The convergence of the discrete solution of the space-time-FDE and the time-FDECLD for refinement of the grid

5.1 Introduction

This chapter is devoted to convergence proofs for random walk models of difference scheme type (discrete in space and time) for the fractional diffusion without and with central linear drift.

For the space-time-fractional diffusion equation (space-time-FDE), we prove the convergence for the general Cauchy problem

$$\underset{t\ *}{D^{\beta}}\, u(x,t) = \underset{0}{D^{\alpha}}\, u(x,t)\ , u(x,0) = \delta(x)\ ,\ 0 < \beta \le 1\ ,\ 0 < \alpha \le 2\ , \tag{5.1.1}$$

where $\underset{0}{D^{\alpha}}$ is called the *Riesz* space-fractional derivative operator and $\underset{t\ *}{D^{\beta}}$ is the Caputo time-fractional derivative operator (see Appendix A for more information). We have worked with such schemes for simulating particle paths and for calculating probability densities evolving in time in the previous chapters. As in Chapter 4 we distinguish the following cases with respect to the orders of α and β

(a) $\alpha = 2$, $\beta = 1$: classical diffusion equation,

(b) $\alpha = 2$, $0 < \beta < 1$: time-FDE,

(c) $0 < \alpha < 2$, $\beta = 1$: space-FDE,

(d) $0 < \alpha < 2$, $\alpha \neq 1$, $0 < \beta < 1$: space-time-FDE,

(e) $\alpha = 1$, $0 < \beta < 1$: a singular case of time-fractional diffusion equation.

Case (a) is formally contained in case (b). By extending all formulas to $\beta = 1$, the proof remains valid. The convergence in this special case is also a well-known fact from classical random walk theory and from numerical analysis. Case (c) has been well treated by Gorenflo & Mainardi (see e. g. [34], [35] and [36]). We treat here cases (b), (d) and (e).

For the time-fractional diffusion equation with central linear drift (time-FDECLD), we prove the convergence for the equation

$$\underset{t\ *}{D^{\beta}}\, u(x,t) = \frac{\partial^2}{\partial x^2}\, u(x,t) + \frac{\partial}{\partial x}\,(x u(x,t))\ ,\ u(x,0) = \delta(x - x^*),\ 0 < \beta \leq 1\ , \tag{5.1.2}$$

where x^* is the initial position of the particle. To avoid confusion with the grid point $x_0 = 0h$, we denote here the initial position not by x_0, but by x^*. As in Chapters 2 and 3, we distinguish the following cases with respect to the value of the order β

(f) $\beta = 1$: classical diffusion with central linear drift (the generalized Ehrenfest model),

(g) $0 < \beta < 1$: time-fractional diffusion with central linear drift (time-FDECLD).

Although case (f) is formally contained in case (g), we treat it separately because the transitions probabilities in this case are completely different from those of case (g).

For all these cases, we show that by properly scaled transition to the limit of vanishing step sizes, in space and time, there is convergence in the Fourier-Laplace domain which then implies convergence in distribution (weak convergence) of the corresponding probability densities for the location of the particle.

This chapter is organized as follows:

In Section 2, discretization of case (b) will be discussed. We give an outline of the theory of the convergence of the discrete-space discrete-time solution to the corresponding fundamental solution.

In Section 3, the general notations for the symmetric space-fractional operators are given and the discretization is described and discussed. Then the convergence of the model is studied and interpreted for cases (d) and (e) separately.

In Section 5, The proof of the convergence for cases (f) and (g) are discussed separately.

5.2 Discretization of the time-FDE and conditions of convergence

In this section we consider case (b) of Section (5.1) and show convergence of the discrete solution of this model to the solution of the Cauchy problem

$$\underset{t\ *}{D^{\beta}}\, u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)\ , u(x,0) = \delta(x)\ ,\ \ 0 < \beta \leq 1\ , \qquad (5.2.1)$$

In the Fourier-Laplace domain equation (5.2.1) reads

$$s^{\beta}\widehat{\widetilde{u}}(\kappa, s) - s^{\beta-1} = -|\kappa|^2 \widehat{\widetilde{u}}(\kappa, s)\ .$$

Solving for $\widehat{\widetilde{u}}(\kappa, s)$, gives

$$\widehat{\widetilde{u}}(\kappa, s) = \frac{s^{\beta-1}}{s^{\beta} + |\kappa|^2}\ ,\ s > 0\ ,\ \kappa \in \mathbb{R}\ , \qquad (5.2.2)$$

where the Laplace transform of a (generalized) function $g(t)$ is defined in (4.2.4) and the Fourier transform of a (generalized) function $f(x)$ is defined in (4.2.9).

The inverse Laplace transform of (5.2.2) gives

$$\widehat{u}(\kappa, t) = E_{\beta}(-|\kappa|^2 t^{\beta})\ \ ,$$

where $E_{\beta}(z)$ is the Mittag-Leffler function of order β. The series and the integral representations of the Mittag-Leffler function are given in Appendix C.

To generate a discrete approximate solution to equation (5.2.1), we discretize the space variable x by using the definition of the grid points (2.3.1) and (2.3.2). The dependent variable is then discretized by introducing $y_j(t_n)$, see (2.3.3).

The discretization of the time-fractional diffusion equation is based on the Grünwald-Letnikov scheme for the Caputo time-fractional derivative operator $\underset{t\ *}{D^{\beta}}$ (see Appendix A) and symmetric approximation to $\frac{\partial^2}{\partial x^2}$. Then the discretization of equation (5.2.1) for all $n \in \mathbb{N}_0$ is

$$\sum_{k=0}^{n+1} (-1)^k \binom{\beta}{k} (y_j(t_{n+1-k}) - y_j(t_0)) =\ \mu\, y_{j+1}(t_n) - 2\mu y_j(t_n) + \mu\, y_{j-1}(t_n)\ , \qquad (5.2.3)$$

and

$$y_j(t_0) = \delta_{j0} = \begin{cases} 1 & j = 0\ , \\ 0 & j \neq 0\ , \end{cases}$$

with the scaling relation

$$\mu = \tau^{\beta}/h^2 \ \leq \beta/2 \ . \tag{5.2.4}$$

This scaling relation ensures, after solving for $y_j(t_{n+1})$, that all the coefficients of $y_j(t_n)$, for all $n \in \mathbb{N}_0$, are non-negative. An often successful method to deal with a system of difference equation as (5.2.3) is the method of generating functions. Therefore, for $n \in \mathbb{N}_0$, we define

$$q_n(z) = \sum_{j\in\mathbb{Z}} y_j(t_n) z^j \ , \tag{5.2.5}$$

for the two-sided sequence of the sojourn probabilities

$$\{\cdots, y_{-2}(t_n), y_{-1}(t_n), y_0(t_n), y_1(t_n), y_2(t_n), \cdots\} \ \ \forall n \in \mathbb{N}_0 \ . \tag{5.2.6}$$

We note here that the sequence (5.2.6) satisfies the conservative and non-negativity preserving conditions because all $y_j(t_n) \geq 0$, and as we have shown in Chapter 3, $\sum\limits_{j=-\infty}^{\infty} y_j(t_n) = 1$. Therefore we obtain $\sum\limits_{j=-\infty}^{\infty} y_j(t_n)|z|^j = 1$, for $|z| = 1$. This means that the series (5.2.5) converges absolutely on the circle $|z| = 1$. From now on we assume $|z| = 1$.

