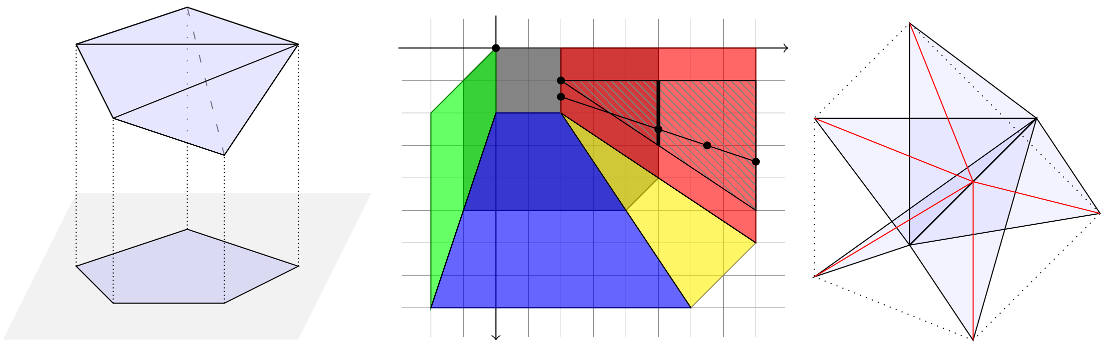
# THREE INTERESTING LATTICE POLYTOPE PROBLEMS

or the story behind these pictures



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# Part I

# INTRODUCTION

# 1 INTRODUCTION

There are many interesting objects in mathematics. But the one that charmed me the most is lattice polytopes. The following three chapters are all based on, or grew out of interesting questions on lattice polytopes. But before we dive into these let's give the objects of our love the introduction they deserve. (If you are not already in love, this will hopefully help to get you excited.) Unless stated otherwise, unproved statements made in this introduction can be found in the introductory book by Ziegler [Zie95] and the upcoming book by Haase, Nill and Paffenholz. [HNP2x] (A preliminary version can be found online.)

But why should you be interested in lattice polytopes? If their inherent beauty is not enough for you, there are plenty more reasons for that. As far as applications go, let me mention two examples. One are reflexive polytopes and their generalisations, Gorenstein polytopes, which correspond to Calabi-Yau varieties in toric geometry, that are used by physicists working on string theory and mirror symmetry. One of the advantages that polytopes bring into the picture, is that they help us classify certain interesting classes; see e.g. [LN15].

Of course a prominent example is linear optimization, where polytopes and, if the problem is unbounded, polyhedra, are the key underlying objects that are studied. If you are interested in integral solutions for your optimization problem, what you are looking at are lattice polytopes and polyhedra. It is worth mentioning that in this case the problem becomes much harder, as the integer linear programming problem is NP-complete [Sch86, Theorem 18.1]. Polytopes also provide a geometric view on the optimization problem, for example if your problem is bounded, the optimal solution is always attained at a vertex and one of the most used algorithms to solve linear optimization problems, the simplex algorithm, works via walking along edges of the respective polytope. It is still an open problem, whether there are rules on how to pick the next edge so that its running time will be polynomial in the input, like in the ellipsoid method and in the interior point method. Polytopes and their properties are used in attempts to solve this question.

Apart from those applications, polytopes have a connection to many other fields inside mathematics, like combinatorics, algebraic geometry, symplectic geometry, statistics and many more.

So without further ado, lets start to get to know them better.

A polytope $P \subset \mathbb{R}^d$ can be defined in two equivalent ways, called the $V$-description and the $H$-description of $P$. The equivalence of both descriptions is known as the *Minkowski-Weyl theorem* or *main theorem for polytopes* [Zie95]. As we will use both, here they are.

**Definition 1.0.1.** A *polytope* $P \subset \mathbb{R}^d$ is the convex hull of finitely many points, i.e. given a set $V = \{v_1, \ldots, v_s\} \subset \mathbb{R}^d$, then

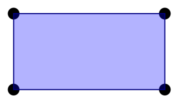
$$P := \operatorname{conv}(V) := \left\{ \sum_{i=1}^{s} \lambda_i v_i \ : \ \sum_{i=1}^{s} \lambda_i = 1 \ \text{ and } \ \lambda_i \geq 0 \right\}.$$

Equivalently, a *polytope* $P \subset \mathbb{R}^d$ is a bounded polyhedron, where a *polyhedron* is defined as the intersection of finitely many halfspaces $H_1^+, \ldots, H_m^+ \subset \mathbb{R}^d$:

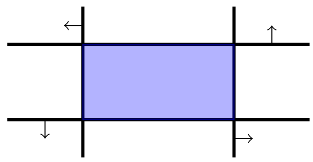
$$P := \bigcap_{i=1}^{m} H_i^+$$

**Example 1.0.2.** Looking for example at Figure 1.1. We depict the two constructions for the same polytope $P \subseteq \mathbb{R}^2$, which is given as

$$P = \operatorname{conv}\begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix} \quad \text{and} \quad P = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : \ \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix} \right\}$$



(a) $V$-description

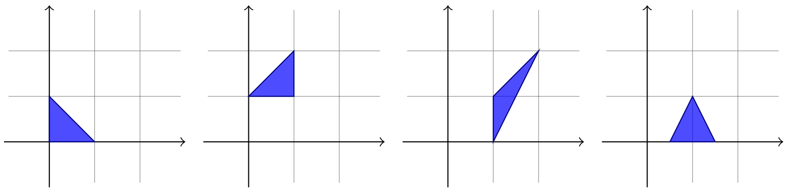


(b) $H$-description

Figure 1.1: Two different ways to define a polytope

The *dimension* of a polytope $P$, $\dim(P)$, is the dimension of its affine span, $\operatorname{aff}(P)$. We often abbreviate $P$ being a $d$-dimensional polytope to $P$ being a $d$-polytope. If $P$ is a $d$-polytope and the convex hull of $d+1$ points, it is called a *simplex* and a special example for one is the *standard simplex* $\Delta_d := \operatorname{conv}(0, e_1, \ldots, e_d)$. A linear inequality $ax \leq b$ is *valid* for $P$, if it is satisfied for all $x \in P$. A *face* $F$ of a polytope $P$, denoted by $F \prec P$ is any subset of $P$, of the form $F = P \cap \{x \ : \ ax = b\}$, where $ax \leq b$ is a valid inequaliy for $P$. If $P$ is a $d$-polytope, then 0-dimensional faces are called *vertices*, 1-dimensional faces are called *edges*, $(d-2)$-dimensional ones *ridges* and $(d-1)$-dimensional ones *facets*. By $P[k]$ we denote the set of $k$-dimensional faces of $P$. Looking back at the two descriptions for polytopes, the first one is called $V$-description, because the set of vertices of $P$, $\operatorname{vert}(P)$, is the unique minimal set of which $P$ is the convex hull of. I.e. for every set $W$, with $P = \operatorname{conv}(W)$ we have $\operatorname{vert}(P) \subseteq W$ and $P = \operatorname{conv}(\operatorname{vert}(P))$. In the second one $H$ stands for hyperplane and similar to the other description we have a unique minimal set, which in this case is coming from hyperplanes that cut out the facets of $P$. One interesting piece of combinatorial data is the number of faces we have in the respective dimensions. Given a $d$-polytope $P$, this is recorded in the *$f$-vector* of $P$, $f(P) = (f_{-1}(P), \ldots, f_d(P))$, with $f_i(P) := |P[i]|$ and setting $\dim(\emptyset) = -1$. Using the empty set might be surprising at first, but it is a face of every polytope, because $0x \leq 1$ is a valid inequality and as a bonus its inclusion also makes certain formulas look nicer. Our focus in this work will be lattice polytopes, where a polytope is called *lattice polytope* if all its vertices lie in a common lattice $\Lambda$. In our case we have $\Lambda \cong \mathbb{Z}^d$ and for us this is not really a restriction, as using a linear map that maps a lattice basis of $\mathbb{Z}^d$ to a basis of an arbitrary lattice $\Lambda$, we can see that properties we study basically remain intact, so that similar statements are also true if we replace $\mathbb{Z}^d$ by any rank $d$ lattice $\Lambda$.

In a lot of cases we do not want to distinguish between polytopes and their translates and also not if we transform them unimodularly. Because of that we call two lattice polytopes $P$ and $Q$ *unimodular equivalent* or just *equivalent* if there is an affine transformation $f$ mapping $P$ to $Q$, that preserves the lattice. Sometimes this is also called $\Lambda$- or $\mathbb{Z}$-equivalence. For our favorite lattice $\mathbb{Z}^d$, this boils down to the existence of an invertible integer matrix $A \in \mathbb{Z}^{dxd}$ and an integer vector $b \in \mathbb{Z}^d$, such that $Q = AP + b$. For example in Figure 1.2 we see some polytopes that are equivalent to the 2-dimensional standard simplex $\Delta_2$ and one that is not.



(a) $\Delta_2$ (b) $A$ (c) $B$ (d) $C$

Figure 1.2: Polytopes $A$ and $B$ are equivalent to $\Delta_2$, $C$ is not.

The simplex $\Delta_2$ is a peculiar example, as it is equivalent to every lattice simplex $S$ such that $\left|S \cap \mathbb{Z}^2\right| = 3$. This stops being the case from dimension 3 onwards.

A lot of interesting properties of a polytope $P$ are preserved by this equivalence, like volume, number of lattice points and the entries of the $f$-vector, in particular all properties that we will investigate further in the coming sections are preserved. Properties that are not preserved are for example angles or the euclidean length of segments or edges. For the latter of the two, there is the notion of *lattice length* of a segment $e$, denoted by $\ell(e)$, which is preserved and which we will use later on. To define it properly let $e$ be the segment connecting two rational points $v, w \in \mathbb{Q}^d$ and let $u$ be the shortest integer vector on the line spanned by $w - v$. Then $e = ku$ for some $k \in \mathbb{Q}$ and $\ell(e) := |k|$. We also consider degenerate segments with $v = w$; in this case we set $\ell(e) = 0$. If on the other hand $e$ is a lattice segment, i.e. $v, w \in \mathbb{Z}^d$, then we can compute $\ell(e)$ by just counting the number of lattice points on $e$ and substracting 1, i.e. $\ell(e) = \left|e \cap \mathbb{Z}^d\right| - 1$.

Another useful object used to study polytopes are fans. To define them we need to look at cones first. A *cone* $C \subseteq \mathbb{R}^d$ of a set $T \subseteq \mathbb{R}^d$ is given by

$$C := \operatorname{cone}(T) := \left\{ \sum_{i=1}^{n} \lambda_i t_i \ : \ \lambda_i \geq 0, \ n \in \mathbb{N}, \ t_i \in T \right\}.$$

If there is a finite set $V$, such that $C = \operatorname{cone}(V)$, then we call $C$ *polyhedral*, as in this case $C$ is also a polyhedron, i.e. the intersection of finitely many halfspaces. The definitions made above for polytopes for the dimension and faces carry over to cones, here the one dimensional faces of a $d$-cone are called *rays* and the $(d-1)$-dimensional ones *facets*. If a $d$-cone $C$ is spanned by $d$ rays it is called *simplicial*. A polyhedral cone $C$ is called *rational*, if there is a finite set $S \subset \mathbb{Q}^d$, spanning $C$. In this case all rays of $C$ are rational cones and given one of these rays $\rho \prec C$, there this a shortest integer vector $v$ contained in $\rho$, which we call *primitive ray generator*. Now given a $d$-cone $C$, if the primitive ray generators of $C$ form a lattice basis of $\mathbb{Z}^d$, then we call $C$ *unimodular*. Note that in this case $C$ has to be simplicial.

A *fan* $\Sigma$ in $\mathbb{R}^d$ is a collection $\Sigma = \{C_1, \dots, C_s\}$ of nonempty polyhedral cones $C_i \subseteq \mathbb{R}^d$, with the following two properties:

1. Every nonempty face of a cone in $\Sigma$ is also a cone in $\Sigma$.
2. The intersection of any two cones in $\Sigma$ is a face of both.

The fan $\Sigma$ is called *complete*, if the so called *support* of $\Sigma$, $|\Sigma| = C_1 \cup \ldots \cup C_s$ covers the entire underlying space, i.e. $|\Sigma| = \mathbb{R}^d$. If all cones in $\Sigma$ are unimodular, we call $\Sigma$ itself *unimodular*. Similar to the polytopal case, $\Sigma[k]$ denotes the set of $k$-dimensional cones in $\Sigma$ and the *$f$-vector* $f(\Sigma)$ records the numbers of the cones in the respective dimensions.

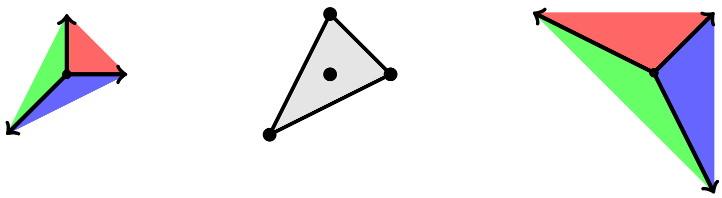
Coming back to polytopes, there are two important fans we can associate with our polytope $P$, the *face fan* $\mathcal{F}(P)$ and the *normal fan* $\mathcal{N}(P)$. Where the face fan of a polytope $P$ with $0 \in \operatorname{relint}(P)$ is given by

$$\mathcal{F}(P) := \{\operatorname{cone}(F) \ : \ F \prec P\}.$$

It is also possible to define the face fan, for polytopes without 0 in the relative interior, if you just use any point $x \in \operatorname{relint}(P)$ and the cones $\operatorname{cone}(F - x)$ to make up $\mathcal{F}(P)$. In this case the face fan is not unique, but there are still many properties that remain unchanged, no matter which point in the relative interior you choose. But as we will only use it in connection with polytopes containing 0 and use lattice properties of the rays, we stick with the first definition. As for the normal fan $\mathcal{N}(P)$, for every nonempty face $F$ of $P$ there exists a linear functional $c_F$, such that $c_F^t x$ is maximal over $P$ if and only if $x \in F$, i.e. $F = P \cap \{x \ : \ c_F x = m\}$, where $m$ is the maximal value $c_F$ achieves on $P$. In this case we also say that $c_F$ defines the face $F$. The set

$$C_F = \left\{ c \ : \ \left\{ z : \max_{x \in P} c^t x = c^t z \right\} \supseteq F \right\}$$

is a polyhedral cone. Then the *normal fan* $\mathcal{N}(P)$ of $P$ is the collection of these cones over all nonempty faces of $P$. The correspondence $F \longleftrightarrow C_F$ is an inclusion reversing bijection, i.e. given two faces $F, F' \prec P$, then $F \subseteq F'$ if and only if $C_{F'} \subseteq C_F$. Both $\mathcal{F}(P)$ and $\mathcal{N}(P)$ are examples for complete fans, but in most cases they will not be unimodular.



(a) $\mathcal{F}(P)$ (b) $P$ (c) $\mathcal{N}(P)$

Figure 1.3: Example of a polytope $P$ with its associated face and normal fan

While there is a unique normal fan and a unique face fan associated to a polytope, it is not a 1 to 1 corresponce between fan and polytope, as for example both the face fan and the normal fan stay the same if we dilate the polytope.

Now that we have covered some of the basic properties of polytopes, in the following three sections we will give three short introductions to the questions and classes of polytopes that will be discussed in more detail in chapters 2,3 and 4, respectively.

## 1.1 CLASSIFYING LATTICE POLYTOPES

When working with lattice polytopes, it is a natural question to ask what kind of lattice polytopes are there and considerable effort has gone into several classification projects for several classes of them, with motivation stemming from different sources. For example:

- A monumental task and now a shining example is the classification of reflexive polytopes up to dimension 4 by Kreuzer and Skarke [KS00], the data for these and other Calabi-Yau manifolds can be found online under `http://hep.itp.tuwien.ac.at/~kreuzer/CY.html`.
- Smooth reflexive polytopes were classified up to dimension 8 by Øbro [Øbr07] and in dimension 9 by Lorenz and Paffenholz [LP08] (see also `https://polymake.org/polytopes/paffenholz/www/fano.html`). This classification led to new discoveries about smooth reflexive polytopes in arbitrary dimension and hereby helped solving long-open problems [AJP14; LN15; NP11; OSY12].
- Lattice polytopes with a single lattice point in their interior (assumed to be the origin) are important in algebraic geometry. They correspond to projective toric

varieties with at most *canonical singularities*, which is why they are called *canonical polytopes*. Canonical polytopes all of whose boundary lattice points are vertices are called *terminal*. Canonical 3-dimensional lattice polytopes were fully enumerated by Kasprzyk [Kas10]. The data for this and a lot more can be found in the graded ring database (http://www.grdb.co.uk).

- Empty simplices, that is, lattice simplices with no lattice points apart from their vertices, are the building blocks into which every lattice polytope can be decomposed, and they correspond to terminal quotient singularities in algebraic geometry. Their classification in dimension three (the so-called "terminal lemma") is by now classical [Whi64]. But in dimension four it is yet not complete, despite efforts coming both from algebraic geometry [Bar+11; Bob09; MMM88; San90] and discrete geometry [HZ00].

- Last but not least, a classification specially useful for us is that of *hollow* lattice polytopes, by which we mean lattice polytopes without interior lattice points. (These include empty simplices, but also other things). In dimension two they consist of the polygons of width one and the second dilation of a unimodular triangle. In dimension three the full classification has recently been completed by Averkov et al. [AWW11] and [AKW17]. See Section 2.5 for details.

All these classifications are modulo unimodular equivalence.

From the point of view of discrete geometry alone, it seems natural to classify, or enumerate, *all* lattice polytopes of a given dimension and with a certain number of lattice points. We call the latter the *size* of a lattice polytope. In dimension 1 this is trivial, since the unique lattice segment of size $n$ is that of length $n-1$. In dimension 2 it is also an easy exercise, since Pick's Theorem implies that there are finitely many different lattice polygons for each size and an enumeration algorithm is straightforward. However in dimension 3 and higher the task is a-priori undoable, since the number is infinite already for the smallest possible case, that of *empty tetrahedra*, i.e. lattice 3-polytopes of size 4. Indeed, the following infinite family of so-called *Reeve tetrahedra* was described 60 years ago by Reeve [Ree57]:

$$T_r := \operatorname{conv}\left\{ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\r\end{pmatrix} \right\}.$$

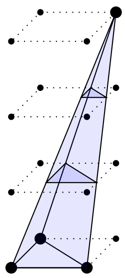


Figure 1.4: The Reeve tetrahedron $T_3$

Still, Blanco and Santos [BS16a] found a way of making sense of the question in dimension 3. They proved that for each size $n$, all but finitely many lattice 3-polytopes have width one and they classified lattice polytopes of width larger than one and of sizes up to eleven [BS16a; BS16b; BS16c].

| # vertices | 4 | 5 | 6 | 7 | 8 | 9 | 10 | total |
|---|---|---|---|---|---|---|---|---|
| size 5 | 9 | 0 | – | – | – | – | – | 9 |
| size 6 | 36 | 40 | 0 | – | – | – | – | 76 |
| size 7 | 103 | 296 | 97 | 0 | – | – | – | 496 |
| size 8 | 193 | 1195 | 1140 | 147 | 0 | – | – | 2675 |
| size 9 | 282 | 2853 | 5920 | 2491 | 152 | 0 | – | 11698 |
| size 10 | 478 | 5985 | 18505 | 16384 | 3575 | 108 | 0 | 45035 |
| size 11 | 619 | 11432 | 48103 | 64256 | 28570 | 3425 | 59 | 156464 |

Table 1.1: Lattice 3-polytopes of width larger than one and size $\leq 11$, classified according to their size and number of vertices.

As a by-product, their result includes a full classification of 3-dimensional *distinct pair sum polytopes*, or dps-polytopes [CLR02; Rez08], since these are known to have size at most $2^d$ in dimension $d$.

Here, the *width* of a lattice polytope $P$ *with respect to a linear functional* $\ell \in (\mathbb{R}^d)^*$ is defined as

$$\text{width}_\ell(P) := \max_{p,q \in P} |\ell \cdot p - \ell \cdot q| \ .$$

The *lattice width*, or simply *width*, of $P$ is the minimum such $\text{width}_\ell(P)$, where $\ell$ ranges over non-zero integer functionals:

$$\text{width}(P) := \min_{\ell \in (\mathbb{Z}^d)^* \setminus \{0\}} \text{width}_\ell(P).$$

For example, $P$ has width one if and only if it lies between two consecutive lattice hyperplanes, as the Reeve tetrahedra mentioned above.

The starting point for our work in chapter 2 is the observation that the finiteness result of Blanco and Santos generalizes as follows:

THEOREM 1.1.1. For each dimension $d$ there is a constant $w \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ the number of lattice $d$-polytopes of size $n$ and width larger than $w$ is finite. Moreover, the minimal such constant satisfies

$$d - 2 \quad \leq \quad w \quad \leq \quad O\left(d^{3/2}\right). \tag{1.1}$$

To prove this, we need the following lemma.

LEMMA 1.1.2. There are only finitely many non-hollow lattice $d$-polytopes of size $n$.

*Proof.* The lemma follows directly from the combination of the following two results: Hensley [Hen83, Thm. 3.6] showed that there is a bound on the volume of non-hollow lattice $d$-polytopes with a given number $k$ of interior points. Taking the maximum of these for $k \in \{1, \ldots, n-d-1\}$ provides a bound for the volume of non-hollow $d$-polytopes of size $n$. Finally, Lagarias and Ziegler [LZ91, Thm. 2] proved finiteness of the number of equivalence classes of lattice $d$-polytopes with a bound on their volume. □

*Proof of Theorem 1.1.1.* The inequality $d - 2 \leq w$ will be proved later in chapter 2 (see part (5) of Theorem 2.1.2 and the comment after it).

For the rest of the statement, observe that hollow polytopes of dimension $d$ have a global bound of $O(d^{3/2})$ for their width [Ban+99, Theorem. 2.4]. With this, the only thing left to prove is that once $d$ and $n$ are fixed there are only finitely many non-hollow lattice $d$-polytopes of size $n$, but that is the exact statement of Lemma 1.1.2. □

Theorem 1.1.1 gives rise to the following defintion, which will introduce the main object of chapter 2.

**Definition 1.1.3.** For each $d \in \mathbb{N}$ we call *finiteness threshold width* in dimension $d$, and denote it $w^\infty(d)$, the minimum constant $w$ such that for every $n \in \mathbb{N}$ the number of lattice $d$-polytopes of size $n$ and width larger than $w$ is finite.

For instance, $w^\infty(1) = w^\infty(2) = 0$ since, as mentioned above, there are only finitely many one or two dimensional lattice polytopes of each size. Blanco and Santos' aforementioned result states that $w^\infty(3) = 1$. In chapter 2 we prove several relations and properties of $w^\infty(d)$ and then use those to prove our main result of chapter 2, the exact value of $w^\infty(4)$:

THEOREM 1.1.4 (Corollary 2.5.5). For each $n \geq 5$ there are only finitely many lattice 4-polytopes of size $n$ and width greater than 2, that is, $w^\infty(4) = 2$.

This result in particular implies the following result:

COROLLARY 1.1.5. There are infinitely many empty 4-simplices of width two but only finitely many of larger width.

*Proof.* Haase and Ziegler [HZ00, Proposition 6] found infinitely many empty 4-simplices of width 2. $w^\infty(4) = 2$ implies there are only finitely many of larger width. □

*Remark* 1.1.6. Corollary 1.1.5 is the main result in Barile et al. [Bar+11], but we have found out that the proof given in that paper is incomplete. More precisely, the authors use a classification of infinite families of empty 4-simplices of width $> 1$, that had been conjectured by Mori et al. [MMM88] and proved by Sankaran [San90] and Bover [Bob09], for simplices whose determinant (i.e., their normalized volume) is a prime number. But when the determinant is not prime other infinite families do arise, such as the following explicit example: the empty 4-simplices with vertices $e_1$, $e_2$, $e_3$, $e_4$ and $(2, N/2 - 1, a, N/2 - a)$, where the determinant $N$ is a multiple of 4 and coprime with $a$. As a conclusion, the proof of Corollary 1.1.5 given in [Bar+11] is valid only for simplices of prime determinant.

