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The variation of the monodromy group in families of stratified bundles in positive characteristic

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In this thesis we study smooth families of stratified bundles in positive characteristic and the variation of their monodromy group. Our aim is, in particular, to strengthen the weak form of the positive equicharacteristic p-curvature conjecture stated and proved by Esnault and Langer in [EL13]. The main result is that if the ground field is uncountable then the strong form holds, in parallel to what happens in characteristic zero. In the case where the ground field is countable we provide a counterexample that shows that the strong form cannot hold in general and prove the weak form of the theorem for non-proper morphism assuming the stratified bundle to be regular singular.

In dieser Doktorarbeit studieren wir glatte Familien stratifizierter Bündel in positiver Charakteristik und das Verhalten der Monodromiegruppen in einer solchen Familie. Insbesondere ist es unser Ziel die schwache Form der p-Krümmungsvermutung in positiver Äquicharakteristik zu verallgemeinern, die von Esnault und Langer in [EL13] formuliert und bewiesen wurde. Unser Hauptresultat ist, dass über überabzählbaren Grundkörpern eine stärkere Version der Vermutung richtig ist, ähnlich der Situation in Charakteristik 0. Über abzählbaren Grundkörpern ist die stärkere Version der Vermutung nicht wahr; wir konstruieren ein Gegenbeispiel. Wir beweisen die schwache Version der Vermutung für nicht-eigentliche Morphismen und regulär singuläre stratifizierte Bündel.

Contents

Introduction

Let (E, ∇) be a vector bundle endowed with a flat connection on a smooth complex variety X. Then there exists a smooth scheme S over $\operatorname{Spec} \mathbb{Z}$ such that $(E, \nabla) = (E_S, \nabla_S) \otimes_S \mathbb{C}$ and $X = X_S \otimes_S \mathbb{C}$ with X_S smooth over S and (E_S, ∇_S) flat connection on X_S relative to S. The p-curvature conjecture of Grothendieck and Katz (see [And04, Conj. 3.3.3]) predicts that if for all closed points s of a dense open sub-scheme $\tilde{S} \subset S$ we have that $E_S \times_S s$ is spanned by its horizontal sections, then (E, ∇) must be trivialized by an étale finite cover of X.

An analogue problem can be studied in equicharacteristic zero, and in fact it reduces the p-curvature conjecture to the number field case. Y. André in [And04, Prop. 7.1.1] and E. Hrushovsky in [Hru02, p. 116] stated and proved the following equicharacteristic zero version of the p-curvature conjecture: let $X \to S$ be a smooth morphism of varieties over a field F of characteristic zero; let (E, ∇) be a flat connection on X relative to S such that, for every closed point s in a dense open $\tilde{S} \subset S$, the flat connection $(E, \nabla) \times_S s$ is trivialized by a finite étale cover. Then, there exists a finite étale cover of the generic geometric fiber over $\bar{\eta}$ that trivializes $(E, \nabla) \times_S \bar{\eta}$, where $\bar{\eta}$ is a geometric generic point of S.

The theorem of André and Hrushovsky translates naturally in positive characteristic, providing a positive equicharacteristic analogue to the p-curvature conjecture. Here, the role of relative flat connections is played by relative stratified bundles. A *stratified bundle on X relative to S* is a vector bundle of finite rank with an action of the ring of differential operators $\mathscr{D}_{X/S}$ on X relative to S.

In [EL13, Cor. 4.3, Rmk. 5.4.1] H. Esnault and A. Langer proved, using an example of Y. Laszlo (see [Las01]), that there exists a projective smooth morphism of $\bar{\mathbb{F}}_2$-varieties $X \to S$ and a stratified bundle over X relative to S which is trivialized by a finite étale cover on each closed fiber but not on the geometric generic one. In particular, this provides a counterexample to the positive equicharacteristic version of André and Hrushovsky's theorem.

Nevertheless, they were able to prove what they call a weak form of the theorem (see [EL13, Thm. 7.2]): let $X \to S$ be a *projective* smooth morphism and let $\mathbb{E} = (E, \nabla)$ be a stratified bundle on X relative to S such that, for all closed points of a dense subset $\tilde{S} \subset S$, the stratified bundle $\mathbb{E} \times_S s$ is trivialized by a finite étale cover *of order prime to p*. Then, if $F \neq \bar{\mathbb{F}}_p$, there exists a finite étale cover *of order prime to p* of the generic geometric fiber that trivializes $\mathbb{E} \times_S \bar{\eta}$. In case $F = \bar{\mathbb{F}}_p$, there exists a finite étale cover *of order prime to p* such that the pullback of $\mathbb{E} \times_S \bar{\eta}$ is a direct sum of stratified line bundles.

In characteristic zero, though, the theorem holds in greater generality and hence two questions naturally arise: the first one is whether we can relax the assumption of coprimality to p of the order of the trivializing covers of the $\mathbb{E} \times_S s$, while keeping the assumption that X is proper over S. The second one is if we can drop this last assumption as well. The main result of this thesis, in particular, is that if F is uncountable then the positive equicharacteristic version of André and Hrushovsky's theorem holds in its full generality.

Leitfaden

In Chapter 1 we describe the main objects of study of this thesis, that is stratified bundles. We prefer to give a very hands-on approach to these objects, choosing the point of view of coherent modules endowed with an action of the sheaf of differential operators instead of the one of Frobenius divided sheaves. In the the same spirit, we describe in the last section of this chapter how to construct explicit examples of stratified bundles.

In Chapter 2 we recall the formalism of Tannakian categories and the definition of monodromy group of an object in such categories. We also list the main properties of the monodromy group of a stratified bundle and establish a fundamental technical result (see Corollary 2.3.11): if F is algebraically closed, L is a field extension and $\mathbb{E}$ is a stratified bundle over X a smooth F-variety, then the monodromy group of $\mathbb{E}$ based changed over L is the base change to L of the monodromy group of $\mathbb{E}$. In other words, the monodromy group of a stratified bundle behaves well under extension of the ground field.

Let assume from now on that F is algebraically closed. Chapter 3 is devoted to study to the core of the problem, providing a complete generalization of the result in [EL13, Thm. 7.2]. Bearing in mind the counterexample of Esnault and Langer ([EL13, Cor. 4.3]) we cannot hope in general to completely eliminate the assumption of coprimality to p of the order of the trivializing covers of the $\mathbb{E} \times_S s$. Still, we prove that it suffices to impose to the power of p dividing the order of such trivializing covers to be bounded:

Theorem 1 (See Theorem 3.1.3)**.** Let F be an algebraically closed field, $X \to S$ a smooth proper morphism of F-varieties and $\mathbb{E} = (E, \nabla)$ a stratified bundle on X relative to S. Assume that for every closed point s in a dense open $\tilde{S} \subset S$ the stratified bundle $\mathbb{E}_s = \mathbb{E} \times_S s$ is trivialized by a finite étale cover *whose order is not divisible by* p^N *for some fixed* $N \geq 0$. Then, if $F \neq \bar{\mathbb{F}}_p$, there exists a finite étale cover of the generic geometric fiber that trivializes $\mathbb{E}_{\bar{\eta}} = \mathbb{E} \times_S \bar{\eta}$. In case $F = \bar{\mathbb{F}}_p$, there exists a finite étale cover such that the pullback of $\mathbb{E}_{\bar{\eta}}$ is the direct sum of stratified line bundles.

The assumption on X being proper over S is more delicate to eliminate; the order of the trivializing covers does not play any role while the cardinality of F becomes the main obstruction:

Counterexample (See Proposition 3.2.1)**.** If F is a countable field, then there exists a stratified bundle on $\mathbb{A}^2_F$ relative to $\mathbb{A}^1_F$ which is trivial on every closed fiber but is not trivialized by any finite étale cover on the generic geometric fiber.

On the other hand the main result of this thesis is that in case F is uncountable the strong version of the theorem holds. Namely, using completely different techniques and the results from Chapter 2 we obtain:

Theorem 2 (See Theorem 3.3.1)**.** Let F be an *uncountable* algebraically closed field, $X \to S$ a smooth morphism of F-varieties and $\mathbb{E} = (E, \nabla)$ a stratified bundle on X relative to S such that, for every closed point s in a dense open $\tilde{S} \subset S$, the stratified bundle $\mathbb{E}_s = \mathbb{E} \times_S s$ is trivialized by a finite étale cover. Then, there exists a finite étale cover of the generic geometric fiber that trivializes $\mathbb{E}_{\bar{\eta}} = \mathbb{E} \times_S \bar{\eta}$.

In Chapter 4 we prove that in the case where F is countable and X is not proper over S there is still room for improvement, using the theory of regular singular stratified bundles (introduced in [Gie75]). Roughly speaking, a stratified bundle is regular singular if it has only mild (that is logarithmic) singularities along the divisor at infinity. In characteristic zero there is a parallel notion of regular singular flat connections, and one of the first steps in the proof of André's theorem is to show that if a relative flat connection (E, ∇) on X over S is regular singular on the fiber over all closed points of a dense subset of S then it is regular singular on the generic fiber (see [And04, Lemma 8.1.1]). In positive characteristic this is no longer true, as our counterexample shows. The converse still holds (see the proof of Lemma 4.0.10): if X admits a good compactification over S and $\mathbb{E} = (E, \nabla)$ is a stratified bundle on X relative to S such that $\mathbb{E}_{\bar{\eta}}$ is regular singular then for every closed point s of some dense open $\tilde{S} \subset S$ the stratified bundle $\mathbb{E}_s$ is regular singular as well. Moreover, assuming $\mathbb{E}_{\bar{\eta}}$ to be regular singular we obtain the same results than in the proper case:

Theorem 3 (See Theorem 4.0.13)**.** Let F be an algebraically closed field of any cardinality and $X \to S$ a smooth morphism of F-varieties. Let $\mathbb{E} = (E, \nabla)$ be a stratified bundle on X relative to S such that, for every closed point s in a dense open $\tilde{S} \subset S$, the stratified bundle $\mathbb{E}_s = \mathbb{E} \times_S s$ is trivialized by a finite étale cover whose order is not divisible by p^N for some fixed $N \geq 0$. Assume moreover that $\mathbb{E}_{\bar{\eta}} = \mathbb{E} \times_S \bar{\eta}$ is *regular singular*. Then, if $F \neq \bar{\mathbb{F}}_p$, there exists a finite étale cover of the generic geometric fiber that trivializes $\mathbb{E}_{\bar{\eta}}$. In case $F = \bar{\mathbb{F}}_p$, there exists a finite étale cover such that the pullback of $\mathbb{E}_{\bar{\eta}}$ is the direct sum of stratified line bundles.

In their way to the proof of [EL13, Thm. 7.2] Esnault and Langer obtained similar results for families of isotrivial vector bundles. For the sake of completeness, in Chapter 5 we apply techniques parallel to the ones of Chapter 3 to broaden these results.

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Notations

i) The letter F will always denote a field of positive characteristic p. Even if some results hold for a general F, all main theorems will require F to be algebraically closed.

ii) A *variety* over F will be a reduced scheme of finite type over $\operatorname{Spec} F$.

iii) Group schemes are always assumed to be affine, a group scheme G over F is *algebraic* if it is locally of finite type.

iv) For a F-group scheme G we denote by $\operatorname{Repf}_F G$ the category of finite dimensional representations of G over F.

v) The notation Vec_F and Vecf_F is used to indicate the categories of vector spaces and finite dimensional vector spaces over F.

vi) For a scheme X, we denote by $\operatorname{QCoh}(X)$ the category of quasi coherent $\mathscr{O}_X$-modules and by $\operatorname{Coh}(X)$ the full subcategory of coherent $\mathscr{O}_X$-modules

vii) In order to keep the notation as simple as possible, we will sometimes write F for $\operatorname{Spec} F$, when no confusion can arise.

viii) In general, if A is an object (for example a scheme, a morphism, a stratified bundle and so on) defined over F and L is a field extension, we will denote by A_L its base change to L.

CHAPTER 1

THE CATEGORY OF STRATIFIED BUNDLES

Let X be a complex variety and let E be a vector bundle of finite rank over X. A *connection* on E is a $\mathbb{C}$-linear action of the derivations on E. A connection is said to be *flat* if its curvature is zero, and this is equivalent to the action of the derivations extends to an action of the whole ring of differential operators ([BO78, Thm. 2.15]). Note that this is true because in characteristic zero the derivations span the ring of differential operators. In positive characteristic, however, this no longer holds and flat connections are not equivalent to vector bundles endowed with an action of the differential operators. This thesis is focusing on object of the latter kind, as it turns out they have a richer and better behaved structure. In this chapter we will recall what is the sheaf of differential operators both in its classic and logarithmic form, and introduce the main object of interest of this thesis, that is, stratified bundles.

1.1 THE SHEAF OF DIFFERENTIAL OPERATORS

Throughout this chapter F will denote a field of positive characteristic p and $u : X \to S$ a smooth morphism of varieties over F, of relative dimension d. We will denote by $\mathscr{D}_{X/S}$ be the quasi-coherent $\mathscr{O}_X$-module of relative differential operators as defined in [EGA4, §16]; we will not go through its general construction as in our relatively smooth situation it has a fairly simple description. In order to give such characterization, let us recall that if $\mathcal{U}$ is an open sub-scheme of X admitting global coordinates $x_1, \dots, x_d$ relative to S, then for every $k \in \mathbb{N}$ there are $\mathscr{O}_S$-linear maps $\partial_{x_l}^{(k)} : \mathscr{O}_{\mathcal{U}} \to \mathscr{O}_{\mathcal{U}}$

given by

$$\partial_{x_l}^{(k)}(x_j^h) = \delta_{lj}\binom{h}{k}(x_j^{h-k})$$

where δ_{lj} is the Kronecker delta. Heuristically, one can think about these maps as $1/k!\partial_{x_l}^k$, even though of course this cannot be given as a formal definition as in principle p may divide $k!$.

The *sheaf of differential operators on X relative to S* is the sub-algebra $\mathscr{D}_{X/S}$ of $\mathrm{End}_{\mathscr{O}_S}(\mathscr{O}_X)$ which is locally generated by the $\partial_{x_l}^{(k)}$, that is, if $\mathcal{U}$ is an open of X as before, we have that

$$\mathscr{D}_{X/S|\mathcal{U}} = \mathscr{O}_{\mathcal{U}}\big[\partial_{x_l}^{(k)} \mid l \in \{1,\ldots,d\}, k \in \mathbb{N}_{>0}\big].$$

Notice that for $k = 1$ the maps $\partial_{x_l} = \partial_{x_l}^{(1)}$ are derivations, for which the Leibniz rule applies. Similarly, for higher differential operators we have an extension of the Leibniz rule, namely if $f, g \in \mathscr{O}_{\mathcal{U}}$ then

$$\partial_{x_l}^{(k)}(fg) = \sum_{\substack{a+b=k\\ a,b\geq 0}} \partial_{x_l}^{(a)}(f)\partial_{x_l}^{(b)}(g). \tag{1.1}$$

When $S = \operatorname{Spec} F$ we will use the notation $\mathscr{D}_{X/F}$ instead of $\mathscr{D}_{X/\operatorname{Spec} F}$ and we will call such a case the *absolute case* in contrast to the situation when S is a generic variety to which we will refer to as the *relative case.*

Remark 1.1.1. If p is the characteristic of the ground field F, then it is easy to see that $(\partial_{x_l})^p \equiv 0$, and hence that the derivatives cannot span the whole ring of differential operators. Notice that in characteristic zero the algebra of differential operators is finitely generated as a $\mathscr{O}_X$-algebra. Instead, in positive characteristic, a countable set of generators is needed, in order to span $\mathscr{D}_{X/S}$ and this will translate in a crucial distinction, in our main theorem, between the situation where F is countable (Proposition 3.2.1) and the one where F is uncountable (Corollary 3.3.5).

The absolute sheaf of differential operators has its logarithmic counterpart. Let X be a smooth variety over F and D a strict normal crossing divisor over X, with ideal sheaf $\mathcal{I}$. Then the *sheaf of differential operators on X relative to S* is the sub-algebra $\mathscr{D}_{X/F}(\log D)$ of $\mathscr{D}_{X/F}$ consisting of the differential operators sending $\mathcal{I}^n$ to itself for every $n \in \mathbb{N}$. As before, it has a local explicit description: if $\mathcal{U}$ is an open of X admitting global coordinates $x_1, \ldots, x_d$ and $D \cap \mathcal{U}$ is cut out by $x_1 = \ldots = x_e = 0$. Then

$$\mathscr{D}_{X/S}(\log D)|_{\mathcal{U}} = \mathscr{O}_{\mathcal{U}}\big[x_l^k\partial_{x_l}^{(k)}, \partial_{x_j}^{(k)} \mid l \in \{1,\ldots,e\}, j \in \{e+1,\ldots,d\}, k \in \mathbb{N}_{>0}\big].$$

We do not wish to give here an exhaustive account of the theory of logarithmic geometry and of logarithmic differential operators. The reader interested in a more complete reference is advised to consult [Kin12b, Chapter 2].

1.2 Stratified bundles

The main object of study of this work are stratified bundle, that are, as suggested in the beginning of this chapter, vector bundles of finite rank with an action of the sheaf of differential operators. Before entering the discussion over such objects, we would like to give to the reader a little caveat: such definition should be in principle used only when $X \to S$ is smooth, as only in this case there is an equivalence with the category of Frobenius divided sheaves, also called flat bundles (see [Gie75, Thm. 1.4]). The latter can be seen as the category to refer to, as it possess the good properties we will need further on, also in the situation when $X \to S$ is not smooth. Nevertheless, we find that the following equivalent definition is more indicated in our situation:

Definition 1.2.1. A *stratified bundle on X relative to S* is a locally free $\mathscr{O}_X$-module of finite rank endowed with a $\mathscr{O}_S$-linear $\mathscr{D}_{X/S}$-action extending the $\mathscr{O}_X$-module structure via the inclusion $\mathscr{O}_X \subset \mathscr{D}_{X/S}$. A *morphism* of stratified bundles is a morphism of $\mathscr{D}_{X/S}$-modules. The category of stratified bundles on X relative to S is denoted by $\mathrm{Strat}(X/S)$. If $\mathbb{E} \in \mathrm{Strat}(X/S)$ is a stratified bundle we will denote by E the underlying vector bundle. If $S = \operatorname{Spec} F$ we will use the subscript X/F instead of $X/\operatorname{Spec} F$.

The simpler example of a stratified bundle is $\mathscr{O}_X$ itself with the natural $\mathscr{D}_{X/S}$-action coming from the inclusion of $\mathscr{D}_{X/S}$ as a sub-ring of $\mathrm{End}_{\mathscr{O}_S}(\mathscr{O}_X)$. We will denote this stratified bundle with $\mathbb{I}_{X/S}$ or simply with $\mathbb{I}$ when no confusion is possible and will say that a stratified bundle $\mathbb{E} \in \mathrm{Strat}(X/S)$ is *trivial* if $\mathbb{E} = \oplus_{i=1}^{r} \mathbb{I}_{X/S}$ for some $r \in \mathbb{N}$.

