commutation relations, where d never appears on the right-hand side, the group thus generated will not use the derivation generator. $\hat{G}$ has the following Iwasawa decomposition

$$\hat{G} = \hat{N}\hat{A}\hat{K}\,, \tag{3.4.2}$$

analogous to (A.3.1), but now $\hat{A}$ is the exponential of the $(\dim(\mathfrak{h})+1)$-dimensional, abelian algebra $\hat{\mathfrak{h}} \equiv \mathfrak{h} \oplus c\mathbb{R}$ only, see [76]. $\hat{G}$ does not include the group generated by the derivation d; we denote by E_9 the group $\hat{E}_8$ with the derivation added to it.

Similar to the definition of the Eisenstein series over finite-dimensional groups, in the infinite-dimensional case one can define in a meaningful manner the series

$$E^{\hat{G}}(\hat{\lambda};\hat{g},v) = \sum_{\hat{\gamma}\in\hat{B}(\mathbb{Z})\backslash\hat{G}(\mathbb{Z})} e^{\langle\hat{\lambda}+\hat{\rho}|\hat{H}(\hat{\gamma}\,v^d\,\hat{g})\rangle}\,, \tag{3.4.3}$$

where v parameterises the group associated with the derivation generator d and is written as e^{-rd} in the notation of [76]. This definition of the Eisenstein series is derived in [76] and the convergence of the series is proven for $\mathrm{Re}\langle\hat{\lambda}|\hat{\alpha}_i\rangle > 1$ for $i = 1,\ldots,\mathrm{rk}(G)+1$. The definition domain can be extended by meromorphic continuation. One important special property of the affine case that enters in (3.4.3) is the definition of the affine Weyl vector $\hat{\rho}$: The usual requirement for the Weyl vector to have inner product $\langle\hat{\rho}|\hat{\alpha}_i\rangle = 1$ with all affine simple roots $\hat{\alpha}_i$ does not fix $\hat{\rho}$ completely; it is only defined up to shifts by the so-called (primitive) null root $\hat{\delta}$ that has vanishing inner product with all $\hat{\alpha}_i$ [88]. We choose the standard convention that $\hat{\rho}$ is the sum of all the fundamental weights, as in [88], i.e., it acts on the derivation d by $\hat{\rho}(d) = 0$.

As in the finite-dimensional case, the Eisenstein series (3.4.3) is an eigenfunction of the full affine Laplacian and has eigenvalue

$$\Delta^{\hat{G}/\hat{K}} E^{\hat{G}}(\hat{\lambda}; \hat{g}, v) = \frac{1}{2}(\langle\hat{\lambda}|\hat{\lambda}\rangle - \langle\hat{\rho}|\hat{\rho}\rangle) E^{\hat{G}}(\hat{\lambda}; \hat{g}, v)\,. \tag{3.4.4}$$

The Laplacian itself is not unambiguously defined because of the ambiguity in $\hat{\rho}$ (related to a rescaling of the overall volume of moduli space). We reiterate that we adopt consistently the convention that $\hat{\rho}$ has no $\hat{\delta}$ part. An important difference to the finite-dimensional case is that there are no higher order polynomial invariant differential operators that help to determine $\hat{\lambda}$ but only transcendental ones [90]. Their detailed action on (3.4.3) was not investigated in our work, but it would certainly be interesting to follow-up on this.

By imposing the additional condition $\hat{\lambda} = 2s\hat{\Lambda}_{i_*} - \hat{\rho}$ on the (minimal) Eisenstein series defined above in (3.4.3) one can, as in (3.3.9), obtain a maximal parabolic, affine Eisenstein series:

$$E^{\hat{G}}_{i_*;s}(\hat{g}, v) := \sum_{\hat{\gamma} \in \hat{P}_{i_*}(\mathbb{Z})\backslash\hat{G}(\mathbb{Z})} e^{2s\langle\hat{\Lambda}_{i_*}|\hat{H}(\hat{\gamma}\, v^d\, \hat{g})\rangle}\,. \tag{3.4.5}$$

3.4.2 More general Kac–Moody Eisenstein series

Turning to more general Kac–Moody groups, we will assume that the Eisenstein series for $E_n(\mathbb{Z})$ with $n > 9$ are defined formally exactly as in (3.2.20). A proof for the validity of this formula (i.e. existence via convergence) is not known to our knowledge but for sufficiently large real parts of λ one should obtain a convergent bounding integral and then continue meromorphically. The definition of the real group and the Chevalley group proceeds along the same lines as in the affine case [90]. The expression for the Laplace eigenvalue is as before in (3.2.30) and is unambiguous for E_n with $n > 9$.

For a discussion of Eisenstein series defined on rank 2 hyperbolic Kac–Moody groups see [91].

3.5 Eisenstein series in string theory

Having introduced all the necessary mathematical background and in particular the definition of a maximal parabolic Eisenstein series, let us now state the Eisenstein series and discuss some of their properties, which appear in the string theory and which shall be of particular interest to us.

As mentioned in chapter 2, we will mostly consider the two lowest orders of string theory curvature corrections in the effective action of four-graviton scattering. They are the $\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$ terms, with coefficients $\mathcal{E}^D_{(0,0)}$ and $\mathcal{E}^D_{(1,0)}$, respectively. It has been found that for the low-energy expansion of four-graviton scattering in type IIB Superstring Theory in $D \geq 3$, $\mathcal{E}^D_{(0,0)}$ and $\mathcal{E}^D_{(1,0)}$ are given by maximal parabolic Eisenstein series, multiplied by a suitable normalisation factor [72, 74]

$$\mathcal{E}^D_{(0,0)} = 2\zeta(3)E^G_{1;3/2}, \quad \text{and} \quad \mathcal{E}^D_{(1,0)} = \zeta(5)E^G_{1;5/2}\,. \tag{3.5.1}$$

Here, as before, ζ is the Riemann-Zeta function. These Eisenstein series are of the general maximal parabolic type $E^G_{i_*;s}$, (3.3.9), introduced in the previous section, where $G = E_{d+1}(\mathbb{R}) = E_{11-D}(\mathbb{R})$ is the duality in group in $D \geq 3$ space-time dimensions on which the Eisenstein series is defined. We remind the reader of the relation $D = 10 - d$. Note that by the restriction $D \geq 3$, the respective duality groups are all finite-dimensional. The extension to the infinite-dimensional Kac–Moody groups is made in the following section 3.4. Evidence for this particular form of the series (3.5.1) in $D < 10$ dimensions is indirect. While for $D = 10$ explicit matrix theory calculations exist, see for instance [25], and confirm the form of the series, for $D < 10$ the form of the series is supported by taking certain physical degeneration limits of $\mathcal{E}^D_{(0,0)}$ and $\mathcal{E}^D_{(1,0)}$, which then provide a consistency check. Such limits will be discussed in detail in section 5.4. Note that in general we do not consider the cases $6 \leq D \leq 9$ where the functions $\mathcal{E}^D_{(0,0)}$ and $\mathcal{E}^D_{(1,0)}$ are more complicated and are given by sums of Eisenstein series.

3.5.1 Kac–Moody Eisenstein series in string theory

The proposal made in article **I** is that the E_9, E_{10} and E_{11} automorphic couplings of the $\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$ terms are given by

$$\begin{aligned} \mathcal{E}^2_{(0,0)} &= 2\zeta(3) v E^{E_9}_{1;3/2} && (\text{i.e., } \hat{\lambda} = 3\hat{\Lambda}_1 + \hat{\delta} - \hat{\rho}), \\ \mathcal{E}^2_{(1,0)} &= \zeta(5) v E^{E_9}_{1;5/2} && (\text{i.e., } \hat{\lambda} = 5\hat{\Lambda}_1 + \hat{\delta} - \hat{\rho}), \end{aligned} \tag{3.5.2}$$

for E_9, by

$$\begin{aligned} \mathcal{E}^1_{(0,0)} &= 2\zeta(3) E^{E_{10}}_{1;3/2} && (\text{i.e., } \lambda = 3\Lambda_1 - \rho), \\ \mathcal{E}^1_{(1,0)} &= \zeta(5) E^{E_{10}}_{1;5/2} && (\text{i.e., } \lambda = 5\Lambda_1 - \rho), \end{aligned} \tag{3.5.3}$$

for E_{10} and by

$$\begin{aligned} \mathcal{E}^0_{(0,0)} &= 2\zeta(3) E^{E_{11}}_{1;3/2} && (\text{i.e., } \lambda = 3\Lambda_1 - \rho), \\ \mathcal{E}^0_{(1,0)} &= \zeta(5) E^{E_{11}}_{1;5/2} && (\text{i.e., } \lambda = 5\Lambda_1 - \rho), \end{aligned} \tag{3.5.4}$$

for E_{11}. Except for the additional factor of v related to the shift of the weight by $\hat{\delta}$ these are straight-forward generalisations of the results of [72–74], i.e. of the serie (3.5.1). In the section 5.4 we will subject the proposals for E_9, E_{10} and E_{11} to consistency checks by expanding the constant terms in different (maximal) parabolic subgroups and comparing to the degeneration limits discussed above. Our checks will only concern the constant terms and so are insensitive to possible cusp forms (which by definition have vanishing constant terms). In the finite-dimensional case, there are good arguments to show that no cusp forms compatible with string theory boundary conditions exist [68, 74].

For a discussion of Kac–Moody Eisenstein series in the context of supergravity and string theory see also [92].

3.6 Adelization of Eisenstein series

When analysing Eisenstein series it is often convenient to not treat them as functions on the real Lie group $G = G(\mathbb{R})$ but instead consider them as functions on the group $G(\mathbb{A})$ over the (rational) adeles $\mathbb{A}$. The validity of this extended viewpoint is guaranteed by the strong approximation theorem, see for example [93, 94]. The advantage of this is that more mathematical tools are available when performing operations on $E(\chi, g)$.

The adeles $\mathbb{A}$ can be roughly thought of as

$$\mathbb{A} = \mathbb{R} \times {\prod_{p<\infty}}' \mathbb{Q}_p, \tag{3.6.1}$$

where $\mathbb{Q}_p$ are the p-adic numbers and the product involves all inequivalent completions of the field $\mathbb{Q}$ of rational numbers. The prime on the product indicates that almost all elements in this infinite product are restricted to the integers (in the appropriate sense). For a concise summary of the theory of p-adic numbers, see the appendix B.

The adelic version of the Eisenstein series (3.2.19), with the definition of the character χ appropriately extended to $\mathbb{A}$, is

$$E(\chi, g_{\mathbb{A}}) = \sum_{\gamma \in B(\mathbb{Q}) \backslash G(\mathbb{Q})} \chi(\gamma g_{\mathbb{A}}) \,, \tag{3.6.2}$$

where the difference to (3.2.19) is that now $g_{\mathbb{A}} \in G(\mathbb{A})$ and the sum is over the diagonally embedded discrete subgroup $G(\mathbb{Q})$. The series (3.2.19) is recovered by restricting the element $g_{\mathbb{A}}$ to lie solely in the real factor:

$$g_{\mathbb{A}} = (g_{\mathbb{R}}, 1, 1, , \ldots) \,. \tag{3.6.3}$$

Evaluating the adelic Eisenstein series for such $g_{\mathbb{A}}$ defines a function on the real group $G(\mathbb{R})$ and this function is equal to (3.2.29) defined above. We will in the sequel drop the subscript on the group element as it will be clear from the context whether g is in $G(\mathbb{A})$ or $G(\mathbb{R})$.

An extensive discussion of the adelic treatment of Eisenstein series is given in the article **IV**.

Chapter 4

Fourier expansions of Eisenstein series

In this chapter we discuss Fourier expansions of Eisenstein series. The discussion in this chapter will be kept general and we aim to make the structure of the expansion of an Eisenstein series as clear as possible. In particular we will set up the relevant Fourier integrals which we will then solve in the two subsequent chapters 5 and 6, for the constant term and the Fourier coefficients, respectively.
Following the definition an Eisenstein series on a group over the adeles $\mathbb{A}$ in section 3.6, all of the following calculations in this section will be made in the adelic context. Let us remark, however, that in order to follow these calculations, it is not necessary to have an in-depth understanding of the theory of adeles. For the most part we will be dealing with manipulations of cosets and the reader may ignore the fact that the groups involved are defined on the adeles.

This chapter contains excerpts from the unpublished articles **II** and **IV**.

4.1 General expansion scheme

In this section we discuss the general structure of Fourier expansions of Eisenstein series $E(\chi, g)$ and set up some of our basic notation. Fourier expansions can be defined with respect to arbitrary unipotent radicals U of the group

G that the Eisenstein series is defined on. The largest such radical will be denoted by N (rather than a general U) and it is the unipotent radical of the (minimal parabolic) Borel subgroup $B \subset G$. This will be the main case of interest to us; however, we begin by developing some of the theory for arbitrary U that we imagine is associated with a parabolic subgroup $P \subset G$, where $P = LU = UL$ is the Levi decomposition and L denotes the Levi factor of the parabolic P.

4.1.1 Fourier coefficients

The central object of interest to our work is the Fourier coefficient F_{ψ_U} associated with a Fourier kernel given by the character (group homomorphism)

$$\psi_U : U(\mathbb{Q})\backslash U(\mathbb{A}) \to U(1)\,. \tag{4.1.1}$$

The notation for the domain indicates that the character is trivial on the discrete subgroup $U(\mathbb{Q})$ in the adelic unipotent $U(\mathbb{A})$ and the image is the circle group of uni-modular complex numbers. The Fourier coefficient F_{ψ_U} of an Eisenstein series $E(\chi, g)$ is then defined by the following integral:

$$F_{\psi_U}(\chi, g) = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\chi, ug)\overline{\psi_U(u)}\mathrm{d}u\,. \tag{4.1.2}$$

In general, the unipotent group U can be non-abelian and therefore the character is trivial on the commutator subgroup $[U, U]$. Hence the character can be thought of as defined on the 'abelianization' of U, $[U, U]\backslash U$. The Lie algebra of this space is called the *character variety.* For this reason, the Fourier coefficient (4.1.2) is sometimes referred to as an abelian Fourier coefficient. It only captures part of the Eisenstein series in the sense that

$$\sum_{\psi_U} F_{\psi_U}(\chi, g) = \int\limits_{[U,U](\mathbb{Q})\backslash[U,U](\mathbb{A})} E(\chi, ug)\mathrm{d}u\,, \tag{4.1.3}$$

where the sum is over all possible characters ψ_U of the type (4.1.1). In other words, the Fourier expansion with respect to characters ψ_U does not reflect the dependence of $E(\chi, g)$ on $[U(\mathbb{A}), U(\mathbb{A})]$ as this is averaged out in (4.1.3).

By writing a group element in the form $g = ulk$ with $u \in U$, $l \in L$ and $k \in K$ one finds that

$$F_{\psi_U}(\chi, g) = \psi_U(u) F_{\psi_U}(\chi, l) \tag{4.1.4}$$

and hence F_{ψ_U} is completely determined by its values on the Levi subgroup $L \subset G$. In the following, we will restrict our analysis to this dependence.

A particular role is played by the trivial character, $\psi_U \equiv 1_U$, given by the identity on U. The corresponding Fourier coefficient represents the zeroth mode of the Fourier expansion, which we denote by $F_{1_U}(g)$. This contribution to the expansion is also referred to as the *constant term*, as was already defined in (1.2.7) for instance. Each non-trivial character, $\psi_U \neq 1_U$, contributes a term F_{ψ_U}, in total making up the so-called abelian Fourier coefficients in the expansion of the series. Then the Fourier expansion takes the general form

$$E(\chi, g) = F_{1_U}(\chi, g) + \sum_{\psi_U \neq 1} F_{\psi_U}(\chi, g) + \dots . \tag{4.1.5}$$

Here, the ellipsis indicates further possible terms associated with the non-zero commutator components of U that are averaged out in (4.1.3). To describe them, one has to study so-called generalized or non-abelian Fourier expansions that are associated with the derived series of U. This was, for example, studied in [23, 95]. In this thesis we will not, however, deal with this part of the expansion. As mentioned already, the constant term in the Fourier expansion can be evaluated using Langlands' formula (4.2.26) or similar formulas derived by Mœglin–Waldspurger [96].

The Fourier coefficients F_{ψ_U} possess the important property that their values along $L(\mathbb{Z})$ orbits are related by a simple formula (see [62, 97]):

$$F_{\gamma \cdot \psi_U}(\chi, g) = F_{\psi_U}(\chi, \gamma g) \quad \text{for } \gamma \in L(\mathbb{Z}). \tag{4.1.6}$$

Here, the action of an element γ of the Levi subgroup $L(\mathbb{Z})$ on a character ψ_U is defined by $(\gamma \cdot \psi_U)(u) = \psi_U(\gamma u \gamma^{-1})$. Realising the character in terms of (the dual of) a Lie algebra element of $[U, U]\backslash U$ one is therefore led to the study of character variety orbits in the terminology of footnote 4.1.1. These orbits have been completely classified for finite-dimensional simple and simply-laced *complex* Lie algebras [97–104]; the finer classification for integral rather than complex orbits has only been carried out in some special cases, see for example [105, 106].

4.1.2 Whittaker vectors and characters on N

The notion of a Fourier coefficient is general and is used for the F_{ψ_U} making up the abelian part of the Fourier expansion with respect to a unipotent subgroup U. From now on we will focus on the case of a minimal parabolic expansion, where $P = B = NA$, such that the unipotent radical is given by N. Therefore characters are now group homomorphisms

$$\psi \,:\, N(\mathbb{Q})\backslash N(\mathbb{A}) \to U(1)\,. \tag{4.1.7}$$

Without a subscript, characters will always refer to the unipotent N in this thesis. To further mark the distinction and in accordance with standard terminology the Fourier coefficients with respect to such characters are defined by

$$W_\psi(\chi, a) = \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi, na)\overline{\psi(n)}\mathrm{d}n \tag{4.1.8}$$

and are also called *Whittaker vectors* (whence the symbol W_ψ). This definition is completely analogous to (4.1.2) and we have already restricted the dependence on G to the Levi factor A of the minimal parabolic B. The Levi factor in this case is identical to the maximal (split) torus.

It will be important to describe and distinguish in more detail the characters (4.1.7) on N. To this end we denote by $N_\alpha(\mathbb{A})$ the restriction of the unipotent group $N(\mathbb{A})$ to the one-parameter subgroup associated with the positive root α, then we can parametrise the space on which characters ψ

depend as

$$[N,N]\backslash N \cong \prod_{\alpha\in\Pi} N_\alpha \,. \tag{4.1.9}$$

(Recall that Π denotes a chosen set of simple roots of G.) The character ψ is only sensitive to the part of N in the 'directions' of the simple roots and we choose to write the character in the following way:

$$\psi\left(\prod_{\alpha\in\Pi} x_\alpha(u_\alpha)\right) = e^{2\pi i(\sum_{\alpha\in\Pi} m_\alpha u_\alpha)}\,, \tag{4.1.10}$$

where $m_\alpha \in \mathbb{Q}$ are $\mathrm{rk}(\mathfrak{g})$ many parameters that define the character completely. In the argument of ψ we have used the Chevalley notation $x_a(u_\alpha) = \exp(u_\alpha E_\alpha)$, where E_α is the (canonically normalised) step operator corresponding to the (one-dimensional) root space of the simple root α. The order of the factors does not matter since ψ is a homomorphism to an abelian group. The m_α parametrise the character variety in this case.

Different values of parameters m_α correspond to certain types of the character ψ. We distinguish the following three basic types. (i) The character is *trivial* if $m_\alpha = 0$ for all $\alpha \in \Pi$ and in this case $\psi \equiv 1_N$, i.e. one obtains the constant term. (ii) If $m_\alpha \neq 0$ for all $\alpha \in \Pi$, we call the character *generic* and (iii) if $m_\alpha = 0$ for at least one, but not all $\alpha \in \Pi$, then the character is *non-generic*. We will later use a subset of simple roots $\Pi' \subset \Pi$ to define the non-trivial directions of ψ, such that $m_\alpha \neq 0$ if $\alpha \in \Pi'$ and is zero otherwise. The character is then said to have support on Π'. Note that in the following, we also sometimes use the term *degenerate* to refer to non-generic characters.

The case of Whittaker vectors W_ψ with non-generic ψ is more complicated than that of generic ones and presents the main focus of our work. We will deal with this case primarily in section 6.1. Let us note at this point that we will also refer to the Whittaker vectors associated with non-generic (degenerate) ψ, as *degenerate* Whittaker vectors.

4.2 The Fourier integral

Let us re-write the expression for the Fourier integral (4.1.8) of an expansion with respect to the minimal parabolic subgroup $P = B = NA$. The first step is to simply substitute the definition of the Eisenstein series (3.6.2)

$$W_\psi(\chi, a) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \chi(\gamma n a)\overline{\psi(n)} \mathrm{d}n \,. \tag{4.2.1}$$

By some coset arithmetic we can then re-write the right-hand side in the following way

$$\begin{aligned} W_\psi(\chi, a) &= \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \chi(\gamma n a)\overline{\psi(n)} \mathrm{d}n \\ &= \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})/B(\mathbb{Q})} \sum_{\delta \in \gamma^{-1}B(\mathbb{Q})\gamma \cap B(\mathbb{Q})\backslash N(\mathbb{A})} \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \chi(\gamma\delta n a)\overline{\psi(n)} \mathrm{d}n \\ &= \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})/B(\mathbb{Q})} \int\limits_{\gamma^{-1}B(\mathbb{Q})\gamma \cap N(\mathbb{Q})\backslash N(\mathbb{A})} \chi(\gamma n a)\overline{\psi(n)} \mathrm{d}n \,. \end{aligned} \tag{4.2.2}$$

In the first line, we have written the sum over γ in terms of cosets over $B(\mathbb{Q})$ on the right which are labelled by δ. Because of the quotient by $B(\mathbb{Q})$ on the left in the original γ sum, we must make sure that we do not overcount the coset representatives δ and this is achieved by the restriction on the δ sum. In addition, we have 'unfolded' the sum over δ to the integration domain by enlarging it. The measure on this larger space is induced from the embedding $N(\mathbb{Q}) \to N(\mathbb{A})$.

As the next step we then use the Bruhat decomposition

$$G(\mathbb{Q}) = \bigcup_{w \in \mathcal{W}} B(\mathbb{Q}) w B(\mathbb{Q}) \tag{4.2.3}$$

to label the double cosets in the γ sum in terms of elements of the Weyl group $\mathcal{W}$. Defining also $N^w(\mathbb{Q}) = w^{-1}B(\mathbb{Q})w \cap N(\mathbb{Q})$ for notational convenience, we arrive at:

$$W_\psi(\chi, a) = \sum_{w \in \mathcal{W}} \int\limits_{N^w(\mathbb{Q})\backslash N(\mathbb{A})} \chi(wna)\overline{\psi(n)}\mathrm{d}n\,, \tag{4.2.4}$$

from which we furthermore define

$$F_{w,\psi}(\chi, a) = \int\limits_{N^w(\mathbb{Q})\backslash N(\mathbb{A})} \chi(wna)\overline{\psi(n)}\mathrm{d}n\,. \tag{4.2.5}$$

4.2.1 Integration range

We begin by analysing in more detail the integration range of the integral in equation (4.2.5). Depending on the character ψ, we will find that $F_{w,\psi}$ will only be non-zero for a restricted subset of Weyl words.

