Appendix B: Stable distributions and domain of attraction

The stable laws were introduced by Paul Lévy [60] during his investigations of the behaviour of the sums of independent identically distributed random variables. Then the theory was developed by many authors and laid out in many books, including for examples: B. V. Gnedenko & A. N. Kolmogorov [23], Feller [20], Lukacs [61], Breiman [8], Chung-Jeh [11], K. L. Chung, [12], Laha & Rohatgi [59], Zolotarev & Uchaikin [104] and Meerschaert & Scheffler [74]. We recall here some basic results without proof. They are needed to simulate the CTRW of the fractional diffusion equation with and without drift (see Chapter 4).

A random variable X is said to have a stable distribution $P(x) = Prob\{X \leq x\}$ if for any $n \geq 2$ there are a positive number c_n and a real number d_n such that

$$X_1 + X_2 + \cdots + X_n \underset{=}{d} \; c_n X + d_n \,, \tag{B.1}$$

where $X_1, X_2, \cdots, X_n$ are mutually independent random variables having the same distribution as X, and $\underset{=}{d}$ represents here and later equality in distribution. We use the abbreviation *iid* for the random variables, when they are mutually independent and have a common distribution shared with a given random variable X. If $d_n = 0$, the distribution is called *strictly stable*.

Feller has shown that the norming constant c_n in equation (B.1) are of the form

$$c_n = n^{1/\alpha}, \;\; 0 \leq \alpha \leq 2 \,,\; n \geq 2 \,. \tag{B.2}$$

The parameter α is called the *characteristic exponent* or the *index* of stability.

Lukacs [61] has given an alternative definition in place of (B.1) as: A random variable X is said to have a stable distribution if for any positive numbers A and B, there are a positive number C and a real number D such that

$$AX_1 + BX_2 \underset{=}{d} CX + D \,, \tag{B.3}$$

where X_1 and X_2 are independent realizations of the random variable X. If $D = 0$ the distribution is called *strictly stable.*

A stable distribution is called *symmetric* if the random variable $-X$ has the same distribution as X.

An important rule for the constants A, B, and C in (B.3) states that: For any stable distribution, there is a number $\alpha \in (0, 2]$ satisfying the relation

$$C^\alpha = A^\alpha + B^\alpha \,. \tag{B.4}$$

The proof of this relation can be found in [91]. This relation is considered as an alternative to Feller's relation (B.2). It is known that a sum of two iid random variables having the same stable distribution function is again stable with the same index α. However, the invariance of this property does not hold for different indices, (i. e. a sum of two iid stable random variables with different α's is not stable) [61].

We need now to define the *domain of attraction.* We say that the random variable X has a domain of attraction, if there exist three sequences of the following kind: a sequence of iid random variables $Y_1, Y_2 \cdots,$. with sums $S_n = Y_1 + Y_2 \cdots + Y_n$, a sequence of positive numbers $c_n \in \mathbb{R}^+$, and a sequence of real numbers $\delta_n \in \mathbb{R}$, such that

$$\frac{S_n}{c_n} + \delta_n \underset{\Rightarrow}{d} X \;,\; n \geq 2 \,, \tag{B.5}$$

where $\underset{\Rightarrow}{d}$ denotes convergence in distribution (see Feller [20], Zolotarev [109] and Uchaikin et al [104]). Trivially a stable random variable X is lying in its own domain of attraction.

If all Y_i are taken to be independent and all have the same distribution as X, then there exists a sequence of norming constants c_n such that

$$\frac{S_n}{c_n} = \frac{Y_1 + Y_2 + \cdots + Y_n}{c_n} \underset{=}{d} X \;\; n \geq 2 \,. \tag{B.6}$$

When X is Gaussian and all Y_i are iid with finite variance, then (B.5) is the statement of the ordinary *Central Limit Theorem.* The domain of attraction is said to be *normal* if the elements of the sequence c_n satisfies the relation (B.2). For more information, see for example: Feller [20], Lukacs [61], Breiman [8] and [59].