Let $\mathbb{E}, \mathbb{F} \in \mathrm{Strat}(X/S)$ be two stratified bundles and E, F the underlying vector bundles. Then $E \otimes F$ carries a natural $\mathscr{D}_{X/S}$-action locally given on sections by

$$\partial_{x_l}^{(k)}(e \otimes f) = \sum_{a=0}^{k} \partial_{x_l}^{(a)}(e) \otimes \partial_{x_l}^{(k-a)}(f).$$

We will denote this stratified bundle by $\mathbb{E} \otimes \mathbb{F}$. In a similar way $E \oplus F$ and $\mathrm{Hom}_{\mathscr{O}_X}(E, E')$ admit a canonical $\mathscr{D}_{X/S}$-module structure. In particular $\mathbb{E} \oplus \mathbb{F}$ and $\mathbb{E}^\vee$ are well defined objects of $\mathrm{Strat}(X/S)$. If $f : Y \to X$ is a morphism of S varieties then $f^*\mathbb{E}$ is in a natural way an object in $\mathrm{Strat}(Y/S)$, if f is finite and étale and $\mathbb{E} \in \mathrm{Strat}(Y/S)$ then $f_*\mathbb{E}$ is in a natural way an object in

$\mathrm{Strat}(X/S)$ (note that f is proper, flat and its fibers are zero-dimensional, in particular the push-forward of a locally free sheaves is again locally free by [MFK94, Sec. 0.5]).

As in the previous paragraph, if $S = \operatorname{Spec} F$ we will call an object in $\mathrm{Strat}(X/F)$ an *absolute* stratified bundle in order to distinguish this case from the generic relative situation.

Remark 1.2.2. If $S = \operatorname{Spec} F$ then any $\mathscr{O}_X$-coherent module endowed with a $\mathscr{D}_{X/F}$-action is locally free (see [BO78, p. 2.17]). Note that this is no longer true for a generic variety S.

We are interested in a specific class of stratified bundles, namely the one that are trivial up to a finite étale cover: a stratified bundle $\mathbb{E} \in \mathrm{Strat}(X/S)$ is called *isotrivial* if there exists a finite étale cover $f: Y \to X$ such that the pullback $f^*\mathbb{E}$ is trivial.

1.3 Explicit constructions

In this last section we want to present some notations that will be used later on in order to construct some explicit examples of stratified bundles. For the rest of this chapter let us assume that X admits global étale coordinates $x_1, \dots, x_d$ relative to S. If E is a vector bundle on X, then a $\mathscr{D}_{X/S}$-module structure on E is by definition a $\mathscr{O}_S$-linear morphism

$$\phi : \mathscr{D}_{X/S} \to \operatorname{End}_{\mathscr{O}_S}(E)$$

extending the $\mathscr{O}_X$-module structure on E; in particular, the image of $\mathscr{O}_X \subset \mathscr{D}_{X/S}$ under ϕ is always fixed. Therefore, to determine the action of the whole $\mathscr{D}_{X/S}$ it is enough to consider the image of the algebra generators $\partial_{x_l}^{(k)}$ under the morphism ϕ.

Assume moreover from now on that E admits a global basis $e_1, \dots, e_r$, and let $A_{k,l} = (a_{ij}^{k,l})$ be given by $\partial_{x_l}^{(k)}(e_i) = \sum a_{ij}^{k,l} e_j$. Then the $A_{k,l}$, for $k \in \mathbb{N}_{>0}$ and $l = 1, \dots, d$, determine the $\mathscr{D}_{X/S}$-action: If $s = \sum_{i=1}^r f_i \cdot e_i$ is a section of E, with $f_i \in \mathscr{O}_X$, then using (1.1) we have that

$$\partial_{x_l}^{(k)}(s) = \sum_{i=1}^{r} \sum_{\substack{a+b=k \\ a,b \geq 0}} \partial_{x_l}^{(a)}(f_i) A_{b,l} \cdot e_i. \tag{1.2}$$

Note that if $e'_1, \dots, e'_r$ is an other basis of E and $U = (u_{ij}) \in \mathrm{H}^0(X, \mathrm{GL}_r)$ is the base change matrix given by $e'_i = \sum u_{ij} e_j$, then, by (1.1), it follows that in this new basis the matrices $A'_{k,l} = (a'^{k,l}_{ij})$ describing the action of $\partial_x^{(k)}$ are given by

$$A'_{k,l} = \Big[\sum_{\substack{a+b=k\\ a,b\geq 0}} \partial_{x_l}^{(a)}(U)A_{b,l}\Big]U^{-1}. \tag{1.3}$$

Summarizing, with the assumptions of this section, a $\mathscr{D}_{X/S}$-module structure on a globally free stratified bundle of rank r is uniquely described by the data of $r \times r$ matrices $A_{k,l}$ with values in $\mathrm{H}^0(X, \mathscr{O}_X)$, for $l = 1, \dots, d$ and $k \in \mathbb{N}_{>0}$.

On the other hand, given a collection of such matrices $A_{k,l}$, in order for them to define a $\mathscr{D}_{X/S}$-action on a globally free stratified bundle, it is necessary and sufficient that they satisfy the relations that hold between the $\partial_{x_l}^{(k)}$ in the $\mathscr{O}_X$-algebra $\mathscr{D}_{X/S}$.

Chapter 2

The Tannakian monodromy group

In this chapter we will briefly recall the notion of Tannakian categories and the main theorem relating such categories and their so called Tannakian fundamental group. The second section is devoted to define the monodromy group associated to an absolute stratified bundle via the Tannakian duality, and to describe its main properties. All these constructions are somehow classic and the majority of the results can be found in [DM82]. Finally, in the last section we show that the monodromy group behaves well with respect to base change to a field extension: that is, extending scalars and taking the monodromy group can be done one after the other in any order giving canonically isomorphic results.

2.1 Tannakian categories

Tannakian categories are abelian categories equipped with all the features that are proper to the categories of finite dimensional representations of an affine group. Let us give first a rather dry:

Definition 2.1.1. A (neutral) *Tannakian category* over F is a pair $(\mathcal{T}, \omega)$ where $\mathcal{T}$ is tensor category and $\omega : \mathcal{T} \to \mathrm{Vec}_F$ is a *fiber functor*, that is a F-linear, exact tensor functor to the category of F-vector spaces.

We will now give a rough explanation of what the terminology we use in the previous definition means, redirecting the reader to [DM82] for a more detailed account. We also warn the reader that what we will often omit the word neutral, that is that what we call Tannakian category is usually called neutral Tannakian category.

A category $\mathcal{T}$ is called

F-linear if for every objects A,B the set $\mathrm{Hom}_{\mathcal{T}}(A, B)$ is a F-vector space and the composition law is F-bilinear;

symmetric monoidal if it has an inner tensor product, that is a functor

$$\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$$

which is associative and commutative, and there exists an *identity object* $\mathbb{I}$ which is the neutral element for the tensor product;

rigid if every object A admits a *dual* object $A^\vee$ and there are evaluation and coevaluation maps $ev : A \otimes A^\vee \to \mathbb{I}$ and $coev : \mathbb{I} \to A \otimes A^\vee$.

tensor if it is abelian, F-linear, symmetric monoidal, rigid and $\mathrm{End}(\mathbb{I}) = F$.

It is clear, for example, that the category Vec_F of F-vector spaces is a F-linear, abelian, symmetric monoidal category, with identity object the one dimensional vector space F. However, it is not rigid, as it does not have coevaluation maps. On the other hand the full subcategory Vecf_F of finite dimensional vector spaces does has a coevaluation map and is hence a tensor category.

In order to unravel completely the terminology, we need a last definition. A functor ω between symmetric monoidal categories is called a *tensor functor* (sometimes abbreviated in $\otimes$-functor) if it respects the tensor structure, that is if $\omega(A \otimes B) \simeq \omega(A) \otimes \omega(B)$ in a functorial way.

The most important example of a Tannakian category is given by the pair $(\mathrm{Repf}_F\, G, \mathrm{for}_G)$, that is the category of finite dimensional representations of an affine F-group scheme G endowed with the forgetful functor $\mathrm{for}_G : \mathrm{Repf}_G\, F \to \mathrm{Vecf}_F$. Its importance is due to the main theorem on Tannakian categories, also called Tannakian duality:

Theorem 2.1.2. [DM82, Thm. 2.11] *Let $(\mathcal{T}, \omega)$ be a (neutral) Tannakian category over F with fiber functor $\omega : \mathcal{T} \to \mathrm{Vecf}_F$. Then there exists an affine F-group scheme $\pi(\mathcal{T}, \omega)$ and an equivalence of Tannakian categories*

$$(\mathcal{T}, \omega) \simeq (\mathrm{Repf}_F\, \pi(\mathcal{T}, \omega), \mathrm{for}_G).$$

Remark 2.1.3. Notice that the previous theorem implies that ω is more that only additive and faithful: it actually inherits all the properties of the forgetful functor, for example a morphism $A \to B$ in $\mathcal{T}$ is injective, surjective or an isomorphism if and only if $\omega(A \to B)$ is.

The F-group scheme $\pi(\mathcal{T},\omega)$ is called the *Tanakian fundamental group* of $(\mathcal{T},\omega)$. For later use, let us recall the description of $\pi(\mathcal{T},\omega)$ via its functor of points. Let $\omega,\omega':\mathcal{T}\to\mathcal{T}'$ be two tensor functors, then a *morphism of tensor functors* is a natural transformation α respecting the tensor structure, that is such that the diagram

$$\begin{array}{ccc} \omega(A\otimes B) & \xrightarrow{\simeq} & \omega(A)\otimes\omega(B) \\ \downarrow{\scriptstyle \alpha_{\omega(A\otimes B)}} & & \downarrow{\scriptstyle \alpha_{\omega(A)\otimes\omega(B)}} \\ \omega'(A\otimes B) & \xrightarrow{\simeq} & \omega'(A)\otimes\omega'(B) \end{array}$$

commutes. Note that as $\mathrm{Vec}_F=\mathrm{QCoh}(\mathrm{Spec}\,F)$ then for every $u:T\to\mathrm{Spec}\,F$ scheme over F we have that the composition $u^*\circ\omega:\mathcal{T}\to\mathrm{QCoh}(T)$ makes sense and is a tensor functor. Then the functor from schemes over F to groups defined by

$$\underline{\mathrm{Aut}}^{\otimes}_F(u:T\to\mathrm{Spec}\,F)=\{\alpha:u^*\circ\omega\simeq u^*\circ\omega \text{ isomorphism of } \otimes\text{-functors}\}$$

is representable by the affine F-group scheme $\pi(\mathcal{T},\omega)$.

Theorem 2.1.2 implies that every affine F-group scheme can be uniquely reconstructed by the category of its finite dimensional representations over F (together with the forgetful functor). More is actually true: there is a correspondence between homomorphisms between groups and functors between the respective categories of finite dimensional representations. Moreover, some properties of a morphism can be inferred from the ones of the corresponding functor, and vice versa, as shown by the next proposition. Let $\mu:G\to G'$ be a homomorphism of affine F-group schemes, then we denote by $M_\mu:\mathrm{Repf}_F\,G'\to\mathrm{Repf}_F\,G$ the induced functor on the categories of finite dimensional representations.

Proposition 2.1.4. [DM82, Cor. 2.9,Prop. 2.21] *Let G,G' be two affine F-group schemes and let $M:\mathrm{Repf}_F\,G'\to\mathrm{Repf}_F\,G$ be a tensor functor such that* $\mathrm{for}_{G'}=\mathrm{for}_G\circ M$. *Then there exists a unique homomorphism $\mu:G\to G'$ such that $M=M_\mu$. Moreover*

- *i) μ is faithfully flat if and only if M_μ is fully faithful and for every $\rho\in\mathrm{Repf}_F\,G'$ every sub-object of $M(\rho)$ is isomorphic to the image of a sub-object of ρ.*
- *ii) μ is a closed immersion if and only if every object of* $\mathrm{Repf}_F\,G$ *is isomorphic to a sub-quotient of $M(\rho)$ for some ρ object of* $\mathrm{Repf}_F\,G'$.

To close this section let us introduce two fundamental notions: a full subcategory $\mathcal{T}' \subset \mathcal{T}$ of a tensor category is called a *tensor subcategory* if $\mathcal{T}'$ is an abelian subcategory, which is closed under isomorphisms, finite tensor products (in particular it contains $\mathbb{I}$) and duals. If A is an object in $\mathcal{T}$ we denote by $\langle A \rangle_\otimes$ the tensor subcategory of $\mathcal{T}$ *spanned* by A, that is the smallest tensor subcategory of $\mathcal{T}$ containing A. It can be described as the full subcategory of $\mathcal{T}$ consisting of all objects that are isomorphic to a sub-quotient of $p(A, A^\vee)$ where $p \in \mathbb{N}[x, y]$. A tensor category $\mathcal{T}$ *has a tensor generator* if it spanned by one of its objects.

Lemma 2.1.5. [DM82, Prop. 2.20]*Let G be an affine F-group scheme. Then*

i) G is finite if and only if there exists an object A of $\operatorname{Repf}_F G$ such that every other object is isomorphic to a sub-quotient of A^n;

ii) G is algebraic if and only if $\operatorname{Repf}_F G$ has a tensor generator.

2.2 The monodromy group and its properties

If X is a smooth connected F-variety, $\operatorname{Strat}(X/F)$ is a tensor category over F and the choice of a rational point $x \in X(F)$ defines a fiber functor ω_x to the category of finite dimensional F-vector spaces,

$$\begin{aligned} \omega_x : \operatorname{Strat}(X/F) &\to \operatorname{Vec}_F \\ \mathbb{E} &\mapsto E_x \end{aligned}$$

where E is the vector bundle underlying $\mathbb{E}$ ([Riv72, §VI.1]). Hence the pair $(\operatorname{Strat}(X/F), \omega_x)$ is a neutral Tannakian category and by Tannakian duality (Theorem 2.1.2) there exists an affine group scheme $\pi_1^{\text{Strat}}(X, x) \doteq \pi(\operatorname{Strat}(X/S), \omega_x)$ over F such that $\operatorname{Strat}(X/F)$ is equivalent via ω_x to the category of finite dimensional representations of $\pi_1^{\text{Strat}}(X, x)$ over F. For every $\mathbb{E} \in \operatorname{Strat}(X/F)$ we denote by $\langle \mathbb{E} \rangle_\otimes \subset \operatorname{Strat}(X/F)$ the full Tannakian subcategory spanned by $\mathbb{E}$ with fiber functor ω_x defined as above. The affine group scheme $\pi(\mathbb{E}, x) \doteq \pi(\langle \mathbb{E} \rangle_\otimes, \omega_x)$ is called the *monodromy group* of $\mathbb{E}$. If $\mathcal{U} \subset X$ is an open dense sub-scheme of X then by [Kin12a, Lemma 2.5(a)] the restriction functor $\rho_\mathcal{U} : \langle \mathbb{E} \rangle_\otimes \to \langle \mathbb{E}_{|\mathcal{U}} \rangle_\otimes$ is an equivalence. Hence, in particular, the monodromy group of $\mathbb{E}$ is invariant under restriction to any dense open sub-schemes. Moreover, when F is algebraically closed, the monodromy group does not depend on the choice of x, up to non unique isomorphism (this can be deduced from [DM82, Thm. 3.2]). In this situation we will hence sometimes use the notation $\pi(\mathbb{E})$ instead of $\pi(\mathbb{E}, x)$.

Definition 2.2.1. We say that $\mathbb{E} \in \mathrm{Strat}(X/F)$ is *finite* if its monodromy group is finite. By what we have just remarked, when F is algebraically closed, this property is independent of the choice of x.

Recall that by definition $\mathbb{E}$ is isotrivial if it is étale trivializable. If F is algebraically closed these two properties are equivalent:

Lemma 2.2.2. *Let assume F is algebraically closed. For a stratified bundle $\mathbb{E} \in \mathrm{Strat}(X/F)$ the following are equivalent:*

i) *$\mathbb{E}$ is isotrivial;*

ii) *$\mathbb{E}$ is finite.*

Moreover, if $\mathbb{E}$ is finite, then there exists an étale $\pi(\mathbb{E},x)$-torsor $h_{\mathbb{E},x} : Y_{\mathbb{E},x} \to X$, called the Picard–Vessiot torsor *of $\mathbb{E}$ such that, for any $\mathbb{E}' \in \mathrm{Strat}(X/F)$, the pullback $h^*_{\mathbb{E},x}\mathbb{E}'$ is trivial if and only if $\mathbb{E}' \in \langle \mathbb{E} \rangle_\otimes$.*

Finally, for a finite étale cover $h : Y \to X$ such that $h^*\mathbb{E}$ is trivial, the following conditions are equivalent:

i) *$h : Y \to X$ is the Picard–Vessiot torsor for $\mathbb{E}$;*

ii) *every finite étale cover trivializing $\mathbb{E}$ factors (non uniquely) through the cover $h : Y \to X$;*

iii) *$h : Y \to X$ is Galois and $\langle \mathbb{E} \rangle_\otimes = \langle h_* \mathbb{I}_{Y/F} \rangle_\otimes$;*

iv) *$h : Y \to X$ is Galois of Galois group $\pi(\mathbb{E},x)(F)$.*

Proof. The first part of the lemma is [EL13, Lemma 1.1]. As for the second part, first notice that point (b) and (f) of [Kin12a, Prop. 2.15], together with [Kin12a, Cor. 2.16] imply that if $h : Y \to X$ is a finite étale cover trivializing $\mathbb{E}$ then $\langle \mathbb{E} \rangle_\otimes \subset \langle h_* \mathbb{I}_{Y/F} \rangle_\otimes$ and that $\langle \mathbb{E} \rangle_\otimes = \langle h_{\mathbb{E},x*} \mathbb{I}_{Y_{\mathbb{E},x}} \rangle_\otimes$. Moreover, if $h : Y \to X$ is Galois of Galois group G, then $\pi(h_* \mathbb{I}_{Y/F}, x)$ is the finite constant group G and if $\tilde{h} : \tilde{Y} \to X$ is an étale cover factoring through h then $\langle h_* \mathbb{I}_{Y/F} \rangle_\otimes \subset \langle \tilde{h}_* \mathbb{I}_{\tilde{Y}/F} \rangle_\otimes$. We are now ready to prove the rest of the lemma.

(i)⇒(ii) Because $\langle \mathbb{E} \rangle_\otimes = \langle h_{\mathbb{E},x*} \mathscr{O}_{Y_{\mathbb{E},x}/F} \rangle_\otimes$, a cover $\tilde{h} : \tilde{Y} \to X$ trivializes $\mathbb{E}$ if and only if it trivializes $h_{\mathbb{E},x*} \mathbb{I}_{Y_{\mathbb{E},x}/F}$. Let $Z = \tilde{Y} \times_X Y_{\mathbb{E},x}$, and let p_1 and p_2 be the projections on the first and second factor. Then by flat base change (notice that the flat base change morphism is compatible with the $\mathscr{D}_{\tilde{Y}/F}$-action) there is an isomorphism of $\mathscr{D}_{\tilde{Y}/F}$-modules $\tilde{h}^* h_{\mathbb{E},x*} \mathbb{I}_{Y_{\mathbb{E},x}/F} \simeq p_{1*} \mathbb{I}_{Z/F}$; hence, the latter is also a trivial stratified bundle. This, together with [Kin12a, Cor. 2.17], implies that $p_1 : Z \to \tilde{Y}$ is a trivial covering. In particular, it admits a section s; hence, $\tilde{h} = s \circ p_2 \circ h_{\mathbb{E},x}$ and $\tilde{h}$ factors through h.