After this work was completed a new proof of Corollary 1.1.5 has been obtained which gives the following more explicit information: there are exactly 179 empty 4-simplices of width larger than two, all of width three except for one of width four [ISep].

We thank O. Iglesias for the computations leading to finding this (and other) families and the authors of [Bar+11] for acknowledging (private communication) their mistake and for helpful discussions about the extent of it.

Chapters 1.1 and 2 are based on a joint paper with Mónica Blanco, Christian Haase and Franscisco Santos [Bla+1x].

We thank Benjamin Nill and Gennadiy Averkov for helpful discussions in the development of our paper [Bla+1x], in particular for pointing us to references [AWW11] and [AKW17].

## 1.2 IDP AND LONG-EDGED POLYTOPES

A polytope $P$ has the *integer decompostion property*, or short IDP, if for every $k \in \mathbb{N}$ the $k$-th dilation $kP = P + \ldots + P$ of $P$ decomposes at the level of lattice points: $kP \cap \mathbb{Z}^d = (P \cap \mathbb{Z}^d) + \ldots + (P \cap \mathbb{Z}^d)$. Note that sometimes polytopes with the IDP are also called *normal*, but in most of the literature there is a distinction between the two and *normal* is defined as follows. Let $\Lambda$ be the lattice generated by $P \cap \mathbb{Z}^d$. Then $P$ is called *normal*, if for every $k \in \mathbb{N}$ the dilation $kP = P + \ldots + P$ of $P$ decomposes on the level of lattice points: $kP \cap \Lambda = (P \cap \Lambda) + \ldots + (P \cap \Lambda)$. So the difference is in the underlying lattice and in the case when $\Lambda = \mathbb{Z}^d$, they are the same. Examples of polytopes that are normal, but do not have the integer decomposition property are empty lattice simplices of large volume. Polytopes with the IDP turn up in many fields of mathematics. The name IDP comes from integer programming. In algebraic geometry these polytopes correspond to projectively normal embeddings of toric varieties. In commutative algebra they are called integrally closed.
So it is natural to ask which polytopes have the IDP. There has been a lot of research concerning this question in recent years. One way to prove a given polytope satisfies the IDP is to cover it with simpler polytopes known to have the IDP. The first approach

would be to use the easiest IDP polytopes, namely unimodular simplicies, and try to show that every polytope with the IDP can be triangulated into unimodular simplices. Although this works in dimension 2, it does not work in general, in fact it already fails in dimension 3 [KS03]. Relaxing triangulations to coverings with unimodular simplices, there is a famous 5-dimensional polytope with the IDP which does not have such a covering [BG99]. On the other hand, one very nice positive result is that given a lattice polytope $P$, if all edge lenghts of $P$ (with respect to the lattice) have a common factor $c \geq d-1$, then $P$ has the IDP [EW91; LTJZ93; BGT97].
The following conjecture proposed during a workshop [HHM07], suggests that this is also true (maybe with a higher bound) in a more generalized setting, where the edge lengths can be independent.

> CONJECTURE 1.2.1. Simple lattice polytopes with long edges have the integer decompostion property, where long means some invariant, uniform in the dimension.

This conjecture was then proved by Gubeladze in the following precise form.

> THEOREM ([Gub12]). Let $P$ be a lattice polytope of dimension $d$. If every edge of $P$ has lattice length at least $4d(d+1)$, then $P$ has the integer decompostion property.

He proves this theorem by first introducing the notion of $k$-convex-normality and proving that a polytope $P$ is $k$-convex-normal, if for all edges $e \prec P$ we have $\ell(e) \geq kd(d+1)$. Then he shows, that 4-convex-normal lattice polytopes have the IDP.

In chapter 3.1 we further examine $k$-convex-normal polytopes and show that if $P$ is a lattice polytope and $k$-convex-normal for some $k \geq 3$, then $P$ is also $m$-convex-normal for all $m \geq 2$ (Theorem 3.1.5). The lemma used to prove this theorem, also allows us to improve Gubeladze's bound to $2d(d+1)$ (Corollary 3.1.10).

In chapter 3.2 we extend the notion of convex-normal polytopes to pairs of polytopes. We show that given two polytopes $P$ and $Q$, the map $(Q \cap \mathbb{Z}^d) \times (P \cap \mathbb{Z}^d) \to (Q+P) \cap \mathbb{Z}^d$ given by $(q,p) \mapsto q+p$ is surjective, if the normal fan of $P$ is a refinement of the normal fan of $Q$ and every edge of $P$ is at least $d$ times as long as its corresponding face (edge or vertex) in $Q$. (Theorem 3.2.9)

Chapter 3 is based on a joint paper with Christian Haase [HH17].

We would like to thank Petra Meyer as the first part of said paper grew out of her master's thesis, that we supervised.

## 1.3 REFLEXIVE POLYTOPES AND THE NUMBER 12

Coming to the final part of our introduction, let's meet the class of reflexive polytopes.

**Definition 1.3.1** (polar dual & reflexive)**.** Let $P \subset \mathbb{R}^d$ be a $d$-dimensional polytope with $0 \in P$. The *polar dual* of $P$ is

$$P^* = \left\{ \alpha \in \left(\mathbb{R}^d\right)^* : \alpha \cdot x \leq 1 \text{ for all } x \in P \right\}.$$

Given a lattice polytope $P$, if $P^*$ is also a lattice polytope, we call $P$ *reflexive*.

Let us first collect some basic facts about reflexive polytopes. [Zie95, Thm 2.11-2.14]

LEMMA 1.3.2. Let $P$ be a reflexive $d$-polytope, then

1. $P$ has exactly one interior point.
2. There is a correspondence between $k$-faces of $P$ and $(d-1-k)$-faces of $P^*$.
3. If $N = \{n_1, \dots, n_s\}$ are the primitive normal vectors of the facets of $P$, then $P^* = \mathrm{conv}(N)$.

In light of this lemma, given a face $F \prec P$, we denote by $F^*$ its corresponding dual face.

Even though the defintion of relexive looks very restrictive at first, the following suprising result of Haase and Melnikov [HM06], shows that it is a rich class nevertheless.

LEMMA 1.3.3. Every lattice polytope is equivalent to a face of some (possibly high-dimensional) reflexive polytope.

Following this result, a natural question is, given a $d$-polytope $P$, what is the lowest dimension $m$ sucht that $P$ is a face of a reflexive $m$-polytope. For more on this so called *reflexive dimension* of $P$ see [HM06]. Our main interest though stems from the following curious result.

THEOREM 1.3.4. Let $P$ be a reflexive 2-polytope and $P^\star$ its polar dual, then the sum of the number of lattice points on the boundary of $P$ and of $P^\star$ is 12, or equivalently expressed in terms of the lattice length of the edges we get

$$\sum_{e \in P[1]} \ell(e) + \sum_{e' \in P^\star[1]} \ell(e') = 12.$$

One way to see that this result is true, is to check the equation for all reflexive lattice polygons by counting the boundary lattice points of the respective pairs. There are 16 of them and they are pictured in Figure 1.5. Those connected with an arrow are dual to each other, those without are self-dual (meaning, they are equivalent to their dual).

There are several more insightful ways to prove this equation than by exhaustion, we will present one at the end of chapter 4. But you should also consider reading the nice paper of Poonen and Rodriguez-Villegas [PRV00] for four different proofs of the equation (using stepping in the space of polygons, toric varieties and modular forms respectively). Their paper also includes a generalization to a class of possibly non-convex lattice polygons, called legal loops, then also taking the winding number into account. Futher interesting applications and generalisations can be found in the interesting papers [HS02],[HS09] and [KN12].

Moving on to dimension three there is a similarly intriguing result.

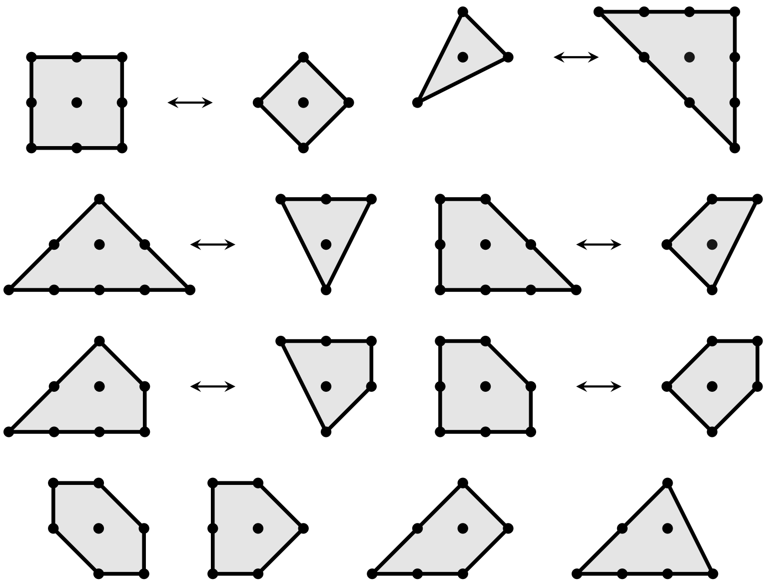


Figure 1.5: All 16 reflexive 2-dimensional polytopes

THEOREM 1.3.5. Let $P$ be a reflexive 3-polytope and $P^\star$ its polar dual, then

$$\sum_{e \in P[1]} \ell(e)\ell(e^\star) = 24.$$

Dimitrios Dais [Bec+08] was first to prove this Theorem by discovering that it actually is a Corollary of a more general result on Hodge-Deligne numbers [DK86].

Using the classification of reflexive 3- and 4-polytopes of Kreuzer and Skarke [KS00], we know all the 4319 reflexive 3-polytopes so a proof by exhaustion would again be possible, but even less favorable than before. When thinking about higher dimensional generalisations of these equations, a quick look table 1.6 shows us that with 473800776 reflexive polytopes already in dimension 4, a proof by exhaustion is not the way to go. For dimensions higher than 4, not even the number of reflexive polytopes is known, even though using that reflexive polytopes contain exactly one interior lattice point, we can apply Lemma 1.1.2 to see that there are only finitely many of them. To get an actual bound, we can use the volume bounds from [LZ91] and get that there are less than $2^{\left(dd!14^{d2^{d+1}}+1\right)^d}$ $d$-polytopes with exactly 1 interior lattice point.

| $d$ | # of reflexive polytopes |
|---|---|
| 1 | 1 |
| 2 | 16 |
| 3 | 4319 |
| 4 | 473.800.776 |
| 5 | ??? |

Figure 1.6: Table showing the # of reflexive polytopes

The question on how to generalise the equations to dimensions higher than 3 was open for quite some years until Godinho, Heymann and Sabatini [GHS16] came along and by restricting themselves to smooth reflexive polytopes, i.e. those having a unimodular normal fan, they were able to generalise it to arbitrary dimension.

THEOREM 1.3.6. [GHS16, Theorem 1.2] Let $P$ be a smooth reflexive polytope of dimension $d \geq 2$, with $f$-vector $f = (f_0, \dots, f_d)$. Then

$$\sum_{e \in P[1]} \ell(e) = 12 f_2 + (5 - 3d) f_1.$$

In chapter 4 we generalize this result to complete unimodular fans (Theorem 4.1.8) and show that this new equation and the well-known Dehn-Sommerville-equations are all the independent equations there are on $(f_0, \dots, f_d, \sum \ell(e))$ (Theorem 4.2.3).

Chapter 4 is based on an ongoing project joint with Christian Haase.

We like to thank Frederik von Heymann for giving an interesting talk about their paper [GHS16] and for the stimulating discussions thereafter, which started this project.

# Part II

## THREE INTERESTING LATTICE POLYTOPE PROBLEMS

# 2 THRESHOLD WIDTH OF LATTICE POLYTOPES

## 2.1 INTRODUCTION

The main goal of this chapter is to show that $w^\infty(4) = 2$, but on the way there, we will learn more about finiteness threshold width $w^\infty(d)$ in general.

We sometimes stratify the threshold width in terms of size, and denote $w^\infty(d,n) \in \mathbb{N} \cup \{\infty\}$ the minimal width $W \geq 0$ such that there exist only finitely many lattice $d$-polytopes of size $n$ and width $> W$. Clearly, $w^\infty(d) = \max_{n\in\mathbb{N}} w^\infty(d,n)$ and in particular, each $w^\infty(d,n)$ is finite.

In order to get bounds on $w^\infty(d)$ we relate it to the maximum widths of hollow and/or empty $d$-polytopes. As already mentioned, a lattice polytope is *hollow* if there is no lattice point in its interior and *empty* if its vertices are the only lattice points it contains.

**Definition 2.1.1.** We denote $w_H(d)$ and $w_E(d)$ the maximum widths of hollow and empty $d$-polytopes, respectively.

Finiteness of $w_H(d)$ (and hence of $w_E(d)$) is usually called the "flatness theorem", dating back to Khinchine (1948); see, e.g., [KL88]. The current best upper bound of $w_H(d) \leq O(d^{3/2})$ (used in the proof of Theorem 1.1.1) is by Banaszczyk et. al [Ban+99, Theorem. 2.4]. As for lower bounds, $w_H(d) \geq d$ follows from hollowness of the $d$-th dilation of a unimodular $d$-simplex, while $w_E(d) \geq 2\lfloor d/2 \rfloor - 1$ was proved by Sebő [Seb99] by slightly modifying this same dilated $d$-simplex to make it empty.

Throughout this chapter, we prove the following properties and bounds of $w^\infty(d,n)$ and $w^\infty(d)$:

THEOREM 2.1.2.

1. $w^\infty(d,n) \leq w^\infty(d,n+1)$ for all $d,n$. (Proposition 2.2.1)
2. $w^\infty(d) \leq w^\infty(d+1)$ for all $d$. (Proposition 2.2.2)
3. $w^\infty(d) \leq w_H(d-1)$. (Lemma 2.2.3)
4. $w_E(d-1) \leq w^\infty(d)$ for $d \geq 3$. (Corollary 2.3.5)
5. $w_H(d-2) \leq w^\infty(d)$. (Corollary 2.3.7)

*Remark* 2.1.3. None of the inequalities $w_H(d-2) \leq w^\infty(d) \leq w_H(d-1)$ or $w_E(d-1) \leq w^\infty(d)$ (for $d \geq 3$) is sharp, as the following table of known values shows.

| $d$ | $w_E(d-1)$ | $w_H(d-2)$ | $w^\infty(d)$ | $w_H(d-1)$ |
|---|---|---|---|---|
| 1 | – | – | 0 | – |
| 2 | 1 | – | 0 | 1 |
| 3 | 1 | 1 | 1 | 2 |
| 4 | 1 | 2 | 2 | 3 |
| 5 | $\geq 4$ | 3 | $\geq 4$ | $\geq 4$ |

The values of $w^\infty(d)$, $d = 1, 2, 3, 4$, have been discussed above. For the others:

- In dimension 1, the unique hollow lattice segment is equivalent to $[0, 1]$, and then $w_E(1) = w_H(1) = 1$.
- In dimension 2, the second dilation of a unimodular triangle is the only hollow lattice polygon of width larger than one (see, e.g., [Tre08]). Hence $w_H(2) = 2$ and, since this polygon is not empty, $w_E(2) = 1$.
- In dimension 3, Howe ([Sca85, Thm. 1.3]) proved that $w_E(3) = 1$. For $w_H(3)$, Averkov et al. ([AWW11, Theorem 2.2] and [AKW17, Theorem 1]) have classified all hollow 3-polytopes and their maximum width is three (see more details in Lemma 2.5.3), so $w_H(3) = 3$.
- In dimension 4, Haase and Ziegler [HZ00] showed $w_E(4) \geq 4$, which implies $w^\infty(5) \geq 4$ by part (4) of Theorem 2.1.2.

The structure of this chapter is as follows. The monotonicity properties stated in parts (1) and (2) of Theorem 2.1.2 are proved at the beginning of Section 2.2. We then prove the upper bound $w^\infty(d) \leq w_H(d-1)$ (Lemma 2.2.3) from the following statement, which combines results of Hensley [Hen83], Lagarias–Ziegler [LZ91] and Nill–Ziegler [NZ11]: all but finitely many lattice $d$-polytopes of bounded size are hollow and project to hollow $(d-1)$-polytopes. This fact implies that to search for an infinite family of lattice $d$-polytopes of bounded size we can focus on lifts (see Definition 2.2.4) of hollow polytopes of one dimension less. The remainder of Section 2.2 is devoted to prove several lemmas on the width of polytopes and the width of lifts of a polytope that we will use later in the chapter. Most importantly, we prove that it is enough to look at *tight* lifts (see Definition 2.2.11), which are inclusion-minimal lifts of a polytope (Corollary 2.2.13).

In Section 2.3 we prove sufficient properties for hollow $(d-1)$-polytopes to have infinitely many lifts of bounded size. In particular, we prove the existence of certain such hollow $(d-1)$-polytopes of widths $w_E(d-1)$ and $w_H(d-2)$, which provides the lower bounds $w_E(d-1) \leq w^\infty(d)$ (Corollary 2.3.5) and $w_H(d-2) \leq w^\infty(d)$ (Corollary 2.3.7). Moreover, we get the following characterization of the finiteness threshold width:

THEOREM 2.1.4 (Theorem 2.3.8 and Corollary 2.2.13). For all $d \geq 3$, $w^\infty(d)$ equals the maximum width of a hollow lattice $(d-1)$-polytope $Q$ for which there are infinitely many (equivalence classes of) lattice $d$-polytopes $P$ of bounded size and projecting to $Q$.

One direction of the theorem is easy, since a $Q$ as in the statement has all but finitely many of its lifts of the same width as $Q$ (Theorem 2.2.10). The other is less obvious since $w^\infty(d)$ might a priori be achieved by the existence of infinitely many hollow $(d-1)$-polytopes $Q$, each with finitely many lifts of size $n$.

**Example 2.1.5.** In dimension 3, the infinite family of Reeve tetrahedra are lifts of size 4 of a unit square, which is a hollow polygon of width one. On the other hand, the unique hollow polygon of width larger than one is the second dilation of the unimodular triangle, which has only finitely many lifts of bounded size (see the proof of Corollary 22 in [BS16a]). Hence $w^\infty(3) = 1$.

In dimension 4, observe that $w^\infty(4) \geq 2$ follows from the fact that the following hollow 3-polytope of width two can be lifted to infinitely many empty 4-simplices (Haase and Ziegler [HZ00, Proposition 6]):

$$Q = \operatorname{conv}\left\{ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}2\\2\\3\end{pmatrix} \right\}.$$

Sections 2.4 and 2.5 are aimed at proving our main result Theorem 1.1.4. By Theorem 2.1.4 and Example 2.1.5, to prove $w^\infty(4) = 2$ it suffices to show that each hollow 3-polytope of width larger than two has finitely many 4-dimensional lifts of bounded size. For this we first prove sufficient conditions for lattice polytopes (in arbitrary dimension) to have only finitely many lifts of bounded size (Section 2.4). Subsequently we apply them to the full list of hollow 3-polytopes of width larger than two. This list, containing only 5 polytopes, is derived from the classification of *maximal* hollow 3-polytopes by Averkov et al. ([AWW11, Theorem 2.2] and [AKW17, Theorem 1]) in Section 2.5.

In light of these results, we ask the following questions.

**Question 2.1.6.** Besides the monotonicity in parts (1) and (2) of Theorem 2.1.2, does $w^\infty(d,n) \leq w^\infty(d+1,n+1)$ always hold? The case $w^\infty(d,d+1) \leq w^\infty(d+1,d+2)$ follows from [HZ00, Proposition 1]: every empty $d$-simplex is a facet of infinitely many empty $(d+1)$-simplices of at least the same width.

**Question 2.1.7.** For all known values ($d \leq 4$) we have $w^\infty(d) = w^\infty(d,d+1)$. That is, the finiteness threshold width for all lattice $d$-polytopes is determined by empty $d$-simplices. Does this hold for arbitrary $d$?

## 2.2 FINITENESS THRESHOLD WIDTH AND LIFTS OF HOLLOW POLYTOPES

### 2.2.1 Monotonicity of the finiteness threshold widths

Parts (1) and (2) of Theorem 2.1.2 have the following proofs:

PROPOSITION 2.2.1. $w^\infty(d,n) \le w^\infty(d,n+1)$ for all $n \ge d+1$.

*Proof.* Let $W \in \mathbb{N}$ be such that there exists an infinite family $\{P_i\}_{i\in\mathbb{N}}$ of lattice $d$-polytopes of size $n$ and width $W$. We are going to show that for each $P_i$ there is a $P'_i$ of size $n+1$ and width $W$ containing $P_i$. To prove this, let $\ell_i$ be an integer functional giving width $W$ to $P_i$, and assume without loss of generality that $\ell_i(P_i) = [0,W]$. Taking any point $q_i \in \mathbb{Z}^d \cap \ell_i^{-1}[0,W] \setminus P_i$ we easily get a $Q_i = \mathrm{conv}(P_i \cup \{q_i\})$ of width $W$ and properly containing $P_i$ (see Figure 2.1). If $Q_i \setminus P_i$ has more than one lattice point, remove them one by one until only one remains (which can always be done; simply choose a vertex $v$ of $Q_i$ not in $P_i$ and replace $Q_i$ to the convex hull of $(Q_i \cap \mathbb{Z}^d) \setminus \{v\}$; then iterate).

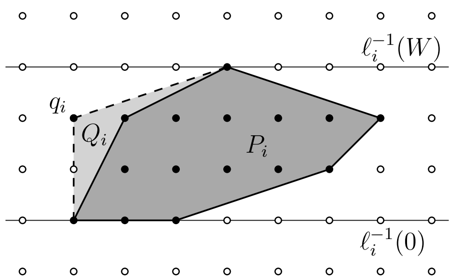



Figure 2.1: The setting of the proof of Proposition 2.2.1.

That implies the lemma except for the fact that different polytopes $P_i$ and $P_j$ may produce isomorphic $P'_i$ and $P'_j$, so it is not obvious that $\{P'_i\}_{i\in\mathbb{N}}$ is an infinite family. But each element of $\{P'_i\}_{i\in\mathbb{N}}$ can only correspond to at most $n+1$ elements from $\{P_i\}_{i\in\mathbb{N}}$ (because $P_i$ is recovered from $P'_i$ by removing one of its $n+1$ lattice points), so the proof is complete. $\square$

PROPOSITION 2.2.2. $w^\infty(d) \le w^\infty(d+1)$, for all $d$.

*Proof.* Let $W \in \mathbb{N}$ be such that, for some $n \in \mathbb{N}$, there is an infinite family $\{P_i\}_{i\in\mathbb{N}}$ of lattice $d$-polytopes of size $n$ and width $W$. Then, $\mathcal{P} = \{P_i \times [0,W]\}_{i\in\mathbb{N}}$ is a sequence of $(d+1)$-polytopes of size $n(W+1)$ and width $W$. A priori two different $P_i$'s can give isomorphic polytopes in $\mathcal{P}$, but each polytope in $\mathcal{P}$ can correspond to only finitely many $P_i$'s since $P_i$ is the projection of $P_i \times [0,W]$ along the direction of an edge. Hence $\mathcal{P}$ is infinite and $w^\infty(d+1) \ge W$. $\square$

The following lemma proves part (3) of Theorem 2.1.2:

LEMMA 2.2.3. Let $d < n \in \mathbb{N}$. All but finitely many lattice $d$-polytopes of size bounded by $n$ are hollow and admit a projection to some hollow lattice $(d-1)$-polytope. In particular, $w^\infty(d) \le w_H(d-1)$ for all $d$.

*Proof.* As argued in the proof of Theorem 1.1.1, the number of non-hollow lattice $d$-polytopes of size bounded by $n$ is finite. Hence, all but finitely many lattice $d$-polytopes of size bounded by $n$ are hollow.

