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Ancient Dynamics of the Einstein Equations and the Tumbling Universe

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Jeder große Fortschritt der Menschheit beginnt mit dem Zweifel und zeigt sich in einem Protest gegen überlieferten Dogmatismus

Gustav von Schmoller

The development of scientific thought may once again take us beyond the present achievement, but a return to the old narrow and restriced scheme is out of the question

Hermann Weyl

We should be on our guard not to overestimate science and scientific methods when it is a question of human problems; and we should not assume that experts are the only ones who have a right to express themselves on questions affecting the organization of society

Albert Einstein

Contents

Introduction

In General Relativity, the Einstein Equations are used to describe the geometry of the four-dimensional spacetime. Today, nearly 100 years after their publication by Albert Einstein [**8, 9**] in 1915[1], suprisingly less is known about the dynamics of these equations towards the initial singularity, i.e. the "big bang".

Achieving progress on this question is an active area of ongoing research in the field of mathematical general relativity, for recent surveys see [**54, 55**]. In a broader scope, the general research program could be called trying to understand "the nature of spacetime singularities", as Rendall has put it in an earlier survey paper also covering material on black holes ([**40**]).

For understanding the approach to the initial singularity, the so-called "BKL-Conjecture" plays a central role. It is named after Belinskiĭ, Khalatnikov and Lifshitz who pioneered the study of oscillatory singularities in the Einstein equations in the 1970s (see [**29, 3, 4**]).

However, the BKL-Conjecture is more a guiding principle how to understand the approach to the initial singularity than a clearly formulated conjecture, that's why Rendall suggested it should better be called the "BKL picture" ([**40, 41**]).

In this picture, the approach to the initial singularity in General Relativity is generically vacuum dominated, local and oscillatory. The first point means that generically, solutions in a cosmological model with matter will converge to the vacuum model towards the big bang - so "matter does not matter" for the approach to the initial singularity.

The "local" part of the BKL picture means that in inhomogeneous cosmologies the evolution of each spatial point will decouple in backwards time. As a result, each spatial point will evolve individually and independently of its neighbors towards the initial singularity as a spatially homogeneous model.

[1]Although he published some core ideas already in 1915 ([**8, 9**]), the article "Die Grundlage der Allgemeinen Relativitätstheorie" [**10**] by Albert Einstein from 1916 is the first comprehensive exposition of general relativity

That's why the spatially homogeneous Bianchi cosmologies play an important role in understanding generic spacelike singularities, as they are believed to capture the essential dynamics of more general solutions. Bianchi models are divided into two groups, labeled as "class A" and "class B". Historically, most research has concerned oscillatory Bianchi models of class A. However, oscillatory Bianchi models of class B play an important role in inhomogenous cosmologies, because of the following reason:

Arguably the simplest cosmological model where all aspects of the BKL-picture can be studied in combination is the G_2-model, because it is both inhomogeneous and oscillatory. But G_2 does not contain any oscillatory Bianchi models of class A. That's why it is important also to study Bianchi models of class B, where the oscillatory Bianchi $VI^*_{-1/9}$ model plays a central role.

In this dissertation, we study oscillatory Bianchi models of both class A and class B and achieve some new results that we will describe now.

On the one hand, we are able to show, for the first time, that for admissible periodic heteroclinic chains in Bianchi IX there exisist C^1-stable - manifolds of orbits that follow these chains towards the big bang. A detailed study of Takens Linearization Theorem and the Non-Resonance-Conditions leads us to this new result in Bianchi class A. More precisely, we can show that there are no heteroclinic chains in Bianchi IX with constant continued fraction development that allow Takens-Linearization at all of their base points. Geometrically speaking, this excludes "symmetric" heteroclinic chains with the same number of "bounces" near all of the 3 Taub Points - the result shows that we have to require some "asymmetry" in the bounces in order to allow for Takens Linearization, e.g. by considering admissible 2-periodic continued fraction developments.

As the second main result of this dissertation, we find an example for a periodic heteroclinic chain in Bianchi $VI^*_{-1/9}$ that allows Takens Linearization at all base points. It will turn out to be a "18-cycle", i.e. involving a heteroclinic chain of 18 different base points at the Kasner circle. We then show that the Combined Linear Local Passage at the 18-cycle is a contraction. This qualifies the 18-cycle as a candidate for proving the first rigorous convergence theorem in Bianchi $VI^*_{-1/9}$.

The structure of this dissertation is as follows: Chapter 2 puts our research in the broader context of mathematical General Relativity and explains the significance of spatially homogeneous models. This is also where Bianchi cosmologies of class A and B are defined and the relevant equations are presented.

Chapter 3 reviews existing results in Bianchi models of class A and describes challenges that occur in models of class B. In recent years, it has turned out that Dynamical Systems techniques can be very helpful to prove rigorous theorems about cosmological models. That's why we briefly review some relevant tools from non-linear dynamics for the use in later chapters.

This leads naturally to Chapter 4, which presents Takens Linearization Theorem. It has been succesfully applied to Bianchi cosmologies (see [**2**]), and will be one of our central tools for our own results. That's why we also review the structure of the proof of the theorem, especially to understand the form of the non-resonance-conditions. Those will play an important role when it comes to linearizing at the base points of periodic heteroclinic chains.

Chapter 5 presents our main results for Bianchi IX: We show that there are C^1- stable - manifolds for admissible periodic heteroclinic chains in Bianchi IX. The first part of the chapter deals in details with the question which chains are admissible, while the second part is devoted to the proof that after linearizing, a C^1- stable - manifold can be obtained by finding a suitable set of points that admits a C^1-hyperbolic stucture, which will give the desired result via a graph transform.

In Chapter 6 we deal with Bianchi $VI^*_{-1/9}$. In order to illustrate our approach, we first consider the Combined Linear Local Passages and Takens Linearization at the 3-cycle in Bianchi $VI^*_{-1/9}$. We show that the Sternberg Non-Resonance Conditions are not satisfied for the 3-cycle, but they are satisfied for the 18-cycle. We then show that the Combined Linear Local Passage at the 18-cycle is a contraction.

We conclude with an outlook on how to proceed further in studying Bianchi cosmologies, and also discuss directions for future research in inhomogeneous (PDE-) cosmological models. This puts our results in a broader perspective. The appendix contains symbolic and numerical computations done by Mathematica for examples discussed throughout the text.

In two recent survey papers ([**54, 54**]), Uggla reviews recent developments concerning generic spacelike singularities. One of his conclusions is "that we are only at the beginning of understanding generic singularities, even though considerable progress has been accomplished during the last few years" ([**54**], p.20). Our aim with this dissertation is to make some contributions to this research program.

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It is a pleasure to thank my advisor Bernold Fiedler for introducing me to the exciting field of the dynamics of cosmological models, for giving me the freedom to choose a topic of my own interest for the dissertation, and for providing a very stimulating working environment in our research group. I would also like to thank him for personal support in sometimes hard emotional times during the last two years, and especially for supporting my transition from cosmology to mathematical economics.

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CHAPTER 1

General Relativity

1. The Einstein Equations

We start by presenting the Einstein Equations that are at the core of General Relativity. They were first published by Albert Einstein in 1915 (see [**8, 9, 10**]). For a modern exposition focussing on analytical aspects see [**43, 41**], while physical aspects can be found e.g. in [**35**]. We consider a 4-dimensional Lorenzian manifold M, i.e. M is endowed with a metric g with signature $(-,+,+,+)$. By convention in mathematical relativity, we use geometrized units and set $8\pi G = c = 1$, where c is the speed of light and G the gravitational constant. Then the Einstein Equations take the following form:

$$R_{ab} + \frac{1}{2} R g_{ab} = T_{ab} \tag{1}$$

The indicies a, b can be interpreted as "abstract indicies", see [**60**], and in this interpretation the Einstein Equations are an equation for tensor fields on the manifold M (for the differential geometry background, see e.g. [**50**] or [**12**]).

On the left side of the Einstein Equations there are the Ricci curvature tensor R_{ab}, the scalar curvature R and the metric tensor g_{ab} of the spacetime, reflecting the geometic properties of space. On the right side of the equations, we have the so-called "stress-energy-tensor" T_{ab}, representing the matter field. A short summary of the resulting behaviour is given by the following famous quote: "Matter tells space how to curve, and space tells matter how to move" by J.A. Wheeler (see e.g. [**61**], p.235). This statement of "general relativity in a nutshell" can be interpreted as follows: By the the distribution of matter, the curvature of space is determined, i.e. "matter tells space how to curve". On the other hand, the matter particles will move along geodesics that are adapted to the curvature of the spacetime, i.e. "space tells matter how to move". More details can be found the now classic works by Wheeler [**35**] or Wald [**60**] .

In order to deal with the Einstein Equations, there are two basic strategies, which we will describe next.

2. The Metric Approach

In the metric approach, the unknown of the Einstein Equations is taken to be the metric tensor g_{ab}[1]. If we chose local coordinates $x_1 ... x_n$ (with $n = 4$ in our case for a four dimensional space-time), it holds that:

$$[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}] = \delta_{ij} \tag{2}$$

for $i, j = 1...4$, where δ_{ij} stands for the Kronecker-Delta and the square bracket denotes the Lie-bracket for vector-field on manifolds (see [**12**]).

As an example for the metric approach, we will discuss an explicit solution of the Einstein Equations called the "Kasner solution" in section 4.

3. The Orthonormal Frame Approach

In the The orthonormal frame approach, the unknown of the Einstein Equations is taken to be the commutators[2] of a chosen orthonormal frame $e_1, ..., e_n$ that forms a basis of the tangent space of the manifold at each point $p \in M$. In this approach, the metric becomes very easy, as by our choice, our frame is orthonormal, meaning that it holds for the metric:

$$g(e_i, e_j) = \delta_{ij} \tag{3}$$

However, now the Lie-bracket of the orthonormal frame elements is more complicated:

$$[e_i, e_j] = \gamma_{ij}^k e_k \tag{4}$$

where we have used the Einstein summation convention of summing over repeated indicies in the last line. Thus, in this case, we solve the Einstein Equations for the commutators γ_{ij}^k - in fact the variables appearing later in the Wainwright-Hsu equations for the Bianchi cosmological models (see section 2) are a neat decomposition of these

[1]to be precise, in the presence of symmetries, the basic variables are the metric components $g_{ab}(t)$ relative to a group-invariant, time-independent frame (see [**58**], p.107).

[2]in the presence of symmetries, we take as basic variables the commutation functions associated with a group-invariant orthonormal frame, see [**58**], p.108).

commutators (for details how to derive the Wainwright-Hsu equations from the Einstein equations, see the appendix of the paper [**42**]).

4. Example: The Kasner Solution

In this section, we consider an example of an explicit solution of the Einstein Equations: For the so-called Kasner solution, the metric has the following form:

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2 \tag{5}$$

where $t \in (0, \infty)$ and x, y and z denote the standard Cartesian coordinates on $\mathbb{R}^3$. The constants p_i are often called "Kasner exponents" as they appear in the exponent in equation (5) and they satisfy the following relations (see e.g. [**41**]):

$$\sum_{i=1}^{3} p_i = 1, \quad \sum_{i=1}^{3} p_i^2 = 1$$

We will later see that the Kasner solution plays an important role in analysing the general Einstein Equations towards the initial singularity.

5. Symmetries and Spatially Homogenous Models

In this section, we follow [**58**], chapter 1.2 in order to define spatially homogenous models. We will also define Bianchi models of class A and class B.

Symmetries of the spacetime can be understood in the framework of group actions. This leads to a hierachy of models, where it has turned out that in order to fully understand one model, it is necessary to go beyond it and investigate it in the context of more general models.

Let (M, g) be an 4-dimensional Lorenzian manifold as before. The set of all isometries of (M, g) forms a Lie group G_r, the so-called isometry group of (M, g). We consider the action of G_r on M and make the following definitions:

At first, we denote by $r := dim(G_r)$ the dimension of the isometry group G_r. Secondly, we define $s := dim(\mathcal{O}(G_r))$ as the dimension of the orbits[3] of the isometry group G_r acting on (M, g).

Finally, we denote by $d := r - s$ the difference between r and s, which equals to the dimension of the isotropy subgroup of isometries that leave the point p fixed. Then we can classify important cosmological models according to the following table:

[3] the orbit $\mathcal{O}(p)$ of a point $p \in M$ under the group G_r is the set of points into which p is mapped when all elements of G_r act on p, see [**58**], p.21.

	r	s	d
FLRW	6	3	3
Bianchi	3	3	0
G_2	2	2	0
Gowdy	2	2	0

DEFINITION 1.1. *(Spatially Homogeneous Cosmological Model) We call a cosmological model spatially homogeneous if it admits an action by a group of isometries G_r with spacelike three dimensional orbits (i.e. $s = 3$ in the table above).*

DEFINITION 1.2. *(Bianchi models of class A and B) A Bianchi cosmology is a model that admits a three-dimensional group of isometries G_r acting simply transitively[4] on spacelike hypersurfaces (i.e. $s = 3$ in the table above, and a simply transitive action on the spacelike group orbits[5]).*

If G_r is unimodular, the Bianchi model is said to be of class A, otherwise it is of class B.

The standard model of cosmology is FLRW (see e.g. [**58**], chapter 2). It is both spatially homogeneous as well as isotropic (the latter meaning that $d = 3$). Bianchi models are also spatially homogeneous, but not isotropic.

The G_2 and Gowdy models are inhomogeneous cosmological models that still have some spatial symmetry, where the difference between the two is that in Gowdy there exists an additional discrete symmetry. This additional symmetry has important dyamical consequences, as general G_2 models are believed to be oscillatory towards the singularity, while Gowdy models are known to be not (see e.g. [**58**] or [**41**]).

That's why arguably the simplest cosmological model where all aspects of the BKL-Conjecture can be studied in combination is the G_2-model, because it is both inhomogeneous and oscillatory. But G_2 does not contain any oscillatory Bianchi models of class A (see e.g. [**21**]). Therfore it is important to study also oscillatory Bianchi models of class B, which means Bianchi $VI^*_{-1/9}$.

[4] a group acts simply transitively on an orbit if the dimension of the orbit equals the dimension of the group

[5] note that there are also the LRS Bianchi models with $r = 4$, $s = 3$ and the G_4 having a subgroup G_3 acting simply transitively on the three dimensional orbits

CHAPTER 2

Bianchi Spacetimes - Existing Results, Challenges and Techniques

1. The Equations of Wainwright and Hsu

In the paper [**59**] by Wainwright and Hsu, a formulation of the Einstein Equations for Bianchi models is presented that has several advantages, one of them beeing that it contains all models of Bianchi class A. Here are the equations, which are used throughout this dissertation:

$$
\begin{aligned}
N_1' &= (q - 4\Sigma_+)N_1, \\
N_2' &= (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2, \\
N_3' &= (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3, \\
\Sigma_+' &= (q-2)\Sigma_+ - 3S_+, \\
\Sigma_-' &= (q-2)\Sigma_- - 3S_-.
\end{aligned}
\tag{6}
$$

with constraint

$$
\Omega = 1 - \Sigma_+^2 - \Sigma_-^2 - K \tag{7}
$$

and abbreviations

$$
\begin{aligned}
q &= 2\left(\Sigma_+^2 + \Sigma_-^2\right) + \tfrac{1}{2}(3\gamma - 2)\Omega, \\
K &= \tfrac{3}{4}\left(N_1^2 + N_2^2 + N_3^2 - 2\left(N_1N_2 + N_2N_3 + N_3N_1\right)\right), \\
S_+ &= \tfrac{1}{2}\left(\left(N_2 - N_3\right)^2 - N_1\left(2N_1 - N_2 - N_3\right)\right), \\
S_- &= \tfrac{1}{2}\sqrt{3}\left(N_3 - N_2\right)\left(N_1 - N_2 - N_3\right).
\end{aligned}
\tag{8}
$$

The fixed parameter γ is related to the choice of matter model (e.g. $\gamma = 1$ represents dust, whereas $\gamma = 4/3$ represents radiation).

The properties of equations above have been studied intensively, for a recent survey see [**17**]. The main goal are rigorous results on the correspondence of iterations of the so-called "Kasner map" f to the dynamics of nearby trajectories to the Bianchi system (6) with reversed time, i.e. in the α-limit $t \to -\infty$. We will introduce the necessary background in the rest of this section.

Bianchi Class	N_1	N_2	N_3
I	0	0	0
II	+	0	0
VI_0	0	+	−
VII_0	0	+	+
VIII	−	+	+
IX	+	+	+

1.1. Vacuum Models of Bianchi Class A. From now on, we will restrict ourselves to the vacuum case $\Omega = 0$. This yields a 4-dimensional model, as we have five variables and one constraint:

DEFINITION 2.1. *(Phase Space of the Vacuum Wainwright-Hsu ODEs)*

$$\mathcal{W} = \{(N_1, N_2, N_3, \Sigma_+, \Sigma_-) \mid 0 = 1 - \Sigma_+^2 - \Sigma_-^2 - K\}$$

As we are interested in the dynamics of the Bianchi system (6) with reversed time, i.e. in the α-limit $t \to -\infty$, we will we denote by X^W the vector field corresponding to this time direction, for use in later chapters[1]. This means X^W stands for the vector field corresponding to the right side of (6), multiplied by -1.

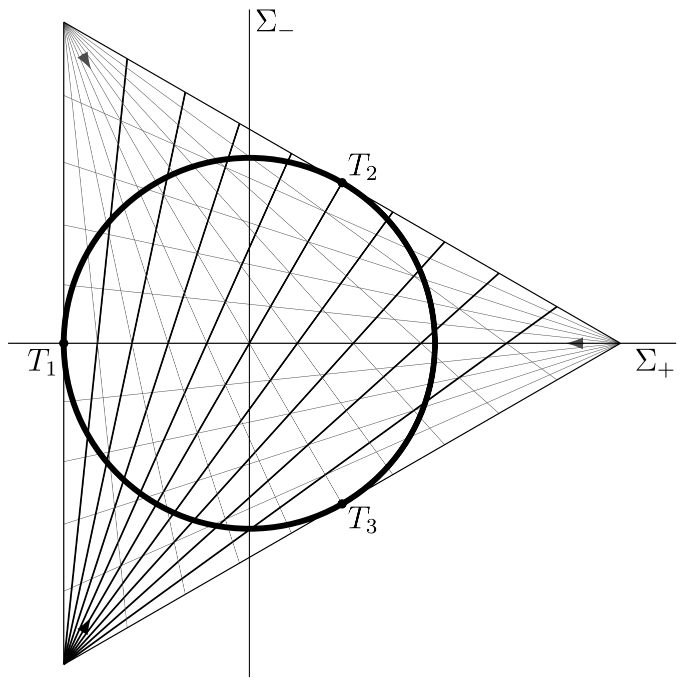
When we look at those equations, we observe that if $N_1 = N_2 = N_3 = 0$, the vector field is zero, as $K = 0$ and $q = 2$ in this case. We denote by $\mathcal{K} = \{N_1 = N_2 = N_3 = 0,\ \Omega = 0\}$ the resulting circle of equilibria: we obtain a circle because the constraint (7) reduces to $\Sigma_+^2 + \Sigma_-^2 = 1$. It is called the "Kasner circle", because the points $p \in \mathcal{K}$ represent the Kasner solution of the Einstein Equations discussed in section 4.

In the classification of spatially homogenous models (based on the classification of 3-dimensional Lie-Algebras by Bianchi [**5**]), these are of Bianchi class I. One advantage of the Wainwright-Hsu equations is that they contain all models of Bianchi class A, with the signs of the N_i determining the type of Bianchi model, see the table above.

If we allow one of the N_i to be non-zero, the resulting half ellipsoids are called "Kasner caps": $C_k = \{N_k > 0,\ N_l = N_m = 0,\ \Omega = 0\}$ with $\{k, l, m\} = \{1, 2, 3\}$. They consist of heteroclinic orbits to equilibria on the Kasner circle and are of Bianchi class II. The projections of the trajectories of Bianchi class-II vacuum solutions onto the $\Sigma_\pm$-plane yield straight lines connecting two points of the Kasner circle.

[1]in the paper by Béguin [**2**], the equations are presented directly with time direction chosen towards the big bang, but we stick to the form of the equations used in the classic reference [**59**], and also in [**42, 27**].

These can be constructed geometrically in the following way: for a point $p \in \mathcal{K}$ of the Kasner circle, identify the nearest corner of the circumscribed triangle, and draw the resulting line as illustrated in the picture below:



Observe that this works for any $p \in \mathcal{K}$ except for the three points where the Kasner circle touches the circumscribed triangle, which we denote by T_1, T_2, T_3 and refer to them as Taub points.

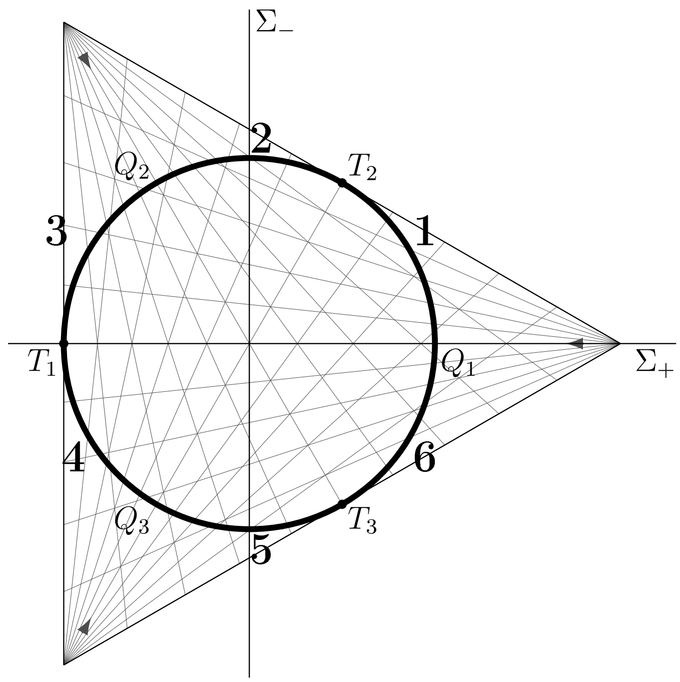
1.2. The Kasner Map. This leads to the definition of the so-called Kasner map $f : \mathcal{K} \to \mathcal{K}$. For each point $p_+ \in (\mathcal{K} \setminus \{T_1, T_2, T_3\})$ there exists a Bianchi class-II vacuum heteroclinic orbit $H(t)$ converging to p_+ as $t \to \infty$. This orbit is unique up to reflection $(N_1, N_2, N_3) \mapsto (-N_1, -N_2, -N_3)$. Its unique α-limit p_- defines the image of p_+ under the Kasner map

$$f(p_+) := p_- \tag{9}$$

Including the three fixed points, $f(T_k) := T_k$, this construction yields a continuous map, $f : \mathcal{K} \to \mathcal{K}$. In fact f is a non-uniformly expanding map and its image $f(\mathcal{K})$ is a double cover of $\mathcal{K}$, which can be seen directly from the geometric description given above.

For use in later chapters, let us denote by $H_{p,f(q)}$ the heteroclinic Bianchi-II-orbit from p to its image point under the Kasner map $f(q)$ and by $H_B = \bigcup_{p \in B} H_{p,f(q)}$ the set of all heterclinic Bianchi-II-orbits connecting two basepoints in the set $B \subset \mathcal{K}$.

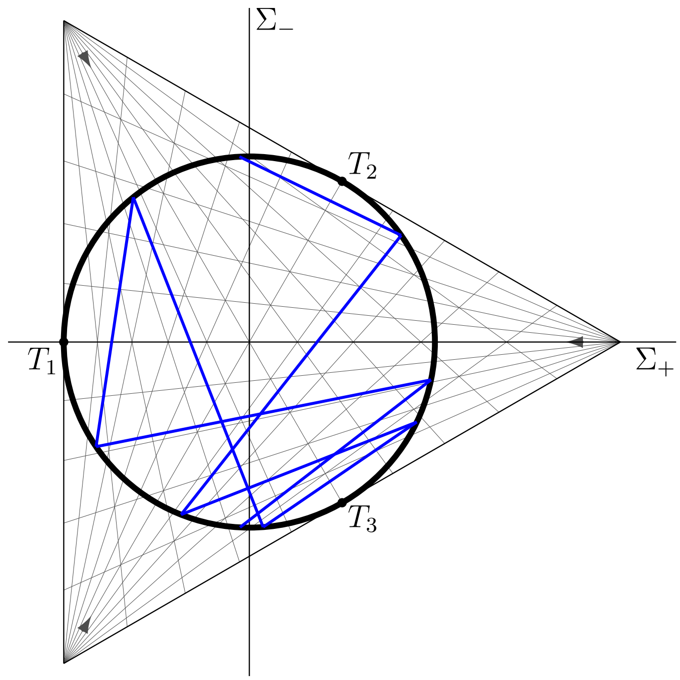
We will now introduce the so-called Kasner-parameter u, as it is a convenient way to parametrize the Kasner circle. Let us divide the Kasner circle into six sectors and label them as follows:



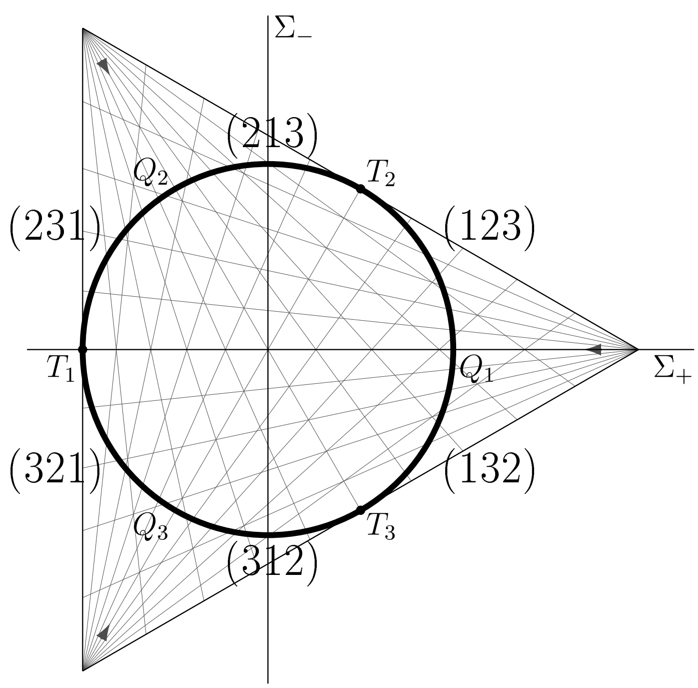
The Kasner parameter ranges in $[1, \infty]$, where it holds that $u = \infty$ at the Taub point T_i and $u = 1$ at the points Q_i shown in the picture. So for each value of u, we get an equivalence class of six points cooresponding to the $p \in \mathcal{K}$ in each sector of the Kasner circle (except for the values $u = \infty$ and $u = 1$, where the equivalence class consists only of three points). Expressed in u, the Kasner map has a very simple form:

$$f(u) \quad = \quad \begin{cases} u - 1 & u \in [2, \infty] \\ \frac{1}{u-1} & u \in [1, 2] \end{cases}$$

In the picture below, the dynamics of the Kasner-map is shown: If you start close to a Taub point (meaning that u is large compared to 1), there are first "bounces" around this Taub point as the value of u is decreased by 1 in each step. Then, after the value of u has fallen below 2, there is a "excursion" to a different part of the Kasner circle:



There is also an interesting connection to the Kasner solution described in section 4. Each sector of the Kasner circle corresponds to a permutation of the Kasner expontents p_i, which can be expressed in the Kasner parameter u:



where sector (321) means e.g. that $p_3 < p_2 < p_1$ which fixes the formula for each of them. As an example, consider the sector 5 or (312) (for details see [**17**], p.8):

$$p_3 = \frac{-u}{1+u+u^2}$$
$$p_1 = \frac{(u+1)}{1+u+u^2}$$
$$p_2 = \frac{u(u+1)}{1+u+u^2}$$

We will need the formulas above, as they will allow us to express the eigenvalues of the linearized vector field at points of the Kasner circle in u, a key step for obtaining our results in the later chapters.

