Appendix A: Important definitions

In this Appendix, we give some important definitions used in the fractional calculus (i.e. the differentiation and the integration of an arbitrary positive order). The fractional calculus is recently widely applied in many fields related to mathematics and real physics problems. A growing number of articles and mathematical and physical books have appeared in the last 25 years (see for example: [79], [72], [87], [80], [90], [37], and [88]). For a very general theory, see [56].

There is no unique definition for the fractional integral operator or for the fractional differential operator. Many versions are applied to functions defined on a half-axis or on the whole real line. The fractional integral usually represents the convolution with a power function, while the fractional derivative is usually defined as the left-inverse to the fractional integration operator.

We give here a survey on fractional integration and differentiation in the interval $0 \leq t < \infty$ as we need for our purpose.

First we consider the Riemann-Liouville fractional integral, denoted by J^β, and defined by

$$J^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(\tau)}{(t-\tau)^{(1-\beta)}} \, d\tau \, , \; t > 0, \; \beta > 0 \, . \tag{A.1}$$

For completeness we set $J^0 f(t) = f(t)$. This means J^0 is an identity operator. The Riemann-Liouville integral satisfies also the semigroup property

$$J^\alpha J^\beta f(t) \; = \; J^\beta J^\alpha f(t) \; = \; J^{\alpha+\beta} f(t) \; , \; \alpha, \beta \geq 0 \, .$$

This can directly be seen by applying the definition and interchange the order of the integration. An interesting example is the integral of the power function $t^\nu, \nu > -1$

$$J^\beta t^\nu = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu+\beta)} t^{\nu+\beta} \, .$$

From the definition (A.1), we can deduce that the operator J^β is a natural generalization of the integration of an integer order, which accordingly to the (n- fold) iterated integration can be shown to be

$$J^n f(t) = \frac{1}{n-1!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau, t > 0 \ . \tag{A.2}$$

From this equation, we deduce that $D^n J^n = I$, but $J^n D^n \neq I$, where $n \in \mathbb{N}$ and I is the identity operator. Actually

$$J^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k \ .$$

Assuming now $m-1 < \beta \leq m \in \mathbb{N}$, and as a consequence of the last discussion, we expect to have $D^\beta \, J^\beta u(t) = u(t)$. Formally we obtain

$$u(t) = D^\beta f(t) \ ,$$

as a solution of the *Abel* integral equation

$$J^\beta u(t) = f(t) \ , \ t > 0 \ , \ \beta > 0 \ ,$$

if $f(t)$ is a sufficiently smooth function. Setting

$$J^m u(t) = J^{m-\beta} J^\beta u(t) = J^{m-\beta} f(t) \ ,$$

letting $\phi(t) = J^{m-\beta} f(t)$, and assuming all $\phi^{(k)}(0) = 0$, for $k = 0, 1, \cdots, m-1$, then we can set

$$D^m \phi(t) = D^m J^{m-\beta} f(t) \ .$$

Now by using this equation and the definition of the identity operator I, we can set

$$u(t) = D^m J^m u(t) = D^m J^{m-\beta} f(t) \ .$$

This means

$$D^\beta f(t) = D^m J^{m-\beta} f(t) \ . \tag{A.3}$$

Therefore, the Riemann-Liouville fractional derivative, the most used fractional derivative operator, for a sufficiently smooth function $f(t)$ given in an interval $[0, \infty)$, is defined as (see for example: [90], [37], and [78])

$$(D^\beta f)(t) := \begin{cases} \frac{1}{\Gamma(m-\beta)} \frac{d^m}{dt^m} \int_0^t \frac{f(\tau)}{(t-\tau)^{\beta+1-m}} \, d\tau \ , & m-1 < \beta < m \ , \\ \frac{d^m}{dt^m} f(t) \ , & m = \beta \ . \end{cases} \tag{A.4}$$

For the important case $0 < \beta < 1$, we have $m = 1$, and the widely used relation

$$D^{\beta} f(t) = D^1 J^{1-\beta} f(t) \ .$$

From the definitions (A.1) and (A.4), we can define the Riemann-Liouville fractional derivative D^{β} as the left inverse of J^{β}, $\beta \geq 0$. Therefore some authors prefer to write the Riemann-Liouville fractional integral operator J^{β} as $D^{-\beta}$ (e.g. [80], [90] and [72]). For completeness, we have $D^0 = I$. The Riemann-Liouville fractional derivative of the power function t^{μ}, gives

$$D^{\beta}\, t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\beta)} t^{\mu-\beta} \ , \beta \geq 0 \ .$$

This means the Riemann-Liouville fractional derivative of a non-zero constant is different from zero. In fact

$$D^{\beta} 1 = D^{\beta} t^0 = \frac{\Gamma(1)}{\Gamma(1-\beta)} t^{-\beta} \ \text{ if } 0 < \beta < 1 \ .$$