On the other hand, Nill and Ziegler [NZ11, Corollary 1.7] proved that all but finitely many hollow $d$-polytopes admit a projection to a hollow $(d-1)$-polytope. And these have width at most that of their projection, which is $\le w_H(d-1)$. □

### 2.2.2 *Finiteness threshold width via polytopes with infinitely many lifts of bounded size*

**Definition 2.2.4.** We say that a lattice polytope $P \subset \mathbb{R}^d$ is a *lift* of a lattice $(d-1)$-polytope $Q$ if there is a lattice projection $\pi$ with $\pi(P) = Q$. Without loss of generality, we will typically assume $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ to be the map that forgets the last coordinate.

Two lifts $P_1$ and $P_2$, with projections $\pi_1 : P_1 \to Q$ and $\pi_2 : P_2 \to Q$ are *equivalent* if there is a unimodular equivalence $f : P_1 \to P_2$ with $\pi_2 \circ f = \pi_1$. That is, if for each $p \in \mathbb{Z}^d$, $f(p) \in \pi_2^{-1}(\pi_1(p))$ (the equivalence maps a point in the fiber of $p$ under $\pi_1$, to a point in the fiber of $p$ under $\pi_2$). See Figure 2.2 for examples of equivalent and non-equivalent lifts.

We say that "$Q$ has finitely many lifts of bounded size" if for every $n \in \mathbb{N}$ there are finitely many equivalence classes of lifts of $Q$ of size $n$. Accordingly, "$Q$ has infinitely many lifts of bounded size" means that there is an $n \in \mathbb{N}$ for which there are infinitely many equivalence classes of lifts of $Q$.

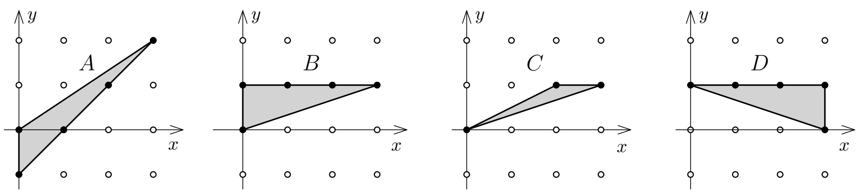



Figure 2.2: Polytopes $A, B, C, D \subset \mathbb{R}^2$ are lifts of $[0,3] \subset \mathbb{R}$ under projection $\pi(x,y) = x$. Only $A$ and $B$ are equivalent lifts. $D$ is equivalent to $A$ and $B$ as a lattice polytope, but not as a lift of $[0,3]$ under $\pi$.

*Remark* 2.2.5. Saying that "$Q \subset \mathbb{R}^{d-1}$ has infinitely many lifts of bounded size" is equivalent to saying that "there are infinitely many (equivalence classes of) lattice polytopes $P \subset \mathbb{R}^d$ of bounded size that have a lattice projection to $Q$". The implication from right to left is trivial, and the implication from left to right follows from the fact that once $P$ is fixed there is a finite number of integer affine projections $P \to Q$ (an overestimate is $q^p$, where $p$ and $q$ are the numbers of lattice points in $P$ and $Q$, respectively; $q^{d+1}$ is also an upper bound, since an affine map is determined by the image of an affine basis).

Our interest in these concepts comes from the following fact, that follows from Theorem 2.2.10 below. Less obvious is the converse, that we prove in Theorem 2.3.8.

> PROPOSITION 2.2.6. For all $d \geq 3$, $w^\infty(d)$ is at least the maximum width of a lattice $(d-1)$-polytope $Q$ that admits infinitely many lifts of bounded size.

For the proof of Theorem 2.2.10 we need a couple of technical lemmas. In the first one, $Q$ does not need to be a lattice polytope, or even a polytope, but only a convex body (compact, convex subset of $\mathbb{R}^n$).

> LEMMA 2.2.7. Let $Q \subset \mathbb{R}^d$ be a full-dimensional convex body, and $W \in \mathbb{N}$. Then, there is only a finite number of functionals $\ell \in (\mathbb{Z}^d)^*$ such that $\mathrm{width}_\ell(Q) \leq W$.

*Proof.* Observe that $\mathrm{width}_\ell(Q)$ equals the maximum value of $\ell$ in the centrally symmetric body $Q - Q$. This, in turn, equals the smallest $\lambda$ with $\ell \in \lambda(Q-Q)^*$, where $(Q-Q)^*$ is the polar of $Q - Q$. Equivalently, the integral functionals giving width $\leq W$ to $Q$ are the lattice points in $W(Q-Q)^*$. Since $Q - Q$ is full-dimensional its polar is bounded, so there are finitely many such lattice points. □

*Remark* 2.2.8. The proof of Lemma 2.2.7 is based on the following interpretation of the lattice width: $\mathrm{width}(Q)$ is the minimum $\lambda$ such that $\lambda(Q-Q)^*$ contains a non-zero lattice point $\ell \in (\mathbb{Z}^d)^*$. This $\lambda$ is usually called the *first successive minimum* of $(Q-Q)^*$ and is defined for every centrally symmetric convex body [Gru07, p.376].

A lift of $Q$ may have the same dimension as $Q$ and still not be unimodularly equivalent to it. For example, the segment $[0,k]$ in $\mathbb{R}^1$ can be lifted to the primitive segment $\mathrm{conv}\{(0,0),(k,1)\}$. However, the number of different such lifts of $Q$ is finite, modulo the equivalence relation in Definition 2.2.4:

> LEMMA 2.2.9. A $(d-1)$-dimensional polytope $Q$ has only finitely many $(d-1)$-dimensional lifts.

*Proof.* Every $(d-1)$-dimensional lift $P$ of $Q$ can be described as follows: there is an affine map $f : \mathbb{R}^{d-1} \to \mathbb{R}$ with

$$P = \mathrm{conv}\{(v, f(v)) : v \text{ is a vertex of } Q\},$$

and such that $f$ is integer in all vertices of $Q$. Assuming, without loss of generality, that $f$ is linear and the origin is a vertex of $Q$, this implies $f \in \Lambda(Q)^*$, where $\Lambda(Q)$ is the lattice spanned by the vertices of $Q$. Two such functionals give equivalent lifts if, and only if, they are in the same class modulo $(\mathbb{Z}^{d-1})^*$. Thus, the number of different lifts equals the index of $(\mathbb{Z}^{d-1})^*$ in $\Lambda(Q)^*$. □

THEOREM 2.2.10. Let $Q \subset \mathbb{R}^{d-1}$ be a lattice $(d-1)$-polytope of width $W$. Then all lifts $P \subset \mathbb{R}^d$ of $Q$ have width $\leq W$. All but finitely many of them have width $= W$.

*Proof.* As a convenient notation, for a given vector $v = (v_1, \dots, v_d)$, we set $\tilde{v} = (v_1, \dots, v_{d-1})$. As projections do not decrease the width, we only need to prove that all but finitely many lifts of $Q$ have width $\geq W$. For this, let $\ell \in (\mathbb{Z}^d)^* \setminus \{0\}$ be a functional and $P$ be some lift of $Q$.

If $\ell_d = 0$, then $\mathrm{width}_\ell(P) = \mathrm{width}_{\tilde{\ell}}(Q) \geq W$, so for the rest of the proof we assume that $\ell_d \neq 0$ for all our functionals.

Let $T \subseteq Q$ be a $(d-1)$-simplex with $\mathrm{vert}(T) \subseteq \mathrm{vert}(Q)$ and $S$ be one of the finitely many $(d-1)$-dimensional lifts of $T$ (see Lemma 2.2.9). Every lift $P$ of $Q$ has to contain one of these $S$, so it suffices to show that there are finitely many such $P$ having width $< W$. Furthermore we can assume that both $P$ and $S$ are contained in $Q \times \mathbb{R}_{\geq 0}$. Let $H$ be the hyperplane containing $S$, and $L_1 := \max\{x_d \,:\, x \in H \cap (Q \times \mathbb{R})\} + 1$. Then $R := \bigcap_{x \in Q \times \{L_1\}} \mathrm{conv}(S \cup \{x\})$ is a full-dimensional polytope (in fact, $R$ is a simplex, because $S$ is), which following Lemma 2.2.7 implies that there are only finitely many integer functionals such that $\mathrm{width}_\ell(R) \leq W$. Let $\ell^1, \dots, \ell^s$ be those functionals and set $D := \max_{x \in Q,\, i \in \{1,\dots,s\}} |\tilde{\ell}^i \cdot x|$ and $L := \max\{L_1, 2D + W\}$.

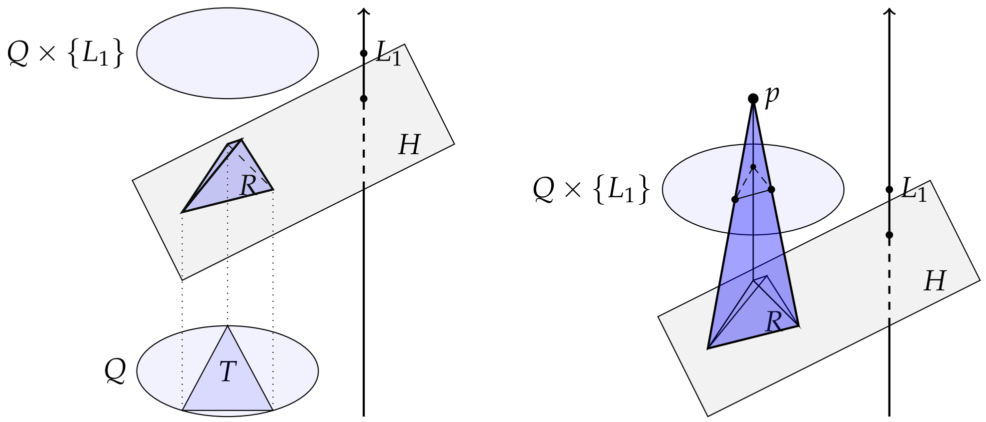



Figure 2.3: Constructing $\hat{R}$ and seeing that $\hat{R} \subseteq P$ (setting of the proof of Theorem 2.2.10).

Now assume that there exists a point $p \in P$ with $p_d \geq 2L$ and let $\hat{R} := R \cup \{p\}$. Then thanks to $p_d \geq L_1$, we have $\hat{R} \subseteq P$, as $R \subseteq \mathrm{conv}(S, p)$ (see Figure 2.3). Suppose $\ell$ is such that $\mathrm{width}_\ell(P) \leq W$, then $\mathrm{width}_\ell(R) \leq \mathrm{width}_\ell(\hat{R}) \leq \mathrm{width}_\ell(P) \leq W$ and hence $\ell = \ell^i$ has to be one of the finitely many integer functionals giving width $\leq W$ to $R$. But if $\ell$ is one of those functionals, given some $r \in R$ we get:

$$\mathrm{width}_\ell(P) \geq \mathrm{width}_\ell(\hat{R}) \geq |\ell \cdot (p - r)| \geq |\ell_d(p_d - r_d)| - |\tilde{\ell} \cdot (\tilde{p} - \tilde{r})| \geq |\ell_d L| - 2D \geq W.$$

That is, $P$ has width exactly $W$.

Hence all lifts $P$ that extend $S$ and contain a point $p$ with $p_d \geq 2L$ have width $W$. That leaves us with those $P$ that are contained in $Q \times [0, 2L]$, but there are only finitely many lattice subpolytopes of $Q \times [0, 2L]$ and hence only finitely many of width $< W$. □

### 2.2.3 *Tight lifts*

We finish this section showing that in order to decide whether a given $Q$ has infinitely many lifts of bounded size it is enough to look at *tight* lifts. This will simplify the work in the rest of the chapter:

**Definition 2.2.11.** Let $Q \subset \mathbb{R}^{d-1}$ be a $(d-1)$-dimensional lattice polytope. We say that a lift $P \subset \mathbb{R}^d$ of $Q$ is *tight* if the projection sending $P$ to $Q$ bijects their sets of vertices. That is, if $P = \text{conv}\{(v, h_v) : v \in \text{vert}(Q)\}$ for some $\mathbf{h} \in \mathbb{Z}^{\text{vert}(Q)}$.

See Figure 2.4 for examples of tight and not tight lifts.

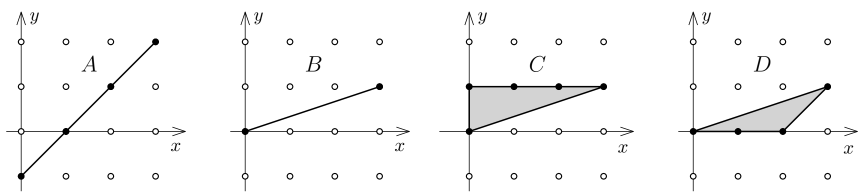



Figure 2.4: Polytopes $A, B, C, D \subset \mathbb{R}^2$ are lifts of $[0, 3] \subset \mathbb{R}$ under the projection $\pi(x, y) = x$. $A$ and $B$ are tight lifts; $C$ and $D$ are not.

Notice that a tight lift is not necessarily full-dimensional and that every lift of $Q$ contains a tight lift.

LEMMA 2.2.12. Let $P \subset \mathbb{R}^d$ be a (not necessarily full-dimensional) lift of a lattice $(d-1)$-polytope $Q$. Then, there are only finitely many lifts of $Q$ of bounded size that contain $P$.

*Proof.* For each $q \in Q \cap \mathbb{Z}^{d-1}$, pick $h_q \in \mathbb{R}$ such that $p_q = (q, h_q) \in P$ (these exist as $P$ projects to $Q$).

Let $P' \subset \mathbb{R}^d$ be any lift of $Q$ that contains $P$. Given $p' \in P' \cap \mathbb{Z}^d$, then $p' = (q, h')$ for some $q \in Q \cap \mathbb{Z}^{d-1}$ and $h' \in \mathbb{Z}$. Without loss of generality assume that $h' \geq h_q$ (the other case is symmetric). Then $P'$ contains the segment $\text{conv}\{p_q, p'\} = \{q\} \times [h_q, h'] \subset P'$, which already contains $h' - \lceil h_q \rceil + 1$ lattice points (see Figure 2.5). Since the size of $P'$ is bounded, there are finitely many possibilities for $h'$ and hence for all points of $P'$. □

The results of this section lead to the following:

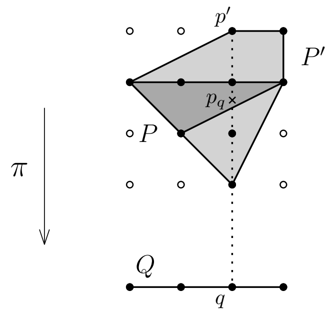



Figure 2.5: The segment $\pi^{-1}(q) \cap P$ has to be bounded for all $q \in Q \cap \mathbb{Z}^d$ in the proof of Lemma 2.2.12.

COROLLARY 2.2.13. Let $Q \subset \mathbb{R}^{d-1}$ be a lattice $(d-1)$-polytope. The following are equivalent:

(1) There are infinitely many (isomorphism classes of) lattice $d$-polytopes of bounded size projecting to $Q$.

(2) $Q$ has infinitely many lifts of bounded size.

(3) $Q$ has infinitely many lifts of bounded size and of the same width as $Q$.

(4) $Q$ has infinitely many tight lifts of bounded size.

In any of those cases, the width of $Q$ is a lower bound for $w^\infty(d)$.

*Proof.* (1) $\implies$ (2), (3) $\implies$ (2), and (4) $\implies$ (2) are obvious. For the converses: (2) $\implies$ (1) is Remark 2.2.5 together with Lemma 2.2.9, (2) $\implies$ (3) follows from Theorem 2.2.10, and (2) $\implies$ (4) comes from Lemma 2.2.12 and the fact that any lift of a polytope contains a tight lift. The fact that $w^\infty(d) \geq \text{width}(Q)$ is Proposition 2.2.6. □

## 2.3 HOLLOW POLYTOPES WITH INFINITELY MANY LIFTS OF BOUNDED SIZE

LEMMA 2.3.1. Let $Q \subset \mathbb{R}^{d-1}$ be a hollow $(d-1)$-polytope and let $v \in \text{vert}(Q)$ be such that $Q' := \text{conv}(\text{vert}(Q) \setminus \{v\})$ is $(d-1)$-dimensional. (That is, $Q$ is not a pyramid with apex at $v$). Suppose that every proper face $F$ with $v \in F$ is either hollow or a pyramid with apex $v$. Then, for every $h \in \mathbb{Z} \setminus \{0\}$ the $d$-dimensional tight lift $P(h) := \text{conv}\left((Q' \times \{0\}) \cup \{(v,h)\}\right)$ of $Q$ has the following properties:

1. $\text{size}(P(h)) \leq \text{size}(Q)$, with equality for infinitely many values of $h$.

2. $\text{width}(P(h)) = \text{width}(Q)$ for every sufficiently large $h$.

See Figure 2.6 for an example of this layout.

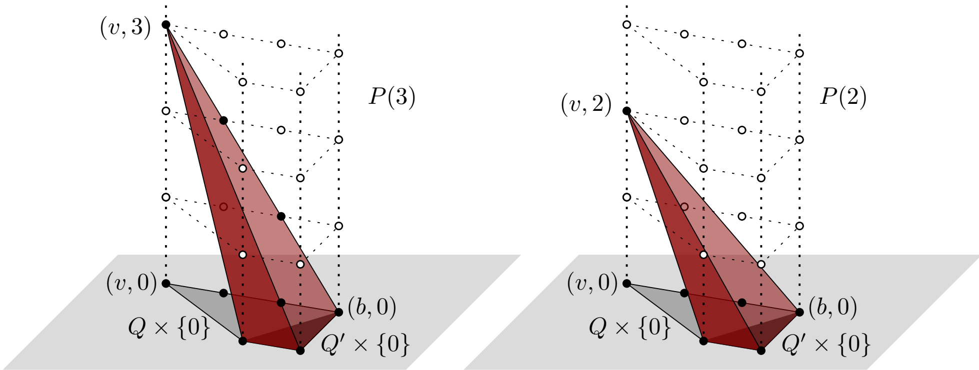



Figure 2.6: The setting of Lemma 2.3.1. The figure shows the hollow polygon $Q$ and two of its tight lifts $P(h)$. One of the proper faces of $Q$ containing $v$ is empty, and the other is a 1-dimensional lattice pyramid over a point $b$, with the distance from $v$ to $b$ being 3. This implies that $h=3$ yields $P(3)$ with as many lattice points as $Q$, and $h=2$ yields $P(2)$ with strictly fewer lattice points.

*Proof.* Fix $h \in \mathbb{Z} \setminus \{0\}$ and let $P = P(h)$ be as in the statement.

For the first statement, let $q \in Q \cap \mathbb{Z}^{d-1}$. We claim that the fiber $\pi^{-1}(q)$ has at most one lattice point in $P$, with equality in many cases. For this, let $F$ be the carrier face of $q$ in $Q$ (that is, the unique face with $q \in \operatorname{relint}(F)$). Since $Q$ is hollow, $F$ is a proper face. By assumption, there are three possibilities for $F$:

- *$F$ does not contain $v$.* Then $\pi^{-1}(F) \cap P = F \times \{0\}$. In particular, $(q,0)$ is the only lattice point of $P$ in the fiber $\pi^{-1}(q)$.
- *$v \in F$ and $F$ is hollow.* Since $q \in \operatorname{relint}(F)$, we must have $F = \{q\} = \{v\}$. In particular, $(v,h)$ is the only lattice point of $P$ in the fiber $\pi^{-1}(q)$.
- *$F$ is a pyramid with apex at $v$.* Let $F'$ be the base of the pyramid. Remember that $v$ is lifted to $(v,h)$ and every other vertex $w$ of $F'$ is lifted to $(w,0)$. In particular, the face $\pi^{-1}(F) \cap P$ of $P$ equals the affine image of $F$ under the map $x \mapsto (x, h\cdot \operatorname{dist}(F',x)/\operatorname{dist}(F',v))$, where $\operatorname{dist}(F',x)$ denotes the lattice distance from $x$ to (the hyperplane spanned by) $F'$. Thus, $(q, h\cdot\operatorname{dist}(F',q)/\operatorname{dist}(F',v))$ is the only point of $P$ in the fiber $\pi^{-1}(q)$. That point will be a lattice point if (but perhaps not only if) $h$ is an integer multiple of $\operatorname{dist}(F',v)$.

In particular, we have $\operatorname{size}(P(h)) = \operatorname{size}(Q)$ for any $h$ that is an integer multiple of $\operatorname{lcm}\{\operatorname{dist}(F',v) : F \text{ face of } Q \text{ that is a pyramid with base } F' \text{ and apex } v\}$.

The second statement follows directly from Theorem 2.2.10. Indeed, the polytopes $P(h)$ are unimodularly non-isomorphic for different values of $|h|$, since their volume is proportional to $|h|$. $\square$

**Corollary 2.3.2.** Let $Q$ be a hollow polytope and not a simplex. If $Q$ is either empty or simplicial then it has infinitely many lifts of the same size and width of $Q$.

*Proof.* Since $Q$ is not a simplex, there is a vertex $v$ such that $Q$ is not a pyramid with apex at $v$. Being empty or simplicial guarantees the conditions of Lemma 2.3.1 for $v$ are met. □

These results give us a way to lower-bound $w^\infty(d)$; if a lattice $(d-1)$-polytope $Q$ is in the conditions of Lemma 2.3.1 or the Corollary 2.3.2, then $w^\infty(d) \geq \mathrm{width}(Q)$.

**Definition 2.3.3.** A hollow lattice $d$-polytope is called *hollow-maximal* if it is maximal under inclusion of hollow lattice $d$-polytopes.

An empty lattice $d$-polytope is called *empty-maximal* if it is maximal under inclusion of empty lattice $d$-polytopes.

LEMMA 2.3.4. Let $Q$ be a hollow-maximal or empty-maximal $d$-polytope, for $d \geq 2$. Then, for every vertex $v$ of $Q$ there is a lattice point $u \in Q$ that is not contained in any facet containing $v$.

*Proof.* Let $v$ be a vertex of $Q$ and suppose that every lattice point of $Q$ is in a facet containing $v$. We claim that this contradicts $Q$ being hollow-maximal or empty-maximal. For this, let $C_v = v + \mathbb{R}_{\geq 0}(Q - v)$ be the cone of $Q$ at $v$, then all the lattice points of $Q$ lie in the boundary of the cone. Let $u \in \mathrm{int}(C_v) \cap \mathbb{Z}^d$ be such that $u$ is the only lattice point of $Q' := \mathrm{conv}(Q, u)$ in the interior of $C_v$. (Such a $u$ can be found, for example, minimizing in $\mathrm{int}(C_v) \cap \mathbb{Z}^d$ any supporting linear functional of $C_v$). Then $Q'$ strictly contains $Q$ and it is still empty or hollow if $Q$ was empty or hollow, respectively (see Figure 2.7). □

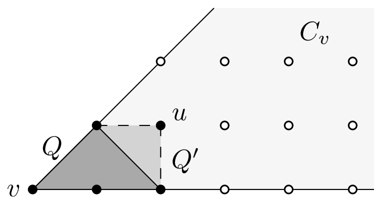



Figure 2.7: Finding vertices $u$ and $v$ not contained in a common facet in the proof of Lemma 2.3.4.

With this we can now prove that $w^\infty(d)$ is at least $\max\{w_E(d-1), w_H(d-2)\}$.

COROLLARY 2.3.5. For every $d \geq 3$ there exists an empty $(d-1)$-polytope of width $w_E(d-1)$ with infinitely many lifts of bounded size. In particular, $w^\infty(d) \geq w_E(d-1)$.

*Proof.* Lemma 2.3.4 implies that $w_E(d-1)$ is achieved by a non-simplex $Q$, and then Corollary 2.3.2 shows $Q$ has infinitely many lifts of bounded size. Proposition 2.2.6 implies then that $w^\infty(d) \geq \mathrm{width}(Q) = w_E(d-1)$. □

We call a polytope *bipyramid*, if there are two vertices $u$ and $v$ such that every facet is a pyramid with apex either $u$ or $v$, and there is no facet containing both. Hollow bipyramids clearly satisfy the conditions of Lemma 2.3.1, hence they have infinitely many lifts of bounded size.

LEMMA 2.3.6. For every $d \geq 2$ there exists a hollow bipyramid of dimension $d$ and width $w_H(d-1)$.