(ii)⇒(iii) Because h trivializes $\mathbb{E}$, we have the inclusion $\langle \mathbb{E} \rangle_{\otimes} \subset \langle h_* \mathbb{I}_{Y/F} \rangle_{\otimes}$. On the other side, by assumption, $h_{\mathbb{E},x} : Y_{\mathbb{E},x} \to X$ factors through $h : Y \to X$; hence, $\langle h_* \mathbb{I}_{Y/F} \rangle_{\otimes} \subset \langle h_{\mathbb{E},x*} \mathbb{I}_{Y_{\mathbb{E},x}/F} \rangle_{\otimes} = \langle \mathbb{E} \rangle_{\otimes}$.

(iii)⇒(iv) As $\langle \mathbb{E} \rangle_{\otimes} = \langle h_* \mathbb{I}_{Y/F} \rangle_{\otimes}$, then we have the equality $\pi(\mathbb{E}, x) = \pi(h_* \mathbb{I}_{Y/F}, x)$ and as $h : Y \to X$ is Galois, then its Galois group is $\pi(h_* \mathbb{I}_{Y/F})(F) = \pi(\mathbb{E}, x)(F)$.

(iv)⇒ (i) By what we already proved there must be a factorization $f : Y \to Y_{\mathbb{E},x}$ such that $h = h_{\mathbb{E},x} \circ f$. Hence, if G is the Galois group of $h : Y \to X$ then $h_{\mathbb{E}} : Y_{\mathbb{E}} \to X$ corresponds to a normal subgroup H of G. But by assumption $G = \pi(\mathbb{E}, x)(F) = H$; hence, $h = h_{\mathbb{E}}$. □

Corollary 2.2.3. *Let F be algebraically closed. If $\mathbb{E} \in \mathrm{Strat}(X/F)$ is finite then the set of finite étale covers of X trivializing $\mathbb{E}$ has a minimal element which is Galois of Galois group $\pi(\mathbb{E}, x)(F)$.*

By [San07, Cor. 12] for every $\mathbb{E} \in \mathrm{Strat}(X/F)$ the group scheme $\pi(\mathbb{E}, x)$ is smooth (which is equivalent to being reduced). Given a finite stratified bundle $\mathbb{E} \in \mathrm{Strat}(X/F)$ it is straightforward to see that for every $L \supset K$ algebraically closed field extension $\mathbb{E}_L = \mathbb{E} \otimes_F L \in \mathrm{Strat}(X_L/L)$ is finite as well. We will prove in next section that more is true, namely that $\pi(\mathbb{E}_L) \simeq \pi(\mathbb{E}) \otimes_F L$. Still, it is interesting to see how, using only the properties of the Picard-Vessiot torsor, it is possible to show the following weaker result:

Lemma 2.2.4. *Let $\mathbb{E} \in \mathrm{Strat}(X/F)$ and let $L \supset F$ be an algebraically closed field extension such that $\mathbb{E}_L$ is finite. Then for every $L' \supset F$ algebraically closed field extension such that there exists an immersion $L \hookrightarrow L'$ which is the identity on F, we have that $\mathbb{E}_{L'}$ is finite. Moreover for any $x \in X(F)$*

$$\pi(\mathbb{E}_L, x)(L) \simeq \pi(\mathbb{E}_{L'}, x)(L'),$$

where we consider $x \in X_L(L)$ via $F \subset L$ and similarly for L'.

Proof. Let L and L' as in the hypothesis, then we can construct an immersion $L \hookrightarrow L'$ which is the identity on F, just by sending any transcendence basis of L to a algebraically independent set in L' over F and using the fact that L' is algebraically closed to see that this extends to an immersion $L \hookrightarrow L'$. Hence, we have reduced the problem to proving that if $\mathbb{E}$ is finite and $L \supset F$

is an algebraically closed field extension then $\mathbb{E}_L$ is finite and has the same monodromy group of $\mathbb{E}$ as abstract groups. In order to do so we need first to establish a result on Galois covers:

Claim. Let $h : Y \to X$ be a Galois cover of Galois group G and let $h_L : Y_L \to X_L$ the extension of scalars of $h : Y \to X$ to L, then h_L is a Galois cover of Galois group G.

Proof. Certainly $h_L : Y_L \to X_L$ is a finite étale morphism as these properties are stable under base change. We are left to check that (i) Y_L is connected, (ii) $\mathrm{Aut}(Y_L/X_L)$ acts transitively on the fiber over some geometric point of X_L and finally (iii) $\mathrm{Aut}(Y_L/X_L) \simeq \mathrm{Aut}(Y/X)$.

i) As F is algebraically closed (hence, in particular, separably closed) Y is connected if and only if Y_L is connected for any field extension $L \supset F$. In particular, Y_L is connected.

ii) Let $x \in X_L(L)$ be any closed (in particular, geometric) point of X_L, then the composition $\bar{x} : \mathrm{Spec}(L) \to X_L \to X$ is a geometric point for X. As $h : Y \to X$ is Galois, $\mathrm{Aut}(Y/X)$ acts transitively on $Y_{\bar{x}} = Y \times_X \bar{x} = Y_L \times_{X_L} x = Y_{L,x}$. Now, the action of $\mathrm{Aut}(Y/X)$ on $Y_{L,x}$ factors through $\mathrm{Aut}(Y_L/X_L)$ via the inclusion $\mathrm{Aut}(Y/X) \subset \mathrm{Aut}(Y_L/X_L)$ defined by $\phi \mapsto \phi_L$. Hence, the action of $\mathrm{Aut}(Y_L/X_L)$ on $Y_{L,x}$ is transitive as well; therefore, $h_L : Y_L \to X_L$ is Galois.

iii) As $Y_{\bar{x}} = Y_{L,x}$ and both h and h_L are Galois, it follows that the order of their Galois group is the same, as it is the cardinality of the respective geometric fibers over $\bar{x}$ and over x. Moreover we have a natural inclusion $\mathrm{Aut}(Y/X) \subset \mathrm{Aut}(Y_L/X_L)$ so as they have the same cardinality they must be equal; hence, $\mathrm{Aut}(Y_L/X_L) = G$. □

Until the end of the proof let us denote by $h_{\mathbb{E},x} : Y \to X$ the Picard–Vessiot torsor of $\mathbb{E}$ (see Lemma 2.2.2), then $h_{\mathbb{E},x} \otimes_F L : Y_L \to X_L$ is a Galois cover trivializing $\mathbb{E}_L$ which is then finite, by Lemma 2.2.2. Recall that $\langle (h_{\mathbb{E},x})_* \mathbb{I}_{Y/F} \rangle_\otimes = \langle \mathbb{E} \rangle_\otimes$. But then in particular, $\langle (h_{\mathbb{E},x})_* \mathbb{I}_{Y/F} \otimes_F L \rangle_\otimes = \langle \mathbb{E}_L \rangle_\otimes$ and as $(h_{\mathbb{E},x})_* \mathbb{I}_{Y/F} \otimes_F L = (h_{\mathbb{E},x} \otimes_F L)_* \mathbb{I}_{Y_L/L}$, it follows that $\langle (h_{\mathbb{E},x} \otimes_F L)_* \mathbb{I}_{Y_L/L} \rangle_\otimes = \langle \mathbb{E}_L \rangle_\otimes$. Hence, by Lemma 2.2.2, we have that $h_{\mathbb{E},x} \otimes_F L : Y_L \to X_L$ is the minimal trivializing cover for $\mathbb{E}_L$. Now, the Galois group of $h_{\mathbb{E},x}$ is the same as that of $h_{\mathbb{E},x} \otimes_F L$ by the previous claim; hence, again by Lemma 2.2.2, we have that $\pi(\mathbb{E}_L, x)(L) = \pi(\mathbb{E}, x)(F)$. □

We will close this section with two lemmas about descend of stratified bundles.

Lemma 2.2.5. *Let $\mathbb{E}$ be defined over F, let $L \supset F$ a field extension and let $\mathbb{F}$ be a sub-object of $\mathbb{E}_L = \mathbb{E} \otimes_F L$. Then $\mathbb{F}$ is defined a finitely generated F-algebra. The same holds for sub-quotients.*

Proof. It is enough to prove the claim on a finite cover of opens of X, hence we can assume that $X = \operatorname{Spec} R$, that there are global étale coordinates and that $\mathbb{E}$ is globally free. In particular, once fixed a basis of $\mathbb{E}$ the action of $\mathscr{D}_{X/F}$ is given by $r \times r$ matrices $A_{i,k}$ describing the action of the operators $\partial_{x_i}^{(k)}$. The same matrices define the action of $\mathscr{D}_{X_L/L}$ on $\mathbb{E}_L$. If we change the basis with base change matrix U we get that in the new basis , by (1.3)

$$A'_{i,k} = \Big[\sum_{\substack{a+b=k \\ a,b\geq 0}} \partial_{x_i}^{(a)}(U) A_{i,b}\Big] U^{-1}.$$

Let now choose a basis of $\mathbb{F}$ and complete it to a basis of $\mathbb{E}_L$, let U be the base change matrix. Let L' be a finite type extension of F such that U and U^{-1} are defined over L'. Then $\mathbb{F}$ is defined over L' also as a $\mathscr{D}_{X_L/L}$-module. To see this, it is enough to prove that the $A'_{i,k}$ are defined over L', this because the action of $\mathscr{D}_{X_L/L}$ over $\mathbb{F}$ is defined by the first $r' \times r'$ minors of the $A'_{i,k}$. By the base change formula, it is enough to prove that $\partial_{x_i}^{(a)}(U)$ are defined over L' for every i and a. The entries of U are in $R \otimes_F L'$ and the operators $\partial_{x_i}^{(a)}$ are linear on the ground field. So if $u \in R \otimes_F L'$ we have that $\partial_{x_i}^{(a)}(u) \in R \otimes_F L'$ as well and this completes the proof. □

Finite stratified bundle have an additional propriety that will turn out to be very useful to prove that some stratified bundle cannot be isotrivial:

Lemma 2.2.6. *Let F be algebraically closed and let $\mathbb{E} \in \operatorname{Strat}(X/F)$ be a finite stratified bundle. Then there exists a subfield $F' \subset F$ of finite type over $\mathbb{F}_p$ over which X and $\mathbb{E}$ are defined; that is, there exists X' smooth variety over F' and $\mathbb{E}' \in \operatorname{Strat}(X'/F')$ such that $X = X' \times_{\operatorname{Spec} F'} \operatorname{Spec} F$ and $\mathbb{E} = \mathbb{E}' \otimes_{F'} F$.*

Proof. Let $h_{\mathbb{E},x} : Y_{\mathbb{E},x} \to X$ be the Picard–Vessiot torsor of $\mathbb{E}$ (see Lemma 2.2.2), and let $\mathbb{H} = (h_{\mathbb{E},x})_* \mathscr{O}_{Y_{\mathbb{E},x}}$, then (see Lemma 2.2.2) $\mathbb{E} \in \langle \mathbb{H} \rangle_\otimes$. Certainly there exists F'' of finite type over $\mathbb{F}_p$ on which $h_{\mathbb{E},x} : Y_{\mathbb{E},x} \to X$ is defined; hence, $\mathbb{H}$ is also defined over F''. Notice that $\mathbb{E}$ is a sub-quotient of $\mathbb{P}$ where $\mathbb{P} \in \mathbb{Z}[\mathbb{H}, \mathbb{H}^\vee]$ (see e.g. [Kin12a, def. 2.4]); that is, $\mathbb{E} \simeq \tilde{\mathbb{P}}/\bar{\mathbb{P}}$ with $\bar{\mathbb{P}} \subset \tilde{\mathbb{P}} \subset \mathbb{P}$. By Lemma 2.2.5 $\tilde{\mathbb{P}}$ and $\bar{\mathbb{P}}$ are defined over some extension F' of finite type of F'' (thus over $\mathbb{F}_p$). Therefore, so does $\mathbb{E} \simeq \tilde{\mathbb{P}}/\bar{\mathbb{P}}$. □

2.3 Geometric Tannakian pairs and the invariance of the monodromy group

Let F be any field of positive characteristic and let X a smooth geometrically connected F-variety having a rational point. Let L be a field extension of F, and let $i : \operatorname{Spec} L \to \operatorname{Spec} F$ denote the morphism corresponding to the inclusion $F \subset L$. Let $X_L = X \otimes_F L$ its base change to L. Let $\mathbb{E} \in \operatorname{Strat}(X/F)$ be the base change of X to L and let $\mathbb{E}_L = \mathbb{E} \otimes_F L \in \operatorname{Strat}(X_L/L)$. In general, we will use the subscript L to indicate the base change of some object from F to L.

Let us fix a rational point x in X: as explained in the previous section, this induces a fiber functor, that we will denote by $\omega : \langle \mathbb{E} \rangle_\otimes \to \mathrm{Vec}_F$. Let $\omega_L : \langle \mathbb{E} \rangle_\otimes \to \mathrm{Vec}_L$ be the fiber functor associated to the closed point of X_L given by the base change of x to L. Let $\pi(\mathbb{E}, \omega)$, respectively $\pi(\mathbb{E}_L, \omega_L)$, the algebraic groups associated to the Tannakian categories $(\langle \mathbb{E} \rangle_\otimes, \omega)$, respectively to $(\langle \mathbb{E}_L \rangle_\otimes, \omega_L)$.

The goal of this section is to prove the following

Theorem 2.3.1. *There is a functorial morphism*

$$\pi(\mathbb{E}_L, \omega_L) \to \pi(\mathbb{E}, \omega) \otimes L,$$

which is a closed immersion.

Moreover, if F is algebraically closed, it is an isomorphism. In particular, we can explicitly describe the category $\operatorname{Repf}(\pi(\mathbb{E}_L, \omega_L))$ *in terms of the category* $\operatorname{Repf}(\pi(\mathbb{E}), \omega)$.

We will prove this theorem in a more general setting, which applies to certain classes of Tannakian categories:

Definition 2.3.2. Let F be any field and $i : F \subset L$ a field extension. A pair consisting of a (neutral) Tannakian category $(\mathcal{T}, \omega)$ over F and a (neutral) Tannakian category $(\mathcal{T}_L, \omega_L)$ over L is a *geometric* (F, L)*-pair* if there exists an additive, exact tensor functor $-_L : \mathcal{T} \to \mathcal{T}_L$, $A \mapsto A_L$ called the *base change* functor such that

(A1) there is a natural equivalence $\omega_L \circ -_L = i^* \circ \omega(A)$.

A geometric (F, L)-pair is said to be *filtered* if there exists a (possibly non exhaustive) filtration of subcategories indexed by the F-algebras of finite type $R \subset L$, denoted $\mathcal{T}_L^R$ such that:

(F0) for every $A \in \mathcal{T}$, $A_L \in \mathcal{T}_L^F$ and for $R \subset R'$ the category $\mathcal{T}_L^R$ is a faithful subcategory of $\mathcal{T}_L^{R'}$;

(F1) for every $A \in \mathcal{T}$ and every $B' \subset A_L$, if $B' \in \mathcal{T}_L^R$ then the immersion $B' \subset A_L$ is in $\mathcal{T}_L^R$ and the restriction of the evaluation functor ω_L factors through $- \otimes_R L : \mathrm{Bun}_R \to \mathrm{Vecf}_L$, where Bun_R is the category of vector bundles over $\operatorname{Spec} R$;

(F2) for every $A \in \mathcal{T}$, the subcategory $\langle A_L \rangle_\otimes$ is exhaustively filtered by $\mathcal{T}_L^R$, that is every object and morphism in $\langle A_L \rangle_\otimes$ is contained in $\mathcal{T}_L^R$ for some R.

As an example, the geometric (F, L)-pair $(\mathrm{Vecf}_F, \mathrm{for})$ and $(\mathrm{Vecf}_L, \mathrm{for})$ admits a natural filtration

$$\mathrm{Vecf}_L^R = \{B \mid B \subset A \otimes_F L, A \in \mathrm{Vecf}_F \text{ and } (B \subset A_L) \text{ is defined over } R\}$$

$$\mathrm{Hom}_{\mathcal{T}_L^R}(A, B) = \{\phi \in \mathrm{Hom}_{\mathrm{Vecf}_L}(A, B) \mid \phi \text{ is defined over } R\}$$

where $B \subset A_L$ is defined over R if there exists $B' \subset A \otimes_F R$ as vector bundles such that $(B' \subset A_R)_L = (B \subset A_L)$ and similarly for homomorphisms. For every F-rational point $z \in \operatorname{Spec} R(F)$ the restriction gives a natural evaluation map $ev_z : \mathrm{Coh}(\operatorname{Spec} R) \to \mathrm{Coh}(\operatorname{Spec} F) = \mathrm{Vecf}_F$. We want a map with similar properties to exist on our filtered (F, L)-pair, hence the following:

Definition 2.3.3. A geometric (F, L)-pair is endowed with an *evaluation* structure if it is filtered and for every F-algebra of finite type $R \subset L$, for every $z \in \operatorname{Spec} R(F)$ there exists an functor $ev_z : \mathcal{T}_L^R \to \mathcal{T}$ preserving direct sums and tensor product such that

(A2) there are natural equivalences $ev_z \circ \omega_L = \omega \circ ev_z : \mathcal{T}_L^R \to \mathrm{Vecf}_F$ and $ev_z \circ -_L = id$.

A geometric (F, L)-pair is *comparable* if it is endowed with an evaluation structure and if

(A3) for every $A \in \mathcal{T}$ the monodromy group $\pi(\langle A \rangle_\otimes, \omega)$ is reduced or, equivalently, smooth.

Remark 2.3.4. Notice that if F is not algebraically closed, it may happen that $\operatorname{Spec} R(F) = \varnothing$ and hence that $(A2)$ is a void condition. In order for the evaluation structure to be of some use we will need F to be algebraically closed, in parallel to what one should expect also on the vector spaces side.

Lemma 2.3.5. *Let $F \subset L$ be any field extension. Let X a smooth geometrically connected variety over F, $x \in X(F)$ a rational point and $x_L \in X(L)$ the induced rational point on $X_L = X \otimes_F L$. Then* $\mathrm{Strat}(X/F)$ *and* $\mathrm{Strat}(X_L/L)$ *are a geometric (F, L)-pair. If moreover F is algebraically closed, then they are comparable.*

Proof. Of course the functor $\mathrm{Strat}(X/F) \to \mathrm{Strat}(X_L/L)$ sending $\mathbb{E}$ to $\mathbb{E}_L$ is additive, exact and respects the tensor structure. We define the stratification $\mathrm{Strat}(X_L/L)^R$ as the sub-objects $\mathbb{F} \subset \mathbb{E}_L$, for $\mathbb{E} \in \mathrm{Strat}(X/F)$, coming by base change from some $\mathbb{F}' \subset \mathbb{E}_R \in \mathrm{Strat}(X_R/R)$ and similarly for homomorphisms. The axioms $(F0)$ and $(F1)$ are clearly satisfied and $(F2)$ comes from Lemma 2.2.5. In particular, we obtain $ev_z : \mathrm{Strat}(X/L)^R \subset \mathrm{Strat}(X \otimes_F R/\operatorname{Spec} R) \to \mathrm{Strat}(X/F)$ which clearly satisfies $(A2)$. The axiom $(A1)$ follows by the definition of ω and ω_L and the axiom $(A3)$ is [San07, Cor. 12]. □

Remark 2.3.6. The very same construction holds for any Tannakian category $\mathcal{T}(X)$ associated to a F-scheme X, with a base change functor $\mathcal{T}(X) \to \mathcal{T}(X_L)$, $A \mapsto A_L$, such that $(A3)$ holds and that for every $A \in \mathcal{T}(X)$, every sub-object $B \subset A_L$ is defined over a finitely generated F-algebra (as a sub-object). The following examples give rise to (comparable) geometric pairs:

i) the category of flat connections over a F-variety X with F (algebraically closed) of characteristic zero and $X(F) \neq \varnothing$;

ii) the category of essentially finite sheaves over a pseudo-proper F-variety X (see [NB13] for the definition of pseudo-proper) for F (algebraically closed) of characteristic zero and $X(F) \neq \varnothing$. Notice that in positive characteristic $(A3)$ may fail;

iii) variations of (i) and (ii) such as the subcategories of unipotent objects, finite tame objects (see [NB13, Def. 12.1]) and so on.

iv) variations of $\mathrm{Strat}(X/F)$, for example its largest semi-simple sub-category, its largest unipotent sub-category and so on.