The integration range of the Fourier integral (4.2.5) for $F_{w,\psi}$ is given by the coset

$$N^w(\mathbb{Q})\backslash N(\mathbb{A}) = w^{-1}B(\mathbb{Q})w \cap N(\mathbb{Q})\backslash N(\mathbb{A}) \tag{4.2.6}$$

The intersection in the denominator of this coset consists of those upper elements (generated by positive root generators) of the minimal parabolic subgroup B, that are also mapped to upper elements under the Weyl group action. For the whole denominator we can therefore write

$$N^w(\mathbb{Q}) \simeq \prod_{\alpha>0|w\alpha>0} N_\alpha(\mathbb{Q})\,. \tag{4.2.7}$$

With this, the integration range then splits up in the following way

$$N^w(\mathbb{Q})\backslash N(\mathbb{A}) \simeq \left(\prod_{\beta>0|w\beta>0} N_\beta(\mathbb{Q})\backslash N_\beta(\mathbb{A})\right) \cdot \left(\prod_{\gamma>0|w\gamma<0} N_\gamma(\mathbb{A})\right)\,. \tag{4.2.8}$$

Let us introduce the following notation. We denote the product in the first parenthesis as

$$N^w_{\{\beta\}} := \left(\prod_{\beta>0|w\beta>0} N_\beta(\mathbb{Q})\backslash N_\beta(\mathbb{A}) \right) \tag{4.2.9}$$

and the product in the second parenthesis as

$$N^w_{\{\gamma\}} := \left(\prod_{\gamma>0|w\gamma<0} N_\gamma(\mathbb{A}) \right) . \tag{4.2.10}$$

Here the root sets $\{\beta\}$ and $\{\gamma\}$ contain precisely those roots which satisfy the conditions imposed on the products in (4.2.9) and (4.2.10), respectively. Writing for the integration variable $n = n_\beta n_\gamma$ in accordance with this splitting of the integration range a contribution $F_{w,\psi}$ then takes the following form:

$$F_{w,\psi}(\chi, a) = \int\limits_{N^w_{\{\beta\}}} \int\limits_{N^w_{\{\gamma\}}} \chi(wn_\beta n_\gamma a)\overline{\psi(n_\beta n_\gamma)}\mathrm{d}n_\beta \mathrm{d}n_\gamma \,. \tag{4.2.11}$$

The two integrals can be disentangled further by inserting $w^{-1}w$ between n_β and n_γ and splitting the Fourier kernel into two factors. One obtains

$$F_{w,\psi}(\chi, a) = \int\limits_{N^w_{\{\beta\}}} \int\limits_{N^w_{\{\gamma\}}} \chi(wn_\beta w^{-1} wn_\gamma a)\overline{\psi(n_\beta)}\,\overline{\psi(n_\gamma)}\mathrm{d}n_\beta \mathrm{d}n_\gamma \,. \tag{4.2.12}$$

As the character χ is left invariant under any element of N and $wn_\beta w^{-1} \in N$ by the definition of the roots β in (4.2.9) we find

$$F_{w,\psi}(\chi, a) = \int\limits_{N^w_{\{\beta\}}} \overline{\psi(n_\beta)}\mathrm{d}n_\beta \cdot \int\limits_{N^w_{\{\gamma\}}} \chi(wn_\gamma a)\overline{\psi(n_\gamma)}\mathrm{d}n_\gamma \,. \tag{4.2.13}$$

We reiterate from (4.2.9) and (4.2.10) that the integration domain $N^w_{\{\beta\}}$ is a compact quotient whereas $N^w_{\{\gamma\}}$ consists of non-compact copies of $\mathbb{A}$ (as many as there are roots $\gamma > 0$ with $w\gamma < 0$).

4.2.2 Conditions for non-zero $F_{w,\psi}(\chi, a)$

The expression (4.2.13) gives a restriction on the Weyl words w that yield a non-zero $F_{w,\psi}(\chi, a)$ for a given ψ. The reason is that the integral over n_β is effectively the average of a character over a full period. If the set $\{\beta\}$ contains one (simple) root along which the character ψ is non-trivial ($m_\beta \neq 0$) then the character averages to zero. Let us analyse in more detail the two cases of non-generic and generic character ψ with parameters m_α as given in (4.1.10).

Let ψ be a character, which is non-trivial along the subset $\Pi' \subset \Pi$ of the set of simple roots Π, i.e. $m_\alpha \neq 0$ if and only if $\alpha \in \Pi' \subset \Pi$. Considering (4.2.13) it is then clear that only those Weyl words w will yield $F_{w,\psi} \neq 0$ which satisfy the condition

$$w\alpha' < 0 \text{ for all simple roots } \alpha' \in \Pi' . \tag{4.2.14}$$

We can therefore write (in short-hand notation)

$$W_\psi(\chi, a) = \sum_{w \in \mathcal{W} | w\Pi' < 0} F_{w,\psi}(\chi, a) , \tag{4.2.15}$$

where in this case

$$F_{w,\psi}(\chi, a) = \int\limits_{N^w_{\{\gamma\}}} \chi(w n_\gamma a) \overline{\psi(n_\gamma)} \mathrm{d}n_\gamma , \tag{4.2.16}$$

since the integral over $N^w_{\{\beta\}}$ in (4.2.13) yields unity. We then distinguish between the generic and the non-generic case according to definitions for ψ given in 4.1.2. Let us discuss the generic case in some more detail.

In this case $\Pi' = \Pi$ and the character ψ is non-trivial along the directions of all simple roots. It is clear that the integral over n_β in (4.2.13) is zero (and hence also $F_{w,\psi}$ will be zero), unless the set $\{\beta\}$ does not contain any simple roots. In other words, by the definition (4.2.9) all simple roots have to be mapped to negative (simple) roots under the action of w. Since all roots are linear combinations of simple roots, this also means that the entire set of positive roots is mapped to negative roots by w. For finite-dimensional G, it is a standard result that this only happens when w is the longest Weyl word, which we denote by w_0, of the Weyl group of G. This means that the sum (4.2.15) has only one, generically non-zero, contribution coming from $w = w_0$, and we have

$$W_\psi(\chi, a) = \int\limits_{N(\mathbb{A})} \chi(w_0 na)\overline{\psi(n)}\mathrm{d}n. \tag{4.2.17}$$

Here, we have used the fact that $N^{w_0}_{\{\gamma\}} = N(\mathbb{A})$ and suppressed the index γ on n in order to match the standard definition of the generic Whittaker vector in the literature. It is for this Whittaker vector that nice simple formulas exist (at the finite places) like the formula of Casselman–Shalika [107].

4.2.3 Character twist

We will now perform some basic further transformation of the integral (4.2.16), in order to extract the a-dependence. So let us consider the integral

$$F_{w,\psi}(\chi, a) = \int\limits_{N^w_{\{\gamma\}}} \chi(wn_\gamma a)\overline{\psi(n_\gamma)}\mathrm{d}n_\gamma. \tag{4.2.18}$$

Inserting a factor of aa^{-1} between w and n_γ in the argument of χ and performing a change of integration variables $n \to a^{-1}na$, under which the measure transforms as $\mathrm{d}n_\gamma \to \delta_w(a)\mathrm{d}n_\gamma$, we obtain

$$F_{w,\psi}(\chi, a) = \chi(waw^{-1})\delta_w(a) \int\limits_{N^w_\gamma} \chi(wn_\gamma)\overline{\psi^a(n_\gamma)}\mathrm{d}n_\gamma\,, \tag{4.2.19}$$

where we have defined the 'twisted' Fourier kernel $\psi^a(n) = \psi(ana^{-1})$ and we have furthermore extracted the a dependence from the argument of χ. The subscript on the Jacobi factor $\delta_w(a)$ serves to indicate that the integration is not over all of $N(\mathbb{A})$ and therefore $\delta_w(a)$ is not equal to the standard modulus character $\delta(a)$ of $N(\mathbb{A})$ (similar to equation (3.2.15)). As just argued, a non-vanishing $F_{w,\psi}$ is expressed solely in terms of an integral over $N^w_{\{\gamma\}}$ and it is this part of all of $N(\mathbb{A})$ that contributes to $\delta_w(a)$. That is, $\delta_w(a)$ is given by the relation $\mathrm{d}(an_\gamma a^{-1}) = \delta_w(a)\mathrm{d}n_\gamma$, where n_γ is an element of $N^w_{\{\gamma\}}$ as before. With (4.2.10), one has

$$\delta_w(a) = a^{\sum_{\gamma>0\,|\,w\gamma<0}\gamma} = a^{\rho-w^{-1}\rho}, \tag{4.2.20}$$

where we have used standard results on the set $\{\gamma > 0\,|\,w\gamma < 0\}$ given in [108], that we also rederive in appendix C for completeness. Using also the expression (3.2.13), we deduce that the prefactor in (4.2.19) is given by

$$\chi(waw^{-1})\delta_w(a) = a^{w^{-1}(\lambda+\rho)+\rho-w^{-1}\rho} = a^{w^{-1}\lambda+\rho}. \tag{4.2.21}$$

In order to ease the notation in the following chapters, let us define the integral

$$\mathcal{F}_{w,\psi}(\chi) = \int\limits_{N^w_\gamma} \chi(wn_\gamma)\overline{\psi(n_\gamma)}\mathrm{d}n_\gamma\,. \tag{4.2.22}$$

In terms of this quantity the full Whittaker vector (4.2.4) is then

$$W_\psi(\chi, a) = \sum_{w\in\mathcal{W}} a^{w^{-1}\lambda+\rho}\mathcal{F}_{w,\psi^a}(\chi), \tag{4.2.23}$$

where the twisted character ψ^a enters. Below we will study in detail the integral (4.2.22) for $\mathcal{F}_{w,\psi}$ for arbitrary ψ and only substitute back the particular twisted character ψ^a at the very end of the calculation.

4.2.4 Whittaker vectors and Kac–Moody groups

An important observation is that in the case of infinite-dimensional Kac–Moody groups, the expression (4.2.17) *never* applies. The reason is that there is no longest Weyl word w_0 or no other word that maps all positive simple roots to negative roots. As a result, $\mathcal{F}_{w,\psi}$ will be zero, whenever the Fourier kernel ψ is generic. The only non-vanishing Whittaker vectors for Kac–Moody groups are therefore those associated with degenerate characters ψ.

4.2.5 Adelic treatment

Let us conclude this section by making some comments about the adelic treatment in the context of Fourier expansions. One virtue of the adelic treatment is that one can write the quantity $M(w, \lambda)$ of (3.2.24), which already appeared in the functional relation satisfied by the Eisenstein series, as

$$M(w^{-1}, \lambda) = \int\limits_{w^{-1}B(\mathbb{Q})w \cap N(\mathbb{Q}) \backslash N(\mathbb{A})} \chi(wn) \mathrm{d}n. \tag{4.2.24}$$

This integral arises naturally when calculating the constant term of the Eisenstein series that is defined by

$$C(g) = \int\limits_{N(\mathbb{Q}) \backslash N(\mathbb{A})} E(\chi, ng) \mathrm{d}n \tag{4.2.25}$$

and this is a function solely on the Cartan torus. Using the Bruhat decomposition (4.2.3) and the integral (4.2.24) one can demonstrate Langlands' constant term formula

$$C(a) = \sum_{w \in \mathcal{W}} M(w, \lambda) a^{w\lambda + \rho}, \tag{4.2.26}$$

where the notation (3.2.13) was used. This formula will be explained in detail in the following chapter 5.

Chapter 5

Constant term of Kac–Moody Eisenstein series

In this chapter we will consider the zeroth order Fourier mode in the expansion of Eisenstein series. This part of the expansion is also commonly referred to as the 'constant term'. Langlands' formula, which we will introduce in the following, provides an easy way to compute this constant term contribution to the expansion. The central achievement presented in this chapter is to demonstrate how Langlands' formula can be applied in the case of special types of Kac–Moody Eisenstein series.

The chapter includes excerpts from article **I**.

5.1 Langlands' constant term formula

Let us note right away that the constant term of an Eisenstein series is obtained in the case when $m_\alpha = 0$ for all $\alpha \in \Pi$ according to the definition in (4.1.10), i.e. when $\psi \equiv 1_N$. In this section we then introduce the constant term formula for this part of the Fourier expansion, first written down by R. P. Langlands in [35].

The constant terms, of a minimal parabolic Eisenstein series, are those terms that do not depend on those G/K coset space coordinates associated with the unipotent radical N in (3.2.1), but only on the Cartan subalgebra coordinates. They are hence obtained by integrating out the unipotent part (using the invariant Haar measure):

$$\int\limits_{N(\mathbb{Z})\backslash N(\mathbb{R})} E^G(\lambda, ng)\mathrm{d}n = \sum_{w\in\mathcal{W}} M(w,\lambda)e^{\langle w\lambda+\rho|H(g)\rangle} . \tag{5.1.1}$$

This is Langlands' formula for the constant term. The integral for the constant term can hence be evaluated in terms of a sum over the Weyl group $\mathcal{W}$ of the group G. The individual summands being the numerical factor $M(w,\lambda)$ times a monomial of the Cartan subalgebra coordinates. For a definition of the map $H(\cdot)$, see (3.2.9). The numerical factors $M(w,\lambda)$ are given explicitly by

$$M(w,\lambda) = \prod_{\substack{\alpha\in\Delta_+ \\ w\alpha\in\Delta_-}} \frac{\xi\left(\langle\lambda|\alpha\rangle\right)}{\xi\left(1+\langle\lambda|\alpha\rangle\right)} = \prod_{\substack{\alpha\in\Delta_+ \\ w\alpha\in\Delta_-}} c\left(\langle\lambda|\alpha\rangle\right) . \tag{5.1.2}$$

The product runs over all positive roots Δ_+, which also satisfy the condition that $w\alpha$ be a negative root (i.e. an element of Δ_-) for the Weyl group element w. The function ξ is the completed Riemann ζ-function and is defined as $\xi(k) \equiv \pi^{-k/2}\Gamma\left(\frac{k}{2}\right)\zeta(k)$. We will discuss crucial properties of this factor in some detail in section 5.2.1.

The expansion in equation (5.1.1) will be referred to as *minimal parabolic expansion* of the constant terms, since the expansion was made with respect to the parabolic subgroup $P = B = NA$, where a is the Borel subgroup. We will also introduce in some detail expansions with respect to maximal parabolics in section 5.4.

In order to make Langlands' formula (5.1.1) more transparent, let us evaluate it in the simplest case, namely for the $SL(2,\mathbb{R})$ series defined in (3.1.7). This will reproduce the constant term part of the expansion (1.2.8).

Example 5.1.1. *Langlands' formula for the* $SL(2,\mathbb{R})$ *Series*

We proceed by evaluating step-by-step the various parts of Langlands' formula (5.1.1) *for the* $SL(2,\mathbb{R})$ *case, where the defining weight* $\lambda = 2s\Lambda_1 - \Lambda_1$ *is that of a maximal parabolic series and for the Weyl vector we have* $\rho = \Lambda_1$*, with* Λ_1 *being the single fundamental weight of the algebra. Let us first con-*

sider the exponential factor in the expansion formula, which in our example takes the form:

$$e^{\langle (w\lambda)+\rho, H(g)\rangle} = e^{\langle (2s-1)\Lambda_1 | H(g)\rangle} .$$

The Weyl group of $SL(2,\mathbb{R})$ has two elements; the identity element, id., and the fundamental Weyl reflection w_1. Hence the two associated exponents in the constant term are given by

- $w = id.$ *(identity element):* $w\lambda + \rho = 2s\Lambda_1$,
- $w = w_1$*:* $w\lambda + \rho = 2(1-s)\Lambda_1$.

Making use of the definition (3.2.9) *of* $H(g) = \log(n^{-1}gk^{-1}) = \log(a)$ *and parameterising the Cartan group element a according to*

$$a = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} ,$$

we find that the constant term then takes the following form

$$M(id., \lambda)y^s + M(w_1, \lambda)y^{1-s} ,$$

Determining the two coefficients, we find

- $w = id.$*:* $M(w, \lambda) = 1$*, since the single positive root α_1 of the algebra does not satisfy the condition for contributing in the product.*
- $w = w_1$*: In this case the root α_1 does contribute to the product and we obtain*

$$M(w_1, \lambda) = \frac{\xi(\langle\lambda, \alpha_1\rangle)}{\xi(1 + \langle\lambda, \alpha_1\rangle)} = \frac{\xi((2s-1)\langle\Lambda_1|\alpha_1\rangle)}{\xi(1 + (2s-1)\langle\Lambda_1|\alpha_1\rangle)} = \frac{\xi(2s-1)}{\xi(2s)} .$$

Putting everything together, we then find the constant term to be given by

$$\int\limits_{N(\mathbb{Z})\backslash N(\mathbb{R})} E^{SL(2,\mathbb{R})}(s, ng)\mathrm{d}n = \int\limits_0^1 E^{SL(2,\mathbb{R})}(s, z)\mathrm{d}x = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} ,$$

where $z = x + iy$. Upon comparison we find that this correctly reproduces the constant term part in (1.2.8).

For finite-dimensional groups the number of terms contributing to the constant term, obtained by the minimal parabolic expansion (5.1.1), is generically equal to the finite order of the Weyl group, $|\mathcal{W}|$. For special choices of λ there can, however, be vast cancellations reducing the number of constant terms [74]. Some of these particular choices of λ are the ones which are also relevant in string theory and were discussed in [74] and article **I**. Let us recall that in these cases λ is of the form $\lambda = 2s\Lambda_{i_*} - \rho$ and hence defines maximal parabolic Eisenstein series. The big reduction in the number of constant terms occurs for special choices of i_* and the parameters s.

In the case of Eisenstein series, defined on a general Kac–Moody group, one would, by formula (5.1.1), generically expect infinitely many contributions to the constant term. This is simply due to the infinite-dimensional nature of a Kac–Moody group and the infinite order of its associated Weyl group. However, as we will show in the following, there are special choices for λ, for which the a priori infinite number of contributions reduces to a *finite* and indeed very small number of terms. We will also refer to this reduction as a 'collapse' of the constant term and are going to present a detailed argument for it in the following. Indeed, the demonstration of the precise mechanism of this collapse represents a central result of this thesis.

5.2 The collapse mechanism

In order to demonstrate the mechanism of collapse for the constant term, we will now restrict ourselves to maximal parabolic Eisenstein series (3.3.9) and (3.4.5) (for affine groups), by making the particular choice, $\lambda = 2s\Lambda_{i_*} - \rho$, for the defining weight. A central role in the argument of collapse for the constant term is played by the coefficient $M(w, \lambda)$ in Langlands formula (5.1.1). Let us therefore introduce some important properties of this factor.

5.2.1 Properties and functional relation of $M(w, \lambda)$

It is clear from (5.1.2) that the coefficients $M(w, \lambda)$ satisfy the multiplicative identity

$$M(w\tilde{w}, \lambda) = M(w, \tilde{w}(\lambda))M(\tilde{w}, \lambda)\,. \qquad (5.2.1)$$

One also has the following functional relation for minimal Eisenstein series

$$E^G(\lambda, g) = M(w, \lambda)E^G(w(\lambda), g)\,, \qquad (5.2.2)$$

which was first stated in [35]. The completed Riemann zeta-function, $\xi(s)$, entering in (5.1.2) satisfies the simple functional equation

$$\xi(k) = \xi(1-k)\,, \qquad (5.2.3)$$

which is at the heart of the meromorphic continuation of the Riemann zeta-function. Defining the function $c(k)$ by

$$c(k) := \frac{\xi(k)}{\xi(1+k)}\,, \qquad (5.2.4)$$

the functional equation (5.2.3) implies

$$c(k)c(-k) = 1. \qquad (5.2.5)$$

The only (simple) zero of $c(k)$ occurs for $k = -1$; consequently $c(k)$ has a (simple) pole at $k = +1$:

$$c(-1) = 0\,, \qquad c(+1) = \infty\,. \qquad (5.2.6)$$

With the definition of the c-function above, $M(w, \lambda)$ then reads

$$M(w, \lambda) = \prod_{\substack{\alpha\in\Delta_+ \\ w\alpha\in\Delta_-}} c\,(k)\;, \qquad (5.2.7)$$

where $k = \langle\lambda|\alpha\rangle$. If, for a given Weyl word w, the product $M(w,\lambda)$ contains more $c(-1)$ than $c(+1)$ factors, then, by (5.2.6), $M(w,\lambda)$ will vanish. This property will be crucial in the following argument where we show that Langlands' formula can even be applied for a restricted class of Kac–Moody Eisenstein series.

The collapse mechanism, which we will provide an argument for, consists of two steps. Each step is implied by a restriction from the Eisenstein series for which we would like to evaluate Langlands' formula. The first restriction comes from the fact that we are considering Eisenstein series of maximal parabolic type with weight, $\lambda = 2s\Lambda_{i_*} - \rho$. The second restriction is imposed for special choices for the values of i_* and s. Let us discuss the effect of these restrictions now.

5.2.2 Restricting to the maximal parabolic

For an Eisenstein series of maximal parabolic type, the defining weight is of the form $\lambda = 2s\Lambda_{i_*} - \rho$. From now on we will only consider such Eisenstein series.

As was already noted above, that the argument of the c-function (5.2.4) appearing in $M(w,\lambda)$, is $k = \langle\lambda|\alpha\rangle$. Then for a simple root $\alpha_i \neq \alpha_{i_*}$

$$k = \langle 2s\Lambda_{i_*} - \rho|\alpha_i\rangle = -1\,. \tag{5.2.8}$$

Therefore, $c(\langle\lambda|\alpha_i\rangle) = 0$ for simple roots $\alpha_i \neq \alpha_{i_*}$. This reduces the number of terms in the constant term considerably. Namely, in order to obtain a non-vanishing $M(w,\lambda)$ factor, it is necessary to restrict to the following subset of Weyl words [74]

$$\mathcal{S}_{\Pi^*} := \{w \in \mathcal{W} | w\alpha > 0 \text{ for all simple roots } \alpha \in \Pi^*\} \subset \mathcal{W}\,. \tag{5.2.9}$$

Here $\Pi^* := \Pi\backslash\{\alpha_{i_*}\}$ denotes the set of all simple roots Π of G, except for the simple root α_{i_*}, which has been removed. Note that in the article **I**, this set of Weyl words was denoted by $\mathcal{S}_{i_*}$. If $w \notin \mathcal{S}_{\Pi^*}$, then there will be at least one simple root α_i included in the product (5.1.2) and consequently

$M(w, \lambda)$ vanishes and the corresponding term in sum (5.1.1) disappears. The zero coming from the simple root cannot be cancelled by $c(+1)$ contributions from other roots; this can be argued by analytic continuation in s [62].

Now we want to give a more manageable description of the set $\mathcal{S}_{\Pi^*}$ in (5.2.9). We will see in the following that elements of $\mathcal{S}_{\Pi^*}$ correspond to carefully chosen representative in the coset $\mathcal{W}/\mathcal{W}^*$. Here $\mathcal{W}^*$ denotes the Weyl group of the subgroup of G associated with the simple roots in Π^*. The Dynkin diagram of the subgroup is obtained by deleting the node corresponding to the root α_{i_*} in the Dynkin diagram of G.

Alternatively, $\mathcal{W}^*$ can also be defined as the stabiliser group of the weight $\Lambda = \Lambda_{i_*}$. In general, the weight Λ is defined as

$$\Lambda = \sum_{\alpha \in \overline{\Pi^*}} \Lambda_\alpha \,. \tag{5.2.10}$$

The set $\overline{\Pi^*}$ denotes the complement of Π^* in the set of simple roots Π and in our case is just given by $\{\alpha_{i_*}\}$.

The quotient $\mathcal{W}/\mathcal{W}^*$ has to arise since any non-trivial element in $\mathcal{W}^*$ maps at least one of the simple roots of Π^* to a negative root and it would therefore also appear in the product for $M(w, \lambda)$. Hence we should remove any $\mathcal{W}^*$ element from the right end of a Weyl word appearing in the sum over Weyl words (5.1.1). Once this is done, the Weyl words appearing in $\mathcal{S}_{\Pi^*}$ all start with the fundamental Weyl reflection w_{i_*} on the right and will never map any of the simple roots Π^* to a negative root, which is what we require for a non-vanishing $M(w, \lambda)$.

A different and more explicit description of this fact can be given by constructively computing the set $\mathcal{S}_{\Pi^*}$ by using the Weyl orbit $\mathcal{O}_{i_*}$ of the dominant weight weight $\Lambda = \Lambda_{i_*}$. The minimal (with respect to word length) Weyl words necessary to compute the orbit $\mathcal{O}_{i_*}$ are exactly those appearing in $\mathcal{S}_{\Pi^*}$. Let us now explain this method in some detail. We will from now refer to it as the 'orbit method'.

5.2.3 Orbit method

In the following we will describe a general method for constructing a set of Weyl words, which satisfy the condition:

$$w\alpha > 0 \text{ for all simple roots } \alpha \in \Pi^* \,, \tag{5.2.11}$$

For the rest of this section, Π^* should be thought of as some general subset of the simple roots Π of G. The construction proceeds in the following way. Consider the dominant weight

$$\Lambda = \sum_{\alpha \in \overline{\Pi^*}} \Lambda_\alpha \,, \tag{5.2.12}$$

already defined above. Its $\mathcal{W}$-orbit points are in bijection with the coset $\mathcal{W}/\mathcal{W}^*$ as $\mathcal{W}^*$ stabilises Λ. We construct its orbit under the action of the Weyl group $\mathcal{W}$ of G iteratively, according to the following standard algorithm:

1. Start with the initial set of orbit points $\mathcal{O} = \{\Lambda\}$.
2. Given a weight $\mu \in \mathcal{O}$, compute its Dynkin labels $p_\alpha = \langle \mu | \alpha \rangle$ for all $\alpha \in \Pi$.
3. For all labels p_α that are strictly positive, construct $\mu' = w_\alpha \mu$ where w_α is the fundamental reflection in the simple root α. Add the resulting μ' to the set $\mathcal{O}$ of orbit points if they are not already in there.
4. If there remains a weight μ in $\mathcal{O}$ for which steps 2. and 3. have not been carried out, repeat them for this μ.

This algorithm constructs the orbit representatives of the $\mathcal{W}$-orbit of Λ. If one remembers for each orbit point μ in the orbit the sequence of fundamental reflections that were needed to obtain it, one thus obtains a set of minimal (with respect to word length) Weyl words that relate the dominant weight Λ to each of its images. Let us provide an example illustrating this method.