*Proof.* By induction on $d$. For $d = 2$, the unit square is a hollow bipyramid of width $1 = w_H(1)$. For higher $d$, let us first see that there exists a hollow $(d-1)$-polytope $Q$ of width $w_H(d-1)$ and having two lattice points $u$ and $v$ not sharing any facet.

- If $w_H(d-1) = w_H(d-2)$ then let $Q$ be a hollow bipyramid of dimension $d-1$ and width $w_H(d-2)$, which exists by induction hypothesis.
- If $w_H(d-1) > w_H(d-2)$ then there are only finitely many hollow $(d-1)$-polytopes of width $w_H(d-1)$ [NZ11]. Hence, there is one such $Q$ that is maximal. By Lemma 2.3.4, there are lattice points $u$ and $v$ in $Q$ not contained in the same facet.

Now consider the convex hull of $(Q \times \{0\}) \cup \{(u,h),(v,-h)\}$. This is a hollow bipyramid of dimension $d$ and, for sufficiently large $h$, it has the same width as $Q$ (by Theorem 2.2.10). □

COROLLARY 2.3.7. For every $d \geq 3$ there exists a hollow $(d-1)$-polytope of width $w_H(d-2)$ with infinitely many lifts of bounded size. In particular, $w^\infty(d) \geq w_H(d-2)$.

*Proof.* Let $Q$ be a hollow $(d-1)$-dimensional bipyramid of width $w_H(d-2)$, which exists by Lemma 2.3.6. Lemma 2.3.1 shows $Q$ has infinitely many lifts of bounded size. Proposition 2.2.6 implies then that $w^\infty(d) \geq \text{width}(Q) = w_H(d-2)$. □

This finally allows us to prove that:

THEOREM 2.3.8. For all $d \geq 3$, $w^\infty(d)$ equals the maximum width of a lattice $(d-1)$-polytope $Q$ that admits infinitely many lifts of bounded size. Moreover, $Q$ is hollow.

*Proof.* In Proposition 2.2.6 we saw that $w^\infty(d)$ is at least the width of any lattice $(d-1)$-polytope with infinitely many lifts of bounded size.

For the other inequality, Corollary 2.3.7 proves the statement in the case when $w^\infty(d) = w_H(d-2)$ (it proves as well that $w^\infty(d) \geq w_H(d-2)$).

The only remaining case is then when $w^\infty(d) > w_H(d-2)$. First of all, since $w^\infty(3) = 1$ and by Proposition 2.2.2, we have that $w^\infty(d) > 0$ for all $d \geq 3$ (this guarantees the existence of infinitely many lattice $d$-polytopes of some fixed size). Let $n$ be such that $W := w^\infty(d) = w^\infty(d,n)$. That is, there is an infinite family $\{P_i\}_{i\in\mathbb{N}}$ of lattice $d$-polytopes of size $n$ and width $W$. Without loss of generality (Lemma 2.2.3) assume all $P_i$'s are hollow and have a hollow lattice $(d-1)$-dimensional projection $Q_i$. Since projecting does not decrease the width, every $Q_i$ has width at least $W$, and since

$W = w^\infty(d) > w_H(d-2)$ no $Q_i$ admits a hollow projection to dimension $d-2$. This implies the family $\{Q_i\}_{i\in\mathbb{N}}$ to be finite as all but finitely many hollow $(d-1)$-polytopes project onto hollow $(d-2)$-polytopes **[NZ11]**, so one of them, call it $Q$, lifts to infinitely many members of the family $\{P_i\}_{i\in\mathbb{N}}$. Theorem 2.2.10 implies then that $Q$ has width *exactly* $W$.

Any $Q$ with infinitely many lifts of bounded size is hollow, by Lemma 2.2.3 and Corollary 2.2.13. □

## 2.4 POLYTOPES WITH FINITELY MANY LIFTS OF BOUNDED SIZE

LEMMA 2.4.1. Let $Q$ be a lattice pyramid with basis $F$ and apex $v$. If $F$ has finitely many lifts of bounded size, then so does $Q$.

*Proof.* Let $Q \subset \mathbb{R}^{d-1}$ be $(d-1)$-dimensional lattice pyramid. Any tight lift of $Q$ is of the form $P(\tilde{F}, h) := \operatorname{conv}(\tilde{F} \cup \{\tilde{v}\})$, where $\tilde{F}$ is a tight lift of $F$ and $\tilde{v} = (v, h)$ is a point in the fiber of $v$. Since $\tilde{F}$ is contained in some hyperplane $H$ orthogonal to $\{x_d = 0\}$ and containing $F \times \{0\}$, $P(\tilde{F}, h)$ is a pyramid with basis $\tilde{F}$ and apex $\tilde{v}$ (see Figure 2.8).

Let $m$ be the distance from $v$ to $F$. Then $P(\tilde{F}, h)$ is equivalent to $P(\tilde{F}, h+m)$ for all $h \in \mathbb{Z}$ (we leave it to the reader to derive the unimodular transformation). That is, there are at most $m$ values of $h$ that give non-equivalent tight lifts $P(\tilde{F}, h)$, for any fixed $\tilde{F}$. By hypothesis, there are only finitely many such $\tilde{F}$ of bounded size, hence finitely many tight lifts of bounded size of $Q$. Corollary 2.2.13 implies the statement. □

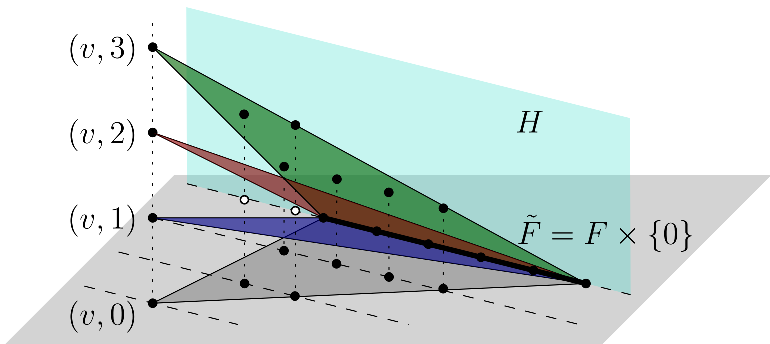



Figure 2.8: The setting of the proof of Lemma 2.4.1. In the figure, the case when $\tilde{F}$ is the tight lift $F \times \{0\}$ is represented. The apex $v$ is at distance 3, hence $(v,3)$ yields equivalent lift as $(v,0)$, while $(v,1)$ and $(v,2)$ do not.

COROLLARY 2.4.2. Lattice simplices have only finitely many lifts of bounded size.

*Proof.* Using induction on the dimension and Lemma 2.4.1, this follows from the fact that a single lattice point has only finitely many lifts of bounded size. □

We now want to show that non-hollow lattice polytopes have only finitely many lifts of bounded size. The following geometric lemma (in which $Q$ need not be a lattice polytope) will be helpful.

LEMMA 2.4.3. Let $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the standard projection that forgets the last coordinate. Let $q$ be a point in the interior of a $(d-1)$-polytope $Q$. Then, there is a $c \in \mathbb{R}$ such that for every $d$-polytope $P \subset \mathbb{R}^d$ with $\pi(P) = Q$ we have

$$\mathrm{vol}(P) \leq c \cdot \mathrm{length}(P \cap \pi^{-1}(q)).$$

*Proof.* Assume without loss of generality that $q$ is the origin and that the vertical segment $P \cap \pi^{-1}(q)$ goes from $(q, 0)$ to $(q, 1)$. This is no loss of generality since the parameter $\mathrm{vol}(P)/\,\mathrm{length}(P \cap \pi^{-1}(q))$ does not change by vertical translation or vertical dilation/contraction of $P$. Notice that the polytope $P$ may be rational. Under these assumptions what we want to show that there is a global upper bound $c$ for the volume of $P$.

By considering respective supporting hyperplanes of $P$ at $(q, 0)$ and $(q, 1)$ we see that $P$ is contained in the region $f_1(x_1, \dots, x_{d-1}) \leq x_d \leq f_2(x_1, \dots, x_{d-1}) + 1$, for some linear functionals $f_1, f_2 \in (\mathbb{R}^{d-1})^*$, and as it only makes $P$ bigger and the argument easier we assume that $P$ actually equals the intersection of $\pi^{-1}(Q)$ with that region (see Figure 2.9). Now, for $\pi(P)$ to equal $Q$ we need $f_1 - f_2 \leq 1$ on $Q$, which is equivalent to saying that $f_1 - f_2$ is in the polar $Q^*$ of $Q$. The volume of $P$ is a continuous function of the functional $f_1 - f_2$. (In fact, it equals the integral in $Q$ of the function $1 + f_2 - f_1$). Since the origin is in the interior of $Q$, $Q^*$ is compact, and there is a global bound on the volume of $P$. $\square$

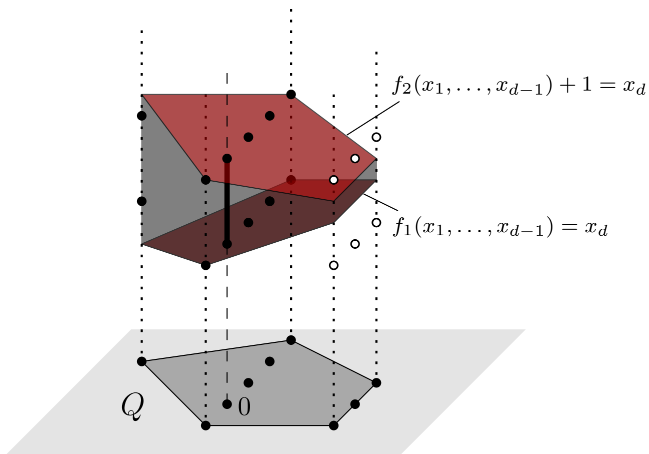



Figure 2.9: The setting of the proof of Lemma 2.4.3. The figure shows the rational $d$-polytope $\pi^{-1}(Q) \cap \{f_1(x_1, \dots, x_{d-1}) \leq x_d \leq f_2(x_1, \dots, x_{d-1}) + 1\}$.

COROLLARY 2.4.4. A non-hollow lattice polytope has only finitely many lifts of bounded size.

*Proof.* Let $Q \subset \mathbb{R}^{d-1}$ be a lattice $(d-1)$-polytope and let $q \in \mathbb{Z}^{d-1}$ be an interior lattice point of $Q$. A bound $n$ for the size of a lift $P$ of $Q$ implies a bound $n+1$ for the length of $\pi^{-1}(q) \cap P$. By Lemma 2.4.3, this gives a bound for the volume of $P$. Since there are only finitely many lattice $d$-polytopes with bounded volume ([Hen83, Thm. 3.6]), the result follows. $\square$

## 2.5 THE FINITENESS THRESHOLD WIDTH IN DIMENSION 4

According to Theorem 2.3.8, $w^\infty(4)$ equals the largest width of a hollow lattice 3-polytope with infinitely many lifts of bounded size. Since $w^\infty(4) \geq 2$ is known (Haase and Ziegler [HZ00, Proposition 6] showed infinitely many empty 4-simplices of width two), we only need to look at hollow 3-polytopes of width at least 3. Let us show that there are only five of them, all of width three (see Lemma 2.5.3 and Figure 2.10).

We start with the following classification of hollow lattice 3-polytopes:

THEOREM 2.5.1 ( [Tre08, Theorem 1.3]). Any hollow lattice 3-polytope falls exactly under one of the following categories:

1. It has width 1. All polytopes of width 1 are hollow and there are infinitely many of them for each size.
2. It has width 2 and admits a projection onto the polygon $2\Delta_2$. There are infinitely of them, although finitely many for each fixed size.
3. It has width $\geq 2$, and does not admit a projection to $2\Delta_2$. There are finitely many of them, regardless the size. They are all contained in hollow-maximal 3-polytopes.

The hollow-maximal 3-polytopes referred to in part (3) have been enumerated in [AWW11; AKW17]. More precisely, Averkov, Wagner and Weismantel [AWW11] classified the hollow lattice 3-polytopes that are not properly contained in any other hollow convex body. Then Averkov, Krümpelmann and Weltge [AKW17] showed that the maximal lattice 3-polytopes in this sense (which they call $\mathbb{R}$-maximal) coincide with the hollow-maximal lattice 3-polytopes in our sense (which they call $\mathbb{Z}$-maximal). It is known that these two notions of maximality for hollow polytopes do not coincide in dimensions four and higher [NZ11].

**Theorem 2.5.2** ( [AWW11, Theorem 2.2] and [AKW17, Theorem 1]). There are 12 hollow-maximal lattice 3-polytopes. Here given by their vertices as column vectors of the following matrices:

$$\mathcal{M}_1 \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \qquad \mathcal{M}_2 \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad \mathcal{M}_3 \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\mathcal{M}_4 \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad \mathcal{M}_5 \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} \qquad \mathcal{M}_6 \begin{pmatrix} 0 & 3 & 1 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\mathcal{M}_7 \begin{pmatrix} 0 & 4 & 1 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \mathcal{M}_8 \begin{pmatrix} 2 & -2 & 0 & 0 & 1 \\ 0 & 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad \mathcal{M}_9 \begin{pmatrix} -1 & 2 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\mathcal{M}_{10} \begin{pmatrix} 1 & 0 & -1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 3 & 3 & 3 \end{pmatrix} \qquad \mathcal{M}_{11} \begin{pmatrix} 1 & -1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 \end{pmatrix}$$

$$\mathcal{M}_{12} \begin{pmatrix} 0 & -1 & 1 & 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \end{pmatrix}$$

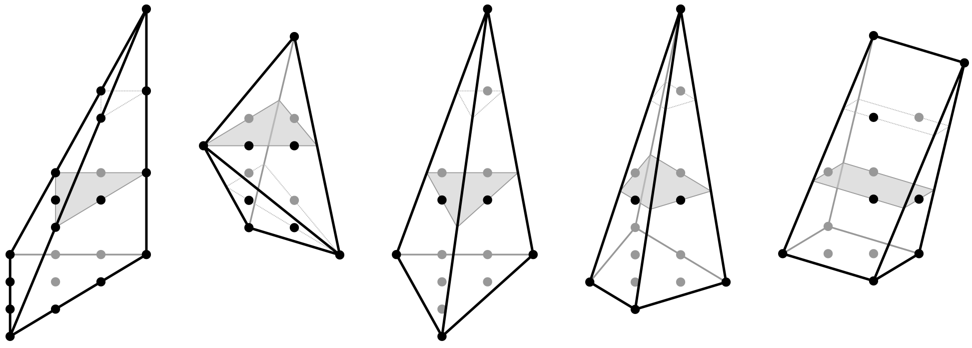
They all have width two except $\mathcal{M}_3$, $\mathcal{M}_5$, $\mathcal{M}_6$, $\mathcal{M}_9$ and $\mathcal{M}_{10}$, which have width three.

In particular, every hollow 3-polytope has width $\leq 3$ and those of width three are contained in one of $\mathcal{M}_3$, $\mathcal{M}_5$, $\mathcal{M}_6$, $\mathcal{M}_9$ and $\mathcal{M}_{10}$. These five polytopes are pictured in Figure 2.10, taken from [AKW17]. (The coordinate system in the figure is not the same as in the definition.)

A priori there could be proper subpolytopes of one of these five that still have width three, but it is not difficult to prove that this is not the case:

**Lemma 2.5.3.** The only lattice hollow 3-polytopes of width $> 2$ are $\mathcal{M}_3$, $\mathcal{M}_5$, $\mathcal{M}_6$, $\mathcal{M}_9$ and $\mathcal{M}_{10}$, and they have width three.

*Proof.* It suffices to check that all the subpolytopes of $\mathcal{M}_3$, $\mathcal{M}_5$, $\mathcal{M}_6$, $\mathcal{M}_9$ and $\mathcal{M}_{10}$ obtained by removing a single vertex have width two (or lower). For this, in turn, it suffices to find for each of the five polytopes and each vertex of it, an integer affine functional having value 3 on that vertex and values 0, 1 or 2 in all the others. Such functionals are specified in the following matrices $\mathcal{F}_3$, $\mathcal{F}_5$, $\mathcal{F}_6$, $\mathcal{F}_9$, and $\mathcal{F}_{10}$, where the $i$-th row of matrix $\mathcal{F}_j$ is the functional corresponding to the vertex that is the $i$-th



$\mathcal{M}_3$ $\mathcal{M}_5$ $\mathcal{M}_6$ $\mathcal{M}_9$ $\mathcal{M}_{10}$

Figure 2.10: The five hollow 3-polytopes of width three. This picture has been taken from Averkov et al [AKW17].

column of the matrix $\mathcal{M}_j$ from Theorem 2.5.2. A row $(a\ b\ c \mid d)$ represents the functional $(x, y, z) \mapsto ax + by + cz + d$:

$$\mathcal{F}_3 \left(\begin{array}{ccc|c} -1 & -1 & -1 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \qquad \mathcal{F}_5 \left(\begin{array}{ccc|c} -1 & 0 & 0 & 3 \\ 2 & -1 & -1 & 1 \\ -2 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \end{array}\right) \qquad \mathcal{F}_6 \left(\begin{array}{ccc|c} -1 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\mathcal{F}_9 \left(\begin{array}{ccc|c} -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \qquad \mathcal{F}_{10} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 2 \\ -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right)$$

In the two that are perhaps less obvious, $\mathcal{M}_5$ and $\mathcal{M}_{10}$, the (linear parts of) the functionals come in pairs of opposite ones. Figure 2.11 shows projections along which these functionals are coordinates (one picture, with two coordinate functionals, for $\mathcal{M}_5$, three pictures with the horizontal coordinate in each picture as one of the functionals, for $\mathcal{M}_{10}$). $\square$

*Remark* 2.5.4. To double-check we have enumerated, using Polymake [GJ00], all subpolytopes of $\mathcal{M}_1, \ldots, \mathcal{M}_{12}$ of width $\geq 2$, ordered by size. There are a total of 3992 such polytopes and Table 2.12 shows how many there are of each size. Our width algorithm is included in releases of Polymake starting with version 3.0 as a property of a polytope with command `LATTICE_WIDTH`. The lists of the subpolytopes and the algorithms we used to compute them can be found at `http://ehrhart.math.fu-berlin.de/`

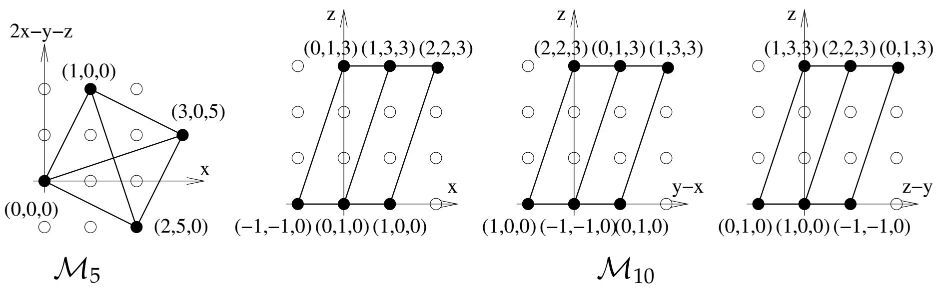



Figure 2.11: Projections showing that all proper subpolytopes of $\mathcal{M}_5$ and $\mathcal{M}_{10}$ have width $< 3$.

Research/Data/. The source code of the algorithms can also be found in chapter A.1 of the appendix.

| # Lattice Points | # Polytopes | # Lattice Points | # Polytopes |
|---|---|---|---|
| 23 | 1 (1) | 14 | 600 (5) |
| 22 | 3 (1) | 13 | 654 |
| 21 | 7 | 12 | 625 (1) |
| 20 | 22 (1) | 11 | 456 |
| 19 | 49 | 10 | 292 (1) |
| 18 | 109 (12) | 9 | 134 |
| 17 | 192 | 8 | 58 |
| 16 | 316 | 7 | 17 |
| 15 | 452 | 6 | 4 |
| 14 | 600 (5) | 5 | 1 |

Figure 2.12: Number of lattice polytopes of width $\geq 2$ contained in the maximal ones. The number in the brackets indicate where the 12 maximal polytopes come into play.

Corollary 2.5.5 (Finiteness Threshold Width in dimension 4). $w^\infty(4) = 2$. That is, for each $n \geq 5$, there exist only finitely many lattice 4-polytopes of size $n$ and width larger than two.

*Proof.* Example 2.1.5 shows that $w^\infty(4) \geq 2$.

In the light of Theorem 2.3.8, in order to prove $w^\infty(4) \leq 2$ we only need to check that no hollow 3-polytope of width larger than two has infinitely many lifts of bounded size. Lemma 2.5.3 tells us that there are only five polytopes to check, depicted in Figure 2.10. $\mathcal{M}_3$, $\mathcal{M}_5$ and $\mathcal{M}_6$ are simplices and hence have only finitely many lifts of bounded size by Corollary 2.4.2. That $\mathcal{M}_9$ and $\mathcal{M}_{10}$ have only finitely many lifts of bounded size is proved in Propositions 2.5.6 and 2.5.7 below. $\square$

PROPOSITION 2.5.6. The pyramid $\mathcal{M}_9$ has a finite number of lifts with bounded size.

*Proof.* The basis of the pyramid is a quadrilateral with three (relative) interior points. This quadrilateral has a finite number of lifts of bounded size by Corollary 2.4.4, and the whole pyramid by Lemma 2.4.1. $\square$

PROPOSITION 2.5.7. The prism $\mathcal{M}_{10}$ has finitely many lifts of bounded size.

*Proof.* Let $u, v, w, u', v'$ and $w'$ be the vertices of the prism, where $uu', vv', ww'$ are edges. Let $Q := \text{conv}\{u, v, w, u', v'\} \subset \mathcal{M}_{10}$. It is a quadrangular pyramid over a polygon with interior points.

Any tight lift of $\mathcal{M}_{10}$ will be of the form $P(\tilde{Q}, \tilde{w}')$ (as defined in Lemma 2.4.1), where $\tilde{Q}$ is a tight lift of $Q$ and $\tilde{w}'$ is a point in the fiber of $w'$. By Lemma 2.4.1 and Corollary 2.4.4, there are only finitely many such $\tilde{Q}$ of bounded size. Fix one, and let us see that there are only finitely many possibilities for $\tilde{w}'$.

Each lift $\tilde{w}'$ (together with the fixed tight lift $\tilde{Q}$) induces a lift of the quadrilateral $R := \text{conv}\{u, w, u', w'\}$. We claim that at most two choices of $\tilde{w}'$ correspond to equivalent lifts of $R$.

By fixing $\tilde{Q}$ we already have fixed a lift of the three vertices $u, w, u'$. These three lifts are contained in a plane $\Pi$. On the other hand, the possible lifts of the point $w'$ are in the line $\pi^{-1}(w')$. This line is not contained in $\Pi$, so these tight lifts of $R$ are all 3-dimensional (except for at most one lift of $\tilde{w}'$), and their volume is proportional to the distance between $\tilde{w}'$ and $\Pi$. That is, each of the possibilities for $\tilde{w}'$ induces non-equivalent tight lifts of the quadrilateral $R$, up to (perhaps) reflection with respect to the plane $\Pi$.

Now, as the quadrilateral $R$ contains interior points, Corollary 2.4.4 implies that it has only finitely many lifts of bounded size. Infinitely many choices of $\tilde{w}'$ would then have unbounded size, and so would $P(\tilde{Q}, \tilde{w}')$. That is, $\mathcal{M}_{10}$ has only finitely many tight lifts of bounded size, and Corollary 2.2.13 implies the statement. $\square$

# 3 CONVEX-NORMAL (PAIRS OF) POLYTOPES

As mentioned in the introduction the aim of this chapter is on the one hand to get to know $k$-convex-normal polytopes and improve Gubeladzes bound and on the other hand to generalise his result to pairs of polytopes.

## 3.1 CONVEX-NORMALITY REVISITED

Let $P \subseteq \mathbb{R}^d$ be a lattice polytope. Then $P$ has the *integer decompostion property (IDP)*, if for all $k \in \mathbb{N}$ and all $z \in kP \cap \mathbb{Z}^d$, there exist $x_1, \ldots, x_k \in P \cap \mathbb{Z}^d$ such that

$$z = x_1 + \cdots + x_k \,.$$

Every one or two dimensional lattice polytope has the integer decompostion property. In dimension 3, however, already simplices do not need to possess the IDP.