1.3. Eigenvalues in Terms of the Kasner Parameter u. When we linearize the vector field corresponding to equations (6) at points of the Kasner circle, we arrive at the following Matrix:

$$\begin{pmatrix} 2-4\Sigma_+ & 0 & 0 & 0 & 0 \\ 0 & 2+2\Sigma_+ + 2\sqrt{3}\Sigma_- & 0 & 0 & 0 \\ 0 & 0 & 2+2\Sigma_+ - 2\sqrt{3}\Sigma_- & 0 & 0 \\ 0 & 0 & 0 & 3(2-\gamma)\Sigma_+^2 & 3(2-\gamma)\Sigma_+\Sigma_- \\ 0 & 0 & 0 & 3(2-\gamma)\Sigma_+\Sigma_- & 3(2-\gamma)\Sigma_-^2 \end{pmatrix}.$$

and we can compute the following eigenvalues to eigenvectors ∂_{N_1}, ∂_{N_2}, ∂_{N_3} tangential to the Bianchi class-II vacuum heteroclinics:

$$\begin{aligned} \mu_1 &= 2-4\Sigma_+, \\ \mu_2 &= 2+2\Sigma_+ + 2\sqrt{3}\Sigma_-, \\ \mu_3 &= 2+2\Sigma_+ - 2\sqrt{3}\Sigma_- \end{aligned}$$

In addition we have the trivial eigenvalue zero to the eigenvector $-\Sigma_-\partial_{\Sigma_+} + \Sigma_+\partial_{\Sigma_-}$ tangential to the Kasner circle $\mathcal{K}$. The fifth eigenvalue $\mu_\Omega = 3(2-\gamma) > 0$ corresponds to the eigenvector $\Sigma_+\partial_{\Sigma_+} + \Sigma_-\partial_{\Sigma_-}$ transverse to the vacuum boundary $\{\Omega = 0\}$.

Now we use that it is possible to express the $\Sigma_{+/-}$-variables in terms of the Kasner exponents p_i (see [**17**], p.7):

$$\Sigma_+ = -\frac{3}{2}p_1 + \frac{1}{2}$$

$$\Sigma_- = -\frac{\sqrt{3}}{2}(p_1 + 2p_2 - 1)$$

Thus, we arrive at the following formulas for the eigenvalues expressed in u:

$$(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{-6u}{1+u+u^2}, \frac{6(1+u)}{1+u+u^2}, \frac{6u(1+u)}{1+u+u^2} \right) \tag{10}$$

As it holds that $u \in [1, \infty]$, we observe that at each point of the Kasner circle, there is one negative and two positive eigenvalues. But recall that we are interested in the time-direction $t \to -\infty$. The negative eigenvalue is unstable towards the past, while the two positive eigenvalues are stable in backwards-time. This means that for our vector field X^W (which has the time-direction already reversed, see beginning of section 1.1) there is one unstable eigenvalue λ_u and two stable eigenvalues λ_s, λ_{ss}, at a point of the Kasner circle. Away from the Taub points it also holds that:

$$|\lambda_u| < |\lambda_s| < |\lambda_{ss}|$$

Finally, also note that in Bianchi IX, the situation in different sectors of the Kasner-circle only differs by a permutation of those 3 formulas for the eigenvalues. This makes it easy to examine the question of resonances of the eigenvalues, which we are trying to exclude when linearizing the vector field.

However, we will also deal with Bianchi $VI^*_{-1/9}$ later, and there the situation is more complicated as the formulas for the eigenvalues at points on the Kasner circle do depend on the sector, so in order to check for resonances, a lot of cases have to be considered. This will be done in chapter 5. In the next section, we will introduce the equations for Bianchi $VI^*_{-1/9}$, which is of class B and not covered by the equations of Wainwright and Hsu.

2. Bianchi $VI^*_{-1/9}$

We now present the equations for Bianchi $VI^*_{-1/9}$, which is the most general model in Bianchi class B, and has a crucial importance for inhomogenous cosmologies (see e.g. [**21**]):

$$\Sigma_+^{'} = (q-2)\Sigma_+ + 3\Sigma_2^2 - 2N_-^2 - 6A^2 \tag{11}$$

$$\Sigma_-^{'} = (q-2)\Sigma_- - \sqrt{3}\Sigma_2^2 + 2\sqrt{3}\Sigma_\times^2 - 2\sqrt{3}N_-^2 + 2\sqrt{3}A^2 \tag{12}$$

$$\Sigma_\times^{'} = (q-2-2\sqrt{3}\Sigma_-)\Sigma_\times - 8N_-A \tag{13}$$

$$\Sigma_2^{'} = (q-2-3\Sigma_+ + \sqrt{3}\Sigma_-)\Sigma_2 \tag{14}$$

$$N_-^{'} = (q+2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_- + 6\Sigma_\times A \tag{15}$$

$$A^{'} = (q+2\Sigma_+)A \tag{16}$$

Abbreviations:

$$q = 2\Sigma^2 + \frac{1}{2}(3\gamma-2)\Omega \tag{17}$$

$$\Sigma^2 = \Sigma_+^2 + \Sigma_-^2 + \Sigma_2^2 + \Sigma_\times^2 \tag{18}$$

Constraints:

$$\Omega = 1 - \Sigma^2 - N_-^2 - 4A^2 \tag{19}$$

$$g = (\Sigma_+ + \sqrt{3}\Sigma_-)A - \Sigma_\times N_- = 0 \tag{20}$$

Auxilliary Equations:

$$\Omega^{'} = [2q-(3\gamma-2)]\Omega \tag{21}$$

$$g^{'} = 2(q+\Sigma_+ - 1)g \tag{22}$$

Note that the auxilliary equations (21) and (22) follow from (11) – (16) and show the invariance of $\Omega = 0$ and $g = 0$, where $\Omega = 0$ results in the vacuum equations.

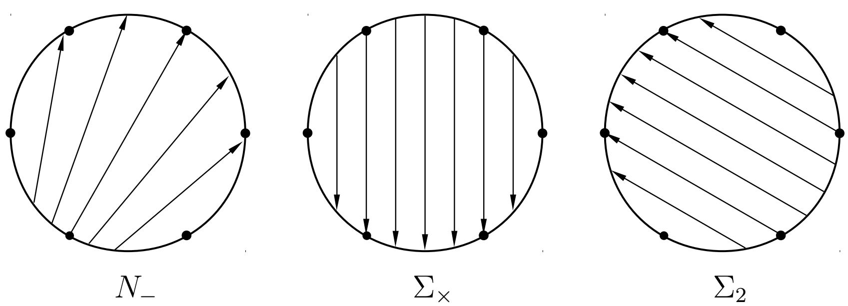
We define the phase space for vacuum Bianchi $VI^*_{-1/9}$ again by requiring that the constraints (19) and (20) are satisfied. This time, we have six variables and two constraints, yielding again a 4-dimensional state space for the Vacuum models, as in Bianchi IX before.

DEFINITION 2.2. *(Phase Space for Vacuum Bianchi $VI^*_{-1/9}$)*

$$\mathcal{B} = \{\Sigma_+, \Sigma_-, \Sigma_\times, \Sigma_2, N_-, A) \mid 0 = 1 - \Sigma^2 - N_-^2 - 4A^2 \text{ and } g = 0\}$$

The equations have been analysed in **[21]**, we give here only a very brief overview about the similarities and differences of Bianchi $VI^*_{-\frac{1}{9}}$ compared to Bianchi IX that are relevant for our own research (see chapter 5).

When we look at the equations, we observe there is also a Kasner circle of fixed points: $\mathcal{K} = \{\Sigma_\times = \Sigma_2 = N_- = A = 0\}$, leading again to $\Sigma_+^2 + \Sigma_-^2 = 1$. Similar to Bianchi IX, we can define caps of heteroclinic orbits connecting points of $\mathcal{K}$, but Bianchi $VI^*_{-\frac{1}{9}}$ is less symmetric than Bianchi IX. The transitions can be illustrated as follows:

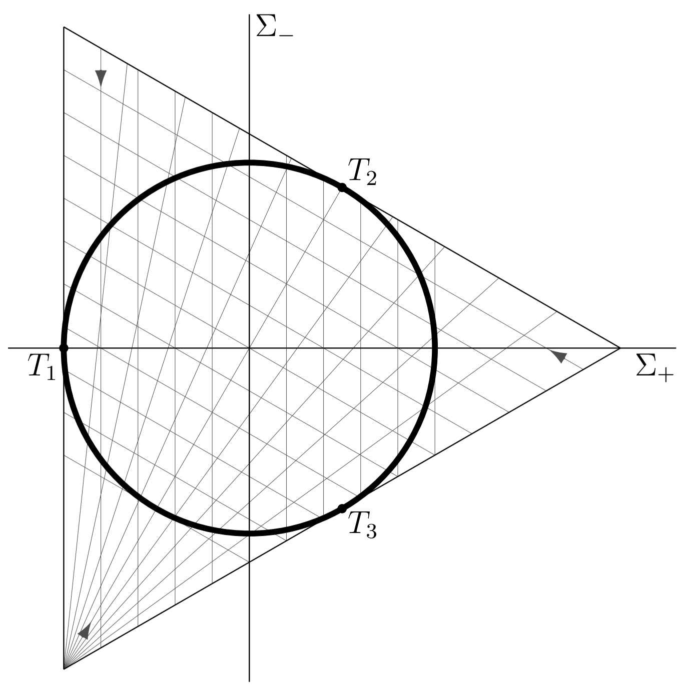


We have the following caps of heteroclinic orbits:

- $C_{N_-} = \{\Sigma_\times = \Sigma_2 = A = 0\}$, which represent transitions in the variable N_-, i.e. curvature transitions. This means the Kasner parameter u changes according to the Kasner map as explained in section 1.2 for Bianchi IX.
- $C_{\Sigma_\times} = \{\Sigma_2 = N_- = A = 0\}$, which represent transitions in the variable Σ_x, i.e. a frame transition. This means the Kasner parameter u is not changed by the transition, it rather connects two points of $\mathcal{K}$ that are in the same equivalence class with the same u in different sectors

- $C_{\Sigma_2} = \{\Sigma_\times = N_- = A = 0\}$ which represent transitions in the variable in the variable Σ_2, also a frame transition.

In addition to the traditional curvature transition in the variable N_- similar to those that also appear in Bianchi class A, we also observe frame transitions of two types, for the variables $\Sigma_\times$ and Σ_2. The fact that the frame transition do not change u can be seen from the fact that the projections of the heteroclinic orbits on the (Σ_+, Σ_-)-plane are parallel lines, and the (inverse) distance to the Taub points (which is one way to interpret u) stays the same after the transition.



Similar to what we did in Bianchi IX, we will also find expressions for the eigenvalues of the transition-variables in terms of the Kasner parameter u. This will be done in chapter 5.

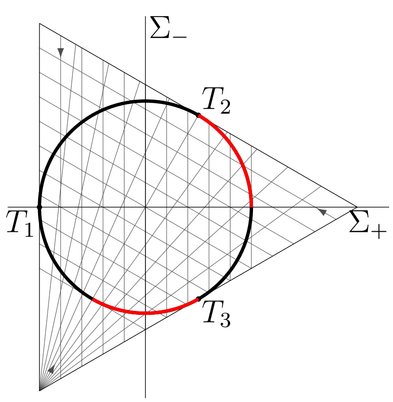
3. Existing Results in Bianchi IX and Difficulties in Bianchi B

Historically, the chaotic oscillations in Bianchi models have first been observed in the Bianchi IX model, see [**29, 3, 4**]. It is also known under the name "Mixmaster", a term coined by Misner [**34**]. The first rigorous theorem on the ancient dynamics in Bianchi IX was proved by Ringström [**42**], based on earlier results by Rendall [**39**]. For a recent survey on the "Facts and Beliefs" concerning the Mixmaster model, see [**17**].

One important research question is relating properties of the Kasner map (which is known to be chaotic, see e.g. the chapter 11 in the book [**58**]) to the real dynamics in Bianchi models. This can be seen as making the BKL-conjecture rigorous for spatially homogenous spacetimes, i.e. by proving the "oscillatory" (and possibly also the "vacuum") part in a case where the model is already "local".

In Bianchi IX, there exist such rigorous convergence results by Liebscher et al [**27**], Béguin [**2**], and Reiterer/Trubowitz [**38**]. For Bianchi VI_0 with a magnetic field as matter, there have been recent results by Liebscher, Rendall and Tschapa [**28**].

Until today, there exist no rigorous convergence results for Bianchi $VI^*_{-1/9}$, which is of class B (see chapter 1, section 5). The reason for this is that Bianchi $VI^*_{-1/9}$ is more difficult than Bianchi class A. One example is that in Bianchi IX, the normal hyperbolicity of the Kasner circle fails only at the three Taub points, while in Bianchi $VI^*_{-1/9}$ this is true for all of the six points that mark the borders of the sectors of the Kasner circle defined in section 2. Also there are non-unique heteroclinic chains because of multiple unstable eigenvalues for some sectors of the Kasner circle, marked in red in the picture below. We will discuss this matter further in chapter 5.



4. Dynamical Systems Techniques

As we have seen in section 2, the Kasner circle plays an important role in the Bianchi IX cosmological model. It is a normally hyperbolic manifold of equilibria, except at three special points, where normal hyperbolicity fails (this situation can be understood in the framework of "bifurcation without parameters" and has been studied intensively by Liebscher [**26**]). As the fixed points of the Kasner circle are not hyperbolic due to the trivial eigenvalue zero, many standard tools are not available.

4.1. Topological Equivalence. In general, linearizing near a fixed point is a difficult endavour. If the co-ordinate change is only required to be C^0, then the theorems of Grobman-Hartman (in the hyperbolic case) and Shoshitahvilii (also in the non-hyperbolic case) provide powerful tools. The theorem of Grobman-Hartman ([**13, 14, 16, 15**]) is well-known and can be found in many textbooks on dynamical systems (e.g. in [**1**]), but the theorem of Shoshitaishvili is less known, that's why we include it here:

Theorem. *(Shoshitaishvili 1972, 1975* [**47, 48**]*) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a vector-field with $f(0) = 0$. Then it exists a vector field $g = g(x_c)$ and a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ that yields a C^0-conjugation between $\dot{x} = f(x)$ and*

$$\begin{aligned} \dot{x}_- &= -x_- \\ \dot{x}_c &= g(x_c) \\ \dot{x}_+ &= x_+ \end{aligned}$$

This means that the dynamics near an equilibrium is topologically equivalent to the direct product of a standard saddle and the dynamics in the centre direction.

4.2. Linearization. For a C^1-linearization, already in the hyperbolic case it is necessary to require certain non-resonance-conditions, as Sternberg ([**51, 52**]) has shown: Linearization can fail if the eigenvalues are in resonance. As we will later consider the more general case of non-hyperbolic fixed points (relevant for Bianchi cosmologies) where a similar but more complicated non-resonance condition will appear (see section 7.2), we label the condition below as "Hyperbolic Sternberg Condition", although this is not standard terminology:

Definition. *(Hyperbolic Sternberg Condition) Consider a hyperbolic fixed point p of a vector field X in $\mathbb{R}^n$ and define that the eigenvalues of the linearization of X at p are in resonance of order $m = \sum_{i=1}^{n} m_i$ iff $\exists(m_1, ..., m_n)$ s.t. $m_i \in \mathbb{N}_0^+$ and $1 \leq j \leq n$:*

$$\sum_{i=1}^{h} \lambda_i m_i - \lambda_j \qquad \begin{cases} = 0 \\ \in \imath\mathbb{R} \end{cases}$$

A consequence of such a resonance is that the non-linear term x^m has the same order of magnitude as the linear term x_j, opposed to the usual local dominace of the linear terms. The non-resonance condition in the non-hyperbollic case will be presented in section 3 that deals with Takens Linearization Theorem.

4.3. Counter-Examples. An example where linearization fails has been constructed by Sell [**45**]:

Theorem. *(Sell 1985) Consider the following vector-field on $\mathbb{R}^3$:*

$$\begin{aligned} \dot{x} &= 2\kappa^2 x \\ \dot{y} &= -\kappa y + x^\kappa z^\kappa \\ \dot{z} &= -(2\kappa^2 + 1) z \end{aligned} \tag{23}$$

with a parameter $\kappa \geq 2$. Then it holds that

- *the system (23) has a $C^{\kappa-1}$-linearization*
- *the system (23) does not have a C^κ-linearization.*
- *the associated linearized system of (23) satisfies a non-resonance-condition of order $(2\kappa - 1)$*

This means that the non-resonance conditions are "sharp". For the case $\kappa = 1$, a similar example already appears in [**16**].

4.3.1. *Invariant Manifold Theorems.* An alternative to linearization is using the invariant manifold theorems, which do not give full equivalence to the linearized system, but instead look for non-linear objects analogue to the linear eigenspaces. For example, using the theory of center-manifolds can provide a considerable reduction of dimension in the non-hyperbolic case.

In order to prove invariant manifold theorems, there are at least two options available: Either the approach based on the "Variations-of-Constants"-formula (see e.g. [**57**]), or the so-called graph-transform method. We use the the second option in order to prove our result on stable manifolds in Bianchi IX in chapter 4. We also comment on how to prove the differentiability of the invariant manifolds in section 4.7.

CHAPTER 3

Takens Linearization Theorem for Partially Hyperbolic Fixed Points

This chapter reviews and explains the results of Floris Takens from his paper [**53**]. They will be essential tools to prove our main theorems in the later chapters of this dissertation. At first, we deal with Takens Linearization Theorem for diffeomorphisms, while the analogous theorem for vector fields is the subject of the later part of this chapter. We will explain key parts of the proof, especially to understand the the so-called "Sternberg-Non-Resonance-Conditions" and the the form of $\alpha(k)$ and $\beta(k)$ (see below) that are the basis of our own results in the later chapters. The notation stays also close to the one of Takens, although we slightly change it at some points if necessary for our needs.

A C^∞-diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with a fixed point $\varphi(0) = 0$ induces a splitting of the tangent space at the origin $T_0\mathbb{R}^n = E^c \oplus E^s \oplus E^u$ with respect to the differential $d\varphi \restriction T_0\mathbb{R}^n$, where the three eigenspaces stand for the eigenvalues with absolute value equal to one (center), <1 (stable), and >1 (unstable). We label the dimensions of these eigenspaces as follows:

$$\begin{aligned} c &= dim(E^c) \\ s &= dim(E^s) \\ u &= dim(E^u) \end{aligned}$$

By a slight abuse of notation, let us label the coordinates of $\mathbb{R}^n$ as (x^c, y^s, z^u), where $x^c = x^c_1 ... x^c_c$, $y^s = y^s_1 ... y^s_s$ and $z^u = z^u_1 ... z^u_u$, where the upper index is just to remind us for which eigenspace the respective coordinate stands for.

DEFINITION. *(Standard Form of a Diffeomorphism) Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞-diffeomorphism with $\varphi(0) = 0$. We say φ is locally in standard form w.r.t. coordinates (x^c, y^s, z^u) if, for some neighbourhood of the origin,*

$$
\begin{aligned}
\varphi(x^c, y^s, z^u) &= [\phi_1(x^c), ..., \phi_c(x^c), \\
&\quad \sum_{i=1}^{s} a_{1i}(x^c)\cdot y_i^s, ..., \sum_{i=1}^{s} a_{si}(x^c)\cdot y_i^s, \\
&\quad \sum_{j=1}^{u} b_{1j}(x^c)\cdot z_j^u, ..., \sum_{j=1}^{u} b_{uj}(x^c)\cdot z_j^u]
\end{aligned}
$$

where:
(1) all eigenvalues of $(\partial\phi_i/\partial x_i^c)$ in $(x^c = 0)$ have absolute value one
(2) all eigenvalues of the matrix $A_{ij}(0)$ have absolute value < 1
(3) all eigenvalues of the matrix $B_{ij}(0)$ have absolute value > 1

1. Sternberg Non-Resonance Conditions

Takens considers a diffeomorphism φ as above, and its linearization at 0, $d\varphi(0)$. He labels the "center-eigenvalues" of $d\varphi(0)$ (corresponding to point (1) above) as $\{\mu_1, ..., \mu_c\}$, and the "hyperbolic eigenvalues" of $d\varphi(0)$ (corresponding to (2) above) as $\{\lambda_1, ..., \lambda_h\}$, where $h := u + s$.

Definition. *(Sternberg-l-Conditions)*

φ satisfies the Sternberg-Conditions of order l (in short the "Sterberg-l-conditions") $\iff$ the following 2 conditions hold:

$$|\prod_{i=1}^{h} \lambda_i^{v_i}| \neq 1 \tag{24}$$

$$|\lambda_j^{-1}\cdot \prod_{i=1}^{h} \lambda_i^{v_i}| \neq 1 \tag{25}$$

$\forall (v_1, ..., v_h)$ s.t. $v_i \in \mathbb{N}_0^+$with $2 \leq \sum_{i=0}^{h} v_i \leq l$ and $1 \leq j \leq h$

Theorem. (*Takens Linearization Theorem for Partially Hyperpolic Fixed Points*)

Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞-diffeomorphism with $\varphi(0) = 0$, and $k \in \mathbb{N}$. If φ satisfies the Sternberg-$\alpha(k)$-Condition, then there exist C^k coordinates (x^c, y^s, z^u) such that φ is locally in standard form with respect to the coordinates(x^c, y^s, z^u)

2. The Formula for $\alpha(k)$ and $\beta(k)$

Takens uses the following notation for the hyperbolic part $\{\lambda_1, ..., \lambda_h\}$ of the spectrum of $d\varphi(0)$: Sort those numbers λ_i s.t. it holds $|\lambda_1| \leq |\lambda_2| \leq ... \leq |\lambda_s| \leq 1 \leq |\lambda_{s+1}| \leq ... \leq |\lambda_h|$ and label:

- $M := |\lambda_h|$, $m := |\lambda_{s+1}|$
- $N := |\lambda_1|^{-1}$, $n := |\lambda_s|^{-1}$

Note that all numbers defined above are ≥ 1. Now the number $\beta(k)$ is defined as the smallest integer s.t. it holds $\forall r \leq k$:

$$N \cdot M^r \cdot n^{(r-\beta)} < 1$$

After that, $\alpha(k)$ is defined to be the smallest integer s.t. it holds $\forall r \leq \beta$:

$$M \cdot N^r \cdot m^{(r-\alpha)} < 1$$

Note that in fact $\alpha = \alpha(k, d\varphi(0))$ and $\beta = \beta(k, d\varphi(0))$, i.e. the numbers α, β depend on the linearization of the vector field at the point where you want to employ the Takens Theorem.