Now for the simulation of the random variable X, we need to search for a general form of the probability density function $p(x)$, therefore we must study first the characteristic function.

The characteristic function

The properties of many distributions are more easily investigated in terms of their characteristic functions. The characteristic function is a variant of the Fourier transform of the applied probability density function. Let us denote the characteristic function of a random variable X with density $p(x) = \frac{d}{dx}P(X \leq x)$ by $\widehat{p}(\kappa)$, defined as

$$\widehat{p}(\kappa) = E[e^{i\kappa X}] = \int_{-\infty}^{\infty} e^{i\kappa x} p(x) dx \ , \ \kappa \in \mathbb{R} \ .$$

The probability density function is said to be symmetric ($P(x) = 1 - P(-x-0)$) iff

$$\widehat{p}(-\kappa) = \widehat{p}(\kappa) \ ,$$

where $P(x)$ is the probability distribution function (see [61]).

In our survey of the theorem of stable probability distributions, we follow Gnedenko and Kolmogorov [23], which is based on the results of Khintchine and Lévy. By using their notation, the characteristic function $\widehat{p}(\kappa)$ belongs to an α-stable distribution, $\alpha \in (0,2]$, if and only if it has the form

$$log\widehat{p}(\kappa) = i\mu'\kappa - c\,|\kappa|^{\alpha}\{1 + i\beta' \frac{\kappa}{|\kappa|} w(|\kappa|, \alpha)\} \ , \qquad \text{(B.7)}$$

where $\kappa \in \mathbb{R}$, $c \geq 0$, $\mu' > 0$, $|\beta'| \leq 1$, and the function $w(\kappa, \alpha)$ is defined as

$$w(|\kappa|, \alpha) = \begin{cases} tan\frac{\pi\alpha}{2} & \text{if } \alpha \neq 1 \ , \\ (2/\pi) log|\kappa| & \text{if } \alpha = 1 \ , \in \mathbb{R} \ , \end{cases} \qquad \text{(B.8)}$$

(see [61] and the references therein). For $\alpha = 2$, we have $w(|\kappa|, \alpha) = 0$ which is the special case of the normal distribution. We note here that the authors who follow Lévy [60] assign the opposite sign to β' in the canonical form (B.7).

Here β' is the symmetry parameter. It determines the skewness of the distribution. $\beta' = 0$ corresponds to a symmetric distribution. c is the scale parameter. It determines the spread of the samples from a distribution around the mean. μ' is the location parameter and $exp(i\mu'\kappa)$ basically corresponds to a shift in the $x-axis$ of the probability density function . For $1 < \alpha \leq 2$, μ' represents the mean and for $0 < \alpha \leq 1$, it represents the median. A stable distribution is said to be standard if $\mu' = 0$ and $c = 1$.

In our work we use another parameterization of the characteristic function. According to Feller the characteristic function of strictly stable densities $p(x;\theta)$ are denoted by $\widehat{p}_\alpha(\kappa;\theta)$ and are defined as

$$\widehat{p}_\alpha(\kappa;\theta) = exp[\,-|\kappa|^\alpha\, e^{\frac{i\theta\pi}{2} sig(\kappa)}]\ . \tag{B.9}$$

The range of the parameters α and θ is restricted to: $0 < \alpha \leq 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\}$, and is visualized by the Feller-Takayasu diamond [65]. The relation between the Gnedenko-Kolmogorov and Feller on the other side is related to the skewness θ of $\widehat{p}_\alpha(\kappa;\theta)$ in equation (B.9) and the skewness β' of $\widehat{p}(\kappa)$ in equation (B.8) as follows

$$\beta' = \frac{tan(\theta\frac{\pi}{2} sig(\kappa))}{tan(\frac{\alpha\pi}{2})} \quad , \ \alpha \neq 1 \tag{B.10}$$

with $\mu' = 0$, $c = 1$.