For example $P = \operatorname{conv}\{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\}$ does not have the IDP as $(1,1,1) \in 2P$ is not the sum of two lattice points in $P$.

Given a rational polytope $Q$ with vertex set $\operatorname{vert}(Q)$ we set

$$G(Q) := \bigcup_{v \in \operatorname{vert}(Q)} (v + \mathbb{Z}^d) \cap Q \,,$$

that is, we base the lattice in one vertex after the other and take the union of those shifted lattices inside $Q$. Note that if $Q$ is a lattice polytope, then $G(Q) = Q \cap \mathbb{Z}^d$.

Following Gubeladze [Gub12], we call a rational polytope $P \subseteq \mathbb{R}^d$ *k-convex-normal* for some $k \in \mathbb{Q}$, if for all rational $c \in [2, k]$:

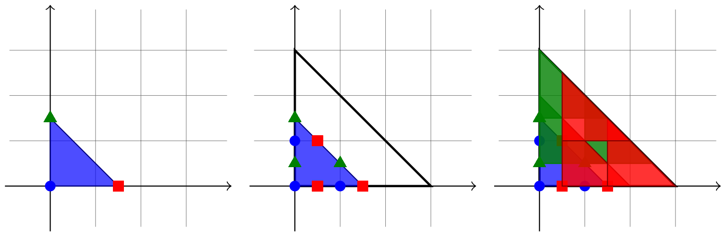
$$cP = G((c-1)P) + P \,. \tag{3.1}$$

Observe that the inclusion $\supseteq$ is always true.

**Example 3.1.1.** In Figure 3.1, where the polytope $Q$ is $\operatorname{conv}\{(0,0), (\frac{3}{2}, 0), (0, \frac{3}{2})\}$ we get

$$G(Q) = \{(0,0), (1,0), (0,1), \quad (\tfrac{3}{2}, 0), (\tfrac{1}{2}, 0), (\tfrac{1}{2}, 1), \quad (0, \tfrac{3}{2}), (0, \tfrac{1}{2}), (1, \tfrac{1}{2})\} \,.$$

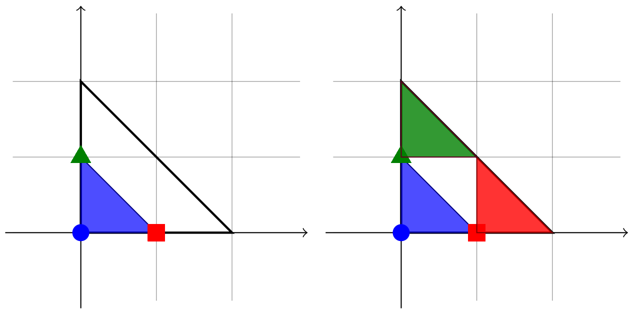
The shapes in the figures encode which vertex produced the base point for the corresponding copy of $Q$ and we can see that $Q$ is 2-convex-normal.



(a) $Q$ (b) $Q$ with $G(Q)$ (c) $G(Q)+Q=2Q$

Figure 3.1: A 2-convex-normal polytope.

**Example 3.1.2.** An easy example of a polytope which is not even 2-convex-normal is the 2-dimensional standard simplex $\Delta_2 = \text{conv}\{(0,0),(1,0),(0,1)\}$ as shown in Figure 3.2



(a) $P$ with $G(P)$ (b) $G(P)+P \neq 2P$

Figure 3.2: A polytope that is not 2-convex-normal.

Our first lemma highlights a special behavior of $G(rP)$, when $P$ is a lattice polytope.

LEMMA 3.1.3. Let $P$ be a lattice polytope and $r \in \mathbb{Q}_{>0}$, then

$$G(rP) + G(P) \subseteq G((r+1)P).$$

*Proof.* Let $x = rv + u \in G(rP)$ and $y = w + u' \in G(P)$ with $v, w \in \text{vert}(P)$ and $u, u', v, w \in \mathbb{Z}^d$. As $x \in rP$ and $y \in P$ it follows that $z = x + y \in (r+1)P$ and also

$$z = x + y = rv + u + w + u' = (r+1)v + (w - v + u + u') \in \text{vert}((r+1)P) + \mathbb{Z}^d$$

so $z \in G((r+1)P)$. $\square$

The other inclusion "$\supseteq$" does not hold in general. In fact, $G(rP) + G(P) \supseteq G((r+1)P)$ holds for all integral $r$ if and only if $P$ has the integer decomposition property. Now we can prove the main lemma of this section. Providing insight concerning equation (3.1) that $P$ has to satisfy to be $k$-convex-normal.

**Lemma 3.1.4.** Let $P$ be a 2-convex-normal lattice polytope and $c > 2$, then

$$G((c-2)P) + P = (c-1)P \text{ implies } G((c-1)P) + P = cP.$$

*Proof.* $G((c-1)P) + P \subseteq cP$ is always true, hence we only have to show the other direction $cP \subseteq G((c-1)P) + P$:

$$cP = (c-1)P + P = (G((c-2)P) + P) + P = G((c-2)P) + 2P$$

but $P$ is 2-convex-normal so that $2P = G(P) + P$ and hence:

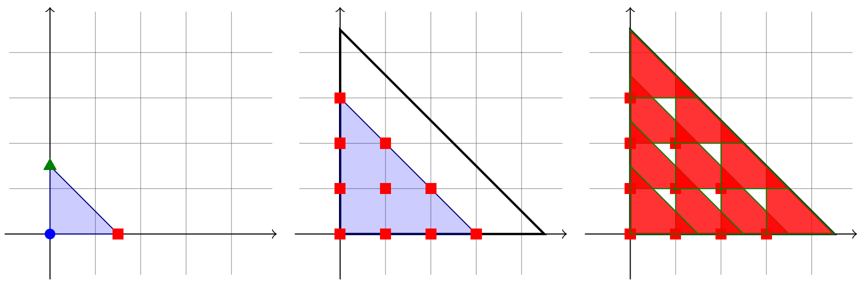
$$cP = G((c-2)P) + 2P = G((c-2)P) + G(P) + P \subseteq G((c-1)P) + P$$

where the inclusion follows from Lemma 3.1.3. □

Now, given a lattice polytope $P$, if $P$ satisfies equation (3.1) for $c = s - 1$, it will also satisfy the equation for $s$. In particular, if $P$ satisfies the equation for all rational $c$ in the interval $[2, 3]$, then $P$ satisfies it for all rational $c \geq 2$. This proves the following theorem.

**Theorem 3.1.5.** Let $P$ be a lattice polytope. If $P$ is 3-convex-normal, then $P$ is also $k$-convex-normal, for all $k \geq 2$.

*Remark* 3.1.6. Note that the implication in Lemma 3.1.4 heavily depends on $P$ being a lattice polytope, as we can see in Figure 3.3. Our polytope from Example 3.1.1, where $Q = 1,5 \cdot \Delta_2$, satisfies $2Q = G(Q) + Q$ (as seen in Example 3.1.1) but on the other hand $G(2Q) + Q \neq 3Q$.



(a) $Q$ (b) $G(2Q)$ (c) $G(2Q) + Q \neq 3Q$

Figure 3.3: $2Q = G(Q) + Q$ does not imply $3Q = G(2Q) + Q$ for rational polytopes.

The reason why this might not be too surprising is that when we think about $G(kQ)$, in the case when $kQ$ is a lattice polytope (or to be more precise, if every vertex of $kQ$ contributes the same points to $G(kQ)$), $G(kQ)$ contains the least amount of points. Meaning that we have to cover $(k+1)Q$ with fewer copies of $Q$, compared to a polytope where the vertices lie in different translated copies of $\mathbb{Z}^d$.

For lattice polytopes, combining this chain of thought with Lemma 3.1.4 we conjecture the following:

> CONJECTURE 3.1.7. Let $P$ be a lattice polytope. If $P$ is 2-convex-normal it is also $k$-convex-normal for all $k \geq 2$.

In the discussion and open problems section (3.3.1), we show that the conjecture is true in dimension 2.

Now we want to use Theorem 3.1.5, to improve Gubeladze's bound. To this end, let $e$ be the edge of a rational polytope $P$ connecting vertices $v$ and $w$. By $\ell(e)$ we denote the lattice length of $e$, i.e., let $u$ be the smallest integer vector on the line spanned by $w - v$ then $e = ku$ for some $k \in \mathbb{Q}$ and $\ell(e) := |k|$. We also consider degenerate edges with $v = w$; in this case we set $\ell(e) = 0$. With this notation we can phrase two of Gubeladzes results in the following way

> THEOREM 3.1.8. [Gub12, Theorem 1.2 and Lemma 6.2] Let $P$ be a rational $d$-polytope. If $\ell(e) \geq kd(d+1)$ for all edges of $P$, then $P$ is $k$-convex-normal.
> If $P$ is a 4-convex-normal lattice polytope, then $P$ has the IDP.

Combining these results with Theorem 3.1.5, implies that a lower bound of $\ell(e) \geq 3d(d+1)$ for every edge $e$ of $P$ would be enough. But using Lemma 3.1.4 directly, we can do even better.

> COROLLARY 3.1.9. Let $P$ be a lattice polytope. If $P$ is 2-convex-normal, then $P$ has the integer decompositions property.

*Proof.* As $P$ is 2-convex-normal, using Lemma 3.1.4 repeatedly we know that $kP = G((k-1)P) + P$ for all $k \in \mathbb{N}$. Now given $z \in kP \cap \mathbb{Z}^d$ for some $k \in \mathbb{N}$, we know that $z = x + y$ with $y \in P$, $x \in G((k-1)P) = (k-1)P \cap \mathbb{Z}^d$ and therefore $y \in P \cap \mathbb{Z}^d$. By induction we can find $x_1, \ldots, x_{k-1} \in P \cap \mathbb{Z}^d$ such that $x = x_1 + \ldots + x_{k-1}$. □

Now combining this Corollary with the previously mentioned result of Gubeladze (Theorem 3.1.8) we get the bound we promised in the introduction.

> COROLLARY 3.1.10. Let $P$ be a lattice polytope. If for every edge $e$ of $P$ the lattice length $\ell(e) \geq 2d(d+1)$, then $P$ has the integer decompostion property.

## 3.2 CONVEX-NORMALITY FOR PAIRS OF POLYTOPES

In this chapter we extend the above definitions and results to pairs of polytopes.

**Definition 3.2.1.** A pair of rational polytopes $(Q, P)$ is called *convex - normal*, if

$$Q + P = G(Q) + P$$

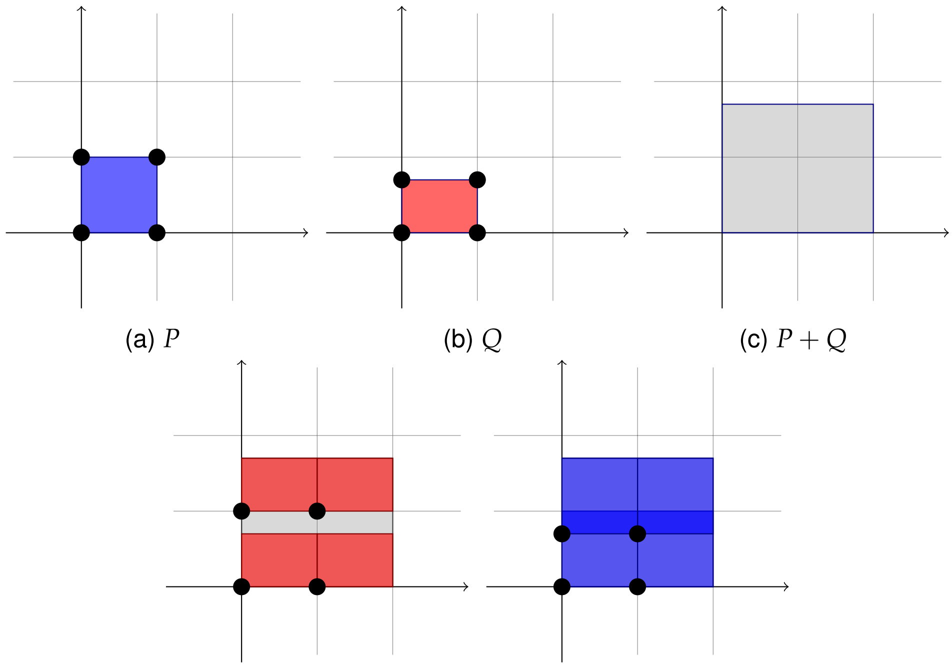
Note, that we only have to show $Q + P \subseteq G(Q) + P$ as the other inclusion is always true since $G(Q) \subset Q$. Furthermore, this notion is invariant under independent translations of $P$ and $Q$ by rational vectors: A small calculation shows that $G(Q-w) = G(Q) - w$. Hence we can set two vertices $v \in \operatorname{vert}(P)$ and $w \in \operatorname{vert}(Q)$ to $\mathbf{0}$. In these terms a single polytope $P$ is $k$-convex-normal, if for all rational $c \in [2, k]$ the pairs $((c-1)P, P)$ are convex-normal.

**Example 3.2.2.** As seen in **Example 3.1.1** the pair $(1.5 \cdot \Delta_2, 1.5 \cdot \Delta_2)$ is convex-normal and the pair $(\Delta_2, \Delta_2)$ is not. More generally, $P$ is 2-convex-normal if and only if $(P, P)$ is convex-normal.

**Example 3.2.3.** Convex-normality is not symmetric. When we set

$$P = \operatorname{conv}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad Q = \operatorname{conv}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0.7 & 0.7 \end{pmatrix}.$$

Figure 3.4 illustrates that $G(Q) + P = Q + P$ but $G(P) + Q \neq P + Q$:



(a) $P$ (b) $Q$ (c) $P+Q$

(d) $G(P) + Q \neq P + Q$ (e) $G(Q) + P = Q + P$

Figure 3.4: Convex-normality of pairs is not symmetric

The second definition we need is an extension of the integer decomposition property to pairs of polytopes:

**Definition 3.2.4.** A pair of polytopes $(Q, P)$ has the *integer decomposition property (IDP)*, if the map

$$\begin{array}{ccccc} (Q \cap \mathbb{Z}^d) & \times & (P \cap \mathbb{Z}^d) & \to & (Q + P) \cap \mathbb{Z}^d \\ (q & , & p) & \mapsto & q + p \end{array}$$

is surjective, that is, if $(P+Q)\cap\mathbb{Z}^d = (P\cap\mathbb{Z}^d)+(Q\cap\mathbb{Z}^d)$.

If the pairs $(P, nP)$ have the integer decomposition property for all $n \in \mathbb{N}$, then $P$ is a lattice polytope and has the IDP.
The pair $(\Delta_2, \Delta_2)$ from the example above has the integer decomposition property, so we see that pairs of polytopes with the IDP are not always convex-normal. But the converse implication is true:

**Lemma 3.2.5.** Let $P$ be a polytope and let $Q$ be a lattice polytope such that $(Q, P)$ is convex-normal. Then $(Q, P)$ has the integer decomposition property.

*Proof.* As $(Q, P)$ is convex-normal, we know that $Q+P = G(Q)+P$.
As $Q$ is a lattice polytope, we have $G(Q) = Q\cap\mathbb{Z}^d$ and hence

$$(Q+P)\cap\mathbb{Z}^d = (G(Q)+P)\cap\mathbb{Z}^d = ((Q\cap\mathbb{Z}^d)+P)\cap\mathbb{Z}^d = (Q\cap\mathbb{Z}^d)+(P\cap\mathbb{Z}^d)\,.$$

□

In the remainder of this chapter we will prove a sufficient condition, based on edge lengths, for a pair $(Q, P)$ to be convex-normal.
In the above examples $P$ and $Q$ had the same normal fan. If we drop this condition, there are pairs of polytopes with arbitrarily long edges lacking the integer decomposition property and not being convex-normal.

**Example 3.2.6.** Set

$$Q = \operatorname{conv}\begin{pmatrix} 0 & 1 & 0 \\ 0 & k & 1 \end{pmatrix} \quad \text{and} \quad P = \operatorname{conv}\begin{pmatrix} 0 & -l & -(l-1) \\ 0 & 1 & 1 \end{pmatrix}.$$

See Figure 3.5.
If we look at $(nQ, nP)$, then both polytopes have edge length $n$ and there are $O(n^4)$ lattice points in $(nP\cap\mathbb{Z}^2)+(nQ\cap\mathbb{Z}^2)$, but $k\cdot l\cdot O(n^2)$ lattice points in $nP+nQ$. Hence for $k, l \gg n$, the pair $(nQ, nP)$ neither has the integer decomposition property nor is it convex-normal.

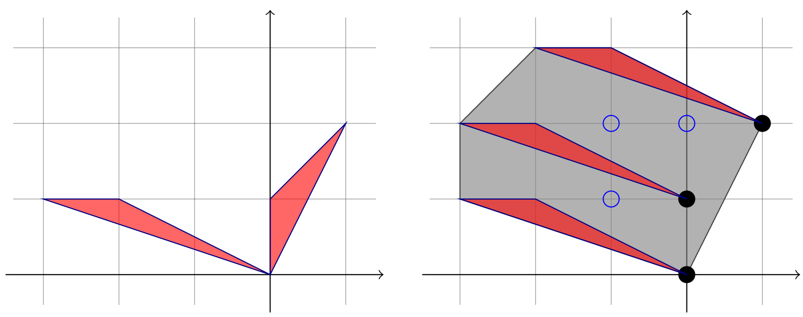


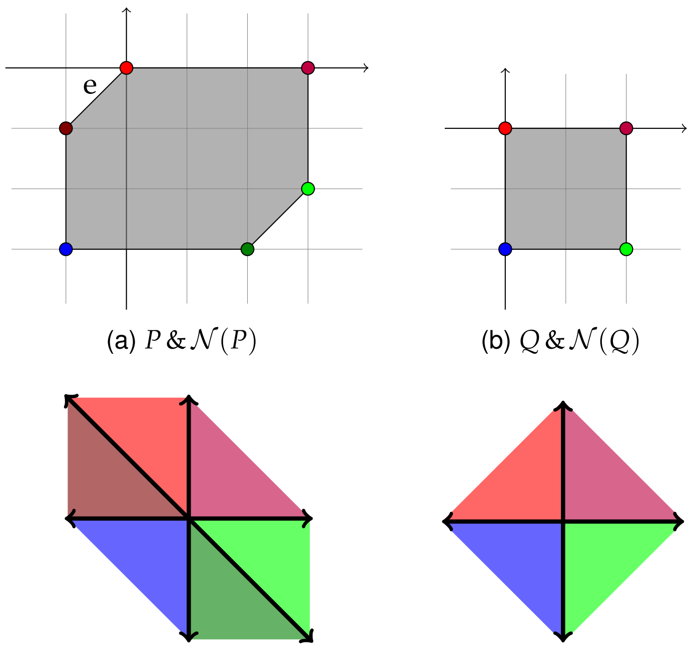
Figure 3.5: $Q+P$ and $G(Q)+P$ for $n=1$, $k=2$ and $l=3$.

For a pair $(Q, P)$ of polytopes to be convex-normal, it is not enough if both polytopes have the integer decomposition property, be $k$-convex-normal or have long edges and the examples suggest that we need a condition on the normal fans of $P$ and $Q$.

Given two $d$-polytopes $Q$ and $P$, the normal fan $\mathcal{N}(P)$ is a *refinement* of $\mathcal{N}(Q)$, if for every cone $C \in \mathcal{N}(P)$ there exists a cone $D \in \mathcal{N}(Q)$ s.t. $C \subseteq D$. In this case we can define a map $\Phi' : \mathcal{N}(P) \to \mathcal{N}(Q)$ s.t. $\Phi'(C)$ is defined as the smallest cone in $\mathcal{N}(Q)$ containing $C$. This map preserves inclusions and has a corresponding map $\Phi : \mathcal{L}(P) \to \mathcal{L}(Q)$ on the face lattices of $P$ and $Q$, taking a face $F \prec P$ with corresponding cone $C_F$ to the face $G \prec Q$ with corresponding cone $C_G = \Phi'(C_F)$.

**Example 3.2.7.** In Figure 3.6 we illustrate the map with

$$P = \operatorname{conv}\begin{pmatrix} 0 & 3 & 3 & 2 & -1 & -1 \\ 0 & 0 & -2 & -3 & -3 & -1 \end{pmatrix} \quad \text{and} \quad Q = \operatorname{conv}\begin{pmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 \end{pmatrix}.$$




(a) $P \,\&\, \mathcal{N}(P)$ (b) $Q \,\&\, \mathcal{N}(Q)$

Figure 3.6: Each face of $P$ corresponds to a face of $Q$

For example the edge $e$ from $(-1,-1)$ to $(0,0)$ in $P$ corresponds to the vertex $(0,0)$ in $Q$, i.e. $\Phi(e) = (0,0)$ because $e$ corresponds to cone $\begin{pmatrix} -1 \\ 1 \end{pmatrix} \in \mathcal{N}(P)$ and the smallest cone of $\mathcal{N}(Q)$ containing it is cone $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, which is the normal cone belonging to $(0,0)$ in $Q$.

### 3.2.1 *A sufficient condition for convex-normality of (Q,P)*

Now that we got all the tools lined up, we can start the proof of our main result with the following lemma, which is the base case for our induction:

**Lemma 3.2.8.** Let $P = [0, p]$ and $Q = [0, m]$ be intervals with $p \geq \min\{1, m\}$, then $(Q, P)$ is convex-normal.

*Proof.* Set $l := \lfloor m \rfloor$. If $l \geq 1$, then $p \geq 1$ and

$$Q + P = [0, p + m] = \left( \bigcup_{i=0}^{l} i + [0, p] \right) \cup (m + [0, p]) \subseteq G(Q) + P.$$

If $l < 1$, then $p \geq l$ and:

$$Q + P = (0 + P) \cup (m + P).$$

□

Now we can prove the main result.

**Theorem 3.2.9.** Let $P$ and $Q$ be rational $d$-polytopes such that $\mathcal{N}(P)$ is a refinement of $\mathcal{N}(Q)$ and such that $\ell(e_P) \geq d \cdot \ell(e_Q)$ for every edge $e_P \prec P$ and corresponding face (edge or vertex) $e_Q = \Phi(e_P) \prec Q$. Then $(Q, P)$ is convex-normal.

*Proof.* Lemma 3.2.8 took care of the base case. Hence let $P$ and $Q$ be $d$-polytopes with $d \geq 2$.