Now, by introducing the generalized function

$$\sum_{j\in\mathbb{Z}} \delta(x - x_j) y_j(t_n) \ , \forall n \in \mathbb{N}_0 \ ,$$

and applying the Fourier-transform, we obtain

$$\mathcal{F}\{\sum_{j\in\mathbb{Z}} \delta(x - x_j) y_j(t_n); \kappa\} \ = \sum_{j\in\mathbb{Z}} e^{i\kappa x_j} y_j(t_n) = q_n(e^{i\kappa h}) \ , \ \kappa \in \mathbb{R} \ . \tag{5.2.7}$$

By comparing equations (5.2.5) and (5.2.7), we see that the Fourier-transform of the sequence (5.2.6) coincides with the generating function $q_n(z)$, if we replace z by $e^{i\kappa h}$. In other words the Fourier transform of the sequence of clumps $y_j(t_n)$, $j \in \mathbb{Z}$, can be represented by $q_n(e^{i\kappa h})$.

Now let us introduce the following *bivariate* (two-fold) generating function

$$Q(z,\zeta) = \sum_{n=0}^{\infty} q_n(z)\zeta^n = \sum_{n=0}^{\infty} (\sum_{j\in\mathbb{Z}} y_j(t_n) z^j) \zeta^n \ , \tag{5.2.8}$$

as a function of ζ for the sequence

$$\{q_0(z), q_1(z)), q_2(z), \cdots\} . \tag{5.2.9}$$

Because all $|q_n(z)| \le 1$, the sequence $Q(z,\zeta)$ converges for $|\zeta| < 1$, and from now on we assume $|\zeta| < 1$.

By introducing the generalized function $\sum_{n=0}^{\infty} \delta(t-t_n) q_n(z)$ and applying the Laplace-transform, we get

$$\mathcal{L}\{\sum_{n=0}^{\infty} \delta(t-t_n) q_n(z); s\} = \sum_{n=0}^{\infty} e^{-st_n} q_n(z) \ , \ s > 0 . \tag{5.2.10}$$

From equation (5.2.7) and equation (5.2.10), we deduce that if we replace z by $e^{i\kappa h}$ and ζ by $e^{-s\tau}$, in equation (5.2.8), we get the Fourier-Laplace transform of the bivariate sequence $\{y_j(t_n)\,|j \in \mathbb{Z},\, n \in \mathbb{N}_0\}$ which is obtained by collecting all the sequences (5.2.6). This means

$$Q(e^{i\kappa h}, e^{-s\tau}) = \sum_{n=0}^{\infty} (\sum_{j\in\mathbb{Z}} y_j(t_n) e^{i\kappa j h}) e^{-ns\tau} \ , \kappa \in \mathbb{R}, \ s > 0 . \tag{5.2.11}$$

Our aim now is to prove that $Q(e^{i\kappa h}, e^{-s\tau})$ is related asymptotically to the Fourier-Laplace transform of $u(x,t)$ which represents the fundamental solution of the time-FDE (5.2.1). By considering (2.3.3), we find that the discretization of the Fourier-transform of $u(x,t)$ formally gives the approximation

$$\widehat{u}(\kappa, t_n) \ \sim \ q_n(e^{i\kappa h}) \ , \ \kappa \in \mathbb{R} .$$

By taking the Laplace-transform of it and imitating a rectangle rule for numerical integration, we get the formal approximation

$$\widehat{\widetilde{u}}(\kappa, s) \sim \tau Q(e^{i\kappa h}, e^{-s\tau}) \ , \ s > 0 . \tag{5.2.12}$$

Our aim now is to find the explicit form of $\tau Q(e^{i\kappa h}, e^{-s\tau})$ in order to show that for a fixed $\kappa \in \mathbb{R}$, for fixed $s > 0$, as $n \to \infty$, and under the condition (5.2.4), we get

$$\lim_{h,\tau\to 0} \tau Q(e^{i\kappa h}, e^{-s\tau}) \ = \ \widehat{\widetilde{u}}(\kappa, s) \ ,$$

(i.e. the discrete solution approximates the Fourier-Laplace transform of the corresponding fundamental solution). To this aim, we construct $Q(z,\zeta)$ with the initial condition $q_0(z) = \widehat{\delta}(\kappa) = 1$. Then by multiplying equation (5.2.3) by z^j and summing over all $j \in \mathbb{Z}$, we get

$$\sum_{k=0}^{n+1}(-1)^k\binom{\beta}{k}(\,q_{n+1-k}(z)-1\,)=\mu\,(z^{-1}-2+z\,)q_n(z)\ ,\ \sum_{j\in\mathbb{Z}}y_j(0)z^j=1\ . \tag{5.2.13}$$

Now multiplying equation (5.2.13) by ζ^n, and summing over all $n \in \mathbb{N}_0$, we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n+1}(-1)^k\binom{\beta}{k}(\,q_{n+1-k}(z)-1\,)\zeta^n=\mu\,(z^{-1}-2+z)\sum_{n=0}^{\infty}q_n(z)\zeta^n\ , \tag{5.2.14}$$

Using the definition (5.2.8), setting $m = n + 1$, and summing the R.H.S of equation (5.2.14) over all $m \in \mathbb{N}$, we get

$$\sum_{m=1}^{\infty}\sum_{k=0}^{m}(-1)^k\binom{\beta}{k}(\,q_{m-k}(z)-1\,)\zeta^m=\mu\,(z^{-1}-2+z)\,Q(z,\zeta)\ . \tag{5.2.15}$$

Since $q_o(z) = 0$, we can begin the summation over m with $m = 0$.

To proceed further, we need the convolution of two general sequences which is equivalent to the multiplication of their generating functions (see Feller [20]). If $\{\alpha_n\}$ and $\{\beta_k\}$ are any two numerical sequences with absolutely convergent power series

$$\alpha(\zeta)=\sum_{n=0}^{\infty}\alpha_n\zeta^n\ ,\ \beta(\zeta)=\sum_{k=0}^{\infty}\beta_k\zeta^k\ ,$$

then

$$\alpha(\zeta).\beta(\zeta)=c(\zeta)=\sum_{r=0}^{\infty}c_r\zeta^r\ ,\ c_r=\sum_{n=0}^{r}\alpha_n\beta_{r-n}$$

This means that the R.H.S of equation (5.2.15) with m ranging from zero to infinity can be represented as a multiplication of the two generating functions

$$\alpha(\zeta)=\sum_{k=0}^{\infty}(-1)^k\binom{\beta}{k}\zeta^k=(1-\zeta)^{\beta}\ ,\ \beta(\zeta)=\sum_{m=0}^{\infty}(\,q_{m-k}(z)-1\,)\zeta^m\ .$$

Setting again $m = n + 1$ and shifting the index of n in the series $\beta(\zeta)$, we get the new version of equation (5.2.15), namely

$$\frac{(1-\zeta)^{\beta}}{\zeta}\,(\,Q(z,\zeta)-\frac{1}{1-\zeta}\,)=\zeta\mu(z^{-1}-2+z)\,Q(z,\zeta)\ . \tag{5.2.16}$$

Solving for $Q(z,\zeta)$, we get

$$Q(z,\zeta) = \frac{(1-\zeta)^{\beta-1}}{(1-\zeta)^{\beta} - \zeta\mu(z^{-1}-2+z)} \,. \tag{5.2.17}$$

Replacing z by $e^{i\kappa h}$, ζ by $e^{-s\tau}$, we get asymptotically for $h \to 0$, $\tau \to 0$

$$(z^{-1}-2+z) \sim -(\kappa h)^2 \;\;,\;\; (1-\zeta)^{\beta} \sim s^{\beta}\mu h \,. \tag{5.2.18}$$