In this more general setting, Theorem 2.3.1 can be restated as follows

Theorem 2.3.7. *Let (T,ω) and (T_L,ω_L) a geometric (F,L)-pair and let $A \in \mathcal{T}$. Then there is a functorial morphism*

$$\mu : \pi(A_L,\omega_L) \to \pi(A,\omega) \otimes_F L$$

which is a closed immersion. If moreover the pair is comparable and both F and L are algebraically closed, then μ is an isomorphism.

We will spend the rest of this section to prove this theorem. Let $(\mathcal{T},\omega)$ and $(\mathcal{T}_L,\omega_L)$ be a geometric (F,L)-pair, and let $A \in \mathcal{T}$. As ω and ω_L are fixed, we will often write $\pi(A)$ for $\pi(A,\omega)$ and $\pi(A_L)$ for $\pi(A_L,\omega_L)$. In order to compare $\pi(A_L)$ and $\pi(A) \otimes L$ as algebraic groups over L we need to exhibit a morphism between them. By construction of the associated Tannakian group, if we consider $\pi(A_L)$ as its functor of points from the category of schemes over L to Groups then

$$\pi(A_L)(s : T \to \operatorname{Spec} L) = \{\alpha : s^*\omega_L \simeq s^*\omega_L \text{ iso. of } \otimes\text{-functors}\}.$$

Similarly as functor of points from the category of schemes over F to Groups

$$\pi(A)(s : T \to \operatorname{Spec} F) = \{\alpha : s^*\omega \simeq s^*\omega \text{ iso. of } \otimes\text{-functors}\}.$$

In our situation we are interested in considering the functor of points $\pi(A) \otimes L$ on schemes over L, that is we want to describe $\pi(A)_L(s : T \to \operatorname{Spec} L)$. Notice that if $s : T \to \operatorname{Spec} L$ is a scheme over L then, considering T as a scheme over F via $i \circ s$, we have that

$$\begin{aligned}\operatorname{Hom}_L(T, \pi(A)_L) = \operatorname{Hom}_F(T, \pi(A)) = \pi(A)(i \circ s : T \to \operatorname{Spec} F) = \\ = \{\alpha : s^*i^*\omega \simeq s^*i^*\omega \text{ iso. of } \otimes\text{-functors}\}.\end{aligned}$$

By definition, $\alpha : s^*\omega_L \simeq s^*\omega_L$ is a compatible collection of automorphisms $\alpha_B \in \operatorname{Aut}(s^*(\omega_L(B)))$ for every object $B \in \langle A_L \rangle_\otimes$, where by compatible we mean that if $f : B \to B'$ is a morphism then the diagram

$$\begin{array}{ccc} s^*\omega_L(B) & \xrightarrow{\alpha_B} & s^*\omega_L(B) \\ \downarrow{\scriptstyle s^*\omega_L(f)} & & \downarrow{\scriptstyle s^*\omega_L(f)} \\ s^*\omega_L(B') & \xrightarrow{\alpha_{B'}} & s^*\omega_L(B') \end{array}$$

commutes. In particular we can give another description of the functor of points, namely

$$\pi(A_L)(s : T \to \operatorname{Spec} L) = \{(\alpha_B)_{B \in \langle A_L \rangle_\otimes} \mid \alpha_B \in \operatorname{Aut}(s^*\omega_L(B)) \text{ comp. coll.}\}.$$

On the other hand, for every object $B \in \langle A \rangle_\otimes$, by the axiom $(A1)$ we have that $i^*\omega(A) = \omega_L(A_L)$. Let us denote by $\mathcal{S} \subset \langle A_L \rangle_\otimes$ the subset of all objects coming from $\langle A \rangle_\otimes$ by the base change functor, that is

$$\mathcal{S} = \{B \in \langle A_L \rangle_\otimes \mid B \simeq B'_L, B' \in \langle A \rangle_\otimes\}.$$

Then, we have that

$$\pi(A)_L(s : T \to \operatorname{Spec} L) = \{(\alpha_B)_{B \in \mathcal{S}} \mid \alpha_B \in \operatorname{Aut}(s^*\omega_L(B)) \text{ comp. collection}\}.$$

Hence, there is a natural map $\pi(A_L)(T) \to \pi(A)_L(T)$ that restricts the collection of automorphisms from the objects of $\langle A_L\rangle_\otimes$ to $\mathcal{S}$. It is clearly a map of group functors and hence by Yoneda it induces a morphism of group schemes $\mu : \pi(A_L) \to \pi(A)_L$.

Lemma 2.3.8. *For every $s : T \to \operatorname{Spec} L$ the homomorphism of groups*

$$\mu(T) : \pi(A_L)(T) \to \pi(A)_L(T)$$

is injective. In particular, $\mu : \pi(A_L) \to \pi(A)_L$ is a closed immersion.

Proof. Let us consider $\alpha \in \pi(A_L)(T)$, that is, a compatible collection (α_B) for $B \in \langle A_L\rangle_\otimes$ and $\alpha_B \in \operatorname{Aut}(s^*\omega_L(B))$. Then $\mu(T)(\alpha)$ is by definition the restriction of the collection (α_B) to the objects of the form B'_L with $B' \in \langle A\rangle_\otimes$. Assume that $\mu(T)(\alpha) = id$, that is $(\mu(T)(\alpha))_{B'_L} = id_{B'_L}$ for every $B' \in \langle A\rangle_\otimes$. Then, in particular we have that $\alpha_{A_L} = id_{A_L}$ and $\alpha_{p(A_L,A_L^\vee)} = id_{p(A_L,A_L^\vee)}$ for every $p \in \mathbb{N}[x,y]$ (notice that as $A \mapsto A_L$ is an additive tensor functor $p(A_L, A_L^\vee) = p(A, A^\vee)_L$).

Now, the collection of automorphisms must be compatible with morphisms between stratified bundles, in particular if $B \subset B'$ is a sub-object (and hence $s^*\omega_L(B) \subset s^*\omega_L(B')$), then $\alpha_B : s^*\omega_L(B) \to s^*\omega_L(B')$ must be the restriction of $\alpha_{B'} : s^*\omega_L(B') \to s^*\omega_L(B')$. Similarly, if B is a quotient of B' then α_B must be the quotient of $\alpha_{B'}$. In particular, as every object is isomorphic to a sub-quotient of $p(A_L, A_L^\vee)$ for some $p \in \mathbb{N}[x,y]$, we obtain $\alpha_B = id_B$ for every $B \in \langle A_L\rangle_\otimes$, that is $\alpha = id$ hence $\mu(T)$ is injective for every T. In particular, μ is a monomorphism hence, by [SGA3, VI_B Cor. 1.4.2], a closed immersion. ☐

Remark 2.3.9. Another way of proving the previous lemma would be to establish explicitly what is the associated functor $M_\mu : \operatorname{Repf}_L \pi(A)_L \to \operatorname{Repf}_L \pi(A_L)$ and use Proposition 2.1.4 to conclude. We prefer to propose this direct approach instead of the more indirect, though more elegant, one.

By definition of geometric Tannakian category, $\pi(A)$ and hence $\pi(A)_L$ are reduced. In particular, when L is algebraically closed, in order to prove that μ is an isomorphism it suffices to show the following:

Lemma 2.3.10. *If F is algebraically closed the morphism of groups $\mu(L)$: $\pi(A_L)(L) \to \pi(A)_L(L)$ is surjective.*

Proof. As before, let $\mathcal{S} \subset \langle A_L\rangle_\otimes$ denote the objects isomorphic to B_L for some $B \in \langle A\rangle_\otimes$.

In order for $\mu(L)$ to be surjective, we need to find, for every $\beta \in \pi(A)_L(L)$ an element $\alpha \in \pi(A_L)(L)$ in the preimage. Unraveling the definition, for every compatible collection $\beta_B \in \mathrm{Aut}(\omega_L(B))$ for $B \in \mathcal{S}$ we want to find a compatible collection $\alpha_B \in \mathrm{Aut}(\omega_L(B))$ for $B \in \langle A_L \rangle_\otimes$ such that $\alpha_B = \beta_B$ for every $B \in \mathcal{S}$.

We will first construct the collection α_B satisfying $\alpha_B = \beta_B$ for every $B \in \mathcal{S}$ and then show it is compatible. In order to construct the collection, as every $B \in \langle A_L \rangle_\otimes$ is a sub-quotient of $p(A_L, A_L^\vee)$ for some $p \in \mathbb{N}[x,y]$, we will show that for every $B \subset p(A_L, A_L^\vee)$ we have that $\omega_L(B)$ is $\beta_{p(A_L,A_L^\vee)}$-invariant. In particular, $\beta_{p(A_L,A_L^\vee)}$ restrict to every $\omega_L(B)$ (and hence also to all sub-quotients) defining a collection α_B which, by construction, satisfies $\alpha_B = \beta_B$ for every $B \in \mathcal{S}$.

Without loss of generality, as all functors preserve direct sum and tensor product, we can assume $p(A_L, A_L^\vee) = A_L$ and reduce ourselves to prove the invariance only for β_{A_L}. By way of contradiction, assume that there exists $B' \subset A_L$ a sub-object of A_L such that $\omega_L(B')$ is not invariant under β_{A_L}.

By $(F2)$, there is some algebra R of finite type over F, on which β_{A_L} is defined (that is, β_{A_L} comes from an endomorphism of $\omega(A) \otimes_F R$ in Bun_R) and such that $(B' \subset A) \in \mathcal{T}_L^R$. In particular by $(F1)$ we have that $\omega_L(B')$, as a sub-object of $\omega_L(A_L) = \omega(A)_L$, comes by base change from a sub-object of $\omega(A) \otimes_F R$ in Bun_R. Notice that for every $z \in \operatorname{Spec} R(F)$ the $ev_z(\beta_{A_L})$ is an automorphism of $\omega(A) = ev_z(\omega_L(A_L))$. Moreover for every $C \subset A$ the subspace $\omega(C)$ is invariant for it: as the β_B form a compatible collection, we have that $\omega_L(B_L)$ is β_{A_L}-invariant and hence $ev_z(\omega_L(C_L)) = ev_z(\omega(C) \otimes_F L) = \omega(C)$ is $ev_z(\beta_{A_L})$-invariant. Notice that this holds if R is *any* F-algebra on which β_{A_L} is defined and such that $(B' \subset A) \in \mathcal{T}_L^R$, in particular by $(F0)$ we can without loss of generality enlarge R along the proof. To get a contradiction, we will show that there exists R such that $(B' \subset A) \in \mathcal{T}_L^R$ and on which β_{A_L} is defined and there exists $z \in \operatorname{Spec} R(F)$ such that $ev_z(B')$ is a sub-object of A but $\omega(ev_z(B'))$ is not $ev_z(\beta_{A_L})$-invariant.

Let $v_1, \ldots, v_r$ be a basis of $\omega(A)$ over F, and let us denote again by $v_1, \ldots, v_r$ the one induced on $\omega_L(A_L) = \omega(A) \otimes_F L$. Let $w_1, \ldots, w_s$ be a basis for $\omega_L(B')$, $w_i = \sum a_{ij} v_j$, with $a_{ij} \in \mathrm{H}^0(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ (recall that $\omega_L(B')$ comes from a sub-object of $\omega(A)_R$ in Bun_R). We can complete it to a basis of $\omega_L(A_L)$, and up to reordering assume it is of the form $w_1, \ldots, w_s, v_{s+1}, \ldots, v_r$. Notice that as being a basis depends on a determinant to be invertible, $ev_z(w_1), \ldots, ev_z(w_s), v_{s+1}, \ldots, v_r$ are a basis for every choice of a closed point $z \in \operatorname{Spec} R$, where $ev_z(w_i) = \sum ev_z(a_{ij}) v_j$. As $\omega_L(B')$ is not invariant under β_{A_L}, then there exists a vector $w = \sum b_i w_i$ such that $\beta_{A_L}(w) \notin \omega_L(B')$, that is $\beta_{A_L}(w) = \sum c_i w_i + \sum d_j v_j$, with $d_j \neq 0$ for some j, say $j = r$. Up to enlarging R we can assume that b_i, c_i and d_j are in $\mathrm{H}^0(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ as well.

As $d_r \neq 0$ up to shrinking $\operatorname{Spec} R$ we have that for every closed point $z \in \operatorname{Spec} R$ the evaluation of d_r at z is not zero. If $z \in \operatorname{Spec} R(F)$, then $ev_z(\omega_L(B')) = \langle ev_z(w_1)\rangle \oplus \cdots \oplus \langle ev_z(w_s)\rangle$ and we have that $ev_z(\beta_{A_L})(ev_z(w)) = ev_z(\beta_{A_L}(w)) = \sum ev_z(c_i) ev_z(w_i) + \sum ev_z(d_j) v_j$, and hence $ev_z(\omega_L(B'))$ is not $ev_z(\beta_{A_L})$-invariant. By $(A2)$ we have that $ev_z(\omega_L(B')) = \omega(ev_z(B'))$. Moreover, up to localizing R, we can assume that for every $z \in \operatorname{Spec} R(F)$ $ev_z(\omega_L(B') = \omega(ev_z(B')) \subset \omega(A)$. In particular, by Remark 2.1.3, it follows that $ev_z(B') \subset A$.

So after possibly substituting R with a localization of $R' \supset R$ we have that, for every $z \in \operatorname{Spec} R(F)$, $ev_z(B') \subset A$ but $\omega(ev_z(B'))$ is not $ev_z(\beta_{A_L})$-invariant. As F is algebraically closed, there always is such a z, yielding to a contradiction.

Now that we have our collection α_B for $B \in \langle A_L\rangle_\otimes$ we need to check that it is compatible, that is for every $f : B \to B'$ in $\langle A_L\rangle_\otimes$ that the diagram

$$\begin{array}{ccc} \omega_L(B) & \xrightarrow{\alpha_B} & \omega_L(B) \\ \downarrow{\scriptstyle \omega_L(f)} & & \downarrow{\scriptstyle \omega_L(f)} \\ \omega_L(B') & \xrightarrow{\alpha_{B'}} & \omega_L(B') \end{array}$$

commutes. Similarly as before, there exists some finite type F-algebra R such that objects and morphisms are in $\mathcal{T}_L^R$ and everything is defined over R. With a similar argument as before, one can chose such a R so that evaluating at closed point (that is, F-rational points) yields to a contradiction if we assume that the diagram is not commuting. □

Corollary 2.3.11. *Let assume that we are in the case where $T = \operatorname{Strat}(X/F)$ (or in one of the examples in Remark 2.3.6). If the field L is not algebraically closed then μ is still an isomorphism.*

Proof. Let us consider $F \subset L \subset \bar{F}$, then by the theorem the composition $\pi(\mathbb{E}_{\bar{L}}) \to \pi(\mathbb{E}_L) \otimes_L \bar{L} \to \pi(\mathbb{E}) \otimes_F \bar{L}$ is an isomorphism. In particular $\mu_{\bar{L}}$ is an isomorphism and hence μ is as well. □

2.4 The structure of $\operatorname{Repf}(\pi(\mathbb{E}_L))$

Let F be algebraically closed and let L be a field extension. By Corollary 2.3.11 we have that $\pi(\mathbb{E}_L) \simeq \pi(\mathbb{E}) \otimes L$ we can now describe explicitly

$\langle \mathbb{E}_L \rangle_\otimes$ in terms of $\langle \mathbb{E} \rangle_\otimes$ or, up to equivalence of categories, $\mathrm{Repf}(\pi(\mathbb{E}_L))$ in terms of $\mathrm{Repf}(\pi(\mathbb{E}))$.

Theorem 2.4.1 (Jordan-Hölder)**.** [Ses67, Thm 2.1] *Let $\mathcal{C}$ be an abelian category and $X \in \mathcal{C}$ an object of finite length. Then X admits a filtration, called* composition series

$$0 = A_r \subset A_{r-1} \subset \cdots \subset A_0 = X$$

such that $S_i = A_i/A_{i+1}$ are simple for every i (and different from zero). Moreover every such two filtration have the same length r and their associated graded objects are isomorphic. The S_i are called the composition factors *of X.*

Lemma 2.4.2. *An object $\mathbb{F}$ in $\langle \mathbb{E} \rangle_\otimes$ is simple if and only if $\mathbb{F}_L$ is simple in $\langle \mathbb{E}_L \rangle_\otimes$.*

Proof. Assume that $\mathbb{F}$ is not simple, and let $0 \neq \mathbb{F}' \subset \mathbb{F}$ a sub-object. Then $0 \neq \mathbb{F}'_L \subset \mathbb{F}_L$ is a sub-object as well. Conversely, assume that $\mathbb{F}_L$ is not simple and let $0 \neq \mathbb{F}' \subset \mathbb{F}_L$ be a sub-object. Let R be a F-algebra of finite type over F such that $\mathbb{F}'$ is defined over R (see the proof of Lemma 2.3.5) then we can evaluate at closed points $z \in \operatorname{Spec} R(F)$ getting $ev_z(\mathbb{F}' \subset \mathbb{F}_L) = ev_z(\mathbb{F}') \to \mathbb{F}$. All we need to show is then that $ev_z(\mathbb{F}' \to \mathbb{F}_L)$ is not the zero morphism for some $z \in \operatorname{Spec} R(F)$. But this must be the case as otherwise $\mathbb{F}' \to \mathbb{F}_L$ would be the zero morphism on every closed point and hence would be zero. □

If $0 = \mathbb{F}_r \subset \cdots \subset \mathbb{F}$ is a composition series for $\mathbb{F}$ then by the previous lemma $0 = \mathbb{F}_r \otimes L \subset \cdots \subset \mathbb{F} \otimes L$ is a composition series for $\mathbb{F}_L$. If $\mathbb{F}'$ is a sub-object or a quotient of $\mathbb{F}$ then its composition factors are isomorphic to a subset of the composition factors of $\mathbb{F}$. This implies in particular that the simple objects in $\langle \mathbb{E}_L \rangle_\otimes$ are the composition factors of $p(\mathbb{E}_L, \mathbb{E}_L^\vee)$ with $p \in \mathbb{N}[x,y]$. As $p(\mathbb{E}, \mathbb{E}^\vee)_L = p(\mathbb{E}_L, \mathbb{E}_L^\vee)$ we have the following

Corollary 2.4.3. *Simple objects in $\langle \mathbb{E}_L \rangle_\otimes$ are of the form $\mathbb{F}_L$ for some $\mathbb{F}$ simple object in $\langle \mathbb{E} \rangle_\otimes$.*

Note that the very same corollary is valid for describing the simple objects in $\mathrm{Repf}(\pi(\mathbb{E}_L))$ in terms of the ones in $\mathrm{Repf}(\pi(\mathbb{E}))$. We are now left to describe the Hom and Ext groups between them. Using the Tannakian duality and [Jan87, Ch. 4] (note that we are dealing with affine algebraic groups by Lemma 2.1.5) we have that $\mathrm{Hom}_{\langle \mathbb{E} \rangle_\otimes}(\mathbb{F}, \mathbb{F}') \otimes L = \mathrm{Hom}_{\langle \mathbb{E}_L \rangle_\otimes}(\mathbb{F}_L, \mathbb{F}'_L)$ and more in general $\mathrm{Ext}^i_{\langle \mathbb{E} \rangle_\otimes}(\mathbb{F}, \mathbb{F}') \otimes L = \mathrm{Ext}^i_{\langle \mathbb{E}_L \rangle_\otimes}(\mathbb{F}_L, \mathbb{F}'_L)$ for every i. Note that this means that we expect that the objects of $\langle \mathbb{E}_L \rangle_\otimes$ are not all coming from $\langle \mathbb{E} \rangle_\otimes$ by base change as new extension will appear as soon as $\mathrm{Ext}_{\langle \mathbb{E} \rangle_\otimes}(\mathbb{F}, \mathbb{F}') \neq 0$.