It is important to say here that in most cases, the inverse Fourier transform of the general canonical form (B.7, B.8) can not be carried out with elementary functions. The most known ones (see [61]) are corresponding to $\alpha = 1$, $\beta' = 0$, and $c = 1$ giving the Cauchy distribution,

$$p_1(x;0) = \frac{1}{\pi}\frac{1}{1+x^2} \quad ,$$

$\alpha = 2$, $\beta' = 0$, and $c = 1$ giving the Gaussian distribution

$$p_2(x;0) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}\ ,$$

and $\alpha = 1/2$ which has been shown by Lévy with $\mu' = 0$, $c = 1$, and $\beta' = -1$ as

$$P_{1/2}(x;-1) = \begin{cases} \frac{1}{\sqrt{2\pi}} x^{-3/2}\, e^{-1/(2x)} & \text{if } x > 0\ , \\ 0 & \text{if } x < 0\ . \end{cases} \tag{B.11}$$

For $\alpha = 1$, $\theta \neq 0$, the stable distributions according to Feller are different from those of Gnedenko-Kolmogorov. In fact in this case Feller's distributions are strictly stable whereas those of Gnedenko-Kolmogorov are not. Feller has shown for $\alpha = 1$, $0 < |\theta| < 1$ that

$$p_1(x;\theta) = \frac{1}{\pi}\frac{cos(\theta\pi/2)}{[x +\ sin(\theta\pi/2)]^2 + [cos(\theta\pi/2)]^2} \quad -\infty < x < \infty\ ,$$

and for $\alpha = 1$, $\theta = \pm 1$,

$$p_1(x;\pm 1) \; = \delta(x \pm 1) \; , \; -\infty < x < \infty \; .$$

Apart from these three cases no stable distribution functions are known whose density functions are elementary functions. The other stable distributions according to Feller can be obtained from equation (B.9) in terms of convergent power series valid for $x > 0$ (see for example: [18], [20] and [65])

(a) $0 < \alpha < 1$ (negative powers), $|\theta| \leq \alpha$

$$p_\alpha(x;\theta) = \frac{1}{\pi x}\sum_{n=1}^{\infty}(-x^{-\alpha})^n \, \frac{\Gamma(n\alpha+1)}{n!} \; sin[\frac{n\pi}{2}(\theta-\alpha)] \; , \qquad \text{(B.12)}$$

(b) $1 < \alpha \leq 2$, $|\theta| \leq 2 - \alpha$

$$p_\alpha(x;\theta) = \frac{1}{\pi x}\sum_{n=1}^{\infty}(-x)^n \, \frac{(1+\frac{n}{\alpha})}{n!} sin[\frac{n\pi}{2\alpha}(\theta-\alpha)] \; . \qquad \text{(B.13)}$$

The values for $x < 0$ can be obtained from (B.12) and (B.13) by using the *symmetry relation*

$$p_\alpha(-x;\theta) = p_\alpha(x;-\theta) \; .$$

For figures exhibiting graphs of $p_\alpha(x;\theta)$ for different values of α and θ, see [65]. See also Fig. [B.1].

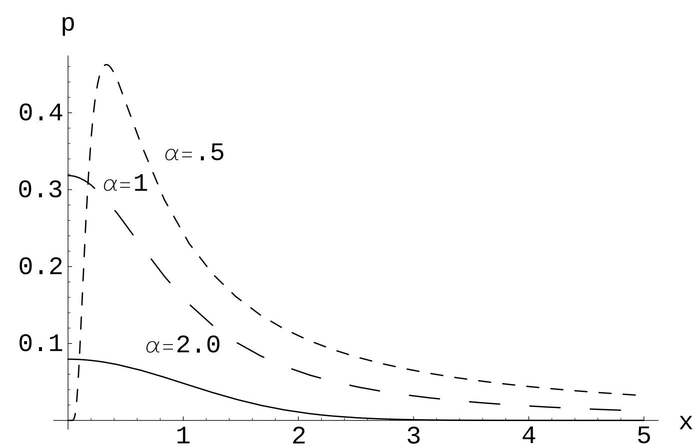

Figure B.1: the comparison of the probability density of Gaussian, Cauchy and the one corresponds to $\alpha = 1/2, \beta' = -1$ equation (B.11).