Now multiplying both sides by τ, using the scaling relation (5.2.4), and passing to the limit, equation (5.2.17) gives

$$\lim_{h\to 0, \tau\to 0} \tau Q(e^{i\kappa h}, e^{-s\tau}) = \frac{s^{\beta-1}}{s^{\beta}+\kappa^2} \;=\; \widehat{\widetilde{u}}(\kappa, s) \,. \tag{5.2.19}$$

Using the continuity theorem of probability theory (see Feller [20] and Lukacs [61]) we deduce from equation (5.2.19) with the scaling relation (5.2.4) that the solution of the difference scheme of the time-FDE converges in distribution to the corresponding fundamental solution. By using the inverse-Laplace transform for equation (5.2.19), we get

$$\lim_{h\to 0} \tau Q(e^{i\kappa h}, t) \to \widehat{u}(\kappa, t) = E_{\beta}(-\kappa^2 t^{\beta}) \;\;,$$

and we can find the behaviour of the second moment of the density $u(x,t)$ as a function of t: $(\sigma(t))^2 = -\frac{\partial^2}{\partial\kappa^2}\widehat{u}(\kappa,t)|_{\kappa=0} = \frac{2\,t^{\beta}}{\Gamma(1+\beta)}$ (see [30]). In the special case $\beta = 1$, $\tau Q(e^{i\kappa h}, t)$, tends to $e^{-\kappa^2 t}$, for $h \to 0$, and consequently $(\sigma(t))^2 = 2t$ (see [34]).

5.3 Discretization of the space-time-FDE and conditions of convergence

In this section, we begin our discussion by considering case (d) of Section (5.1). This means that we discuss the outcome of replacing the second order space-derivative in equation (5.2.1) by the *Feller operator* [18] in the symmetric case with order $0 < \alpha \le 2$.

In equation (5.1.1) the operator ${}_{x}D_{0}^{\alpha}$ is called the *Riesz* space-fractional differentiation operator. We adopt, for simplicity, the notation introduced by Zaslavski [89]. We recognize that ${}_{x}D_{0}^{\alpha}$ formally is a power of the positive definitive operator ${}_{x}D^{2} = -\frac{d^2}{dx^2}$ and must not be confused with a power of the first order differential operator ${}_{x}D$ (see [35] for a detailed presentation of the theory of this operator and related operators). Since $-|\kappa|^\alpha = -(\kappa^2)^{\alpha/2}$, we can set ${}_xD_0^\alpha = -(-\frac{d^2}{dx^2})^{\alpha/2}$, which proves that the Riesz derivative is a symmetric fractional generalization of the second derivative. For more information about the Fourier transform and the pseudo-differential operators as semi groups of linear operators, see e. g. [25], [48] and [49].

The Fourier transform of ${}_xD_0^\alpha$ reads

$$\mathcal{F}\{{}_xD_0^\alpha\,\Phi(x)\,;\kappa\} = -|\kappa|^\alpha\widehat{\phi}(\kappa)\ ,\ 0<\alpha\le 2\ ,\kappa\in\mathbb{R}\ , \tag{5.3.1}$$

while

$$\mathcal{F}\{\frac{d^n}{dx^n}\,\Phi(x)\,;\kappa\} = -(i\kappa)^n\widehat{\phi}(\kappa)\ ,\ n\in\mathbb{N}\ ,\ \kappa\in\mathbb{R}\ . \tag{5.3.2}$$

This means that, in Zaslavski's notations, we have

$${}_xD_0^\alpha\,\Phi(x) = \frac{d^\alpha}{d|x|^\alpha}\ ,\ \ 0<\alpha\le 2\ . \tag{5.3.3}$$

From (5.3.1-5.3.3), we easily see that in the case $\alpha = 1$

$$\mathcal{F}\{{}_xD_0^1\phi(x);\kappa\}\ \neq\ \mathcal{F}\{\frac{d\phi(x)}{dx};\kappa\}\ \ .$$

Then the Fourier-Laplace transform of the space-time-FDE (5.1.1), for $0<\beta<1$ and $0<\alpha\le 2$, reads

$$s^\beta\widehat{\widetilde{u}}(\kappa,s) - s^{\beta-1} = -|\kappa|^\alpha\widehat{\widetilde{u}}(\kappa,s)\ .$$

Solving for $\widehat{\widetilde{u}}(\kappa,s)$, gives

$$\widehat{\widetilde{u}}(\kappa,s) = \frac{s^{\beta-1}}{s^\beta + |\kappa|^\alpha}\ ,\ s>0\ ,\ \kappa\in\mathbb{R}\ , \tag{5.3.4}$$

and the inverse Laplace transform of this equation gives

$$\widehat{u}(\kappa,t) = E_\beta(-|\kappa|^\alpha t^\beta)\ \ .$$

From this relation we can deduce that in the case $0<\alpha<2$, the second moment of the density $u(x,t)$ is infinite for all $t>0$ (see [30]).

We express now the operator ${}_xD_0^\alpha$ as the negative inverse of the suitable integral operator (*Riesz potential*) I_0^α whose symbol is $(|\kappa|^{-\alpha})$. We write

$${}_xD_0^\alpha = -I_0^{-\alpha}\ ,\ 0<\alpha\le 2\ ,\ \alpha\neq 1\ , \tag{5.3.5}$$

where $I_0^{-\alpha}$ is the inverse of the *symmetric Riesz* potential (see Samko & Marichev [90])

$$I_0^\alpha \Phi(x) = c_-(\alpha)\, I_+^\alpha \Phi(x) + c_+(\alpha)\, I_-^\alpha \Phi(x)\ ,\ \alpha \neq 1\ , \qquad (5.3.6)$$

where

$$c_-(\alpha) = c_+(\alpha) = 1/(2cos\frac{\alpha\pi}{2}) \quad .$$

Therefore, the symmetric Riesz potential can be written as

$$\underset{0}{I}^{\alpha}\, \Phi(x) = \frac{1}{2cos\frac{\alpha\pi}{2}}(I_+^\alpha \Phi(x) + I_-^\alpha \Phi(x))\ , 0 < \alpha \leq 2\ ,\ \alpha \neq 1\ ,\ (5.3.7)$$

where $I_\pm^\alpha$ denote the *Riemann-Liouville* fractional integral operators, by some people called *Weyl* integrals. They are defined as

$$\begin{aligned} I_+^\alpha \Phi(x) &= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-\xi)^{\alpha-1}\Phi(\xi) d\xi\ , \\ I_-^\alpha \Phi(x) &= \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (\xi - x)^{\alpha-1}\Phi(\xi) d\xi\ . \end{aligned} \qquad (5.3.8)$$

The symmetric Riesz potential is then defined as

$$I_0^\alpha \Phi(x) = \frac{1}{2\Gamma(\alpha)cos(\alpha\pi/2)} \int_{-\infty}^{\infty} |x-\xi|^{\alpha-1}\Phi(\xi) d\xi\ . \qquad (5.3.9)$$

The Riesz potential operator is well defined if the index is located in the range $(0,1)$ and we have the semi group property,

$$I_0^{\alpha+\beta} = I_0^\alpha I_0^\beta\ ,\ 0 < \alpha < 1\ ,\ 0 < \beta < 1\ ,\ \alpha + \beta < 1 \quad .$$

Now we can extend our definitions according to Feller [18] and Samko [90] to introduce the inverse Riesz potential operator in the whole range $0 < \alpha \leq 2$ as

$$\underset{x\ 0}{D}^{\alpha} = \frac{-1}{2cos(\alpha\pi/2)}\left[I_+^{-\alpha} + I_-^{-\alpha}\right] , 0 < \alpha \leq 2\ ,\ \alpha \neq 1\ , \qquad (5.3.10)$$

where $I_\pm^{-\alpha}$ are the inverses of the operators $I_\pm^\alpha$. This means the operators $I_\pm^{-\alpha}$ are obtained from the definitions of $I_\pm^\alpha$ equation (5.3.8) by changing the sign of α. Off course, appropriate assumptions are required for the functions to which these operators are applied. In the Fourier domain, I_0^α is represented by the symbol $-|\kappa|^\alpha$, and we see that ${}_{x}D_{0}^{\alpha}$ is also meaningful as a pseudo-differential operator if $\alpha = 1$, but is different from $D = \frac{d}{dx}$.