2.5 THE COMPARISON MORPHISM ON $\pi_1^{\text{STRAT}}(X)$

The very same construction via functor of points that we did for the monodromy group of some $\mathbb{E} \in \text{Strat}(X/F)$ can be carried out for the whole category $\text{Strat}(X/F)$, hence giving a morphism

$$\mu : \pi_1^{\text{strat}}(X_L) \to \pi_1^{\text{strat}}(X) \otimes L,$$

or, equivalently, a $\otimes$-functor

$$M : \text{Repf}_L(\pi_1^{\text{strat}}(X) \otimes L) \to \text{Repf}_L(\pi_1^{\text{strat}}(X_L)).$$

Lemma 2.5.1. *The morphism μ is faithfully flat. In particular, the functor M is fully faithful.*

Proof. Let G be any affine group scheme over a field F. Then, by [Mil12, VIII,Thm. 8.1], G is the projective limit of its algebraic quotients or, equivalently, if $G = \operatorname{Spec} A$ for A and Hopf algebra, A is the direct union of its finitely generated Hopf sub-algebras. Moreover, by [Mil12, VI,Thm. 11.1] every inclusion of Hopf algebras $A \subset B$ over a field is faithfully flat, hence every affine group scheme over a field F is the projective limit of its faithfully flat algebraic quotients. If A_i and A_j are two Hopf sub-algebras of A that are finitely generated over F then their generators taken together span a finitely generated Hopf sub-algebra A_k of A containing both A_i and A_j.

Summarizing, every Hopf algebra A over a field F is the directed limit (that is, union) of its finitely generated Hopf sub-algebras A_i such that A is faithfully flat over A_i.

Let now $G = \pi_1^{\text{strat}}(X)$ and A its associated Hopf algebra over F. Then, by what just remarked

$$A = \varinjlim A_i$$

where the union is taken over the finitely generated Hopf sub-algebras A_i such that A is faithfully flat over A_i. As it is a directed limit, tensor product commutes with it, hence if L is a field extension of F we have that

$$A \otimes_F L = (\varinjlim A_i) \otimes_F L = \varinjlim (A_i \otimes_F L).$$

Now, by [DM82, Prop. 2.21], as $G \to \operatorname{Spec} A_i$ is faithfully flat, then the induced $\otimes$-functor $M_i : \text{Repf}_F(A_i) \to \text{Repf}_F(G)$ must be fully faithful and if ρ is an object of $\text{Repf}_F(A_i)$ every sub-object of $M_i(\rho)$ must come from a sub-object of ρ. Hence, we can consider $\text{Repf}_F(A_i)$ as a full subcategory of $\text{Repf}_F(G)$ which is closed under sub-quotient and tensor product. Up to category equivalence we can also assume that $\text{Repf}_F(A_i)$ is closed under isomorphisms. Moreover, by [DM82, Prop. 2.20] $\mathrm{Repf}_F(A_i)$ is $\otimes$-generated by one object, hence it will be equivalent to $\langle \mathbb{E} \rangle_{\otimes}$ for some $\mathbb{E} \in \mathrm{Strat}(X/F)$. On the other hand for every $\mathbb{E} \in \mathrm{Strat}(X/F)$ we have that $\pi(\mathbb{E}) = \mathrm{Spec}\, A_{\mathbb{E}}$ for $A_{\mathbb{E}} \subset A$ a Hopf sub-algebra on which A is faithfully flat. Using that $\pi(\mathbb{E}_L) = \pi(\mathbb{E}) \otimes_F L$, we get

$$\pi_1^{\mathrm{strat}}(X) \otimes_F L = \varprojlim \pi(\mathbb{E}) \otimes_F L = \varprojlim \pi(\mathbb{E}_L).$$

In the language of Hopf algebras, if $\pi_1^{\mathrm{strat}}(X_L) = \mathrm{Spec}\, B$ and B_j are its Hopf sub-algebras that are finitely generated over L, then for every $\mathbb{E} \in \mathrm{Strat}(X/F)$ we have that $A_{\mathbb{E}_L} = B_j$ for some j, in particular

$$A \otimes L = \varinjlim A_{\mathbb{E}} \otimes L = \varinjlim A_{\mathbb{E}_L} \subset \varinjlim B_j = B.$$

Hence, using again [Mil12, VI,Thm. 11.1] we have that B is faithfully flat over A and by [DM82, Prop. 2.21] we have that M is fully faithful. □

Remark 2.5.2. The proof and the conclusion of the previous lemma apply to every geometric comparable (F, L)-pair on which Theorem 2.3.7 applies. In general, though, μ is not an isomorphism. For example [San07, Cor. 23] provides a counterexample for $\pi_1^{strat}(X)$ even when X is projective, but already in [Del89, Par. 10.35] Deligne showed that the construction of the Tannakian fundamental group of the category of (regular singular) flat connections is not compatible with a transcendental extension of the ground field.

Chapter 3

The positive equicharacteristic p-curvature conjecture

The chore of this work is devoted to the study of the behavior of the monodromy group of stratified bundles in families. There are interesting results and conjectures about this question, both in mixed and zero characteristic, that we briefly summarize in what follows. Let (E, ∇) be a vector bundle endowed with a flat connection on a smooth complex variety X. Then there exists a smooth scheme S over $\operatorname{Spec}\mathbb{Z}$ such that $(E, \nabla) = (E_S, \nabla_S) \otimes_S \mathbb{C}$ and $X = X_S \otimes_S \mathbb{C}$ with X_S smooth over S and (E_S, ∇_S) flat connection on X_S, relative to S. The p-curvature conjecture of Grothendieck and Katz (see [And04, Conj. 3.3.3]) predicts that if for all closed points s of a dense open sub-scheme $\tilde{S} \subset S$ the vector bundle $E_S \times_S s$ is spanned by its horizontal sections, then (E, ∇) must be trivialized by an étale finite cover of X.

An analogue problem can be studied in equicharacteristic zero, and in fact reduces the p-curvature conjecture to the number field case. Y. André in [And04, Prop. 7.1.1] and E. Hrushovsky in [Hru02, p. 116] stated and proved the following equicharacteristic zero version of the p-curvature conjecture: let $X \to S$ be a smooth morphism of varieties over a field F of characteristic zero; let (E, ∇) be a flat connection on X relative to S such that, for every closed point s in a dense open $\tilde{S} \subset S$, the flat connection $(E, \nabla) \times_S s$ is trivialized by a finite étale cover. Then, there exists a finite étale cover of the generic geometric fiber over $\bar{\eta}$ that trivializes $(E, \nabla) \times_S \bar{\eta}$, where $\bar{\eta}$ is a geometric generic point of S.

The theorem of André and Hrushovsky translates naturally to the case where $X \to S$ is a morphism of $\mathbb{F}_p$-schemes, providing a positive equicharacteristic analogue to the p-curvature conjecture, which will be the central argument of this chapter.

In this and next chapter F will always be algebraically closed.

3.1 Families of finite stratified bundles

Let $\mathbb{E} \in \mathrm{Strat}(X/S)$ be a relative stratified bundle. Then we can see $\mathbb{E}$ as a family of stratified bundles parametrized by the points of S. In particular, for every $s \in S(F)$, let $\mathbb{E}_s \in \mathrm{Strat}(X_s/k(s))$ denote the restriction of $\mathbb{E}$ on X_s and $\mathbb{E}_{\bar{\eta}} \in \mathrm{Strat}(X_{\bar{\eta}}/k(\bar{\eta}))$ its restriction on the geometric generic fiber given by a choice of an algebraic closure $\overline{k(S)}$ of $k(S)$. It is natural then to ask how the property of being isotrivial behaves in families: the main question we want to study is whether it is true that if $\mathbb{E}_s$ is finite for every $s \in S(F)$ then so is $\mathbb{E}_\eta$. As already mentioned, this is true in characteristic by [And04, Prop. 7.1.1]. In positive characteristic, following an idea of Laszlo, in [EL13, Cor. 4.3, Rmk. 5.4.1] the authors proved that there exists $X \to S$ a projective smooth morphism of varieties over $\bar{\mathbb{F}}_2$ and a stratified bundle on X relative to S which is finite on every closed fiber but not on the geometric generic one (recall that by 2.2.2 as the ground field is algebraically closed then finiteness and isotriviality are two equivalent notions for a stratified bundle). Nevertheless, assuming X to be projective over S and imposing a coprimality to p condition on the order of the monodromy group on the closed fibers, they proved the following:

Theorem 3.1.1. [EL13, Thm. 7.2] *Let $X \to S$ be a smooth* projective *morphism of F-varieties with geometrically connected fibers, let $\mathbb{E} \in \mathrm{Strat}(X/S)$. Assume that there exists a dense subset $\tilde{S} \subset S(F)$ such that, for every $s \in \tilde{S}$, the stratified bundle $\mathbb{E}_s$ has finite monodromy* of order prime to p. *Then*

i) there exists $f_{\bar{\eta}} : Y_{\bar{\eta}} \to X_{\bar{\eta}}$ a finite étale cover of order prime to p *such that $f^*\mathbb{E}_{\bar{\eta}}$ decomposes as direct sum of stratified line bundles;*

ii) if $F \neq \bar{\mathbb{F}}_p$ then $\mathbb{E}_{\bar{\eta}}$ is trivialized by a finite étale cover of order prime to *p.*

Note that the cover of order prime to p in the second point of the theorem factors through the Picard–Vessiot torsor of $\mathbb{E}_{\bar{\eta}}$ by its minimality (see Lemma 2.2.2). In particular, this implies that the order of the monodromy group of $\mathbb{E}_{\bar{\eta}}$ is prime to p.

In this chapter we will determine how the assumptions of X being projective over S and of the order of the monodromy groups to be prime to p

can or cannot be relaxed in order to get similar results, in order to get a full parallelism with the results in characteristic zero. A first strengthening of the theorem comes rather directly from the ideas in the proof of Theorem 3.1.1. In order to prove it we need first to establish the following

Lemma 3.1.2. *Let $h : X \to S$ be a proper flat separable morphism of connected varieties with geometrically connected fibers over an algebraically closed field F and suppose it has a section $\sigma : S \to X$. Let $\tilde{S} \subset S(F)$ be any subset of the closed points of S, let $N \in \mathbb{N}$ and let us fix for every $s \in \tilde{S}$ a finite étale cover $g_s : Z_s \to X_s$ of degree less than N. Then there exists an open sub-scheme $\mathcal{U} \subset S$ and a finite étale cover $f : W \to X \times_S \mathcal{U}$ dominating all the g_s for $s \in \tilde{S} \cap \mathcal{U}$; that is, for every $s \in \tilde{S} \cap \mathcal{U}$ the finite étale cover $f_s : W_s \to X_s$ factors through $g_s : Z_s \to X_s$.*

Proof. The proof of this lemma is a generalization of the construction that one can find in the beginning of the proof of [EL13, Thm. 5.1].

First notice that if the order of the $g_s : Z_s \to X_s$ is bounded by N then the order of their Galois closures is bounded by $N!$, hence we can assume all the $g_s : Z_s \to X_s$ to be Galois. Moreover if S' is connected and $S' \to S$ is étale and generically finite then there is a non-trivial open $\mathcal{U}$ over which $S' \times_S \mathcal{U} \to \mathcal{U}$ is finite and étale. As $X \to S$ is smooth its image is open; hence, by shrinking S, we can assume that $X \to S$ is surjective.

Let $S' \to S$ be finite étale, then so is $X' = X \times_S S' \to X$. Let $s \in \tilde{S}$ and $s' \in S'(F)$ lying over s. Assume that we have found $f' : W \to X'$ such that $f'_s : W_s \to X'_s$ factors through $g_{s'} : Z_{s'} = Z_s \times_{k(s)} k(s') \to X'_{s'}$ then the composition $f : W \to X' \to X$ is a finite étale cover of X and $f_s : W_s \to X_s$ factors through g_s.

As $h : X \to S$ has geometrically connected fibers so does $h' : X' \to S'$ as if $s' \in S$ lies over $s \in S$ then $X'_{s'} = X_s \otimes_{k(s)} k(s')$. Therefore, $h' : X' \to S'$ is proper, flat, separable and has geometrically connected fibers. To summarize, if S' is connected the morphism $h : X' \to S'$ together with the section $\sigma' : S' \to X'$ induced by $\sigma : S \to X$ satisfy the assumptions of the theorem and without loss of generality we only need to prove the theorem for $h : X' \to S'$. Moreover, taking for every $s \in S(F) - \tilde{S}$ the cover g_s to be the identity we can assume $\tilde{S}$ to be the whole S.

For any $s \in S$ let $G_s \subset \pi_1^{\text{ét}}(X_s, \sigma(s))$ be the open normal subgroup corresponding via Galois duality to the cover $g_s : Z_s \to X_s$. Let $\bar{\eta}$ be a generic geometric point of S given by the choice of an algebraic closure $\overline{k(S)}$ of $k(S)$. The fibers of $X \to S$ are geometrically connected and the morphism is proper, flat and separable; hence, the specialization map $\pi_1^{\text{ét}}(X_{\bar\eta},\sigma(\bar\eta)) \twoheadrightarrow \pi_1^{\text{ét}}(X_s,\sigma(s))$ is surjective. Composing it with the quotient of $\pi_1^{\text{ét}}(X_s,\sigma(s))$ by G_s we get

$$\rho_s : \pi_1^{\text{ét}}(X_{\bar\eta},\sigma(\bar\eta)) \twoheadrightarrow \pi_1^{\text{ét}}(X_s,\sigma(s)) \twoheadrightarrow \pi_1^{\text{ét}}(X_s,\sigma(s))/G_s.$$

Notice that the index of $\ker(\rho_s)$ in $\pi_1^{\text{ét}}(X_{\bar\eta},\sigma(\bar\eta))$ is bounded by N. Let $\tau : S' \to S$ be any finite étale cover and let $s' \in S$ lying over s. As F is algebraically closed, then $k(s') \simeq k(s)$ and hence $X'_{s'} \simeq X_s$. In particular, the natural morphism $\pi_1^{\text{ét}}(X'_{s'},\sigma'(s')) \to \pi_1^{\text{ét}}(X_s,\sigma(s))$ is an isomorphism. Let $G_{s'} \subset \pi_1^{\text{ét}}(X'_{s'},\sigma'(s'))$ is the open subgroup corresponding to $g_{s'} : Z_{s'} \to X'_{s'}$, that is, the preimage of G_s under this isomorphism and let us denote by

$$\rho_{s'} : \pi_1^{\text{ét}}(X_{\bar\eta},\sigma(\bar\eta)) \twoheadrightarrow \pi_1^{\text{ét}}(X'_{s'},\sigma'(s')) \twoheadrightarrow \pi_1^{\text{ét}}(X'_{s'},\sigma'(s'))/G_{s'},$$

then $\ker(\rho_{s'}) = \ker(\rho_s) \subset \pi_1^{\text{ét}}(X_{\bar\eta},\sigma(\bar\eta))$.

As $X_{\bar\eta}$ is a projective $\overline{k(S)}$-variety, then $\pi_1^{\text{ét}}(X_{\bar\eta},\sigma(\bar\eta))$ is topologically finitely generated and hence has finitely many subgroups of index less than N, which are all opens by Nikolov–Segal theorem ([NS07, Thm 1.1]), the intersection of which we denote by G: It is a normal open subgroup and it has finite index. Moreover

$$G \subset \bigcap_{s\in\tilde S} \ker(\rho_s) = \bigcap_{s'\in\tau^{-1}(\tilde S)} \ker(\rho_{s'}).$$

At this point, we need an additional step before concluding similarly than in the proof of [EL13, Thm. 5.1]. By Galois duality G corresponds to a finite étale cover $Z_{\bar\eta} \to X_{\bar\eta}$. Let $k(S)^{\text{sep}}$ be the separable closure of $k(S)$ in $\overline{k(S)}$. The base change functor from the category of finite étale covers over $X \otimes_S k(S)^{\text{sep}}$ to the one of finite étale covers over $X_{\bar\eta}$ is an equivalence. Hence, $Z_{\bar\eta}$ is defined over some separable extension of $k(S)$. In particular, there exists an étale generically finite cover $S' \to S$ such that $Z_{\bar\eta}$ descends to a finite étale cover of $X' = X \times_S S'$.