The alternative fractional derivative operator is the *Caputo* fractional derivative of order $\beta > 0$ (see [37]). It can be defined by interchanging the operators in the R. H. S. of equation (A.3). This gives

$$\underset{*}{D}^{\beta} f(t) = J^{m-\beta}\, D^m\, f(t) \ , \ m-1 < \beta \leq m \ ,$$

and more explicitly

$$\underset{*}{D}^{\beta} f(t) \ = \ \begin{cases} \frac{1}{\Gamma(m-\beta)} \{ \int\limits_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\beta+1-m}} d\tau & \text{for } m-1 < \beta < m \ , \\ \frac{d^m}{dt^m} f(t) & \text{for } \beta = m \ . \end{cases} \tag{A.5}$$

The Caputo fractional derivative $\underset{*}{D}^{\beta}$ can also be defined through its image in the Laplace transform domain, which is

$$\mathcal{L}\{\underset{*}{D}^{\beta} f(t); s\} = s^{\beta} \widetilde{f}(s) - s^{\beta-1} f(0) - \grave{f}(0) s^{\beta-2} - \cdots - f^{(m-1)}(0) s^{\beta-m} \ , \ s > 0 \ .$$

The definition (A.5) is more restrictive than (A.4) because it requires that $f^{(m)}(t)$ exists. It is convenient for our work to mention the relation between the Riemann-Liouville fractional derivative and integral operators and the Caputo fractional derivative of order β, in the special case $0 < \beta \leq 1$. Since we have

$$\underset{*}{D}^{\beta} f(t) = J^{1-\beta} D f(t) = \underset{*}{D}^{\beta} \left(f(t) - f(0^+) \right) \ , \tag{A.6}$$

and

$$D^{\beta}(f(t)-f(0)) = DJ^{1-\beta}(f(t)-f(0)) = D^{\beta}f(t) - \frac{f(0)}{\Gamma(1-\beta)}t^{-\beta} \,, \tag{A.7}$$

we can deduce that

$$\underset{*}{D^{\beta}} f(t) = D^{\beta}(f(t)-f(0)) \,,\; 0<\beta\le 1 \,, \tag{A.8}$$

which in the Laplace domain reads

$$\mathcal{L}\{\underset{*}{D^{\beta}} f(t); s\} = s^{\beta}\widetilde{f}(s) - s^{\beta-1}f(0)\; s>0 \,.$$

Equation (A.8) represents the relation between the Riemann-Liouville and the Caputo fractional derivative, and the dependence on the initial conditions. This equation is important for solving the fractional differential equations.

In what follows, we consider the finite difference scheme for a function $f(t)$ which is differentiable up to an integer order $n \in \mathbb{N}$, or up to a fractional order $\beta \in \mathbb{R}^{+}$. The backward finite difference operator of an integer order $n \in \mathbb{N}$ is defined as

$$(\Delta_{\tau}^{n} f)(t) = \sum_{k=0}^{n}(-1)^{k}\binom{n}{k} f(t-k\tau) \,.$$

If the function is differentiable up to an integer order $n \in \mathbb{N}$, then we have

$$f^{(n)}(t) = \lim_{\tau\to 0}\frac{(\Delta_{\tau}^{n} f)(t)}{\tau^{n}} \,. \tag{A.9}$$

By generalizing formula (A.9) to $\beta > 0$ instead of n and taking ∞ as the upper limit of the summation, we get the definition of the *Grünwald-Letnikov* fractional derivative of order $\beta > 0$. With the fractional finite difference operator of a positive order β

$$(\Delta_{\tau}^{\beta} f)(t) = \sum_{k=0}^{\infty}(-1)^{k}\binom{\beta}{k} f(t-k\tau) \,, \tag{A.10}$$

for a sufficiently smooth function $f(t)$, defined on the whole line (see [90]), we define

$$f^{(\beta)}(t) = \lim_{\tau\to 0^{+}}\frac{(\Delta_{h}^{\beta} f)(t)}{\tau^{\beta}} \,. \tag{A.11}$$

We note that for $\beta > 0$, the series $\sum\limits_{k=0}^{\infty}\left|\binom{\beta}{k}\right|$ converges absolutely. We note also that

$$\sum_{k=0}^{n}\binom{n}{k} = 2^{n} \,.$$

$f^{(\beta)}(t)$ is called the *Grünwald-Letnikov* fractional derivative of order $\beta > 0$ for a function $f(t)$. It can also be defined on a half line $t \geq 0$ by the finite difference operator

$$(\Delta_\tau^\beta f)(t) = \sum_{k=0}^{\frac{t-a}{\tau}} (-1)^k \binom{\beta}{k} f(t - k\tau) \ , \tag{A.12}$$

where $t > a$. Then we have

$$f^{(\beta)}(t) = \lim_{\tau \to 0} \frac{1}{\tau^\beta} \sum_{k=0}^{\frac{t-a}{\tau}} (-1)^k \binom{\beta}{k} f(t - k\tau) \ , \tag{A.13}$$

where $\tau = \frac{b-a}{n} \to 0$ (see [80] and [90]). We have used this definition, with $a = 0$, to discretize the model discussed in Chapter 3.

Because of the importance of the Grünwald-Letnikov operator it is worth to say that this operator has been modified by Vu Kim Tuan and Gorenflo [103] in order to show that the error committed by approximating $(D^\beta f)(t)$ by $\tau^{-\beta}(\Delta_\tau^\beta f)(t)$ possesses an asymptotic expansion of integer powers of the step length τ (as $\tau \to 0$) if f is sufficiently smooth. For more information see [103].