Example 5.2.1. *E_8 Weyl orbit*

We illustrate the procedure for the specific example of E_8 and choose $\Pi^* = \Pi \backslash \{\alpha_{i_*}\}$, with $i_* = 1$. Then the dominant weight $\Lambda = \Lambda_1$, with its Dynkin

Weyl words	Weights in Orbit
id.	$\Lambda_1 = [1, 0, 0, 0, 0, 0, 0, 0]$ (dominant weight)
$w_{i_*} = w_1$	$[-1, 0, 1, 0, 0, 0, 0, 0]$
$w_3 w_1$	$[0, 0, -1, 1, 0, 0, 0, 0]$
$w_4 w_3 w_1$	$[0, 1, 0, -1, 1, 0, 0, 0]$
$w_2 w_4 w_3 w_1$; $w_5 w_4 w_3 w_1$	$[0, -1, 0, 0, 1, 0, 0, 0]$; $[0, 1, 0, 0, -1, 1, 0, 0]$
$\vdots$	$\vdots$

Table 5.1: *Weyl words and weights in the Weyl orbit of Λ_1 for E_8. Note that the Dynkin labels of the weights follow our standard Bourbaki convention, cf. the Dynkin diagram 2.3.*

labels given by $\Lambda_1 = [1, 0, 0, 0, 0, 0, 0, 0]$ (for remarks on this notation see appendix A.1). The only fundamental Weyl reflection that acts non-trivially on Λ_1 is w_1, yielding the weight $[-1, 0, 1, 0, 0, 0, 0, 0]$. In order to create a new weight we can only act with w_3, yielding $[0, 0, -1, 1, 0, 0, 0, 0]$. Then one can only act with w_4, giving $[0, 1, 0, -1, 10, 0, 0]$. At this point we have two possibilities of fundamental Weyl reflections to act with, namely w_2 and w_5, giving us $[0, -1, 0, 0, 1, 0, 0, 0]$ and $[0, 1, 0, 0, -1, 1, 0, 0]$ respectively. We continue in this way iteratively until we are left with weights with entries being only -1 or 0. This only happens for finite-dimensional Weyl groups and the final element in the orbit is the negative of a dominant weight. The first few Weyl words generated in this way are summarised in Table 5.1. In this way one computes efficiently all the elements of $\mathcal{S}_{\Pi^*}$ from the orbit of Λ_1.

The size $|\mathcal{O}_{i_*}|$ of the Weyl orbit of Λ_{i_*} in the finite-dimensional case is given by

$$|\mathcal{O}_{i_*}| = \frac{|\mathcal{W}|}{|stab(\Lambda_{i_*})|} = \frac{|\mathcal{W}|}{|\mathcal{W}^*|} \,. \tag{5.2.13}$$

In our example, we have the stabiliser subgroup $stab(\Lambda_1) = \mathcal{W}(D_7)$ and the size of the orbit is 2160. Therefore we have 2160 distinct Weyl words in the left column of Table 5.1.

In article **I** an inductive proof is given that each Weyl word that generates an element of the orbit $\mathcal{O}$, by the method outlined above, does indeed satisfy the condition

$$w\alpha > 0 \text{ for all simple roots } \alpha \in \Pi^* . \tag{5.2.14}$$

This then establishes a one-to-one correspondence between elements in the orbit $\mathcal{O}$ and Weyl words of $\mathcal{S}_{\Pi^*}$. Since it does not yield to any new insights, necessary for in the following presentation, we will omit this proof here and refer the reader to section 3.3 of article **I**.

In summary, there is a one-to-one correspondence between the elements of $\mathcal{S}_{\Pi^*}$ and Weyl words that make up the orbit $\mathcal{O}$. This correspondence also gives a very manageable way of constructing the set $\mathcal{S}_{\Pi^*}$ by starting from the dominant weight Λ and computing its Weyl orbit as a rooted and branched tree of Weyl words of increasing length. There is a natural partial order induced on the constant terms from the Weyl orbit; this can be used to display the constant term structure in terms of a Hasse diagram. By the multiplicative identity (5.2.1), one obtains that when going down the tree one has that if $M(\tilde{w}, \lambda)$ vanishes, the subsequent $M(w\tilde{w}, \lambda)$ will also vanish. Therefore one can stop the construction of the tree along a given branch once the factor $M(w, \lambda)$ on a vertex vanishes. Again, it cannot happen that the zero of $M(\tilde{w}, \lambda)$ gets balanced by a diverging $M(w, \tilde{w}(\lambda))$.

5.2.4 Restricting to special i_* and s values

We note that in contrast to the finite-dimensional groups, in the case of Kac–Moody groups, the Weyl orbit of Λ is of infinite size and the algorithm has to be truncated at some point in practice. Such a truncation would occur, if one can show that there exists a point along each branch of the tree of Weyl words, at which the associated $M(w, \lambda)$ coefficient vanishes. In the following section we will show that for the maximal parabolic Eisenstein series with defining weight $\lambda = 2s\Lambda_{i_*} - \rho$, there are particular choices for i_* and the parameter s, for which this is precisely the case.

Restricting to such special values in the case of Eisenstein series defined on finite-dimensional groups leads to a similar collapse in the constant term. However, the collapse in this case is not as 'drastic' as in the infinite-dimensional case, since it is a collapse from a finite, large number to a very small number of terms [74], as opposed to the infinite-dimensional case, where the collapse yields a reduction from an infinite number to a finite, very small number of terms.

We will now explain how in the case of maximal parabolic Kac–Moody Eisenstein series, one can obtain a collapse of the constant term, by restricting to special values of i_* and s. For this we will consider first Eisenstein series on affine groups, cf. (3.4.3), and we will in particular treat the case of the affine simplest affine group $\widehat{SL(2,\mathbb{R})}$ explicitly in an example. Based on our findings for the affine Kac-Moody groups, we will argue that these results similarly generalise other types of Kac–Moody groups, such as the hyperbolic E_{10} group.

To begin, let us state the form of Langlands' formula for the case of Eisenstein series on affine groups

$$\int\limits_{\hat{N}(\mathbb{Z})\backslash\hat{N}(\mathbb{R})} E^G(\hat{\lambda};\hat{g},v)\mathrm{d}\hat{n} = \sum_{\hat{w}\in\widehat{\mathcal{W}}} M(\hat{w},\hat{\lambda})e^{\langle\hat{w}\hat{\lambda}+\hat{\rho}|\hat{H}(v^d\,\hat{g})\rangle}\,. \tag{5.2.15}$$

This generalises equation (5.1.1) to the affine case. For explanations on the particular form of this formula we refer the reader to [76] and article **I**. Let us just note again here that the use of hats indicates that we are now dealing with objects associated to an affine algebra.

The only way to reduce from an infinite to a finite number of contributions in the constant term of an affine Eisenstein series is if for all but a finite number of terms in (5.2.15), the coefficients $M(\hat{w},\hat{\lambda})$ vanish. The coefficients $M(\hat{w},\hat{\lambda})$, given by (5.1.2), will vanish as before if they include more $c(-1)$ than $c(+1)$ factors.

In order to exhibit that almost all $M(\hat{w},\hat{\lambda})$ vanish for special $\hat{\lambda}$ we need to introduce some more notation and results on the affine root system [88]. The following calculation, even though it is slightly technical in nature, represents a crucial ingredient towards an understanding of the Kac–Moody case.

Let G be a simple, simply-laced and maximally split Lie group as before; let $r = \mathrm{rk}(G)$ and denote by α_i $(i = 1,\dots,r)$ a choice of simple roots. In this basis the unique highest root of G is written as

$$\theta = \sum_{i=1}^{r} \theta_i \alpha_i = (\theta_i, \theta_2, \dots, \theta_r) \,. \tag{5.2.16}$$

The affine extension of the root system is obtained adding a simple root α_0. From now on roots carrying a hat will be associated with roots of the affine group $\hat{G}$ whereas roots without a hat belong to G. A general affine root is then of the form

$$\hat{\alpha} = n_0\alpha_0 + n_1\alpha_1 + ... + n_r\alpha_r = n_0\hat{\delta} + \vec{\Delta}\cdot\vec{A} \,, \tag{5.2.17}$$

where we have used the standard definition of the null root

$$\hat{\delta} = \alpha_0 + \theta \tag{5.2.18}$$

and introduced some further shorthand notation for finite-dimensional part of the root. The quantity n_0 is called the affine level and the vector $\vec{\Delta}$ is given by

$$\vec{\Delta} = (n_1 - n_0\theta_1, n_2 - n_0\theta_2, ..., n_r - n_0\theta_r) \tag{5.2.19}$$

and corresponds to a root vector of G or vanishes. Vanishing $\vec{\Delta}$ corresponds to imaginary roots of the algebra; they can never contribute to constant terms and therefore we will assume $\vec{\Delta} \neq 0$ in the following.

Consider the expression $\langle\hat{\lambda}|\hat{\alpha}\rangle$ that appears in (5.1.2) for $\hat{\lambda} = 2s\hat{\Lambda}_{i_*} - \hat{\rho}$ and the affine Weyl vector $\hat{\rho}$

$$\langle\hat{\lambda}|\hat{\alpha}\rangle = 2s\langle\hat{\Lambda}_{i_*}|\hat{\alpha}\rangle - \langle\hat{\rho}|\hat{\alpha}\rangle = 2s\langle\hat{\Lambda}_{i_*}|\hat{\alpha}\rangle - \mathrm{ht}(\hat{\alpha}) \,, \tag{5.2.20}$$

with the height $\mathrm{ht}(\hat{\alpha}) = \sum_{i=0}^{r} n_i$. We are interested in the condition $\langle\hat{\lambda}|\hat{\alpha}\rangle = \pm 1$, where '+' corresponds to a $c(+1)$ factor and '−' to a $c(-1)$ factor in $M(\hat{w}, \hat{\lambda})$. The condition $\langle\hat{\lambda}|\hat{\alpha}\rangle = \pm 1$, together with the requirement that $\hat{\alpha} > 0$ defines two sets of roots

$$\Delta_s(\pm 1) := \left\{ \hat{\alpha} \; : \; \langle\hat{\lambda}|\hat{\alpha}\rangle = \langle 2s\hat{\Lambda}_{i_*} - \hat{\rho}|\hat{\alpha}\rangle = \pm 1 \right\} . \tag{5.2.21}$$

Solving $\langle\hat{\lambda}|\hat{\alpha}\rangle = \pm 1$ for s we obtain

$$s = \frac{\mathrm{ht}(\hat{\alpha}) \pm 1}{2\langle\hat{\Lambda}_{i_*}|\hat{\alpha}\rangle} = \frac{\mathrm{ht}(\hat{\alpha}) \pm 1}{2n_{i_*}} . \tag{5.2.22}$$

We can express the height of $\hat{\alpha}$ as

$$\mathrm{ht}(\hat{\alpha}) = n_0 \mathrm{ht}(\hat{\delta}) + \mathrm{ht}(\vec{\Delta} \cdot \vec{A}) = n_0 \left(1 + \sum_{i=1}^{r} \theta_i \right) + \sum_{i=1}^{r} \Delta_i \, . \tag{5.2.23}$$

Further we note that when $i_* \neq 0$, then $n_{i_*} = \Delta_{i_*} + n_0 \theta_{i_*}$. Inserting both expressions into (5.2.22) and solving for n_0 we obtain

$$n_0 = \frac{2s\Delta_{i_*} - \sum_{j=1}^{r} \Delta_j \mp 1}{\mathrm{ht}(\hat{\delta}) - 2s\theta_{i_*}} \tag{5.2.24}$$

For a particular choice of the parameter s and simple root α_{i_*}, we can use this formula to determine the affine levels n_0 on which roots producing $c(\pm 1)$ factors can occur. Since $-\theta_i \leq \Delta_i \leq \theta_i$, we see from the formula that there exists a maximum value of n_0, such that no roots producing $c(-1)$ or $c(+1)$ factors can exist on higher affine levels.[1] In other words, both sets $\Delta_s(1)$ and $\Delta_s(-1)$ only contain a finite number of elements. The result and formula (5.2.24) remain true if $i_* = 0$ and one declares $\theta_0 = 0$.

Having determined the roots which may possibly cause the coefficient factor $M(\hat{w}, \hat{\alpha})$ to vanish we now determine for which $\hat{w}$ they actually contribute in the product running over positive roots. A root $\hat{\alpha} \in \Delta_s(\pm 1)$ will only appear in the product defining $M(\hat{w}, \hat{\lambda})$, if for a particular Weyl word

[1] We assume that the denominator does not vanish. This is true in all cases of interest later.

$\hat{w}$, the condition $\hat{w}(\hat{\alpha}) < 0$ is satisfied. In order to analyse this condition, we need to consider the general action of an affine Weyl group element $\hat{w}$.

The Weyl group $\widehat{\mathcal{W}}$ of an affine algebra can be written as a semi-direct product of the classical Weyl group $\mathcal{W}$ and a translational part $\mathcal{T} \cong \mathbb{Z}^r$ (where r is the rank of the underlying finite-dimensional algebra)

$$\widehat{\mathcal{W}} = \mathcal{W} \ltimes \mathcal{T} \,. \tag{5.2.25}$$

We will write an element of $\widehat{\mathcal{W}}$ as $\hat{w} = (w, t_\beta)$, where $w \in \mathcal{W}$ and $t_\beta \in \mathcal{T}$ with β an element of the finite-dimensional root lattice. It should be noted that in general β is *not* a root of the algebra. The action of $\hat{w}$ on a general root $\hat{\alpha} = n_0\hat{\delta} + \vec{\Delta} \cdot \vec{A}$ is then given by

$$\begin{aligned} \hat{w}(\hat{\alpha}) = (w, t_\beta)(\hat{\alpha}) &= w\left(t_\beta(\hat{\alpha})\right) \\ &= w\left(\hat{\alpha} - \langle \vec{\Delta} \cdot \vec{A} | \beta \rangle \hat{\delta}\right) \\ &= w\left(\vec{\Delta} \cdot \vec{A} + (n_0 - \langle \vec{\Delta} \cdot \vec{A} | \beta \rangle)\hat{\delta}\right) \\ &= w(\vec{\Delta} \cdot \vec{A}) + \left(n_0 - \sum_{i=1}^{r} \Delta_i \langle \alpha_i | \beta \rangle\right)\hat{\delta} \,. \end{aligned} \tag{5.2.26}$$

From the last line of (5.2.26), we conclude that for a β of sufficient height (corresponding to $\hat{w}$ of sufficient length) and appropriate direction, the coefficient of the null root $\hat{\delta}$ will be negative and therefore we have $\hat{w}(\hat{\alpha}) < 0$. Then the root $\hat{\alpha}$ will appear in the product expression for $M(\hat{w}, \hat{\lambda})$ and will produce a $c(\pm 1)$-factor. The conditions on β will always be satisfied for almost all $\hat{w}$ that contribute to the constant term. We now show this in an example.

Example 5.2.2. *The constant term of the $\widehat{SL(2,\mathbb{R})}$ Eisenstein Series*

In the following we consider the maximal parabolic Eisenstein series $E^{\hat{A}_1}_{\alpha_{i_*};s}$ for the affine extension $\hat{A}_1$ of $A_1 = SL(2,\mathbb{R})$. In this example we will choose α_{i_*} to be determined by $i_* = 1$. The root system of $\hat{A}_1$ is given by

$$\hat{\alpha} = n_0\alpha_0 + n_1\alpha_1 = n_0\hat{\delta} + \Delta_1\alpha_1 \,, \tag{5.2.27}$$

Weyl words	Weights in Orbit
id	$\Lambda_1 = [0,1]$ (dominant weight)
$w_{i_*} = w_1$	$[2,-1]$
$w_0 w_1$	$[-2,3]$
$w_1 w_0 w_1$	$[4,-3]$
$w_0 w_1 w_0 w_1$	$[-4,5]$
$\vdots$	$\vdots$

Table 5.2: *Affine Weyl orbit of* $\Lambda_{i_*=1}$.

with integers n_0 *and* n_1 *such that* $n_0 - n_1 \in \{-1, 0, 1\}$. *Here,* $\hat{\delta} = \alpha_0 + \alpha_1$ *and* n_0 *counts the affine level. The height is* $ht(\hat{\alpha}) = n_0 + n_1$.

In order to gain some intuition let us briefly consider the affine Weyl group orbit $\mathcal{O}_{i_*}$. *Starting with the fundamental weight* Λ_{i_*} *we construct its Weyl orbit in a similar way to the one already described for the case of finite-dimensional groups. We obtain Table 5.2.*

It is easy to see that we obtain an infinite number of weights in this orbit. The Weyl words in the left column of the table make up the set $\mathcal{S}^{\infty}_{i_*=1}$ *and satisfy the condition* $\hat{w}(\alpha_0) > 0$ *for all* $\hat{w} \in \mathcal{S}^{\infty}_1$. *Here, we have added* ∞ *to indicate that* $\mathcal{S}^{\infty}_1$ *contains an infinite number of elements.*

In the notation introduced above, the set of elements $\mathcal{S}^{\infty}_1$ *is given by*

$$\mathcal{S}^{\infty}_1 = \{(id, t_{k\alpha_1})\}_{k\in\mathbb{Z}_{\geq 0}} \cup \{(w_1, t_{k\alpha_1})\}_{k\in\mathbb{Z}_{\geq 0}} \,, \tag{5.2.28}$$

where $t_{\alpha_1} = w_0 w_1$. *From equation* (5.2.26) *we see that the action of an element* $\hat{w} \in \mathcal{S}^{\infty}_1$ *becomes*

$$\hat{w}(\hat{\alpha}) = w(\Delta_1 \alpha_1) + (n_0 - 2\Delta_1 k)\hat{\delta} \,. \tag{5.2.29}$$

From the second term in this equation we conclude that $\hat{w}(\hat{\alpha})$ *will be a negative root for* $\Delta_1 = 1$ *and long enough Weyl words* $\hat{w}$ *(large enough* k*). For* $\Delta_1 = 1$ *we see from* (5.2.22) *that we will get* $c(-1) = 0$ *factors in* $M(\hat{w}, \hat{\lambda})$ *for* $s = (n_1 - 1)/n_1$ *with* $n_1 \in \mathbb{Z}_{>0}$, *i.e.* $s = 0, 1/2, 2/3, 3/4, 4/5, \ldots$. *For these* *choices of s the constant term will contain a finite number of terms since there are no cancellations from $c(+1)$ factors.*

We denote the set of Weyl words w that have a non-zero factor $M(w,\lambda)$ by

$$\mathcal{C}_\lambda = \{w \in \mathcal{W} \,|\, M(w,\lambda) \neq 0\}\,, \tag{5.2.30}$$

where we also allow potentially infinite values. These always appear in combinations such that the sum over them has a well-defined limit. We will use this particular set of Weyl words later on in chapter 6.

5.2.5 E_9, E_{10} and beyond

In the case of E_9 it is not so simple to write down the set $\mathcal{S}^\infty_{\Pi^*}$ in an equally explicit way as was done for the case of $\hat{A}_1$ in (5.2.28). However, the argument we gave in (5.2.26), that a root $\hat{\alpha}$ will become negative when acting on it with a long enough Weyl word from the set $\mathcal{S}^\infty_{\Pi^*}$ still holds. From relation (5.2.24) one can then see again that both sets $\Delta_{s,i_*}(\pm 1)$ contain a finite number of roots. In practice, one can first compute the finite sets $\Delta_{s,i_*}(\pm 1)$ and then construct the set $\mathcal{S}^\infty_{\Pi^*}$ iteratively from the Weyl orbit $\mathcal{O}_{i_*}$ and check whether after a finite number of steps it happens that more elements from $\Delta_{s,i_*}(-1)$ than from $\Delta_{s,i_*}(+1)$ contribute to $M(\hat{w},\hat{\lambda})$. By the multiplicative identity (5.2.1) one then can terminate the calculation of $\mathcal{S}_{\Pi^*}$ along the branch of the orbit where this happened. If $\hat{\lambda}$ is chosen appropriately only a (small) finite number of Weyl words remain in $\mathcal{S}^\infty_{\Pi^*}$ and give contributions to the constant terms.

Due to the absence of the nice affine level structure, the situation for hyperbolic Kac-Moody algebras is much harder to analyse. It is not possible to use a formula similar to (5.2.24) to see that the sets $\Delta_{s,i_*}(\pm 1)$ only contain a finite number of elements. Instead one can use the following procedure for Eisenstein series with weight $\lambda = 2s\Lambda_{i_*} - \rho$. Note that now, all the quantities refer to the hyperbolic algebra, but we refrain from putting additional decorations on the symbols to avoid cluttering the notation. The relevant inner product is

$$\langle\lambda|\alpha\rangle = 2sn_{i_*} - \mathrm{ht}(\alpha)\,. \tag{5.2.31}$$

The height of a root grows much faster than the component along a given root n_{i_*}. It is hence clear that for moderately small s, roots of sufficient height will have inner products $\langle\lambda|\alpha\rangle < -1$ and therefore will not belong to $\Delta_{s,i_*}(\pm1)$. Therefore, computing the set of 'dangerous' roots $\Delta_{s,i_*}(\pm1)$ is a finite computational problem. More precisely, we can denote by $\Delta(n_{i_*})$ the set of positive real roots $\alpha = \sum_i n_i\alpha_i$ with a given n_{i_*}. This set is finite as long as the removal of the node i_* from the Dynkin diagram leaves the diagram of a finite-dimensional algebra. For E_{10} this means $i_* \neq 10$. We will assume this in the following. Then we can define

$$h(n_{i_*}) := \min\left\{\mathrm{ht}(\alpha)\ :\ \alpha \in \Delta(n_{i_*})\right\}\,. \tag{5.2.32}$$

This is a monotonous function of n_{i_*}. Its rate of growth with n_{i_*} is roughly equal to the height of the affine null root of the underlying affine algebra divided by its Kac label. For moderately small s – like those of interest to us – this is greater than the rate of growth of $2sn_{i_*}$. Therefore we can construct $\Delta(n_{i_*})$ by increasing height and terminate the construction of roots when $2sn_{i_*} - h(n_{i_*}) < -1$ for some n_{i_*}. To be on the safe side computationally, one can check the next few steps after this inequality is satisfied for the first time. From the resulting finite set of roots we can select those α that belong to $\Delta_{s,i_*}(\pm1)$.

The next step is to determine those Weyl words that contribute to the constant terms. This is done in the same way as before: One constructs the Weyl words from the orbit of Λ_{i_*} and checks whether more elements from $\Delta_{s,i_*}(-1)$ than from $\Delta_{s,i_*}(+1)$ contribute to $M(w,\lambda)$. For generic s this will of course result in an infinite number of Weyl words. However, if s is chosen appropriately, this leaves a finite number of Weyl words and hence a finite number of summands in the constant term. These are the cases that we will focus on in the following.

5.3 Explicit constant terms

Now consider the minimal parabolic expansion of maximal parabolic Eisenstein series. The explicit expressions for the minimal parabolic expansions of $E^G_{1;\frac{3}{2}}$ and $E^G_{1;\frac{5}{2}}$ with $G = E_9$, E_{10} and E_{11} can be found in appendix D. These expressions are directly obtained by evaluating (5.2.15), without the additional parameter v of the derivation, for E_{10} and E_{11}. We point out that these series develop logarithmic and (logarithm)^2 terms from taking limits in the ξ-functions entering $M(w, \lambda)$.

In a general expansion of $E^G_{1;s}$, it is instructive to count the number of Weyl words in the sum on the right-hand side of (5.2.15), for which the corresponding factors $M(w, \lambda)$ are non-vanishing (but possibly infinite). We do this for a range of values of the parameter s and for the $E_{n\geq 6}$ groups, i.e., in dimensions $0 \leq D \leq 5$. The results are shown in Table 5.3 which shows the number of contributing Weyl words as a function of s for the various E_n groups. This is evaluated in the normalisation of the Eisenstein series $E^{E_n}_{1;s}$ that we have been using throughout the paper. It is possible to explain some of the number patterns found in this table in terms of the root system of the E_n series. We will however not go into the these details here and instead refer the interested reader to section 5.6 of article **I**.

For the finite-dimensional groups it is clear that when increasing the value of s, one will eventually always reach a threshold value. For larger values of s the number of Weyl words yielding non-vanishing $M(w, \lambda)$ factors will always be equal to the dimension of the Weyl orbit $\mathcal{O}_{i_*}$. The reason for this is that for large enough values of s no positive root α exists which satisfies $\langle\alpha|\lambda\rangle = -1$. Hence all possible terms will be present in the sum over elements of $\mathcal{S}_{\Pi^*}$. For the infinite-dimensional groups the situation regarding this issue is less clear, since for these groups there are roots of arbitrary height available. In a sporadic check for some values of $s \geq 7/2$, the calculation on a computer of the constant term did not terminate within a reasonably short period of time (in contrast with the computations for $s < 7/2$). This can be taken as an tentative indication that in these cases the number of Weyl words contributing is actually infinite. Physically, this may be related to curvature

s	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5	$\frac{11}{2}$	6	$\frac{13}{2}$
E_6	1	2	27	7	12	27	$\cdots$							
E_7	1	2	126	8	14	35	56	126	91	126	$\cdots$			
E_8	1	2	2160	9	16	44	72	408	534	1060	1460	1795	2160	$\cdots$
E_9	1	2	∞	10	18	54	90	∞	$\cdots$					
E_{10}	1	2	∞	11	20	65	110	∞	$\cdots$					
E_{11}	1	2	∞	12	22	77	132	∞	$\cdots$					

Table 5.3: *The table shows the number of Weyl words with non-vanishing coefficients $M(w,\lambda)$ in a minimal parabolic expansion of $E^{E_{d+1}}_{1;s}$ in dimensions $1 \leq D \leq 5$ and for a range of values for the parameter s. The ellipsis signifies that the row is continued with the last number explicitly written out (for $D \leq 2$ this is conjectural).*

correction terms unprotected by supersymmetry. This is the reason why we put ∞ for the corresponding entries in Table 5.3.