We want now to discretize the Riesz fractional operator D_{0}^{α} by a suitable finite difference scheme to approximate the solution of the space-time fractional diffusion equation (5.1.1), excluding the case $\alpha = 1$ which we treat later. To do so we approximate the inverse operators $I_{\pm}^{-\alpha}$ by the Grünwald-Letnikov scheme (see Oldham & Spanier [79], Ross & Miller [72], and recently Gorenflo & Mainardi (e.g. [36], [35], [34] and [27]). Recalling now our discretization (2.3.1 and 2.3.3) the inverse of the Riemann-Liouville integrals can formally be obtained as the limit

$$I_{\pm}^{-\alpha} = \lim_{h \to 0} {}_{h}I_{\pm}^{-\alpha} \,, \tag{5.3.11}$$

where ${}_{h}I_{\pm}^{-\alpha}$ denote the approximating Grünwald-Letnikov scheme which reads
(A) $0 < \alpha < 1$

$$I_{\pm}^{-\alpha}\, \phi(x) = \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} (-1)^{k} \binom{\alpha}{k} \phi(x_j \mp kh) \,, \tag{5.3.12}$$

(B) $1 < \alpha \le 2$

$$I_{\pm}^{-\alpha}\, \phi(x) = \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} (-1)^{k} \binom{\alpha}{k} \phi(x_j \mp (k-1)h) \,. \tag{5.3.13}$$

The shift in the index j in (5.3.13) is required to obtain a scheme with all coefficients which are non-negative in the final formula for $y_j(t_{n+1})$, see equation (5.3.26).

So far the discretization of the Riesz potential operator is as follows

$${}_{h}D_{0}^{\alpha} y_j(t_n) = \frac{-1}{2cos\frac{\alpha\pi}{2}} \left({}_{h}I_{+}^{-\alpha} + {}_{h}I_{-}^{-\alpha} \right) y_j(t_n) \,,\; 0 < \alpha \le 2 \,, \alpha \neq 1 \,, j \in \mathbb{Z} \,. \tag{5.3.14}$$

We must distinguish the discretization of ${}_{h}I_{\pm}^{-\alpha}$ with respect to the value of α:

$${}_{h}I_{\pm}^{-\alpha} y_j(t_n) = \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} (-1)^{k} \binom{\alpha}{k} y_{j \mp k} \,,\; 0 < \alpha \le 1 \,, \tag{5.3.15}$$

while

$${}_{h}I_{\pm}^{-\alpha} y_j(t_n) = \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} (-1)^{k} \binom{\alpha}{k} y_{j \pm 1 \mp k} \,,\; 1 < \alpha \le 2 \,, \tag{5.3.16}$$

Now we adjoin the discretization of $\underset{h\,0}{D^{\alpha}}$, for case (A), with the discretization of $\underset{\tau\,*}{D^{\beta}}$, with the infinite sequence $\{y_j(t_n)\}$, $j \in \mathbb{Z}$, to get the discretization of the space-time-FDE for these two cases.

First for the case (A):
By using equation (5.3.12), the discretization of equation (5.1.1) reads

$$\tau^{-\beta}\sum_{m=0}^{n+1}(-1)^m\binom{\beta}{m}\{y_j(t_{n+1-m}) - y_j(0)\}$$
$$= \frac{-h^{-\alpha}}{2cos\frac{\alpha\pi}{2}}\{\sum_{k\neq 0}(-1)^k\binom{\alpha}{|k|}y_{j-k}(t_n) + 2y_j(t_n)\}\ . \quad (5.3.17)$$

Introducing now the scaling parameter

$$\mu = \frac{\tau^{\beta}}{h^{\alpha}}\ , \quad (5.3.18)$$

and solving for $y_j(t_{n+1})$, we get

$$y_j(t_{n+1}) = \sum_{m=0}^{n}(-1)^m\binom{\beta}{m}y_j(0) + \sum_{m=2}^{n}(-1)^{m+1}\binom{\beta}{m}y_j(t_{n+1-m})$$
$$+ (\beta - \frac{\mu}{cos\frac{\alpha\pi}{2}})y_j(t_n) - \frac{\mu}{2cos\frac{\alpha\pi}{2}}\sum_{k\neq 0}^{\infty}(-1)^k\binom{\alpha}{|k|}y_{j-k}(t_n)\ . \quad (5.3.19)$$

In order to have a random walk of the model, all the coefficients in the L.H.S must be non-negative for all values of $n \geq 0$. This means that the following two conditions must be satisfied. For the coefficient of $y_j(t_n)$, we have the condition

(1) $(\beta - \frac{\mu}{cos\frac{\alpha\pi}{2}}) \geq 0$
which is equivalent the condition related to the scaling parameter (5.3.18)

$$0 < \mu \leq \beta\, cos\frac{\alpha\pi}{2}\ ,\ 0 < \alpha < 1\ . \quad (5.3.20)$$

Furthermore (2) and (3) hold:
(2) $(-1)^{k+1}\frac{\mu}{2cos\frac{\alpha\pi}{2}}\binom{\alpha}{|k|} \geq 0\ , \alpha \neq 1\ ,\ k \neq 0\ ,$

For the scaling relation (5.3.20), we observe that the scheme (5.3.19) has the further properties of non-negativity and conservation.

(3) $\sum_{m=0}^{n}(-1)^m\binom{\beta}{m} \geq 0$, $\sum_{m=2}^{n}(-1)^{m+1}\binom{\beta}{m} \geq 0$, and

$$\sum_{m=0}^{n}(-1)^m\binom{\beta}{m} + \sum_{m=1}^{n}(-1)^{m+1}\binom{\beta}{m} = 1 \ \forall n \geq 1 \, , 0 < \beta < 1 \, .$$

Therefore equation (5.3.19) represents a random walk model of the space-time-FDE as time proceeds from t_n to t_{n+1}. If we use the initial condition $y_j(0) = \delta_{j0}$ and use the last three properties, we can prove by induction that $\sum_{j\in\mathbb{Z}} y_j(t_n) = 1 \ \forall n \geq 1$.