**STEP 1 - SUBDIVIDING $Q + P$:**

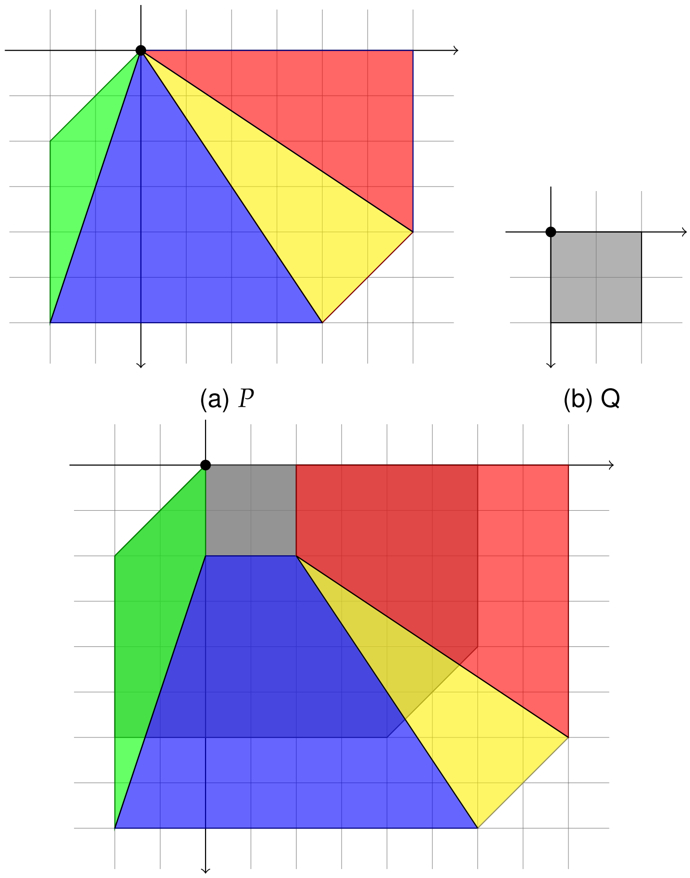
Without loss of generality we assume $\mathbf{0} \in \text{vert}(P)$ and $\mathbf{0} = \Phi(\mathbf{0}) \in \text{vert}(Q)$ and start by subdividing $Q + P$ by assigning weights/heights to the vertices of $P$ and $Q$. Vertices of $Q$ and the vertex $\mathbf{0}$ of $P$ get height 0 and all the other vertices of $P$ get height 1. We use those heights to define new polytopes $P'$ and $Q'$ in $\mathbb{R}^{d+1}$ as follows.

$$\begin{aligned} Q' &:= \text{conv}\{(w, 0) : w \in \text{vert}(Q)\} \quad \text{and} \\ P' &:= \text{conv}\left((\mathbf{0}, 0) \cup \{(u, 1) : u \in \text{vert}(P) \setminus \{\mathbf{0}\}\}\right). \end{aligned}$$

Then the projection of $P' + Q'$ onto the first $d$ coordinates is $P + Q$ and the lower boundary of $P' + Q'$ induces a subdivision of $P + Q$ into the following pieces.

$$0 + Q \quad \text{and} \quad F_Q + (\text{conv}(0, F_P))$$

for faces $F_Q \prec Q$ and faces $F_P \prec P$, with $0 \notin F_P$ and $\Phi(F_P) = F_Q$. Compare Figure 3.7.



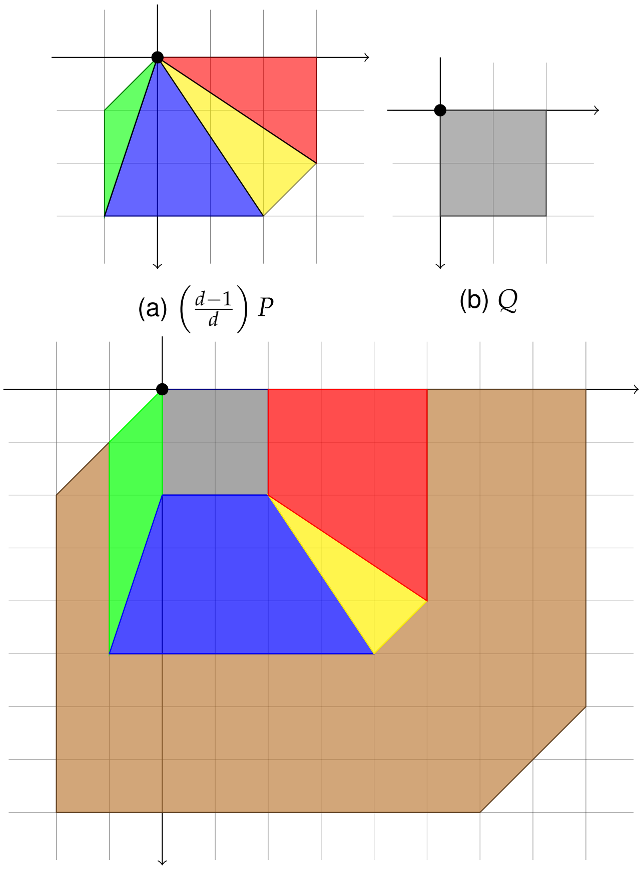
(a) $P$ (b) Q

Figure 3.7: $Q+P$ subdivided into $0+Q$ and $F_Q+(\mathrm{conv}(0,F_P))$.

Another decomposition of $P+Q$ we will be using, is the following:

$$I := \left(\frac{d-1}{d}\right) P + Q \quad \text{and} \quad B := \overline{(P+Q)\setminus I}$$

Where $I$ stands for the "inner" part of $P+Q$ and $B$ stands for the "boundary" part of $P+Q$; see Figure 3.8



(a) $\left(\frac{d-1}{d}\right) P$ (b) $Q$

Figure 3.8: $Q + P$ divided into $I$ and $B$.

In the next step we will be using our first sudivision to cover the boundary part. We will then show that covering $I$ is easy because it lies in $0 + P$.

**STEP 2.1 - COVERING B:**

Let $x \in B$, then $x \notin Q$ and hence we can find facets $F_P \prec P$ and $F_Q \prec Q$ such that $x \in F_Q + (\mathrm{conv}(0, F_P))$ coming from our subdivision in STEP 1. Hence $x$ can be written as $x = q + \mu p$, with $q \in F_Q \prec Q$, $p \in F_P \prec P$ and $0 \le \frac{d-1}{d} \le \mu \le 1$. Then $z := q + \frac{d-1}{d} p$ is contained in $\frac{d-1}{d} F_P + F_Q$. Furthermore $(F_Q, \frac{d-1}{d} F_P)$ is convex-normal by induction, as $\mathcal{N}(\frac{d-1}{d} F_P)$ is a refinement of $\mathcal{N}(F_Q)$ and given edges $e_{F_Q} \prec F_Q$ and $\frac{d-1}{d} e_{F_P} \prec \frac{d-1}{d} F_P$ ($\Leftrightarrow e_{F_P} \prec F_P$) we have

$$\ell\left(\frac{d-1}{d} e_{F_P}\right) = \left(\frac{d-1}{d}\right) \ell\left(e_{F_P}\right) \ge \left(\frac{d-1}{d}\right) \cdot d\ell(e_{F_Q}) = (d-1)\ell(e_{F_Q}).$$

Hence we can find a point $g \in G(F_Q)$ such that $z \in g + \frac{d-1}{d} F_P$, and since $p \in F_P \subseteq \mathrm{conv}(0, F_P)$ we get $x \in g + \mathrm{conv}(0, F_P) \subseteq g + P$, as illustrated in Figure 3.9.

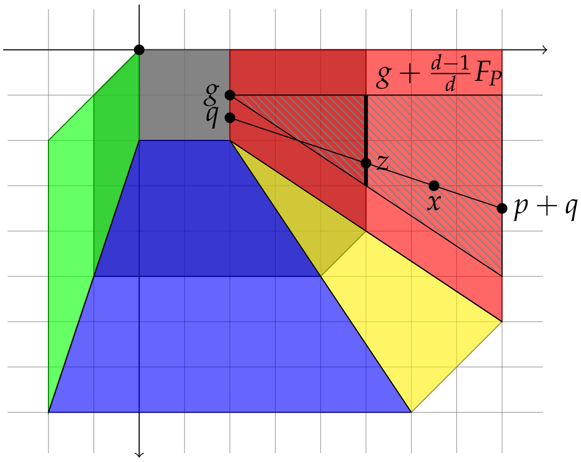



Figure 3.9: Covering $B$ using induction.

**STEP 2.2 - COVERING I:**

Now we are left with covering the points in the inner part $I$ of $P+Q$. We claim that $I \subseteq P$, which implies $I \subseteq 0 + P \subseteq G(Q) + P$. First we reformulate the problem by using that $I = \left(\frac{d-1}{d}\right) P + Q \subseteq P$ is equivalent to $Q \subseteq \frac{1}{d}P$.

To show the latter, suppose $Q \not\subseteq \frac{1}{d}P$, then there exists a vertex $u$ of $Q$ that does not lie in $\frac{1}{d}P$. This implies that there exists a functional $c$ such that $c^t u = b$ and $c^t x < b$ for all $x \in \frac{1}{d}P$. When we use the simplex method to maximize $c$ over $\frac{1}{d}P$ starting in 0, we get a monotone edge path from 0 to an optimal vertex $u'$. As $\mathcal{N}(\frac{1}{d}P)$ is a refinement of $\mathcal{N}(Q)$ we have an inclusion-preserving map $\mathcal{L}(\frac{1}{d}P) \to \mathcal{L}(Q)$ between the two face lattices. Using this map, we get a corresponding edge path in $Q$, which also ends in an optimal vertex $u''$, as $c \in C_{u'} \subseteq C_{u''}$. But as every edge in $\frac{1}{d}P$ is at least as long as the corresponding face (edge or vertex) in $Q$, we have

$$c^t u' \geq c^t u'' = c^t u \quad \lightning$$

Hence no vertex of $Q$ is lying outside of $\frac{1}{d}P$, so that $Q \subseteq \frac{1}{d}P$ which finishes our proof. $\square$

Theorem 3.2.9 requires $Q$ to be a lot smaller than $P$. But in conjunction with the following Lemma, it can be used in certain cases where $Q$ is allowed to be big.

LEMMA 3.2.10. Let $P$ be a rational polytope and $Q$ be a lattice polytope, with

$$Q = Q_1 + \ldots + Q_s$$

where the $Q_i$ are lattice polytopes such that the pairs $(Q_i, P)$ are convex-normal for all $i$. (For example, they could satisfy the conditions of the previous Theorem.) Then $(Q, P)$ is convex-normal.

*Proof.* As $(Q_i, P)$ are convex-normal we get:

$$\begin{aligned} Q + P &= (Q_1 + \ldots + Q_s) + P \\ &= G(Q_1) + \ldots + G(Q_s) + P \\ &\subseteq G(Q_1 + \ldots + Q_s) + P \\ &= G(Q) + P \end{aligned}$$

where the second equality is true because the Minkowski sum is commutative and associative and the inclusion is true because the $Q_i$ are lattice polytopes. $\square$

In particular, if $Q$ is a lattice polytope and $(Q, P)$ is convex-normal, then $(kQ, P)$ is convex-normal for all $k \in \mathbb{N}$. Putting together Lemma 3.2.5, Theorem 3.2.9 and Lemma 3.2.10 we get the following corollary.

COROLLARY 3.2.11. Let $P$ and $Q$ be rational polytopes, where $\mathcal{N}(P)$ is a refinement of $\mathcal{N}(Q)$. If $Q$ has a decomposition into lattice polytopes $Q = Q_1 + \ldots + Q_s$ and every edge of $P$ is at least $d$ times as long as the corresponding edge in $Q_i$ for all $i$, then $(Q, P)$ has the integer decomposition property.

## 3.3 DISCUSSION OF QUESTIONS AND OPEN PROBLEMS

In this section we state and discuss some questions and open problems, some of which we already mentioned in the two previous sections. For some we also state partial results, but in particular this section is meant to point the interested reader to some question, where we would love to know the answer.

### 3.3.1 *Does being 2-convex-normal imply being k-convex-normal for lattice polytopes?*

We already know from Lemma 3.1.4 that it is enough to show that the equation is true for all $c \in [2,3)$, because

$$cP = G((c-1)P) + P \quad \Longrightarrow \quad (c+1)P = G(cP) + P \quad \forall c \geq 2$$

This Lemma also implies, that if $P$ is 2-convex-normal in addition to $2P = G(P) + P$ we get that $nP = G((n-1)P) + P \quad \forall n \in \mathbb{N}$.

So given a 2-convex-normal lattice polytope $P$ and some $c \in [2,3)$. Suppose without loss of generality that $0 \in P$, that way $2P$ is automatically covered as $G(P) \subset G((c-1)P)$. ( In fact here $G(P) = P \cap \mathbb{Z}^d$ ).
Hence the strip that is left to cover is $cP \setminus 2P$.

LEMMA 3.3.1. For a point $z \in cP \setminus 2P$, if a vertex of $z - (c-2)P$ lies in $2P$, we can cover $z$.

*Proof.* Let $(c-2)v$ be a vertex of $(c-2)P$, s.t. $x-(c-2)v \in 2P$. As we can cover $2P$, there exists a $g \in G(P)$ s.t. $z-(c-2)v \in g+P$, here $g = w' + z'$ with $w' \in \text{vert}(P)(\cap\mathbb{Z}^d)$ and $z' \in \mathbb{Z}^d$.
<u>We claim:</u> $g + (c-2)v \in G((c-1)P)$
Clearly $g + (c-2)v \in (c-1)P$ as $g \in P$ and $(c-2)v \in (c-2)P$. So we have to show that $g+(c-2)v = w+z$ with $w \in \text{vert}((c-1)P)$ and $z \in \mathbb{Z}^d$. Set $v' := \frac{(c-1)}{(c-2)}(c-2)v$. This is a vertex of $(c-1)P$ and $v' - (c-2)v = v$ is the corresponding vertex on $P$ and as $P$ is a lattice polytope also $v \in \mathbb{Z}^d$. But then

$$g + (c-2)v = g + (c-1)v - v = v' + \underbrace{(g-v)}_{\in\mathbb{Z}^d}$$

□

The following lemma gives us a condition on $c$, that guarantees that we will find such a vertex.

LEMMA 3.3.2. If $c \leq \frac{2}{d} + 2$ and $x \in cP \setminus 2P$, then a vertex of $x - (c-2)P$ is contained in $2P$.

*Proof.* Suppose all vertices of $Q := x - (c-2)P$ are outside of $2P$. And without loss of generality suppose that $P$ is a simplex with $\text{vert}\, P = \{0, v_1, \ldots, v_d\}$, then $\text{vert}\, Q = \{x, x-(c-2)v_1, \ldots, x-(c-2)v_d\}$ and $2P = \text{conv}(0, 2v_1, \ldots, 2v_d)$.

So given $x \in cP \setminus 2P \implies x = \sum_{i=1}^{d} \lambda_i c v_i$ with $\sum_{i=1}^{d} \lambda_i \leq 1$ and $\lambda_i \geq 0$. $x - (c-2)v_k = \lambda_1 c v_1 + \ldots + \lambda_{k-1} c v_{k-1} + (\lambda_k c - (c-2))v_k + \ldots + \lambda_d c v_d$ and the question is, when is this equal to $\sum_{i=1}^{d} \mu_j 2v_j$ with $\sum_{i=1}^{d} \mu_j \leq 1$ and $\mu_j \geq 0$.

For $x - (c-2)v_k$ not to be in $2P$ we know that $\lambda_k c - (c-2) < 0$, which implies that $\lambda_k < \frac{c-2}{c}$. As we assumed that all vertices of $Q$ are outside of $2P$, this has to be true for all $k$. But if we now rewrite $x$ as a conic combination of $2v_1, \ldots, 2v_d$ we get $x = \sum_{i=1}^{d} \lambda_i \frac{c}{2}(2v_i)$ and for the sum of coefficients we get

$$\sum_{i=1}^{d} \frac{c}{2}\lambda_i < \frac{c}{2}(d \cdot \frac{c-2}{c}) = \frac{d}{2}(c-2) \text{ which is } \leq 1 \text{ if and only if } c \leq \frac{2}{d} + 2$$

But if the sum is less or equal to 1, this implies that $x \in 2P$, which contradicts our assumption. □

Now combining the two previous lemmata we get that a 2-convex-normal polytope is never just 2-convex-normal but always a tad more, albeit the extra is decreasing with dimension.

**Corollary 1.** *Let $P$ be a 2-convex-normal lattice polytope. Then $P$ is also $k$-convex-normal for $k \leq \frac{2}{d} + 2$.*

But if we look at dimension 2, then this corollary implies that a 2-convex-normal polytope, is also 3-convex-normal. Combining this with Lemma3.1.4, we get a positive answer to Conjecture 3.1.7 at least for dimension 2

**Corollary 2.** *Let $P$ be a 2-dimensional 2-convex-normal polytope, then $P$ is $k$-convex-normal for all $k$.*

### 3.3.2 *Classes of convex-normal polytopes*

In the section about convex-normal pairs we already saw that we can translate both polytopes without losing the convex-normal property. The same holds true for a $k$-convex-normal polytope.

LEMMA 3.3.3. Let $P$ be a $k$-convex-normal polytope and $v \in \mathbb{Q}^d$, then $v + P$ is $k$-convex-normal.

*Proof.* If $P$ is $k$-convex-normal, then for all $r \in [2, k]$, $rP$ is covered by certain copies of $P$. If we now translate $P$ by $v$, then $rP$ is translated by $rv$, which translates the vertices of

$$v + G(Q) = G(v + Q)$$

$$\begin{aligned} r(v+P) &= rv + rP &&= rv + G((r-1)P) + P \\ &= (r-1)v + G((r-1)P) + (v+P) &&= G((r-1)(v+P)) + (v+P) \end{aligned}$$

□

In Lemma 3.2.8 we saw, that a pair of two not to short intervals ( 1-dimensional parallelepipeds ) is convex-normal, a similar result holds in the general case. The following lemma shows, that also in the non-pair case we can show that parallelepipeds with side length at least 1 are $k$-convex-normal.

LEMMA 3.3.4. Let $P$ be an orthogonal parallelepiped, assume without loss of generality $P = \prod_{i=1}^{d}[0, r_i]$. If all edges have length at least 1, i.e. if $r_i \geq 1$ $\forall i$, then $P$ is $k$-convex-normal for all $k$.

*Proof.* Let $c \in [2, k] \cap \mathbb{Q}$, then $cP = \prod_{i=1}^{d}[0, cr_i]$ and let $z \in cP$. We then find the base point $g \in G((c-1)P)$ in the following way. We can write $z = (a_1 r_1, \ldots, a_d r_d)$ with $0 \leq a_i \leq c$ and set

$$g := (g_1, \ldots, g_d), \text{ where } g_i := \begin{cases} (c-1)r_i, & \text{if } a_i \geq c-1 \\ \lfloor a_i r_i \rfloor, & \text{otherwise} \end{cases}$$

This implies that $g \in G((c-1)P)$ as without loss of generality we assume that the first $l$ entries fall under the second category and the last $d-l$ under the first and then

$$\begin{array}{rcccc} g & = & (\lfloor a_1 r_1 \rfloor, \ldots, \lfloor a_l r_l \rfloor, (c-1)r_{l+1} \ldots, (c-1)r_d) & & \\ & = & (0, \ldots, 0, (c-1)r_{l+1} \ldots, (c-1)r_d) & + & (\lfloor a_1 r_1 \rfloor, \ldots, \lfloor a_l r_l \rfloor, 0, \ldots, 0) \\ & \in & \text{vert}((c-1)P) & + & \mathbb{Z}^d \end{array}$$

and $g \in (c-1)P$ is clear as $0 \leq g_i \leq (c-1)r_i$. But with that we are finished as now $z \in g + P$.

□

For polytopes with the integer decomposition property it is known, that if you can cover a polytope $P$ with polytopes that have the IDP, then $P$ will also have the integer decomposition property. For $k$-convex-normal polytopes this is not so clear, but in the case of 2-convex-normal polytopes, we get a weaker version of the aforementioned statement.

LEMMA 3.3.5. If $P$ is covered by 2-convex-normal polytopes $Q_1, \ldots, Q_r$, such that $\text{vert}(Q_i) \subset G(P)$ , then $P$ is 2-convex-normal itself.

*Proof.* Given $P$ with $P = \bigcup_{i=1}^{r} Q_i$, then $\text{vert}(Q_i) \subset G(P)$, implies $G(Q_i) \subseteq G(P)$ and hence

$$2P = \bigcup_{i=1}^{r} 2Q_i = \bigcup_{i=1}^{r} (G(Q_i) + Q_i) \subseteq G(P) + \bigcup_{i=1}^{r} Q_i = G(P) + P,$$

which implies that $P$ is 2-convex-normal. □

### 3.3.3 *Finding polytopes with long edges that are not k-convex-normal*

Theorem 3.1.8 by Gubeladze tells us, that if $l(e) \geq kd(d+1)$ for all edges $e$ of $P$, then $P$ is $k$-convex-normal. In the case of lattice polytopes this bound can be improved to $3d(d+1)$ using Lemma 3.1.4 (see Theorem 3.1.5) and so a natural question is how low this bound could get and for this we need examples of polytopes with long edges that are not $k$-convex-normal.

One family of examples we already used a lot were simplices and they indeed provide us with a first lower bound.

Lemma 3.3.6. Let $\Delta_d = \operatorname{conv}(0, e_1, \ldots, e_d)$ be the standard simplex. Then

- $(d-1)\Delta_d$ is not 2-convex-normal.
- $r\Delta_d$ is $k$-convex-normal for all $r \geq d$ (and all $k$).

*Proof.* Starting with the first part, take $z = (1 - \frac{1}{d+1}, \ldots, 1 - \frac{1}{d+1})$, a point in $2(d-1)\Delta_d$ that is not covered by a copy of $(d-1)\Delta_d$, as in this case $G(\Delta_d) = \Delta_d \cap \mathbb{Z}^d$ and the only possible base point for $z$ is $\mathbf{0}$, because all other points in $G(\Delta_d)$ have at least one entry $\geq 1$. But $\sum z_i = d(1 - \frac{1}{d+1}) = d - \frac{d}{d+1} > d - 1$ and hence $z \notin \mathbf{0} + (d-1)\Delta_d$.

We show the second part by constructing a suitable base point for an arbitrary point. So given $z \in c(r\Delta_d)$ with $c \in [2,k] \cap \mathbb{Q}$, we have to find a $g \in G((c-1)r\Delta_d)$ with $z \in g + r\Delta_d$. Set $e = (c-1)r - \lfloor (c-1)r \rfloor$. If $\lfloor z \rfloor \in (c-1)r\Delta_d$ we are finished, as $\sum z_i - \lfloor z_i \rfloor \leq d \leq r$ and hence $z \in \lfloor z \rfloor + r\Delta_d$. So suppose that this is not a case and further we assume that $z_1$ is the maximal entry of $z$. Now set

$$g^0 = (\lfloor z_1 \rfloor - 1 + e, \lfloor z_2 \rfloor, \ldots, \lfloor z_d \rfloor)$$

As $\lfloor z \rfloor \notin (c-1)r\Delta_d$, $g^0$ is our first candidate and we iterate from $g^i$ to $g^{i+1}$ by subtracting 1 at some non-zero entry. Let $g = g^j$ the the iteration, s.t. $\sum g_i = (c-1)r$. Then $g \in G((c-1)r\Delta_d$ and $z \in g + r\Delta_d$. □

There is still a lot of room between $d-1$ and $kd(d+1)$, so it would be interesting to find more examples like this. One big problem in finding nice examples is that testing if something is $k$-convex-normal is a non-trivial task, at least for everything of dimension bigger than 2. Having a good algorithm to decide convex-normality would be great which is our next open problem.

### 3.3.4 *Finding a (better) algorithm to decide, if a polytope is 2-convex-normal or a pair is convex-normal*

The following is an algorithm for deciding, if a polytope $P$ is 2-convex-normal. We restrict ourselves to 2-convex-normality because from $(2+\epsilon)$-convex-normality onwards we have infinitely many cases to check and so a good algorithm to decide 2-convex-normality and convex-normality of pairs is a good place to start. The polymake source code for this algorithm can be found in the appendix, in chapter A.2.

**Algorithm 1:** Checking if a polytope $P$ is 2-convex-normal

**Input** : Lattice polytope $P$
**Output**: Yes - if $P$ is 2-convex-normal; No - otherwise.

1 Compute $G(P)$
2 Compute $P_I := \bigcap_{i \in I} g_i + P$ for all subsets of $I \subset G(P)$
3 Compute $V = \sum_{I \subset G(P)} (-1)^{|I|+1} \operatorname{vol}(P_I)$
4 Check if $\operatorname{vol}(2P) = V$, if positive, then $P$ is 2-convex-normal

As for why this algorithm works, suppose we count some part $n$ times, i.e. $n$ copies of $P$ contain this part. Then we add it's volume $\binom{n}{1}$ times by adding the volumes of copies of $P$ containing it. We then remove it $\binom{n}{2}$ times by subtracting the pairwise intersections of said copies containing it. Continuing like this we count it exactly $\sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} = 1$ times. ( The equation follows from the binomial theorem $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ )

The same algorithm of course works for pairs $(Q, P)$, we just have to substitute $G(Q)$ for $G(P)$ and compare the volume to $\operatorname{vol}(Q+P)$ in the end.