Looking at Table 5.3 it is tempting to interpret it as a strong sign for the special properties associated with the small values of s in the set

$$s \in \{0, 1/2, 1, 3/2, 2, 5/2, 3\} \ . \tag{5.3.1}$$

More precisely, by requiring the constant term to only encode a *finite* number of perturbative effects as required by supersymmetry, the range of possible values that s can take, gets reduced from a previously infinite set to a finite number of possible values. It would be desirable to make these statements more precise and to prove them rigorously.

5.4 Degeneration limits

A Fourier expansion, different from the minimal parabolic expansion that we have discussed up to now, can be used to check the consistency of the automorphic couplings $\mathcal{E}^D_{(p,q)}$ in the low-energy expansion. Namely, the functions (3.5.1) are subject to a number of strong consistency requirements [73, 109] that arise from the interplay of string theory in various dimensions. The consistency conditions are typically phrased in terms of three (maximal parabolic) limits, corresponding to different combinations of the torus radii (in appropriate units) and the string coupling becoming large. The three standard limits correspond to

(i) decompactification from D to $D+1$ dimensions, where one torus circle becomes large,

(ii) string perturbation theory, where the D-dimensional string coupling is small, and

(iii) the M-theory limit, where the whole torus volume becomes large.

In terms of the E_{d+1} Dynkin diagram 2.3 this means singling out the nodes $d+1$, 1 or 2, respectively. Each of these limits corresponds to going into a special 'corner' of moduli space. Mathematically, these limits are tantamount to computing the constant terms of the Eisenstein series in different maximal parabolic expansions, which we will explain in detail now.

Taking a maximal parabolic Fourier expansion of an Eisenstein series corresponds to choosing a maximal parabolic subgroups defined by a node $j_\circ$ of the Dynkin diagram, as in appendix (A.2.8). In order to introduce it, let us remark that the Levi component $L_{j_\circ}$ of such a maximal parabolic subgroup can be written as the product of two groups, namely

$$L_{j_\circ} = GL\,(1) \times G_d\,, \tag{5.4.1}$$

where G_d is the subgroup of E_{d+1} which is determined by our choice of a simple root $\alpha_{j_\circ}$ in the Dynkin diagram 2.3 of E_{d+1}. The Dynkin diagram of G_d is given by the diagram which is left once one has deleted the node associated with $\alpha_{j_\circ}$ from the Dynkin diagram of E_{d+1}. The one-parameter group $GL\,(1)$ can be parameterised by a single variable $r \in \mathbb{R}^\times$.

The corresponding arrangement of the constant term then highlights the dependence on only one of the parameters, namely r, corresponding to the single node $j_\circ$ (say, a decompactifying circle) and maintains the invariance under the remaining group G_d in the decomposition (5.4.1). In that case the constant terms can be packaged using cosets of the Weyl group $\mathcal{W}$. Denoting the Weyl group of the maximal parabolic subgroup $P_{j_\circ}$ by $\mathcal{W}_{P_{j_\circ}}$, the constant terms read [74, 96, 110, 111]

$$\int\limits_{N_{P_{j_\circ}}(\mathbb{Z})\backslash N_{P_{j_\circ}}(\mathbb{R})} E^G(\lambda, g)\mathrm{d}n = \sum_{w\in\mathcal{W}_{j_\circ}\backslash\mathcal{W}} M(w,\lambda)e^{\langle(w\lambda+\rho)_{\|j_\circ}|H(g)\rangle} E^{G_d}\left((w\lambda)_{\perp j_\circ}, g\right) . \tag{5.4.2}$$

Let us explain some of the notation introduced here. For a weight λ, $(\lambda)_{\|j_\circ}$ is a projection operator on the component of λ proportional to the fundamental weight $\Lambda_{j_\circ}$, and $(\lambda)_{\perp j_\circ}$ is orthogonal to $\Lambda_{j_\circ}$, i.e., a linear combination of the simple roots of G_d. The Eisenstein series on the right-hand side of the equation does not depend on the $GL(1)$ factor in (5.4.1) since the dependence on the abelian group is explicitly factored out using the projections. The expression (5.4.2) does not depend solely on the Cartan subalgebra coordinates but also retains dependence on some of the positive step operators that appear in the Eisenstein series defined with respect to the reductive factor G_d. Even though indicated as depending on $g\in G$, the Eisenstein series on the right-hand side of (5.4.2) effectively depends only on $g\in G_d$. This type of expansion is called *maximal parabolic expansion* of the constant terms of an Eisenstein series.

The restriction of the sum in the Langlands formula to special representatives of the coset $\mathcal{W}/\mathcal{W}_{i_*}$ for the constant terms expanded in the minimal parabolic subalgebra also has consequences for the expansion in maximal parabolic algebras as described by formula (5.4.2). The constant terms in this case are described by certain representatives of the *double cosets* via

$$\begin{aligned}\int\limits_{N_{P_{j_\circ}}(\mathbb{Z})\backslash N_{P_{j_\circ}}(\mathbb{R})} & E^G(\lambda, g)\mathrm{d}n \\ &= \sum_{w\in\mathcal{W}_{j_\circ}\backslash\mathcal{W}/\mathcal{W}_{i_*}} M(w,\lambda)e^{\langle(w\lambda+\rho)_{\|j_\circ}|H(g)\rangle} E^{G_d}\left((w\lambda)_{\perp j_\circ}, g\right) ,\end{aligned} \tag{5.4.3}$$

see also [111]. These are typically very few in number. The rooted tree of Weyl words mentioned above can be contracted further in this case thanks to the double coset structure.

5.4.1 Degeneration limits for $D > 2$

In or above $D = 3$ dimensions (i.e., up to and including E_8), the functions in (3.5.1) have been successfully subjected to the consistency requirements [62, 73, 74]. There are also direct checks of their correctness for some dimensions and parts of their expansions (see [25, 26, 29] and references therein) and general considerations on perturbative expansions for functions constructed from lattice sums (not necessarily satisfying a Laplace equation) [112]. We will provide a heuristic derivation of the parameters entering (3.5.1) below.

As mentioned above, the three limits are referred to as the decompactification, perturbative and the semi-classical M-theory limit; and we restrict ourselves to taking the limit for the constant terms of the Eisenstein series. What 'taking the limit' means is to calculate the constant term of an Eisenstein series with respect to a particular maximal parabolic subgroup $P_{j_\circ}$. Formally, this corresponds to integrating out all the components of the unipotent radical $N_{j_\circ}$ of $P_{j_\circ}$ as in (5.4.2) and (5.4.12). We will use the following abbreviated notation for this integration

$$\int_{j_\circ} \mathcal{E}^D_{(p,q)} \equiv \int_{N_{j_\circ}/G(\mathbb{Z})\cap N_{j_\circ}} \mathcal{E}^D_{(p,q)} \mathrm{d}n \,. \tag{5.4.4}$$

For $D \geq 3$, the parameter r of the $GL(1)$ factor in the decomposition (5.4.1), acquires a different physical meaning in each of the three degeneration limits, and can be expressed in terms of fundamental string theory quantities. In [72–74] general expressions for the three degeneration limits of $\mathcal{E}^D_{(0,0)}$ and $\mathcal{E}^D_{(1,0)}$ were given for $D \geq 3$, which we will now summarise.

Decompactification limit:

In this limit $r_d/\ell_{D+1} \gg 1$, which corresponds to making one of the circles of the torus very large in units of the $(D+1)$-dimensional Planck scale. In terms of maximal parabolic subgroups this limit corresponds to singling out the node $d+1$ in figure 2.3, i.e., $j_\circ = d+1$, leading to $G_d = E_d$. One has the standard relation between Planck scales $\ell_{D+1}^{D-1} = \ell_D^{D-2} r_d$. The constant terms of the coefficients $\mathcal{E}^D_{(0,0)}$ and $\mathcal{E}^D_{(1,0)}$ behave in the following way under expansion with respect to the parabolic subgroup $P_{\alpha_{d+1}}$ [72–74]

$$\int_{d+1} \mathcal{E}^D_{(0,0)} \simeq \frac{\ell_{D+1}^{8-D}}{\ell_D^{8-D}} \left(\frac{r_d}{\ell_{D+1}} \mathcal{E}^{D+1}_{(0,0)} + \left(\frac{r_d}{\ell_{D+1}}\right)^{8-D} \right) \tag{5.4.5}$$

and

$$\int_{d+1} \mathcal{E}^D_{(1,0)} \simeq \frac{\ell_{D+1}^{12-D}}{\ell_D^{12-D}} \left(\frac{r_d}{\ell_{D+1}} \mathcal{E}^{D+1}_{(1,0)} + \left(\frac{r_d}{\ell_{D+1}}\right)^{6-D} \mathcal{E}^{D+1}_{(0,0)} + \left(\frac{r_d}{\ell_{D+1}}\right)^{12-D} \right) , \tag{5.4.6}$$

where the $\simeq$ symbol indicates that numerical factors in front of each term are not shown explicitly. The first terms on the right hand sides of the equations (5.4.5) and (5.4.6) are easily understood from decompactification from D to $D+1$ dimensions; the other terms are threshold effects [72]. Since one can relate ℓ_{D+1}/ℓ_D to r_d/ℓ_{D+1}, the expansion on the right hand side is in terms of a single variable that parameterises the $GL(1)$ in the Levi factor $M_{d+1} = GL(1) \times E_d$. In our conventions we have $r = (r_d/\ell_{D+1})^{(D-1)/(D-2)} = r_d/\ell_D$. This yields the following decompactification rules

$$\begin{aligned} \int_{d+1} \mathcal{E}^D_{(0,0)} &\simeq r^{6/(D-1)} \mathcal{E}^{D+1}_{(0,0)} + r^{8-D} , \\ \int_{d+1} \mathcal{E}^D_{(1,0)} &\simeq r^{10/(D-1)} \mathcal{E}^{D+1}_{(1,0)} + r^{D(D-7)/(1-D)} \mathcal{E}^{D+1}_{(0,0)} + r^{12-D} . \end{aligned} \tag{5.4.7}$$

These have to be fulfilled by the automorphic forms for $D \geq 3$. The coefficients of the last terms, that we call pure threshold terms, are known to be proportional to $\xi(8-D)$ and $\xi(12-D)$ respectively [72, 73].

Perturbative limit:

This corresponds to the weak string coupling expansion in D dimensions $y_D \to 0$. The D-dimensional string coupling y_D is given by $y_D = \ell_D^{D-2}/\ell_s^{D-2}$ and the string scale ℓ_s is kept fixed. Then one requires [72–74]

$$\int_1 \mathcal{E}^D_{(0,0)} \simeq \frac{\ell_s^{8-D}}{\ell_D^{8-D}} \left(\frac{2\zeta(3)}{y_D} + E^{SO(d,d)}_{d+1;\frac{d}{2}-1} \right) \tag{5.4.8}$$

and

$$\int_1 \mathcal{E}^D_{(1,0)} \simeq \frac{\ell_s^{12-D}}{\ell_D^{12-D}} \left(\frac{\zeta(5)}{y_D} + E^{SO(d,d)}_{d+1;\frac{d}{2}+1} + y_D E^{SO(d,d)}_{3;2} \right) , \tag{5.4.9}$$

respectively. Here, the first terms are fixed by string tree level calculations and the $SO(d,d)$ Eisenstein series on the right-hand side are maximal parabolic Eisenstein series as in (3.3.9) and our (non-standard) labelling convention for the $SO(d,d)$ series is induced from removing node 1 from the E_{d+1} Dynkin diagram 2.3. That is, the d nodes are labelled 2 through to $d+1$. Again, one can recombine the pre-factors by using the definition of the string coupling and then expand in terms of a single variable which is associated to the $GL(1)$ factor in the Levi decomposition. We choose here $r = (\ell_s/\ell_D)^2 = y_D^{2/(2-D)}$. We note that the string coupling y_D can be defined alternatively in terms of the ten-dimensional string coupling g_s and the string compactification volume V_d via $y_D = g_s^2 \ell_s^d / V_d$.

Semi-classical M-theory limit:

In this limit one takes the volume of the whole M-theory torus large. In terms of the E_{d+1} Dynkin diagram 2.3 this corresponds to the maximal parabolic associated with node 2. The relevant conditions on the Eisenstein series are then [72–74]

$$\int_2 \mathcal{E}^D_{(0,0)} \simeq \frac{\mathcal{V}_{d+1}}{\ell_{11}^3 \ell_D^{8-D}} \left(4\zeta(2) + \left(\frac{\ell_{11}^{d+1}}{\mathcal{V}_{d+1}} \right)^{\frac{3}{d+1}} E^{SL(d+1)}_{1;\frac{3}{2}} \right) \tag{5.4.10}$$

and

$$\int_2 \mathcal{E}^D_{(1,0)} \simeq \frac{\ell_{11}\mathcal{V}_{d+1}}{\ell_D^{12-D}} \left(\left(\frac{\mathcal{V}_{d+1}}{\ell_{11}^{d+1}}\right)^{\frac{1}{d+1}} E^{SL(d+1)}_{1;-\frac{1}{2}} + \left(\frac{\ell_{11}^{d+1}}{\mathcal{V}_{d+1}}\right)^{\frac{5}{d+1}} E^{SL(d+1)}_{1;\frac{5}{2}} \right.$$
$$\left. + \left(\frac{\ell_{11}^{d+1}}{\mathcal{V}_{d+1}}\right)^{\frac{8}{d+1}} E^{SL(d+1)}_{3;2} \right). \tag{5.4.11}$$

The first term in (5.4.10) for the R^4 term is determined by a one-loop computation in $D = 11$ supergravity [26], there is no similar term for $\partial^4 R^4$ in (5.4.11) since this term does not exist as a curvature correction term in $D = 11$.

The parameter r of the $GL(1)$ in the Levi factor of the maximal parabolic defined by node 2 of the E_{d+1} Dynkin diagram is then given by either $r = (\mathcal{V}_{d+1}/\ell_D^{d+1})^{1/3} = (\mathcal{V}_{d+1}/\ell_{11}^{d+1})^{3/(D-2)}$, where $\ell_D^{D-2} = \ell_{11}^9/\mathcal{V}_{d+1}$, or equivalently $r^2 = \mathcal{V}_{d+1}/\ell_{11}^3\ell_D^{8-D}$. Here, $\mathcal{V}_{d+1}$ denotes the volume of the M-theory torus (in contrast to the string theory torus V_d).

5.4.2 Degeneration limits of Kac–Moody Eisenstein series

Through a generalisation of formulae (5.2.15) and (5.4.2) one deduces that the constant term in the expansion of an affine Eisenstein series with respect to a particular maximal parabolic subgroup $P_{j_\circ}$ is given by

$$\int\limits_{N_{P_{j_\circ}}(\mathbb{Z})\backslash N_{P_{j_\circ}}(\mathbb{R})} E^G(\hat\lambda; \hat g, v) \mathrm{d}n$$
$$= \sum_{\hat w \in \mathcal{W}_{j_\circ}\backslash \widehat{\mathcal{W}}} M(\hat w, \hat\lambda) e^{\langle (\hat w\hat\lambda + \hat\rho)_{\parallel j_\circ} | \hat H(v^d \hat g)\rangle} E^{G_d}\left(\left(\hat w \hat\lambda\right)_{\perp j_\circ}, \hat g\right). \tag{5.4.12}$$

where in all the formulæ $\hat\lambda = 2s\hat\Lambda_{i_*} - \hat\rho$, so that we are again restricting to maximal parabolic Eisenstein series. Note that the Levi factor in this case is a finite-dimensional group. The projections $(\hat\lambda)_{\parallel j_\circ}$ and $(\hat\lambda)_{\perp j_\circ}$ are different now from those in (5.4.2) since the Cartan subalgebra includes the additional direction d. $(\hat{\lambda})_{\perp j_\circ}$ has to be a weight of the Levi factor $L_{j_\circ}$ and has two directions less than $\hat{\lambda}$; it is a combination of the simple roots of G_d. By contrast, $(\hat{\lambda})_{\| j_\circ}$ is a combination of the fundamental weight $\hat{\Lambda}_{j_\circ}$ and the null root $\hat{\delta}$. The Levi factor explicitly reads

$$L_{j_\circ} = GL(1) \times GL(1) \times G_d \tag{5.4.13}$$

and the pre-factor of the Eisenstein series in (5.4.12) now depends on the two parameters of the $GL(1)$ factors. One of them is v and we will call the other one r below.

In the affine case the expressions above follow from [76]. We will assume that they also hold *mutatis mutandis* in the general Kac–Moody case (where one does not need v and they are therefore similar to 5.1.1 and 5.4.2) and provide some consistency checks on this assumption with our calculations. The validity of (5.2.15), i.e. convergence of the series, is in proven [76] for the affine case. In particular it was proven that (5.2.15) possesses a meromorphic continuation, which extends the convergence condition stated for equation (3.4.3) to $\mathrm{Re}\langle\hat{\lambda}|\hat{\delta}\rangle > -\mathrm{ht}(\hat{\delta})$.

5.4.3 Degeneration limits for $D = 2$

When $D \leq 3$ the limits above require additional care. This is due to the absence of a natural Planck length in $D = 2$ space-time dimensions as normally defined through the two-derivative Einstein–Hilbert action; nor is it possible to define a Kaluza–Klein reduction from $D = 3$ to $D = 2$ such that one ends up in $D = 2$ Einstein frame, since the gravitational action is conformally invariant. Higher derivative terms on the other hand are of course accompanied by length scales.

Decompactification limit:

In order to understand the decompactification limit from $D = 3$ to $D = 2$ one has to properly understand the relation between three-dimensional and two-dimensional gravity theories. This has been well-studied for example in the context of the Geroch group that describes the infinite symmetries of $D = 2$

(super-)gravity (such as E_9). The set-up was pioneered in [52, 59, 113, 114] and reviewed for example in [115, 116].

The three-dimensional metric decomposes as (setting to zero the off-diagonal pieces for simplicity)

$$ds_3^2 = \lambda^{-2} ds_2^2 + \rho^2 \left(dx^3\right)^2 . \tag{5.4.14}$$

Here, λ^{-1} is the conformal factor of the two-dimensional metric and x^3 is the compactifying direction. It is not possible to choose λ such that the $D = 2$ theory is in Einstein frame. One necessarily obtains *two* new parameters just like going from E_8 to E_9 enlarges the Cartan subalgebra by two generators. In the context of the Geroch group, λ is associated with the central extension and ρ with the derivation [58, 114, 117]. The same is true here. The two parameters in (5.4.14) are given by

$$\lambda = \frac{\ell_3}{\ell_2}, \quad \rho = \frac{r_d}{\ell_3}, \tag{5.4.15}$$

where r_d is the size of the decompactifying circle and we will refer to ℓ_2 as the two-dimensional Planck scale. The two-derivative Einstein–Hilbert term in $D = 2$ is not accompanied by the (arbitrary) length scale ℓ_2, but the higher derivative terms are. The decompactification limit now consists in sending $\rho \to \infty$ and we choose to keep λ fixed.

Performing the usual analysis of higher derivative couplings we obtain for the Eisenstein series the decompactification relations

$$\begin{aligned}
&\int_{d+1} \mathcal{E}^2_{(0,0)} \simeq \lambda^6 \left(\rho \mathcal{E}^3_{(0,0)} + \rho^6\right) , \\
&\int_{d+1} \mathcal{E}^2_{(1,0)} \simeq \lambda^{10} \left(\rho \mathcal{E}^3_{(1,0)} + \rho^4 \mathcal{E}^3_{(0,0)} + \rho^{10}\right) ,
\end{aligned} \tag{5.4.16}$$

where we have again suppressed numerical coefficients and augmented them by threshold terms as in (5.4.5). The decompactifying node is $d + 1 = 3$ and unlike in other dimensions it is not possible to relate λ and ρ. The precise numerical coefficients can be found in the detailed expansions of the

Eisenstein series below where we will also see that requirement (5.4.16) forces us to modify the naive guess for the $D = 2$ Eisenstein series.

Perturbative limit:

The definition of the string coupling as above (5.4.8) fails in $D = 2$; instead one should use the one at the end of that paragraph, i.e. $y_2 = g_s^2 \ell_s^8 / V_8$. Similar to the decompactification limit, there is no way of relating the two-dimensional string coupling y_2 to the two-dimensional Planck length ℓ_2, both appear as independent parameters. The perturbation limit on the automorphic form in terms of the $SO(d,d)$ invariant parameters y_2 and ℓ_s/ℓ_2 is then

$$\begin{aligned} \int_1 \mathcal{E}^2_{(0,0)} &\simeq \left(\frac{\ell_s}{\ell_D}\right)^6 \left(\frac{2\zeta(3)}{y_2} + E^{SO(8,8)}_{9;3}\right), \\ \int_1 \mathcal{E}^2_{(1,0)} &\simeq \left(\frac{\ell_s}{\ell_D}\right)^{10} \left(\frac{\zeta(5)}{y_2} + E^{SO(8,8)}_{9;5} + y_2 E^{SO(8,8)}_{3;2}\right). \end{aligned} \tag{5.4.17}$$

Semi-classical M-theory limit:

The relations (5.4.10) and (5.4.11) remain valid except that it is again impossible to relate the two-dimensional Planck length ℓ_2 to the other variables and there are two independent $SL(9,\mathbb{Z})$ invariant expansion parameters, namely ℓ_2/ℓ_{11} and the volume of the M-theory 9-torus $\mathcal{V}_9/\ell_{11}^9$:

$$\int_2 \mathcal{E}^2_{(0,0)} \simeq \left(\frac{\ell_{11}}{\ell_2}\right)^6 \left(4\zeta(2)\frac{\mathcal{V}_9}{\ell_{11}^9} + \left(\frac{\mathcal{V}_9}{\ell_{11}^9}\right)^{\frac{2}{3}} E^{SL(9)}_{1;\frac{3}{2}}\right) \tag{5.4.18}$$

and

$$\int_2 \mathcal{E}^2_{(1,0)} \simeq \left(\frac{\ell_{11}}{\ell_2}\right)^{10} \left(\left(\frac{\mathcal{V}_9}{\ell_{11}^9}\right)^{\frac{10}{9}} E^{SL(9)}_{1;-\frac{1}{2}} + \left(\frac{\mathcal{V}_9}{\ell_{11}^9}\right)^{\frac{4}{9}} E^{SL(9)}_{1;\frac{5}{2}} + \left(\frac{\mathcal{V}_9}{\ell_{11}^9}\right)^{\frac{1}{9}} E^{SL(9)}_{3;2}\right). \tag{5.4.19}$$

5.4.4 Degeneration limits for $D = 1$

Since the dimension of the Cartan subalgebra of E_{10} is equal to the number of simple roots of the algebra most limits are easier to describe than in the E_9 case.

(Double) decompactification limit:

The first limit we study is the decompactification limit which is the only problematic case since it involves two-dimensional gravity and the associated problems of conformal invariance. Equivalently, the maximal parabolic is the affine E_9. For the algebraic relation between E_9 and E_{10} see also [118]. More precisely, it is again impossible to relate the ratio r_d/ℓ_2 to the ratio of Planck scales ℓ_1/ℓ_2 since the two-dimensional Planck scale is ill-defined. But we note that the (pure) threshold terms in (5.4.5) and (5.4.6) are well-defined here since ℓ_2 drops out. We did not determine from first principles the decompactification limit from $D = 1$ to $D = 2$ but instead a direct decompactification of two directions from $D = 1$ to $D = 3$. The general rule for this double decompactification (as implied for instance by (5.4.5)) is

$$\int_{d+1,d} \mathcal{E}^{D}_{(0,0)} \simeq v_2^6 \mathcal{E}^{D+2}_{(0,0)} + v_2^{D(7-D)} r^{D-6} + r^{8-D} \,, \tag{5.4.20}$$

$$\begin{aligned}\int_{d+1,d} \mathcal{E}^{D}_{(1,0)} \simeq\ & v_2^{10} \mathcal{E}^{D+2}_{(1,0)} + v_2^{(D+1)(6-D)} r^{D-4} \mathcal{E}^{D+2}_{(0,0)} + v_2^{D(11-D)} r^{D-10} \\ & + v_2^6 r^{6-D} \mathcal{E}^{D+2}_{(0,0)} + v_2^{D(7-D)} + r^{12-D} \,,\end{aligned} \tag{5.4.21}$$

where the expansion parameters are given in terms of the 2-torus volume

$$v_2 = \left(\frac{\mathrm{vol}(T^2) \ell_{D+2}^{6-D}}{\ell_D^{8-D}} \right)^{1/6} = \left(\frac{\mathrm{vol}(T^2) \ell_{D+2}^{10-D}}{\ell_D^{12-D}} \right)^{1/10} \tag{5.4.22}$$

and one of the circles with $r = r_d/\ell_D$ as before. In the case $D = 1$, these relations do not make explicit reference to the Planck length in two dimensions and remain well-defined. We will use the relation (5.4.20) to check our proposal for the $E_{10}(\mathbb{Z})$ Eisenstein series. Relating the $E_{10}(\mathbb{Z})$ series to $E_9(\mathbb{Z})$ we will also derive a single decompactification rule for $D = 1$ that will turn out to be subtly different from (5.4.6) in the twelve derivative case. A double decompactification corresponds to a parabolic subgroup that is not maximal.