3. The Overall Structure of the Proof

The proof is split into 3 Propositions:

Proposition 3.1. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞-diffeomorphism with $\varphi(0) = 0$ satisfying the Sterberg-l-condition. Then, for any integer N, there is a neighbourhood U of 0 in $\mathbb{R}^n$ and C^N-coordinates (x^c, y^s, z^u) such that $\varphi \restriction U = S\varphi \restriction U + R\varphi \restriction U$ and the following holds:

(1) $S\varphi \restriction U$ is in standard form w.r.t. coordinates (x^c, y^s, z^u)
(2) $R\varphi \restriction U$ is zero up to order l along $W^c = E^c = \{y^s = z^u = 0\}$
(3) $W^{cu} = E^{cu} = \{y^s = 0\}$ is invariant under $R\varphi \restriction U$

Proposition 3.2. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^N-diffeomorphism with $\varphi(0) = 0$, and $N \geq \alpha(k)$. Suppose that $\varphi \restriction U = S\varphi \restriction U + R\varphi \restriction U$ as in the conclusion of Proposition 1, with respect to coordinates (x^c, y^s, z^u) and $l = \alpha(k)$. Then there are a $C^{\beta(k)}$-coordinates $(\tilde{x}^c, \tilde{y}^s, \tilde{z}^u)$ s.t., in a neighbourhood of 0, we have $\varphi \restriction U = \tilde{S}\varphi \restriction U + \tilde{R}\varphi \restriction U$ where

(1) $\tilde{S}\varphi \restriction U$ is in standard form with respect to coordinates $(\tilde{x}^c, \tilde{y}^s, \tilde{z}^u)$

(2) $\tilde{R}\varphi \restriction U$ is zero up to order $\beta(k)$ along $W^{cu} = E^{cu} = \{y^s = 0\}$

Proposition 3.3. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^{\beta(k)}$-diffeomorphism with $\varphi(0) = 0$. Suppose that $\varphi \restriction U = \tilde{S}\varphi \restriction U + \tilde{R}\varphi \restriction U$ as in the conclusion of Proposition 2 with respect to coordinates (x^c, y^s, z^u). Then there are a C^k-coordinates $(\tilde{x}^c, \tilde{y}^s, \tilde{z}^u)$ such that, in a neighbourhood of 0, such that φ is locally in standard form w.r.t $(\tilde{x}^c, \tilde{y}^s, \tilde{z}^u)$.

The content of Propositions 1-3, can be illustrated by the following table:

	Smoothness	afterwards good jet on	NRCs -> good jet to order
Prop1	$\infty \to N$	W^c	$l \to l$ (“constant”)
Prop2	$\alpha \to \beta$	W^{cu}	$\alpha \to \beta$
Prop3	$\beta \to k$	$\mathbb{R}^n$	$\beta \to k$

In the first column of the table above, the smoothness of the coordinates before and after the respective proposition is shown, e.g. in the second row we start with a C^α-Diffeomorphism in the requirements of Proposition 2, and we will get C^β-coordinates for the “good jet” on W^{cu}, which means that $R\varphi$ vanishes up to order β along W^{cu} and we have to require Non-Resonance-Conditions of order α for this to work out.

4. Proof of Proposition 1

4.1. General Idea of the Proof. The aim of Proposition 1 is to find a map α conjugating our initial diffeomorphism φ and the “standard form” $S\varphi$ (here on W^c, later on the whole $\mathbb{R}^n$), up to order l:

$$\alpha^{-1} \circ \varphi \circ \alpha = S\varphi$$

Takens uses (manifolds of) jets as the appropiate setting for theses “coordinate changes up to order l” are , and we will introduce them below. He then “lifts” φ to the space of jets, at first only for the hyperbolic directions, and observe that a fixed point of the “lifted” map φ_l means that

$$[\alpha]_l = \varphi_l([\alpha]_l)$$

The key idea of the proof of Proposition 1 is to consider a center-manifold W^{*c} for an extended φ_l^* in the space of jets as well as the projection of W^{*c} “down” to the center-manifold of our inital diffeomorphism φ.

$$p_c : W^{*c} \to W^c$$

The invariance of W^{*c} allows the choice of the desired coordinates, which will prove Proposition 1.

4.2. Usage of Sternberg Non-Resonance Conditions. In the proof of Proposition 1, the Sternberg conditions are used in the following form: In the proof, it is necessary to show that a certain map is hyperbolic, and the relevant eigenvalues are:

- $\mu_p \cdot \prod_{i=1}^{h} \lambda_i^{-v_i}$
- $\lambda_q \cdot \prod_{i=1}^{h} \lambda_i^{-v_i}$

None of these eigenvalues has absolute value one because of the Sternberg condition, hence the map under consideration is hyperbolic, which is an essential step in the proof of Proposition 1.

4.3. Choices and Definitions. At first, Takens chooses coordinates such that the following submanifolds of $\mathbb{R}^n$ are invariant for φ (i.e. he "straightens out" the non-linear invariant manifolds locally such that they coincide with the linear eigenspaces defined below, see e.g. [**46**]):

$$
\begin{aligned}
W^u &= E^u = \{x^c = y^s = 0\} \\
W^s &= E^s = \{x^c = z^u = 0\} \\
W^c &= E^c = \{y^s = z^u = 0\} \\
W^{cu} &= E^{cu} = \{y^s = 0\} \\
W^{cs} &= E^{cs} = \{z^u = 0\}
\end{aligned}
$$

Definition. *(the spaces V_r and the map φ_r) Let $\tilde{V}_r$ be the manifold of r-jets of embeddings $(\mathbb{R}^h, 0) \to (\mathbb{R}^h, 0)$, where this notation means that $\alpha(0) = 0$ for $\alpha \in \tilde{V}_r$. V_r is obtained from $\tilde{V}_r$ by the following identifications: α_1 and $\alpha_2 \in \tilde{V}_r$ are identified in $\tilde{V}_r$ if there is a linear map $\alpha : (\mathbb{R}^h, 0) \to (\mathbb{R}^h, 0)$ such that $\alpha_1 \circ \alpha = \alpha_2$. This induces a transformation $\varphi_r : V_r \to V_r$ on the jets:*

$$
\begin{aligned}
\varphi_r : V_r &\to V_r \\
[\alpha]_r &\to [\varphi \circ \alpha]_r
\end{aligned}
$$

There is also a natural projection $\pi_r : V_r \to V_{r-1}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
V_r & \xrightarrow{\varphi_r} & V_r \\
\big\downarrow \pi_r & & \big\downarrow \pi_r \\
V_{r-1} & \xrightarrow{\varphi_{r-1}} & V_{r-1}
\end{array}
$$

4.4. Lemma 2.2.

LEMMA. *Under the assumption (as in Proposition 1) that φ satisfies the Sternberg l-condition, there is for each $1 \leq r \leq l$ a unique element $[\alpha]_r \in V_r$ such that:*

(1) $[\alpha]_r$ can be represented by an embedding $(\mathbb{R}^h, 0) \to (\mathbb{R}^h, 0)$ with image tangent to $T^s \oplus T^u$
(2) $[\alpha]_r$ is a hyperbolic fixed point of φ_r

PROOF. The proof is done by induction on r. In the induction step, the eigenvalues mentioned above are calculated, of a modified map $\bar{\varphi}_r : \pi^{-1}([\alpha]_{r-1}) \to \pi^{-1}([\alpha]_{r-1})$, by considering the following basis of eigenvectors for $\bar{\varphi}_r$ (as before, the numbers $c, h \in \mathbb{N}$ stand for the dimensions of the center/hyperbolic eigenspaces of φ):

$$B = \{v_i^{i_1,\dots,i_h}, w_j^{i_1,\dots,i_h} | i = 1...c\,;\, j = 1...h\,;\, , i_1, ...i_\nu \geq 0\,;\, \sum_{\nu=1}^{h} i_\nu = r\}$$

where □

- $v_i^{i_1,\dots,i_h}$ is represented by $(w_1, ..., w_h) \mapsto (p_1, ..., p_c, w_1, ..., w_h)$ with $p_i = w_1^{i_1} \cdot ... \cdot w_h^{i_h}$ and $p_k = 0$ for $k \neq i$
- $\tilde{v}_j^{i_1,\dots,i_h}$ is represented by $(w_1, ..., w_h) \mapsto (0, ..., 0, w_1+q_1, ..., w_h+q_h)$ with $q_j = w_1^{i_1} \cdot ... \cdot w_h^{i_h}$ and $q_k = 0$ for $k \neq j$

The formulas above result in the following eigenvalues:

- $v_i^{i_1,\dots,i_h}$ yields an eigenvalue $\mu_i \cdot \lambda_1^{-i_1} \cdot ... \cdot \lambda_h^{-i_h}$
- $\tilde{v}_j^{i_1,\dots,i_h}$yields an eigenvalue $\lambda_j \cdot \lambda_1^{-i_1} \cdot ... \cdot \lambda_h^{-i_h}$

, which can be checked directly using the following formula for $\bar{\varphi}_r([\beta]) = \varphi \circ \beta \circ A - \varphi \circ \vartheta \circ A + \vartheta$ for $[\beta] \in \pi^{-1}([\alpha]_{r-1})$ and the follwing additional definitions/assumptions:

- $(d\varphi)_0$ is first assumed to be in diagonal form, meaning that φ can be written as $\varphi(x^c, y^s, z^u) = \varphi(x_1, ..., x_c, y_1, ..., y_s, z_1, ..., z_u) = (\mu_1 x_1, ..., \mu_c x_c, \lambda_1 y_1, ..., \lambda_h z_u)$[1]
- $A : (\mathbb{R}^h, 0) \to (\mathbb{R}^h, 0)$ defined by $A(w_1, ..., w_h) := (\lambda_1^{-1} w_1, ..., \lambda_h^{-1} w_h)$
- $\vartheta : (\mathbb{R}^h, 0) \to (\mathbb{R}^n, 0)$ defined by $\vartheta(w_1, ..., w_h) := (0, ..., 0, w_1, ..., w_h)$

Now the Sternberg Non-Resonance Conditions show that none of the eigenvalues of $\bar{\varphi}_r$ has absolute value one, which yields the hyperbolicity of $\bar{\varphi}_r$ and is the essential step in the proof of the Lemma.

[1]Later the general case is reduced to the diagonal case by complexifying and using the Jordan Normal Form Theorem

4.5. Construction of the Required Coordinate System.

Definition. *(the space* V_r^* *and the map* φ_r^**) Let* $\tilde{V_r^*}$ *be the manifold of r-jets of embeddings* $(\mathbb{R}^h, 0) \to (\mathbb{R}^n, W^c)$*, where this notation means that* $\alpha(0) \in W^c$ *for* $\alpha \in \tilde{V_r^*}$*.* V_r^* *is obtained from* $\tilde{V_r^*}$ *by the analogous identification as above. We can extend the map* $\varphi_r : V_r \to V_r$ *to a map* φ_r^* *defined on a neighbourhood of* V_r *in* V_r^**, using the local invariance of* W^c *under* φ*.*

In order to make this precise, Takens considers the following projections.

Definition. *(the projections p and* p_c*) Observe that* $V_r \subset V_r^*$ *and define* $p : V_r^* \to W^c$ *by* $p(\alpha) := \alpha(0) \in W^c$ *for* $\alpha \in V_r^*$*. Note that* $p^{-1}(0) = V_r$*. We define* p_c *as the resctriction of p to* $W^{*c} \subset V_r^*$ *defined below:* $p_c := p \restriction W^{*c}$*.*

By the Lemma above, there is a hyperbolic fixed point $[\alpha]_l$ of φ_l in V_l. Takens notes that $[\alpha]_l$ is also a fixed point for φ_l^*, but it is not hyperbolic: the set of eigenvalues for $(d\varphi_l^*)_{[\alpha]_l}$ is the union of the set of eigenvalues of $(d\varphi_l)_{[\alpha]_l}$ and the set of eigenvalues of $d(\varphi \restriction W^c)_0$. He then constructs a center-manifold W^{*c} for $[\alpha]_l$ in V_l^*, and claims that the projection $p_c : W^{*c} \to W^c$ defined above is a diffeomorphism restricted to a small neighbourhood of $[\alpha]_l$, using that $[\alpha]_l$ is a hyperbolic fixed point for the non-center directions. This means that we can define $p_c^{-1}(P)$ for P sufficiently close to the origin, obtaining a class of l-jets of embeddings $(\mathbb{R}^h, 0) \to (\mathbb{R}^n, P)$, and choose the coordinate system (x^c, y^s, z^u) such that, for each $P \in W^c$, close enough to the origin, $p_c^{-1}(P)$ is represented by the affine embedding:

$$(y^s, z^u) \mapsto (x^c, y^s, z^u)$$

where

$$P = (x^c, 0)$$

By the local invariance of W^{*c} it follows that also the image $\varphi_l^*([\alpha]_l)$ can be represented as described above. Takens concludes the proof by noting that this means, for some neighborhood U_1 of the origin, $\Phi \restriction U_1 = S\Phi \restriction U_1 + R \restriction U_1$ where $S\Phi \restriction U_1$ is in standard form with respect to coordinates (x^c, y^s, z^u) and $R \restriction U_1$ is zero up to order l along $W^c = E^c = \{y^s = z^u = 0\}$, which proves Proposition 1 (compare **[53]**, p. 138).

5. Proof of Proposition 2

Assume that the map φ, the coordinates (x^c, y^s, z^u) and the "splitting $\varphi = S\varphi + R\varphi$" are as in the assumptions of Proposition 2.

5.1. General Idea of the Proof. We would like to find a map σ conjugating our initial diffeomorphism φ and the "standard form" $S\varphi$ (here on W^{cu}, later on the whole $\mathbb{R}^n$), i.e. $\sigma^{-1} \circ \varphi \circ \sigma = S\varphi$. The idea is now to write this in a slightly different way:

$$\sigma = \varphi \circ \sigma \circ (S\varphi)^{-1} := \Phi(\sigma)$$

and to find a suitable setting where Φ is a contraction, yielding a fixed point which will be our desired change of coordinates. An appropiate setting for this to work out are Jet bundles, and they are introduced in the next section.

5.2. Jet bundles.

Definition. *(Jet-bundles and Φ_r) The elements of the Jet-bundle are equivalence classes, which can be represented by pairs $[p, \sigma]_r$, where $p \in \mathbb{R}^n$ and σ is a C^r-map from a neighbourhood of p to $\mathbb{R}^n$. Then Takens defines the map Φ_r as follows, inspired by the idea described above:*

$$\begin{aligned} \Phi_r : J^r(\mathbb{R}^n, \mathbb{R}^n) &\rightarrow J^r(\mathbb{R}^n, \mathbb{R}^n) \\ [p, \sigma]_r &\rightarrow [S\varphi(p), \varphi \circ \sigma \circ (S\varphi)^{-1}]_r \end{aligned}$$

$J^r(\mathbb{R}^n, \mathbb{R}^n)$ is fibred over $J^{r-1}(\mathbb{R}^n, \mathbb{R}^n)$, and $\pi_r : J^r(\mathbb{R}^n, \mathbb{R}^n) \rightarrow J^{r-1}(\mathbb{R}^n, \mathbb{R}^n)$ is the projection. The following diagramm commutes:

$$\begin{array}{ccc} J^r(\mathbb{R}^n, \mathbb{R}^n) & \xrightarrow{\Phi_r} & J^r(\mathbb{R}^n, \mathbb{R}^n) \\ \downarrow{\scriptstyle \pi_r} & & \downarrow{\scriptstyle \pi_r} \\ J^{r-1}(\mathbb{R}^n, \mathbb{R}^n) & \xrightarrow{\Phi_{r-1}} & J^{r-1}(\mathbb{R}^n, \mathbb{R}^n) \end{array}$$

5.3. Metric on the Fibre Bundle. At first, Takens defines appropriate fiber metrics of π_r, in order to be able to talk about convergence in the space of jets. He defines a metric $\rho_r([p, \sigma_1]_r, [p, \sigma_2])$ between two jets $[p, \sigma_1]$ and $[p, \sigma_2]$ in the same fiber of π_r, where he labels the common base point $\sigma(p) := \sigma_1(p) = \sigma_2(p)$:

$$\rho_r([p, \sigma_1]_r, [p, \sigma_2]) := \lim_{a \to 0}\left(\sup_{\|X|=a} \left(\frac{|\hat{\sigma}_{1,2}(X)|}{|X|^r} \right)\right)$$

where

$$
\begin{array}{rcl}
\hat{\sigma}_{1,2} : T_p(\mathbb{R}^n) & \to & T_{\sigma(p)}(\mathbb{R}^n) \\
X & \to & \hat{\sigma}_{1,2}(X) = exp_{\sigma(p)}^{-1} \circ \sigma_1 \circ exp_p(X) - exp_{\sigma(p)}^{-1} \circ \sigma_2 \circ exp_p(X)
\end{array}
$$

and $exp_q : T_q(\mathbb{R}^n) \to \mathbb{R}^n$ stands for the exponential map on manifolds.

The norm $|\cdot|$ that appears in the definition of $\rho_r([p,\sigma_1]_r, [p,\sigma_2])$ is with respect to a specific metric on $\mathbb{R}^n$ which will be explained in the next section.

5.4. Metric on $\mathbb{R}^n$. Define for $p \in W^{cu}$ the following numbers:

$$
\begin{array}{rcl}
\tilde{M}_p & = & ||(d\varphi)_p|| \\
\tilde{N}_p & = & ||(d(S\varphi)^{-1})_p||
\end{array}
$$

and

$$
\tilde{m}_p = \left\{ \begin{array}{ll} \text{for } p \notin W^c & = \rho(S\varphi(p), W^c) \cdot (\rho(p, W^c))^{-1} \\ \text{for } p \in W^c & = \liminf_{q \in W^{cu}, q \notin W^c, q \to p} (\tilde{m}_p) \end{array} \right\}
$$

Now recall that it holds $\forall r \leq \beta$:

$$
M \cdot N^r \cdot m^{(r-\alpha)} < 1
$$

It follows that we can chose a metric on $\mathbb{R}^n$ such that (for $p = 0$, i.e. the origin)

$$
\tilde{M}_0 \cdot \tilde{N}_0^{\ r} \cdot m_0^{(r-\alpha)} < \mu
$$

for some fixed μ and $\forall r \leq \beta$.

5.5. Lemma 3.5. Let K_δ be the closed δ neighbourhood of W^c in W^{cu}, and chose δ so small that $\forall p, q, v \in K_\delta$ it holds that

$$
\tilde{M}_p \cdot \tilde{N}_q^{\ r} \cdot m_v^{(r-\alpha)} < \mu
$$

, where $\mu < 1$ stands for the constant that appear in the previous section when defining our metric on $\mathbb{R}^n$. Let us label $\rho_p^k := \rho(p, W^c)^k$ in order to formulate the next Lemma:

LEMMA. *(3.5 in Takens Paper)*
Let $[p,\sigma_1]$ and $[p,\sigma_2]$ be two jets in the same fiber of π_r with $p, S\varphi(p), \sigma_1(p), \sigma_2(p) \in K_\delta$ but $p \notin W^c$, and $r \leq \beta$. Then it holds:

$$
\rho_r(\Phi_r[p,\sigma_1], \Phi_r[p,\sigma_1]) \cdot (\rho(S\varphi(p), W^c)^{r-\alpha} < \mu \cdot \rho_r([p,\sigma_1]_r, [p,\sigma_2]) \cdot (\rho(p, W^c))^{r-\alpha}
$$

or

$$\rho_r(\Phi_r[p,\sigma_1],\Phi_r[p,\sigma_1]) < \mu\cdot\rho_r([p,\sigma_1]_r,[p,\sigma_2])\cdot\frac{\rho_p^{r-\alpha}}{\rho_{S\varphi(p)}^{r-\alpha}}$$

In order to get the desired contraction, we have to get rid of the factor $\frac{\rho_p^{r-\alpha}}{\rho_{S\varphi(p)}^{r-\alpha}}$. This will be done in the last section of this chapter, by defining appropriate spaces $\mathcal{F}_i$ s.t. $\Phi_i:\mathcal{F}_i\to\mathcal{F}_i$ is a contraction with respect to an adapted metric $\tilde{\rho}_i$ on $\mathcal{F}_i$ (see Proposition 2'). But first we would like to show how the proof of Proposition 2 is finished once this has been done.

5.6. End of the Proof of Proposition 2. Takens finally proves Proposition 2 by giving a convergent sequence of "jets of coordinates alon W^{cu}" the limit of which is invariant under Φ. In order to do this, he defines a sequence $\{F_i\}_i^\infty$ of C^β-maps $(\mathbb{R}^n,W^{cu})\to(\mathbb{R}^n,W^{cu})$ inductively as follows: $F_o=id$, and for a given F_i, the F_{i+1} is obtained from $\phi\circ F_i\circ(S\phi)^{-1}$ by omitting the terms of order $>\beta$ in $y_1,\dots,y_s$. Then it holds that:

(1) $F_i(x^c,z^u)=\sum f^i_{j_1,\dots,j_s}(x^c,z^u)\cdot y_1^{j_1}\cdot\ldots\cdot y_s^{j_s}$, where the sum is taken over all $(j_1,\dots,j_s)$ with $j_\nu\geq 0$ and $\sum j_\nu\leq\beta$. Note that $f^i_{j_1,\dots,j_s}$ takes values in $\mathbb{R}^n$, where $f^i_{0,0,\dots,0}$ takes values in W^{cu};

(2) the jet of $(\Phi_\beta)^i\vartheta_\beta$ in $p\in W^{cu}$ can be represented by $[p,F_i]_\beta$.

By Proposition 2' (see below), $\{F_i\}_i^\infty$ has a limit which is of the form

$$F(x^c,z^u)=\sum f_{j_1,\dots,j_s}(x^c,z^u)\cdot y_1^{j_1}\cdot\ldots\cdot y_s^{j_s}$$

where the summation is taken over the same $(j_1,\dots,j_s)$ as above, but now each $f_{j_1,\dots,j_s}$ is only a $C^{\beta-\sum j_\nu}$-function. According to Whitney's extension theorem, there is a C^β-function $\tilde{F}:(\mathbb{R}^n,W^{cu})\to(\mathbb{R}^n,W^{cu})$ such that, for each $p\in W^{cu}$, $[p,\tilde{F}]_\beta$ represents the jet of $(\lim_{i\to\infty}\Phi^i_\beta\vartheta^i)$ in p. Takens notes that $\tilde{F}$ induces a diffeomorphism from W^{cu} to itself that can be extended to a diffeomorphism from the whole $\mathbb{R}^n$ to itself. By definition, $S\varphi$ and $\tilde{F^{-1}}\circ\varphi\circ\tilde{F}$ have the same β-jet along W^{cu}, so $\tilde{F}$ defines the desired coordinate system, as indicated the beginning of this chapter, which proves Proposition 2 (compare [**53**], p. 144).

5.7. Proposition 2'. The appropriate setting for "good coordinates along W^{cu} up to order β" is the jet bundle $J^\beta((\mathbb{R}^n,W^{cu}),(\mathbb{R}^n,W^{cu}))\to W^{cu}$, but this section is formulated slightly more general for any finite order r, because we will have to do a similar step in the proof of Proposition 3 (see below). That's why Takens considers the bundle

$$\Pi_r = \pi_o \circ ... \circ \pi_{r-1} \circ \pi_r : J^r((\mathbb{R}^n, W^{cu}), (\mathbb{R}^n, W^{cu})) \to W^{cu}$$

To a section of Π_r , it is possible to apply the transformation Φ_r as follows: If $\kappa : W^{cu} \to J^r((\mathbb{R}^n, W^{cu}), (\mathbb{R}^n, W^{cu}))$ is a section of Π_r, then $\Phi_r\kappa$ is the section which assigns to $p \in W^{cu}$ the jet $(\Phi_r\kappa)p := \Phi_r(\kappa(q))$, where $q := (S\varphi)^{-1}(p)$. In addition, define the map ϑ_r : $J^r((\mathbb{R}^n, W^{cu}),(\mathbb{R}^n, W^{cu}))$ by $\vartheta_r(p) = [p, id]_r$ for all $p \in W^{cu}$. ϑ_r is a cross-section in the bundle Π_r.

Proposition. *(2') The sequence of sections of* Π_r, *defined by* $\{(\Phi_r)^i\vartheta_r\}_i^\infty$ *converges to a continious section of* Π_r *for* $r \leq \beta$

In the proof of Proposition 2', Takens also uses that $R\varphi$ is zero up to order α along W^c (which is the result of Proposition 1 and the key assumption in Proposition 2) and the Fibre Contraction Theorem (see **[24]**, p. 25) as the main technical tool. He then constructs suitable spaces $\mathcal{F}_i$ s.t. $\Phi_i : \mathcal{F}_i \to \mathcal{F}_i$ is a contraction yielding the claimed convergence.

6. Proof of Proposition 3

6.1. Sketch of Proof. The proof of Proposition 3 is only sketched in Takes paper, as it is analogous to the proof of Proposition 2. In the latter, Takens started with the "good jet" along W^c and ended with the "good jet" along W^{cu}, making essential use of the fact that, in W^{cu}, Φ was "expanding away" from W^c. In order to apply the same method in obtaining the "good jet" over all of $\mathbb{R}^n$, it is necessary to replace Φ by Φ^{-1}, which is expanding away from W^{cu}:

This implies that M,m,n,N are replaced by N,n,m,M, which is reflected in the definitions of $\alpha(k)$ and $\beta(k)$, see above. Because in the proof of Proposition 2, the fact that $d\Phi \restriction T_o(W^c)$ has only eigenvalues of absolute value one was not used, Takens concludes that the analogy is complete (see **[53]**, p. 144).