We wrote a script using the above algorithm in polymake and looking at table 3.10 we see that the computations take very long even in the case of very small and easy polytopes, like the dilated standard simplex. The fields with "–" indicate times when we canceled the computations before completion after already running it for more than three days.

| $k$ | $k\Delta_2$ | $k\Delta_3$ |
|---|---|---|
| 1 | 0.23 | 0.35 |
| 2 | 0.71 | 1.96 |
| 3 | 3.75 | 29.27 |
| 4 | 31.26 | 1390 |
| 5 | 362 | – |
| 6 | – | – |

Figure 3.10: Computation times in seconds for checking if $k\Delta_2$ and $k\Delta_3$ are 2-convex-normal

One better idea for an algorithm might be to use the fact that we are only looking at shifted copies of $P$, so that the inequalities are all basically the same, so we could create sort of a "subdivision" of $Q+P$ (might not cover all of it) and keep track of the single parts, so we do not have to compute all the intersections and all the volumes, but only the ones actually appearing.

# 4 12, 24 AND BEYOND FANIFIED

In the introduction we have seen an exhaustive proof for the following Theorem.

THEOREM 4.0.1. Let $P$ be a reflexive 2-polytope and $P^\star$ its polar dual, then

$$\sum_{e \in P[1]} \ell(e) + \sum_{e' \in P^\star[1]} \ell(e') = 12.$$

As promised, we will provide a nicer proof for this result and its 3-dimensional counterpart later in section 4.3. But first we want to generalise Theorem 1.3.6 of Godinho, Heymann and Sabatini [GHS16] to complete unimodular fans.

## 4.1 GENERALIZING IT TO FANS

Consider a $(d-1)$-cone $\tau$ in a complete unimodular fan $\Sigma$ in $\mathbb{R}^d$ and a primitive generator $v$ of a ray $\rho$ that together with $\tau$ is forming a $d$-cone $\sigma \in \Sigma[d]$. Because $\Sigma$ is unimodular, $v$ has to lie in a hyperplane $H$ that is parallel to that generated by $\tau$, which has lattice distance 1 to it, i.e. there is no lattice point between the two hyperplanes. This leads to the following key observation.

LEMMA 4.1.1. Let $\Sigma$ be a complete unimodular fan in $\mathbb{R}^d$. Every $(d-1)$-cone $\tau \in \Sigma[d-1]$ with primitive generators $v_1, \dots, v_{d-1}$ is contained in precisely two $d$-cones $\sigma = \operatorname{cone}(\tau, v_d)$ and $\sigma' = \operatorname{cone}(\tau, v'_d)$ in $\Sigma$.
In this situation, there are unique integers $a(\tau, v_i)$ such that

$$v_d + v'_d = \sum_{i=1}^{d-1} a(\tau, v_i) v_i.$$

In dimension 2 we sometimes abbreviate $a(\tau, v_i)$ to $a(\tau)$ or $a_i$, because in this case the sum has only one summand.

**Example 4.1.2.** As an example take the fan $\Sigma$ pictured in Figure 4.1. There are three $(d-1=1)$-dimensional cones $\tau_1 = \operatorname{cone}(e_1), \tau_2 = \operatorname{cone}(e_2)$ and $\tau_3 = \operatorname{cone}(-e_1-e_2)$ and three 2-dimensional cones $\operatorname{cone}(e_1, e_2)$, $\operatorname{cone}(e_1, -e_1-e_2)$ and $\operatorname{cone}(e_2, -e_1-e_2)$.

- For $\tau_1$ we get $e_2 + (-e_1 - e_2) = a(\tau_1, e_1)e_1$ and hence $a(\tau_1, e_1) = -1$,
- for $\tau_2$ we get $e_1 + (-e_1 - e_2) = a(\tau_2, e_2)e_2$ and hence $a(\tau_2, e_2) = -1$ and finally
- for $\tau_3$ we get $e_1 + e_2 = a(\tau_3, -e_1 - e_2)(-e_1 - e_2)$ and hence $a(\tau_3, -e_1 - e_2) = -1$ as well.

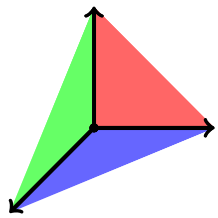


Figure 4.1: Fan used in Example 4.1.2

The base to our endeavours is the 2-dimensional case.

THEOREM 4.1.3. Let $\Sigma$ be a complete unimodular fan in $\mathbb{R}^2$. Then

$$\sum_{\tau \in \Sigma[1]} (3 - a(\tau)) = 12. \tag{4.1}$$

We are going to prove it by showing that the set of complete unimodular 2-dimensional fans is connected by the operation of barycentric subdivisions and their inverses. That the same theorem is true in dimension $d$ is much harder to prove and now known as the toric case of the weak factorization theorem [AMR99, Corollary 7.11]. Having the connectedness and using that the equation holds for the fan in example 4.1.2, we only need to show, that the equation holds for a fan $\Sigma$ if and only if it holds for a barycentric stellar subdivisions $\Sigma'$ of $\Sigma$.

**Definition 4.1.4** (compare [Hud69]). Let $\Sigma$ be a complete unimodular fan and $\sigma := \mathrm{cone}(v_1, \ldots, v_k) \in \Sigma[k]$ a cone, with primitive ray generators $v_1, \ldots, v_k$ and set $v := \sum_{i=1}^{k} v_i$. Then we define

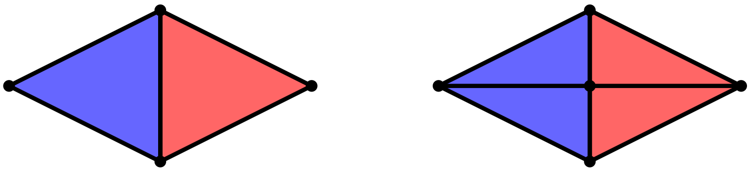
- $\mathrm{star}(\sigma, \Sigma) = \{\tau \in \Sigma \ : \ \exists F \in \Sigma \text{ with } \tau \prec F \text{ where } \sigma \prec F\}$ the *(closed) star* of $\sigma$ in $\Sigma$, where we also just write $\mathrm{star}(\sigma)$ if it is clear which fan we are talking about.

- $\mathrm{lk}(\sigma, \Sigma) = \{\tau \in \mathrm{star}(\sigma, \Sigma) \ : \ \sigma \cap \tau = \emptyset\}$ the *link* of $\sigma$ in $\Sigma$.

- $\mathrm{st}_\sigma(\Sigma) = \{\Sigma \setminus \mathrm{star}(\sigma)\} \cup \{\mathrm{cone}(w, F, G) : w \in \{0, v\}, F \precneqq \sigma \text{ and } G \in \mathrm{lk}(\sigma)\}$ the *barycentric stellar subdivision* of $\Sigma$ in $\sigma$.

- An *inverse barycentric stellar subdivision* is the process of going from $\mathrm{st}_\sigma(\Sigma)$ to $\Sigma$.

As all our subdivisions in this chapter will be barycentric stellar subdivisions, we sometimes abbreviate it to just *subdivision*. In Figure 4.2, we see (a 2-dimensional picture) of a stellar subdivision of a 2-cone $\sigma$ in 3-dimensional fan.



(a) $\operatorname{star}(\sigma,\Sigma)$ (b) $\{\operatorname{cone}(w,F,G) : w \in \{0,v\},$
$F \precneqq \sigma$ and $G \in \operatorname{lk}(\sigma)\}$

Figure 4.2: Example for a subdivision of a 2-cone in a 3-dimensional fan

LEMMA 4.1.5. Given a complete unimodular fan $\Sigma$ in $\mathbb{R}^2$, there is a complete unimodular fan $\Sigma'$ containing $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$ as primitive ray generators, which refines $\Sigma$ and that we can reach from $\Sigma$ by a series of barycentric stellar subdivisions.

*Proof.* We show it for $(0,1)$, the other ones work the same. Given $\Sigma$ and suppose $\operatorname{cone}(0,1)$ is not a ray of $\Sigma$, then there are to rays $\rho_1, \rho_2 \in \Sigma[1]$ with ray generators $v_1, v_2$ such that $(0,1) \in \sigma = \operatorname{cone}(v_1,v_2)$ and say $v_1 = (a,b)$ and $v_2 = (c,d)$ with $a < 0$ and $c > 0$. Then subdividing the fan in this cone and setting $v_3 = (a+c, b+d)$ we get a new ray $\rho_3 = \operatorname{cone}(v_3)$. Then $(0,1)$ is either in $\sigma_1 = \operatorname{cone}(v_1,v_3)$ or $\sigma_2 = \operatorname{cone}(v_2,v_3)$. Iterating this process we see that the absolute value of the sum in the first coordinate decreases monotonically. Reaching $0$ we get $(0,1)$. $\square$

LEMMA 4.1.6. Given a complete unimodular fan $\Sigma$, that contains $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$ as primitive ray generators, then using a series of inverse subdivisions we can transform it into the fan $\Sigma'$ only containing the aforementioned rays, the suitable 2-cones and the 0-cone, i.e. $\Sigma' = \{\operatorname{cone}(0), \operatorname{cone}(e_1), \operatorname{cone}(e_2), \operatorname{cone}(-e_1), \operatorname{cone}(-e_2),$ $\operatorname{cone}(e_1,e_2), \operatorname{cone}(e_2,-e_1), \operatorname{cone}(-e_1,-e_2), \operatorname{cone}(-e_2,e_1)\}$

*Proof.* We show that we can drop all rays in $\operatorname{cone}((0,1),(1,0))$. The other quadrants work similarly. Given three adjacent rays $\rho_1,\rho_2,\rho_3 \in \Sigma[1]$, with primitive generators $v_1, v_2, v_3$. We can drop $\rho_2$, if $v_1 + v_3 = v_2$ (i.e. $a_2 = 1$ in this case), as the resulting fan will still be complete and unimodular. So suppose $\rho_0 = \operatorname{cone}((0,1)), \rho_1, \dots, \rho_k, \rho_{k+1} \in \Sigma[1]$ are in order the rays of $\Sigma$ contained in $\operatorname{cone}((0,1),(1,0))$ and $v_1, \dots, v_{k+1}$ are their primitive generators respectively. Suppose there is no combination of rays $\rho_{i-1}, \rho_i, \rho_{i+1}$ with $a_i = 1$. $a_i < 1$ is impossible here, as $v_i > 0$ componentwise for all $1 \leq i \leq k$. So say $a_i \geq 2$ for all $i$. Looking at all equations we get in this quadrant: $v_0 + v_2 = a_1 v_1, v_1 + v_3 = a_2 v_2, \dots, v_{k-1} + v_{k+1} = a_k v_k$. If we sum the equations up we get $v_0 + v_1 + 2v_2 + \ldots + 2v_{k-1} + v_k + v_{k+1} = \sum_{i=1}^{k} a_i v_i$. But this is a contradiction, as it implies

$$(1,1) = v_0 + v_{k+1} = (a_1 - 1)v_1 + (a_k - 1)v_k + \sum_{i=2}^{k-1} (a_i - 2)v_i > (1,1),$$

where $>$ is componentwise here. So there always has to be a triple with $a_i = 1$ and hence we can drop one ray after the other until only the ones generated by $(0,1)$ and $(1,0)$ remain. □

LEMMA 4.1.7. Equation 4.1 holds for a complete unimodular fan $\Sigma$ in $\mathbb{R}^2$ if and only if it holds for a barycentric stellar subdivision $\Sigma'$ of $\Sigma$.

*Proof.* Let $\sigma = \operatorname{cone}(v_1, v_2)$ be the cone we want to subdivide, where $v_1$ and $v_2$ are primitive ray generators and set $v := v_1 + v_2$. Then subdividing means replacing the cone $\sigma$ by two 2-cones $\sigma_1 = \operatorname{cone}(v_1, v)$, $\sigma_2 = \operatorname{cone}(v, v_2)$ and a new ray $\rho = \operatorname{cone}(v)$.

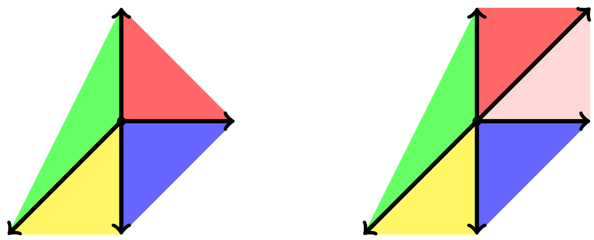


Figure 4.3: Example of a 2-dimensional complete unimodular fan and a subdivision of it

To compute the changes in the $a'$s, we need to compare their values at the two consecutive rays of $\Sigma$ generated by the primitive ray generators $v_1, v_2$ to the rays in $\Sigma'$ generated by the primitive ray generators $v_1, v_1 + v_2, v_2$. The subdivision changes $v_0 + v_2 = a_1 v_1$ to $v_0 + v = a'_1 v_1$ which implies $a'_1 = a_1 + 1$ and similarly $a'_2 = a_2 + 1$. Additionally we get $v_1 + v_2 = a_v v$, where $a_v = 1$. The other $a$'s do not change. Altogether we get $\sum a'_i = \sum a_i + 3$, but we also got one more ray, so that the increase in the sum of the $a$'s gets canceled and hence

$$\sum_{\tau \in \Sigma[1]} (3 - a(\tau)) = \sum_{\tau \in \Sigma'[1]} (3 - a(\tau)).$$

□

With these foundations we can now prove a the complete unimodular fan-version of the result of Godinho, Heymann and Sabatini [GHS16].

THEOREM 4.1.8. Let $\Sigma$ be a complete unimodular fan in $\mathbb{R}^d$, with $f$-vector $f = (f_0, \ldots, f_d)$. Then

$$\sum_{\tau \in \Sigma[d-1]} \sum_{i=1}^{d-1} -a(\tau, v_i) = 12 f_{d-2} - 3(d-1) f_{d-1}\,. \tag{4.2}$$

We want to use the 2-dimensional result to prove this theorem. Therefore we need a Lemma that is a $d$-dimensional equivalent of Lemma 6.1.22 in [HNP2x].

LEMMA 4.1.9. Let $\Sigma$ be a complete unimodular fan in $R^d$, and let $\theta \in \Sigma[d-2]$ be a $(d-2)$-cone with generators $v_1, \ldots, v_{d-2}$. Then the projection $\pi : \mathbb{R}^d \to \mathbb{R}^d/\mathbb{R}\theta$ maps $\mathrm{star}(\theta, \Sigma)$ to a complete unimodular fan $\Sigma/\theta$. If $\tau \in \mathrm{star}(\theta; \Sigma)$ is a $(d-1)$-cone with primitive generators $v_1, \ldots, v_{d-2}, v_{d-1}$, then the corresponding ray $\pi(\tau)$ of $\Sigma/\theta$ has parameter $a(\pi(\tau)) = a(\tau, v_{d-1})$.

*Proof.* Let $v_d$ and $v'_d$ be the additional primitive generators of the $d$-cones containing $\tau$. Then $v_d + v'_d = \sum_{i=1}^{d-1} a(\tau, v_i) v_i$. Applying $\pi$ yields $\pi(v_d) + \pi(v'_d) = a(\tau, v_{d-1})\pi(v_{d-1})$ as all the other $v_i$'s are sent to 0. □

Now we can prove Theorem 4.1.8.

*Proof of Theorem 4.1.8.* Given a complete unimodular fan $\Sigma$, we have

$$\begin{aligned}
\sum_{\tau \in \Sigma[d-1]} \sum_{i=1}^{d-1} -a(\tau, v_i) &= \sum_{\theta \in \Sigma[d-2]} \sum_{\substack{w \in \Sigma[1] \\ \tau = \mathrm{cone}(\theta, w) \in \Sigma[d-1]}} -a(\tau, w) \\
&= \sum_{\theta \in \Sigma[d-2]} 12 - 3\deg(\theta) \\
&= 12 f_{d-2} - 3 \sum_{\theta \in \Sigma[d-2]} \deg(\theta) \\
&= 12 f_{d-2} - 3(d-1) f_{d-1}
\end{aligned}$$

Where the *degree* of $\theta$ is defined by $\deg(\theta) := |\{\tau \in \Sigma[d-1] \ : \ \theta \subseteq \tau\}|$ and the first equality is true, because given any $a(\tau, v_i)$, with $\tau = \mathrm{cone}(v_1, \ldots, v_d)$ on the left side, then $\theta = \mathrm{cone}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{d-1}) \in \Sigma[d-2]$ and $w = v_i \in \Sigma[1]$ and every $a$ only appears once. We get the second equality by looking at $\Sigma/\theta$ and using Lemma 4.1.9 and Theorem 4.1.3. □

## 4.2 THERE ARE NO MORE RELATIONS

One can ask the question if there are more equations relating the $f$-vector and the sum of the $a(\tau, v_i)$. One family of such equations on the $f$-vector are the well-known Dehn-Sommerville equations. To prove these for complete unimodular fans, we will first introduce the $h$-vector of a complete unimodular fan. For that we need a to know half-open decompositions first. Let $\Sigma$ be a complete unimodular fan in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$ a generic vector, where in our case *generic* means that for every $\tau \in \Sigma[d-1]$, $x$ does not lie in the hyperplane generated by $\tau$. Then $x$ induces a *half-open decomposition* $\Pi_x$ of $\Sigma$ in the following way: Given a $(d-1)$-cone $\tau \in \Sigma[d-1]$, then there are two $d$-cones $\sigma, \sigma' \in \Sigma[d]$ with $\tau = \sigma \cap \sigma'$. Furthermore let $n$ be a normal vector of the hyperplane generated by $\tau$ that is non-negative on $\sigma$. Then in our half-open decompostion $\Pi_x$, $\tau$ lies in $\sigma$, if $n^T x > 0$. This way every face lies in exactly one $d$-cone. Note, that sometimes faces lying in $\sigma$ in $\Pi_x$ are also called *visible* faces of $\sigma$.

**Definition 4.2.1.** Let $\Sigma$ be a complete unimodular fan and $\Pi$ a half-open decomposition of $\Sigma$. Then the *h-vector* of $\Sigma$ is given by $h(\Sigma) = (h_0(\Sigma), \ldots, h_d(\Sigma))$, where $h_i(\Sigma)$ counts the number of $d$-cones $\sigma$ in $\Pi$ that contain $i$ many $(d-1)$-cones.

For this notion to be well-defined, the $h$-vector has to stay the same whatever half-open decomposition we choose for $\Sigma$. We get this from the following connection of the $h$-vector with the $f$-vector. Given any $d$-cone $\sigma$ in a complete unimodular fan, then $\sigma$ is simplicial and if we remove $k$ many $(d-1)$-cones from $\sigma$, it still contains $\binom{d+1-k}{j-k}$ many $j$-cones. But with that we can compute the $f$-vector from the $h$-vector as for all $0 \leq j \leq d$, we have

$$f_j = \sum_{k=0}^{d} h_k \binom{d-k}{j-k}. \tag{4.3}$$

As this is in fact an invertible linear map, we can also express the $h$-vector in terms of the $f$-vector and as the $f$-vector does not change, we see that the $h$-vector is well-defined. Another neat way to express this connection is in terms of polynomials. We call $h_\Sigma(t) := h_d(\Sigma) + h_{d-1}(\Sigma)t + \ldots + h_0(\Sigma)t^d$ the *h-polynomial* of $\Sigma$ and similarly $f_\Sigma(t) := f_d(\Sigma) + f_{d-1}(\Sigma)t + \ldots + f_0(\Sigma)t^d$ the *f-polynomial* of $\Sigma$. The connection between the two then turns into

$$f_\Sigma(t) = h_\Sigma(t+1).$$

With these preparations it is now easy to show the following theorem.

THEOREM 4.2.2 (Dehn-Sommerville equations for complete unimodular fans). Let $(h_0, h_1, \ldots, h_d)$ be the $h$-vector of complete unimodular fan $\Sigma$. Then

$$h_k = h_{d-k} \quad \forall k = 0, \ldots, d$$

where the first $\left\lceil \frac{d}{2} \right\rceil$ are linearly independent. Converted to the $f$-vector we get

$$\sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} f_j = (-1)^{d-1} f_k \qquad \forall k = -1, \ldots, d-2$$

where for $d$ even the equations for $k = 0, 2, 4, \ldots, d-2$ and $k = -1, 1, \ldots, d-3$ and for $d$ odd the equations for $k = -1, 1, 3, \ldots, d-2$ and $k = -1, 0, 2, 4, \ldots, d-3$ are independent sets.

*Proof.* Let $(h_0(\Sigma), \ldots, h_d(\Sigma))$ be the $h$-vector of $\Sigma$ coming from a half-open decomposition induced by a generic vector $x$. Then $-x$ is also generic, and looking at the half-open decomposition it induces every $(d-1)$-cone now changes sides to the other $d$-cone it is contained in. That means every $d$-cone that before contained $k$ many $(d-1)$-cones now contains $d-k$ many and hence $h_k = h_{d-k}$. □

We now want to show that apart from the 12&24-equation (4.2) and the Dehn-Sommerville equations there are no more equations. To make this more precise, we call $\left(h_0(\Sigma), \ldots, h_d(\Sigma), \sum_{\tau\in\Sigma[d-1]} \sum_{i=1}^{d-1} -a(\tau, v_i)\right)$ the *extended h-vector* of $\Sigma$ and denote with $\mathcal{H}$ and $\mathcal{H}^e$ the space of all possible $h$-vectors of complete unimodular fans and all possible extended $h$-vectors of complete unimodular fans, respectively. With these definitions, the 12&24-equation (4.2), with the $f_i$'s on the right-hand side replaced by the appropriate sum of $h_i$'s using equation (4.3), and the Dehn-Sommerville equations hold for all elements of $\mathcal{H}^e$ and the rest of this section is devoted to prove the following theorem.

THEOREM 4.2.3. Every equation that holds for all points in $\mathcal{H}^e$ is a linear combination of the 12&24-equation (4.2) and the Dehn-Sommerville equations.

To show that, we will prove that the dimension of $\mathcal{H}$ is big enough by finding a large enough set of independent $h$-vectors. A similar theorem is true in the polytopal case, where the $h$-vectors of cyclic polytopes do the job. Unfortunately their fans cannot to be unimodular, so we have to find another family of fans.

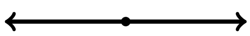
But before we can describe that family, we have to define the product of two fans. For that let $\Sigma_1$ be a fan with support in $\mathbb{R}^m$ and $\Sigma_2$ a fan with support in $\mathbb{R}^n$, then we define their product in the following way:

$$\Sigma = \Sigma_1 \times \Sigma_2 := \left\{ A \times B \subseteq \mathbb{R}^{m+n} \ : \ A \in \Sigma_1, B \in \Sigma_2 \right\}.$$

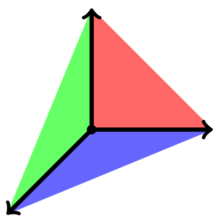
THEOREM 4.2.4. Given the two complete unimodular fans from Figure 4.4:

- $\Sigma := \{\text{cone}(0), \text{cone}(e_1), \text{cone}(-e_1)\}$ and
- $\hat{\Sigma} := \{\text{cone}(0), \text{cone}(e_1), \text{cone}(e_2), \text{cone}(-e_1 - e_2), \text{cone}(e_1, e_2),$
  $\text{cone}(e_1, -e_1 - e_2), \text{cone}(e_2, -e_1 - e_2)\}$.

Setting $\Sigma_k = \Sigma^{d-2k} \times \hat{\Sigma}^k$, the $\left\lfloor \frac{d}{2} + 1 \right\rfloor$ $h$-vectors given by $h(\Sigma_k)$ for $0 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor$ are affinely independent and hence the space of $h$-vectors of complete unimodular fans has at least dimension $\left\lfloor \frac{d}{2} + 1 \right\rfloor$.



(a) $\Sigma$



(b) $\hat{\Sigma}$

Figure 4.4: The two fans $\Sigma$ and $\hat{\Sigma}$

To prove this theorem it is very helpful to use the polynomial description of the $h$-vector.

LEMMA 4.2.5. Given two fans $\Sigma_1$ and $\Sigma_2$ and let $\Sigma = \Sigma_1 \times \Sigma_2$, then the $h$-polynomial of $\Sigma$ is given by the product of the $h$-polynomials of $\Sigma_1$ and $\Sigma_2$: $h_\Sigma(t) = h_{\Sigma_1}(t)h_{\Sigma_2}(t)$.