Perturbative limit:

In this limit, the maximal parabolic subgroup has as semi-simple part the finite-dimensional $D_9 = SO(9,9)$ T-duality group. The definitions of the expansion parameters in the cases $D > 3$ continue to hold so that we immediately deduce

$$\begin{aligned}\int_1 \mathcal{E}^1_{(0,0)} &\simeq \frac{\ell_s^7}{\ell_1^7}\left(\frac{2\zeta(3)}{y_1} + E^{SO(9,9)}_{10;\frac{7}{2}}\right)\\ &\simeq 2\zeta(3) y_1^6 + y_1^7 E^{SO(9,9)}_{10;\frac{7}{2}}\,,\end{aligned} \tag{5.4.23}$$

and

$$\begin{aligned}\int_1 \mathcal{E}^1_{(1,0)} &\simeq \frac{\ell_s^{11}}{\ell_1^{11}}\left(\frac{\zeta(5)}{y_1} + E^{SO(9,9)}_{10;\frac{11}{2}} + y_1 E^{SO(9,9)}_{3;2}\right)\\ &\simeq \zeta(5) y_1^{10} + y_1^{11} E^{SO(9,9)}_{10;\frac{11}{2}} + y_1^{12} E^{SO(9,9)}_{3;2}\,,\end{aligned} \tag{5.4.24}$$

where $y_1 = \ell_s/\ell_1$ was used. Our expansion parameter r below is related to y_1 via $r = y_1^2$.

Semi-classical M-theory limit:

The maximal parabolic has now semi-simple factor $A_9 = SL(10)$. The expressions (5.4.10) and (5.4.11) are still valid and become

$$\begin{aligned}\int_2 \mathcal{E}^1_{(0,0)} &\simeq \frac{\mathcal{V}_{10}}{\ell_{11}^3 \ell_1^7}\left(4\zeta(2) + \left(\frac{\ell_{11}^{10}}{\mathcal{V}_{10}}\right)^{\frac{3}{10}} E^{SL(10)}_{1;\frac{3}{2}}\right)\\ &\simeq 4\zeta(2)\left(\frac{\mathcal{V}_{10}}{\ell_1^{10}}\right)^{2/3} + \left(\frac{\mathcal{V}_{10}}{\ell_1^{10}}\right)^{7/10} E^{SL(10)}_{1;\frac{3}{2}}\end{aligned} \tag{5.4.25}$$

and

$$\int_2 \mathcal{E}^1_{(1,0)} \simeq \frac{\ell_{11}\mathcal{V}_{10}}{\ell_1^{11}}\left(\left(\frac{\mathcal{V}_{10}}{\ell_{11}^{10}}\right)^{\frac{1}{10}} E^{SL(10)}_{1;-\frac{1}{2}} + \left(\frac{\ell_{11}^{10}}{\mathcal{V}_{10}}\right)^{\frac{5}{10}} E^{SL(10)}_{1;\frac{5}{2}} + \left(\frac{\ell_{11}^{10}}{\mathcal{V}_{10}}\right)^{\frac{8}{10}} E^{SL(10)}_{3;2}\right)$$
$$\simeq \left(\frac{\mathcal{V}_{10}}{\ell_1^{10}}\right)^{11/10} E^{SL(10)}_{1;-\frac{1}{2}} + \left(\frac{\mathcal{V}_{10}}{\ell_1^{10}}\right)^{7/6} E^{SL(10)}_{1;\frac{5}{2}} + \left(\frac{\mathcal{V}_{10}}{\ell_1^{10}}\right)^{6/5} E^{SL(10)}_{3;2} \,. \tag{5.4.26}$$

Our expansion parameter r below is related to the fundamental quantities via $r = (\mathcal{V}_{10}/\ell_1^{10})^{1/3}$.

For explicit expressions (including numerical coefficients) of the three degeneration limits of the $\mathcal{E}^D_{(0,0)}$ and the $\mathcal{E}^D_{(1,0)}$ in $D = 2, 1$ and 0, we refer the reader to the appendix E.

Chapter 6

Fourier coefficients of Kac–Moody Eisenstein series

This chapter deals with the Fourier coefficients of Eisenstein series, as defined in chapter 4. In particular we will derive a *reduction formula* which allows one to compute explicit expression for degenerate Whittaker vectors. We then provide explicit examples of such Whittaker vectors for maximal parabolic Kac–Moody Eisenstein series defined on E_9 and E_{10}, and in the appendix F also for Eisenstein series on the finite-dimensional groups E_6, E_7 and E_8.

The chapter contains excerpts from the unpublished article **II**.

6.1 Degenerate Whittaker vectors

In this section we will present a method for calculating Whittaker vectors $W_\psi(\chi, a)$, when the Fourier kernel ψ is non-generic, cf. section 4.1.2. For this we will discuss the general schematics of the integral for $\mathcal{F}_{w,\psi}$, which allows us to derive a reduction formula that expresses W_ψ in terms of non-degenerate Whittaker vectors of subgroups $G' \subset G$ determined by the degenerate character ψ.

6.1.1 Parametrising the contributing Weyl words

According to the discussion in section 4.2.2, the contributing set of Weyl words for which $\mathcal{F}_{w,\psi}$ can be non-zero, is given by

$$\mathcal{C}_\psi := \{w \in \mathcal{W} \,|\, w\Pi' < 0\}\,, \tag{6.1.1}$$

where $\Pi' \subset \Pi$ denotes the simple set of roots α for which $m_\alpha \neq 0$ (cf. also (4.1.10)). We have added the subscript ψ as a reminder that the definition of the set depends on ψ.

We are now going to characterise the special set $\mathcal{C}_\psi$ of Weyl words (6.1.1) in a more practical way. The set of simple roots Π', together with its complement $\overline{\Pi'}$, partition the set of simple roots Π of G. The subgroup $G_{\Pi'}$ of the full invariance group G is then defined by the Dynkin diagram given by Π'. The Weyl group associated with Π' is denoted by $\mathcal{W}'$.

The statement that we are going to prove in the following is that the elements of our special set of words can be written in the following form

$$w \in \mathcal{C}_\psi \quad \Longleftrightarrow \quad w = w_c w_0'\,, \tag{6.1.2}$$

where w_0' is the longest Weyl word of $\mathcal{W}'$ and w_c is a carefully chosen representative of the coset $\mathcal{W}/\mathcal{W}'$. We refer to the construction method of the particular coset representative that we require as the *orbit method* which was explained in detail in section 5.2.3. We recall that a given Weyl element $w \in \mathcal{W}$ has of course many seemingly different representations in terms of products of other elements. What we are claiming here is that all w that satisfy $w\Pi' < 0$ have a representation in the form given.

Let us first characterise the representative w_c required in (6.1.2). By its very definition, the action of the longest Weyl word w_0' of $\mathcal{W}'$ makes all roots of G', and in particular the simple roots in Π', negative. In order to satisfy the condition (6.1.1), one can then add further Weyl words w_c to the left of w_0', provided they satisfy the condition

$$w_c\alpha' > 0 \text{ for all } \alpha' \in \Pi'\,. \tag{6.1.3}$$

Then the combined word $w = w_c w_0'$ will map all simple roots in Π' to negative roots. Weyl words w_c satisfying (6.1.3) can be constructed as carefully chosen representatives of the coset $\mathcal{W}/\mathcal{W}'$.

6.1.2 Reduction formula

We now return to evaluating the contribution $\mathcal{F}_{w,\psi}$ to a degenerate Whittaker vector given by (cf. (4.2.13)) and use that $w = w_c w_0' \in \mathcal{C}_\psi$ with the particular construction of the preceding section. Associated with the word $w = w_c w_0'$ is a parametrisation of the elements n in the integration domain $N^w_{\{\gamma\}}$, cf. (4.2.10). We write

$$n_\gamma = n_c n' \quad \text{with} \quad n_c \in N_c(\mathbb{A}) \text{ and } n' \in N'(\mathbb{A})\,, \tag{6.1.4}$$

where $N'(\mathbb{A})$ is the unipotent subgroup of the minimal Borel $B'(\mathbb{A})$ of the subgroup $G'(\mathbb{A})$ determined by the set of simple roots Π' that indicate the directions on which the degenerate character ψ depends non-trivially. The set $N_c(\mathbb{A})$ involves the remaining positive roots γ that are mapped to negative roots by the action of w but that are not positive roots of the subgroup G'. The degenerate character ψ does not depend on n_c since $N_c(\mathbb{A})$ does not involve any of the simple roots of G'. We can thus write the integral for $\mathcal{F}_{w,\psi}$ as

$$\mathcal{F}_{w,\psi}(\chi) = \int\limits_{N_c(\mathbb{A})} \int\limits_{N'(\mathbb{A})} \chi(w_c w_0' n_c n') \overline{\psi(n')} \mathrm{d}n_c \mathrm{d}n'\,. \tag{6.1.5}$$

The argument of the character χ can be rewritten as

$$\chi(w n_c n') = \chi(w n_c w^{-1} w n') = \chi(w n_c w^{-1} w_c \tilde{n}\tilde{a}), \tag{6.1.6}$$

where we have used $w = w_c w_0'$. Moreover, $\tilde{n}\tilde{a}\tilde{k} = w_0' n'$ arises from the Iwasawa decomposition in the group G' and we have used the right-invariance of χ under $\tilde{k}$ in the last step. Now, it is important that w_c satisfies the condition (6.1.3) which implies that, even though it is no longer in $G'(\mathbb{A})$, the element $w_c \tilde{n}\tilde{a} w_c^{-1} = \hat{n}\hat{a}$ is an element of the Borel subgroup $B(\mathbb{A})$ and in particular $\hat{a} = w_c \tilde{a} w_c^{-1}$. In the next step we conjugate both $\hat{n}$ and $\hat{a}$ through to the left in the argument of χ. For $\hat{n}$ this induces a uni-modular change of integration variables for n_c. By contrast, for $\hat{a}$ this generates a non-trivial Jacobi factor when passing past $w n_c w^{-1}$ that we can determine in a way similar to (4.2.20). The relevant manipulation is:

$$\begin{aligned}\int\limits_{N_c(\mathbb{A})} \chi(\hat{n} w n_c w^{-1} \hat{a}) \mathrm{d}n_c &= \int\limits_{N_c(\mathbb{A})} \chi(\hat{n}\hat{a} w n_c w^{-1}) \hat{a}^{w_c\rho-\rho} \mathrm{d}n_c \\ &= \int\limits_{N_c(\mathbb{A})} \chi(w n_c w^{-1}) \tilde{a}^{w_c^{-1}\lambda+\rho} \mathrm{d}n_c\,, \end{aligned} \tag{6.1.7}$$

where we have used $\chi(\hat{n}\hat{a}) = \hat{a}^{\lambda+\rho}$ and $\hat{a} = w_c \tilde{a} w_c^{-1}$. Now, $\tilde{a}$ does not depend on n_c and we can take it out of the $N_c(\mathbb{A})$ integral. None of these transformations have any impact on the argument of the character $\psi(n')$.

The result of these steps is that $\mathcal{F}_{w,\psi}$ factorises according to

$$\mathcal{F}_{w,\psi}(\chi) = \int\limits_{N_c(\mathbb{A})} \chi(w_c w_0' n_c) \mathrm{d}n_c \cdot \int\limits_{N'(\mathbb{A})} \chi'(w_0' n') \overline{\psi(n')} \mathrm{d}n', \tag{6.1.8}$$

where the character $\chi' : B'(\mathbb{A}) \to \mathbb{C}^*$ is given by the (projection of the) weight $w_c^{-1}\lambda + \rho$. The Jacobi factor arising from $\hat{a}$ has been transformed back into the expression $\chi'(w_0' n')$ in the second integral.

The two separate integrals in (6.1.8) are of well-known types. The $N_c(\mathbb{A})$ integral is identical to the integral that determines (the numerical coefficient of) the contribution of the Weyl word w_c to the constant term (in the minimal parabolic) (4.2.24) and therefore yields the factor $M(w_c^{-1}, \lambda)$. Referring back to (4.2.17), we recognise the second integral as the non-degenerate Whittaker vector for the subgroup $G'(\mathbb{A}) \subset G(\mathbb{A})$ for a *generic* Fourier character ψ on N' of the Eisenstein series determined by the weight $w_c^{-1}\lambda + \rho$, projected orthogonally to $G'(\mathbb{A})$, and evaluated at the identity group element. The expression (6.1.8) for $\mathcal{F}_{w,\psi}$, then evaluates to

$$\mathcal{F}_{w,\psi}(\lambda) = M(w_c^{-1}, \lambda) W'_\psi(w_c^{-1}\lambda, \mathrm{id})\,. \tag{6.1.9}$$

Equation (6.1.9) is the expression for an arbitrary character ψ. For the Whittaker vector W_ψ we require (6.1.9) evaluated at the twisted character ψ^a according to (4.2.23). Combining all elements as prescribed by (4.2.23) we obtain the following final expression for the degenerate Whittaker vector

$$W_\psi(\lambda, a) = \sum_{w_c w_0' \in \mathcal{C}_\psi} a^{(w_c w_0')^{-1}\lambda+\rho} M(w_c^{-1}, \lambda) W'_{\psi^a}(w_c^{-1}\lambda, \mathrm{id}) \,, \tag{6.1.10}$$

where the factor in front of $\mathcal{F}_{w,\psi^a}$ was determined in (4.2.21). Here, W'_{ψ^a} denotes a Whittaker function of the $G'(\mathbb{A})$ subgroup of $G(\mathbb{A})$. Therefore, Whittaker vectors of non-generic characters ψ can be evaluated as sums over Whittaker vectors of subgroups on which the character is generic. At local places this then can be evaluated using the Casselman–Shalika formula.

6.2 The collapse for degenerate Whittaker vectors

As the next step, we will now combine the collapse mechanism with the formula (6.1.10) for degenerate Whittaker vectors. This will allow us to calculate explicitly some degenerate Whittaker vectors for Kac–Moody groups. The following applies to (maximal) parabolic Eisenstein series.

Looking at the reduction formula (6.1.10), we first construct the set $\mathcal{C}_\lambda$ defined in (5.2.30). In the context of (6.1.10) the elements of $\mathcal{C}_\lambda$ should be interpreted as w_c^{-1}. We also know that all words $w \in \mathcal{C}_\psi$ contributing to (6.1.10) are of the form $w = w_c w_0'$, cf. (6.1.2). Therefore we form set

$$\mathcal{C}_{\lambda,\psi} = \left\{ w_c^{-1} \in \mathcal{C}_\lambda \,|\, w_c w_0' \in \mathcal{C}_\psi \right\} = \mathcal{C}_\psi \cap (\mathcal{C}_\lambda)^{-1} w_0'. \tag{6.2.1}$$

This set typically contains a very small number of elements for the special values of λ, i.e. i_* and s, that are relevant in string theory. The sum in the reduction formula is restricted to $\mathcal{C}_{\lambda,\psi}$.

This is not the only simplification that arises. Further terms can be absent for a degenerate Whittaker vector when the factor $W'_{\psi^a}((w_c^{-1}\lambda)_{G'}, \mathrm{id})$ vanishes. This always happens for example when the projected weight $(w_c^{-1}\lambda)_{G'}$

is equal to $-\rho'$ (ρ' being the Weyl vector of G'). The reason is that for this case one is computing the Whittaker vector of a constant function which vanishes. Similar cases can arise when $(w_c^{-1}\lambda)_{G'}$ is such that generic Whittaker vector on G' vanishes, i.e., the projected weight corresponds to a degenerate principal series representation.

6.2.1 Explicit results for some Kac–Moody groups

We now present the results that we obtained by implementing the formalism above for E_9, E_{10} and E_{11} for the special cases $i_* = 1$ and $s = 3/2$ and $s = 5/2$ in (3.3.8). See figure 2.3 for our labelling convention of the E_n Dynkin diagram. We note that in appendix F we give examples of Whittaker vectors of Eisenstein series defined on the finite-dimensional groups E_6, E_7 and E_8.

In all cases, we denote an element of the maximal torus by

$$a = \prod_{i=1}^{n} v_i^{H_i}, \tag{6.2.2}$$

where H_i is the standard Chevalley generator of the simple root α_i in Bourbaki labelling and n is the rank of the group.

What we will display in the sequel are the degenerate Whittaker vectors that have only support along one simple root, such that the set Π' of section 6.1 contains only one simple root. The subgroup G' is then of type $SL(2)$ and its Whittaker vectors are modified Bessel functions. For a degenerate character ψ with only non-zero charge m_α for a single simple root α we write the corresponding Whittaker vector as

$$\begin{aligned} W'_{\psi^a}(\chi', \mathrm{id}) &:= B_{s',m_\alpha}(a^\alpha) \\ &:= \frac{2}{\xi(2s')} |a_\alpha|^{s'-1/2} |m_\alpha|^{1/2-s'} \sigma_{2s'-1}(m_\alpha) K_{s'-1/2}(2\pi |m_\alpha| a^\alpha). \end{aligned} \tag{6.2.3}$$

Here, s' parametrises the projected character χ' on $SL(2)$ by $\lambda' = 2s'\Lambda' - \rho'$ where Λ' is the unique fundamental weight of $SL(2)$. In some of the examples below, it will also be useful to have the definition, $\tilde{B}_{s',m} = \xi(2s')B_{s',m}$, of a differently normalised Bessel function at hand.

We present the result in table form and use some short-hand notations for the Whittaker vectors of the subgroups A_1. While most of the examples presented are for Whittaker vectors associated with one non-trivial charge, we also include one example with two-nontrivial charges.

E_9 with $s = 3/2$

Whittaker vectors of the $E^{E_9}_{1;3/2}$ Eisenstein series, associated with one non-trivial charge.

ψ	$W_\psi(\chi_{3/2}, a)$
$(m,0,0,0,0,0,0,0,0)$	$v_3^2 v_1^{-1} B_{3/2,m}\left(v_1^2 v_3^{-1}\right)$
$(0,m,0,0,0,0,0,0,0)$	$\frac{v_2^2 \tilde{B}_{0,m}(v_2^2 v_4^{-1})}{\xi(3)}$
$(0,0,m,0,0,0,0,0,0)$	$\frac{\xi(2) v_4 B_{1,m}\left(v_3^2 v_1^{-1} v_4^{-1}\right)}{\xi(3)}$
$(0,0,0,m,0,0,0,0,0)$	$\frac{v_4 \tilde{B}_{1/2,m}(v_4^2 v_2^{-1} v_3^{-1} v_5^{-1})}{\xi(3)}$
$(0,0,0,0,m,0,0,0,0)$	$\frac{v_5^2 \tilde{B}_{0,m}(v_5^2 v_4^{-1} v_6^{-1})}{\xi(3) v_6}$
$(0,0,0,0,0,m,0,0,0)$	$\frac{\xi(2) v_6^3 B_{-1/2,m}\left(v_6^2 v_5^{-1} v_7^{-1}\right)}{\xi(3) v_7^2}$
$(0,0,0,0,0,0,m,0,0)$	$v_7^4 v_8^{-3} B_{-1,m}\left(v_7^2 v_6^{-1} v_8^{-1}\right)$
$(0,0,0,0,0,0,0,m,0)$	$\frac{\xi(4) v_8^5 v_9^{-4} B_{-3/2,m}\left(v_8^2 v_7^{-1} v_9^{-1}\right)}{\xi(3)}$
$(0,0,0,0,0,0,0,0,m)$	$\frac{\xi(5) v_9^6 v^{-5} B_{-2,m}\left(v_9^2 v_8^{-1} v^{-1}\right)}{\xi(3)}$

E_{10} **with** $s = 3/2$

Whittaker vectors of the $E^{E_{10}}_{1;3/2}$ Eisenstein series, associated with one non-trivial charge.

ψ	$W_\psi(\chi_{3/2}, a)$
$(m,0,0,0,0,0,0,0,0,0)$	$v_3^2 v_1^{-1} B_{3/2,m}\left(v_1^2 v_3^{-1}\right)$
$(0,m,0,0,0,0,0,0,0,0)$	$\frac{v_2^2 \tilde{B}_{0,m}(v_2^2 v_4^{-1})}{\xi(3)}$
$(0,0,m,0,0,0,0,0,0,0)$	$\frac{\xi(2) v_4 B_{1,m}\left(v_3^2 v_1^{-1} v_4^{-1}\right)}{\xi(3)}$
$(0,0,0,m,0,0,0,0,0,0)$	$\frac{v_4 \tilde{B}_{1/2,m}(v_4^2 v_2^{-1} v_3^{-1} v_5^{-1})}{\xi(3)}$
$(0,0,0,0,m,0,0,0,0,0)$	$\frac{v_5^2 \tilde{B}_{0,m}(v_5^2 v_4^{-1} v_6^{-1})}{\xi(3) v_6}$
$(0,0,0,0,0,m,0,0,0,0)$	$\frac{\xi(2) v_6^3 B_{-1/2,m}\left(v_6^2 v_5^{-1} v_7^{-1}\right)}{\xi(3) v_7^2}$
$(0,0,0,0,0,0,m,0,0,0)$	$v_7^4 v_8^{-3} B_{-1,m}\left(v_7^2 v_6^{-1} v_8^{-1}\right)$
$(0,0,0,0,0,0,0,m,0,0)$	$\frac{\xi(4) v_8^5 v_9^{-4} B_{-3/2,m}\left(v_8^2 v_7^{-1} v_9^{-1}\right)}{\xi(3)}$
$(0,0,0,0,0,0,0,0,m,0)$	$\frac{\xi(5) v_9^6 v_{10}^{-5} B_{-2,m}\left(v_9^2 v_8^{-1} v_{10}^{-1}\right)}{\xi(3)}$
$(0,0,0,0,0,0,0,0,0,m)$	$\frac{\xi(6) v_{10}^7 B_{-5/2,m}\left(v_{10}^2 v_9^{-1}\right)}{\xi(3)}$

There is obviously a simple pattern associated with this set of degenerate Whittaker vectors.

We have also verified that the Whittaker vectors for less degenerate characters ψ are all identically zero. All the non-vanishing Whittaker vectors are associated with subgroups of type $A_1 \cong SL(2)$.

E_{10} with $s = 5/2$

Whittaker vectors of the $E^{E_{10}}_{1;5/2}$ Eisenstein series, associated with one non-trivial charge. The expressions for the $SL(2)$-type degenerate Whittaker vectors are much longer in this case and we do not give all of them. An illustrative example is obtained for the instanton charge vector $(m,0,0,0,0,0,0,0,0,0)$. There we have

$$\begin{aligned}W_\psi(\chi_{5/2},a) = {}& v_1^3 v_{10}^5 B_{-\frac{1}{2},m}\left(\frac{v_1^2}{v_3}\right) + \frac{\xi(2)v_1^3v_5^2B_{-\frac{1}{2},m}\left(\frac{v_1^2}{v_3}\right)}{\xi(5)v_4} + \frac{\xi(3)v_1^3v_7^2B_{-\frac{1}{2},m}\left(\frac{v_1^2}{v_3}\right)}{\xi(5)v_8}\\ &+ \frac{\xi(4)v_1^3v_2^4B_{-\frac{1}{2},m}\left(\frac{v_1^2}{v_3}\right)}{\xi(5)v_3^2} + \frac{\xi(3)v_1^3v_4^3B_{-\frac{1}{2},m}\left(\frac{v_1^2}{v_3}\right)}{\xi(5)v_2^2v_3^2} + \frac{v_3^4B_{\frac{5}{2},m}\left(\frac{v_1^2}{v_3}\right)}{v_1^3}\\ &+ \frac{\xi(3)v_1^3v_8^3B_{-\frac{1}{2},m}\left(\frac{v_1^2}{v_3}\right)}{\xi(5)v_9^2} + \frac{\xi(4)v_1^3v_9^4B_{-\frac{1}{2},m}\left(\frac{v_1^2}{v_3}\right)}{\xi(5)v_{10}^3}\\ &+ \frac{6\xi(3)(\gamma_{\mathrm{E}} - \log(4\pi v_5) + 2\log(v_6) - \log(v_7))v_1^3v_6B_{-\frac{1}{2},m}\left(\frac{v_1^2}{v_3}\right)}{\pi\xi(5)}\end{aligned} \tag{6.2.4}$$

E_{10} with $s = 5/2$ and two non-trivial charges

Whittaker vectors of the $E^{E_{10}}_{1;5/2}$ Eisenstein series, associated with two non-trivial charge. We note that all the non-vanishing Whittaker vectors are of type $A_1 \times A_1$.