6.2. Understanding the form of $\alpha(k)$ and $\beta(k)$. At first, we recall that $\alpha(k)$ is defined to be the smallest integer s.t. it holds $\forall r \leq \beta$:

$$M \cdot N^r \cdot m^{(r-\alpha)} < 1$$

where N, M, n are the "spectral boundaries" of our diffeomorphism φ (see above). This definition was used in the proof of Proposition 2, where similar numbers are defined that are closely related (for $p \notin W^c$):

$$\begin{aligned}\tilde{M}_p &= ||(d\varphi)_p|| \\ \tilde{N}_p &= ||(d(S\varphi)^{-1})_p|| \\ \tilde{m}_p &= \rho(S\varphi(p), W^c)\cdot(\rho(p,W^c))^{-1}\end{aligned}$$

When estimating the distance of two image points $\rho_r(\Phi_r[p,\sigma_1],\Phi_r[p,\sigma_1])$ under the map Φ_r $([p,\sigma]_r) = [S\varphi(p), \varphi\circ\sigma\circ(S\varphi)^{-1}]_r$, the chain rule yields (for $[p,\sigma_1]$ and $[p,\sigma_2]$ in the same fibre of π_r and $\sigma_1(p)=\sigma_2(p)=q$, compare Lemma 3.4 of the Takens paper [**53**]):

$$\rho_r(\Phi_r[p,\sigma_1],\Phi_r[p,\sigma_1]) \leq ||(d(S\varphi)^{-1})_{S\varphi(p)}||^r \cdot ||(d\varphi)_q|| \cdot \rho_r([p,\sigma_1]_r,[p,\sigma_2])$$

, where $||(d\varphi)_q|| = \sup_{|X|=1, X\in T_q(\mathbb{R}^n)}(|d\varphi(X)|)$. The number $\tilde{m}_p$ appears in Lemma 3.5 to deal with $\sigma_1(p)\neq\sigma_2(p)$ (but $\sigma_1(p),\sigma_2(p)\in K_\delta$, see above), and the the exponent $(r-\alpha)$ becomes clear only in the end of the proof of Proposition 2' and is related to the definition of the spaces $\mathcal{F}_i$ (see [**53**], p. 143).

The "symmetry" of the definitions of $\alpha(k)$ and $\beta(k)$ comes from the fact that in the proof of Proposition 3, the numbers M,m,n,N are replaced by N,n,m,M and the same argument is carried out for Φ^{-1} instead of Φ.

7. Takens Linearization Theorem for Vector Fields

7.1. Partially Hyperbolic Fixed Points. In a later part of the paper [**53**] (p. 144), Floris Takens proves a Linearization Theorem for Vector fields - this will be the version of the Theorem that we will apply to Bianchi cosmologies.

Definition (Standard Form of a Vector field). *Let X be a vector-field on $\mathbb{R}^n$ which is zero at the origin. We say X is locally in standard form w.r.t. coordinates (x^c, y^s, z^u) $\iff$ for some neighbourhood of the origin,*

$$X = \sum_{i=1}^{c}\phi(x_i^c)\frac{\partial}{\partial x_i^c} + \sum_{i,j=1}^{s} A_{ij}(x^c)y_j^s\frac{\partial}{\partial y_i^s} + \sum_{i,j=1}^{u} B_{ij}(x^c)z_j^u\frac{\partial}{\partial z_i^u}$$

where:
(1) all eigenvalues of $(\partial\phi_i/\partial x_i^c)$ in $(x^c=0)$ have real part zero
(2) all eigenvalues of the matrix $A_{ij}(0)$ have real part <0
(3) all eigenvalues of the matrix $B_{ij}(0)$ have real part >0

THEOREM (Floris Takens, 1971). Let X be a C^∞-vectorfield on $\mathbb{R}^n$ which is zero at the origin, and $k \in \mathbb{N}$. If X satisfies the Sternberg-$\alpha(k)$-Condition, then there exist C^k coordinates (x^c, y^s, z^u) such that X is locally in standard form w.r.t.(x^c, y^s, z^u)

7.2. Sternberg Non-Resonance Conditions[2]**.** Consider a vectorfield X as above, and its linearization at 0, $L := DX(0)$. Let us label the "center-eigenvalues" of L (corresponding to point (1) above) as $\{\mu_1, ..., \mu_c\}$, and the "hyperbolic eigenvalues" of L (corresponding to (2) and (3) above) as $\{\lambda_1, ..., \lambda_h\}$, where $h := s + u$.

DEFINITION (Sternberg Non-Resonance-Conditions of order l). *X satisfies the Sternberg Non-Resonance-Conditions of order l (in short "SNC of order l") iff the following two conditions hold:*

$$\sum_{i=1}^{h} \lambda_i m_i \neq 0 \tag{26}$$

$$-\lambda_j + \sum_{i=1}^{h} \lambda_i m_i \neq 0 \tag{27}$$

$\forall (m_1, ..., m_h)$ s.t. $m_i \in \mathbb{N}_0^+$with $2 \leq \sum_{i=1}^{h} m_i \leq l$ and $1 \leq j \leq h$

7.3. The Formula for $\alpha(k)$ and $\beta(k)$. We use the following notation for the hyperbolic part $\{\lambda_1, ..., \lambda_h\}$ of the spectrum of L: Sort those numbers λ_i s.t. it holds $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_s < 0 < \lambda_{s+1} \leq ... \leq \lambda_h$ and label:

- $M := \lambda_h$, $m := \lambda_{s+1}$
- $N := -\lambda_1$, $n := -\lambda_s$

Note that all numbers defined above are ≥ 0. Now, according to the Takens-Paper, the number $\beta(k)$ is defined as the smallest integer s.t. it holds $\forall r \leq k$:

$$N + rM + (r - \beta)n < 0$$

After that, $\alpha(k)$ is defined to be the smallest integer s.t. it holds $\forall r \leq \beta$:

$$M + rN + (r - \alpha)m < 0$$

Clearly, if the conditions above are satisfied for the maximal $r = k$ resp. $r = \beta$, then they hold for all smaller r, too. This results in the following formulas for $\alpha(k)$ and $\beta(k)$:

[2] only for real eigenvalues here, as this is the case relevant for Bianchi models

$$\beta = Ceiling[\frac{N + k(M+n)}{n}]$$

$$\alpha = Ceiling[\frac{M + \beta(N+m)}{m}]$$

where $a := Ceiling[b]$ means that a is the smallest integer for which i$a \geq b$. Note that in fact $\alpha = \alpha(k, L)$, i.e. the number α depends on the linearization of the vector field at the point where you want to employ the Takens Theorem.

7.4. The Structure of the Proof. The proof is split into 2 Propositions, that are the analogue of the Propositions 1 and 2 for Diffeomorphisms discussed above:

7.5. Proposition 1. There exist coordinates (x^c, y^s, z^u) and vectorfields SX and RX and a neighbourhood U of the origin s.t. $X = SX + RX$ on U and the following holds:
(1) SX is in standart form w.r.t. (x^c, y^s, z^u)
(2) RX is zero up to any finite order along $W^c = E^c = \{y^s = z^u = 0\}$
(3) RX is tangent to $W^{cu} = E^{cu} = \{y^s = 0\}$

Proving Proposition 1 requires essentially an application of the centermanifold theorem, as done the proof of Proposition 1 for Diffeomorphisms.

7.6. Proposition 2. There exist coordinates (x^c, y^s, z^u) and vectorfields SX and RX and a neighbourhood U of the origin s.t. $X = SX + RX$ on U and the following holds:
(1) SX is in standart form w.r.t. (x^c, y^s, z^u)
(2) RX is zero up to any finite order along $W^{cu} = E^{cu} = \{y^s = 0\}$

In the proof of Proposition 2 for Diffeomorphisms, we modified Φ (outside a neighbourhood of the origin) and chose an Euclidean metric ρ on $\mathbb{R}^n$ such that certain inequalities were satisfied. In order to prove the above proposition for vector fields, modify X (outside a neighbourhood of the origin) such that the time-t-map for the flow of X has with respect to ρ the same properties as the modified Φ. That's why we can "linearize X along $W^{cu} = E^{cu} = \{y^s = 0\}$ in the z^u directions. Takens concludes the proof by noting that "linearizing in the y^s directions can be done by a procedure completely analogous to the linearization in the z^u directions in $y^s = 0$" ([**53**], p.145). This argument is similar to the proof of Proposition 3 for Diffeomorphisms.

8. Concluding Remarks

The aim of this chapter was to explain key parts of the proof of Takens Linearization Theorem, especially to understand the "Sternberg Non-Resonance Conditions" and the the form of $\alpha(k)$ and $\beta(k)$ that are the basis of our own results in the later chapters.

However, we did not intend to fill all the gaps and present a complete proof in all details. In order to do so, a lot more effort and a deep insight into global analysis on manifolds is necessary. One good reference is the book [**6**], which also includes an extensive presentation of the proof for a linearization theorem in the vicinity of arbitrary compact invariant manifolds that is a generalization of Takens Linearization Theorem (compare [**6**],p.342). As it is stated in the introduction to [**6**], surprisingly enough, the proofs for a rest point are not much easier as compared with the general case.

CHAPTER 4

C^1- Stable - Manifolds for Periodic Heteroclinic Chains in Bianchi IX

In this chapter, we prove the following Theorem, which is new and can be seen as the main result of this dissertation:

THEOREM 4.1. *There exist C^1 - stable - manifolds of co-dimension one for admissible periodic heteroclinic chains in vacuum Bianchi IX.*

Here co-dimension one means the stable manifolds are of dimension three, as the phase space for vacuum Bianchi IX is four dimensional, see chapter 2. We will give a detailed description of admissible periodic chains in section 3 of this chapter, and are able to prove for example:

THEOREM 4.2. *Let $u = [a, b, a, b, ...]$ be an (infinite) periodic continued fraction development with minimal period two s.t. $a, b > 1$ and neither $a \mid b$ nor $b \mid a$.*

Then there exists a three dimensional C^1- stable - manifold of initial conditions such that the corresponding vacuum Bianchi IX - solutions converge to the periodic heteroclinic chain generated by u towards the big bang.

This means e.g. that the Hausdorff distance between the heteroclinic orbits that are part of the chain and the respective piece of the Bianchi IX - solution tends to zero as $t \to -\infty$.

The main point of the proof of Theorems 4.1 and 4.2 is to check the applicability of the Takens Linearization Theorem for periodic heteroclinic chains in Bianchi IX. This is the aim of sections (1)-(3) of this chapter.

We will then obtain the desired stable manifold via a graph transform for the global return map, which possesses a hyperbolic structure. This is done in section 4, very similar to the construction by Francois Béguin in [**2**].

When combining our results with those by Béguin ([**2**]), the limit of the analysis presented here can be formulated as follows, again for vaccum Bianchi IX:

THEOREM 4.3. *(C^1 - Stable - Manifolds for Admissible Chains Keeping a Distance from Taub Points)*

Let B^T_ϵ be the set of base points of the Kasner circle that satisfy the Non-Resonsonance-Conditions in order to allow for Takens Lineraization and keep a minimum distance of ϵ from the Taub Points. Then $\forall \epsilon > 0, \forall p$ s.t. $\overline{\{f^n(p)\}} \subseteq B^T_\epsilon$, there exist C^1 - stable - manifolds $W^s(p)$ with the following properties:

- *$W^s(p)$ is of co-dimension one and depends continiously on the base point p in the C^1 - topology*
- *for any point $r \in W^s(p)$, there is an increasing sequence of times $(t_n)_{n\geq 0}$ such that the Hausdorff distance between the piece of the Bianchi IX - orbit starting from r for the time-intervall $[t_n, t_{n+1}]$ and the heteroclinic orbit $H_{f^n(p),f^{n+1}(p)}$ tends to 0 for $n \to \infty$ (where f stands for the Kasner map)*

Now we will sketch how the proof of Theorem 4.1 will work. That's why we introduce the following definition:

DEFINITION 4.4. *(C^1-Hyperbolic Structure) Let $\Phi : M \to M$ be a C^1-map on a smooth manifold M. A hyperbolic structure of Φ on a compact, Φ-invariant subset $C \subset M$ is a familiy of subspaces $\{X_p, Y_p\}$ with $p \in C$ and $X_p, Y_p \subset T_pM$ s.t. $\forall p \in C\ \exists 0 < \lambda < 1$:*

1. *Splitting: $T_pM = X_p \oplus Y_p$, with continuous dependence of X_p and Y_p on the base point p*
2. *Invariance: $d_p\Phi(X_p) \subset X_{\Phi(p)}$ and $d_p\Phi(Y_p) \subset Y_{\Phi(p)}$*
3. *Contraction/Expansion:*
   - *$|d_p\Phi(v)| < \lambda|v|$ for all $v \in X_p$*
   - *$|d_p\Phi(w)| > \frac{1}{\lambda}|w|$ for all $w \in Y_p$*

Note that we use the following notation for differentials on manifolds: If $f : M \to N$ is a map between two manifolds, then we denote by $d_pf : T_pM \to T_{f(p)}N$ the differential at point p between the respective tangent spaces.

PROOF. (Sketch of Proof for Theorem 4.1)
The idea of the proof is to use Takens Linearization Theorem to linearize the vectorfield near the base-points of an admissible periodic heteroclinic chain. For this it is necessary to check in detail which resonances must be excluded in order to be able to apply the Takens Theorem.

In order to prove Theorem 4.1, it is strictly speaking sufficient to find one example of an admissible infinite periodic heteroclinic chain that satisfies the necessary conditions to apply the Takens-Linearization

Theorem at each of its base points. Such examples are u=[7,5,7,5,...] or u=[3,3,2,3,3,2,...], as can be checked directly (see Appendix 1) , but we also give a detailed combinatorial description of admissible periodic chains in section 3.

After this linearization is done, the result follows from combining the linzearized local passage with the global passage. Applying a graph-transform (see e.g. [**36**]), we then obtain the stable manifold. Our approach can be seen as a combination of the techniques used by Liebscher et al and Béguin [**27, 2**]. Before we come to the detailed proof, let us sketch the main idea here.

We have a choice of sections when we decompose the global return map into smaller pieces. We use the following decomposition:

- Local Passage (near the Kasner circle): $\Phi^{loc} : \Sigma^{in} \to \Sigma^{out}$
- Global Passsage (near the heteroclinic orbit): $\Phi^{glob} : \Sigma^{out} \to \Sigma^{in}$

We then consider the following global return map:

$$\Phi^{return} = \Phi^{glob} \circ \Phi^{loc} : \Sigma^{in} \to \Sigma^{in}$$

The hyperbolic structure necessary for the graph transform looks as follows: The expansion is given by the Kasner map, which is an expanding map acting on the Kasner circle (which is the center direction of the local passage map) away from the Taub points. The contraction is given by the hyperbolic directions of the local passage map, which is a short explicit calculation after the linearization has been done. Applying the graph transfrom then yields a fixed point in the space of graphs, which is the desired stable manifold for the heteroclinic chain. As we worked in a C^1-setting, i.e. the Takens-linearization and all other involved maps are at least C^1, the resulting hyperbolic structure and thus also stable manifold is as well of regularity C^1, which proves the theorem.

□

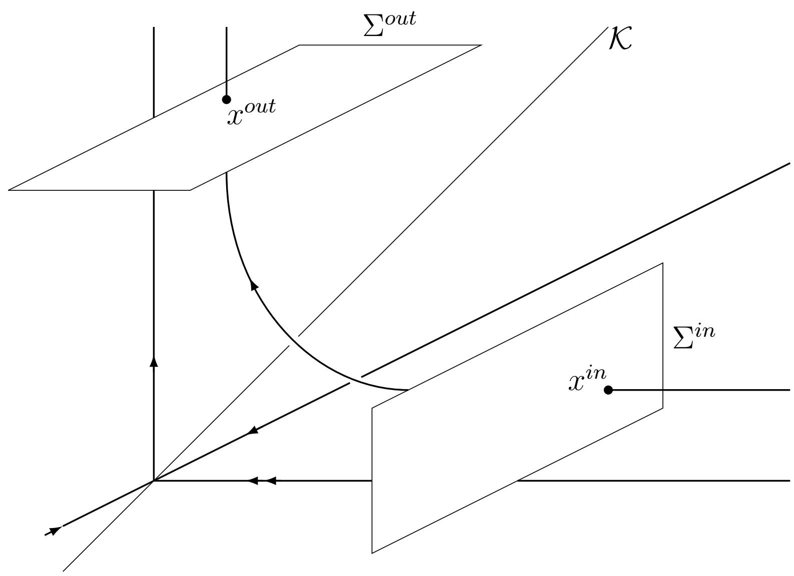


FIGURE 1. Local passage Φ^{loc}.

Figure 1 illustrates the local passage near the Kasner circle. We see that we come in via the direction of the strong-stable eigenvalue (indicated by the double arrow towards the equilibrium), while there is an additional stable direction (single arrow towards the equilibrium) as well as the outgoing direction (single arrow away from the equilibrium) and the center direction (i.e. the Kasner circle $\mathcal{K}$ of equliibria, indicated by the additional line without arrows).

1. Resonances for Periodic Chains in Bianchi IX

We are interested in periodic heteroclinic chains in the Bianchi IX cosmological model. These can be represented by a Kasner parameter $u \in \mathbb{R}$ with an infinite periodic continued fraction representation, e.g. $u = [a, b, c, a, b, c, ...]$.

1.1. Infinite Periodic Continued Fractions.

From the theory of continued fractions, we know that it holds:

THEOREM. *$u \in \mathbb{R}$ has an infinite periodic continued fraction representation $\iff$ $u \in \mathbb{R}$ is a "quadratic irrational" $\iff$ u is a real but irrational root of a quadratic equation with integer coefficients, i.e. $\exists : c_1, c_2, c_3 : c_1 + c_2 u + c_3 u^2 = 0$ (with $c_i \in \mathbb{Z}$).*

Here we are only interested in the direction "$\Rightarrow$", which follows directly for the general formulas for continued fractions in section 3.3, see below. The other direction is a bit more elaborate (see e.g. **[37]** §19 or **[25]** §10).

For the argument we will carry out later, it is of crucial importance that, up to a common scaling factor $z \in \mathbb{Z}$, there is exactly one quadratic equation satisfied by a quadratic irrational u.

As this is very important when considering the resonances of the eigenvalues in Bianchi models, we include a proof of this fact here (and we assume that the c_i do not have a common factor because we will be interested in the smallest possible coefficients, where this is clear, see section 1.5):

LEMMA 4.5. *For a (fixed) quadratic irrational u, let $c_i \in \mathbb{Z}, i = 1...3$ be s.t. $c_1 + c_2 u + c_3 u^2 = 0$ and $\gcd(c_i) = 1$, i.e. the c_i do not have a common factor. Now assume that $d_1 + d_2 u + d_3 u^2 = 0$ also holds with $d_i \in \mathbb{Z}$. Then it follows that*

$$\exists z : d_i = z * c_i, \text{ for } (i = 1...3) \text{ with } z \in \mathbb{Z}$$

PROOF. Multiplying the equation with coefficients c_i with d_1 and the other one with c_1 results in the following two equations:

$$\begin{aligned} d_1 c_1 + d_1 c_2 u + d_1 c_3 u^2 &= 0 \\ c_1 d_1 + c_1 d_2 u + c_1 d_3 u^2 &= 0 \end{aligned}$$

Subtracting the second from the first equation leads to

$$(28) \qquad u(d_1c_2 - c_1d_2 + (d_1c_3 - c_1d_3)u) \;=\; 0$$

and, as $u \neq 0$, we conclude that

$$u = \frac{c_1d_2 - d_1c_2}{d_1c_3 - c_1d_3}$$

if $d_1c_3 - c_1d_3 \neq 0$, which leads to a contradiction because $u \notin \mathbb{Q}$ was assumed.

If, on the other hand, $d_1c_3 - c_1d_3 = 0$, it follows from (28) that also $d_1c_2 - c_1d_2 = 0$, which leads to the conclusion that $\frac{d_1}{c_1} = \frac{d_2}{c_2} = \frac{d_3}{c_3} := z$ with $z \in \mathbb{Z}$. Note that $z \in \mathbb{Q}$ would lead to a contradiction because we assumed that the c_i do not have a common factor. □

1.2. The Case of Bianchi IX. In order to check the (SNC) for the linearized vectorfield at a point on the Kasner circle, observe that $DX(p)$ is diagonal and that there are three hyperbolic eigenvalues for all points of the Kasner circle except for the Taub points.

In terms of the Kasner parameter u, the following formulas hold for those three eigenvalues (see section 2):

$$(29) \qquad (\lambda_1, \lambda_2, \lambda_3) = \left(\frac{-6u}{1+u+u^2}, \frac{6(1+u)}{1+u+u^2}, \frac{6u(1+u)}{1+u+u^2} \right)$$

All 3 hyperbolic eigenvalues are real. A resonance thus means in this case: $\exists k = (k_1, k_2, k_3), k_i \in \mathbb{Z}$ s.t.

$$(30) \qquad k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 = 0$$

where either all of the k_i must have the same sign, or the one of the k_i that has a different sign than the other two must be equal to ± 1. Because this "sign condition" will play an important role later on, let us make the following definition:

Definition. *A triple $k = (k_1, k_2, k_3), k_i \in \mathbb{Z}$ satisfies the Resonance Sign Condition (RSC) $\iff$ either all of the k_i must have the same sign, or the one of the k_i that has a different sign than the other two must be equal to ± 1*

Only if a triple fulfils the RSC, it qualifies as a coefficient-triple for a resonance that prevents the application of Takens Linearization Theorem. This means that if we can show that resonant coefficients do not fulfill the RSC, they do not matter and Takens-Linearization is still possible. Note that a simple way of showing that the RSC is not satisfied is to show that one coefficient is strictly bigger than one, while a different one is strictly less than minus one.

1.3. SNC for Infinite Periodic Heteroclinic Chains. In preparation for further generalizations to Bianchi-models of class B, this section is formulated a bit more general that it would be necessary for discussing only the case of Bianchi IX. As seen above, the eigenvalues of the linearized vectorfield in BIX for points of the Kasner circle can be expressed in the Kasner parameter u:

$$\lambda_i = \frac{l^i_1 + l^i_2 u + l^i_3 u^2}{1 + u + u^2} \tag{31}$$

Combining (30) and (31), one gets

$$k_1(l^1_1 + l^1_2 u + l^1_3 u^2) + k_2(l^2_1 + l^2_2 u + l^2_3 u^2) + k_3(l^3_1 + l^3_2 u + l^3_3 u^2) = 0 \tag{32}$$

or, equivalently,

$$(k_1 l^1_1 + k_2 l^2_1 + k_3 l^3_1) + (k_1 l^1_2 + k_2 l^2_2 + k_3 l^3_2)u + (k_1 l^1_3 + k_2 l^2_3 + k_3 l^3_3)u^2 = 0 \tag{33}$$

As discussed above, for infinite periodic heteroclinic chains, there are (up to a common scaling factor) unique coefficients $c_i \in \mathbb{Z}$ s.t.

$$c_1 + c_2 u + c_3 u^2 = 0 \tag{34}$$

Comparing (32) to (33), one sees that (SNC) does not hold if $\exists k = (k_1, k_2, k_3)$ as above and $z \in \mathbb{Z}$ s.t.

$$M * \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = z * \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{35}$$

with

$$M = \begin{pmatrix} l^1_1 & l^2_1 & l^3_1 \\ l^1_2 & l^2_2 & l^3_2 \\ l^1_3 & l^2_3 & l^3_3 \end{pmatrix}$$

where we will solve (35) for (k_1, k_2, k_3) in order to check the order of the first resonance.

1.4. Conclusions for Bianchi IX.

It can be seen easily that the formulas (37) imply that for Bianchi IX we have

$$M_{BIX} = \begin{pmatrix} 0 & 6 & 0 \\ -6 & 6 & 6 \\ 0 & 0 & 6 \end{pmatrix} = 6 * \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$M_{BIX}^{-1} = \frac{1}{6} * \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Observe that we have a choice of the factor z on the right hand side of (35), and that a choice of $z = 6$ will result in an integer resonance with the smallest possible order:

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \frac{1}{6} * \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} * 6 * \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 - c_2 + c_3 \\ c_1 \\ c_3 \end{pmatrix} \tag{36}$$

If the entries of the vector on the right hand side of (36) do not have a common factor, then the first resonance will occur at order $l := |k_1| + |k_2| + |k_3| = |c_1 - c_2 + c_3| + |c_1| + |c_3|$

1.5. Uniqueness of the Resonance.

For the argument we will carry out later, it is of crucial importance that we find the order l of the **first** resonance, meaning that we can exclude all resonances with order $\tilde{l} < l$.

In order to do this, we will need the Lemma 4.5 on the uniquness of the coefficients for the quadratic equation for quadratic irrationals.

We claim that if we choose the smallest possible coefficients c_i for the equation in u (meaning that the c_i do not have a common factor), this will lead to the smallest resonance $l := |k_1| + |k_2| + |k_3|$.

This is true because of the linear dependence of the k_i on the c_i in (36), meaning that we can exclude all resonances with order $\tilde{l} < l$.