Let $\bar\eta'$ be the geometric generic point of S' given by $k(S) \subset k(S') \subset \overline{k(S)}$. Then $X'_{\bar\eta'} = X_{\bar\eta}$ and $\sigma(\bar\eta) = \sigma'(\bar\eta')$. Hence, the following diagram commutes:

$$\begin{array}{ccc} \pi_1^{\text{ét}}(X'_{\bar\eta},\sigma'(\bar\eta)) & \longrightarrow & \pi_1^{\text{ét}}(X',\sigma'(\bar\eta)) \\ \| & & \downarrow \\ \pi_1^{\text{ét}}(X_{\bar\eta},\sigma(\bar\eta)) & \longrightarrow & \pi_1^{\text{ét}}(X,\sigma(\bar\eta)). \end{array}$$

Let K' be the kernel of $\pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta})) \to \pi_1^{\text{ét}}(X', \sigma'(\bar{\eta}))$. As $X \to S$ is projective we have the following exact sequence:

$$
\begin{array}{ccccccccc}
 & & \pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta})) & & & & & & \\
 & & \Big\downarrow{\scriptstyle q} & \overset{\alpha}{\searrow} & & & & & \\
\{1\} & \longrightarrow & \pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta}))/K' & \xrightarrow{\ i\ } & \pi_1^{\text{ét}}(X', \sigma'(\bar{\eta})) & \underset{\longrightarrow}{\overset{\sigma'_*}{\longleftarrow}} & \pi_1^{\text{ét}}(S', \bar{\eta}) & \longrightarrow & \{1\}.
\end{array}
$$

By [SGA1, V Cor 6.7] and the fact that $Z_{\bar{\eta}} = Z' \times_{S'} \bar{\eta}$ for $Z' \to X'$ an étale cover, we have the inclusion $G \supset K'$. Moreover if we denote by $\Pi_{K'} = \pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta}))/K'$, then the section σ'_* induces a split

$$\pi_1^{\text{ét}}(X', \sigma(\bar{\eta})) \simeq \Pi_{K'} \rtimes \sigma'_*(\pi_1^{\text{ét}}(S', \bar{\eta}))$$

as abstract groups. It is also a split of topological groups (see for example [Bou98, §2.10 Prop. 28] and following discussion). In particular, the topology on $\pi_1^{\text{ét}}(X', \sigma(\bar{\eta}))$ is the product topology. Note that $\overline{G} = q(G)$ is invariant by the action of $\sigma'_*(\pi_1^{\text{ét}}(S', \bar{\eta}))$; hence, we can define

$$H = \overline{G} \rtimes \sigma'_*(\pi_1^{\text{ét}}(S', \bar{\eta})).$$

By definition, and as $G \supset K'$, we have that $\alpha^{-1}(H) = G$, and H has finite index in $\pi_1^{\text{ét}}(X', \sigma(\bar{\eta}))$. It is also open: $\pi_1^{\text{ét}}(X', \sigma(\bar{\eta}))$ is endowed with the product topology, and $\overline{G}$ is open because $G = q^{-1}(\overline{G})$ is open as well. Hence, H corresponds to a finite étale cover $W \to X'$ and as the composition $H \subset \pi_1^{\text{ét}}(X', \sigma(\bar{\eta})) \twoheadrightarrow \pi_1^{\text{ét}}(S', \bar{\eta})$ is surjective, then W has geometrically connected fibers over S'. In particular, the specialization map is again surjective. Let $z \in W$ be a point lying over s' and let $\bar{\zeta}$ be a geometric generic point lying over $\sigma(\bar{\eta})$, then we have the following commutative diagram

$$
\begin{array}{ccccc}
 & & \overset{0}{\frown} & & \\
\pi_1^{\text{ét}}(W_{\bar{\eta}}, \bar{\zeta}) & \longrightarrow & \pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta})) & \longrightarrow & \pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta}))/G \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\pi_1^{\text{ét}}(W_s, z) & \longrightarrow & \pi_1^{\text{ét}}(X'_{s'}, \sigma'(s')) & \longrightarrow & \pi_1^{\text{ét}}(X'_{s'}, \sigma'(s'))/G_{s'}.
\end{array}
$$

Using the surjectivity of the specialization map on W it follows that the composition of the morphisms on the second line is zero as well; hence, if $\tilde{G}_{s'} \subset \pi_1^{\text{ét}}(X'_{s'}, \sigma'(s'))$ is the open normal subgroup corresponding to $f_{s'} : W_{s'} \to X'_{s'}$ then $\tilde{G}_s \subset G_{s'}$. Therefore, $f_{s'}$ factors through $g_{s'}$. □

We can summarize the previous lemma by saying that with the assumptions of the theorem, up to shrinking S every family of finite étale covers of the closed fibers with bounded order can be dominated by a finite étale cover of X (notice that the existence of the section $\sigma : X \to S$ is not essential for the proof as we can always replace S by S').

Theorem 3.1.3. *Let $X \to S$ be a smooth* proper *morphism of F-varieties with geometrically connected fibers and $\mathbb{E} \in \mathrm{Strat}(X/S)$ of rank r. Assume that there exists a dense subset $\tilde{S} \subset S(F)$ such that, for every $s \in \tilde{S}$, the stratified bundle $\mathbb{E}_s$ has finite monodromy and that the highest power of p dividing $|\pi(\mathbb{E}_s)|$ is bounded over $\tilde{S}$. Then*

i) there exists $f_{\bar{\eta}} : Y_{\bar{\eta}} \to X_{\bar{\eta}}$ a finite étale cover such that $f^*\mathbb{E}_{\bar{\eta}}$ decomposes as direct sum of stratified line bundles;

ii) if $F \neq \bar{\mathbb{F}}_p$ then $\mathbb{E}_{\bar{\eta}}$ is finite.

Proof. We will reduce this theorem to Theorem 3.1.1. By the invariance of the monodromy group it suffices to prove the theorem for the $f^*\mathbb{E}$ where $f : Y \to X$ is a morphism of smooth S-varieties which is generically finite étale. By Chow's lemma and using de Jong alterations ([dJ96]) there exists $f : Y \to X$ projective and generically finite étale, hence we can assume X to be projective. Up to taking an étale open of S we can assume that there exists a section $\sigma : S \to X$. For any $s \in \tilde{S}$ let $\Gamma_s = \pi(\mathbb{E}, \sigma(s))$ and $h_s : Y_s \to X_s$ the Picard–Vessiot torsor of $\mathbb{E}_s$ (see Lemma 2.2.2). Let $G_s \subset \pi_1^{\text{ét}}(X_s, \sigma(s))$ be the normal open subgroup corresponding via Galois duality to the cover h_s. By Tannakian duality $\mathbb{E}_s$ corresponds to the image of an r-dimensional representation of $\pi_1^{\text{Strat}}(X_s, \sigma(s))$ ([DM82, Prop. 2.21]) and as $\mathbb{E}_s$ is finite by [San07, Prop. 13] this representation factors through the étale fundamental group, considered as a constant group scheme

$$\pi_1^{\text{Strat}}(X_s, \sigma(s)) \twoheadrightarrow \pi_1^{\text{ét}}(X_s, \sigma(s)) \twoheadrightarrow \pi_1^{\text{ét}}(X_s, \sigma(s))\big/G_s = \Gamma_s \subset GL_r(F)$$

where r is the rank of $\mathbb{E}$. By Brauer–Feit generalization of Jordan's theorem [BF66, Theorem], as the orders of the Sylow-p-subgroups of every G_s are bounded by p^N, there exists an integer $M = f(r, N)$ and, for every $s \in \tilde{S}$, a normal abelian subgroup A_s such that $|\Gamma_s : A_s| < M$. This gives for every $s \in S(F)$ a Galois cover $g_s : Z_s \to X_s$ of order bounded by M and a factorization

$$Y_s \to Z_s \to X_s$$

where $Y_s \to Z_s$ is Galois of Galois group A_s. Therefore, by Lemma 3.1.2, up to shrinking S there exists a cover $g' : Z' \to X$ such that $g'_s : Z'_s \to X_s$ factors through $g_s : Z_s \to X_s$. In particular, if $\mathbb{E}'$ is the pullback of $\mathbb{E}$ via g' then $\pi(\mathbb{E}'_s)$ is abelian for every s. Up to taking an étale open of S the section $\sigma : S \to X$ extends to a section $\sigma' : S \to Z'$. Let $\Gamma'_s = \pi(\mathbb{E}', \sigma'(s))$, as we just noticed for every $s \in \tilde{S}$ we have that Γ'_s is abelian; hence, we can write it as the direct product of its p part with its prime to p part:

$$\Gamma'_s = \Gamma_s^p \times \Gamma_s^{p'}$$

and $\Gamma_s^{p'}$ corresponds to a Galois cover over Z'_s whose index is by assumption bounded by p^N for some $N \in \mathbb{N}$. Applying Lemma 3.1.2 and up to shrinking S we get a Galois cover $g'' : Z'' \to Z'$ dominating all such covers. Let $\mathbb{E}''$ be the pullback of $\mathbb{E}'$ along g'', then $\pi(\mathbb{E}''_s)$ is (abelian) of order prime to p for every $s \in \tilde{S}$. Therefore, we have reduced the problem to Theorem 3.1.1. □

3.2 A COUNTEREXAMPLE OVER COUNTABLE FIELDS

Our next aim is to drop the assumption of X being projective over S. However, before getting to the positive results, let us present a counterexample to understand what we can reasonably expect to hold without this assumption. Assume for the rest of this section F to be an algebraically closed countable field. Let $X = \mathbb{A}^2_F$, $S = \mathbb{A}^1_F$ and let $X \to S$ be given by $F[y] \to F[x, y]$. The main result of this section is the following:

Proposition 3.2.1. *There exists $\mathbb{E} \in \mathrm{Strat}(X/S)$ such that $\mathbb{E}_s$ is trivial for every point $s \in S(F)$ but $\mathbb{E}_{\bar{\eta}}$ is not isotrivial.*

The rest of the section will be spent constructing such a stratified bundle and proving it satisfies the proposition.

As x is a global coordinate of X relative to S, it follows that

$$\mathscr{D}_{X/S} = \mathscr{O}_X[\partial_x^{(k)} \mid k \in \mathbb{N}_{>0}].$$

Moreover any vector bundle is free over X. We are hence in the assumptions of Section 1.3, hence as explained there a $\mathscr{D}_{X/S}$-structure on any E vector bundle over X of rank r is given by $r \times r$ matrices $A_k = (a_{ij}^k)$, $k \in \mathbb{N}_{>0}$, with values in $H^0(X, \mathscr{O}_X)$, satisfying the relations between the $\partial_x^{(k)}$.

In order to construct our example, let us fix a bijection $n \mapsto a_n$ between the natural numbers and $F = S(F)$. Let $\mathbb{E} \in \mathrm{Strat}(X/S)$ be the rank-two relative stratified bundle $E = \mathscr{O}_X \cdot e_1 \oplus \mathscr{O}_x \cdot e_2$ with $\mathscr{D}_{X/S}$-action given by $\partial_x^{(k)}(e_1) = 0$ and

$$\partial_x^{(k)}(e_2) = \begin{cases} \prod_{i=0}^{h}(y-a_i)\cdot e_1 & \text{if } k = p^h, \\ 0 & \text{else.} \end{cases} \tag{3.1}$$

Equivalently, the matrices $A_k = (a_{ij}^k)$ are 2×2 strictly upper triangular matrices, and the only nonzero entry is a_{12}^k which is $\prod_{i=0}^{h}(y-a_i)$ if $k = p^h$ and zero otherwise. In particular along the closed fiber over a_n the matrices A_k are zero for $k > p^{n-1}$.

In order to prove Proposition 3.2.1 we need to show that this actually defines an action of $\mathscr{D}_{X/S}$ over E and that $\mathbb{E}$ satisfies the two properties of the proposition, namely that it is trivial on every closed fiber and not isotrivial on the geometric generic fiber.

Lemma 3.2.2. *The formulae in* (3.1) *define a $\mathscr{D}_{X/S}$-module structure on E.*

Proof. As we already noticed, as we fixed the action of the generators of $\mathscr{D}_{X/S}$, for it to extend to a $\mathscr{D}_{X/S}$-action we only need to check that the relations of the generators in the ring of differential operators are satisfied by their images in $\mathrm{End}_{\mathscr{O}_X}(E)$. By [Bav10, Cor. 2.5] the only relations are

$$[\partial_x^{(l)}, \partial_x^{(k)}] = 0$$

$$\partial_x^{(k)} \circ \partial_x^{(l)} = \binom{k+l}{k}\partial_x^{(k+l)}$$

$$[\partial_x^{(k)}, x] = \partial_x^{(k-1)}.$$

Let us begin with the second relation: for $k, l > 0$

$$\partial_x^{(k)} \circ \partial_x^{(l)}(e_1) = 0$$

$$\partial_x^{(k)} \circ \partial_x^{(l)}(e_2) = \begin{cases} \partial_x^{(k)}(\prod_{i=0}^{h}(y-a_i)\cdot e_1) = 0 & \text{if } l = p^h \\ 0 & \text{else} \end{cases}$$

Hence, we just need to verify that if $k + l = p^h$ then $\binom{k+l}{k} = 0$ but this holds by Lucas's theorem and the first relation follows immediately. Moreover by (1.1) we have

$$\partial_x^{(k)} \cdot x(e_i) = \partial_x^{(k)}(x \cdot e_i) = \sum_{\substack{a+b=k \\ a,b\geq 0}} \partial_x^{(a)}(x)\partial_x^{(b)}(e_i) = x\partial_x^{(k)}(e_i) + \partial_x^{(k-1)}(e_i);$$

hence, the third relation trivially holds. ∎

In order to prove that $\mathbb{E}_s$ is trivial for every closed fiber, let us fix $n \in \mathbb{N}$ and let $s = a_n \in S(F)$, that is, $X_s = \{y = a_n\} \subset X$. Let us consider the basis change on $\mathscr{O}_{X_s} \cdot e_1 \oplus \mathscr{O}_{X_s} \cdot e_2 = \mathbb{E}_s$ given by $e_1' = e_1$ and

$$e_2' = e_2 - \Big[(y - a_0)x + (y - a_0)(y - a_1)x^p + \cdots + \big[\prod_{i=0}^{n-1}(y - a_i)\big]x^{p^{n-1}}\Big] \cdot e_1$$

then by (1.3) in this new basis the action of $\mathscr{D}_{X_s/k(s)}$ is given by $\partial_x^{(k)}(e_1') = \partial_x^{(k)}(e_2') = 0$ hence is the trivial action.

We are now left to prove that $\mathbb{E}_{\bar{\eta}}$ is not isotrivial:

Lemma 3.2.3. *Let $\mathbb{E} \in \mathrm{Strat}(X/S)$ be the stratified bundle defined by (3.1), then $\mathbb{E}_{\bar{\eta}}$ is not isotrivial.*

Proof. In order to prove that $\mathbb{E}_{\bar{\eta}}$ is not isotrivial it suffices by Lemma 2.2.6 to show that it cannot be defined over any F' of finite type over $\mathbb{F}_p$. Remark that $\mathbb{E}_{\bar{\eta}}$ is a stratified bundle over $\mathbb{A}^1_{\bar{\eta}}$ and the latter is obviously coming by base change from $\mathbb{A}^1_{F'}$ for every $F' \subset F$. By the way of contradiction assume then that there exists F' of finite type over $\mathbb{F}_p$ such that $\mathbb{E}_{\bar{\eta}}$ is defined over F' and let $\mathbb{E}'$ be its descent over $\mathbb{A}^1_{F'}$. This means that there is a basis e_1', e_2' of $\mathbb{E}_{\bar{\eta}}$ such that the matrices A_k' in this new basis take values in $F'[x] = H^0(\mathbb{A}^1_{F'}, \mathscr{O}_X)$.

Let $U \in H^0(\mathbb{A}^1_{\bar{\eta}}, GL_2)$ be the basis change matrix between e_i and e_i', then U is defined over some F'' of finite type over F', hence over $\mathbb{F}_p$. Hence, by (1.3) we have that $\prod_{i=0}^h (y - a_i) \in F''[x]$. In particular, if we denote by $\mathcal{A} = \mathbb{F}_p[\prod_{i=0}^h (y - a_i) \mid h \in \mathbb{N}]$, our assumption implies that $\mathcal{A} \subset F''[x]$.

To see that this leads to a contradiction it suffices to show that $\mathcal{K} \nsubseteq F''(x)$ where $\mathcal{K}$ is the quotient field of $\mathcal{A}$. Note that $F \subset \mathcal{K}$; therefore, it is enough to prove that for every F' of finite type over $\mathbb{F}_p$ we have that $F \nsubseteq F'(x)$. As F is algebraically closed, then $\bar{\mathbb{F}}_p \subset F$ and it is sufficient to show $\bar{\mathbb{F}}_p \nsubseteq F'(x)$, which follows from the following:

Claim. Let $\mathbb{F}_q$ be a finite field with $q = p^n$ for some $n \in \mathbb{N}$ and let $F \supset \mathbb{F}_q$ an algebraic extension such that $[F : \mathbb{F}_q] = +\infty$. Then for every $m \in \mathbb{N}$ and every $\varepsilon_1, \dots, \varepsilon_m$ non-algebraic over $\mathbb{F}_q$ we have that

$$F \nsubseteq \mathbb{F}_q(\varepsilon_1, \dots, \varepsilon_m).$$

Proof. By induction on m, the case $m = 0$ being evident. Let $m = 1$, and $\gamma \in F - \mathbb{F}_q$ and let $\mu_\gamma(t)$ its minimal polynomial over $\mathbb{F}_q$. By way of contradiction assume $\gamma \in \mathbb{F}_q(\varepsilon_1)$; then $\gamma = f(\varepsilon_1)/g(\varepsilon_1)$ and

$$g(\varepsilon_1)^{\deg \mu_\gamma} \cdot \mu_\gamma\Big(\frac{f(\varepsilon_1)}{g(\varepsilon_1)}\Big) = 0$$

gives an algebraic dependence of ε_1 over $\mathbb{F}_q$ which is a contradiction with our assumption that ε_1 is not algebraic over $\mathbb{F}_q$. Let now $m \geq 1$, by induction step we know that for every $n \in \mathbb{N}$, $q = p^n$, then no infinite algebraic extension of $\mathbb{F}_q$ is contained in $\mathbb{F}_q(\varepsilon_1, \dots, \varepsilon_{m-1})$; hence, there exists an r such that $\mathbb{F}_{q^r} = \mathbb{F}_q(\varepsilon_1, \dots, \varepsilon_{m-1}) \cap F$. Then

$$\mathbb{F}_q(\varepsilon_1, \dots, \varepsilon_{m-1})(\varepsilon_m) \cap F \subset \mathbb{F}_{q^r}(\varepsilon_m) \neq F$$

by the $m = 1$ step applied to $q = p^{rn}$. In particular, $F \nsubseteq \mathbb{F}_q(\varepsilon_1, \dots, \varepsilon_m)$. □

Note that if F' is of finite type over $\mathbb{F}_p$ then F' can be always be written as $\mathbb{F}_q(\varepsilon_1, \dots, \varepsilon_m)$ for some $q = p^n$ and ε_i non-algebraic over $\mathbb{F}_q$; hence, $\bar{\mathbb{F}}_p \nsubseteq F'(x)$. Therefore, $\mathbb{E}$ cannot be defined over any F' of finite type over $\mathbb{F}_p$ and by Lemma 2.2.6 it cannot be finite. □

Remark 3.2.4. Let us observe that if F is uncountable the same construction provides an example of a relative stratified bundle $\mathbb{E} \in \mathrm{Strat}(\mathbb{A}^2_F/\mathbb{A}^1_F)$ and a dense subset $\tilde{S} \subset \mathbb{A}^1_F(F)$ such that $\mathbb{E}_s$ is trivial for every $s \in \tilde{S}$ but $\mathbb{E}_{\bar{\eta}}$ is not isotrivial. Therefore, the density condition on $\tilde{S}$ of Theorem 3.1.1 will not be sufficient for our purposes, in parallel with the similar problem that one encounters in the equicharacteristic zero case (see [And04, Rmk. 7.2.3]).

3.3 The main theorem

From the example in previous section it appears that in the case where X is not projective over S the situation is significantly different from the one in Theorem 3.1.1. In the latter one big obstruction for the theorem to hold was related to p dividing the order of the monodromy group on the closed fibers. In the counterexample of Section 3.2 these are trivial and the obstruction seems more related to the cardinality of F. As noticed in Chapter 2, as we are assuming F to be algebraically closed, the monodromy group does not depend (up to a non-unique isomorphism) on the choice of $x \in X$. Therefore, in this section we will denote the monodromy group of a stratified bundle $\mathbb{E}$ simply by $\pi(\mathbb{E})$.

In order to phrase the statement of the main theorem let us introduce the following notation: we will denote by $(X, S; \mathbb{E})$ (and call it *a triple over* F) any triple consisting of $X \to S$ smooth morphism of F-varieties with geometrically connected fibers and $\mathbb{E} \in \mathrm{Strat}(X/S)$. We denote furthermore by $F' = F'(X, S; \mathbb{E})$ a (minimal) algebraically closed subfield of F such that $(X, S; \mathbb{E})$ is defined over F', and by $(X', S'; \mathbb{E}')$ the descent of the triple $(X, S; \mathbb{E})$ to F'. Then the following result holds

Theorem 3.3.1. *Let $(X, S; \mathbb{E})$ be a triple over F, let $F' = F'(X, S; \mathbb{E}) \subset F$ and $(X', S'; \mathbb{E}')$ the descent of the triple to F'. Let $k(S')$ be the function field of S'. Let us assume:*

$$\exists \quad i : k(S') \hookrightarrow F \text{ extending } F' \subset F. \qquad (*)$$

If for every $s \in S(F)$ we have that $\mathbb{E}_s$ is finite, then so is $\mathbb{E}_{\bar{\eta}}$. More specifically there exists $s \in S(F)$ such that

$$\pi(\mathbb{E}_s)(F) \simeq \pi(\mathbb{E}_{\bar{\eta}})(\overline{k(S)})$$

in particular:

i) $|\pi(\mathbb{E}_s)|$ is bounded over $S(F)$;

ii) if $p \nmid |\pi(\mathbb{E}_s)|$ for every $s \in S(F)$ then $p \nmid |\pi(\mathbb{E}_{\bar{\eta}})|$;

iii) any group property holding for $\pi(\mathbb{E}_s)$ for every $s \in S(F)$ holds for $\pi(\mathbb{E}_{\bar{\eta}})$.