ψ	$W_\psi(\chi_{5/2}, a)$
$(m_1, m_2, 0, 0, 0, 0, 0, 0, 0, 0)$	$\frac{\xi(3) v_1^3 v_2^4 B_{-1/2,m_1}\left(\frac{v_1^2}{v_3}\right) B_{-1,m_2}\left(\frac{v_2^2}{v_4}\right)}{\xi(5) v_3^2}$
$(m_1, 0, 0, m_2, 0, 0, 0, 0, 0, 0)$	$\frac{\xi(2) v_1^3 v_4^3 B_{-1/2,m_1}\left(\frac{v_1^2}{v_3}\right) B_{-1/2,m_2}\left(\frac{v_4^2}{v_2 v_3 v_5}\right)}{\xi(5) v_2^2 v_3^2}$
$(m_1, 0, 0, 0, 0, m_2, 0, 0, 0, 0)$	$\frac{v_1^3 v_6 B_{-1/2,m_1}\left(\frac{v_1^2}{v_3}\right) \tilde{B}_{1/2,m_2}\left(\frac{v_6^2}{v_5 v_7}\right)}{\xi(5)}$
$(m_1, 0, 0, 0, 0, 0, 0, m_2, 0, 0)$	$\frac{\xi(2) v_1^3 v_8^3 B_{-1/2,m_1}\left(\frac{v_1^2}{v_3}\right) B_{-1/2,m_2}\left(\frac{v_8^2}{v_7 v_9}\right)}{\xi(5) v_9^2}$
$(m_1, 0, 0, 0, 0, 0, 0, 0, m_2, 0)$	$\frac{\xi(3) v_1^3 v_9^4 B_{-1/2,m_1}\left(\frac{v_1^2}{v_3}\right) B_{-1,m_2}\left(\frac{v_9^2}{v_8 v_{10}}\right)}{\xi(5) v_{10}^3}$
$(m_1, 0, 0, 0, 0, 0, 0, 0, 0, m_2)$	$\frac{\xi(4) v_1^3 v_{10}^5 B_{-1/2,m_1}\left(\frac{v_1^2}{v_3}\right) B_{-3/2,m_2}\left(\frac{v_{10}^2}{v_9}\right)}{\xi(5)}$
$\ldots$	$\ldots$

Chapter 7

Conclusion

In this final chapter, let us offer some reflections on the results about Kac–Moody Eisenstein series and their role in string theory presented in this thesis. We will also formulate some open questions for future research and give a short outline of the *quantum cosmological billiards* approach, in which automorphic forms, and possibly Eisenstein series, also play a role.

In this thesis we have defined Eisenstein series on Kac–Moody groups and in particular on the infinite-dimensional U-duality groups. We have done so, not only for the affine Kac–Moody group E_9, but also for the hyperbolic group E_{10} and the Lorentzian group E_{11}. It should be noted that our approach does not, of course, satisfy a mathematician's definition of mathematical rigour. For instance we have only provided a rather heuristic argument for the convergence of the Eisenstein series that we consider. It would certainly be desirable to obtain a full proof of convergence in the future. Such a proof would likely also stimulate mathematicians' interest in the subject of Kac–Moody Eisenstein series and foster more and fruitful exchange between the two fields.

A major part of our work was to demonstrate that special (maximal parabolic) Kac–Moody Eisenstein series have a clearly defined Fourier expansion in complete analogy with Eisenstein series defined on finite-dimensional groups. We have given proof of this through explicit calculation of the zeroth and higher order Fourier modes for different examples. In our opinion this should be seen as a first step towards a more complete understanding of the structure of the Fourier expansion of such Eisenstein series and will potentially contribute to a better understanding of the non-perturbative structure of string theory.

It should be clear from the bulk of this thesis that the results we have presented are of a rather mathematical nature. It still remains open for future work to investigate how these results can be fully accommodated and interpreted within string theory. Let us now mention more specifically some open problems which we think pose some interesting questions for future research.

As mentioned before, a central motivation for our work is to obtain control over the non-perturbative regime of string theory. Taking the calculations presented in chapter 6 as a starting point, one could for example try to extract instanton measures, like the one in (1.2.9), for the case of higher rank finite-dimensional U-duality groups, but possibly also for the Kac–Moody cases.

We explained in section 2.2.1 that the particular Kac–Moody Eisenstein series that we study appear in string scattering amplitudes. It would therefore be interesting to see if it is possible to develop a deeper physical understanding of scattering processes in $D = 2$ and 1 dimensions. The central question here is, of course, what the relevant physical degrees of freedom are in these dimensions.

In compactified type IIB string theory or M theory, the number of gauge fields which appear grows with the number of toroidally compactified directions. In $D \geq 4$ dimensions, the charged particles associated with the gauge fields can be explained in terms of the wrapping of branes on the compactified directions [21]. For $D < 4$ dimensions the associated branes would be required to have tensions of the order of the third, fourth power or higher of the inverse string coupling g_s^{-1}. This would, however, contradict the fact that the tension of a brane can at most be the square of g_s^{-1}, for its surrounding space-time to be non-singular in the weak coupling limit. Therefore an explanation in terms of wrapped branes does not apply and the associated states have been termed *exotic* [21,119]. Upon quantisation, the charges associated with the gauge fields are expected to obey a Dirac-Schwinger-Zwanziger type quantisation condition, see for instance [49], thus forming a discrete charge lattice. In $D \geq 4$ this method of quantisation holds, but runs into unresolved problems in $D < 4$ dimensions, linked to the appearance of the NUT charge, the dual charge of the mass, similar to the electric-magnetic duality. At present no method of quantising the NUT charge exists. However, linking back to the discussion of Eisenstein series, it has been shown for some cases that it is possible to define these series as sums over certain subsets of the lattice charges [120]. It would therefore be interesting to investigate the charge lattice for the case $D = 3$. This includes studying the possibility of quantising the NUT charge and writing down conditions which restrict the expected 248 charges of $E_8(\mathbb{Z})$ to appropriate subsets.

Finally we would like to mention one other possible application where automorphic forms, and possibly Eisenstein series, play a role. Namely, this is in the context of the quantum cosmological billiard approach [121].

The classical version of this approach, based on early investigations of Belinski, Khalatnikov and Lifshitz (BKL) [32, 122, 123], states that the behaviour of space-time near a space-like singularity is described through the motion of a fictitious, massless particle (the 'billiard ball') in the abstract Wheeler-DeWitt superspace. The motion of the particle is geodesic and constrained to a convex region of superspace whose boundary is determined by infinitely sharp potential walls. The geometry of the billiard domain depends on the type of theory analysed. Surprisingly, it turns out that the billiard domain is given by the fundamental Weyl chamber of a hyperbolic Kac–Moody group [61]. In the case of eleven-dimensional supergravity, the Kac–Moody group is found to be E_{10}. Upon quantising the classical cosmological billiard system, the particle is promoted to a wave function which satisfies the Wheeler-DeWitt equation. It was shown in [34] that when restricting to the Cartan subalgebra degrees of freedom this wave function is an automorphic form with respect to the Weyl group of E_{10}. In order to take into account the full infinite tower of degrees of freedom, the wave function should be an automorphic form of the full E_{10} group. Therefore, it might be interesting to study Eisenstein series defined on E_{10}, and possibly other automorphic forms, as candidate wave functions for the quantum cosmological billiard system.

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Appendix A

Algebras and roots

In this chapter we briefly present some of the mathematical background, in particular definitions concerning Lie algebras, which are frequently being made use of throughout the thesis. For further details we refer the reader to the following list of literature [88, 124, 125].

A.1 Lie algebras

A Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ is defined via a Cartan matrix A_{ij} and a set of generators of the algebra, satisfying a certain set of relations. The Cartan matrix is an $r \times r$ matrix, where r is the rank for the algebra and the components A_{ij} satisfy the rules: $A_{ii} = 2$, $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$, $A_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$. For finite-dimensional Lie algebras $\mathfrak{g}(A)$ the Cartan matrix is positive definite, i.e. $A > 0$. For general Kac-Moody algebras the assumption of positive definiteness is dropped.

The generators of the algebra are given by Chevalley-Serre generators, which are formed by triplets of generators $\{E^i_+, E^i_-, H^i\}$ which satisfy the standard relation of an $\mathfrak{sl}(2, \mathbb{R})$ algebra:

$$[E^i_+, E^i_-] = H^i \,, \quad [H^i, E^i_\pm] = \pm 2 E^i_\pm \,. \tag{A.1.1}$$

For a general Lie algebra of rank r, there are r such triplets which generate the entire algebra. In this case, the relations (A.1.1) satisfied by the triplets are generalised to

$$[E^i_+, E^j_-] = \delta_{ij} H_j \quad [H^i, E^j_\pm] = \pm A_{ij} E^i_\pm \quad \text{and} \quad [H^i, H^j] = 0\,, \tag{A.1.2}$$

which then also specify commutations amongst generators of different Chevalley triplets. Finally, there are also the Serre relations which are satisfied

$$(\text{ad}\, E^i_\pm)^{1-A_{ij}}(E^j_\pm) = 0 \quad (i \neq j)\,. \tag{A.1.3}$$

The set of all H^i, with $i = 1, ..., r$ span the Cartan subalgebra $\mathfrak{h}$ (CSA), which constitutes the maximal set of commuting generators, satisfying

$$[H^i, H^j] = 0\,. \tag{A.1.4}$$

Simple roots α_i are defined as the eigenvalues which appear in the adjoint action of the generators H^i of the CSA on the raising and lowering operators $E^i_\pm$ respectively

$$[H^j, E^i_\pm] = \pm\alpha_i(H^j) E^i_\pm\,. \tag{A.1.5}$$

Since the CSA is r-dimensional, the α_i are vectors with r components, which constitute a basis of the root space and they are elements of the dual space of $\mathfrak{h}$, which we denote by $\mathfrak{h}^\star$. The components of the Cartan matrix are defined as products of simple roots according to

$$A_{ij} = \frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}\,, \tag{A.1.6}$$

where the brackets, $(\cdot|\cdot)$ denote the symmetric, bilinear form of the algebra $\mathfrak{g}$. As a map it is defined as

$$(\cdot|\cdot)\,:\,\mathfrak{h}^\star \times \mathfrak{h}^\star \to \mathbb{C}\,. \tag{A.1.7}$$

In this thesis we are only dealing with simply-laced algebras (all roots have the same length), we define that $(\alpha_i|\alpha_i) = 2$. Furthermore, there is a nice diagrammatic way to depict all the information contained in the Cartan matrix. This is known as a Dynkin diagram. The first step is to draw a dot (node) for each simple root. In the second step, one adds links between the nodes. In the simplest case, relevant in thesis, two nodes corresponding to the simple roots α_i and α_j are connected by a single line if $|A_{ij}| = 1$, otherwise the nodes are unconnected.

Apart from the simple roots there are also roots α of the algebra which are non-simple and which can be written as a linear combinations of the simple roots

$$\alpha = \sum_i m^i \alpha_i \,, \tag{A.1.8}$$

with either all $m^i \geq 0$ or all $m^i \leq 0$, such that the root α is called a positive or a negative root, respectively. The raising an lowering operators $E^\alpha_\pm$, corresponding to some non-simple root α, are defined in terms of commutators of the form $[E^i_\pm, E^j_\pm]$ with $i \neq j$ and the components of α follow from

$$[H^i, E^\alpha_\pm] = \pm\alpha(H^i) E^\alpha_\pm \,. \tag{A.1.9}$$

Employing the symmetric, bilinear form, we define the set of fundamental weights $\Lambda_i \in \mathfrak{h}$, such that

$$(\alpha_i | \Lambda_j) = \delta_{ij} \,. \tag{A.1.10}$$

Alternatively to (A.1.8), we can also express a roots α as a sum of the fundamental weights as

$$\alpha = \sum_i p^i \Lambda_i \,. \tag{A.1.11}$$

In this linear combination, the coefficients p^i are referred to as the Dynkin labels and a root is sometimes expressed through the short-hand notation $[p_1, p_2, ..., p_r]$.

A.1.1 The Weyl group

The entire set of roots Δ of $\mathfrak{g}$ can be generated from the set of simple roots Π through the action of the Weyl group, $\mathcal{W} = \mathcal{W}(\mathfrak{g})$ of $\mathfrak{g}$. The Weyl group is a reflection group with generators w_i, where $i = 1, ..., n$. The generators w_i are called the fundamental Weyl reflections. Some general Weyl word w can be built-up from the successive action of several generators. For example $w = w_{i_r}...w_{i_2}w_{i_1}$, where the length of the Weyl $\ell(w)$ in this case is r. Geometrically speaking, the Weyl group allows one to reflect a weight λ on the hyperplane which is orthogonal to some root α of the algebra. This reflection can be calculated from the following formula:

$$w_\alpha(\lambda) = \lambda - 2\frac{(\lambda|\alpha)}{(\alpha|\alpha)}\alpha\,. \tag{A.1.12}$$

A.2 Subalgebras

In this section we will discuss the definitions of some subalgebras and group decompositions that are particularly relevant in the work presented in this thesis. For this it is useful to define the subspace $\mathfrak{g}_\alpha$ of $\mathfrak{g}$, according to

$$\mathfrak{g}_\alpha = x \in \mathfrak{g} \,:\, [H, x] = \alpha(H)x\,,\, \forall\, H \in \mathfrak{h}\,. \tag{A.2.1}$$

Then we can write

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha\,. \tag{A.2.2}$$

A.2.1 Borel and parabolic subalgebras

The *Borel subalgebra* $\mathfrak{b}$ of an algebra $\mathfrak{g}$ is defined as

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha\,. \tag{A.2.3}$$

A (standard) *parabolic subalgebra* $\mathfrak{p}$ is a subalgebra of $\mathfrak{g}$ that contains $\mathfrak{b}$. Parabolic subalgebras $\mathfrak{p}$ decompose in general as the direct sum of the so-called Levi subalgebra $\mathfrak{l}$ and the unipotent radical $\mathfrak{u}$

$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}\,. \tag{A.2.4}$$

A convenient construction of parabolic subalgebras is obtained by selecting a subset Π_1 of the set of simple roots Π. This induces a corresponding subset Γ_1 of the set of positive roots Δ_+, where the Γ_1 are those positive roots that are linear combinations of the simple roots in Π_1 only. The Levi subalgebra and unipotent radical are then defined as

$$\mathfrak{l}(\Pi_1) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma_1 \cup -\Gamma_1} \mathfrak{g}_\alpha \tag{A.2.5}$$

and

$$\mathfrak{u}(\Pi_1) = \bigoplus_{\alpha \in \Delta_+ \backslash \Gamma_1} \mathfrak{g}_\alpha \tag{A.2.6}$$

respectively and the parabolic subalgebra is given by

$$\mathfrak{p}(\Pi_1) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+ \cup (-\Gamma_1)} \mathfrak{g}_\alpha = \mathfrak{b} \oplus \bigoplus_{\alpha \in (-\Gamma_1)} \mathfrak{g}_\alpha \,. \tag{A.2.7}$$

There are two types of parabolic subalgebras which are of importance for us. The first is the *minimal parabolic* case, which is obtained, when $\Pi_1 = \emptyset$ and corresponds to the Borel subalgebra $\mathfrak{b}$. The second is the *maximal parabolic* case for which $\Pi_1 = \Pi \backslash \{\alpha_{i_*}\}$, where α_{i_*} is a (single) simple root. Using an abbreviated notation we denote maximal parabolic subalgebras by $\mathfrak{p}_{i_*}$. We will denote the group associated with the subalgebra $\mathfrak{p}_{i_*}$ by P_{i_*}. Similar to the decomposition of $\mathfrak{p}$ shown in (A.2.4) we also have

$$P_{i_*} = L_{i_*} U_{i_*} \,, \tag{A.2.8}$$

where L_{i_*} and U_{i_*} are the groups associated with the subalgebras $\mathfrak{l}_{i_*}$ and $\mathfrak{u}_{i_*}$.

Example A.2.1. *Langlands' decomposition of $GL(n, \mathbb{R})$*

We illustrate the concept of different parabolic subgroups for the example of $GL(n, \mathbb{R})$, see .e.g [126]. Each standard parabolic subgroup of this group is associated to a partition of the integer n, i.e. $n = n_1 + n_2 + ... + n_r$, where r is the rank of the parabolic subgroup $P_{n_1, n_2, \ldots, n_r}$, represented by a block-diagonal *matrix*

$$P_{n_1,n_2,\ldots,n_r} = \begin{pmatrix} \mathfrak{l}_{n_1} & \star & \ldots & \star \\ & \mathfrak{l}_{n_1} & \ldots & \star \\ & & \ddots & \star \\ & & & \mathfrak{l}_{n_r} \end{pmatrix} . \tag{A.2.9}$$

In Langlands decomposition, we can write this matrix as a product of the Levi subgroup L and the unipotent radical U

$$P_{n_1,n_2,\ldots,n_r} = L_{n_1,n_2,\ldots,n_r} \, U_{n_1,n_2,\ldots,n_r} \tag{A.2.10}$$

where

$$U_{n_1,n_2,\ldots,n_r} = \begin{pmatrix} I_{n_1} & \star & \ldots & \star \\ & I_{n_1} & \ldots & \star \\ & & \ddots & \star \\ & & & I_{n_r} \end{pmatrix} , \quad L_{n_1,n_2,\ldots,n_r} \begin{pmatrix} \mathfrak{l}_{n_1} & 0 & \ldots & 0 \\ & \mathfrak{l}_{n_1} & \ldots & 0 \\ & & \ddots & 0 \\ & & & \mathfrak{l}_{n_r} \end{pmatrix} . \tag{A.2.11}$$

The I_{n_i} are $n_i \times n_i$ unit matrices and $\mathfrak{l}_{n_i} \in GL(n_i, \mathbb{R})$.
In the case of a minimal parabolic subgroup the partition of n is $n_i = 1$ for all $i = 1, 2, ..., r = n$ and for the maximal parabolic subgroup we have $r = 2$ and either $n_1 = 1$, $n_2 = n - 1$ or $n_1 = n - 1$, $n_2 = 1$. Then the matrices for the two cases are

$$P_{\underbrace{1,\ldots,1}_{n \text{ times } 1}} = \begin{pmatrix} \star & \star & \ldots & \star \\ & \star & \ldots & \star \\ & & \ddots & \vdots \\ & & & \star \end{pmatrix} \tag{A.2.12}$$

and

$$P_{1,n-1} = \begin{pmatrix} \star & \star & \dots & \star \\ & \star & \dots & \star \\ & \vdots & \ddots & \vdots \\ & \star & \dots & \star \end{pmatrix}, \qquad P_{n-1,1} = \begin{pmatrix} \star & \dots & \star & \star \\ \vdots & \dots & \vdots & \vdots \\ \star & \dots & \star & \star \\ & & & \star \end{pmatrix}, \tag{A.2.13}$$

for the minimal and maximal parabolic subgroup, respectively.

A.3 Iwasawa decomposition

The general form of the Iwasawa decomposition of a group G is

$$G = NAK\,, \tag{A.3.1}$$

where N is an unipotent groups, A is an abelian group and K an orthogonal group.

Let us specifically state the Iwasawa decomposition of $SL(2,\mathbb{R})$ and thereby define a canonical parameterisation for it, which appears at various places in this thesis. A group element $g \in SL(2,\mathbb{R})$ decomposes according to

$$g = nak = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{A.3.2}$$

Appendix B

Summary on p-adic numbers

In this appendix, which is an excerpt from article **IV**, we provide a short introduction to the theory of p-adic numbers.

B.1 p-adic numbers

Let p be a prime number. The p-adic integers $\mathbb{Z}_p$ are formal power series in p with coefficients between 0 and $p-1$

$$x \in \mathbb{Z}_p \quad \Leftrightarrow \quad x = x_0 p^0 + x_1 p^1 + \dots \quad \text{with } x_i \in \mathbb{Z}/p\mathbb{Z} \cong \{0, 1, \dots, p-1\} \tag{B.1.1}$$

They are a ring. Since all coefficients in the expansion are positive it is maybe not so obvious how the additive inverse(=subtraction) works. As an example consider the equation $x + 1 = 0$ that should have a solution over $\mathbb{Z}_p$. The inverse is given by the infinite power series where all coefficients are equal to $p - 1$.

The associated number field is given by the p-adic numbers $\mathbb{Q}_p$ that are formal Laurent series in p with finite polar part. They can be thought of as the completion of $\mathbb{Q}$ with respect to the *p-adic norm* $|x|_p$ that is given by

$$|x|_p = p^{-k} \quad \Leftrightarrow \quad x = x_k p^k + x_{k+1} p^{k+1} + \dots \quad \text{with } x_k \neq 0. \tag{B.1.2}$$

The integers in this space correspond to

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \,|\, |x|_p \leq 1\}\,. \tag{B.1.3}$$

They correspond to exponents $k \geq 0$ of p. This shows that the p-adic integers are compactly embedded in $\mathbb{Q}_p$. The norm is multiplicative and satisfies a stronger triangle inequality than generic norms, namely

$$|x + y|_p \leq \max(|x|_p, |y|_p)\,. \tag{B.1.4}$$

This property is called ultrametric property and a space with a norm of this type is called *non-archimedean* in contrast with archimedean spaces satisfying the usual archimedean triangle inequality. The p-adic norm of 0 is $|0|_p = 0$. The p-adic norm is also referred to as a *valuation* of $\mathbb{Q}$ or $\mathbb{Q}_p$.

Another way of defining the p-adic norm is through the following definition for an ordinary rational number $x \in \mathbb{Q}$:

$$|x|_p = p^{-k}\,, \tag{B.1.5}$$

where $k \in \mathbb{Z}$ is the largest integer such that $x = p^k y$ with $y \in \mathbb{Q}$ not containing any powers of p in its numerator or denominator (in cancelled form); this is often stated as p^k divides x. It is from this construction that one obtains $\mathbb{Q}_p$ as the completion of $\mathbb{Q}$ and one obtains an embedding of $\mathbb{Q}$ into $\mathbb{Q}_p$. The definition implies that for a prime q and $k \in \mathbb{Z}$

$$|q^k|_p = \left\{ \begin{array}{cc} p^{-k} & \text{if } p = q \\ 1 & \text{otherwise} \end{array} \right) \tag{B.1.6}$$

Example B.1.1. *We consider the p-adic expansion of the rational number $x = \frac{1}{2} \in \mathbb{Q}$ for $p = 2$ and $p = 3$.*

For $p = 2$ one has $|x|_2 = 2^1 = 2$ and hence $\frac{1}{2}$ is not a 2-adic integer. As an element of $\mathbb{Q}_2$ one finds $\frac{1}{2} = 1 \cdot 2^{-1}$ as the expansion.

For $p = 3$ one has $|x|_3 = 3^0 = 1$ and hence $\frac{1}{2}$ is a 3-adic integer. Its expansion of the form (B.1.2) is $\frac{1}{2} = 2 \cdot 3^0 + \sum_{k>0} 3^k$.

The set of multiplicatively invertible elements in $\mathbb{Z}_p$ will be denoted by

$$\mathbb{Z}_p^* = \left\{x \in \mathbb{Z}_p \,\middle|\, x^{-1} \text{ exists in } \mathbb{Z}_p\right\} = \{x \in \mathbb{Z}_p \,|\, |x|_p = 1\} = \{x \in \mathbb{Q}_p \,|\, |x|_p = 1\}\,. \tag{B.1.7}$$

They correspond to those x in (B.1.1) for which $x_0 \neq 0$. Similarly, one defines the set of multiplicatively invertible elements in $\mathbb{Q}_p$ as $\mathbb{Q}_p^*$.

The case $p = \infty$ is typically associated with standard calculus via

$$\mathbb{Q}_\infty = \mathbb{R}. \tag{B.1.8}$$

$p < \infty$ is sometimes referred to as the non-archimedean place and $p = \infty$ as the archimedean place.

Appendix C

γ parameterisation

We require a parameterisation of $N^w_{\{\gamma\}}$. In other words we seek a construction of the set of roots $\{\gamma\}_w$, for given w. We employ a construction obtained in [108], which we will describe in the following.