2. Continued Fraction Expansion for Quadratic Irrationals

We will use the following notation for continued fractions:

$$u = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots}}} =: [a_0, a_1, a_2, ...]$$

In this section, we will consider 3 classes of examples, namely $u \in \mathbb{R}$ with constant, 2-periodic and 3-periodic continued fraction expansions, i.e either $u = [a, a, ...]$ or $u = [a, b, a, b, ...]$ or $u = [a, b, c, a, b, c, ...]$ for $a, b, c \in \mathbb{N}$. We also recall from section 2 that the Kasner map has the following form:

$$u = \begin{cases} u - 1 & u \in [2, \infty] \\ \frac{1}{u-1} & u \in [1, 2] \end{cases}$$

2.1. Constant Continued fraction. Because of the form of the Kasner-map, starting with $u = [a, a, ...]$ will result in the following base-points on the Kasner-circle:

$$\begin{aligned} u_0 &= [a, a, a, ...] \\ u_1 &= [a-1, a, a, ...] \\ u_2 &= [a-2, a, a, ...] \\ &\dots \\ u_{a-1} &= [1, a, a, ...] \\ u_a &= [a, a, a, ...] \\ &\dots \end{aligned}$$

That's why we have to check the Non-Resonance-Conditions at all points with $u = [m, a, a, ...]$ for $m = 1...a$. Now note that for $u = [m, a, a, ...]$ it holds that

$$\frac{1}{u - m} - a = u - m$$

which means that

$$(m^2 - am - 1) + (a - 2m)u + u^2 = 0$$

resulting in a coefficient vector

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} m^2 - am - 1 \\ a - 2m \\ 1 \end{pmatrix}$$

Now we can use equation (36) to compute the coefficients for the resonance of the eigenvectors (we set $s = -1$ in order to match the condition (27)):

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = -1 * \begin{pmatrix} c_1 - c_2 + c_3 \\ c_1 \\ c_3 \end{pmatrix} = \begin{pmatrix} -m^2 + (a-2)m + a \\ -m^2 + am + 1 \\ -1 \end{pmatrix}$$

2.2. 2-Periodic Continued Fraction Expansion. For $u = [a, b, a, b, ...]$, we have to check the base-points with $u = [m, b, a, b, a, ...]$ with $m = 1...a$ and $u = [m, a, b, a, b, ...]$ for $m = 1...b$. Applying the same procedure as above, we note that that u satisfies

$$\frac{1}{\frac{1}{u-m} - a} - b = u - m \ \ \& \ \ \frac{1}{\frac{1}{u-m} - b} - a = u - m$$

when $u = [m, a, b, a, b, ...]$ and $u = [m, b, a, b, a, ...]$, respectively, and get the following coefficient vectors for u:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -am^2 + abm + b \\ 2am - ab \\ -a \end{pmatrix} \ \& \ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -bm^2 + abm + a \\ 2bm - ab \\ -b \end{pmatrix}$$

resulting in these coefficient vectors for the eigenvalues (we set $s = 1$ this time):

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -am^2 + (ab - 2a)m + ab - a + b \\ -am^2 + abm + b \\ -a \end{pmatrix}$$

and

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -bm^2 + (ab - 2b)m + ab + a - b \\ -bm^2 + abm + a \\ -b \end{pmatrix}$$

2.3. 3-Periodic Continued Fraction Expansion. In complete analogy to the computations above, we find the following formulas, for the 3 relevant cases. Note that we show the coefficient vectors for u below, and in all three cases we have to compute the coefficient vectors for the eigenvalues as done before:

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} c_1 - c_2 + c_3 \\ c_1 \\ c_3 \end{pmatrix}$$

u=[m,b,c,a,...] for m=1...a.

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} m^2 + mc + m^2bc - bm - am - ac - abcm - 1 \\ abc + a + b - c - 2m - 2mbc \\ 1 + bc \end{pmatrix}$$

u=[m,c,a,b,...] for m=1...b.

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} m^2 + ma + m^2ca - cm - bm - ba - abcm - 1 \\ abc + b + c - a - 2m - 2mca \\ 1 + ca \end{pmatrix}$$

u=[m,a,b,c,...] for m=1...c.

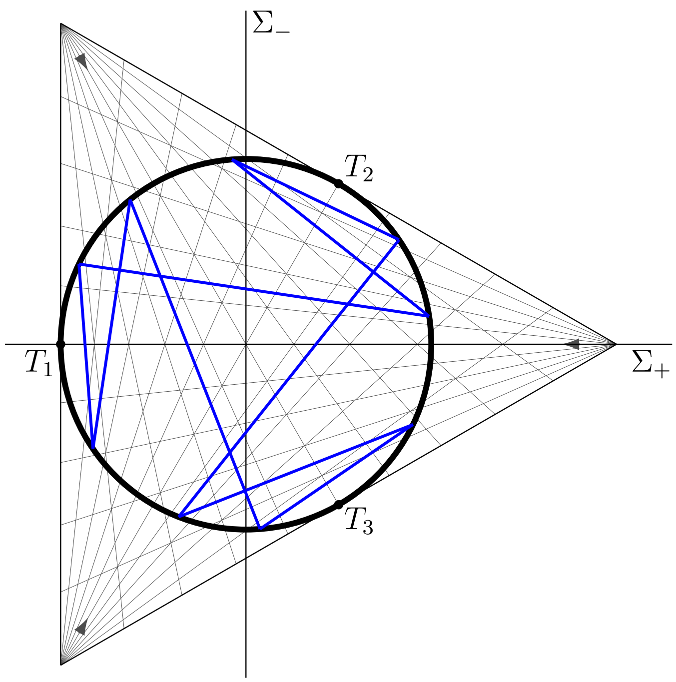
$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} m^2 + mb + m^2ab - am - cm - cb - abcm - 1 \\ abc + c + a - b - 2m - 2mab \\ 1 + ab \end{pmatrix}$$

3. Results on Admissibility of Periodic Heteroclinic Chains in Bianchi IX

In this section, we will concretely check the Sternberg Resonance Conditions for periodic heteroclinic chains in BIX.

We will prove some general theorems, while more concrete examples can be found in the Appendix A.

3.1. Constant Continued Fraction Development. We will give a proof of the fact that there are no infinite periodic heteroclinic chains with constant continued fraction development that allow Takens - Linearization at their base points. More geometrically, this excludes "symmetric" heteroclinic chains with the same number of "bounces" near all of the 3 Taub Points - the result shows that we have to require some "asymmetry" in the bounces in order to allow for Takens-Linearization. Below, there is an illustration of the heteroclinic chain belonging to $u = [3, 3, ...]$ which does not allow for Takens-Linearization:



Theorem 4.6. *For any heteroclinic chain with constant continued fraction development, Takens-Linearization fails at some base point.*

Proof. As we have seen above, a periodic heteroclinic chain has a periodic continued fraction development, leading to a resonance, and let us call the coefficients for that resonance $k = (k_1, k_2, k_3)$. The first thing we have to check is if k satisfies the Resonance Sign Condition (RSC) defined above.

LEMMA. *For constant continued fraction development,* $(u = [a, a, ...])$, *the coefficient vector* $k = (k_1, k_2, k_3)$ *satisfies the Resonance Sign Condition (RSC) at all base points.*

PROOF. To prove the Lemma, we observe the following when looking at the formulas for constant continued fraction development in section 2.1:

- for $m = a$, it holds that $k = (1, a, -1)$
- for $m = a - 1$, $k = (-a, 1, -1)$
- for $1 \geq m < a-1$ and $k = (k_1, k_2, k_3)$, it holds that $k_1, k_2 > 0$, while $k_3 = -1$

Thus, the RSC are satisfied in all cases, and the coefficient vector would qualify.

□

To prove Theorem 4.6, we have to compare two things:

- the order of the resonance of the eigenvalues at the basepoints, expressed first in the Kasner-parameter $(u = [a, a, ...])$ and then directly in a
- the required SNC for C^1-stable-manifolds, i.e. $\alpha(1)$ at all base points

The base points of a infinite periodic heteroclinic chain with $u = [a, a, ...]$ are $u = [m, a, ...]$ for $m = 1...a$. To prove the Theorem, it is enough to show the violation of the Sternberg Non-Resonance Conditions at one base point. Consider the case $m = a - 1$ and start with the formulas for the coefficient vectors, as computed above:

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -m^2 + (a-2)m + a \\ -m^2 + am + 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ a \\ -1 \end{pmatrix}$$

Therefor, it holds that $|k| = a + 2$, i.e. we have linear growth of $|k|$ in a.

On the other hand, re-consider the formulas for the eigenvalues in BIX:

$$(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{-6u}{1+u+u^2}, \frac{6(1+u)}{1+u+u^2}, \frac{6u(1+u)}{1+u+u^2} \right) \tag{37}$$

and order them according to magnitude (with the notation from the SNC's from the Takens-Theorem):

$$\begin{pmatrix} N \\ n \\ m = M \end{pmatrix} = \begin{pmatrix} |\lambda_1| \\ |\lambda_2| \\ |\lambda_3| \end{pmatrix}$$

Insert in the formulas for α, β and compute:

$$\beta = Ceiling[\frac{N + k(M+n)}{n}] \geq \frac{u^2 + 3u + 1}{u + 1}$$

$$\alpha = Ceiling[\frac{M + \beta(N+m)}{m}] \geq \frac{u^3 + 5u^2 + 8u + 3}{u + 1}$$

This shows quadratic growth for α in u. In fact, for $u = [a - 1, a, ...] = a - 1 + \frac{1}{a + \frac{1}{a + \frac{1}{...}}}$, it holds $\forall a > 0 : |k| < \alpha(1)$, i.e. the SNCs are violated and Takens-Linearization is not possible, which proves the Theorem. For consistency, also compare to Appendix A, where we used Mathematica to compute $\alpha(1)$ and $|k|$ for $u = [m, a, ...]$ for $m = 1...a$ and $a = 1...9$.

□

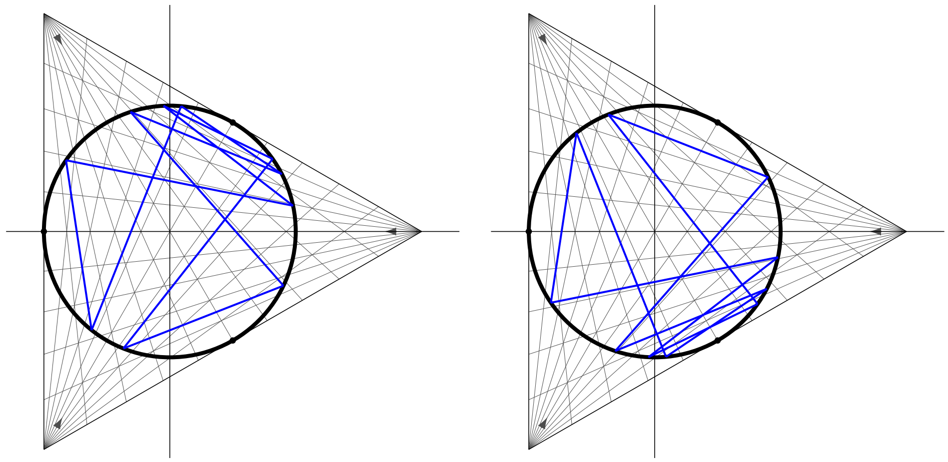
3.2. 2-Periodic Continued Fraction Development. In this section, we will prove the following Theorem:

THEOREM 4.7. *For admissible heteroclinic chains with 2-periodic continued fraction development, Takens Linearization is possible at all base points.*

Here, admissible means that the continued fraction developments has minimal period 2 and the entries are strictly bigger than one (even after cancelling out a possible common factor). To be precise, we define an admissible 2-periodic continued fraction development as follows:

DEFINITION 4.8. *A 2-periodic $u = [a, b, a, b, ...]$ is called admissible $\iff$ $a, b > 1$ and neither $a \mid b$ nor $b \mid a$.*

Note that from the condition above, it follows in particular that $a \neq b$, beeing consistent with the results in the section above about constant contiuned fractions. Two examples of such a heteroclinic chains are illustrated below, with u=[3,2,3,2,...] and with u=[2,3,2,3,...], which are 10-cycles (also compare Appendix 1.2):



PROOF. The Theorem will directly follow from the following Lemma:

LEMMA 4.9. *For admissible 2-periodic continued fraction developments, the coefficient vector $k = (k_1, k_2, k_3)$ violates the Resonance Sign Condition (RSC) at all base points*

PROOF. When we look at the formulas for 2-periodic continued fraction development in section 2.2, we can observe the following:

- for $u = [m, a, b, a, b, ...]$ and $m = 1...b$, it holds that $k_3 = -a < -1$ and $k_2 \geq b > 1$ as $bm \geq m^2$
- for $u = [m, b, a, b, a, ...]$ and $m = 1...a$, it holds that $k_3 = -b < -1$ and $k_2 \geq a > 1$ as $am \geq m^2$

This means that the RSC are violated at all base points of the heteroclinic chain, and the lemma is proven. Note that we need $a, b > 1$, and that if we had $a \mid b$ or $b \mid a$, then coefficients k_1, k_2, k_3 would have a common factor we could cancel, leading to an earlier resonance. That's why we need to restrict to admissible 2-periodic continued fraction developments as defined above.

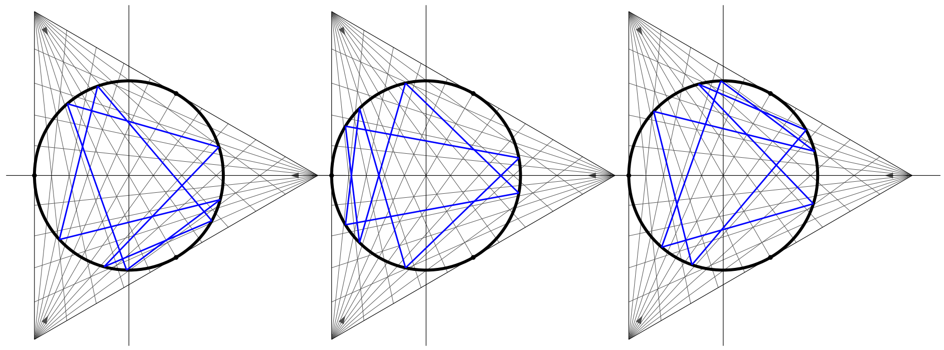
□

The Lemma shows that, for "sign reasons", the occuring resonaces are excluded and do not matter for the application of the Takens Theorem. Therefor Takens Linerarization is possible, as claimed in Theorem 4.7.

□

3.3. Continued Fraction Development with Higher Periods. The idea behind the proof of Lemma 4.9 can be generalized to continued fraction developments with higher periods. However, it is not so easy anymore to find conditions that assure in general that the resulting coefficients do not have a common factor. We will comment on this matter further at the end of the section.

At first, consider 3-periodic continued fractions. Three examples of such a heteroclinic chains are illustrated below, with u=[1,1,2,1,1,2,...], u=[1,2,1,1,2,1,...] and u=[2,1,1,2,1,1,...] which are 8-cycles and arguably the simplest examples of periodic heteroclinic chains where our method works (this can be checked directly for the concrete examples above, see Appendix 1.3). They all start in sector 5, and the different position of the number "2" in the contiued fraction development leads to bounces around the different Taub points which can be seen in the pictures below:



LEMMA 4.10. *Consider a continued fraction development with minimal period 3, i.e. with* $u = [a, b, c, a, b, c, ...]$ *and not* $a = b = c$*. Then the corresponding coefficient vector* $k = (k_1, k_2, k_3)$ *violates the Resonance Sign Condition (RSC) at all base points if the* k_i *do not have a common factor .*

PROOF. When we look at the formulas for 3-periodic continued fraction development in section 2.3, we can observe the following:

- for $u = [m, b, c, a, b, c, a, ...]$ and $m = 1...a$, it holds that $k_2 = c_1 \leq -bm - 1 < -1$ and $k_3 = c_3 = 1 + bc > 1$
- for $u = [m, c, a, b, c, a, b, ...]$ and $m = 1...b$, it holds that $k_2 = c_1 \leq -cm - 1 < -1$ and $k_3 = c_3 = 1 + ca > 1$
- for $u = [m, a, b, c, a, b, c, ...]$ and $m = 1...c$, it holds that $k_2 = c_1 \leq -am - 1 < -1$ and $k_3 = c_3 = 1 + ab > 1$

This means that the RSC are violated at all base points of the heteroclinic chain if we know that neither $k_2 \mid k_3$ nor $k_3 \mid k_2$. This is true in particular if the k_i do not have a common factor as we have assumed for convenience, so Lemma 4.10 is proven.

Note that if we had $a = b = c$, then coefficients k_1, k_2, k_3 would have a common factor, resulting in an earlier resonance as explained above. Also compare to Appendix 1.3 for a consistency check. □

We now try to generalize the argument above to higher periodic continued fractions. In order to do this let us make some general definitions and observations (following [**37**] §19[1], compare also [**25**] §10):

For continued fractions of the form

$$u = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots}}} =: [a_0, a_1, a_2, ...]$$

we define the following numbers A_k, B_k recursively:

$$A_k = A_{k-1}a_k + A_{k-2}$$
$$B_k = B_{k-1}a_k + B_{k-2}$$

with $A_{-1} = 1, A_{-2} = 0$ and $B_{-1} = 0, B_{-2} = 1$, leading to $A_0 = a_1$, $A_1 = a_0a_1 + 1$ and $B_0 = 1$, $B_1 = a_1$.

For an (infinite) continued fraction, we define the "tails" as follows:

$$\xi_k := [a_k, a_{k+1}, ...]$$

Then we have the following general recursion formula for convergent infinite continued fractions $u = [a_0, a_1, ..., a_{k-1}, \xi_k]$ (and $k \geq 0$):

$$u = \xi_0 = \frac{A_{k-1}\xi_k + A_{k-2}}{B_{k-1}\xi_k + B_{k-2}}$$

, which can be proved by induction.[2]. Also compare [**25**] §2 and §3.

Now consider pre-periodic continued fractions with pre-period h and minimal period p, as made precise in the following definition:

Definition. *We call u an h-pre-periodic continued fraction with pre-period h, minimal period p* $\iff u = [a_0, ...a_{h-1}, \overline{a_h, a_{h+1}, ..., a_{h+p-1}}]$ *with* $a_\nu = a_{\nu+p} \forall \nu \geq h$ *and* $\nexists \tilde{p} < p$ *s.t.* $a_\nu = a_{\nu+\tilde{p}} \forall \nu \geq h$

Note that it also holds that $\xi_\nu = \xi_{\nu+p} \forall \nu \geq h$. Thus we can get the following formulas (set $k = h$ and $k = h + p$):

---

[1]but note we have a different labelling of the coefficients as we do not consider continued fractions with enumerators different from one

[2]For $k = 0$, the formula holds by definition: $\xi_0 = \frac{A_{-1}\xi_0 + A_{-2}}{B_{-1}\xi_0 + B_{-2}} = \frac{\xi_0}{1}$.

$$\xi_0 = \frac{A_{h-1}\xi_h + A_{h-2}}{B_{h-1}\xi_h + B_{h-2}}$$

and

$$\xi_0 = \frac{A_{h+p-1}\xi_{h+p} + A_{h+p-2}}{B_{h+p-1}\xi_{h+p} + B_{h+p-2}} = \frac{A_{h+p-1}\xi_h + A_{h+p-2}}{B_{h+p-1}\xi_h + B_{k+p-2}}$$

By solving both equations for ξ_h, we get the following quadratic equation for ξ_0 :

$$c_3\xi_0^2 + c_2\xi_0 + c_1 = 0$$

with (we abbreviate $g = h + p$)

$$\begin{aligned} c_3 &= B_{h-2}B_{g-1} - B_{h-1}B_{g-2} \\ c_2 &= B_{h-1}A_{g-2} + A_{h-1}B_{g-2} - A_{h-2}B_{g-1} - B_{h-2}A_{g-1} \\ c_1 &= A_{h-2}A_{g-1} - A_{h-1}A_{g-2} \end{aligned}$$

The formulas above specialize to (for $h = 0$, this corresponds to the formula for periodic continued fractions without pre-period)

$$\begin{aligned} c_3 &= B_{p-1} \\ c_2 &= B_{p-2} - A_{p-1} \\ c_1 &= -A_{p-2} \end{aligned}$$

and for $h = 1$ to[3]

$$\begin{aligned} c_3 &= -B_{p-1} \\ c_2 &= A_{p-1} + a_0B_{p-1} - B_p \\ c_1 &= A_p - a_0A_{p-1} \end{aligned}$$

Now we are in a position to state the main aim of this section:

CONJECTURE 4.11. *Let $u = [a_0, a_1, ...]$ be an (infinite) periodic continued fraction with minimal period $p \geq 3$. Then the corresponding heteroclinic chain allows Takens-Linearization at all base points.*

PROOF. (idea of proof, but note the remark below)
Let $u = [a_0, a_1, ...]$ be an (infinite) periodic continued fraction. We need to show that the NRC's are satisfied at all base points of the heteroclinic chain. Because of the form of the Kasner-map, we have to check all

[3]compare to the formulas for p=1,2,3 presented in section 2.3, as a consistency check

Kasner-parameters of the form $u = u_m = [m, \overline{a_1, a_2, ..., a_{p-1}, a_p}]$, i.e. it holds that $a_0 = m$ (with $1 \leq m \leq a_p$) and $a_\nu = a_{\nu+p}$, but now only $\forall \nu \geq 1$. From the formulas above (case $h = 1$) we observe the following for the corresponding coefficients of the resonances of the eigenvalues:

$$k_3 = c_3 = -B_{p-1} < -1$$

where we need our assumption that $p \geq 3$ as $B_1 = a_1$ which might be one, but $B_2 = a_2 a_1 + 1$ which is bigger than one. Also

$$k_2 = c_1 = A_p - a_0 A_{p-1} = (a_p - a_0) A_{p-1} + A_{p-2} > 1$$

because we know that $a_0 = m \leq a_p$ and $A_1 = a_0 a_1 + 1$ is bigger than one. That's why the "Resonance Sign Condition" is violated at all base points, and Takens-Linearization is possible. □

The reason why we don't call the Conjecture above a Theorem is that we are not able to exclude in general that c_1 divides c_3 or vice versa, which is essential for the proof above to work out. We believe it is possible to prove this in general for most periodic continued fraction with minimal period $p \geq 3$, probably with a small set of exceptions, but this is an issue for further research.

4. Details on the Proof for Stable Manifolds

In this section, we complete the proof of Theorem 4.1 by showing that there is a C^1-hyperbolic structure for the return map in Bianchi IX after linearizing at all base points of a heteroclinic chain. This then leads to a C^1-stable manifold, as claimed.

We procced along the lines and very close to the paper of Béguin [**2**], but we adapt the notation to our needs and the situation of a periodic chain that Béguin does not consider.

Also compare to the papers by Liebscher et al. [**27, 28**], where they work in a Lipschitz-setting without linearizing at the Kasner circle. There, the following return maps are considered

$$\Phi_k^{return} = \Phi_k^{glob} \circ \Phi_k^{loc} : \Sigma_k^{in} \to \Sigma_{k+1}^{in}$$

where the index k stands for the base points on the Kasner circle of the heteroclinic chain, i.e. Φ_k^{return} maps from one In-section to the next. It is shown that those maps satisfy the necessary cone conditions to allow for a graph-transform on Lipschitz-graphs on a subset of Σ^{in} including the origin (which stands for the heteroclinic orbit). This then leads to the stable manifold result.

However, like Béguin [**2**], we will use a collection Φ_B^{return} of these return maps for all base points of the set $B \subset \mathcal{K}$. We then show that there exists a C^1- hyperbolic structure for a suitable subset of the corresponding In-sections Σ_B^{in}. This results in a C^1-stable manifold.

4.1. Application of Takens Theorem. Let $B = \{p_1, ..., p_n\}$ be the collection of base points on the Kasner circle of the periodic heteroclinic chain we are looking at. Then, as we have chosen an admissible periodic chain that satisfies the necessary Non-Resonance-Conditions by assumption, we can chose co-ordinates near each point $p_k \in B$ such that the vector field has the form described by the Takens Theorem, i.e. it is essentially linear in a neighbourhood U_{p_k}. More precisely, the application of Takens Linearization Theorem is done in the following form (compare Béguin, p.10):

THEOREM 4.12. *Let $p \in B$ be any point of the set of admissible base points B. Then there exists a Takens-Neighbourhood U_p of p in the phase-space of the Wainwright-Hsu ODEs $\mathcal{W}$ and a C^1-coordinate-system on U_p such that the Wainwright-Hsu vector field X^W can be written as*

$$X^W(x^c, x^s, x^{ss}, x^u) = \lambda_s(x^c)x^s \frac{\partial}{\partial x^s} + \lambda_{ss}(x^c)x^{ss} \frac{\partial}{\partial x^{ss}} + \lambda_u(x^c)x^u \frac{\partial}{\partial x^u}$$

where $\lambda_{ss}(x^c) < \lambda_s(x^c) < 0 < \lambda_u(x^c)$ *for all* x^c.