Now we convert equation (5.3.19) into a single equation by using the generating function $q_n(z)$, defined as in equation (5.2.5). As we have done in the previous section, we multiply both sides of equation (5.3.19) by z^j and sum over all $-\infty < j < \infty$. We get

$$\sum_{n=0}^{n+1}(-1)^m\binom{\beta}{m}\{q_{n+1-m}(z) - 1\} = \frac{-\mu q_n(z)}{2cos\frac{\alpha\pi}{2}}\{(1-\frac{1}{z})^\alpha + (1-z)^\alpha\} \, , \tag{5.3.21}$$

where $q_0(z) = 1$ and the series is convergent at $|z| = 1$. Now we use the bivariate (two-fold) generating function $Q(z,\zeta)$ defined as in equation (5.2.8) by multiplying both sides of equation (5.3.21) by ζ^n then sum over $n \in \mathbb{N}_0$. We get

$$Q(z,\zeta) = \frac{(1-\zeta)^{\beta-1}}{(1-\zeta)^\beta + \frac{\zeta\mu}{2cos\frac{\alpha\pi}{2}}\{(1-\frac{1}{z})^\alpha + (1-z)^\alpha\}} \, . \tag{5.3.22}$$

Replacing now z by $e^{i\kappa h}$ and ζ by $e^{-s\tau}$, where $\kappa \in \mathbb{R}$ and $s > 0$, we shall prove that $\tau Q(e^{i\kappa h}, e^{-s\tau})$ tends to the Fourier-Laplace transform of the solution of the space-time-FDE (5.1.1). To do so, we use the asymptotic relations:
First as $h \to 0$, we have $(1-e^{-i\kappa h})^\alpha \sim (i\kappa h)^\alpha$, $(1-e^{i\kappa h})^\alpha \sim (-i\kappa h)^\alpha$, which imply

$$(-i\kappa h)^\alpha + (i\kappa h)^\alpha = 2(|\kappa|h)^\alpha cos\frac{\alpha\pi}{2} \, .$$

Second as $\tau \to 0$, we have $(1-e^{-s\tau})^\beta \sim (s\tau)^\beta$.
Now substituting back with these asymptotics in equation (5.3.22), multiplying both sides by τ, using the scaling relation (5.3.18), and comparing the result with equation (5.3.4), we get

$$\lim_{h\to 0,\tau\to 0} \tau Q(e^{i\kappa h}, e^{-s\tau}) \to \frac{s^{\beta-1}}{s^\beta + |\kappa|^\alpha} = \hat{\tilde{u}}(\kappa, s) \, , \ \kappa \in \mathbb{R} \, , s > 0 \, , \tag{5.3.23}$$

and the inverse-Laplace transform of $\tau Q(e^{i\kappa h}, e^{-s\tau})$ gives

$$\widehat{u}(\kappa, t) = E_\beta(-|\kappa|^\alpha t^\beta) \ , \ 0 < \alpha < 1 \ , \ t > 0 \ . \tag{5.3.24}$$

From this equation we can deduce that the second moment of the density $u(x,t)$ is infinite for $t > 0$ (see [30]).

Second for the case (B):
For $1 < \alpha < 2$, we use the discretization of $\underset{h\,0}{D^\alpha}$, at equation (5.3.13), together with the discretization of $\underset{\tau\ *}{D^\beta}$, for the infinite sequence $\{\cdots + y_{-2}(t_n), y_{-1}(t_n), y_0(t_n), y_1(t_n), \cdots\}$, to discretize equation (5.1.1). We get

$$\tau^{-\beta} \sum_{m=0}^{n+1} (-1)^m \binom{\beta}{m} \{y_j(t_{n+1-m}) - y_j(t_0)\}$$
$$= \frac{-h^{-\alpha}}{2cos\frac{\alpha\pi}{2}} \sum_{k\in\mathbb{Z}} (-1)^k \binom{\alpha}{k} \{y_{j+1-k}(t_n) + y_{j-1+k}(t_n)\} \ . \tag{5.3.25}$$

Let us use the scaling parameter (5.3.18), shift the indices of $y_{j\pm1\mp k}(t_n)$, and solve for $y_j(t_{n+1})$. We get

$$y_j(t_{n+1}) = \sum_{m=0}^{n} (-1)^m \binom{\beta}{m} y_j(0) \ + \sum_{m=2}^{n} (-1)^{m+1} \binom{\beta}{m} y_j(t_{n+1-m})$$
$$+ \ (\beta + \frac{\alpha\mu}{cos\frac{\alpha\pi}{2}}) y_j(t_n) - \frac{\mu}{2cos\frac{\alpha\pi}{2}} \{y_{j+1}(t_n) + y_{j-1}(t_n)$$
$$+ \sum_{k\neq 0,1\in\mathbb{Z}} (-1)^k \binom{\alpha}{|k+1|} y_{j-k}(t_n)\} \ . \tag{5.3.26}$$

Let us remark that the discretization in the special case $\alpha = 2$ reduces the scheme to the symmetric difference approximation of $\frac{\partial^2 u(x,t)}{\partial x^2}$, and so we get the time-FDE, treated in Section 2.

Again (5.3.26) is to represent the random walk for space-time fractional diffusion equation, for $1 < \alpha \leq 2$, as time proceeds from t_n to t_{n+1}, the coefficients of $y_j(t_n)$ must be non-negative and furthermore $\sum_{j\in\mathbb{Z}} y_j(t_n) = 1 \, \forall n \in \mathbb{N}_0$. For this aim we have the following condition:

(1) $\quad (\beta + \frac{\alpha\mu}{cos\frac{\alpha\pi}{2}}) \geq 0$,

which leads to the following inequality related to the scaling relation

$$0 < \ \mu \ \leq \ \frac{-\beta \, cos\frac{\alpha\pi}{2}}{\alpha} \ , \ 1 < \alpha \leq 2 \ , \tag{5.3.27}$$

which can be satisfied because $cos\frac{\alpha\pi}{2} < 0$, as $1 < \alpha \leq 2$. By comparing equation (5.3.26) with equation (5.3.19), we see that conditions (2,3) for $0 < \alpha < 1$, are also satisfied for $1 < \alpha \leq 2$. With the scaling relation (5.3.27), and with conditions (2,3), we find that equation (5.3.26) can simulate the random walk of the model.

Now we shall prove the convergence of the discrete solution corresponding to case (B). We multiply equation (5.3.25) by z^j and sum over all $j \in \mathbb{Z}$. We get

$$\sum_{m=0}^{n+1} (-1)^m \binom{\beta}{m} \{q_{n+1-m}(z) - 1\}$$
$$= \frac{-\mu}{2cos\frac{\alpha\pi}{2}}\, q_n(z)\, \{z^{-1}(1-z)^\alpha \,+\, z(1-\frac{1}{z})^\alpha\}\,. \quad (5.3.28)$$

Now multiplying both sides of this equation by ζ^n and summing over all $n \in \mathbb{N}_0$, we get, for $|z| = 1$, $|\zeta| < 1$,

$$Q(z,\zeta) = \frac{(1-\zeta)^{\beta-1}}{(1-\zeta)^\beta \,+\, \frac{\zeta\mu}{2cos\frac{\alpha\pi}{2}}\{z(1-\frac{1}{z})^\alpha + z^{-1}(1-z)^\alpha\}}\,. \quad (5.3.29)$$

We follow now the same proceedure as in case (A) to prove that if we replace z by $e^{i\kappa h}$ and ζ by $e^{-s\tau}$, where $\kappa \in \mathbb{R}$ and $s > 0$, $\tau Q(e^{i\kappa h}, e^{-s\tau})$ tends to the Fourier-Laplace transform of the solution of the space-time-FDE (5.1.1). To do so, we use the asymptotic relations of case (A), besides the following one

$$\frac{1}{2cos(\frac{\alpha\pi}{2})}\{e^{-i\kappa h}(1-e^{i\kappa h})^\alpha$$
$$+e^{i\kappa h}(1-e^{-i\kappa h})^\alpha\} \sim \frac{(|\kappa|h)^\alpha}{cos(\frac{\alpha\pi}{2})}cos(\frac{\alpha\pi}{2} - \kappa h)$$
$$\sim (|\kappa|h)^\alpha\{cos(\kappa h) + tan(\frac{\alpha\pi}{2})sin(\kappa h)\}$$
$$\sim (|\kappa|h)^\alpha \text{ as } h \to 0\,. \quad (5.3.30)$$

Substituting back in equation (5.3.29) after multiplying by τ, we get

$$\tau Q(e^{i\kappa h}, e^{-s\tau}) \sim \frac{(\tau)^\beta s^{\beta-1}}{(s\tau)^\beta \,+\, \mu(\kappa h)^\alpha}\,,\ h \to 0\,, \tau \to 0\,. \quad (5.3.31)$$

Then using the scaling parameter (5.3.18), we get again the result (5.3.23), and with the inverse Laplace transform, again (5.3.24).