C.1 Definition and identities

For this we fix a reduced expression $w = w_{i_1} w_{i_2} \cdots w_{i_\ell}$ for the Weyl word w of length ℓ. The subscripts refer to the nodes of the Dynkin diagram of G and w_i are the fundamental reflections that generate the Weyl group. Then one can explicitly enumerate all positive roots that are mapped to negative roots by the action of w as follows. Define

$$\gamma_k = w_{i_\ell} w_{i_{\ell-1}} \cdots w_{i_{k+1}} \alpha_{i_k} \tag{C.1.1}$$

where α_{i_k} is the i_kth simple root. We note in particular $\gamma_\ell = \alpha_{i_\ell}$. That this gives a valid description of the positive roots generating $N^w_{\{\gamma\}}$ can be checked easily by induction. Therefore we have

$$\{\gamma\}_w = \{\alpha > 0 \,|\, w\alpha < 0\} = \{\gamma_i \,:\, i = 1, \ldots, \ell(w)\}\,. \tag{C.1.2}$$

We also record that there is a simple expression for the sum of all these roots in terms of the Weyl vector ρ

$$\gamma_1 + \ldots + \gamma_\ell = \rho - w^{-1}\rho \tag{C.1.3}$$

which can again be checked by induction. Furthermore, we record that

$$w(\gamma_1 + ... + \gamma_n) = w'\rho - \rho\,, \tag{C.1.4}$$

where $w' = w_{i_1} w_{i_2} \cdots w_{i_n}$, with $n < \ell$.

It is important to emphasise that by the construction (C.1.1), the set inherits a canonical order of the roots, such that $\{\gamma\}_w = \{\gamma_1, \gamma_2, ..., \gamma_\ell\}$ denotes an ordered set.

Appendix D

Minimal parabolic expansions in $D \leq 2$

In this appendix, we give the minimal parabolic expansions of the E_9, E_{10} and E_{11} maximal parabolic Eisenstein series with $s = 3/2$ and $s = 5/2$. Note that in each case the Eisenstein series which we expand do not include the additional normalisation factors of $2\zeta(3)$ and $\zeta(5)$ shown in (3.5.2)–(3.5.4). In the expressions below, γ_{E} is the Euler-Mascheroni constant and A denotes the Glaisher-Kinkelin constant. We note that the 'number' of terms here does not need to strictly agree with table 5.3 since taking the limits to $s = 3/2$ and $s = 5/2$ in the factors $M(w, \lambda)$ can produce several terms out of a single Weyl word w. The first terms in all expressions is that of the identity Weyl word and corresponds to the string perturbation tree level term.

The variables r_i in the expressions below are defined by parameterising the Cartan subalgebra via a basis of simple roots. More precisely, we let the function $H(\cdot)$ of (3.2.9) be $H(a) = \sum_{i=1}^{r} r_i \alpha_i$, where r is the rank of the algebra (excluding the derivation in the E_9 case) and α_i are the simple roots. With this choice, the (minimal parabolic) constant term of the maximal parabolic Eisenstein series with weight $\lambda = 2s\Lambda_1 - \rho$ starts out with r_1^{2s}. r_1 is the string coupling, the other r_i are different combinations of the physical parameters.

D.1 E_9 Eisenstein series

The constant terms of the maximal parabolic Eisenstein series $E^{E_9}_{1;3/2}$ in the minimal parabolic read

$$r_1^3 + \frac{r_6^3}{r_7^2} + \frac{\pi^3 r_7^4}{45 r_8^3 \zeta(3)} + \frac{2\pi\gamma_{\mathrm{E}} r_4}{\zeta(3)} - \frac{2\pi r_4 \log(4\pi r_5)}{\zeta(3)} + \frac{4\pi r_4 \log(r_4)}{\zeta(3)} - \frac{2\pi r_4 \log(r_3)}{\zeta(3)}$$
$$+ \frac{2\pi r_4 \log(r_2)}{\zeta(3)} + \frac{\pi^2 r_5^2}{3 r_6} + \frac{\pi^2 r_3^2}{3 r_1} + \frac{\pi^2 r_2^2}{3} + \frac{4\pi^4 r_9^6}{945 v^5 \zeta(3)} + \frac{3 r_8^5 \zeta(5)}{2\pi r_9^4 \zeta(3)} \tag{D.1.1}$$

The constant terms of the maximal parabolic Eisenstein series $E^{E_9}_{1;5/2}$ in the minimal parabolic read

$$r_1^5 + \frac{\pi r_4^5}{15 r_5^4} + \frac{\pi r_8^5}{15 v^3} - \frac{4\zeta(3)\log(r_7) r_6 r_1^3}{\zeta(5)} + \frac{2\zeta(3)^2 r_4^3 r_1^3}{\pi r_3^2 r_2^2 \zeta(5)} + \frac{4\pi^7 r_7^8}{70875 r_8^7 \zeta(5)} + \frac{2\pi^3 r_7^2 r_5^2}{9 r_8 r_6 \zeta(5)}$$
$$+ \frac{8\pi^6 r_5^6}{42525 r_6^5 \zeta(5)} + \frac{2\pi^4 r_3^4}{135 r_1^3 \zeta(5)} + \frac{2\pi^3 r_7^2 r_3^2}{9 r_8 r_1 \zeta(5)} - \frac{4\pi^2 \log(r_7) r_6 r_3^2}{3 r_1 \zeta(5)} + \frac{2\pi^3 r_5^2 r_3^2}{9 r_4 r_1 \zeta(5)}$$
$$+ \frac{2\pi^3 r_7^2 r_2^2}{9 r_8 \zeta(5)} + \frac{2\pi^3 r_5^2 r_2^2}{9 r_4 \zeta(5)} + \frac{2\pi^5 r_3^4 r_2^4}{2025 r_4^3 \zeta(5)} + \frac{2\pi^4 r_2^4}{135 r_1 \zeta(5)} + \frac{32\pi^8 r_9^{10}}{1403325 v^9 \zeta(5)}$$
$$+ \frac{2\pi^5 r_7^4 r_9^4}{2025 r_8^3 v^3 \zeta(5)} + \frac{2\pi^4 r_5^2 r_9^4}{135 r_6 v^3 \zeta(5)} + \frac{2\pi^4 r_3^2 r_9^4}{135 r_1 v^3 \zeta(5)} + \frac{2\pi^4 r_2^2 r_9^4}{135 v^3 \zeta(5)} + \frac{2\pi r_6^3 \zeta(3)}{3 r_8 \zeta(5)}$$
$$+ \frac{2\pi r_4^3 \zeta(3)}{3 r_3^2 \zeta(5)} + \frac{2\pi r_7^2 r_1^3 \zeta(3)}{3 r_8 \zeta(5)} + \frac{2\pi r_5^2 r_1^3 \zeta(3)}{3 r_4 \zeta(5)} + \frac{2\pi^2 r_3^4 \zeta(3)}{45 r_2^2 \zeta(5)} + \frac{2\pi r_4^3 \zeta(3)}{3 r_1 r_2^2 \zeta(5)} + \frac{2\pi^2 r_1^3 r_2^4 \zeta(3)}{45 r_3^2 \zeta(5)}$$
$$+ \frac{2\pi^2 r_7^4 \zeta(3)}{45 r_9^2 \zeta(5)} + \frac{2\pi r_8^3 r_5^2 \zeta(3)}{3 r_6 r_9^2 \zeta(5)} + \frac{2\pi r_8^3 r_3^2 \zeta(3)}{3 r_1 r_9^2 \zeta(5)} + \frac{2\pi r_8^3 r_2^2 \zeta(3)}{3 r_9^2 \zeta(5)} + \frac{2\pi^2 r_6^3 r_9^4 \zeta(3)}{45 r_7^2 v^3 \zeta(5)}$$
$$+ \frac{2\pi^2 r_1^3 r_9^4 \zeta(3)}{45 v^3 \zeta(5)} + \frac{2 r_8^3 r_6^3 \zeta(3)^2}{\pi r_7^2 r_9^2 \zeta(5)} + \frac{2 r_8^3 r_1^3 \zeta(3)^2}{\pi r_9^2 \zeta(5)} + \frac{r_6^7 \zeta(7)}{6 r_7^6 \zeta(5)} + \frac{7 r_8^9 \zeta(9)}{12\pi r_9^8 \zeta(5)}$$
$$+ \frac{4\pi^2 r_6 r_3^2}{3 r_1 \zeta(5)} \Big(\gamma_{\mathrm{E}} + 2\log(r_6) - \log(4\pi r_5)\Big) + \frac{4 r_6 r_1^3 \zeta(3)}{\zeta(5)} \Big(\gamma_{\mathrm{E}} + 2\log(r_6) - \log(4\pi r_5)\Big)$$
$$+ \frac{4\pi^2 r_6 r_2^2}{3\zeta(5)} \Big(\gamma_{\mathrm{E}} - \log(4\pi r_7) + 2\log(r_6) - \log(r_5)\Big)$$
$$+ \frac{4\pi^3 r_4 r_9^4}{45 v^3 \zeta(5)} \Big(\gamma_{\mathrm{E}} - \log(4\pi r_5) + 2\log(r_4) - \log(r_3) - \log(r_2)\Big)$$
$$+ \frac{4\pi^2 r_7^2 r_4}{3 r_8 \zeta(5)} \Big(\gamma_{\mathrm{E}} - \log(4\pi r_5) + 2\log(r_4) - \log(r_3) - \log(r_2)\Big)$$

$$
\begin{aligned}
&+\frac{4\pi^2 r_5^2}{3\zeta(5)}\Big(2\gamma_{\rm E}-24\log(A)-\log(r_7)+2\log(r_5)-\log(r_3)-\log(r_2)\Big)\\
&+\frac{4r_8^3 r_4\zeta(3)}{r_9^2\zeta(5)}\Big(\gamma_{\rm E}-\log(4\pi r_5)+2\log(r_4)-\log(r_3)-\log(r_2)\Big)\\
&+\frac{8\pi r_6 r_4}{\zeta(5)}\Big(\gamma_{\rm E}^2-4\gamma_{\rm E}\log(2)+4\log(2)^2+\log(\pi)^2+2(-\gamma_{\rm E}+\log(4))\log(\pi r_5)\\
&\quad+\log(r_7)(-\gamma_{\rm E}+\log(4\pi r_5))+2(\gamma_{\rm E}-\log(4\pi r_7))\log(r_4)\\
&\quad+2\log(r_6)(\gamma_{\rm E}-\log(4\pi r_5)+2\log(r_4)-\log(r_3)-\log(r_2))\\
&\quad-(\gamma_{\rm E}-\log(4\pi r_7))(\log(r_3)+\log(r_2))+\log(r_5)(2\log(\pi)+\log(r_5)\\
&\quad-2\log(r_4)+\log(r_3)+\log(r_2))\Big) \qquad \text{(D.1.2)}
\end{aligned}
$$

D.2 E_{10} Eisenstein series

The constant terms of the maximal parabolic Eisenstein series $E^{E_{10}}_{1;3/2}$ in the minimal parabolic read

$$
\begin{aligned}
&r_1^3+\frac{r_6^3}{r_7^2}+\frac{4\pi^4 r_9^6}{945 r_{10}^5\zeta(3)}+\frac{\pi^3 r_7^4}{45 r_8^3\zeta(3)}+\frac{\pi^2 r_5^2}{3r_6\zeta(3)}+\frac{2\pi\gamma_{\rm E} r_4}{\zeta(3)}+\frac{\pi^2 r_3^2}{3r_1\zeta(3)}-\frac{2\pi r_4\log(4\pi r_5)}{\zeta(3)}\\
&+\frac{4\pi r_4\log(r_4)}{\zeta(3)}-\frac{2\pi r_4\log(r_3)}{\zeta(3)}-\frac{2\pi r_4\log(r_2)}{\zeta(3)}+\frac{\pi^2 r_2^2}{3\zeta(3)}+\frac{3r_8^5\zeta(5)}{2\pi r_9^4\zeta(3)}+\frac{15 r_{10}^7\zeta(7)}{4\pi^2\zeta(3)} \qquad \text{(D.2.1)}
\end{aligned}
$$

The constant terms of the maximal parabolic Eisenstein series $E^{E_{10}}_{1;5/2}$ in the minimal parabolic read

$$
\begin{aligned}
&r_1^5+\frac{4}{315}\pi^2 r_9^6+\frac{\pi r_8^5}{15 r_{10}^3}+\frac{\pi r_{10}^5 r_7^4}{15 r_8^3}+\frac{r_{10}^5 r_5^2}{r_6}+\frac{\pi r_4^5}{15 r_5^4}+\frac{r_{10}^5 r_3^2}{r_1}+r_{10}^5 r_2^2+\frac{3r_{10}^5 r_6^3\zeta(3)}{\pi^2 r_7^2}\\
&+\frac{3r_{10}^5 r_1^3\zeta(3)}{\pi^2}+\frac{2\pi^5 r_9^4 r_7^4}{2025 r_{10}^3 r_8^3\zeta(5)}+\frac{4\pi^7 r_7^8}{70875 r_8^7\zeta(5)}-\frac{4\zeta(3)\log(r_7) r_6 r_1^3}{\zeta(5)}+\frac{2\zeta(3)^2 r_4^3 r_1^3}{\pi r_3^2 r_2^2\zeta(5)}\\
&+\frac{32\pi^8 r_9^{10}}{1403325 r_{10}^9\zeta(5)}+\frac{2\pi^4 r_9^4 r_5^2}{135 r_{10}^3 r_6\zeta(5)}+\frac{2\pi^3 r_7^2 r_5^2}{9r_8 r_6\zeta(5)}+\frac{8\pi^6 r_5^6}{42525 r_6^5\zeta(5)}+\frac{2\pi r_6^3\zeta(3)}{3r_8\zeta(5)}\\
&+\frac{2\pi^4 r_3^4}{135 r_1^3\zeta(5)}+\frac{2\pi^4 r_9^4 r_3^2}{135 r_{10}^3 r_1\zeta(5)}+\frac{2\pi^3 r_7^2 r_3^2}{9r_8 r_1\zeta(5)}-\frac{4\pi^2\log(r_7) r_6 r_3^2}{3r_1\zeta(5)}+\frac{2\pi^3 r_5^2 r_3^2}{9r_4 r_1\zeta(5)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{2\pi^4 r_9^4 r_2^2}{135 r_{10}^3 \zeta(5)} + \frac{2\pi^2 r_9^4 r_1^3 \zeta(3)}{45 r_{10}^3 \zeta(5)} + \frac{2\pi r_7^2 r_1^3 \zeta(3)}{3 r_8 \zeta(5)} + \frac{2\pi^3 r_7^2 r_2^2}{9 r_8 \zeta(5)} + \frac{r_6^7 \zeta(7)}{6 r_7^6 \zeta(5)} + \frac{7 r_8^9 \zeta(9)}{12\pi r_9^8 \zeta(5)} \\
&+ \frac{21 r_{10}^{11} \zeta(11)}{8\pi^2 \zeta(5)} + \frac{2\pi^3 r_5^2 r_2^2}{9 r_4 \zeta(5)} + \frac{2\pi^5 r_3^4 r_2^4}{2025 r_4^3 \zeta(5)} + \frac{2\pi^4 r_2^4}{135 r_1 \zeta(5)} + \frac{2\pi^2 r_7^4 \zeta(3)}{45 r_9^2 \zeta(5)} + \frac{2\pi^2 r_9^4 r_6^3 \zeta(3)}{45 r_{10}^3 r_7^2 \zeta(5)} \\
&+ \frac{2\pi r_8^3 r_5^2 \zeta(3)}{3 r_9^2 r_6 \zeta(5)} + \frac{2\pi r_4^3 \zeta(3)}{3 r_3^2 \zeta(5)} + \frac{2\pi r_8^3 r_3^2 \zeta(3)}{3 r_9^2 r_1 \zeta(5)} + \frac{2\pi r_5^2 r_1^3 \zeta(3)}{3 r_4 \zeta(5)} + \frac{2\pi^2 r_3^4 \zeta(3)}{45 r_2^2 \zeta(5)} + \frac{2\pi r_4^3 \zeta(3)}{3 r_1 r_2^2 \zeta(5)} \\
&+ \frac{2\pi r_8^3 r_2^2 \zeta(3)}{3 r_9^2 \zeta(5)} + \frac{2\pi^2 r_1^3 r_2^4 \zeta(3)}{45 r_3^2 \zeta(5)} + \frac{2 r_8^3 r_6^3 \zeta(3)^2}{\pi r_9^2 r_7^2 \zeta(5)} + \frac{2 r_8^3 r_1^3 \zeta(3)^2}{\pi r_9^2 \zeta(5)} + \frac{9 r_{10}^5 r_8^5 \zeta(5)}{2\pi^3 r_9^4} \\
&+ \frac{4 r_6 r_1^3 \zeta(3)}{\zeta(5)} \Big(\gamma_{\mathrm{E}} + 2\log(r_6) - \log(4\pi r_5)\Big) + \frac{4\pi^2 r_6 r_3^2}{3 r_1 \zeta(5)} \Big(\gamma_{\mathrm{E}} + 2\log(r_6) - \log(4\pi r_5)\Big) \\
&+ \frac{4 r_8^3 r_4 \zeta(3)}{r_9^2 \zeta(5)} \Big(\gamma_{\mathrm{E}} - \log(4\pi r_5) + 2\log(r_4) - \log(r_3) - \log(r_2)\Big) \\
&+ \frac{4\pi^2 r_6 r_2^2}{3\zeta(5)} \Big(\gamma_{\mathrm{E}} - \log(4\pi r_7) + 2\log(r_6) - \log(r_5)\Big) \\
&+ \frac{4\pi^3 r_9^4 r_4}{45 r_{10}^3 \zeta(5)} \Big(\gamma_{\mathrm{E}} - \log(4\pi r_5) + 2\log(r_4) - \log(r_3) - \log(r_2)\Big) \\
&+ \frac{4\pi^2 r_7^2 r_4}{3 r_8 \zeta(5)} \Big(\gamma_{\mathrm{E}} - \log(4\pi r_5) + 2\log(r_4) - \log(r_3) - \log(r_2)\Big) \\
&+ \frac{6 r_{10}^5 r_4}{\pi} \Big(\gamma_{\mathrm{E}} - \log(4\pi r_5) + 2\log(r_4) - \log(r_3) - \log(r_2)\Big) \\
&+ \frac{4\pi^2 r_5^2}{3\zeta(5)} \Big(2\gamma_{\mathrm{E}} - 24\log(A) - \log(r_7) + 2\log(r_5) - \log(r_3) - \log(r_2)\Big) \\
&+ \frac{8\pi r_6 r_4}{\zeta(5)} \Big(\gamma_{\mathrm{E}}^2 - 4\gamma_{\mathrm{E}}\log(2) + 4\log(2)^2 + \log(\pi)^2 + 2(-\gamma_{\mathrm{E}} + \log(4))\log(\pi r_5) \\
&\quad + \log(r_7)(-\gamma_{\mathrm{E}} + \log(4\pi r_5)) + 2(\gamma_{\mathrm{E}} - \log(4\pi r_7))\log(r_4) \\
&\quad + 2\log(r_6)(\gamma_{\mathrm{E}} - \log(4\pi r_5) + 2\log(r_4) - \log(r_3) - \log(r_2)) \\
&\quad - (\gamma_{\mathrm{E}} - \log(4\pi r_7))(\log(r_3) + \log(r_2)) \\
&\quad + \log(r_5)(2\log(\pi) + \log(r_5) - 2\log(r_4) + \log(r_3) + \log(r_2))\Big) \qquad \text{(D.2.2)}
\end{aligned}
$$

D.3 E_{11} Eisenstein series

The constant terms of the maximal parabolic Eisenstein series $E^{E_{11}}_{1;3/2}$ in the minimal parabolic read

$$\begin{aligned}
&r_1^3+\frac{r_6^3}{r_7^2}+\frac{2\pi^5r_{11}^8}{1575\zeta(3)}+\frac{4\pi^4r_9^6}{945\zeta(3)r_{10}^5}+\frac{\pi^3r_7^4}{45\zeta(3)r_8^3}+\frac{\pi^2r_5^2}{3\zeta(3)r_6}+\frac{2\gamma_{\rm E}\pi r_4}{\zeta(3)}+\frac{\pi^2r_3^2}{3\zeta(3)r_1}\\
&+\frac{\pi^2r_2^2}{3\zeta(3)}+\frac{3r_8^5\zeta(5)}{2\pi r_9^4\zeta(3)}+\frac{15r_{10}^7\zeta(7)}{4\pi^2r_{11}^6\zeta(3)}-\frac{6\pi r_4\log(4\pi r_5)}{3\zeta(3)}+\frac{12\pi r_4\log(r_4)}{3\zeta(3)}\\
&-\frac{6\pi r_4\log(r_3)}{3\zeta(3)}-\frac{6\pi r_4\log(r_2)}{3\zeta(3)}
\end{aligned}\tag{D.3.1}$$

The constant terms of the maximal parabolic Eisenstein series $E^{E_{11}}_{1;5/2}$ in the minimal parabolic read

