Appendix C: The Mittag-Leffler function

The Mittag-Leffler function $E_\alpha(z)$ is an entire function defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}, \ \alpha \in \mathbb{R}, \ z \in \mathbb{C}\,. \tag{C.1}$$

This is the original form which is studied by Mittag-Leffler (see for example [17], [75], [42] and [76]). The Mittag-Leffler function arises naturally in the solution of the fractional integral equations (see [63] and [65]). Actually it appears as the solution of the Abel integral equation of the second type (see for example [102], [26] and [42]). This function has many applications specially in the study of the fractional generalization of the kinetic equation, *random walks*, *Lévy flights*, and the so called superdiffusive transport (see [45], [70], [3] and [100]). The generalized Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+k\alpha)}, \ \alpha \in \mathbb{R}, \ \beta \in \mathbb{R}, \ z \in \mathbb{C}\,. \tag{C.2}$$

It is known that this function has various applications in the theory of fractional differential equations, see for example [72] and [90]. We remark that $E_{\alpha,1}(z) = E_\alpha(z) = exp(z)$. This means the Mittag-Leffler function generalizes the exponential function (see [46], [47]). For more details about the analytical properties of the Mittag-Leffler function (see [17] and [63]). It is known that the Mittag-Leffler function $E_\alpha(-x)$, $x \in \mathbb{R}$ is a completely monotonic function for all $0 < \alpha \leq 1$ (see for example [37] and [81]). This proof of completely monotonicity was extended to $E_{\alpha,\beta}(-x)$ in [73] and [95], where it was proved for $0 < \alpha \leq 1$, $\beta \geq \alpha$. It is recently proved that $E_{\alpha,\beta}(1/x)$ is also a completely monotonic function for all $\alpha > 0$ and $\beta > 0$ [74]. The computation of the generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ and its derivative are carried out in [32].

In our treatment of the CTRW, the Mittag-Leffler function appears in the special form $E_\beta(-t^\beta)$ which represents the survival probability function $\Psi(t)$ (see Chapter 4). For $0 < \beta < 1$ and $1 < \beta < 2$ the functions of the form $E_\beta(-t^\beta)$ appear in certain relaxation and oscillation processes called *fractional relaxation* and *fractional oscillation* processes, respectively. The series expansions and the asymptotic representations of $\Psi(t) = E_\beta(-t^\beta)$ and the negative sign of its derivative $\psi(t) = -\frac{d}{dt}E_\beta(-t^\beta)$ which represents for us, in the case $0 < \beta < 1$, the waiting time probability density function are :

$$\Psi(t) = E_\beta(-t^\beta) \begin{cases} = \sum_{n=0}^{\infty}(-1)^n \frac{t^{n\beta}}{\Gamma(n\beta+1)}\,, & t \geq 0\,, \\ \sim \frac{sin\,\beta\pi}{\pi}\frac{\Gamma(\beta)}{t^\beta}\,, & t \to \infty\,, \end{cases} \tag{C.3}$$

and

$$\psi(t) = -\frac{d}{dt}E_\beta(-t^\beta) \begin{cases} = \frac{1}{t^{1-\beta}} \sum_{n=0}^{\infty}(-1)^n \frac{t^{n\beta}}{\Gamma(n\beta+1)}\,, & t \geq 0\,, \\ \sim \frac{sin\,\beta\pi}{\pi}\frac{\Gamma(\beta+1)}{t^{\beta+1}}\,, & t \to \infty\,, \end{cases} \tag{C.4}$$

see [67]. The expression of $\psi(t)$ is equivalent to the one obtained in [45] in terms of the generalized Mittag-Leffler function in two parameters. In the limiting case $\beta = 1$, we have $\Psi(t) = \psi(t) = exp(-t)$ and our memory process reduces to a memoryless process. The integral representation of $\Psi(t)$ and $\psi(t)$ are (see for example [37] and [65])

$$\Psi(t) = \frac{sin\beta\pi}{\pi}\int_0^\infty \frac{x^{\beta-1}e^{-xt}}{x^{2\beta}+2x^\beta cos\beta\pi\ +1}\,dx\,,\ t \geq 0\,, \tag{C.5}$$

and

$$\psi(t) = \frac{sin\beta\pi}{\pi}\int_0^\infty \frac{x^{\beta}e^{-xt}}{x^{2\beta}+2x^\beta cos\beta\pi\ +1}\,dx\,,\ t \geq 0\,. \tag{C.6}$$