This proceedure proves that the discrete solution of the space-time-FDE converges in distribution to the Fourier-Laplace transform of the corresponding one, for $0 < \alpha < 1$ and $1 < \alpha \leq 2$.

Third for case (e):
It remains to prove the convergence also for $\alpha = 1$, related to the Cauchy distribution. In what follows we give the proof of the convergence at $\alpha = 1$ to complete the theory [34] . We rewrite equation (5.1.1) for $\alpha = 1$ as

$$\underset{t\ *}{D}^{\beta}\, u(x,t) = \underset{x\ 0}{D}^{1}\, u(x,t)\ , u(x,0) = \delta(x)\ ,\ 0 < \beta \le 1\ , \qquad (5.3.32)$$

with the scaling parameter

$$\mu = \tau^{\beta}/h\ . \qquad (5.3.33)$$

As in the previous sections we take Fourier-Laplace transform of both sides, and get

$$\widehat{\widetilde{u}}(\kappa, s) = \frac{s^{\beta-1}}{s^{\beta} + |\kappa|}\ .$$

The inverse Laplace transform gives

$$\widehat{u}(\kappa, t) = E_{\beta}(-|\kappa| t^{\beta})\ .$$

We cannot use the Grünwald-Letnikov discretization of $\underset{x\ 0}{D}^{\alpha}$ at equations (5.3.15, 5.3.16) because the denominator in equation (5.3.14) is zero for $\alpha = 1$. Instead of the Grünwald-Letnikov discretization, we use the discretization used in [34]. The authors of [34] replaced the factor $(-1)^k \binom{\alpha}{k}$, $k \in \mathbb{Z}$, in equation (5.3.12) by $\frac{-2}{\pi h}$ for $k = 0$, and $\frac{1}{\pi h |k|(|k|+1)}$ for $\neq 1\, k \in \mathbb{Z}$. Therefore by using the scaling parameter (5.3.33), we have

$$\sum_{m=0}^{n+1} (-1)^m \binom{\beta}{m} \{y_j(t_{n+1-m}) - y_j(0)\} = -\frac{2\mu}{\pi} y_j(0)$$
$$+ \frac{\mu}{\pi} \sum_{\neq 0 k \in \mathbb{Z}} \frac{1}{k(k+1)} y_{j+k}(t_n) + \frac{\mu}{\pi} \sum_{\neq 0 k \in \mathbb{Z}} \frac{1}{k(k+1)} y_{j-k}(t_n)\ , \qquad (5.3.34)$$

and solving for $y_j(t_{n+1})$, we get

$$y_j(t_{n+1}) = (\beta - \frac{2\mu}{\pi}) y_j(t_n) + \sum_{m=2}^{n+1} (-1)^{m+1} \binom{\beta}{m} y_j(t_{n+1-m})$$
$$+ \sum_{m=0}^{n+1} (-1)^m \binom{\beta}{m} y_j(0) + \frac{\mu}{\pi} \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \{y_{j-k}(t_n) + y_{j+k}(t_n)\}\ .$$
$$(5.3.35)$$

This equation represents the random walk of equation (5.3.32) under the restriction

$$\beta - \frac{2\mu}{\pi} \geq 0 \ ,$$

which is equivalent to $0 < \mu \leq \frac{2\mu}{\beta}$. Furthermore, we have

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1 \ ,$$

and Condition (3) of the cases (A) and (B) is also satisfied. By using these conditions and properties, it can be easily proved by induction that

$$\sum_{j=-\infty}^{\infty} y_j(t_n) = \sum_{j=-\infty}^{\infty} y_j(0) \ \ \forall n \in \mathbb{N}_0$$

Now to prove the convergence of this model, we multiply both sides of equation (5.3.34) by z^j, then sum over all $j \in \mathbb{Z}$, and use the relation

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} z^k = 1 - (1 - z^{-1}) log(1 - z) \ , |z| = 1 \ .$$

We get

$$\sum_{m=0}^{n+1} (-1)^m \binom{\beta}{m} \{q_{n+1-m}(z) - 1\}$$
$$= \ -q_n(z) \frac{\mu}{\pi} \{(1 - z^{-1}) log(1 - z) + (1 - z) log(1 - z^{-1})\} \ , \quad (5.3.36)$$

where the generating function $q_n(z)$, is defined in equation (5.2.5). Now multiplying both sides of equation (5.3.36) by ζ^n, and summing over $n \in \mathbb{N}_0$, we get, for $|\zeta| < 1$,

$$Q(z, \zeta) = \frac{(1 - \zeta)^{\beta - 1}}{(1 - \zeta)^{\beta} + \frac{\zeta\mu}{\pi} \{(1 - z^{-1}) log(1 - z) + (1 - z) log(1 - z^{-1})\}} \ , \quad (5.3.37)$$

where the two-fold generating function $Q(z, \zeta)$, is defined in equation (5.2.8). Now we want to prove that $\tau Q(e^{i\kappa h}, e^{-s\tau})$ tends to the Fourier-Laplace transform of the solution of the space-time fractional diffusion equation (5.3.32). To do so, we replace in equation (5.3.37) z by $e^{i\kappa h}$, ζ by $e^{-s\tau}$. Then we use the well known relation for a complex number $W = x + iy$, $W \in \mathbb{C}$, $x \in \mathbb{R}$, $y \in \mathbb{R}$,

$$logW = log|W| + i \, arctan \frac{y}{x} \ ,$$

and get

$$
\begin{aligned}
&(1-e^{-i\kappa h})log(1-e^{i\kappa h})+(1-e^{i\kappa h})log(1-e^{-i\kappa h}) \\
&\quad = 2\{(1-cos\kappa h)log|1-e^{i\kappa h}|+sin\kappa h\, arctan\frac{sin\kappa h}{1-cos\kappa h}\} \, .
\end{aligned}
\tag{5.3.38}
$$

Now we use: $\lim_{u\to\pm\infty} arctan\, u = \pm\frac{\pi}{2}$, and get after multiplying both sides of equation (5.3.37) by τ

$$
\tau Q(e^{i\kappa h}, e^{-s\tau}) \sim \frac{\tau^\beta s^{\beta-1}}{\tau^\beta s^\beta + \mu|\kappa|h} \, , h\to 0, \; \tau \to 0 \, . \tag{5.3.39}
$$

Finally, by using the scaling parameter (5.3.33) we get

$$
\frac{\tau^\beta s^{\beta-1}}{\tau^\beta s^\beta + \mu|\kappa|h} = \frac{s^{\beta-1}}{s^\beta + |\kappa|} = \widehat{\widetilde{u}}(\kappa, s) \;\; ,
$$

and applying the inverse Fourier transform, we get

$$
\widehat{u}(\kappa, t) = E_\beta(-|\kappa|t^\beta) \;\; , \; t>0 \, . \tag{5.3.40}
$$

As we have seen the approximate solutions of the space-fractional diffusion and the space-time-fractional diffusion tend in the scaled limits $h\to 0$, $\tau \to 0$, to the Fourier-Laplace transform of the analytic solution $u(x,t)$. We have seen also that the Mittag-Leffler function (see Appendix c) plays an important rule in the detailed analysis of the process. What we have proved are in agreement with the *continuity theorems* (see [61]). For concluding our discussion about the convergence of the discrete model to the space-time-fractional diffusion, we note that the continuity theorems for inversion of the Fourier and Laplace transform are required (see [61]and [20]).