$$\begin{aligned}
&r_1^5+\frac{4\pi^2r_9^6}{315r_{11}^4}+\frac{\pi r_8^5}{15r_{10}^3}+\frac{4\pi^2r_{11}^6r_8^5}{315r_9^4}+\frac{\pi r_{10}^5r_7^4}{15r_{11}^4r_8^3}+\frac{r_{10}^5r_5^2}{r_{11}^4r_6}+\frac{\pi r_4^5}{15r_5^4}+\frac{r_{10}^5r_3^2}{r_{11}^4r_1}+\frac{r_{10}^5r_2^2}{r_{11}^4}\\
&+\frac{3r_{10}^5r_6^3\zeta(3)}{\pi^2r_{11}^4r_7^2}+\frac{3r_{10}^5r_1^3\zeta(3)}{\pi^2r_{11}^4}-\frac{4\zeta(3)\log(r_7)r_6r_1^3}{\zeta(5)}+\frac{2\zeta(3)^2r_4^3r_1^3}{\pi\zeta(5)r_3^2r_2^2}\\
&-\frac{4\pi^2\log(r_7)r_6r_3^2}{3r_1\zeta(5)}+\frac{22112\pi^9r_{11}^{12}}{1915538625\zeta(5)}+\frac{32\pi^7r_{11}^6r_9^6}{893025r_{10}^5\zeta(5)}+\frac{32\pi^8r_9^{10}}{1403325r_{10}^9\zeta(5)}\\
&+\frac{8\pi^6r_{11}^6r_7^4}{42525r_8^3\zeta(5)}+\frac{2\pi^5r_9^4r_7^4}{2025r_{10}^3r_8^3\zeta(5)}+\frac{4\pi^7r_7^8}{70875r_8^7\zeta(5)}+\frac{2\pi r_7^2r_1^3\zeta(3)}{3r_8\zeta(5)}+\frac{8\pi^5r_{11}^6r_5^2}{2835r_6\zeta(5)}\\
&+\frac{2\pi^4r_9^4r_5^2}{135r_{10}^3r_6\zeta(5)}+\frac{2\pi^3r_7^2r_5^2}{9r_8r_6\zeta(5)}+\frac{8\pi^6r_5^6}{42525r_6^5\zeta(5)}+\frac{2\pi^4r_3^4}{135r_1^3\zeta(5)}+\frac{8\pi^5r_{11}^6r_3^2}{2835r_1\zeta(5)}\\
&+\frac{2\pi^4r_9^4r_3^2}{135r_{10}^3r_1\zeta(5)}+\frac{2\pi^3r_7^2r_3^2}{9r_8r_1\zeta(5)}+\frac{2\pi^3r_5^2r_3^2}{9r_4r_1\zeta(5)}+\frac{8\pi^5r_{11}^6r_2^2}{2835\zeta(5)}+\frac{2\pi^4r_9^4r_2^2}{135r_{10}^3\zeta(5)}+\frac{2\pi^3r_7^2r_2^2}{9r_8\zeta(5)}\\
&+\frac{2\pi^3r_5^2r_2^2}{9r_4\zeta(5)}+\frac{2\pi^5r_3^4r_2^4}{2025r_4^3\zeta(5)}+\frac{2\pi^4r_2^4}{135r_1\zeta(5)}+\frac{2\pi^2r_7^4\zeta(3)}{45r_9^2\zeta(5)}+\frac{2\pi r_6^3\zeta(3)}{3r_8\zeta(5)}+\frac{8\pi^3r_{11}^6r_6^3\zeta(3)}{945r_7^2\zeta(5)}\\
&+\frac{2\pi^2r_9^4r_6^3\zeta(3)}{45r_{10}^3r_7^2\zeta(5)}+\frac{2\pi r_8^3r_5^2\zeta(3)}{3r_9^2r_6\zeta(5)}+\frac{2\pi r_4^3\zeta(3)}{3r_3^2\zeta(5)}+\frac{2\pi r_8^3r_3^2\zeta(3)}{3r_9^2r_1\zeta(5)}+\frac{8\pi^3r_{11}^6r_1^3\zeta(3)}{945\zeta(5)}\\
&+\frac{2\pi^2r_9^4r_1^3\zeta(3)}{45r_{10}^3\zeta(5)}+\frac{2\pi r_5^2r_1^3\zeta(3)}{3r_4\zeta(5)}+\frac{2\pi^2r_3^4\zeta(3)}{45r_2^2\zeta(5)}+\frac{2\pi r_4^3\zeta(3)}{3r_1r_2^2\zeta(5)}+\frac{2\pi r_8^3r_2^2\zeta(3)}{3r_9^2\zeta(5)}\\
&+\frac{2\pi^2r_1^3r_2^4\zeta(3)}{45r_3^2\zeta(5)}+\frac{2r_8^3r_6^3\zeta(3)^2}{\pi r_9^2r_7^2\zeta(5)}+\frac{2r_8^3r_1^3\zeta(3)^2}{\pi r_9^2\zeta(5)}+\frac{9r_{10}^5r_8^5\zeta(5)}{2\pi^3r_{11}^4r_9^4}+\frac{2\pi r_{10}^7\zeta(7)}{63\zeta(5)}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{r_6^7 \zeta(7)}{6 r_7^6 \zeta(5)} + \frac{4 r_8^3 r_4 \zeta(3)}{r_9^2 \zeta(5)} \Big(\gamma_{\mathrm{E}} - \log(4\pi r_5) + 2\log(r_4) - \log(r_3) - \log(r_2) \Big) \\
&+ \frac{7 r_8^9 \zeta(9)}{12\pi r_9^8 \zeta(5)} + \frac{4\pi^2 r_6 r_2^2}{3\zeta(5)} \Big(\gamma_{\mathrm{E}} - \log(4\pi r_7) + 2\log(r_6) - \log(r_5) \Big) \\
&+ \left(\frac{4 r_6 r_1^3 \zeta(3)}{\zeta(5)} + \frac{4\pi^2 r_6 r_3^2}{3 r_1 \zeta(5)} \right) \Big(\gamma_{\mathrm{E}} + 2\log(r_6) - \log(4\pi r_5) \Big) \\
&+ \frac{4\pi^2 r_5^2}{3\zeta(5)} \Big(2\gamma_{\mathrm{E}} - 24\log(A) - \log(r_7) + 2\log(r_5) - \log(r_3) - \log(r_2) \Big) \\
&+ \frac{21 r_{10}^{11} \zeta(11)}{8\pi^2 r_{11}^{10} \zeta(5)} + \frac{6 r_{10}^5 r_4}{\pi r_{11}^4} \Big(\gamma_{\mathrm{E}} - \log(4\pi r_5) + 2\log(r_4) - \log(r_3) - \log(r_2) \Big) \\
&+ \frac{16\pi^4 r_{11}^6 r_4}{945\zeta(5)} \Big(\gamma_{\mathrm{E}} - \log(4\pi r_5) + 2\log(r_4) - \log(r_3) - \log(r_2) \Big) \\
&+ \frac{4\pi^3 r_9^4 r_4}{45 r_{10}^3 \zeta(5)} \Big(\gamma_{\mathrm{E}} - \log(4\pi r_5) + 2\log(r_4) - \log(r_3) - \log(r_2) \Big) \\
&+ \frac{4\pi^2 r_7^2 r_4}{3 r_8 \zeta(5)} \Big(\gamma_{\mathrm{E}} - \log(4\pi r_5) + 2\log(r_4) - \log(r_3) - \log(r_2) \Big) \\
&+ \frac{8\pi r_6 r_4}{\zeta(5)} \Big(\log(r_7)(-\gamma_{\mathrm{E}} + \log(4\pi r_5)) + 4\log(2)^2 + 2(-\gamma_{\mathrm{E}} + \log(4))\log(\pi r_5) \\
&\quad \gamma_{\mathrm{E}}^2 - 4\gamma_{\mathrm{E}}\log(2) + 2\log(r_6)(\gamma_{\mathrm{E}} - \log(4\pi r_5) + 2\log(r_4) - \log(r_3) - \log(r_2)) \\
&\quad + 2(\gamma_{\mathrm{E}} - \log(4\pi r_7))\log(r_4) - (\gamma_{\mathrm{E}} - \log(4\pi r_7))(\log(r_3) + \log(r_2)) \\
&\quad + \log(\pi)^2 + \log(r_5)(2\log(\pi) + \log(r_5) - 2\log(r_4) + \log(r_3) + \log(r_2)) \Big)
\end{aligned}
\tag{D.3.2}
$$

Appendix E

Maximal parabolic expansions in $D \leq 2$

In this appendix we state the explicit expressions for the constant terms in the various maximal parabolic expansions of maximal parabolic Eisenstein series invariant under $E_9(\mathbb{Z})$, $E_{10}(\mathbb{Z})$ and $E_{11}(\mathbb{Z})$. In the course of this investigation, we determine in particular the precise numerical coefficients in the various degeneration limits. The results of this section were obtained by implementing the algorithms described in section 5.4.2 on a standard computer. We use the shorthand (5.4.4) throughout.

When writing down the expressions below one finds that for some terms it is important to consider which particular Weyl word is used to represent an element of the double coset $\mathcal{W}_{j_\circ}\backslash\mathcal{W}/\mathcal{W}_{i_*}$, appearing in the sum on the right-hand side of (5.4.3) or (5.4.12). Although the sum (5.4.3) is independent of the choice of representative, some Weyl words used as coset representatives can yield coefficients $M(w,\lambda)$ that appear to be infinite. In this case, the corresponding Eisenstein series goes to zero so that the product is finite. This choice of having different possible coset representatives also manifests itself in the functional relation (5.2.2). We have verified that our choice of representative gives finite Eisenstein series contributions.

E.1 E_9 Eisenstein series

All maximal parabolic expansions of the E_9 Eisenstein series (3.5.2) will necessarily have two expansion parameters, namely r coming from the choice of the maximal parabolic and v that enters the definition (3.4.3). The additional factor of v in (3.5.2) is crucial here for obtaining the right result in all cases.

E.1.1 Decompactification limit:

$$\int_{d+1} \mathcal{E}^2_{(0,0)} = r^6 v \mathcal{E}^3_{(0,0)} + \frac{4\zeta(6)}{3\zeta(2)} r^6 v^6 \,, \tag{E.1.1}$$

$$\int_{d+1} \mathcal{E}^2_{(1,0)} = r^{10} v \mathcal{E}^3_{(1,0)} + \frac{2}{15}\zeta(2) r^{10} v^4 \mathcal{E}^3_{(0,0)} + \frac{16\zeta(10)}{45\zeta(2)} r^{10} v^{10} \,. \tag{E.1.2}$$

These agree perfectly with (5.4.16) when the expansion parameters are identified as $r = \lambda$ and $v = \rho$. The final terms are consistent with the expected behaviour [72, 73].

E.1.2 Perturbation limit:

$$\int_1 \mathcal{E}^2_{(0,0)} = 2\zeta(3) v r^3 + \frac{16}{21}\zeta(4) r^3 E^{SO(8,8)}_{9;3} \,, \tag{E.1.3}$$

$$\int_1 \mathcal{E}^2_{(1,0)} = \zeta(5) v r^5 + \frac{64}{297}\zeta(8) r^5 E^{SO(8,8)}_{9;5} + \frac{7\zeta(6)}{3\zeta(2)} r^5 v^{-1} E^{SO(8,8)}_{3;2} \,. \tag{E.1.4}$$

These are consistent with (5.4.17) when the expansion parameters are identified as $r = (\ell_s/\ell_2)^2$ and $v = 1/y_2$.

E.1.3 Semi-classical M-Theory limit:

$$\int_2 \mathcal{E}^2_{(0,0)} = 4\zeta(2) r^2 v + 2\zeta(3) r^2 v^{2/3} E^{SL(9)}_{1;\frac{3}{2}} \,, \tag{E.1.5}$$

$$\int_2 \mathcal{E}^2_{(1,0)} = \zeta(5) r^{10/3} v^{4/9} E^{SL(9)}_{1;\frac{5}{2}} + \frac{4}{15}\zeta(3)\zeta(2) r^{10/3} v^{1/9} E^{SL(9)}_{3;2} + \frac{7\zeta(6)}{3\zeta(2)} r^{10/3} v^{10/9} E^{SL(9)}_{1;-\frac{1}{2}} . \tag{E.1.6}$$

These are perfectly consistent with (5.4.18) when the expansion parameters are identified as $r = (\ell_{11}/\ell_2)^3$ and $v = \mathcal{V}_9/\ell_{11}^9$.

E.2 E_{10} Eisenstein series

We now turn to the expansion of the E_{10} Eisenstein series (3.5.3) in the three limits of section 5.4.4.

E.2.1 (Double) decompactification limit:

Mathematically, there is no difficulty with performing the expansion of the E_{10} Eisenstein series in its E_9 parabolic. We give the results thus obtained as well as those of an expansion in its E_8 parabolic, corresponding to a double decompactification. The first E_{10} Eisenstein series (3.5.3) satisfies

$$\begin{aligned}\int_{10} \mathcal{E}^1_{(0,0)} &= v^{-1}\mathcal{E}^2_{(0,0)} + \frac{5\zeta(7)}{4\zeta(2)} v^{-7} ,\\ \int_{10,9} \mathcal{E}^1_{(0,0)} &= a^6 \mathcal{E}^3_{(0,0)} + \frac{4\zeta(6)}{3\zeta(2)} a^6 v^5 + \frac{5\zeta(7)}{4\zeta(2)} v^{-7} ,\end{aligned} \tag{E.2.1}$$

where a is the second parameter that arises in the double expansion. We see that this behaviour is consistent with (5.4.20) for $D = 1$ when the expansion parameters are identified as $a = v_2$ and $v = 1/r$. We also note that the single decompactification is consistent with a naive application of (5.4.5) to $D = 1$ when ignoring the pre-factor. Performing the same analysis for the $\partial^4 R^4$ series in (3.5.3) one obtains

$$\begin{aligned}\int_{10} \mathcal{E}^1_{(1,0)} &= v^{-1}\mathcal{E}^2_{(1,0)} + \frac{\zeta(5)}{4\zeta(2)} v^{-6} \mathcal{E}^2_{(0,0)} + \frac{7\zeta(11)}{16\zeta(2)} v^{-11} ,\\ \int_{10,9} \mathcal{E}^1_{(1,0)} &= a^{10}\left(\mathcal{E}^3_{(1,0)} + \frac{2\zeta(2)}{15} v^3 \mathcal{E}^3_{(0,0)} + \frac{16\zeta(10)}{45\zeta(2)} v^9\right)\end{aligned}$$

$$+ a^6 v^{-5} \left(\frac{\zeta(5)}{4\zeta(2)} \mathcal{E}^3_{(0,0)} + \frac{\zeta(5)\zeta(6)}{3\zeta(2)\zeta(2)} v^5 \right) + \frac{7\zeta(11)}{16\zeta(2)} v^{-11} \,. \tag{E.2.2}$$

This is again in agreement with (5.4.20) with the same identifications as above. However, now there is a difference that is related to the single decompactification limit: The term involving the two-dimensional R^4 contribution $\mathcal{E}^2_{(0,0)}$ does not appear with the right power of v to be consistent with (5.4.6) without the prefactor. More precisely, the v pre-factors from (5.4.6) should be v^{-1}, v^{-5} and v^{-11} rather than v^{-1}, v^{-6} and v^{-11}. This cannot be compensated by the additional factor of v appearing in (3.5.2) since it affects both the first two terms. It would be interesting to investigate whether this means that this particular threshold contribution in $D = 2$ behaves differently from higher dimensions. We also note that the final terms are consistent with the expected behaviour [72, 73].

The double decompactification in the second lines of (E.2.1) and (E.2.2) is naturally also consistent (mathematically) with applying the E_9 decompactification of (E.1.1) to the first lines.

E.2.2 Perturbation limit:

$$\int_1 \mathcal{E}^1_{(0,0)} = 2\zeta(3) r^3 + \frac{5\zeta(7)}{4\zeta(2)} r^{7/2} E^{SO(9,9)}_{10;\frac{7}{2}} \,, \tag{E.2.3}$$

$$\int_1 \mathcal{E}^1_{(1,0)} = \zeta(5) r^5 + \frac{7\zeta(11)}{16\zeta(2)} r^{11/2} E^{SO(9,9)}_{10;\frac{11}{2}} + \frac{7\zeta(6)}{3\zeta(2)} r^6 E^{SO(9,9)}_{3;2} \,. \tag{E.2.4}$$

This is consistent with (5.4.23) for $r = y_1^2$.

E.2.3 Semi-classical M-Theory limit:

$$\int_2 \mathcal{E}^1_{(0,0)} = 4\zeta(2) r^2 + 2\zeta(3) r^{21/10} E^{SL(10)}_{1;\frac{3}{2}} \,, \tag{E.2.5}$$

$$\int_2 \mathcal{E}^1_{(1,0)} = \frac{7\zeta(6)}{3\zeta(2)} r^{33/10} E^{SL(9)}_{1;-\frac{1}{2}} + \zeta(5) r^{7/2} E^{SL(10)}_{1;\frac{5}{2}} + \frac{4}{15} \zeta(2)\zeta(3) r^{18/5} E^{SL(10)}_{3;2} \,. \tag{E.2.6}$$

Looking at (5.4.25) we find perfect agreement for $r = (\mathcal{V}_{10}/\ell_1^{10})^{1/3}$.

E.3 E_{11} Eisenstein Series

In this appendix, we give for completeness the maximal parabolic expansions of the E_{11} Eisenstein series (3.5.4) using the shorthand (5.4.4).

E.3.1 Decompactification limit

The decompactification limits corresponding to the Levi factor $GL(1) \times E_{10}$ for the $s = 3/2$ and $s = 5/2$ series are

$$\begin{aligned}\int_{11} \mathcal{E}^0_{(0,0)} &= r^{-6}\mathcal{E}^1_{(0,0)} + \frac{12\zeta(6)}{5\pi} r^8\,,\\ \int_{11} \mathcal{E}^0_{(1,0)} &= r^{-10}\mathcal{E}^1_{(1,0)} + \frac{2\zeta(6)}{3\pi\zeta(2)}\mathcal{E}^1_{(0,0)} + \frac{16\zeta(12)}{9\pi\zeta(2)} r^{12}\,.\end{aligned} \tag{E.3.1}$$

The powers of r and the structure of the resulting Eisenstein series are in agreement with (5.4.7) applied naively to $D = 0$ when one replaces the '0-dimensional Planck length' ℓ_0 by the radius of the first direction and ℓ_1 according to the standard Kaluza–Klein rules. The final terms are consistent with the expected behaviour [72, 73].

E.3.2 Perturbative limit

$$\begin{aligned}\int_{1} \mathcal{E}^0_{(0,0)} &= 2\zeta(3) r^3 + \frac{12\zeta(6)}{5\pi} r^4 E^{SO(10,10)}_{11;4}\,,\\ \int_{1} \mathcal{E}^0_{(1,0)} &= \zeta(5) r^5 + \frac{16\zeta(12)}{9\pi\zeta(2)} r^6 E^{SO(10,10)}_{11;6} + \frac{4\zeta(4)}{3} r^7 E^{SO(10,10)}_{3;2}\,.\end{aligned} \tag{E.3.2}$$

The powers of r and the structure of the $SO(10,10)$ Eisenstein series are in agreement with the naive application of (5.4.8).

E.3.3 Semi-classical M-theory limit

$$
\begin{aligned}
\int_2 \mathcal{E}^0_{(0,0)} &= 4\zeta(2) r^2 + r^{24/11} E^{SL(11)}_{1;3/2} \,, \\
\int_2 \mathcal{E}^0_{(1,0)} &= \frac{4\zeta(4)}{3} r^{36/11} E^{SL(11)}_{1;1/2} + \zeta(5) r^{40/11} E^{SL(11)}_{1;5/2} + \frac{4\zeta(2)}{15} \zeta(3) r^{42/11} E^{SO(10,10)}_{3;2} \,.
\end{aligned} \tag{E.3.3}
$$

The powers of r and the structure of the $SL(11)$ Eisenstein series are in agreement with the naive application of (5.4.10) and (5.4.11).

E.3.4 Four-dimensional limit

As a final application, we consider the Levi decomposition of E_{11} with Levi factor $SL(4)\times GL(1)\times E_7$ as appropriate for an interpretation in $D = 4$. This corresponds to removing node 8 from the Dynkin diagram. Expanding the constant terms of the Eisenstein series (3.5.4) under the associated maximal parabolic one obtains

$$
\begin{aligned}
\int_8 \mathcal{E}^0_{(0,0)} &= r^3 \mathcal{E}^4_{(0,0)} + \frac{3\zeta(5)}{\pi} r^2 E^{SL(4)}_{9;-2} \,, \\
\int_8 \mathcal{E}^0_{(1,0)} &= r^5 \mathcal{E}^4_{(1,0)} + \frac{\zeta(3)}{\pi} r^{9/2} E^{SL(4)}_{9;-1} \mathcal{E}^4_{(0,0)} + \frac{\pi\zeta(5)}{15} r^{7/2} E^{SL(4)}_{10;-3/2} \\
&\quad + \frac{7\zeta(9)}{12\pi} r^3 E^{SL(4)}_{9;-2} \,.
\end{aligned} \tag{E.3.4}
$$

Here, $r = (\mathrm{vol}(T^4)\ell_0^8/\ell_4^4)^{1/3}$ parameterises the $GL(1)$ factor in the Levi part as usual and the (maximal) Eisenstein series on the right-hand side belong to $SL(4)\times E_7$ and we have factorized them. Note again that our (non-standard) labelling for E_n subgroups is obtained from diagram 2.3 by removing nodes. Here, this means that $SL(4)$ inherits the three nodes labelled 11, 10 and 9 while E_7 has nodes 1 up to 7. The leading terms are the pure E_7 Eisenstein series as they appear in $D = 4$ and we have used the relation 3.5.1.

Let us point out that the work [127] also studies parameters related to 'middle' nodes of the E_n diagram (like our r here) and deduces the first terms in our two expansions (E.3.4).

The constant terms of the $SL(4)$ Eisenstein series can now be analysed in their minimal parabolic, leaving only dependence on four dilatonic scalars (including r) and E_7 Eisenstein series. Then one sees more clearly the expected feature that the E_{11} series knows about the relevant series in $D = 4$ but also about threshold contributions. As always with derivative corrections the term with the highest number of derivatives (here $\partial^4 R^4$) in D dimensions induces the terms with up to that number of derivatives in higher space-time dimensions. In this sense, going to higher rank E_n groups combines the information of derivative corrections of different orders in single objects.

Appendix F

Degenerate Whittaker vectors in $D > 2$

In this appendix, we apply the formula (6.1.10) to some finite-dimensional cases of physical interest. These are associated with the groups $E_{6(6)}$, $E_{7(7)}$ and $E_{8(8)}$ and the particular choices of character χ that arises in string theory. The same convention outlined in section 6.2.1 for labelling the Dynkin diagram, defining the Bessel functions, etc. ..., also apply in this appendix. We give the results for $s = 3/2$ only and denote the associated character as $\chi_{3/2}$, but of course the method is applicable to any value of s. The resulting expressions just get longer. All examples are for a charge vector, with a single non-trivial charge.

F.1 E_6 with $s = 3/2$

Whittaker vectors of the $E^{E_6}_{1;3/2}$ Eisenstein series, associated with one non-trivial charge.

ψ	$W_\psi(\chi_{3/2}, a)$
$(m,0,0,0,0,0)$	$v_3^2 v_1^{-1} B_{3/2,m}\left(v_1^2 v_3^{-1}\right)$
$(0,m,0,0,0,0)$	$\frac{v_2^2 \tilde{B}_{0,m}(v_2^2 v_4^{-1})}{\xi(3)}$
$(0,0,m,0,0,0)$	$\frac{\xi(2) v_4 B_{1,m}\left(v_3^2 v_1^{-1} v_4^{-1}\right)}{\xi(3)}$
$(0,0,0,m,0,0)$	$\frac{v_4 \tilde{B}_{1/2,m}(v_4^2 v_2^{-1} v_3^{-1} v_5^{-1})}{\xi(3)}$
$(0,0,0,0,m,0)$	$\frac{v_5^2 \tilde{B}_{0,m}(v_5^2 v_4^{-1} v_6^{-1})}{\xi(3) v_6}$
$(0,0,0,0,0,m)$	$\frac{\xi(2) v_6^3 B_{-1/2,m}\left(v_6^2 v_5^{-1}\right)}{\xi(3)}$

F.2 E_7 with $s = 3/2$

Whittaker vectors of the $E^{E_7}_{1;3/2}$ Eisenstein series, associated with one non-trivial charge.

ψ	$W_\psi(\chi_{3/2}, a)$
$(m,0,0,0,0,0,0)$	$v_3^2 v_1^{-1} B_{\frac{3}{2},m}\left(v_1^2 v_3^{-1}\right)$
$(0,m,0,0,0,0,0)$	$\frac{v_2^2 \tilde{B}_{0,m}(v_2^2 v_4^{-1})}{\xi(3)}$
$(0,0,m,0,0,0,0)$	$\frac{\xi(2) v_4 B_{1,m}\left(v_3^2 v_1^{-1} v_4^{-1}\right)}{\xi(3)}$
$(0,0,0,m,0,0,0)$	$\frac{v_4 \tilde{B}_{1/2,m}(v_4^2 v_2^{-1} v_3^{-1} v_5^{-1})}{\xi(3)}$
$(0,0,0,0,m,0,0)$	$\frac{v_5^2 \tilde{B}_{0,m}(v_5^2 v_4^{-1} v_6^{-1})}{\xi(3) v_6}$
$(0,0,0,0,0,m,0)$	$\frac{\xi(2) v_6^3 v_7^{-2} B_{-1/2,m}\left(v_6^2 v_5^{-1} v_7^{-1}\right)}{\xi(3)}$
$(0,0,0,0,0,0,m)$	$v_7^4 B_{-1,m}\left(v_7^2 v_6^{-1}\right)$

F.3 E_8 with $s = 3/2$

Whittaker vectors of the $E^{E_8}_{1;3/2}$ Eisenstein series, associated with one non-trivial charge.

ψ	$W_\psi(\chi_{3/2}, a)$
$(m,0,0,0,0,0,0,0)$	$v_3^2 v_1^{-1} B_{3/2,m}\left(v_1^2 v_3^{-1}\right)$
$(0,m,0,0,0,0,0,0)$	$\frac{v_2^2 \tilde{B}_{0,m}(v_2^2 v_4^{-1})}{\xi(3)}$
$(0,0,m,0,0,0,0,0)$	$\frac{\xi(2) v_4 B_{1,m}\left(v_3^2 v_1^{-1} v_4^{-1}\right)}{\xi(3)}$
$(0,0,0,m,0,0,0,0)$	$\frac{v_4 \tilde{B}_{1/2,m}(v_4^2 v_2^{-1} v_3^{-1} v_5^{-1})}{\xi(3)}$
$(0,0,0,0,m,0,0,0)$	$\frac{v_5^2 \tilde{B}_{0,m}(v_5^2 v_4^{-1} v_6^{-1})}{\xi(3) v_6}$
$(0,0,0,0,0,m,0,0)$	$\frac{\xi(2) v_6^3 v_7^{-2} B_{-1/2,m}\left(v_6^2 v_5^{-1} v_7^{-1}\right)}{\xi(3)}$
$(0,0,0,0,0,0,m,0)$	$\frac{v_7^4 B_{-1,m}\left(v_7^2 v_6^{-1} v_8^{-1}\right)}{r_8^3}$
$(0,0,0,0,0,0,0,m)$	$\frac{\xi(4) v_8^5 B_{-3/2,m}\left(v_8^2 v_7^{-1}\right)}{\xi(3)}$

List of publications

P. Fleig, H. A. P. Gustafsson, A. Kleinschmidt, D. Persson, *A physicists' invitation to: Adelic Eisenstein series and automorphic representations*, in preparation.

P. Fleig, A. Kleinschmidt, and D. Persson, *Fourier expansions of Kac–Moody Eisenstein series and degenerate Whittaker vectors*, ArXiv e-prints (Dec., 2013), `arXiv:1312.3643 [hep-th]`.

P. Fleig and H. Nicolai, *Hidden Symmetries: from BKL to Kac-Moody*, to appear in the proceedings of the 'Marcel Grossmann 13' meeting.

P. Fleig and A. Kleinschmidt, *Perturbative terms of Kac-Moody–Eisenstein series*, submitted to the proceedings of the 'String–Math 2012' conference, ArXiv e-prints (Nov., 2012), `arXiv:1211.5296 [hep-th]`.

P. Fleig and A. Kleinschmidt, *Eisenstein series for infinite-dimensional U-duality groups*, *Journal of High Energy Physics* **6** (June, 2012) 54, `arXiv:1204.3043 [hep-th]`.

P. Fleig, M. Koehn, and H. Nicolai, *On Fundamental Domains and Volumes of Hyperbolic Coxeter-Weyl Groups*, Letters in Mathematical Physics **100** (June, 2012) 261, `arXiv:1103.3175 [math.RT]`.

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