In the special case $\beta = 1/2$ the Mittag-Leffler function is related to the error function by the formula

$$E_{1/2}(-t^{1/2}) = e^t\,\mathrm{erfc}\,(t^{1/2}) = e^t\frac{2}{\sqrt{\pi}}\int_{\sqrt{t}}^\infty e^{-u^2}du\,,\ t \geq 0\,, \tag{C.7}$$

where erfc denotes the *complementary error function.*

The infinite series in equation (C.3) exhibits a behaviour similar to that of a stretched exponential for $0 < \beta < 1$ and for small values of t

$$E_\beta(-t^\beta) \approxeq 1 - \frac{t^\beta}{\Gamma(\beta+1)} \approxeq exp\{-t^\beta/\Gamma(\beta+1)\}\ ,\ 0 \ll t \ll 1\ . \quad \text{(C.8)}$$

Whereas for large t, it has the asymptotic representation

$$E_\beta(-t^\beta) \sim \frac{sin(\beta\pi)}{\pi}\frac{\Gamma(\beta)}{t^\beta}\ ,\ t \to \infty\ . \quad \text{(C.9)}$$

At the end of our survey, we present some figures to show the behaviour of the Mittag-Leffler function as a completely monotonic function for different values of $\beta \in (0,1)$ and t. First, we plot the function e^{-t} at figure [C.1] to show its fast decay. Figure [C.2] shows the behaviour of the functions $E_\beta(-t^\beta)$ computed by the aid of the integral representation (C.5) for $0 \leq t \leq 15$. Figure [C.3] exhibits the same function in a small interval (i. e. $0 \leq t \leq 1$). The stretched exponential function (C.8) is plotted in figure [C.4]. Finally figures [C.5,C.6] represent the Mittag-Leffler function $E_\beta(-t^\beta)$ at a and the stretched exponential function $exp(\frac{-t^\beta}{\Gamma(1+\beta)})$ at b for $0 \leq t \leq 1$. These two figures show that the stretched exponential and the Mittag-Leffler function have the same behaviour for t near zero, but when t increases the stretched exponential function decays faster than the Mittag-Leffler function.

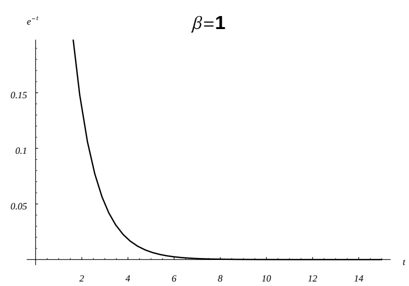

Figure C.1: exp(-t)

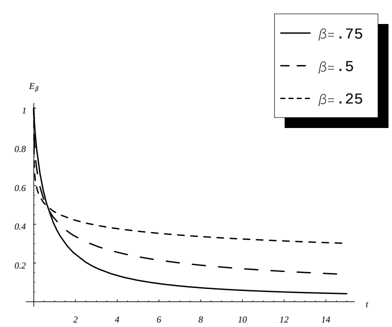

Figure C.2: $t : 0 \to 15$

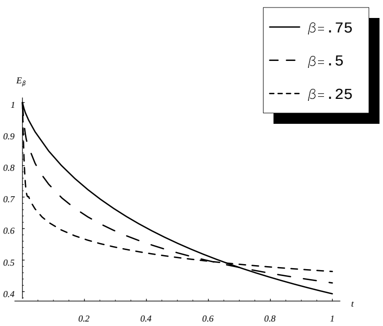

Figure C.3: $t : 0 \to 1$

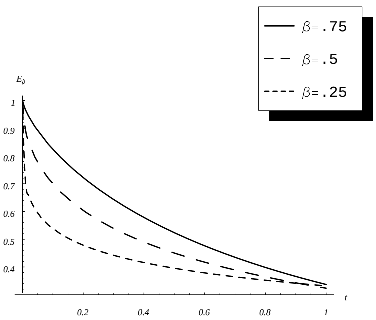

Figure C.4: $exp(-t^{\beta/\Gamma(1+\beta)})$

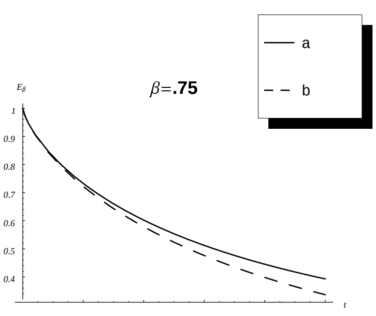

Figure C.5: see a and b

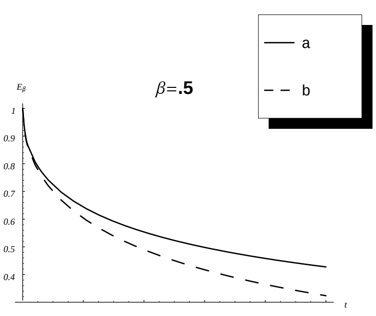

Figure C.6: see a and b