5.4 The convergence of the time-FDECLD

We begin this section by considering case (f) of Section (5.1). The discrete solution of this case has been already discussed in Chapter 2. For this case, i.e. for $\beta = 1$, equation (5.1.2) in the Fourier-Laplace space reads

$$
\kappa \frac{\partial}{\partial \kappa}\widehat{\widetilde{u}}(\kappa, s) \, + \, (\kappa^2 + s)\widehat{\widetilde{u}}(\kappa, s) \, = \, e^{i\kappa x^*} \;\; , \;\; \widehat{\widetilde{u}}(0,s) = 1/s \;\; . \tag{5.4.1}
$$

Now, we recall the discretization of equation (5.1.2) for $\beta = 1$ from Chapter 2. For our purpose, it is convenient to write this discretization for all $n \in \mathbb{N}$, in the form

$$y_j(t_{n+1}) - y_j(t_n) = \mu(y_{j+1}(t_n) - 2y_j(t_n) + y_{j-1}(t_n)) + \frac{\mu h^2}{2}\{(j+1)y_{j+1}(t_n) - (j-1)y_{j-1}(t_n)\} \quad , \quad (5.4.2)$$

where μ is defined in equation (2.3.5). As in Chapter 2, we restrict the index j to the range $\{-R, -R+1, \ldots, R-1, R\}$ where $\frac{2}{h^2} = R \in \mathbb{N}$. This means the two-sided sojourn probability vector of this model is defined as

$$y(t_n) = \{y_R(t_n), y_{-R+1}(t_n), \cdots, y_0(t_n), \cdots, y_{R-1}(t_n), y_R(t_n)\} \ , \forall n \in \mathbb{N}_0 \ . \quad (5.4.3)$$

The initial column vector which is suitable with the initial condition of equation (5.1.2), namely $u(x,0) = \delta(x - x^*)$, is

$$y_j(t_0) = \delta_{jm} = \begin{cases} 1 & j = m \ , \\ 0 & j \neq m \ , \end{cases} \quad (5.4.4)$$

where $m \in [-R, R]$ and $x^* = mh$. Recall equation (2.3.7) representing the random walk of this model, in which the particle sitting on the point $x_j, \ j \in (-R, R)$ at the time instant t_n has the opportunity as the time proceeds to t_{n+1} to jump to x_{j-1} with transition probability λ_j, to the point x_j with transition probability ν, or finally to x_{j+1} with transition probability ρ_j. The transition probabilities λ_j, ν and ρ_j are defined as in equation (2.3.8) but with $a = b = 1$, and satisfy the same conditions (2.3.9) and (2.3.10). Using condition (2.3.10) in (5.4.2) we have proved in Chapter 2 that

$$y_j(t_n) = 0 \ , \ \forall |j| \geq R + 1 \ , \ n \in \mathbb{N}_0 \ . \quad (5.4.5)$$

This condition guarantees that we have a closed Markov chain. This means there is no jump to outside the interval $[-R, R]$. We have proved also in Chapter 2, with the aid of condition (2.3.9), that the vector (5.4.3) satisfies

$$\sum_{j=-R}^{R} y_j(t_n) = \sum_{j=-R}^{R} y_j(t_0) = 1 \quad , \quad (5.4.6)$$

where $y_j(t_n) \geq 0 \ \forall j \in [-R, R]$. Using equation (5.4.5), we can extend equation (5.4.6) to

$$\sum_{j \in \mathbb{Z}} y_j(t_n) = \sum_{j \in \mathbb{Z}} y_j(t_0) = 1 \quad . \quad (5.4.7)$$

Therefore we can use the definition of the generating function $q_n(z)$ defined in (5.2.5) for the infinitely extended sequence of sojourn probabilities

$$y(t_n) = \{\cdots, 0, y_R(t_n), y_{-R+1}(t_n), \cdots, y_0(t_n), \cdots, y_{R-1}(t_n), y_R(t_n), 0, \cdots\}\ , \tag{5.4.8}$$

$\forall n \in \mathbb{N}_0$. Here $q_n(z)$ is also convergent on the circle $|z| = 1$. Again we can use the definition of the bivariate (two-fold) generating function $Q(z, \zeta)$ defined in (5.2.8) for the sequence (5.2.9) which is also convergent for $|\zeta| < 1$.

Our aim now is to prove that if we replace z by $e^{i\kappa h}$ and ζ by $e^{-s\tau}$, then with the limit as $h \to 0$ and $\tau \to 0$, the sequence $\tau Q(e^{i\kappa h}, e^{-s\tau})$ satisfies also the ordinary differential equation (5.4.1). To do so, we will find the explicit form of $\tau Q(e^{i\kappa h}, e^{-s\tau})$. Therefore, we multiply each side of (5.4.2) by z^j and sum over all $z \in \mathbb{Z}$. We get

$$\begin{aligned} q_{n+1}(z) - q_n(z) = \mu q_n(z) \{(z^{-1} - 2 + z) - \frac{y_{-R}}{z^{R+1}} - y_R z^{R+1}\} + \\ \frac{\mu h^2}{2} \{(z^{-1} - z) \sum_{j\in\mathbb{Z}} j y_j(t_n) z^j + \frac{R y_{-R}}{z^{R+1}} + R y_R z^{R+1}\}\ . \end{aligned} \tag{5.4.9}$$

After using the definition of R and the identity $\sum_{j\in\mathbb{Z}} j y_j(t_n) z^j = z \frac{d}{dz} \sum_{j\in\mathbb{Z}} y_j(t_n) z^j$, we get

$$\begin{aligned} q_{n+1}(z) - q_n(z) = \mu q_n(z)\,(z^{-1} - 2 + z) + \\ \frac{\mu h^2}{2}\,(z^{-1} - z)\ z\ \frac{d}{dz} q_n(z)\ . \end{aligned} \tag{5.4.10}$$

Now multiplying both sides by ζ^n and summing over all $n \in \mathbb{N}$ we get

$$\begin{aligned} \frac{1}{\zeta}\,(Q(z,\zeta) - z^m) - Q(z,\zeta) = \mu Q(z,\zeta)\,(z^{-1} - 2 + z) + \\ \frac{\mu h^2}{2}\,(1 - z^2)\,\frac{d}{dz} Q(z,\zeta)\ . \end{aligned} \tag{5.4.11}$$

Then solving for $Q(z, \zeta)$, we get

$$\frac{\mu h^2}{2}\,(1 - z^2)\,\frac{d}{dz} Q(z,\zeta) + \{\zeta\mu\,(z^{-1} - 2 + z) - (1 - \zeta)\} Q(z,\zeta) = -z^m\ . \tag{5.4.12}$$

Replacing now z by $e^{i\kappa h}$, ζ by $e^{-s\tau}$, and taking the limit as $h \to 0$, $\tau \to 0$, we can use equation (5.2.18). Therefore we get after multiplying both sides by τ

$$\kappa \frac{d}{d\kappa}(\tau Q(e^{i\kappa h}, e^{-s\tau}) + (\kappa^2 + s)(\tau Q(e^{i\kappa h}, e^{-s\tau}) = e^{i\kappa m h} \quad . \qquad (5.4.13)$$

So far, we have proved that $\tau Q(e^{i\kappa h}, e^{-s\tau}$ is also a solution for the ordinary differential equation (5.4.1) exactly as $\widehat{\widetilde{u}}(\kappa, s)$. Therefore the discrete solution of the classical diffusion equation with central linear drift tends asymptotically as $h \to 0$, and $\tau \to 0$, to the corresponding analytic solution in the Fourier-Laplace domain in spite of vanishing outside a finite interval which however exhausts for h tending to zero.

