Chapter 3

Adaptive Finite Element Method

3.1 Introduction

As already pointed out in chapter 2, singularities occur in interface problems. When discretizing the problem (2.2.1) with Finite Elements, the singularities impair the approximation properties of Finite Elements. To reduce the approximation error we will use an *a-posteriori* approach relying on a-posteriori error estimates. This approach has the advantage, that it allows for the treatment of the singularities in two and three space dimensions and that it works without knowledge of parameters depending on the singularities. These estimates are reliable and efficient and for a large class of problems also robust. Robustness means, that variations of the diffusion coefficient, i.e. the amount of the jump discontinuity, does not enter in the error bounds. The adaptive procedure consists in refining the mesh on the basis of the a-posteriori error estimators. Numerical experiments show convergence rates measured in terms of nodes of the Finite Element approximation scheme of the same order as for regular problems.
For the derivation of the error estimators we will use certain robust interpolation properties of Finite Element spaces. Therefore, it is necessary to restrict to the class of quasi-monotone diffusion coefficients [62] [24]. In this class we prove robustness of the a-posteriori error estimators. Recall from chapter 2, that quasi-monotone diffusion coefficients guarantee regularity $H^{1+1/4}$ independent of the bounds of the diffusion coefficient k.
The outline of this chapter is as follows. The problem setting and the notation used are introduced in section 3.2. We discuss approximation properties of Finite Elements on uniform grids (section 3.3) and on adapted grids (section 3.4).
To prepare the proof for the upper bound of the error by the error estimator, we need interpolation results (section 3.5). In section 3.5.2 we extend the definition of the quasi-monotonicity given in the chapter 2 to the 3D case. A robust interpolation operator, which is a slight modification of that from [24], will be defined in section 3.5.3. We address the open question about connections between regularity properties and interpolation properties in section 3.5.4.
The main results of this chapter are presented in section 3.6. We introduce residual based error estimators in section 3.6.2. Further we discuss estimators which are based on the solution of local problems and propose a new estimator in section 3.6.3.

We also discuss an approach which relies on hierarchical bases in section 3.7.1 and a Zienkiewicz-Zhu type estimator in section 3.7.2.
Extensions to problems with a mass term and Cauchy boundary conditions are given in section 3.8. The application of the a-posteriori techniques to a parabolic problem is demonstrated in section 3.9.

3.2 Problem setting

3.2.1 Approximation with Finite Elements

The continuous problem is essentially problem 3.2.2 of chapter 2, but as we allow for non-homogeneous boundary conditions we repeat the definition of the continuous problem.
Let Ω be a Lipschitz domain in $R^d, d = 2, 3$ with polygonal (polyhedral) boundary. The domain Ω can be partitioned into a finite sum of subdomains Ω_i with polygonal (polyhedral) boundary. As before there is a function k being constant on each subdomain and fulfilling the bounds

$$\delta \leq k(x) \leq \delta^{-1} \quad \forall x \in \Omega \tag{3.2.1}$$

for some $\delta > 0$. In distinction to section 2.2 we now do not demand that the subdomains Ω_i are Lipschitz.
Let the boundary $\partial\Omega$ be decomposed into $\partial\Omega = \Gamma_D \cup \Gamma_N, meas_{d-1}(\Gamma_D \cap \Gamma_N) = 0$ and $meas_{d-1}(\Gamma_D) > 0$. Let $g_D \in H^{1/2}(\Gamma_D)$ be given. There is an extension of g_D onto a function defined in $H^1(\Omega)$ and having g_D as trace on Γ_D. Let us denote this extension also by g_D. Let $g_N \in L^2(\Gamma_N)$ and $f \in L^2(\Omega)$ be given. We define the space $V := \left\{u \in H^1(\Omega) : u|_{\Gamma_D} = 0\right\}$.
The variational form of the problem is as follows: seek $u \in g_D + V$ satisfying:

$$\int_\Omega k(x)\nabla u(x)\nabla v(x)\;dx \;=\int_\Omega f(x)\;v(x)\;dx + \int_{\Gamma_N} g_N(x)\;v(x)\;dx \quad \forall v \in V \quad . \tag{3.2.2}$$

We introduce a discrete problem through Finite Element spaces $V_h \subset V$ with continuous and piecewise linear functions. The underlying triangulation is refered by $\mathcal{T}_h$. We assume that the triangulation is aligned with the partition of Ω, that means that the boundary $\partial\Omega_i$ is made up of faces from simplices in $\mathcal{T}_h$.
Let $g_{D,h}$ a Finite Element approximation of g_D on Γ_D. Then the solution of the discrete problem u_h satisfies $u_h \in g_{D,h} + V_h$ and

$$\int_\Omega k(x)\nabla u_h(x)\nabla v_h(x)\;dx \;=\int_\Omega f(x)\;v_h(x)\;dx + \int_{\Gamma_N} g_N(x)\;v_h(x)\;dx \quad \forall v_h \in V_h \quad . \tag{3.2.3}$$

We disregard problems arising from the approximation of non-homogeneous boundary conditions, that means we set $g_D = g_{D,h}$ on Γ_D and assume that g_N is piecewise constant on faces F of simplices T with $meas_{d-1}(\partial T \cap \Gamma_N) > 0$. As Γ_D has positive measure uniqueness and solvability of problems (3.2.2) and (3.2.3) is given by Riesz's theorem.