Proof. The inclusion in $(*)$ allows us to consider $X_s \to \operatorname{Spec} F$ both as a closed fiber of $X \to S$ and as a geometric generic one of $X' \to S'$ and then to conclude using the results of Section 2.3. Let $\Delta : S' \to S' \times_{\operatorname{Spec} F'} S'$ be the diagonal morphism and $i : k(S') \hookrightarrow F$ be an immersion as in $(*)$. Then the base change along

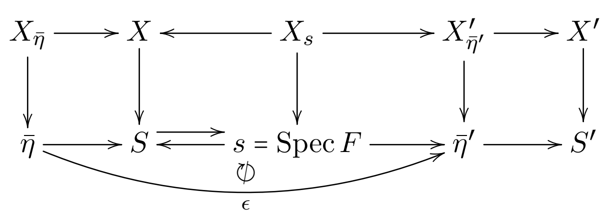
$$\operatorname{Spec} F \xrightarrow{i} \operatorname{Spec} k(S') \longrightarrow \operatorname{Spec} S'$$

induces $(s : \operatorname{Spec} F \to S) \in S(F)$ such that $X' \otimes_{k(S')} F \simeq X_s$ and

$$i^* \mathbb{E}' = \mathbb{E}_s,$$

where we are considering $i : k(S') \hookrightarrow F$ as a geometric generic point of S'. In particular, $\pi(i^*\mathbb{E}') = \pi(\mathbb{E}_s)$. Let $\bar{\eta}'$ be a generic geometric point of S' given by the algebraic closure of $k(S')$ into $\overline{k(S)}$. By 2.3.1, $\pi(\mathbb{E}_s) = \pi(i^*\mathbb{E}') = \pi(\mathbb{E}'_{\bar{\eta}'}) \otimes_{\overline{k(S')}} F$ and $\pi(\mathbb{E}_{\bar{\eta}}) = \pi(\mathbb{E}'_{\bar{\eta}'} \otimes_{\overline{k(S')}} \overline{k(S)}) = \pi(\mathbb{E}'_{\bar{\eta}'}) \otimes_{\overline{k(S')}} \overline{k(S)}$. In particular as abstract groups they all agree and hence $\pi(\mathbb{E}_s)(F) \simeq \pi(\mathbb{E}_{\bar{\eta}})(\overline{k(S)})$, and the rest of the theorem follows. □

Remark 3.3.2. One can picture the previous proof via the following diagram



with the caveat that the composition $\bar{\eta} \to S \to \operatorname{Spec} F \to \bar{\eta}'$ on the bottom row is *not* coming from the natural inclusion $\epsilon : \overline{k(S')} \subset \overline{k(S)}$. All the squares are Cartesian and $X_s \to s$ is, with respect to the left hand side of the diagram, a closed fiber while, with respect to the right hand side of the diagram, it is a geometric generic fiber (notice that the inclusion $i : k(S') \hookrightarrow F$ extends naturally to an inclusion of $\overline{k(S')}$). So we can use the base change properties of the monodromy group to go from properties of $\mathbb{E}_s$ to the ones of $\mathbb{E}'_{\bar{\eta}'}$ and then, by base change via ϵ, to the ones of $\mathbb{E}_{\bar{\eta}}$.

Remark 3.3.3. the theorem holds also asking for the finiteness property to hold on all closed points of some open $\tilde{S}$ if this open is defined over any field extension of F' such that $(*)$ still holds.

The proof actually tell us that the theorem is true for more general properties of the monodromy group than finiteness:

Corollary 3.3.4. *Let $(X, S; \mathbb{E})$ be a triple over F, let $F' = F'(X, S; \mathbb{E}) \subset F$ and $(X', S'; \mathbb{E}')$ the descent of the triple to F'. Let $k(S')$ be the function field of S'. Let us assume:*

$$\exists \quad i : k(S') \hookrightarrow F \text{ extending } F' \subset F. \qquad (*)$$

Then there exists a point $s \in S(F)$ such that

$$\pi(\mathbb{E}_s) = \pi(\mathbb{E}'_{\bar{\eta}'}) \otimes_{\overline{k(S')}} F \qquad \pi(\mathbb{E}_{\bar{\eta}}) \pi(\mathbb{E}'_{\bar{\eta}'}) \otimes_{\overline{k(S')}} \overline{k(S')}$$

where on the left hand side the tensor is via the inclusion i and on the right hand side via the natural inclusion. In particular any group scheme property invariant for base change (on algebraically closed fields) that holds for all $\pi(\mathbb{E}_s)$ holds for $\pi(\mathbb{E}_{\bar{\eta}})$.

As promised, the main result of this chapter is the following:

Corollary 3.3.5. *If F is uncountable then the assumption $(*)$, hence the theorem, always holds.*

Proof. Let F be uncountable, then it suffices to show that for any triple $(X, S; \mathbb{E})$ over F there exists $F' = F'(X, S; \mathbb{E})$ and an inclusion $k(S') \subset F$ extending $F' \subset F$. But it is easy to check that a triple $(X, S; \mathbb{E})$ is defined by countably many data; hence, we can choose F' such that it has countable transcendence degree over $\mathbb{F}_p$. As F is uncountable, it has infinite transcendence degree over F'; hence, there always exists $k(S') \subset F$ as in $(*)$. □

Remark 3.3.6. Notice that if the smooth morphism $X \to S$ does not have geometrically connected fibers then we lose the notion of monodromy group on the closed and geometric generic fibers: if X is a F-variety which is not connected and $\mathbb{I}_{X/F}$ is the trivial stratified bundle on X then $\mathrm{End}(\mathbb{I}_{X/F}) \neq F$, hence $\mathrm{Strat}(X/F)$ is not a Tannakian category. Nevertheless, if we do not assume $X \to S$ to have geometrically connected fibers, the same proof shows that if $\mathbb{E}_s$ is finite when restricted to every connected component of X_s, then the same holds for $\mathbb{E}_{\bar{\eta}}$ on every connected component of $X_{\bar{\eta}}$.

Chapter 4

Regular singularity and a refinement of the theorem

Regardless of the example in Section 3.2, there is a way to broaden the theorem in the case where F is countable, making the additional assumption that the stratified bundle is regular singular on the geometric generic fiber. As in last chapter, F will be assumed to be algebraically closed.

Let X be a smooth variety over F and let $(X, \overline{X})$ be a *good partial compactification* of X; that is: $\overline{X}$ is a smooth variety over F such that $X \subset \overline{X}$ is an open sub-scheme and $D = \overline{X} \backslash X$ is a strict normal crossing divisor. Let $\mathscr{D}_{\overline{X}/F}(\log D) \subset \mathscr{D}_{\overline{X}/F}$ the sub-algebra generated by the differential operators that locally fix all powers of the ideal of definition of D. If $\mathcal{U} \subset \overline{X}$ admits global coordinates $x_1, \dots, x_d$ and D is smooth and given by $\{x_1 = 0\}$ then

$$\mathscr{D}_{\overline{X}/F}(\log D)_{|\mathcal{U}} = \mathscr{O}_{\mathcal{U}}\big[x_1^k \partial_{x_1}^{(k)}, \partial_{x_i}^{(k)} \mid i \in \{2, \dots, d\}, k \in \mathbb{N}_{>0}\big].$$

Definition 4.0.7. A stratified bundle $\mathbb{E} \in \mathrm{Strat}(X/F)$ is called $(X, \overline{X})$-*regular singular* if it extends to a locally free $\mathscr{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-module $\overline{\mathbb{E}}$ on $\overline{X}$. It is *regular singular* if it is $(X, \overline{X})$-regular singular for every partial good compactification $(X, \overline{X})$.

Remark 4.0.8. There is a parallel notion of regular singularities in characteristic zero. Despite the fact that isotrivial implies regular singular over the complex numbers, this is not longer true in positive characteristic, due to the existence of wild coverings (for a more precise statement, see [Kin12a, Thm. 1.1]).

For a $(X, \overline{X})$-regular singular stratified bundle $\mathbb{E}$ we have a theory of exponents (see [Gie75, §3]) of $\mathbb{E}$ along D: it is a finite subset $\mathrm{Exp}_D(\mathbb{E}) \subset \mathbb{Z}_p/\mathbb{Z}$ given by the following:

Proposition 4.0.9. [Gie75, Lemma 3.8],[Kin12a, Prop. 4.12] *Let $\overline{X} = \operatorname{Spec} A$ be a smooth variety over $F = \bar{F}$ with global coordinates $x_1, \dots, x_d$ and let D be the smooth divisor defined by $\{x_1 = 0\}$. Let $\mathbb{E} \in \operatorname{Strat}(X/F)$ a $(X, \overline{X})$-regular singular stratified bundle and $\overline{\mathbb{E}}$ a locally free $\mathscr{D}_{\overline{X}/F}(\log D)$-module extending $\mathbb{E}$. Then there exists a decomposition of $\overline{\mathbb{E}}_{|D} = \bigoplus F_\alpha$ with $\alpha \in \mathbb{Z}_p$ such that $x_1^k \partial_{x_1}^{(k)}$ acts on F_α by multiplication by $\binom{\alpha}{k}$. The image in $\mathbb{Z}_p/\mathbb{Z}$ of the $\alpha \in \mathbb{Z}_p$ such that $F_\alpha \neq 0$ are called the* exponents *of $\mathbb{E}$ along D and do not depend on the choice of $\overline{\mathbb{E}}$.*

If D is not smooth $\mathrm{Exp}_D(\mathbb{E})$ is defined to be the union of the exponents along all the irreducible components of D. By [Kin12a, Cor. 5.4] $\mathbb{E}$ extends to a stratified bundle $\overline{\mathbb{E}}$ on $\overline{X}$ if and only if its exponents are zero. In particular, [Kin12a, Prop. 4.11] implies that if $\mathbb{E}$ is finite then its exponents are torsion. Moreover:

Lemma 4.0.10. *Let $\mathbb{E}$ be a $\mathscr{D}_{X/S}$-module such that $\mathbb{E}_s$ is finite for every $s \in \tilde{S}$ a dense subset of $S(F)$. If $\mathbb{E}_{\bar{\eta}}$ is regular singular then the exponents of $\mathbb{E}_{\bar{\eta}}$ with respect to any partial good compactification of $X_{\bar{\eta}}$ are torsion.*

Proof. Let us fix $(X_{\bar{\eta}}, \overline{X}_{\bar{\eta}})$ a partial good compactification and let $D_{\bar{\eta}} = \overline{X}_{\bar{\eta}} \backslash X_{\bar{\eta}}$. As the exponents can be checked locally, we can shrink $\overline{X}_{\bar{\eta}}$ around the generic point of one of the irreducible components of $D_{\bar{\eta}}$ at a time. Moreover in order to prove the lemma we are allowed to take a generically finite étale open S' of S and substitute X by $X \times_S S'$ (and $\tilde{S}$ by its preimage) as the geometric generic fiber is the same. Finally for every $s \in S(F)$ we have that X'_s is either empty or a finite union of copies of X_s; hence, we will still denote by s any point $s' \in S'$ lying over it.

Hence, without loss of generality, we can assume that we are in the following situation: the partial good compactification $(X_{\bar{\eta}}, \overline{X}_{\bar{\eta}})$ is the restriction of a relative good partial compactification $(X, \overline{X})$ defined on the whole S, $\overline{X}$ is the spectrum of a ring A, with global relative coordinates $x_1, \dots, x_d$ over S and finally $D = \overline{X} \backslash X$ is defined by $\{x_1 = 0\}$. Moreover we can assume that $\mathbb{E}$ is globally free and that on the geometric generic fiber $\mathbb{E}_{\bar{\eta}}$ extends to a globally free $\mathscr{D}_{X_{\bar{\eta}}/\overline{k(S)}}$-module $\overline{\mathbb{E}}_{\bar{\eta}}$.

Let $s \in \tilde{S}$ be any point such that $X_s \cap D \neq \varnothing$, and let us consider the globally free $\mathscr{O}_X$-module $\bar{E} = \mathscr{O}_{\overline{X}} \bar{e}_1 \oplus \cdots \oplus \mathscr{O}_{\overline{X}} \bar{e}_r$. Then the $\bar{e}_i$ induce a basis on the restriction of $\overline{E}$ to the closed fiber over s (as well as to the geometric generic one) and to the boundary divisor (as well as to its complement) as in the following commutative diagram:

$$\begin{array}{ccccc}
E_s = \bigoplus_{i=1}^r \mathscr{O}_{X_s} e_i^s & \xleftarrow{\otimes k(s)} & E = \bigoplus_{i=1}^r \mathscr{O}_X e_i & \xrightarrow{\otimes \overline{k(S)}} & E_{\bar\eta} = \bigoplus_{i=1}^r \mathscr{O}_{X_{\bar\eta}} \varepsilon_i \\
\uparrow{\scriptstyle |X_s} & & \uparrow{\scriptstyle |X} & & \uparrow{\scriptstyle |X_{\bar\eta}} \\
\overline{E}_s = \bigoplus_{i=1}^r \mathscr{O}_{\overline{X}_s} \bar e_i^s & \xleftarrow{\otimes k(s)} & \overline{E} = \bigoplus_{i=1}^r \mathscr{O}_{\overline{X}} \bar e_i & \xrightarrow{\otimes \overline{k(S)}} & \overline{E}_{\bar\eta} = \bigoplus_{i=1}^r \mathscr{O}_{\overline{X}_{\bar\eta}} \bar\varepsilon_i \\
\downarrow{\scriptstyle |D_s} & & \downarrow{\scriptstyle |D} & & \downarrow{\scriptstyle |D_{\bar\eta}} \\
\overline{E}_{|D_s} = \bigoplus_{i=1}^r \mathscr{O}_{D_s} \tilde e_i^s & \xleftarrow{\otimes k(s)} & \overline{E}_{|D} = \bigoplus_{i=1}^r \mathscr{O}_D \tilde e_i & \xrightarrow{\otimes \overline{k(S)}} & \overline{E}_{|D_{\bar\eta}} = \bigoplus_{i=1}^r \mathscr{O}_{D_{\bar\eta}} \tilde\varepsilon_i .
\end{array}$$

Consider the first line of the diagram: on the first (respectively second and third) column there is an action of $\mathscr{D}_{X_s/k(s)}$ (respectively $\mathscr{D}_{X/S}$ and $\mathscr{D}_{X_{\bar\eta}/\overline{k(S)}}$), compatible with each other. On the last column this action extends to a logarithmic action on $\overline{E}_{\bar\eta}$ that we want to extend compatibly to $\overline{E}$.

Similarly as in Section 3.2, let $A_{i,k}$ be the matrices describing the action of $\partial_{x_i}^{(k)} \in \mathscr{D}_{X/S}$ in the basis e_i, then the same ones describe the action of $\partial_{x_i}^{(k)} \in \mathscr{D}_{X_{\bar\eta}/\overline{k(S)}}$ in the basis ε_i. By regular singularity of $\mathbb{E}_{\bar\eta}$ this action extends to a $\mathscr{D}_{\overline{X}_{\bar\eta}/\overline{k(S)}}(\log D_{\bar\eta})$-action. Therefore, there is a second basis $\varepsilon'_1, \dots, \varepsilon'_d$ on the geometric generic fiber such that in the new basis the matrices $A'_{i,k}$ have no poles in x_1 for $i \neq 1$ and logarithmic poles for $i = 1$. Let $U \in \mathrm{H}^0(X_{\bar\eta}, \mathrm{GL}_r)$ the basis change matrix from ε_i to ε'_i. Taking a generically finite étale open of S we can assume that U is defined on the whole S; hence, the $A'_{i,k}$ are defined over the whole S as well and this defines an action of

$$\mathscr{D}_{X/S}(\log D) \doteq \mathscr{O}_X\left[x_1^k \partial_{x_1}^{(k)}, \partial_{x_i}^{(k)} \mid i \in \{2, \dots, d\}, k \in \mathbb{N}_{>0}\right]$$

on $\overline{E}$, compatible with the logarithmic action on the fibers over $\bar\eta$. In particular, this induces a $\mathscr{D}_{\overline{X}_s}(\log D_s)$-action on $\overline{E}_s$; hence, $\mathbb{E}_s$ is $(X_s, \overline{X}_s)$-regular singular (notice that if S' is an étale open of S then for $s \in S(F)$ the fiber X'_s of $X' = X \times_S S'$ is either empty or the disjoint union of finitely many copies of X_s).

We want now to compare $\mathrm{Exp}_{D_{\bar\eta}}(\mathbb{E}_{\bar\eta})$ and $\mathrm{Exp}_{D_s}(\mathbb{E}_s)$. By Proposition 4.0.9 we have that $\overline{\mathbb{E}}_{\bar\eta|D_{\bar\eta}} = \oplus F_\alpha$; hence, there exists $\tilde\varepsilon_i$ a basis of $\overline{E}_{|D_{\bar\eta}}$ such that the matrices $\tilde B_k$ defining the action of $x_1^k \partial_{(k),x_1}$ are diagonal with values $\binom{\alpha}{k} \in \mathbb{F}_p$. Let $\bar\varepsilon_i$ be a lift of $\tilde\varepsilon_i$, then up to taking an étale generically finite open of S we can assume that $\bar\varepsilon_i$ is a restriction of a basis $\bar e_i$ of $\overline{E}$ over

$\overline{X}$. In particular, the decomposition extends as well and $\overline{\mathbb{E}}_{|D} = \oplus F_\alpha$ induces a decomposition on $\overline{\mathbb{E}}_{s|_{D_s}}$. This decomposition must coincide with the one given by Proposition 4.0.9; hence, the exponents must be the same of the ones of $\mathbb{E}_{\bar\eta}$. As $\mathbb{E}_s$ is isotrivial, its exponents are torsion; hence, so must be the ones of $\mathbb{E}_{\bar\eta}$. □

Remark 4.0.11. While the previous proof shows that if $\mathbb{E}_{\bar\eta}$ is regular singular so are the $\mathbb{E}_s$ for every $s \in S(F)$, the example in Section 3.2, together with Theorem 4.0.13, shows that the converse does not hold in general (however, one can prove it is the case when F is uncountable). On the contrary, in characteristic zero it is always true that if a relative flat connection is regular with respect of some smooth good compactification on the fibers over a dense set of points of S, then it is regular on the geometric generic fiber, as proven in [And04, Lemma 8.1.1].