We consider now case (g) of Section (5.1), i. e. the time-FDECLD, whose discrete solution is discussed in Chapter 3. For this case we have also a closed interval with the same $R = \frac{2}{h^2}$ but the scaling relation of this case is defined in equation (3.2.3). For this case, i.e. for $0 < \beta < 1$, equation (5.1.2) in the Fourier-Laplace space reads

$$\kappa \frac{\partial}{\partial\kappa}\widehat{\widetilde{u}}(\kappa, s) \;+\; (\kappa^2 + s^\beta)\widehat{\widetilde{u}}(\kappa, s) \;=\; e^{i\kappa x^*} s^{\beta-1} \;,\;\; \widehat{\widetilde{u}}(0, s) = 1/s \;. \qquad (5.4.14)$$

The solution of this equation gives a complicated expression for $\widehat{\widetilde{u}}(\kappa, s)$ that we do not write down here. To discuss the convergence of this model we join equations (3.2.1) and (3.2.2) for the special case $a = b = 1$, to get

$$\sum_{k=0}^{n+1}(-1)^k \binom{\beta}{k} \left(y_j(t_{n+1-k} - y_j(t_0)\right) = \mu(y_{j+1}(t_n) - 2y_j(t_n) + y_{j-1}(t_n)) + \frac{\mu h^2}{2}\{(j+1)y_{j+1}(t_n) - (j-1)y_{j-1}(t_n)\} \quad . \qquad (5.4.15)$$

As we have shown in Chapter 3, this equation represents, for all $n \in \mathbb{N}$, a diffusion process with a memory and the random walk of this process is modelled by equation (3.6.1). For a quick review we interpret equation (3.6.1) as follows: for all $n \in \mathbb{N}$, we suppose that the particle is sitting in the point x_j. When the time proceeds from the time instant t_n to t_{n+1}, it has the opportunity to jump to x_{j-1} with the transition probability λ_j, to the point x_j with the transition probability ν, to x_{j+1} with transition probability ρ_j, to go back to its previous position $x_j(t_{n+1-k})$ with the transition s_k or back to x_j at (t_0) with transition probability b_n. This means that the particle remembers always all its history $x_j \in [-Rh, Rh]$ where b_n and s_k are defined in (3.2.5) and (3.6.2) respectively. But for $n = 0$, it behaves typically as in the case $\beta = 1$. Again we use equation (5.4.5) and the initial condition (5.4.4). Then with the aid of equation (3.2.7), we can prove that the extended vector (5.4.8) representing this case satisfies also the conservation condition (5.4.7).

Now we can use the definition of the generating function $q_n(z)$ defined in (5.2.5) for the extended sequence of the sojourn probabilities (5.4.8) which is also convergent on the circle $|z| = 1$. Again we can use the definition of the bivariate (two-fold) generating function $Q(z, t)$ defined in (5.2.8) for the sequence (5.2.9) which is also convergent on $|\zeta| < 1$.

Our aim now is to prove that if we replace z by $e^{i\kappa h}$ and ζ by $e^{-s\tau}$, $\tau Q(e^{i\kappa h}, e^{-s\tau})$ satisfies asymptotically the ordinary differential equation (5.4.14). To do so, we multiply first each side of (5.4.2) by z^j and sum over all $z \in \mathbb{Z}$. We get

$$\sum_{k=0}^{n+1}(-1)^k \binom{\beta}{k} \left(q_{n+1-k}(z) - z^m \right) = \mu q_n(z) \left\{ (z^{-1} - 2 + z) - \frac{y_{-R}}{z^{R+1}} - y_R z^{R+1} \right\}$$
$$+ \frac{\mu h^2}{2} \left\{ (z^{-1} - z) \sum_{j \in \mathbb{Z}} j y_j(t_n) z^j + \frac{R y_{-R}}{z^{R+1}} + R y_R z^{R+1} \right\} . \quad (5.4.16)$$

Again by using the definition of R and the identity $\sum_{j\in\mathbb{Z}} j y_j(t_n) z^j = z \frac{d}{dz} \sum_{j\in\mathbb{Z}} y_j(t_n) z^j$, we get

$$\sum_{k=0}^{n+1}(-1)^k \binom{\beta}{k} \left(q_{n+1-k}(z) - z^m \right) = \mu q_n(z) \left(z^{-1} - 2 + z \right) +$$
$$\frac{\mu h^2}{2} \left(z^{-1} - z \right) z \frac{d}{dz} q_n(z) \quad . \quad (5.4.17)$$

Now multiplying equation (5.4.17) by ζ^n, summing over all $n \in \mathbb{N}_0$, and using equation (5.2.16), we get

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n+1} (-1)^k \binom{\beta}{k} \left(q_{n+1-k}(z) - z^m \right) \zeta^n = \mu Q(z, \zeta) \left(z^{-1} - 2 + z \right) +$$
$$\frac{\mu h^2}{2} \left(1 - z^2 \right) \frac{d}{dz} Q(z, \zeta) \quad . \quad (5.4.18)$$

Since the summation in the R.H.S of this equation is the same as the R.H.S of equation (5.2.16), we have

$$\frac{(1-\zeta)^{\beta}}{\zeta}\left(Q(z,\zeta) - \frac{z^m}{1-\zeta} \right) = \mu Q(z,\zeta)\,(z^{-1}-2+z) + \frac{\mu h^2}{2}\,(1-z^2)\,\frac{d}{dz}Q(z,\zeta) \quad . \qquad (5.4.19)$$

Solving for $Q(z,\zeta)$, we get

$$\frac{\zeta\mu h^2}{2}\,(1-z^2)\,\frac{d}{dz}Q(z,\zeta) + Q(z,\zeta)\,\left(\zeta\mu(z^{-1}-2+z) - (1-\zeta)^{\beta}\right) = -\,z^m\,(1-\zeta)^{\beta-1} \quad . \qquad (5.4.20)$$

Replacing now z by $e^{i\kappa h}$, ζ by $e^{-s\tau}$, taking the limit as $h \to 0$ and $\tau \to 0$, under the scaling relation (3.2.3), we can use equation (5.2.18), and get after multiplying both sides by τ

$$\kappa\frac{d}{d\kappa}(\tau Q(e^{i\kappa h},e^{-s\tau})) + (\kappa^2+s^{\beta})(\tau Q(e^{i\kappa h},e^{-s\tau})) \sim s^{\beta-1}\,e^{i\kappa mh} \quad ,\kappa > 0 \quad , \qquad (5.4.21)$$

where $\tau Q(0,s) = 1/s$. So far, we found that the asymptotic ordinary differential equation (5.4.21) with $x^* = mh$ is structured like the ordinary differential equation (5.4.14) and $\tau Q(e^{i\kappa h},e^{-s\tau})$ represents an approximation to $\widehat{\widetilde{u}}(\kappa,s)$. We can interpret this result in the following words: the Fourier-Laplace transform of the discrete solution of equation (5.1.2) satisfies the same ordinary differential equation (5.4.14) asymptotically as $h \to 0$ and $\tau \to 0$. The results of this section are in agreement with the numerical results of Chapter 2, for the case $\beta = 1$, and with the numerical results of Chapter 3, for the case $0 < \beta < 1$.