PROOF. A direct application of the Takens-Theorem 7.1 (from chapter 3) gives the existence of a coordinate system $(x^c, x^{s1}, x^{s2}, x^u)$ in U_p s.t. X^W has the following form in these coordinates:

$$X^W(x^c, x^{s1}, x^{s2}, x^u) = \phi(x^c)\frac{\partial}{\partial x^c} + \sum_{i,j=1}^{2} a_{ij}(x^c)y^{si}\frac{\partial}{\partial y^{sj}} + b(x^c)x^u\frac{\partial}{\partial x^u}$$

For the vector field X^W in the original coordinates, the set $\mathcal{K}\cap U_p$ is the local center-manifold in the neighbourhood U_p at the point p, and it consists of equilibria. As the vector field above vanishes on $K = \{x^{s1} = x^{s2} = x^u = 0\}$ and nowhere else, it follows that $K = \mathcal{K} \cap U_p$. This also means that $\phi \equiv 0$ in the neighbourhood U_p, i.e. there is no drift at all in the center-direction. Now fix $\{x^c = \xi\}$. As can be seen from the formula above, the vector field $X^W(x^c, x^{s1}, x^{s2}, z^u)$ is linear on the restruction to this submanifold. A linear change of coordinates then diagonalizes the 2×2-matrix $(a_{ij})_{i,j\in\{1,2\}}$, as we have 2 distinct real stable eigenvalues of X^W at the point $(\xi, 0, 0, 0)$, and this diagonalization can be done simultaneously, as eigenvalues and eigendirections depend in a smooth way on ξ. Label these new coordinates (x^c, x^s, x^{ss}, x^u) and observe that we have found the claimed local form of the vector field

$$X^W(x^c, x^s, x^{ss}, x^u) = \lambda_s(x^c)x^s\frac{\partial}{\partial x^s} + \lambda_{ss}(x^c)x^{ss}\frac{\partial}{\partial x^{ss}} + \lambda_u(x^c)x^u\frac{\partial}{\partial x^u}$$

□

For the rest of the section, we will use the follwing coordinates: Near the Kasner-circle, we take the coordinates given by the Takens-Linearization-Theorem, at each base point p_k of the heteroclinic chain, and otherwise, we stick to the coordinates $N_i, \Sigma_{+/-}$ of the Wainwright-Hsu-System. The different coordinate systems give rise to the following metrics: the Riemanian metric $g_p = dx^c \wedge dx^c + dx^s \wedge dx^s + dx^{ss} \wedge dx^{ss} + dx^u \wedge dx^u = (dx^c)^2 + (dx^s)^2 + (dx^{ss})^2 + (dx^u)^2$ for the Takes-coordinates in a neighbourhood U_p near a point p of the Kasner circle, and the Riemanian metric $h = dN_1^2 + dN_2^2 + dN_3^2 + d\Sigma_+^2 + d\Sigma_-^2$. Later we use a "global" Riemannian metric adapted to our set of base points B by defining g_B such that

$$g_B \restriction U_p = g_p \forall p \in B \tag{38}$$

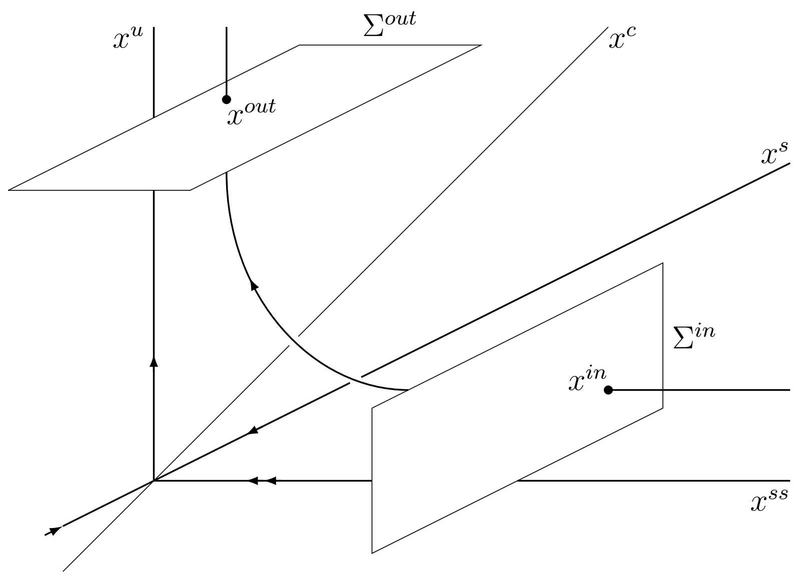


FIGURE 2. Local passage Φ^{loc}.

For the local passage, which we will consider next, we are entirely in the neighbourhood U_p and can use the "local" metric $g_p = (dx^c)^2 + (dx^s)^2 + (dx^{ss})^2 + (dx^u)^2$.

4.2. Local Passage. Our next step is to deal with the local passage near an equilibrium of the Kasner circle $\mathcal{K}$. Figure 2 shows a graphic illustration of the situation in Bianchi IX - note that we are in the lucky situation here that the incoming stable eigenvalue is always stronger that the outgoing unstable eigenvalue, this will change in $BVI_{-\frac{1}{9}}$ that we deal with in chapter 5.

Now we come to the definition of the local In- and Out-Sections illustrated in the picture above: For a point $p_k \in B$, we first define the box V_p as $V_p = V_p(\alpha, \beta, \epsilon) := \{q = (x^c_q, x^s_q, x^{ss}_q, x^u_q) \in U_p | 0 \leq x^s_q, x^{ss}_q, x^u_q \leq \epsilon, \alpha \leq x^c_q \leq \beta\}$ and α, β, ϵ are chosen so small the box V_p lies completely inside the Takens-neighbourhood U_p. Denote their union by $V_B = \bigcup_{k=1}^n V_{p_k}$.

Then define the sections by $\Sigma_k^{in,ss} := V_{p_k} \cap \{x^{ss} = \epsilon\}$ and $\Sigma_k^{out} := V_{p_k} \cap \{x^u = \epsilon\}$. Finally define the "collections" of sections for the whole set of base-points B: $\Sigma_B^{in,s} = \bigcup_{k=1}^n \Sigma_k^{in,s}$,$\Sigma_B^{in,ss} = \bigcup_{k=1}^n \Sigma_k^{in,ss}$ and $\Sigma_B^{out} = \bigcup_{k=1}^n \Sigma_k^{out}$, and finally $\Sigma_B^{in} = \Sigma_B^{in,s} \cup \Sigma_B^{in,ss}$.

We need some more notation before we can introduce the main theorem of this section. Decompose the tangent spaces of the sections defined above into the parts of the hyperbolic direction (V^h, W^h), on the one hand, and the center-component (V^c,W^c), on the other hand. For this, let $q \in \Sigma_B^{in,ss}$ and $r \in \Sigma_B^{out}$:

- $T_q\Sigma_B^{in,ss} = V_q^h \oplus V_q^c$, where $V_q^h = span\{\frac{\partial}{\partial x^s}(q), \frac{\partial}{\partial x^u}(q)\}$, i.e. the additional stable direction and the unstable direction, and $V_q^c = span\{\frac{\partial}{\partial x^c}(q)\}$
- $T_r\Sigma_B^{out} = W_r^h \oplus W_r^c$, where $W_r^h = span\{\frac{\partial}{\partial x^{ss}}(r), \frac{\partial}{\partial x^s}(r)\}$, i.e. the both stable directions, because we are in the out-section, and $W_r^c = span\{\frac{\partial}{\partial x^c}(r)\}$

Note that one point of this construction is to "collect" also the tangent spaces like the other objects before, i.e. to talk about the decomposition of the tangent bundle of the set Σ_B^{out}, which is possible because all object depend smoothly on the base point:

- $T\Sigma_B^{out} = V^h \oplus V^c$
- $T\Sigma_B^{out} = W^h \oplus W^c$

Now we are in the position to state the theorem about the local passage. Recall that H_B stands for the set of all heterclinic Bianchi-II-orbits connecting base points of the set B (see chapter 2, section 1.2):

THEOREM 4.13. *Assume that, for all $p \in B$, the vector field has been according brought to the form as in the conclusion of Theorem 4.12. The local passage map $\Phi_B^{loc} : \Sigma_B^{in} \to \Sigma_B^{out}$ is a C^1-map that satifies, for $q \in H_B \cap \Sigma_B^{in}$:*

- *Φ_B^{loc} contracts super-linearly in the hyperbolic directions, i.e. $d\Phi_B^{loc}(q)(v) = 0 \; \forall v \in V_q^h$*
- *Φ_B^{loc} is the identity in the center-direction, i.e.*
  (1) $d\Phi_B^{loc}(q)(V_q^c) = W^c_{\Phi_B^{glob}(q)}$
  (2) $||d\Phi_B^{loc}(q)(v)||_{g_p} = ||v||_{g_p} \forall v \in V_q^c$

PROOF. Let $p \in B$ be a point from the set of admissible base points. Because of Theorem 4.12, the local passage near the Kasner circle Φ_p^{loc} in a neighbourhood U_p can be calculated explicitly (with $x_{in}^{ss} = 1$ in Σ_p^{in} and $x_{out}^u = 1$ in Σ_p^{out} after approriate scaling):

$$x_{out}^s = e^{\lambda_s t_{loc}} \cdot x_{in}^s = (x_{in}^u)^{-\frac{\lambda_s}{\lambda_u}} \cdot x_{in}^s \tag{39}$$

$$x_{out}^{ss} = e^{\lambda_{ss} t_{loc}} \cdot x_{in}^{ss} = (x_{in}^u)^{-\frac{\lambda_{ss}}{\lambda_u}} \tag{40}$$

$$x_{in}^u = e^{-\lambda_u t_{loc}} \cdot x_{out}^u \tag{41}$$

By solving the third equation for the local passage time t_{loc}, one obtains the following formulas for $\Phi_p^{loc} : \Sigma_p^{in} \to \Sigma_p^{out}$ (when $x^u > 0$):

$$\Phi_p^{loc}(x^c, x^s, 1, x^u) = (x^c, (x_{in}^u)^{-\frac{\lambda_s}{\lambda_u}} \cdot x_{in}^s, (x_{in}^u)^{-\frac{\lambda_{ss}}{\lambda_u}}, 1)$$

and for $x^u = 0$, we get (when following the heteroclinic orbit)

$$\Phi_p^{loc}(x^c, x^s, 1, 0) = (x^c, 0, 0, 1)$$

As the above equations show, the main point for understanding the local passage is the relation of the eigenvalues. In Bianchi IX, we know that it holds (away from the Taub points):

$$|\lambda_u| < |\lambda_s| < |\lambda_{ss}|$$

, i.e. the absolute value of the unstable eigenvalue is strictly smaller than the absolute value of the two stable eigenvalues. This can be seen from the formulas (37) expressing the eigenvalues in terms of the Kasner parameter u, see chapter 2, section 1.3). That's why it holds for the fractions which appear in the exponents of the formulas above:

$$-\frac{\lambda_s}{\lambda_u}, -\frac{\lambda_{ss}}{\lambda_u} > 1$$

and observe that both are necessarily positive because stable and unstable eigenvalues have opposite signs (note that this is even indepent of the chosen time direction towards/away from the big bang). This yields the claimed C^1-map and the super-linear contraction in the hyperbolic directions for the map Φ_p^{loc}.

As the vector field is completely linear in the Takens-neighbourhood, it trivially holds that $x^c_{out} = e^0 \cdot x^c_{in}$, i.e. we have not drift and Φ^{loc} is just the identity in the center-direction.

These observations hold for the local passage $\Phi_p^{loc} : \Sigma_p^{in} \to \Sigma_p^{out}$ at any admissible base point $p \in B$, and therfor also for the collection $\Phi_B^{loc} : \Sigma_B^{in} \to \Sigma_B^{out}$. □

4.3. Global Passage. Now we deal with the global passage. For the proof of the main theorem in this section, we consider two maps which map from the respective sections onto the Kasner circle by following the heteroclinic orbit (compare [**2**], p.19):

$\alpha : H_B \cap \Sigma_B^{out} :\to \mathcal{K} \cap V_B$

$\omega : H_B \cap \Sigma_B^{in} :\to \mathcal{K} \cap V_B$

where we recall that H_B stands for the Bianchi-II-heteroclinics and V_B is the collection of Takens-neighbourhoods (or the boxes, more precisely) constructed above when dealing with the local passage. At this point, we recall how we defined our global metric g_B, see (38). It is composed of the Riemanian metric $g_p = (dx^c)^2 + (dx^s)^2 + (dx^{ss})^2 + (dx^u)^2$ for the Takes-coordinates in a neighbourhood U_p near a point $p \in B$ of the Kasner circle, and the Riemanian metric $h = dN_1^2 + dN_2^2 +$

$dN_3^2 + d\Sigma_+^2 + d\Sigma_-^2$ otherwise. We may assume that both metics coincide when restricted to $\mathcal{K} \cap U_{p_i}$, because the local vector field has no center-component at the Kasner circle, i.e. one can replace center coordinate x by $\phi(x)$ for a diffeo ϕ without changing the vector field.

This means that both maps α, ω are local C^1-isometries for the metrics induced by the global metric g_B on the sets above, and we will use this fact in our proof below.

THEOREM 4.14. *There exits a neighbourhood $\mathcal{V}$ of $H_B \cap \Sigma_B^{out}$ in Σ_B^{out} such that the global passage map*

$$\begin{array}{rcl} \Phi_B^{glob} : \Sigma_B^{out} & \to & \Sigma_B^{in} \\ \mathcal{V} & \to & \Phi_B^{glob}(\mathcal{V}) \end{array}$$

is a C^1-map on $\mathcal{V}$ and a diffeomorphism onto its image.

Φ_B^{glob} expands in the center direction, i.e. for $r \in H_B \cap \Sigma_B^{out}$, it satisfies

(1) $d\Phi_B^{glob}(r)(W_q^c) = V^c_{\Phi_B^{glob}(r)}$

(2) $\exists \kappa > 1 : ||d\Phi_B^{glob}(r)(w)||_{g_B} \geq \kappa ||w||_{g_B} \forall w \in W_r^c$

PROOF. We know that for Ordinary Differential Equation with differentiable (C^k-)vector field, there is a differentiable (C^k-)dependence of the solution on the initial conditions (see e.g. [**1**]). This means that in general, for any "time-t-map" of a differentiable flow, for fixed $t = t^*$ and an open subset $U \subset \mathbb{R}^n$ of the phase space, we get a diffeomorphism onto its image:

$$\begin{array}{rcl} \phi_{t^*} : \mathbb{R}^n & \to & \mathbb{R}^n \\ U & \to & \phi_{t^*}(U) \end{array}$$

The Wainwright-Hsu vector field X^W is polynomial, hence analytic, that's why its flow $\phi_t(x_0)$ does depend in a differential (and even analytic) way on the inital condition. This means that the map $\Phi_p^{glob} : \Sigma_p^{out} \to \Sigma_{f(p)}^{in}$ is a C^1-map and a diffeomorphism onto its image, as claimed for the hyperbolic directions. We are left to show the second part of the theorem, dealing with the center directions. Now let $q \in H_B \cap \Sigma_B^{out}$. Then we observe that $\omega(\Phi^{glob}(q) = \omega(q) = f(\alpha(q)$, where f stands for the Kasner map. Because we have shown that both α and ω are local C^1-isometries w.r.t. g_B, we are left to prove that

$$\exists \kappa > 1 : \forall p \in B, \forall v \in T_p\mathcal{K} : ||df(p)(v)||_g \geq \kappa \cdot |v|_g$$

, which follows directly from the definition of the Kasner map, as we consider a periodic chain which clearly keeps a minimal distance from the Taub points, where f is not expanding.

These observations hold for the global passage $\Phi_p^{glob} : \Sigma_p^{out} \to \Sigma_{f(p)}^{in}$ at any admissible base point $p \in B$, and therfor also for the collection $\Phi_B^{loc} : \Sigma_B^{in} \to \Sigma_B^{out}$. □

4.4. The Return Map and the Hyperbolic Structure. As a consequence, we get the following result:

THEOREM 4.15. *The return map* $\Phi_B^{return} = \Phi_B^{glob} \circ \Phi_B^{loc} : \Sigma_B^{in} \to \Sigma_B^{in}$ *is a* C^1*-map that satisfies, for* $q \in H_B \cap \Sigma_B^{in}$

- Φ_B^{return} contracts super-linearly in the hyperbolic directions, i.e. $d\Phi_B^{return}(q)(v) = 0 \forall v \in V_q^h$
- Φ_B^{return}expands in the center direction, i.e. $\exists \kappa > 1 :$ $||d\Phi_B^{return}(q)(v)||_{g_B} \geq \kappa ||v||_{g_B} \forall v \in V_q^c$

PROOF. We recall the main idea behind our construction: We have shown that for the hyperbolic directions, the local passage is a contraction, while the global passsage is a diffeomorphism. Because of the differential dependence of a solution of an ODE on the initial conditions, the passage time for global passage near a heteroclinic orbit depends in a C^1-way on the base point considered. When approaching the attactor, it remains bounded, while the passage time for the local passage tends to infinity. That's why the local passage dominates, and we get a contraction in the hyperbolic directions. In the center direction, the local passage is the identity in our local coordinate system, which yields the claimed expansion when combined with the global passage which expands the center direction. More formally, we use the chain rule $d\Phi_B^{return}(v) = d\Phi_B^{glob}(\Phi_B^{loc}) \circ d\Phi_B^{loc}(v)$ to get the claims directly from our theorems above, for $q \in H_B \cap \Sigma_B^{in}$:

$d\Phi_B^{return}(q)(v) = 0 \; \forall v \in V_q^h$

$||d\Phi_B^{return}(q)(v)||_{g_B} \geq \kappa ||v||_{g_B} \forall v \in V_q^c$ □

The theorem above means that our return map Φ_B^{return} has a C^1-hyperbolic structure on the set $H_B \cap \Sigma_B^{in}$, i.e. that it is a hyperbolic set. Via Theorem 4.17 (described below), this C^1-hyperbolic structure leads to a C^1-stable-manifold.

To make this more precise, consider a point $p \in B$ and observe that the heteroclinic orbit $H_{p,f(p)}$ intersects Σ_B^{in} in exactly one point that we denote by q. We also note that $q \in (H_B \cap \Sigma_B^{in})$, i.e. it belongs to our hyperbolic set. Theorem 4.17 yields a C^1-embedded 2-dimensional stable manifold $W_\epsilon^s(\Phi, q)$ in Σ_B^{in}. And as we know that the orbits of the Bianchi IX flow are transversal to Σ_B^{in}, we obtain a 3-dimensional stable manifold for the base point p on the Kasner circle as claimed (compare [**2**], p. 22).

In summary, we arrive at the following theorem, which is equivalent to Theorem 4.1:

THEOREM 4.16. *(Stable Manifolds for Points in B)*
Let $p \in B$, where B is the set of base points of a periodic heteroclinic chain that satisfies the Sternberg Non-Resonsonance-Conditions. Then there exists a three dimensional C^1-stable manifold $W^s(p)$ of initial conditions such that the corresponding vacuum Bianchi IX - solutions converge to the periodic heteroclinic chain towards the big bang.

Combining this with Theorem 4.7 and Definition 4.8 on the admissibility of 2-periodic continued fraction developments leads immediately to Theorem 4.2.

Untill now, we have only dealt with periodic heteroclinic chains, as this was the "missing case" in the paper by Béguin, who was treating aperiodic chains. When we combine the two results, we can get C^1-stable manifolds for any points $p \in \mathcal{K}$ that do not contain "forbidden" base points in the closure of the orbit of p under the Kasner map f, i.e. $\overline{\{f^n(p)\}} \subseteq B_\epsilon^T$. For this we define B_ϵ^T to be the set of base points that satisfies the Non-Resonsonance-Conditions in order to allow for Takens Lineraization and keeps a minimum distance of ϵ from the Taub points. This second condition is trivially fullfilled for periodic chains and necessary in order to achive uniform rates of expansion/contraction for the hyperbolic structure. The reason is that both the expansion of the Kasner map as well as the contraction of the local passage breaks down at the Taub points.

We can also elaborate a bit about what it means that solutions of Bianchi IX converges to a heteroclinic chain towards the big bang. For example, we can show that the Hausdorff distance between the heteroclinic orbits that are part of the chain and the respective piece of the Bianchi IX-orbit tends to zero. This follows from the continuity of the flow and the properties of the stable manifold (see [**2**], p.21). Thus the limit of the analysis presented here can be formulated as in Theorem 4.3.

4.5. C^1-Stable Manifolds for C^1-Hyperbolic Sets. We have shown that the global return map admits a C^1-hyperbolic structure. Béguin then uses the following Theorem (see [**2**], p.18) to prove the existence of a C^1-stable manifold: Theorem 4.17 shows that a C^1-Hyperbolic Structure leads to a C^1- stable manifold, where the "index s" of the hyperbolic set stands for the dimension of the stable subbundle of the tangen bundle TM (i.e. $s = dim(X_p)$ in the notation of Definition 4.4). In addition, the theorem specifies the dependence of this manifold on the base point as well as the convergence rate:

THEOREM 4.17. *Let $\Phi : M \to M$ be a C^1map on a manifold M, and C be a compact subset of M which is a hyperbolic set of index s for the map Φ. Then, for every ϵ small enough, for every $q \in C$, the set*

$$W^s_\epsilon(\Phi, q) := \{r \in M | dist(\Phi^n(r), \Phi^n(q)) \leq \epsilon \, for \, every \, n \geq 0\}$$

is a C^1embedded s-dimensional disc, tangent to F^s_q at q, depending continuously on q (for the C^1topology on the space of embeddings). Moreover, if μ is a contraction rate for Φ on C, then there exists a constant κ such that, for every ϵ small enough, for every $q \in C$ and every $r \in W^s_\epsilon(\Phi, q)$

$$dist_g(\Phi^n(r), \Phi^n(q)) \leq \kappa\mu^n$$

Béguin names the book [**36**] by Palis and Takens (page 167) as a reference for Theorem 4.17. In this section of the Appendix “Hyperbolicity: Stable Manifolds and Foliations”, the authors deal with hyperbolic sets for endomorphisms, but results are only sketched and no proofs included. However, there are classic sources for stable manifold theorems of hyperbolic sets: Partly based on an earlier paper ([**22**]), Hirsch and Pugh prove such a theorem in [**23**], which is a chapter of the book “Global Analysis” collecting the proceedings a symposium held on the topic in Berkeley, California, in 1968, and seems to be the first time such a result is proved. We will introduce the theorem by Hirsch/Pugh below, it can be used instead of 4.17 in order to prove our Theorem 4.16.

4.6. Generalized Stable Manifold Theorem by Hirsch/Pugh.

THEOREM. *(Generalized Stable Manifold Theorem) Let U be an open set in a smooth manifold $M(dim < \infty)$ and $f : U \to M$ a C^1-map. Let $\Lambda \subset U$ be a compact hyperbolic set and call the invariant splitting $T_\Lambda M = E_1 \oplus E_2$. Then there is a neighbourhood V of Λ, and* *submanifolds* $W^s(x), W^u(x)$ *tangent to* $E_2(x)$ *and* $E_1(x)$ *respectively for each* $x \in \Lambda$ *such that*

$$W^s(x) = \{y \in V | \lim_{n\to\infty} d(f \restriction V)^n y, f \restriction V)^n x) = 0\}$$

If f *is* C^k*, so is* $W^s(x)$*, and it depends continously on* f *in the* C^k*-topology. Moreover,* $W^s(x)$ *and its derivatives along* $W^s(x)$ *up to order* k *depend continously on* x*. In addition, there exist numbers* $K > 0, \lambda < 1$ such that if $x \in \Lambda, z \in W_x$ and $n \in \mathbb{Z}_+$ then *the following holds:*

$$d(f^n(x), f^n(z) \leq K\lambda^n$$

In **[23]**, the proof of the generalized stable manifold theorem is outlined as follows:

(1) Let $E = E_1 \times E_2$ be a Banach space; $T : E \to E$ a hyperbolic linear map expanding along E_1 and contracting along E_2; $E(r) \subset E$ the ball of radius r, and $f : E(r) \to E$ a Lipschitz pertubation of $T \restriction E(r)$. The unstable manifold W for f will be the graph of a map $g : E_1(r) \to E_2(r)$ which satisfies $W = f(W) \cap E(r)$. Then the following map Γ_f is considered (in a suitable function space G of maps g):

$$graph[\Gamma_f(g)] = E(r) \cap f(graph[g])$$

i.e. Γ_f is the graph transform of g by f. The fixed point g_0 of Γ_f gives the unstable manifold of f - its existence is proved by the contracting map principle if f is sufficiently close to T pointwise, and the Lipschitz constant of $f - T$ is small enough.

(2) If f is C^k so is g_0, which is proved by induction on k. The successive approximations $\Gamma_f^n(g)$ converge C^k to g_0 - here the Fibre Contraction Theorem is used.

(3) Let $\Gamma \subset U$ be a hyperbolic set. Let $\mathcal{M}$ be the Banach manifold of bounded maps $\Lambda \to M$, and $i \in \mathcal{M}$ the inclusion of Λ. Let $\mathcal{U} = \{h \in \mathcal{M} | h(\Lambda) \subset U\}$. Define $f_* : \mathcal{U} \to \mathcal{M}$ by

$$f_*(h) = f \circ h \circ f^{-1}$$

Then f_*has a hyperbolic fixed point at i. By the first point, f_* hast a stable manifold $\mathcal{W}^s \subset \mathcal{M}$. For each $x \in \mathcal{M}$, define $W^s(x) = ev_x(\mathcal{W}^s) = \{y \in M | y = \gamma(x)$ for some $\gamma \in \mathcal{W}^s\}$. This yields a system of stable manifolds for f along Λ

Point (1) of the outline above involves a graph-transform of Lipschitz-graphs (see e.g. [**44**], and compare also [**27, 28**], where it is described in detail how a graph transform can be used to prove Lipschitz-stable-manifolds in Bianchi models even without linearizing at the Kasner circle). Point (3) reduces the proof of a stable manifold for a hyperbolic set to the case of a fixed point, in a suitable chosen infinite-dimensional space (compare also [**36**], p.157).

4.7. Differentiability of the Stable Manifold. In step (2) above, the differentiability of the stable manifold is proved by the Fibre Contraction Principle (see [**23**], p.136 or [**24**], p.25). As the differentiability of the stable manifold is the main point of our Theorem 4.1, we will comment a bit how this is done. For the invariant section (which will be the desired stable manifold) to be differentiable, it is not enough to obtain a fibre contraction. One important point is that it may not contract more along the base space than along the fibres (compare [**24**], p.26), otherwise there are examples where there is no differentiable invariant section (see e.g. [**44**], p. 435). That's why we need additional conditions that assure that the contraction on fibres is stronger than the contraction in the base space to prove a "C^r section Theorem" ([**44**], p. 436).

An alternative approach is the method of cones (e.g. taken by Robinson [**44**], p.185). As above, a stable manifold that is only Lipschitz is obtained in a first step, and then it is shown that the obtained manifold is infact C^k if the original map has this smoothness property ([**44**], p.194).

Finally, the book [**49**] also contains stable manifold theorems both for fixed points (chapter 5) and hyperbolic sets (chapter 6), in an abstract setting similar to [**23**], and also deals with the differentiability question (see [**49**], p.39).