Before stating and proving the main theorem of this section we need to prove the existence of Kawamata coverings in positive characteristic. Analogously to the original construction in characteristic zero ([Kaw81, Thm. 17]) we have the following

Theorem 4.0.12. *Let X be a projective smooth variety of dimension d over an algebraically closed field F of characteristic p and let D be a simple normal crossing divisor on X. Let $m \in \mathbb{N}$ prime to p, then there exist a projective smooth variety Y and a finite surjective mapping $f : Y \to X$ such that $(f^*D)_{red}$ is a simple normal crossing divisor on Y and if $f^*D = \sum m_i\tilde{D}_i$ is the decomposition in irreducible components with $\tilde{D}_i \neq \tilde{D}_j$ for $i \neq j$ then $m \mid m_i$ for all i and m_i are all prime to p.*

Proof. The proof follows the one of the original theorem ([Kaw81, Thm. 17]). Let $D = \sum_{i=1}^t D_i$ the decomposition in irreducible components of D, we will construct Y in t steps $Y = Y_t \to \cdots \to Y_1 \to X$ taking care of one of the components of D at a time. Let us start with D_1: as X is projective there exists an ample line bundle $\mathcal{M}$ on X and $N \gg 0$ such that $N\mathcal{M} - D_1$ is very ample, moreover we can choose N so that $m \mid N$ and $(N, p) = 1$. Fix a presentation $\mathcal{M} = (\mathcal{U}_s, a_s)$, let $a_{st} = a_s/a_t \in \Gamma(\mathcal{U}_s \cap \mathcal{U}_t, \mathcal{O}_X^*)$. Then we claim that we can choose

i) $H_1, \dots, H_d \in |N\mathcal{M} - D_1|$ general elements such that $\sum H_i + D$ is a simple normal crossing divisor;

ii) $\phi_{i,s}$ local equations in $\mathcal{U}_s$ of $D_1 + H_i$ such that $\phi_{i,s} = a_{st}^N \phi_{i,t}$.

For point (i), $N\mathcal{M} - D_1$ is very ample; hence, so is its restriction to any closed sub-scheme of X. A very ample line bundle defines a closed immersion in the projective space; hence, by [Kle74, Cor. 12], any general member it is regular, hence smooth. In particular, we can choose H_1 to be smooth and to intersect smoothly D, and recursively choose the H_i so that $\sum H_i + D$ is a normal crossing divisor. As for (ii), let $\phi_{i,s}$ any local equations for $D_1 + H_i$, then we have that $\mathscr{O}_X(D_1 + H_i) = N\mathcal{M}$; hence, $\phi_{i,s}/\phi_{i,t} = a_{st}^N \psi_{i,s}/\psi_{i,t}$ with $\psi_{i,s} \in \mathscr{O}_X(\mathcal{U}_s)^*$. Therefore, it suffices to replace $\phi_{i,s}$ with $\phi_{i,s}\psi_{i,s}$.

Now notice that by (ii) the field extension of $k(X)$ given by

$$L_s = k(X)(\phi_{1,s}^{1/N}, \dots, \phi_{d,s}^{1/N})$$

does not depend on s; hence, we can consider Y_1 the normalization of X in $L = L_s$.

To show that Y_1 is regular, it enough to show it fiberwise; for every $x \in X$ let $B_x = \mathscr{O}_{X,x} \times_X Y_1$. It is a semi-local ring, more precisely the normalization of $\mathscr{O}_{X,x}$ in L. Let $x \in \mathcal{U}_s$ and let us denote from now on $\phi_i = \phi_{i,s}$. Clearly $\mathscr{O}_{X,x} \subset B'_x = \mathscr{O}_{X,x}[\phi_1^{1/N}, \dots, \phi_d^{1/N}] \subset B_x$, moreover as $L = Frac(B'_x)$ then B_x is also the normalization of B'_x. Now, as $\sum H_i + D_1$ is a simple normal crossing divisor then $\bigcap_{i=1}^d H_i \cap D_1 = \varnothing$ as in every point there can be at most d irreducible components intersecting. Hence, for every point $x \in X$, if $x \in D_1$ then $x \notin H_i$ for some i, so one of the two following situations happens:

case 1: $x \notin D_1$, $x \notin H_i$ for $i = 1, \dots, l$ and $x \in H_i$ for $i = l+1, \dots, d$, with l possibly zero. Then ϕ_i are units in $\mathscr{O}_{X,x}$ for $i = 1, \dots, l$ and part of a regular system of parameters for $i = l+1, \dots, d$. Therefore, by the [GM71, Lemma 1.8.6] B'_x is a semi-local regular ring, hence normal, hence equal to B_x which is then a semi-local regular ring.

case 2: $x \in D_1$, $x \notin H_i$ for $i = 1, \dots l$ and $x \in H_i$ for $i = l+1, \dots, d$ but this time l is at least 1. Then ϕ_i/ϕ_1 are units in $\mathscr{O}_{X,x}$ for $i = 1, \dots, l$, while ϕ_i/ϕ_1 are, together with ϕ_1, part of a regular system of parameters of $\mathscr{O}_{X,x}$ which is, up to multiplication by units, the one coming from the local equation defining $H_{l+1}, \dots, H_d, D_1$. Remark that in this case

$$B'_x \subset B''_x = \mathscr{O}_{X,x}[\phi_1^{1/N}, \phi_2^{1/N}/\phi_1^{1/N}, \dots, \phi_d^{1/N}/\phi_d^{1/N}] \subset B_x$$

and again by [GM71, Lemma 1.8.6], B''_x is a semi-local regular ring, hence normal, hence equal to B_x which is again a semi-local regular ring.

To verify that the $(f_1^* D_i)_{red}$ are nonsingular and cross normally it suffices to apply the same argument to the D_i instead of X, and $f_1 : Y_1 \to X$ clearly ramifies along D_1 with index N; hence, $f^* D_1 = N(f^* D_1)_{red}$. We can now apply the same construction to Y_1 and $(f_1^* D_2)_{red}$ obtaining Y_2 and by recurrence we get $Y \to X$ which has the required properties. □

We can now state and prove the following

Theorem 4.0.13. *Let $X \to S$ be a smooth morphism of F-varieties with geometrical connected fibers and let $\mathbb{E} \in \mathrm{Strat}(X/S)$. Assume that there exists a dense subset $\tilde{S} \subset S(F)$ such that, for every $s \in \tilde{S}$, the stratified bundle $\mathbb{E}_s$ has finite monodromy and that the highest power of p dividing $|\pi(\mathbb{E}_s)|$ is bounded over $\tilde{S}$. Assume moreover that $\mathbb{E}_{\bar{\eta}}$ is regular singular, then*

i) there exists $f_{\bar{\eta}} : Y_{\bar{\eta}} \to X_{\bar{\eta}}$ a finite étale cover such that $f^* \mathbb{E}_{\bar{\eta}}$ decomposes as direct sum of stratified line bundles;

ii) if $F \neq \bar{\mathbb{F}}_p$ then $\mathbb{E}_{\bar{\eta}}$ is finite.

Proof. Let $\mathcal{U} \subset X$ be a dense open, then by invariance of the monodromy group it is enough to show the theorem for $\mathbb{E}_{|\mathcal{U}}$ moreover it is enough to prove finiteness for its pullback along any finite étale cover. Therefore, we can always work up to generically finite étale covers. Using [dJ96] we can find an alteration generically finite étale $f : X' \to X$ such that X' admits a good projective compactification relative to S. By [Kin12a, Prop. 4.4] the pullback of a regular singular stratified bundle is again regular singular. Hence, without loss of generality, we can assume that X admits a good projective compactification $\overline{X}$ relative to S. We will denote by $D = \overline{X} \backslash X$ the divisor at infinity.

Let $\mathrm{Exp}_D(\mathbb{E}_{\bar{\eta}}) \subset \mathbb{Z}_p/\mathbb{Z}$ be the finite set of exponents of $E_{\bar{\eta}}$ along $D_{\bar{\eta}}$ (as defined in Lemma 4.0.9). As $\mathbb{E}_{\bar{\eta}}$ is regular singular then by Lemma 4.0.10 the exponents of $\mathbb{E}_{\bar{\eta}}$ are torsion; let $m \in \mathbb{N}$ an integer prime to p killing the torsion of $\mathrm{Exp}_D(\mathbb{E}_{\bar{\eta}})$ and let $f : \overline{Y}_{\bar{\eta}} \to \overline{X}_{\bar{\eta}}$ be the Kawamata covering constructed in Theorem 4.0.12: it ramifies on a simple normal crossing divisor $\tilde{D}_{\bar{\eta}}$ containing the divisor at infinity $D_{\bar{\eta}} = \overline{X}_{\bar{\eta}} - X_{\bar{\eta}}$ and it is Kummer on $\overline{X}_{\bar{\eta}} - \tilde{D}_{\bar{\eta}}$. As m divides the ramification order along $D_{\bar{\eta}}$ by [Kin12a, Prop. 4.11] the exponents of the pullback of $\mathbb{E}_{\bar{\eta}}$ along $(f^* D_{\bar{\eta}})_{red}$ are zero; hence, it extends to the whole $Y_{\bar{\eta}}$. Up to taking an étale open of S and using a similar argument as in the proof of Lemma 4.0.10 we can assume that this extension is defined on the whole S. Therefore, we have reduced the problem to Theorem 3.1.3. □

Chapter 5

On isotrival vector bundles

In this last chapter we use similar techniques as in Chapter 3 to obtain results of isotriviality on families of vector bundles. The setup is parallel to the one of the case of stratified bundles, and the proof will follow similar ideas.

5.1 Finite vector bundles

The notion of isotriviality also has relevance in the category of vector bundles over a proper smooth F-variety, even though it is not equivalent to the notion of finiteness (see [Nor76, Lemma 3.1] and following definition) for vector bundles, at least in positive characteristic. In order to prove Theorem 3.1.1, Esnault and Langer proved in the same paper the following:

Theorem 5.1.1. [EL13, Thm. 5.1] *Let $X \to S$ be a smooth projective morphism of F-varieties with geometrically connected fibers and let E be a locally free sheaf over X. Assume that there exists a dense subset $\tilde{S} \subset S(F)$ such that, for every $s \in \tilde{S}$, there is a finite étale Galois cover $h_s : Y_s \to X_s$* of order prime to *p such that $h_s^*(E_s)$ is trivial. Then*

i) there exists $f_{\bar{\eta}} : Y_{\bar{\eta}} \to X_{\bar{\eta}}$ a finite étale cover of order prime to *p such that $f^* E_{\bar{\eta}}$ decomposes as direct sum of line bundles;*

ii) if $F \neq \bar{\mathbb{F}}_p$ then $E_{\bar{\eta}}$ is trivialized by a finite étale cover of order prime to *p.*

Then, a reasoning similar to the proof of Theorem 3.1.3 proves the following:

Theorem 5.1.2. *Let $X \to S$ be a smooth* projective *morphism of F-varieties with geometrically connected fibers and let E be a locally free sheaf over X.*

Assume that there exists a dense subset $\tilde{S} \subset S(F)$ such that, for every $s \in \tilde{S}$, there is a finite étale Galois cover $h_s : Y_s \to X_s$ such that $h_s^*(E_s)$ is trivial and that the highest power of p dividing the order of such covers is bounded over $\tilde{S}$. Then

i) there exists $f_{\bar{\eta}} : Y_{\bar{\eta}} \to X_{\bar{\eta}}$ a finite étale cover such that $f^* E_{\bar{\eta}}$ decomposes as direct sum of line bundles;

ii) if $F \neq \bar{\mathbb{F}}_p$ then $E_{\bar{\eta}}$ is trivialized by a finite étale cover.

Proof. We will reduce this theorem to Theorem 5.1.1. By taking an étale open of S we can assume there exists a section $\sigma : S \to X$. Let r be the rank of E and fix s a closed point in S. As X_s is a smooth $k(s)$-variety, then (see [EL13, Definition 3.2] and following discussion) every étale trivializable vector bundle is Nori semi-stable. In particular, the Galois cover $h_s : Y_s \to X_s$ corresponds to a representation of rank r of the Nori fundamental group scheme $\pi_1^N(X_s, \sigma_s)$ (for the definition of the Nori group scheme see [Nor76]) that factors through the étale fundamental group:

$$\pi_1^N(X_s, \sigma(s)) \twoheadrightarrow \pi_1^{\text{ét}}(X_s, \sigma(s)) \twoheadrightarrow \Gamma_s \subset \mathrm{GL}_r(F),$$

where Γ_s is the Galois group of $h_s : Y_s \to X_s$. The rest of the proof follows exactly as in Theorem 3.1.3. □

If the morphism $X \to S$ is not projective but only smooth the connection with finite bundles is lost. In particular, we do not have a notion of monodromy group of a vector bundle in this case. Nevertheless, we get a similar result to Corollary 3.3.5:

Theorem 5.1.3. *Let F be an algebraically closed field of positive characteristic with infinite transcendental degree over $\mathbb{F}_p$. Let $X \to S$ be a smooth morphism of varieties over F and E a vector bundle over X. Assume that there exists a dense open $\tilde{S} \subset S$ such that E_s is isotrivial for every $s \in \tilde{S}(F)$, then so is $\mathbb{E}_{\bar{\eta}}$.*

Proof. There exists F' a sub-field of F of finite type over $\mathbb{F}_p$ such that $X \to S$ and E descend to $X' \to S'$ and E'. Moreover as F has infinite transcendence degree over $\mathbb{F}_p$ there exists an immersion $k(S') \hookrightarrow F$ over F' and a point $s \in S(F)$, like in the proof of Theorem 3.3.1, such that the morphism $i : \operatorname{Spec} F \to \operatorname{Spec} k(S')$ given by $k(S') \subset F$ is a geometric generic point of S', and on $X' \otimes_{k(S')} F \simeq X_s$

$$i^* E' = E_s.$$

Note that as F has infinite transcendence degree over F', there exists an immersion $\iota : F \hookrightarrow \overline{k(S)}$ (which is not the natural one given by the fact that S is a F-variety) that is the identity on $k(S')$. One can construct this immersion by choosing an (infinite) basis of transcendent elements of F over $\overline{k(S')}$ and sending it injectively to one of $\overline{k(S)}$ (note that the two basis have the same cardinality). Hence, via ι we have that $X_{\bar{\eta}} \simeq X' \otimes_{k(S')} \overline{k(S)} \simeq X_s \otimes_F \overline{k(S)}$. In particular, if we continue to consider F as a sub-field of $\overline{k(S)}$ via the immersion ι, then $h_s \otimes_F \overline{k(S)} : Y_s \times_{\operatorname{Spec} F} \operatorname{Spec} \overline{k(S)} \to X_{\bar{\eta}}$ trivializes $E_{\bar{\eta}}$. $\square$

Bibliography

[And04] Y. André. "Sur la conjecture des p-courbures de Grothendieck-Katz et un problème de Dwork". In: *Geometric aspects of Dwork theory. Vol. I, II.* Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 55–112.

[Bav10] V. V. Bavula. "Generators and defining relations for the ring of differential operators on a smooth affine algebraic variety in prime characteristic". In: *J. Algebra* 323.4 (2010), pp. 1036–1051.

[BF66] R. Brauer and W. Feit. "An analogue of Jordan's theorem in characteristic p". In: *Ann. of Math.* 84 (1966), pp. 119–131.

[BO78] P. Berthelot and A. Ogus. "Notes on crystalline cohomology". In: (1978), pp. vi+243.

[Bou98] N. Bourbaki. *General Topology: Chapters 1-4.* Elements of mathematics. Springer, 1998.

[Del89] P. Deligne. "Le groupe fondamental de la droite projective moins trois points". In: *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987).* Vol. 16. Math. Sci. Res. Inst. Publ. Springer, New York, 1989, pp. 79–297.

[dJ96] A.J. de Jong. "Smoothness, semi-stability and alterations". In: *Publ. Math. IHES* 83 (1996), pp. 51–93.

[DM82] P. Deligne and J. Milne. Tannakian Categories*, in Hodge cycles, motives, and Shimura varieties.* Vol. 900. Lecture Notes in Mathematics. Available at `http://www.jmilne.org/math/xnotes/tc.html`. Springer–Verlag, Berlin–New York, 1982, pp. ii+414.

[EGA4] A. Grothendieck and J. Dieudonné. *Éléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas.* Vol. 32. Institut des Hautes Études Scientifiques. Publications Mathématiques. 1967.

[EL13] H. Esnault and A. Langer. "On a positive equicharacteristic variant of the p-curvature conjecture". In: *Doc. Math.* 18 (2013), pp. 23–50.

[Gie75] D. Gieseker. "Flat vector bundles and the fundamental group in non-zero characteristics". In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* 2.1 (1975), pp. 1–31.

[GM71] A. Grothendieck and J.P. Murre. *The tame fundamental group of a formal neighbourhood of a divisor with normal crossings on a scheme*. Lecture Notes in Mathematics, Vol. 208. Springer-Verlag, Berlin-New York, 1971, pp. viii+133.

[Hru02] E. Hrushovsky. "Computing the Galois group of a linear differential equation". In: *Differential Galois Theory, Banach Center Publications* 58 (2002), 97—138.

[Jan87] J.C. Jantzen. *Representations of algebraic groups*. Vol. 131. Pure and Applied Mathematics. Academic Press Inc., Boston, MA, 1987, pp. xiv+443.

[Kaw81] Y. Kawamata. "Characterization of abelian varieties". In: *Comp. Math.* 43 (1981), pp. 253–276.

[Kin12a] L. Kindler. "Regular singular stratified bundles and tame ramification". to appear in Transaction of the AMS. Oct. 2012.

[Kin12b] L. Kindler. *Regular singular stratified bundles in positive characteristic (Dissertation)*. Available at `www.mi.fu-berlin.de/users/kindler`. 2012.

[Kle74] S.L. Kleiman. "Characterization of abelian varieties". In: *Comp. Math.* 28 (3) (1974), pp. 287–297.

[Las01] Y. Laszlo. "A non-trivial family of bundles fixed by the square of Frobenius". In: *C. R. Acad. Sci. Paris Sér. I Math.* 333.7 (2001), pp. 651–656.

[MFK94] D. Mumford, J. Fogarty, and F.C. Kirwan. *Geometric Invariant Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, 1994. ISBN: 9780387569635.

[Mil12] J.S. Milne. *Basic Theory of Affine Group Schemes*. Available at `www.jmilne.org/math/CourseNotes`. 2012.

[NB13] A. Vistoli N. Borne. "The Nori fundamental gerbe of a fibered category". to appear in Transaction of the AMS. Dec. 2013.

[Nor76] M.V. Nori. "On the representations of the fundamental group". In: *Comp. Math.* 33.1 (1976), pp. 29–41.

[NS07] N. Nikolov and D. Segal. "On finitely generated profinite groups I. Strong completeness and uniform bounds". In: *Ann. of Math.* 165 (2007), pp. 171–238.

[Riv72] N. Saavedra Rivano. *Catégories Tannakiennes*. Lecture Notes in Mathematics, Vol. 265. Springer-Verlag, Berlin-New York, 1972, pp. ii+418.

[San07] J.dos Santos. "Fundamental group schemes for stratified sheaves". In: *J. Algebra* 317 (2) (2007), pp. 691–713.

[Ses67] C. S. Seshadri. "Space of unitary vector bundles on a compact Riemann surface". In: *Ann. of Math. (2)* 85 (1967), pp. 303–336.

[SGA1] *Revêtements étales et groupe fondamental (SGA 1)*. Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3. Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin]. Société Mathématique de France, Paris, 2003, pp. xviii+327.

[SGA3] *Schémas en groupes. I: Propriétés générales des schémas en groupes*. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Math., Vol. 151. Springer-Verlag, Berlin-New York, 1970, pp. xv+564.

I, Giulia Battiston, declare that this thesis titled, "The variation of the monodromy group in families of stratified bundles in positive characteristic" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
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