*Proof.* Let $x_1$ be a generic point for $\Sigma_1$ and $x_2$ a generic point for $\Sigma_2$, then $x = (x_1, x_2)$ is a generic point for $\Sigma$ and we claim that $h_i(\Sigma) = \sum_{j=0}^{i} h_j(\Sigma_1)h_{i-j}(\Sigma_2)$. A $d$-cone $C \in \Sigma$ is by definition a combination of an $m$-cone $A$ of $\Sigma_1$ and an $n$-cone $B$ of $\Sigma_2$, with $d = n + m$. Therefore, a facet $F$ of $C$ comes from a facet in one of the two cones and the full other cone combined. (i.e. $F = G \times B$ or $F = A \times H$ with $G, H$ facets of $A$ resp. $B$). This implies that a visible facet in $C$ comes from a visible facet in $A$ and the full cone $B$ combined or the full cone $A$ combined with a visible facet of $B$. So the number of visible facets in $C$ is exactly the number of visible facets in $A$ plus the number of visible facets in $B$. Hence the number of full-dimensional cones with $i$ visible facets is given by $h_i(\Sigma) = \sum_{j=0}^{i} h_j(\Sigma_1)h_{i-j}(\Sigma_2)$, as we combine any two $d$-cones in the product of fans. This implies $h_\Sigma(t) = h_{\Sigma_1}(t)h_{\Sigma_2}(t)$. □

With that lemma we can now prove the theorem.

*Proof of Theorem 4.2.4.* We have $h_\Sigma(t) = 1 + t$ and $h_{\hat{\Sigma}}(t) = 1 + t + t^2$, then $h_{\Sigma_k}(t) = (1 + t + t^2)^k(1 + t)^{d-2k}$. So to show that the $\lfloor \frac{d}{2} \rfloor + 1$ $h$-vectors $h(\Sigma_k)$ are affinely independent, we can show the same for the polytopes as their coefficient vectors are those $h$-vectors. To see that, look at the linear map,

$$h_{\Sigma_k}(t) \mapsto \left( h_{\Sigma_k}(t), h_{\Sigma_k}^{(1)}(t), \ldots, h_{\Sigma_k}^{\left(\lfloor \frac{d}{2} \rfloor\right)}(t) \right),$$

where $h_{\Sigma_k}^{(i)}$ is the $i$-th derivative of $h_{\Sigma_k}$. Evaluating the polynomials in these vectors does not increase the dimension of the space they are spanning. Evaluating them at a third root of unity $\xi$, which in particular implies $h_{\hat{\Sigma}}(\xi) = 0$, we get an upper-triangular $\left(\lfloor \frac{d}{2} \rfloor + 1\right) \times \left(\lfloor \frac{d}{2} \rfloor + 1\right)$-matrix and hence the polynomials and vectors are independent. □

With these preparations we can now prove Theorem 4.2.3

*Proof of Theorem 4.2.3.* Combining the previous results we have $\lfloor \frac{d}{2} \rfloor + 1$ independent vectors, $\lceil \frac{d}{2} \rceil$ Dehn-Sommerville equations and one 12&24-equation. As $\mathcal{H}^e \subseteq \mathbb{R}^{d+2}$ it has at most dimension

$$d + 2 - \left\lceil \frac{d}{2} \right\rceil - 1 = \left\lfloor \frac{d}{2} \right\rfloor + 1.$$

We see that the dimension of $\mathcal{H}^e$ is $\left\lfloor \frac{d}{2} \right\rfloor + 1$ and that every other equation that is valid on $\mathcal{H}^e$, has to be a linear combination of the Dehn-Sommerville equations and the 12&24-equation. □

## 4.3 PROVING 12 & 24 USING THE FAN EQUATION

Now that we have proven the equation for fans and shown that we now know all equations on the extended $h$-vector, let us take a step back and show that it actually generalizes the equations for 12 and 24 for reflexive polytopes. For both cases we have to relate the length of the edges of $P$ and $P^*$ to information we have on our fan, namely the $a's$ and for that we use the following lemma.

LEMMA 4.3.1. Given a reflexive polytope $P$, whose boundary $\delta P$ has a unimodular triangulation $T$. Then the complete fan $\Sigma = \{C : C = \text{cone}(F)\,,\, F \text{ face of } T\}$ is also unimodular and given a $(d-2)$-face $F$ in $P$, the edge $e'$ dual to $F$ has length $\ell(e') = 2 - \sum_{i=1}^{d-1} a(\tau, v_i)$, where $\tau$ is any $(d-1)$-cone in cone$(F)$.

*Proof.* Let $\tau = \text{cone}(v_1, \ldots, v_{d-1})$ be any $(d-1)$-cone inside cone$(F)$. Then $\tau$ is contained in two $d$-cones $\sigma_1 = \text{cone}(\tau, v_d)$ and $\sigma_2 = \text{cone}(\tau, v'_d)$ and there are $a(\tau, v_1), \ldots, a(\tau, v_{d-1})$ such that $v_d + v'_d = \sum_{i=1}^{d-1} a(\tau, v_i) v_i$. As $P$ is reflexive, $\Sigma$ is a unimodular fan, so that $v_1, \ldots, v_d$ form a basis of $\mathbb{Z}^d$ and from now on we will look at our polytope in this basis. Furthermore, let $v_1^*, \ldots, v_d^*$ be its dual basis. Then the dual face of the facet $\text{conv}(v_1, \ldots, v_{d-1}, v_d)$ is the vertex $w := v_1^* + \ldots + v_d^*$. The dual face of $\text{conv}(v_1, \ldots, v_{d-1}, v'_d)$ is the vertex $w' := v_1^* + \ldots + v_{d-1}^* + \left(\sum_{i=1}^{d-1} a(\tau, v_i) - 1\right) v_d^*$, because $w' \cdot v_i = 1$ for $1 \le i \le d-1$ and also $w' \cdot v'_d = w' \cdot \left(-v_d + \sum_{i=1}^{d-1} a(\tau, v_i) v_i\right) = 1$. Written in our new basis we have $w = (1, \ldots, 1)$ and $w' = \left(1, \ldots, 1, -1 + \sum_{i=1}^{d-1} a(\tau, v_i)\right)$. Then the edge $e'$ dual to $\tau$, connects $w$ and $w'$. Hence its length is

$$\ell(e') = \left|(1, \ldots, 1) - \left(1, \ldots, 1, -1 + \sum_{i=1}^{d-1} a(\tau, v_i)\right)\right| = \left|2 - \sum_{i=1}^{d-1} a(\tau, v_i)\right|. \qquad \square$$

THEOREM 4.3.2. Let $P$ be a reflexive 2-polytope and $P^\star$ its polar dual, then the sum of the number of lattice points on the boundary of $P$ and of $P^\star$ is 12, or equivalently expressed in terms of the lattice lengths of the edges we have

$$\sum_{e \in P[1]} \ell(e) + \sum_{e' \in P^\star[1]} \ell(e') = 12.$$

*Proof.* Let $T$ and $\Sigma$ be as in Lemma 4.3.1, i.e. in $\Sigma$ we have a ray through every lattice point on the boundary of $P$ and hence $\sum_{e \in P[1]} \ell(e) = |\Sigma[1]|$ and using Lemma 4.3.1 we get $\sum_{e' \in P^\star[1]} \ell(e') = \sum_{\tau \in \Sigma[1]} (2 - a(\tau))$. Combining those two identities and using Theorem 4.1.3 we get

$$\sum_{e \in P[1]} \ell(e) + \sum_{e' \in P^\star[1]} \ell(e') = |\Sigma[1]| + \sum_{\tau \in \Sigma[1]} (2 - a(\tau)) = \sum_{\tau \in \Sigma[1]} (3 - a(\tau)) = 12.$$

$\square$

THEOREM 4.3.3. Let $P$ be a reflexive 3-polytope and $P^\star$ its polar dual, then

$$\sum_{e \in P[1]} \ell(e)\ell(e^\star) = 24.$$

*Proof.* As $P$ is a 3-dimensional polytope, the boundary is 2-dimensional and hence we can find a unimodular triangulation $T$ of it. Because $P$ is also reflexive, the complete fan $\Sigma$ generated by $T$, i.e. $\Sigma = \{C : C = \mathrm{cone}(F)\,,\, F \text{ face of } T\}$, is also unimodular. And hence our fan equation (4.1.8) is valid on $\Sigma$, i.e., in this case we get $\sum_{\tau \in \Sigma[2]} \sum_{i=1}^{2} -a(\tau, v_i) = 12f_1 - 6f_2$. Now that we have the right unimodular cone, we have to relate the lengths of the edges of $P$ and $P^*$, to the $a(\tau, v_i)$ of $\Sigma$.

$$\begin{aligned} \sum_{e \in P[1]} \ell(e)\ell(e^\star) &= \sum_{\tau \in \Sigma[2]} 2 - (a(\tau, v_1) + a(\tau, v_2)) \\ &= 12f_1 - 4f_2 = 12(f_1 - f_2 + f_3) \\ &= 24 \end{aligned}$$

For the first equation we first use Lemma 4.3.1 to get $\ell(e^*) = 2 - (a(\tau, v_1) + a(\tau, v_2))$. Then instead of going through all edges of $P$, we look at all the edges of the unimodular triangulation $T$ or rather the 2-cones $\tau \in \Sigma[2]$ corresponding to those edges. Because of that $\ell(e)$ disappears, as it is replaced by $\ell(e)$ many edges of $T$ of length 1. A priori we now have too many summands, because of cones that corresponds to an edge of the triangulation that is not part of an original edge of $P$. But as we will see now, these will contribute 0 to the sum. Suppose that $e$ is an edge of $T$ between points $v_1$ and $v_2$ and is not part of an edge of $P$. Looking at the fan $\Sigma$ and the cone $\tau = \mathrm{cone}(v_1, v_2)$ corresponding to $e$, both $v_3$ and $v_3'$ s.t. $v_3 + v_3' = a(\tau, v_1)v_1 + a(\tau, v_2)v_2$ lie in the same facet of $P$ as $e$. As $P$ is reflexive, this facet has lattice distance 1 from the origin. If we now add $v_3$ and $v_3'$, their sum will lie in a hyperplane parallel to that facet at lattice distance 2 from the origin and hence $a(\tau, v_1) + a(\tau, v_2) = 2$. Therefore, in this case we get $\ell(e^*) = 0$. The third equation holds because in a unimodular triangulation we have $3f_3 = 2f_2$ by double counting, as every 3-cone sees three 2-cones and every 2-cone sees two 3-cones and the last equation is using the Euler characteristic $f_1 - f_2 + f_3 = 2$. $\square$

# Part III

## APPENDIX

# A ALGORITHMS

All algorithms listed here are written for the excellent software, polymake [GJ00], which is a great tool for studying polytopes. Documentations and introductions into polymake can be found at `www.polymake.org`.

## A.1 ALGORITHMS FOR THE FINITENESS THRESHOLD CHAPTER

In this section you will find the algorithms that were used in chapter 2. As mentioned before the algorithm computing the lattice width is now part of polymake and can be called as a property of $P$ with the command `P->LATTICE_WIDTH`. We also list the algorithms used to compute the subpolytopes of the 14 maximal hollow 3-polytopes, that have width $\geq 2$.

### A.1.1 *Algorithm computing the lattice width of a Polytope P*

```
use application 'polytope';
use Benchmark qw(:all);

#Bounded and full_dimensional
sub width {
my ($P) = @_;

my $direction = unit_vector($P->CONE_DIM,1);
my $lp=new Polytope(VERTICES=>$P->VERTICES, LP=>(new
   LinearProgram(LINEAR_OBJECTIVE=>$direction)));
my $width=$lp->LP->MAXIMAL_VALUE - $lp->LP->MINIMAL_VALUE;

my $S=scale(polarize(minkowski_sum(1,$P, -1,$P)),$width);
my $B=$S->LATTICE_POINTS;

  for(my $i=0; $i<$B->rows(); ++$i){
   my $current_direction = 0|$B->row($i)->slice(1);
   next if($current_direction->[1] < 0); #because of symmetry
   next if($current_direction ==
      zero_vector($current_direction->dim)); #because 0 is a
      stupid direction

   #TODO multiple lps
   $lp=new Polytope(VERTICES=>$P->VERTICES, LP=>(new
      LinearProgram(LINEAR_OBJECTIVE=>$current_direction)));
```

```
    my $current_width=$lp->LP->MAXIMAL_VALUE
        -$lp->LP->MINIMAL_VALUE;
    if ($current_width<$width){
      $width=$current_width;
      $direction=$current_direction;
    }
  }
print "width: $width vector: ". dense($direction);

return $width;
}
```

### A.1.2 Algorithm computing all subpolytopes of width $\geq 2$ of a given list of polytopes

```
use application 'polytope';
use Benchmark qw(:all);


# prints a progressbar
# @param Int got : the amount of stuff you have already got
# @param Int total : the total amount of stuff
# @param String text : additional text behind the bar.
sub progress_bar {
    my ( $got, $total, $text) = @_;
    my $width = 10;
    my $char = '=';
    local $| = 1;
    printf "|%-${width}s| $got/$total | %-80s \r",
    $char x (($width)*$got/$total). '>', $text;
}


# Creates a list of polytopes by removing one vertex and taking
# the convex hull over all the remaining lattice points.
# @param Set<Matrix<Rational>> input_polys a set containing
    all the vertices of the input polytopes
# @return Set<Matrix<Rational>> ouput_polys a set containing
    all the vertices of the ouput polytopes,
#                     but with one vertex removed
sub subpolylist_step($) {
my ($input_polys) = @_;

# variables for progress bar and timing
my $numPolys = $input_polys->size;
my $date=localtime(time);
```

```
print "subpolylist: started $date\n";
my $t0=Benchmark->new;

# Initializing the output set
my $output_polys = new Set<Matrix<Rational>>();

# For every polytope do:
#for (my $l=0; $l<scalar(@input_polys); ++$l){
my $l=0;
foreach my $vertices (@{$input_polys}){
   ++$l;
   # calc the lattice points
   my $tmpPoly = new Polytope(POINTS=>$vertices);
   prefer_now "projection"; my $latticePoints= new
      Set<Vector>(rows($tmpPoly->LATTICE_POINTS));
   progress_bar($l,$numPolys, "start removing vertices --
      $date");

   # For every vertex do:
   my $num_verts = $vertices->rows();
   my $vert_i = 0;
   foreach my $v (@{rows($vertices)}){
  # progress bar printing
  ++$vert_i;
  $date=localtime(time);
  progress_bar($l,$numPolys, "removing vertex $vert_i from
     $num_verts -- $date");

  # create the convex hull over all lattice points
  # but without this vertex and check if [[LATTICE_WIDTH]] is
     1
  my $tmpPoly = new
     LatticePolytope(POINTS=>($latticePoints-$v), BOUNDED=>1);
  next if (!$tmpPoly->FEASIBLE || $tmpPoly->LATTICE_WIDTH <=
     1);

  # calc the affine lattice normal form and add the polytope
  # to our list (if it is not already there)
  my $outVertNorm = new
     Matrix(affine_lattice_normal_form($tmpPoly));
        # here we add $vertTnorm to the set
    # but since we deal with sets, we do not take care of
       douplicates
  $output_polys += $outVertNorm;
   }
}

# timing stops
my $t1=Benchmark->new;
```

```
my $t=timediff($t1,$t0);
printf "subpolylist: stopped %-80s\n", "$date";
print " time: ".timestr($t)."\n";

# making the hash to a perl array
return $output_polys;
}

# Creates all lists of polytopes by successive removing one
   vertex and taking
# the convex hull over all the remaining lattice points.
# @param Set<Matrix<Rational>> input_polys a set containing
   all the vertices of the input polytopes.
#                               They need to have the same
   number of lattice points
# @param Set<Matrix<Rational>> additional a set containing
   some polytopes which should be added at
#                               the right time
# @param String path the path where all the lists should be
   saved
# @return void
sub subpolylist($$$){
   my ($list, $add_list, $path) = @_;

   # prepare a hash, with the polytopes associated to the
      number of lattice points
   my %hash = ();
   foreach my $verts (@{$add_list}){
  my $p = new LatticePolytope(POINTS=>$verts, BOUNDED=>1);
  my $n_lattice = $p->N_LATTICE_POINTS;
  if(exists($hash{"$n_lattice"})){
     $hash{"$n_lattice"} += $verts;
  }else{
     $hash{"$n_lattice"} = new Set<Matrix<Rational>>($verts);
  }
   }

   # check with how many lattice points we start
   my $n_lattice = new Polytope(POINTS=>$list->[0],
      BOUNDED=>1);
   $n_lattice = new Integer($n_lattice->N_LATTICE_POINTS);

   # iterate till the end
   while($list->size != 0){
  print "\n=== $n_lattice -> ".($n_lattice-1)." ===\n";

  # check if there are some polytopes to add
  $list += $hash{"$n_lattice"}
     if(exists($hash{"$n_lattice"}));
```

```
    $list = subpolylist_step($list);
    --$n_lattice;

    save_data($list, $path."/".$n_lattice."_polys.data");
    print "found ".$list->size." polytopes.\n";
     }
  }
```

## A.2 ALGORITHMS FOR THE CONVEX-NORMAL CHAPTER

In this section you will find the algorithms that were used in chapter 3. First an algorithm that computes the volume of a union of polytopes. This was written by Constantin Fischer and is as of now not included in polymake. We are most grateful to him for letting us use it. Next is the algorithm that checks if a polytope $P$ is 2-convex-normal.

### *A.2.1 Algorithm computing the volume of a union of polytopes*

```
## VOLUME via inclusion exclusion
sub vol_union(@) {
   my @plist = @_;
   my @queue;
   my $Vol = 0;
   my %data = (); #stores the polytopes constructed in the
       calculation which are full-dimensional
   my @tmp = ();
   my $indices;
   my $candidate;
   my $flag;
   my $plistlength = scalar @plist-1;

   # Initialisation of the queue
   @queue = map{$data{"$_"}=$plist[$_];
       [$_];}(0..$plistlength);

   while (@queue) {
  $indices = shift(@queue);

  # adding or substracting according to intersection size
  $Vol += (-1)**((scalar
      @{$indices}+1)%2)*$data{join(",",@{$indices})}->VOLUME;

      # generate the new intersection-polytope candidates for
          a given index set
```

```
    for (my $i= @{$indices}[scalar @{$indices}-1]+1; $i <=
        $plistlength; $i++) {
        $flag = 1;
        if (scalar @{$indices} > 1) {
       for(my $j=0; $j < scalar @{$indices}; ++$j){
           # make sure to only care about full dimensional
               intersections
           if (exists($data{join(",",@{$indices}[(0..$j-1,
               $j+1..scalar @{$indices}-1)]).",$i"}) == 0){
          $flag = 0;
          last;
           }
       }
        }

        # all subintersections are contained in the hashtable
        if ($flag) {
       $candidate =
           intersection($data{join(",",@{$indices})},$plist[$i]);
       if ($candidate->VOLUME != 0) {
           @tmp = (@{$indices},$i);
           push(@queue,[@tmp]);
           $data{join(",",@tmp)} = $candidate;
       }
        }
    }
     }
     return $Vol;
}
```

### A.2.2 *Algorithm checking if P is 2-convex-normal*

```
sub conv_norm {
    my ($p) = @_;
    my $G = new Set<Vector<Rational>>();

    foreach my $v (@{rows($p->VERTICES->minor(All,~[0]))}){
   my $tmp = new Matrix(translate($p,
       -$v)->LATTICE_POINTS->minor(All,~[0]));
   my $translate = repeat_row($v,$tmp->rows());
   $tmp += $translate;
   $G += new Set<Vector<Rational>>(@$tmp);
    }

    my @polys =();
    foreach my $g (@$G){
```

```
  push(@polys, translate($p, $g));
   }

   return scale($p,2)->VOLUME == vol_union(@polys);
}
```

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# ZUSAMMENFASSUNG

Den Kern der vorliegenden Arbeit bilden, wie der Titel deutlich macht, drei interessante Gitterpolytopprobleme, in den Kapiteln 2-4 behandelt werden.

Kapitel 2 dreht sich um das Problem der Klassifizierung von Gitterpolytopen. Wir zeigen hier, dass in jeder Dimension $d$ eine Zahl $w^\infty(d)$, die *Endlichkeitsgrenzweite*, existiert, so dass es nur endlich viele $d$-Gitterpolytope mit einer gegebenen Anzahl an Gitterpunkten gibt, deren Gitterweite größer als $w^\infty(d)$ ist. Insbesondere zeigen wir, dass $d-1 \leq w^\infty(d) \leq O\left(d^{\frac{3}{2}}\right)$. In Dimension 3 war schon bekannt, dass $w^\infty(3) = 1$. Im Laufe des Kapitels erarbeiten wir dann Voraussetzungen, unter denen ein hohles $d$-dimensionales Gitterpolytop endlich bzw. unendlich viele Rückziehungen hat. Diese ermöglichen uns dann die Frage auch für Dimension 4 zu beantworten. In diesem Fall ist die Endlichkeitsgrenzweite 2.

Hinter Kapitel 3 steht die Frage, ob $d$-dimensionale Gitterpolytope mit langen Kanten immer ganz abgeschlossen sind. Diese Frage hat Gubeladze mit ja beantwortet, indem er gezeigt hat, dass wenn in einem Polytop jede Kante mindestens Gitterlänge $4d(d+1)$ hat, das Polytop zwangsläufig ganz abgeschlossen ist. Hierzu führte er den Begriff der Konvex-Normalität ein. Dieser wird hier näher beleuchtet und wir können einige grundlegende Aussagen dazu treffen. Daraus folgt mit dem Hauptresultat des ersten Teils von Kapitel 3 eine Verbesserung der Schranke auf $2d(d+1)$. Danach betrachten wir Paare von Polytopen und verallgemeinern hierfür den Begriff der Konvex-Normalität und können damit das folgende Resultat zeigen. Gegeben seien zwei $d$-dimensionale Gitterpolytope $P$ und $Q$. Wenn der Normalenfächer von $P$ eine Verfeinerung des Normalenfächers von $Q$ ist, und zusätzlich jede Kante in $P$ mindestens $d$-mal so lang ist, wie die dazu korrespondierende Seite (Kante oder Ecke) von $Q$, dann gilt $(Q+P) \cap \mathbb{Z}^d = (Q \cap \mathbb{Z}^d) + (P \cap \mathbb{Z}^d)$.

Im Kapitel 4 greifen wir zwei verblüffende Sätze auf, die erst vor Kurzem eine Verallgemeinerung erfahren haben. Der Erste besagt, dass für ein reflexives Polytop $P$ in Dimension 2, die Summe aus der Anzahl der Gitterpunkte auf dem Rand von $P$ und der Anzahl der Gitterpunkte auf dem Rand von $P^*$ stets 12 ergibt. Für den Zweiten, bezeichne $\ell(e)$ die Gitterlänge eines Gittersegments $e$. Dann gilt für ein reflexives Polytop $P$ in Dimension 3, dass $\sum_{e \in P[1]} \ell(e)\ell(e^\star) = 24$. Für diese beiden Gleichungen für reflexive 2- bzw. 3-Polytope wurde vor Kurzem eine Verallgemeinerung für glatte reflexive $d$-dimensionale Polytope gefunden. In Kapitel 4 zeigen wir zweierlei. Erstens, dass diese neue Gleichung auch für vollständige unimodulare Fächer gilt. Und zweitens, dass es abgesehen von dieser neu gefundenen Gleichung und den wohlbekannten Dehn-Sommerville Gleichungen keine weiteren unabhängigen Gleichungen gibt, die für alle erweiterten $h$-Vektoren von vollständigen unimodularen Fächern gelten.

# SELBSTÄNDIGKEITSERKLÄRUNG

Hiermit vesichere ich, dass ich alle Hilfsmittel und Hilfen angegeben und auf dieser Grundlage die Arbeit selbständig verfasst habe. Zudem wurde die Arbeit auch noch nicht in einem früheren Promotionsverfahren eingereicht.

*Berlin, February 15, 2018*

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