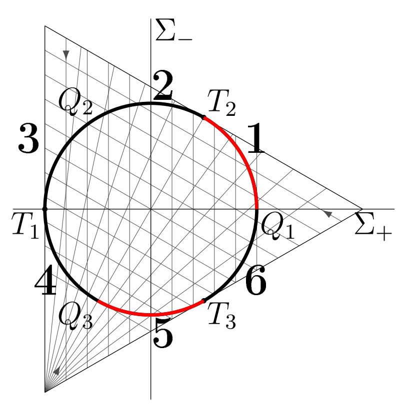
CHAPTER 5

Takens Linearization and Combined Linear Local Passage at the 18-cycle in Bianchi $VI^*_{-1/9}$

In this section, we show how some of the techniques developed in the chapter before can be applied to Bianchi $VI^*_{-1/9}$, where no rigorous convergence result exists to date. We construct an example for a periodic heteroclinic chain in Bianchi $VI^*_{-1/9}$ that allows Takens Linearization at all Base points. It will turn out to be a "18-cycle", i.e. involving a heteroclinic chain of 18 different base points at the Kasner circle. We then show that the combined linear local passage at the 18-cycle is a contraction. This qualifies it as a possible candidate for proving a rigorous convergence theorem in Bianchi $VI^*_{-1/9}$.

The situation in Bianchi $VI^*_{-1/9}$ is more involved than in Bianchi IX, as there are sectors of the Kasner circle with more than one unstable eigenvalue (towards the big bang, the time direction we are interested in). This means that even by starting with the same Kasner-parameter, one can have different realizations in terms of heteroclinic chains.

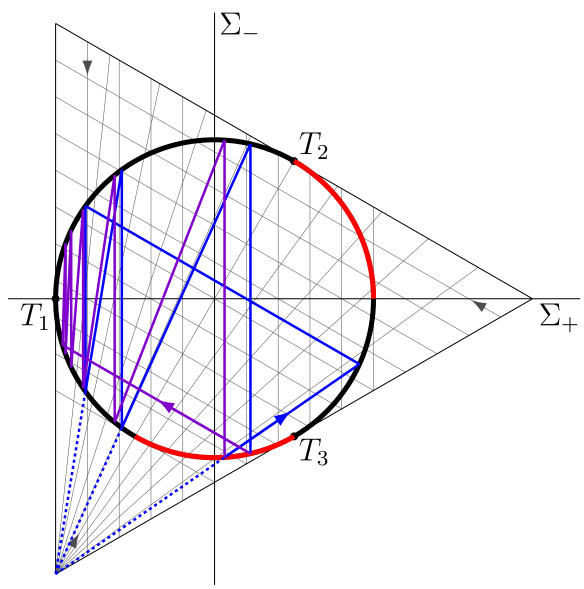
If we label the six sectors of the Kasner circle counter-clockwise as it is often done in Bianchi models (starting in the positive quadrant), this is true e.g. for sector 5, which is part of the 18-cycle. In the graphics below, the sectors with multiple unstable eigenvalues are marked in red color. The different families of heteroclinic orbits in Bianchi $VI^*_{-1/9}$ are illustrated by the lines in light gray in the background (see chapter 2, section 2 for details):



We will illustrate this ambiguity that can arise from the same continued fraction development $u = [3, 5, 3, 5, ...]$ by showing two possible "18-cycles" (details are discussed in section 3.1):

- the "classic" 18-cycle, with sector sequence 54343-425-43434343-425, where we go "left" to sector 4 both times in sector 5
- the "advanced" 18-cycle, with sequence 5634342-543434343425, where we go first "right" to sector 6, and the second time "left" to sector 4

The graphics below shows the "advanced 18 cycle" starting in sector 5. The bounces around the Taub-point can be seen clearly - the blue color shows the part with 3 bounces, while the purple color exhibits 5 bounces. The label "advanced 18-cylce" means that we first follow the blue part of the heteroclinic chain and then the purple one, reflecting our choice of u=[3,5,3,5,...].



Now we are in a position to state the two main results of this chapter:

THEOREM 5.1. *Both the classic and the advanced 18-cycle allow Takens-Linearization at all base points.*

LEMMA 5.2. *The Combined Linear Local Passage at the classic 18-cycle is a contraction.*

The proof of these results is the aim of this chapter, and we will now comment a little on how the rest of chapter is organized.

In order to illustrate our approach, we start by considering the 3-cycle in Bianchi $VI^*_{-1/9}$, as it is easier to handle and the method we develop applies also to longer cycles.

First we derive the formulas for the relevant eigenvalues of the linearized vector field at the base-points of the 3-cycle in Bianchi $VI^*_{-1/9}$ in terms of the Kasner Parameter u. We then consider the Combined Linear Local Passages and Takens Linearization at the 3-cycle. We are able to show that the Sternberg Non-Resonance Conditions are not satisfied for the 3-cycle, but they are satisfied for the 18-cycles.

This means that both the classic and the advanced 18-cycles are periodic heteroclinic chains in Bianchi $VI^*_{-1/9}$ that allow Takens Linearization at all of their Base points. In addition, we give evidence that the Combined Linear Local Passage at the classic 18-cycle is a contraction. This qualifies it as a candidate for proving a rigorous convergence theorem in Bianchi $VI^*_{-1/9}$.

In order to progress further and turn Lemma 5.2 into a rigorous convergence theorem, a better understanding of the global passages in Bianchi $VI^*_{-1/9}$ is necessary. We will comment on possible ideas how to do this in the section "Conclusion and Outlook".

As the formulas for the eigenvalues at points on the Kasner circle in Bianchi $VI^*_{-1/9}$ depend on the sector, many cases have to be checked. That's why we use Mathematica in order to do the necessary computations for the 18-cycles (see Appendix 2).

1. Eigenvalues in Terms of the Kasner Parameter u

1.1. General Formulas for Points on the Kasner Circle. At first, we recall that in Bianchi IX, the formula for the eigenvalues of the linearized vector field at points of the Kasner circle have an easy expression in terms of the Kasner parameter u, compare section 2, equation (37). In Bianchi IX, the situation in different sectors of the Kasner-circle only differs by a permutation of those 3 formulas for the eigenvalues (see [**17**], p.8), which does not matter for the question of resonances.

But in Bianchi $VI^*_{-1/9}$, the situation is more complicated. Here, the formulas for the eigenvalues at points on the Kasner circle do depend on the sector, so in order to check for resonances, a lot of cases have to be considered. Each sector corresponds to a permutation of the Kasner expontents p_i, where sector (321) means e.g. that $p_3 < p_2 < p_1$which fixes the formula for each of them. For this is it important to note that $u \in [1, \infty]$. As an example, consider the sector 5 or (312), compare section 2):

$$p_3 = \frac{-u}{1+u+u^2}$$

$$p_1 = \frac{(u+1)}{1+u+u^2}$$

$$p_2 = \frac{u(u+1)}{1+u+u^2}$$

The general formuals for the relevant eigenvalues (i.e. $\lambda_\times$, λ_2, λ_- corresponding to the variables involved in the heteroclinic chain: $\Sigma_\times$, Σ_2, $N-$, see section 2) in terms of Σ_+, Σ_-resp. the p_i are (see [**17**], p.7):

$$\Sigma_+ = -\frac{3}{2}p_1 + \frac{1}{2}$$

$$\Sigma_- = -\frac{\sqrt{3}}{2}(p_1 + 2p_2 - 1)$$

$$\lambda_\times = -2\sqrt{3}\Sigma_-$$

$$\lambda_2 = -3\Sigma_+ + \sqrt{3}\Sigma_-$$

$$\lambda_- = 2 + 2\Sigma_+ + 2\sqrt{3}\Sigma_-$$

1.2. Eigenvalues at the 3-Cycle. In the following, we present the formulas for the sectors that are involved in the 3-cycle with $u =$ golden mean $= \frac{1+\sqrt{5}}{2}$ and the sector-sequence "5-1-2-5". For the other sectors, similar formulas can be derived analogously (see Appendix 2.2).

Base Point B_1 in sector 5, i.e. (312).

$$\lambda_2 = \frac{3 - 3u^2}{1 + u + u^2}$$

$$\lambda_\times = \frac{6u + 3u^2}{1 + u + u^2}$$

$$\lambda_- = \frac{-6u}{1 + u + u^2}$$

Base Point B_2 in sector 1, i.e. (123).

$$\lambda_2 = \frac{-3 - 6u}{1 + u + u^2}$$

$$\lambda_\times = \frac{3 - 3u^2}{1 + u + u^2}$$

$$\lambda_- = \frac{6u + 6u^2}{1 + u + u^2}$$

Base Point B_3 in sector 2, i.e. (213).

$$\lambda_2 = \frac{3 + 6u}{1 + u + u^2}$$

$$\lambda_\times = \frac{-6u - 3u^2}{1 + u + u^2}$$

$$\lambda_- = \frac{6u + 6u^2}{1 + u + u^2}$$

2. The 3-Cycle in Bianchi $VI^*_{-1/9}$

2.1. (Non-)Resonance and Takens Linearization. In this section, we shortly check the Sternberg-Non-Resonance-Conditions (SNC) for the 3-cycle in Bianchi $VI^*_{-1/9}$. The procedure necessary to do this is described in detail in section 1. The parameter-value at the 3-cylce is $u = g = \frac{1+\sqrt{5}}{2}$, which satisfies

$$1 + \frac{1}{u} = u \implies 1 + u - u^2 = 0$$

The equation for checking the (SNC) thus reads:

$$M * \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = z * \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

According to the formulas above, we observe that

$$M_{B_1} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & -6 \\ \text{-3} & 3 & 0 \end{pmatrix}, M_{B_2} = \begin{pmatrix} -3 & 3 & 0 \\ -6 & 0 & 6 \\ 0 & -3 & 6 \end{pmatrix}, M_{B_3} = \begin{pmatrix} 3 & 0 & 0 \\ 6 & -6 & 6 \\ 0 & -3 & 6 \end{pmatrix}$$

, which are all invertible, and give the following results (with $z = 6$ for the earliest possible resonance):

$$k_{B_1} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, k_{B_2} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}, k_{B_3} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

This means that (SNC) does hold at the Base points of the 3-cycle up to order 3, which is not enough to allow Takens-Linearization, as the the order necessary in the Takens-Theorem, named $\alpha(1)$, is bigger than 10 in all of the sectors involved (see Appendix 2.1). That's why Takens Linearization Theorem may not be employed at the base points of the 3-cycle, and we have to look for a different (longer) cycle. Nevertheless, we will stick to the 3-cycle in the following chapter in order to illustrate our method for calculating the Combined Linear Local Passage, as the method applies to longer cycles as well.

2.2. Combined Linear Local Passages. In this section, we use the linearized vectorfield at the base points of the 3-cycle to explicitly compute the local passages as was done for Bianchi IX before (see 39-41). Be aware that Takens-Linearization is not allowed at the 3-cycle in Bianchi $VI^{*}_{-1/9}$, so this is only a formal calculation in this case to illustrate what we mean by "Combined Local Linear Passage". Later, for the 18-cycles, the calculation will be justified as Takens-Linearization is possible there.

The ratio of the relevant Eigenvalues at the Base-points B_1,B_2,B_3 of the 3-cycle (corresponding to the value $u = \frac{1+\sqrt{5}}{2}$ in the 3 sectors) is given by the following:

Near $B_1 = (-\frac{1}{4}, -\frac{\sqrt{15}}{4})$:

$$r_1 = -\frac{\lambda_\times}{\lambda_n} = \frac{u+2}{2} = 1.8090 > 0$$

$$r_2 = -\frac{\lambda_2}{\lambda_n} = -\frac{u^2-1}{2u} = -0.5000 < 0$$

Near $B_2 = (\frac{1+3\sqrt{5}}{8}, \frac{\sqrt{15}-\sqrt{3}}{8})$:

$$r_3 = -\frac{\lambda_n}{\lambda_2} = \frac{2u(u+1)}{2u+1} = 2 > 0$$

$$r_4 = -\frac{\lambda_u}{\lambda_{uu}} = -\frac{\lambda_\times}{\lambda_2} = -\frac{u^2-1}{2u+1} = -0.3820 < 0$$

Near $B_3 = (-\frac{1}{4}, \frac{\sqrt{15}}{4})$:

$$r_5 = -\frac{\lambda_2}{\lambda_\times} = \frac{2u+1}{u(u+2)} = 0.7236 > 0$$

$$r_6 = -\frac{\lambda_n}{\lambda_\times} = \frac{2(u+1)}{u+2} = 1.4472 > 0$$

We start our combined linear local passage at the In-Section of the local passage at B_1 in sector 5. Thus $\Sigma_\times$ is the incoming variable and we define $a := \Sigma_2^{in}$, $b := N_-^{in}$ (the two remaining relevant variables in the section).

Then we do the calculations for the 3 local passages involved at the 3-cycle near the B_1,B_2 and B_3 as in Bianchi IX before (see 39-41). We arrive at the following formulas for the combined linear local passage near the 3-cycle:

$$\tilde{a} = ([b^{r_2}a]^{r_4}b^{r_1})^{r_5}$$

$$\tilde{b} = ([b^{r_2}a]^{r_4}b^{r_1})^{r_6}(b^{r_2}a)^{r_3}$$

Taking Logarithms on both sides yields the following:

$$\begin{pmatrix} \log \tilde{a} \\ \log \tilde{b} \end{pmatrix} = M_{3-cycle} * \begin{pmatrix} \log a \\ \log b \end{pmatrix}$$

with the following matrix $M_{3-cycle}$:

$$M_{3-cycle} = \begin{pmatrix} r_5 r_4 & r_5 r_4 r_2 + r_5 r_1 \\ r_6 r_4 + r_3 & r_6 r_4 r_2 + r_6 r_1 + r_3 r_2 \end{pmatrix}$$

As mentioned before, Takens-Linearization is not allowed at the 3-cycle in Bianchi $VI^*_{-1/9}$. However, for the 18-cycles, an analogous calculation will be justified as Takens-Linearization is possible there. We will discuss the properties $M_{18-cycle}$ in order to prove Theorem 5.2 in section 3.5.

3. The 18 Cycles in Bianchi $VI^*_{-1/9}$

3.1. Possible Passages in Bianchi $VI^*_{-1/9}$. When we look at the different transitions possible in Bianchi $VI^*_{-1/9}$ (see chapter 2, section 2), we are able to understand which sequence of sectors can arise when solutions converge to their corresponding heteroclinic chains. For the classification below, we have chosen to put our section always before the next "Curvature Transition", i.e. when we leave from sector 4 or 5, that's why all the passages start and end in one of these sectors:

- Passage A: Sectors 4-3-4
- Passage B1: Sectors 4-2-5
- Passage B2: Sectors 4-2-5-4
- Passage C1: Sectors 5-1-2-5
- Passage C2: Sectors 5-1-2-5-4
- Passage D: Sectors 5-6-3-4
- Passage E: Sectors 5-1-6-3-4

For the 18-cycles discussed below, only a few of the passages will occur, namely A, B1, B2 and D.

3.2. The Classic 18-Cycle. We now start with u=[3,5,3,5,...] in Sector 4 and prescribe the follwing dynamics:

u=	[3,5,...]	[2,5,...]	[2,5,...]	[1,5,...]	[1,5,...]	[5,3,...]	[5,3,...]
sector	4	3	4	3	4	2	5

u=	[5,3,...]	[4,3,...]	[4,3,...]	[3,3,...]	[3,3,...]	[2,3,...]	[2,3,...]
sector	4	3	4	3	4	3	4

u=	[1,3,...]	[1,3,...]	[3,5,...]	[3,5,...]	[3,5,...]	...	...
sector	3	4	2	5	4	...	...

This means we have the following sequence of Passages: A A B2 A A A A B2, and this pattern continues arbitrarily often. This involves 18 global passages, that's why I call it an "18-cycle".

Observe that both 18-cycles mentioned in the introduction to this chapter started in sector 5. This was done for illustrative purposes as there is an ambiguity how to continue in this sector. From now on, we refer to the "classic 18-cycle" with the sequence of sectors as illustrated in the table above, which means we have started in sector 4.

Note that we could also derive a different sequence of passages for the same u, as we have a choice in sector 5 either to go to sector 4 via a frame transition (as done above) or to go via curvature transition to sector 6, as done at the first transition for the advanced 18-cycle in the next section.

3.3. The Advanced 18-Cycle. Now we start with u=[3,5,3,5,...] in Sector 5 and prescribe the follwing dynamics:

u=	[3,5,...]	[2,5,...]	[2,5,...]	[2,5,...]	[1,5,...]	[1,5,...]	[5,3,...]
sector	5	6	3	4	3	4	2

u=	[5,3,...]	[5,3,...]	[4,3,...]	[4,3,...]	[3,3,...]	[3,3,...]	[2,3,...]
sector	5	4	3	4	3	4	3

u=	[2,3,...]	[1,3,...]	[1,3,...]	[3,5,...]	[3,5,...]	...	...
sector	4	3	4	2	5	...	...

This means we have the following sequence of passages: D A B2 A A A A B1, which defines the "advanced 18-cycle".

3.4. (Non-)Resonance and Takens Linearization at the 18-Cycle. We now show that both of the 18 cycles are infinite periodic heteroclinic chains in Bianchi $VI^{*}_{-1/9}$ that allows Takens Linearization at all of its base points. The Mathematica-output in Appendix 2.2 shows that for $u = [3, 5, 3, 5, ...]$ the Sternberg-Non-Resonance-Conditions (SNC) are satisfied, because the $\alpha(1)$ that is necessary for a C^1-linearization at each point is always smaller than the sum of the absolute value of the coefficients in the vector $k = \{k_1, k_2, k_3\}$. That's why we can employ the Takens Linearization Theorem for both of the 18-cycles mentioned above, and Theorem 5.1 is proven.

3.5. Combined Linear Local Passage at the 18-Cycle. Our Results from Mathematica (see Appendix 2.3) indicate that we get the following matrix for the combined linear local passage of the classic 18-cycle when we apply the same algorithm that we outlined for the 3-cycle in section 2.2:

$$M_{18-cycle} = \begin{pmatrix} 267.54 & 110.78 \\ 595.16 & 247.51 \end{pmatrix} =: \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$$

$$\mu_1 = 514.49 \text{ with Eigenvector } v_1 = (0.45, 1)$$

$$\mu_2 = 0.5578 \text{ with Eigenvector } v_2 = (-0.41, 1)$$

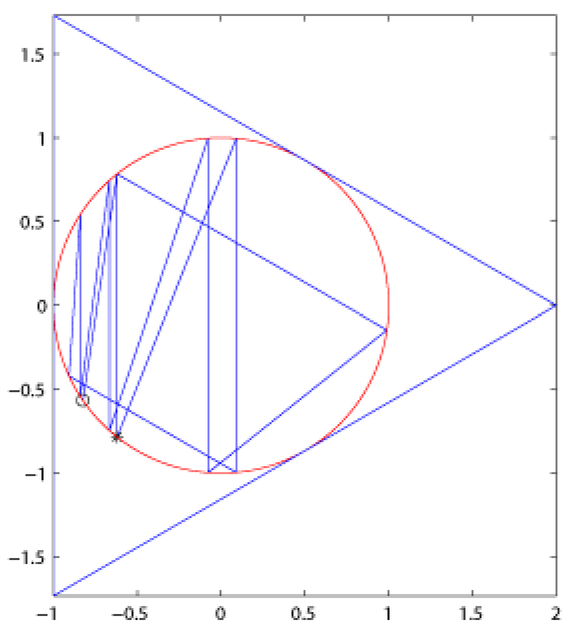
This implies that if we start with small positive a,b (i.e. log a, log b $<<$ 0), the combined linear local passage will bring us closer to the origin - this is what we mean by the term "contraction" in Lemma 5.2.

4. Numerical Simulation

The picture below shows a numerical simulation (with Matlab) of a periodic heteroclinic chain in Bianchi $VI^*_{-1/9}$, here a 13-cycle. According to the terminology developed above, it would be named the "advanced" 13 cycle, as both directions are taken from sector 5.

Although a much more detailed numerical analysis is necessary, our simulation shows that at least there are cases where both directions are taken from sector 5. Thus this possibility seems to really occur in the equations, at least numerically there are Bianchi $VI^*_{-1/9}$-solutions following the "advanced" 13 cycle towards the big bang.

Of course much more effort is needed in order to set up the numerics in an appropriate way instead of just using a built-in Matlab ODE solver[1]. One idea could be to use the explicit linear flow near the equilibria of the Kasner circle, where most of the time is spent, in order to achieve a higher precision.



[1]for producing the picture above, we have used the "ode113" solver, which is a variable order Adams-Bashforth-Moulton PECE solver (according to the Matlab documentation [**33**]). We thank Woei Chet Lim for providing us with some Matlab code that we used in order to carry out our numerical simulations.

Conclusion and Outlook

This dissertation deals with the ancient dynamics of the Einstein Equations, i.e. the dynamics in backwards time towards the big bang. As mentioned in the introduction, the BKL-picture is belived to capture the essence of this approach to the initial singularty, but it is not even a clearly formulated conjecture, not to mention a proven mathematical theorem. However, the aim of mathematically understanding spacetime singularities and increasing the rigour of the BKL-picture is a formidable task, to which we have contributed in two ways:

At first, we have shown that there are periodic heteroclinic chains in Bianchi IX for which there exisist C^1- Stable - Manifolds of orbits that follow these chains towards the big bang. This result is new, and should be compared with the two existing rigorous results on stable manifolds for orbits of the Kasner map in Bianchi IX: Béguin showed the existence of C^1- stable- manifolds for aperiodic orbits of the Kasner map ([**2**]), while Liebscher and co-authors ([**27, 28**]) showed the existence of Lipschitz-stable-manifolds for arbitrary orbits of the Kasner map not accumulating at one of the Taub points (Béguin also had to demand the latter condition).

Our result significantly extends Béguins results, who had to exclude all orbits that are perioidic or accumulate on any periodic orbit, a limitation which we were able to overcome. The techniques by Liebscher et al are able to treat both periodic and aperiodic chains, but yielded only Lipschitz-manifolds, i.e. the leaves of the foliation have less regularity.

But be aware that even though the stable manifolds constructed by Béguin and ourselves are C^1, this concerns only the regularity of the leaves of the foliation, and not the dependence on the base point. We do not get a C^1-foliation which would mean a C^1-dependece on the base point, but only a C^0-dependence of the (C^1-)leaves in the C^1-topology.

These aspects play a crucuial role when discussing the genericity of the foliation-results in BIX, i.e. how generic the set of initial conditions is both "down on the Kasner circle", as well as in the full space of trajectories. This involves delicate distinctions between topological vs.

measure-theoretic genericity, and is subject of current research (for partial results[2], see [**38**][3]).

Secondly, we have also constructed, for the fist time, the 18-cycle as a relatively simple example for a periodic heteroclinic chain in Bianchi $VI^*_{-1/9}$ that allows Takens Linearization at all Base points. In addition, we were able to show that the Combined Linear Local Passage at the classic 18-cycle is a contraction. This could be seen as a first step for proving a rigorous convergence theorem in Bianchi $VI^*_{-1/9}$.

In order to progress further, a better understanding of the global passages in Bianchi $VI^*_{-1/9}$ is necessary. This is not an easy task, as there are less invariant subspaces than in Bianchi IX that restrict the signs of the heteroclinic orbits, so much more complicated transitions are possible. A first step could be to check in detail which sequence of signs for the different transitions occurs numerically, leading to a classification of possible cases. Speculatively, one could think about the possibility to prove a theorem that for heteroclinic chains with periodic continued fraction developments and with a clearly defined seqence of signs for the transitions there exist solutions of the Bianchi $VI^*_{-1/9}$ equations that show this behaviour. But this matter requires further investigation.

Until now, the application of Dynamical Systems Techniques to spatially homogeneous cosmological models yielding Ordinary Differential Equations has been discussed. However, there has also been the attempt to apply such techniques to inhomogeneous cosmologies, yielding Partial Differential Equations. The reason is that in a way the main point[4] of the BKL-picture is the question of "locality", asking if

[2]in [**38**] it is shown that there are trajectories converging to every formal sequence given by a Kasner parameter u with at most polynomially bounded continued fraction expansion. This covers a set of full measure on the Kasner circle, but this does not mean that the set of coresponding initial conditions in a neighborhood of the Kasner circle has full measure. The reason is that there are counterexamples, i.e. it is possible to construct foliations where a countable set of "leaves" is attached to a set of base points that has full measure in the base space.

[3]Uggla comments as follows ([**54**], p. 11): "[**38**] uses quite different mathematical techniques than the other rigorous papers in this area. As a consequence the results seem to be somewhat controversial in the research community, although the claims are arguably quite plausible".

[4]let us again quote the recent survey paper by Uggla on this issue ([**54**],p.2): "However, arguably the most central, and controversial, assumption of BKL is their ‘locality’ conjecture. According to BKL, asymptotic dynamics toward a generic spacelike singularity in inhomogeneous cosmologies is ‘local,’ in the sense that each spatial point is assumed to evolve toward the singularity individually and independently of its neighbors as a spatially homogeneous model"

the "complicated" Einstein Equations that are PDEs[5] can be approximated by "simpler/less complicated" ODEs towards the big bang. Until today, mostly numerical and heuristic results exist in inhomogeneous cosmologies, but very few rigorous mathematical theorems.

In the paper "The past attractor in inhomogeneous cosmologies" ([**56**]), it is outlined how it could be achieved to make the "local" part of the BKL-picture more rigorous, compare also [**18**]. After some results have been proven in the oscillatory spatially homogeneous setting of Bianchi IX and in an inhomogeneous, but non-oscillatory setting of Gowdy-spacetimes (see e.g. [**41**]), the logical next step seems to be to consider inhomogeneous oscillatory cosmological models. Arguably the simplest case is given by the G_2-cosmologies, that's why it has received rising attention in recent years ([**11, 30, 32, 7**]). However, there is not a single rigorous convergence theorem comparable to the results that could be achieved in spatially homogeneous models.

A particular complication in inhomogeneous models is the occurrence of spikes, i.e. the formation of spatial structure. Numerical experiments support the conjecture that spikes form the non-local part of the generalized Mixmaster attractor ([**32**]). Lim has also found explicit spike solutions that are compatible with the usual Bianchi II - transitions, giving rise to a "non-local" version of the mimaster dynamics involving "spike transitions" ([**31**]). Recent progress has been achieved by Heinzle and Uggla, who report more in detail about the role of the spike solutions as building blocks of such an extended non-local mixmaster-dynamics ([**19**]). In addition, they have done a statistical analysis on the spikes in G_2-models ([**20**]).

A first step towards achieving rigorous results in inhomogeneous oscillatory models could be to investigate the process of spike formation in G_2-models. A good understanding of the underlying spatially homogeneous model (which is Bianchi $VI^*_{-1/9}$) is probably necessary for this project, but as Uggla writes in his recent survey: "Unfortunately, there exist no rigorous mathematical results concerning their past asymptotic dynamics", refering to Bianchi $VI^*_{-1/9}$ ([**54**], p.11).

In this context, our result in Bianchi $VI^*_{-1/9}$ is an (admittedly small) step forward. But it is clear that progress in this direction is necessary in order to increase our understanding of the ancient dynamics of the Einstein Equations towards the big bang - in order to shed a little more light on our "tumbling universe" at birth.

[5] for General Relativity from the viewpoint of Partial Differential Equations see [**41, 43**]

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APPENDIX A

Symbolic Computations with Mathematica

1. Results on Admissibility of Periodic Heteroclinic Chains in Bianchi IX

1.1. Constant Continued Fraction Expansion. u=[a,a,...]

For u=[m,a,a,...] and m=1...a, AND a= 1
m= 1 alpha= 16 beta= 4 k1= -1 k2= 1 k3= -1

For u=[m,a,a,...] and m=1...a, AND a= 2
m= 1 alpha= 12 beta= 3 k1= 1 k2= 2 k3= -1
m= 2 alpha= 24 beta= 5 k1= -2 k2= 1 k3= -1

For u=[m,a,a,...] and m=1...a, AND a= 3
m= 1 alpha= 11 beta= 3 k1= 3 k2= 3 k3= -1
m= 2 alpha= 19 beta= 4 k1= 1 k2= 3 k3= -1
m= 3 alpha= 33 beta= 6 k1= -3 k2= 1 k3= -1

For u=[m,a,a,...] and m=1...a, AND a= 4
m= 1 alpha= 11 beta= 3 k1= 5 k2= 4 k3= -1
m= 2 alpha= 18 beta= 4 k1= 4 k2= 5 k3= -1
m= 3 alpha= 28 beta= 5 k1= 1 k2= 4 k3= -1
m= 4 alpha= 45 beta= 7 k1= -4 k2= 1 k3= -1

For u=[m,a,a,...] and m=1...a, AND a= 5
m= 1 alpha= 11 beta= 3 k1= 7 k2= 5 k3= -1
m= 2 alpha= 18 beta= 4 k1= 7 k2= 7 k3= -1
m= 3 alpha= 27 beta= 5 k1= 5 k2= 7 k3= -1
m= 4 alpha= 39 beta= 6 k1= 1 k2= 5 k3= -1
m= 5 alpha= 59 beta= 8 k1= -5 k2= 1 k3= -1

For u=[m,a,a,...] and m=1...a, AND a= 6
m= 1 alpha= 11 beta= 3 k1= 9 k2= 6 k3= -1
m= 2 alpha= 18 beta= 4 k1= 10 k2= 9 k3= -1
m= 3 alpha= 27 beta= 5 k1= 9 k2= 10 k3= -1

m= 4 alpha= 38 beta= 6 k1= 6 k2= 9 k3= -1
m= 5 alpha= 59 beta= 8 k1= 1 k2= 6 k3= -1
m= 6 alpha= 75 beta= 9 k1= -6 k2= 1 k3= -1

For u=[m,a,a,...] and m=1...a, AND a= 7
m= 1 alpha= 11 beta= 3 k1= 11 k2= 7 k3= -1
m= 2 alpha= 18 beta= 4 k1= 13 k2= 11 k3= -1
m= 3 alpha= 27 beta= 5 k1= 13 k2= 13 k3= -1
m= 4 alpha= 38 beta= 6 k1= 11 k2= 13 k3= -1
m= 5 alpha= 51 beta= 7 k1= 7 k2= 11 k3= -1
m= 6 alpha= 67 beta= 8 k1= 1 k2= 7 k3= -1
m= 7 alpha= 93 beta= 10 k1= -7 k2= 1 k3= -1

For u=[m,a,a,...] and m=1...a, AND a= 8
m= 1 alpha= 11 beta= 3 k1= 13 k2= 8 k3= -1
m= 2 alpha= 18 beta= 4 k1= 16 k2= 13 k3= -1
m= 3 alpha= 27 beta= 5 k1= 17 k2= 16 k3= -1
m= 4 alpha= 38 beta= 6 k1= 16 k2= 17 k3= -1
m= 5 alpha= 51 beta= 7 k1= 13 k2= 16 k3= -1
m= 6 alpha= 66 beta= 8 k1= 8 k2= 13 k3= -1
m= 7 alpha= 84 beta= 9 k1= 1 k2= 8 k3= -1
m= 8 alpha= 113 beta= 11 k1= -8 k2= 1 k3= -1

For u=[m,a,a,...] and m=1...a, AND a= 9
m= 1 alpha= 11 beta= 3 k1= 15 k2= 9 k3= -1
m= 2 alpha= 18 beta= 4 k1= 19 k2= 15 k3= -1
m= 3 alpha= 27 beta= 5 k1= 21 k2= 19 k3= -1
m= 4 alpha= 38 beta= 6 k1= 21 k2= 21 k3= -1
m= 5 alpha= 51 beta= 7 k1= 19 k2= 21 k3= -1
m= 6 alpha= 66 beta= 8 k1= 15 k2= 19 k3= -1
m= 7 alpha= 83 beta= 9 k1= 9 k2= 15 k3= -1
m= 8 alpha= 103 beta= 10 k1= 1 k2= 9 k3= -1
m= 9 alpha= 135 beta= 12 k1= -9 k2= 1 k3= -1

1.2. 2-Periodic Continued Fraction Expansion. u=[a,b,...]

Now use a= 2 and b= 3

For u=[m,a,b,a,b,...] and m=1...b
m= 1 alpha= 15 beta= 4 k1= 7 k2= 7 k3= -2
m= 2 alpha= 24 beta= 5 k1= 3 k2= 7 k3= -2
m= 3 alpha= 34 beta= 6 k1= -5 k2= 3 k3= -2

For u=[m,b,a,b,a,...] and m=1...a
m= 1 alpha= 11 beta= 3 k1= 2 k2= 5 k3= -3
m= 2 alpha= 19 beta= 4 k1= -7 k2= 2 k3= -3

Now use a= 3 and b= 5

For u=[m,a,b,a,b,...] and m=1...b
m= 1 alpha= 11 beta= 3 k1= 23 k2= 17 k3= -3
m= 2 alpha= 23 beta= 5 k1= 23 k2= 23 k3= -3
m= 3 alpha= 33 beta= 6 k1= 17 k2= 23 k3= -3
m= 4 alpha= 46 beta= 7 k1= 5 k2= 17 k3= -3
m= 5 alpha= 60 beta= 8 k1= -13 k2= 5 k3= -3

For u=[m,b,a,b,a,...] and m=1...a
m= 1 alpha= 11 beta= 3 k1= 13 k2= 13 k3= -5
m= 2 alpha= 18 beta= 4 k1= 3 k2= 13 k3= -5
m= 3 alpha= 27 beta= 5 k1= -17 k2= 3 k3= -5

Now use a= 1 and b= 2

For u=[m,a,b,a,b,...] and m=1...b
m= 1 alpha= 16 beta= 4 k1= 2 k2= 3 k3= -1
m= 2 alpha= 25 beta= 5 k1= -1 k2= 2 k3= -1

For u=[m,b,a,b,a,...] and m=1...a
m= 1 alpha= 12 beta= 3 k1= -3 k2= 1 k3= -2

Now use a= 2 and b= 4

For u=[m,a,b,a,b,...] and m=1...b
m= 1 alpha= 15 beta= 4 k1= 12 k2= 10 k3= -2
m= 2 alpha= 24 beta= 5 k1= 10 k2= 12 k3= -2
m= 3 alpha= 34 beta= 6 k1= 4 k2= 10 k3= -2

m= 4 alpha= 47 beta= 7 k1= -6 k2= 4 k3= -2

For u=[m,b,a,b,a,...] and m=1...a
m= 1 alpha= 11 beta= 3 k1= 2 k2= 6 k3= -4
m= 2 alpha= 18 beta= 4 k1= -10 k2= 2 k3= -4

1.3. 3-Periodic Continued Fraction Expansion. u=[a,b,c,...]

Use a= 1 and b= 1 and c=2

For u=[m,b,c,a,...] and m=1...a
m= 1 alpha= 16 beta= 4 k1= 5 k2= -2 k3= 3

For u=[m,c,a,b,...] and m=1...b
m= 1 alpha= 12 beta= 3 k1= 2 k2= -3 k3= 3

For u=[m,a,b,c,...] and m=1...c
m= 1 alpha= 16 beta= 4 k1= -3 k2= -5 k3= 2
m= 2 alpha= 24 beta= 5 k1= 3 k2= -3 k3= 2

Use a= 3 and b= 3 and c=2

For u=[m,b,c,a,...] and m=1...a
m= 1 alpha= 11 beta= 3 k1= -23 k2= -22 k3= 7
m= 2 alpha= 19 beta= 4 k1= -10 k2= -23 k3= 7
m= 3 alpha= 33 beta= 6 k1= 17 k2= -10 k3= 7

For u=[m,c,a,b,...] and m=1...b
m= 1 alpha= 15 beta= 4 k1= -22 k2= -23 k3= 7
m= 2 alpha= 24 beta= 5 k1= -7 k2= -22 k3= 7
m= 3 alpha= 34 beta= 6 k1= 22 k2= -7 k3= 7

For u=[m,a,b,c,...] and m=1...c
m= 1 alpha= 11 beta= 3 k1= -7 k2= -17 k3= 10
m= 2 alpha= 23 beta= 5 k1= 23 k2= -7 k3= 10

Use a=b=c=1 (consistency check):

For u=[m,b,c,a,...] and m=1...a
m= 1 alpha= 16 beta= 4 k1= 2 k2= -2 k3= 2

For u=[m,c,a,b,...] and m=1...b
m= 1 alpha= 16 beta= 4 k1= 2 k2= -2 k3= 2

For u=[m,a,b,c,...] and m=1...c
m= 1 alpha= 16 beta= 4 k1= 2 k2= -2 k3= 2

Use a=b=c=3 (consistency check):

For u=[m,b,c,a,...] and m=1...a
m= 1 alpha= 11 beta= 3 k1= -30 k2= -30 k3= 10
m= 2 alpha= 19 beta= 4 k1= -10 k2= -30 k3= 10
m= 3 alpha= 33 beta= 6 k1= 30 k2= -10 k3= 10

For u=[m,c,a,b,...] and m=1...b
m= 1 alpha= 11 beta= 3 k1= -30 k2= -30 k3= 10
m= 2 alpha= 19 beta= 4 k1= -10 k2= -30 k3= 10
m= 3 alpha= 33 beta= 6 k1= 30 k2= -10 k3= 10

For u=[m,a,b,c,...] and m=1...c
m= 1 alpha= 11 beta= 3 k1= -30 k2= -30 k3= 10
m= 2 alpha= 19 beta= 4 k1= -10 k2= -30 k3= 10
m= 3 alpha= 33 beta= 6 k1= 30 k2= -10 k3= 10

1.4. Pre-Periodic Sequences. u=[m,b,a,b,a,...] and u=[m,a,b,c,...]

Now use a= 3, b= 2 and M= 5 for u=[m,b,a,b,a,...], m=1...M

m= 1 alpha= 15 beta= 4 k1= 7 k2= 7 k3= -2
m= 2 alpha= 24 beta= 5 k1= 3 k2= 7 k3= -2
m= 3 alpha= 34 beta= 6 k1= -5 k2= 3 k3= -2
m= 4 alpha= 47 beta= 7 k1= -17 k2= -5 k3= -2
m= 5 alpha= 61 beta= 8 k1= -33 k2= -17 k3= -2

Now use a= 1, b= 1 and c= 2 and M= 3 for u=[m,a,b,c,...], m=1...M

m= 1 alpha= 16 beta= 4 k1= -3 k2= -5 k3= 2
m= 2 alpha= 24 beta= 5 k1= 3 k2= -3 k3= 2
m= 3 alpha= 35 beta= 6 k1= 13 k2= 3 k3= 2

We can summarize our results from Mathematica as follows, where the smoothness of the coordinate change is set to one ($k = 1$ in the Takens Theorem):

- for constant contiued fraction expansions the conditions are violated in the cases $u = [m, a, a, ...]$ for $a = 1...9$, so there is no simple infinite periodic heteroclinic chain with constant continued fraction development. We see, for example, in the case $u = [1, 1, ...]$ of the 3-cycle, it holds that $(k_1, k_2, k_3) = (-1, 1, -1)$, which means that $\lambda_2 = \lambda_1 + \lambda_3$, which can be checked directly and serves as a consistency check.
- for 2-periodic continued fractions like $u = [2, 3, 2, 3, ...]$ or $u = [3, 5, 3, 5, ...]$, the Resonance Sign Condition (RSC) is violated, i.e. Takens-Linearization is possible. But note that for this argument to work, we have to require the coefficients to be greater than one, even after cancelling out a possible common factor. This is illustrated by the examples $u = [1, 2, 1, 2, ...]$ and $u = [2, 4, 2, 4, ...]$.
- For $u = [1, 1, 2, 1, 1, 2, ...]$, the RSC is also violated, illustrating the fact that we don't have to require the coefficients to be greater than one if the period is greater than two. For $u = [3, 3, 2, 3, 3, 2, ...]$, the RSC is also violated. However, even without using this fact, the chain would qualify for Takens Linearization, as the sum of the order of the resonances is always greater than the required α at all base points.
- We have also included the examples for $u = [a, b, c, a, b, c, ...]$ with $a = b = c = 1$ and $a = b = c = 3$ as consistency check: the formulas remain correct, but due to a common factor in the resulting coefficients, there is an earlier resonance that we already found in the section on constant contined fraction expansions.
- the 1-pre-periodic sequences $u = [3, 1, 1, 2, 1, 1, 2, ...]$ and $u = [5, 3, 2, 3, 2, ...]$ show that if the first coefficient m is bigger than the ones that follow, it cannot be assured that the NRC's are met: In the first case, this fails for $m = 3$, in the second case for $m = 4$ and $m = 5$, which means that Takens Linearization is not possible.

2. Results on Non-Resonance-Conditions and CLLP for Heteroclinic Cycles in Bianchi $VI^{*}_{-1/9}$

2.1. Takens Linearization at the Base Points of the 3-Cycle. For u=[m,a,a,...] and m=1...a with the following PARAMETERS:
a= 1, k= 1 (k=1 means smoothness is C^1)

m= 1
Sektor 1 alpha= 22 beta= 3 k= 2,0,1
Sektor 2 alpha= 14 beta= 5 k= -2,0,1
Sektor 3 alpha= 9 beta= 3 k= 2,4,1
Sektor 4 alpha= 15 beta= 6 k= 2,0,1
Sektor 5 alpha= 16 beta= 3 k= -2,0,1
Sektor 6 alpha= 17 beta= 8 k= 2,4,1

2.2. Takens Linearization at the Base Points of the 18-Cycle. We first give the coefficient matricies for the eigenvalues in the other sectors not part of the 3-cycle (see chapter 5, section 1.2):

$$M_{S_3} = \begin{pmatrix} 0 & -3 & 6 \\ 6 & -6 & 6 \\ 3 & 0 & 0 \end{pmatrix}, M_{S_4} = \begin{pmatrix} -3 & 3 & 0 \\ 0 & 6 & -6 \\ 3 & 0 & 0 \end{pmatrix}, M_{S_6} = \begin{pmatrix} 0 & -3 & 6 \\ -6 & 0 & 6 \\ -3 & 3 & 0 \end{pmatrix}$$

For u=[m,a,b,a,b,...] and m=1...b with the following PARAMETERS:
a= 3 b= 5 k= 1 (k=1 means smoothness is C^1)

m= 1
Sektor 1 alpha= 34 beta= 3 k= {-46,-80,-37}
Sektor 2 alpha= 11 beta= 4 k= {-34,-80,-37}
Sektor 3 alpha= 8 beta= 3 k= {6,-40,-37}
Sektor 4 alpha= 25 beta= 10 k= {6,-28,-37}
Sektor 5 alpha= 25 beta= 3 k= {-34,-28,-37}
Sektor 6 alpha= 30 beta= 14 k= {-46,-40,-37}
m= 2
Sektor 1 alpha= 15 beta= 3 k= {-46,-92,-43}
Sektor 2 alpha= 17 beta= 6 k= {-46,-92,-43}
Sektor 3 alpha= 13 beta= 4 k= {6,-40,-43}
Sektor 4 alpha= 10 beta= 4 k= {6,-40,-43}
Sektor 5 alpha= 11 beta= 3 k= {-46,-40,-43}
Sektor 6 alpha= 10 beta= 5 k= {-46,-40,-43}
m= 3
Sektor 1 alpha= 16 beta= 3 k= {-34,-80,-37}
Sektor 2 alpha= 22 beta= 8 k= {-46,-80,-37}
Sektor 3 alpha= 15 beta= 4 k= {6,-28,-37}

Sektor 4 alpha= 12 beta= 4 k= {6,-40,-37}
Sektor 5 alpha= 13 beta= 3 k= {-46,-40,-37}
Sektor 6 alpha= 9 beta= 5 k= {-34,-28,-37}
m= 4
Sektor 1 alpha= 20 beta= 3 k= {-10,-44,-19}
Sektor 2 alpha= 26 beta= 9 k= {-34,-44,-19}
Sektor 3 alpha= 21 beta= 5 k= {6,-4,-19}
Sektor 4 alpha= 14 beta= 4 k= {6,-28,-19}
Sektor 5 alpha= 15 beta= 3 k= {-34,-28,-19}
Sektor 6 alpha= 11 beta= 6 k= {-10,-4,-19}
m= 5
Sektor 1 alpha= 23 beta= 3 k= {26,16,11}
Sektor 2 alpha= 31 beta= 11 k= {-10,16,11}
Sektor 3 alpha= 23 beta= 5 k= {6,32,11}
Sektor 4 alpha= 19 beta= 5 k= {6,-4,11}
Sektor 5 alpha= 17 beta= 3 k= {-10,-4,11}
Sektor 6 alpha= 13 beta= 7 k= {26,32,11}

For u=[m,b,a,b,a,...] and m=1...a with the following
PARAMETERS: a= 3 b= 5 k= 1 (k=1 means smoothness is C^1)

m= 1
Sektor 1 alpha= 50 beta= 3 k= {-26,-52,-21}
Sektor 2 alpha= 11 beta= 4 k= {-26,-52,-21}
Sektor 3 alpha= 11 beta= 4 k= {10,-16,-21}
Sektor 4 alpha= 38 beta= 15 k= {10,-16,-21}
Sektor 5 alpha= 37 beta= 3 k= {-26,-16,-21}
Sektor 6 alpha= 47 beta= 21 k= {-26,-16,-21}
m= 2
Sektor 1 alpha= 16 beta= 3 k= {-6,-32,-11}
Sektor 2 alpha= 17 beta= 6 k= {-26,-32,-11}
Sektor 3 alpha= 12 beta= 4 k= {10,4,-11}
Sektor 4 alpha= 10 beta= 4 k= {10,-16,-11}
Sektor 5 alpha= 12 beta= 3 k= {-26,-16,-11}
Sektor 6 alpha= 12 beta= 6 k= {-6,4,-11}
m= 3
Sektor 1 alpha= 16 beta= 3 k= {34,28,19}
Sektor 2 alpha= 20 beta= 7 k= {-6,28,19}
Sektor 3 alpha= 14 beta= 4 k= {10,44,19}
Sektor 4 alpha= 11 beta= 4 k= {10,4,19}
Sektor 5 alpha= 13 beta= 3 k= {-6,4,19}
Sektor 6 alpha= 9 beta= 5 k= {34,44,19}

2.3. CLLP for the Classic 18-Cycle. The classic 18-cycle has the sector sequence 4343-425-43434343-425, i.e. we start in sector 4 with u=[3,5,3,5,...] which is around 3.18819:

Sector 4, u= 3.18819
Passage A: GP from Sector 4 to 3. In sector 4, the eigenvalues are:
lc= 1.5418 lt= 1.91557 ln= -1.33278 la= 0.209019
Passage A: GP from Sector 3 to 4. In sector 3, the eigenvalues are:
lc= -2.02211 lt= 3.44689 ln= 2.39822 la= 0.37611
The Eigenvalues are {1.4570,1.0000}
The Eigenvectors are {{0.167325,1.0000},{0,1.0000}}

Sector 4, u= 2.18819
Passage A: GP from Sector 4 to 3. In sector 4, the eigenvalues are:
lc= 2.02211 lt= 1.42478 ln= -1.646 la= 0.37611
Passage A: GP from Sector 3 to 4. In sector 3, the eigenvalues are:
lc= -2.81366 lt= 3.15683 ln= 3.64699 la= 0.833333
The Eigenvalues are {1.8416,1.0000}
The Eigenvectors are {{0.62500,1.0000},{0,1.0000}}

Sector 4, u= 1.18819
Passage B2: GP from Sector 4 to 2.In sector 4,the eigenvalues are:
lc= 2.81366 lt= 0.343171 ln= -1.98032 la= 0.833333
Passage B2: GP from Sector 2 to 5.In sector 2,the eigenvalues are:
lc= -3.37457 lt= 1.00965 ln= 5.82633 la= 2.45176
Passage B2: tGP from Sector 5 to 4.In sector 5,the eigenvalues are:
lc= 3.37457 lt= -2.36492 ln= -0.922814 la= 2.45176
The Eigenvalues are {3.1761,-1.9879}
The Eigenvectors are {{1.01227,1.0000},{-0.239459,1.0000}}

Sector 4, u= 5.31366
Passage A: GP from Sector 4 to 3. In sector 4, the eigenvalues are:
lc= 1.00965 lt= 2.36492 ln= -0.922814 la= 0.0868342
Passage A: GP from Sector 3 to 4. In sector 3, the eigenvalues are:
lc= -1.20737 lt= 3.41557 ln= 1.33278 la= 0.125411
The Eigenvalues are {1.2318,1.0000}
The Eigenvectors are {{0.045618,1.0000},{0,1.0000}}

Sector 4, u= 4.31366
Passage A: GP from Sector 4 to 3. In sector 4, the eigenvalues are:
lc= 1.20737 lt= 2.2082 ln= -1.08196 la= 0.125411
Passage A: GP from Sector 3 to 4. In sector 3, the eigenvalues are:

lc= -1.49614 lt= 3.45384 ln= 1.6923 la= 0.196156
The Eigenvalues are {1.3018,1.0000}
The Eigenvectors are {{0.075222,1.0000},{0,1.0000}}

Sector 4, u= 3.31366
Passage A: GP from Sector 4 to 3. In sector 4, the eigenvalues are:
lc= 1.49614 lt= 1.9577 ln= -1.29998 la= 0.196156
Passage A: GP from Sector 3 to 4. In sector 3, the eigenvalues are:
lc= -1.94792 lt= 3.45473 ln= 2.29407 la= 0.346154
The Eigenvalues are {1.4322,1.0000}
The Eigenvectors are {{0.150000,1.0000},{0,1.0000}}

Sector 4, u= 2.31366
Passage A: GP from Sector 4 to 3. In sector 4, the eigenvalues are:
lc= 1.94792 lt= 1.50681 ln= -1.60176 la= 0.346154
Passage A: GP from Sector 3 to 4. In sector 3, the eigenvalues are:
lc= -2.69398 lt= 3.23295 ln= 3.43668 la= 0.742693
The Eigenvalues are {1.7612,1.0000}
The Eigenvectors are {{0.49035,1.0000},{0,1.0000}}

Sector 4, u= 1.31366
Passage B2: GP from Sector 4 to 2.In sector 4, the eigenvalues are:
lc= 2.69398 lt= 0.538969 ln= -1.95129 la= 0.742693
Passage B2: GP from Sector 2 to 5.In sector 2, the eigenvalues are:
lc= -3.45737 lt= 1.5418 ln= 5.58196 la= 2.12459
Passage B2: GP from Sector 5 to 4.In sector 5, the eigenvalues are:
lc= 3.45737 lt= -1.91557 ln= -1.33278 la= 2.12459
The Eigenvalues are {2.8062,-1.4925}
The Eigenvectors are {{2.03045,1.0000},{-0.262732,1.0000}}

Sector 4, u= 3.18819
The Eigenvalues are {514.49,0.55783}
The Eigenvectors are {{0.448591,1.0000},{-0.414926,1.0000}}

Zusammenfassung: Wir haben gezeigt, dass es für zulässige periodische heterokline Ketten Lösungen des Bianchi IX kosmologischen Modells gibt, welche gegen die Kette in Richtung Urknall konvergieren, genauer eine C^1-stabile-Mannigfaltigkeit solcher Lösungen. Des weiteren konnten wir die Existenz einer periodischen heteroklinen Kette im Bianchi $VI^*_{-1/9}$ zeigen, welche die notwendigen Nicht-Resonanz-Bedingungen erfüllt, um eine Linearisierung mittels des Takens Linearisierungssatzes an allen Basispunkten zuzulassen.

Hiermit erkläre ich, dass ich die vorgelegte Dissertation eigenständig verfasst und keine anderen als die im Literaturverzeichnis angegebenen Quellen benutzt habe.

(Johannes Buchner)