3.2.2 Assumptions and Notation

The space dimension will be denoted by d. We use the terminus “2D case” or “3D case” to indicate that $d = 2$ or $d = 3$. Unless stated otherwise all results cover the 2D and the 3D case simultaneously.
We define the weighted (semi-)norm

$$|v|^2_{kH^1(\Omega')} \quad := \int_{\Omega'} k(x)\,(\nabla v(x))^2 \; dx \quad .$$

In the following all integrals are over the space variable x and for the sake of shortness we do not explicitly denote the dependence on x.
The triangulation consists of simplices which intersect at most at a common face, edge, or vertex. Such a triangulation is called *admissible* [15].
We make the following assumptions for the discrete problem (3.2.3): The family of triangulations $\{\mathcal{T}_h\}$ is shape regular but not necessarily uniform [15]. The diameter of a simplex T is denoted by h_T. To ease notation we may drop the subindex T and take the value of h from the simplex under consideration. We will use also the term *mesh* or *grid* instead of *triangulation*.
The triangulation defines a set Ω_h, which coincides with Ω. Further $\bar{\Gamma}_D$ and $\bar{\Gamma}_N$ consists of whole faces of simplices from $\mathcal{T}_h$. To avoid ambiguity we suppose that Γ_D is a closed set.

Definition 3.1 *For functions a, b depending on the data $k, f, u, u_h, g_N, g_{D_h}$ with values in R^1 we use the notation*

(3.2.4) $$a \preceq b$$

to indicate, that it holds $a \leq c\; b$ for a further not specified constant c, that does not depend on the data but only on the shape regularity parameter of the finite element mesh $\mathcal{T}_h$. The bound (3.2.4) will be called robust.
We write $a \approx b$, if it holds $a \preceq b$ and $b \preceq a$.

The set of nodes of the triangulation will be denoted by $\mathcal{N}_h$. We denote by $\mathcal{F}_h$ all faces from simplices from $\mathcal{T}_h$ that are not contained in $\bar{\Gamma}_D$. The set of all edges will be denoted by $\mathcal{E}_h$. All simplices $T \in \mathcal{T}_h$, faces $F \in \mathcal{F}_h$, and edges $E \in \mathcal{E}_h$ are closed sets. For a subset $S \subset \bar{\Omega}$ we denote by $\mathcal{N}_h(S)$ the nodes contained in S and with $\mathcal{E}_h(S)$ the edges in S.
For a node x resp. an edge E or face F we define ω_x resp ω_E or ω_F as the union of all simplices T, which have x resp. E or F in common.
The value of k on a simplex T will be denoted by k_T. For a face F denote by T_F a simplex from ω_F such that $k_{T_F} = \max_{T \subset \omega_F} k_T$. We define $k_F := \sum_{T \subset \omega_F} k_T$. Clearly $k_F \approx k_{T_F}$.
We define a simplex-wise constant approximation f_h of f. For instance one may choose the average on the simplex T, $f_h|_T := |T|^{-1} \int_T f$.

Definition 3.2 *Let $F \in \mathcal{F}_h$ be not contained in Γ_N. Denote by n_T and $n_{T'}$ the outward normal of $F \subset \partial T$ resp. $F \subset \partial T'$. Let u_h be the solution of problem (3.2.3) The jump of the normal fluxes across the face F is defined as*

$$j_F := k_T \frac{\partial u_h}{\partial n_T} + k_{T'} \frac{\partial u_h}{\partial n_{T'}} \quad .$$

Let F be a face on the Neumann boundary. Denote by n_F the outward normal of $F \subset \partial\Omega$. The jump of the normal fluxes across this face F is defined as

$$j_F := g_N|_F - k_T \frac{\partial u_h}{\partial n_F} \quad .$$

The Finite Element shape function taking on the value 1 at the node x_i and vanishing on all other nodes of the triangulation are denoted by λ_i. We define so-called *bubble functions*. These are non-negative shape functions with small support and which are not contained in V_h. An *element bubble function* can be defined as $\phi_T := (d+1)^{d+1} \prod_{x_i} \lambda_i$, where the product is taken over all nodes of x_i on T. A *face bubble function* ϕ_F for the face F will have ω_F as support. We define $\phi_F = d^d \prod_{x_i} \lambda_i$, where the product is taken over all nodes x_i on the face F.

3.3 Finite Element Method on uniform grids

Let us assume in this section that the mesh is quasi-uniform. That means, that the local cell diameter $h(T) \approx h$ is of the same order for each simplex T. Then the approximation error can be measured in terms of the mesh diameter h.
Usually the convergence rate of the solution u_h of (3.2.3) to the solution u of (3.2.2) depends on the number s which yields *global* regularity $u \in H^{1+s}(\Omega)$. We show that in the 2D case it is the *piecewise* regularity $u \in H^{1+s}(\Omega_i)$ and not the global regularity that bounds the convergence rate. For a proof in 3D see [11].
Recall that regularity results from [33] [31] show $u \in H^{1+s}(\bar{\Omega})$ for a certain $s > 0$. The dependence of s from δ is given in 2D in Theorem 2.17. By Sobolev embeddings we know that u is continuous in $\bar{\Omega}$ [1]. Thus, the interpolation operator $I_N : V \rightarrow V_h$ which is given by taking the values in the nodal points is well defined. Exploiting arguments given in Example 3 in [25], one has the following interpolation results in fractional Sobolev spaces $H^{1+s}, s > 0$:

$$|u - I_N(u)|_{H^1(T)} \preceq h^s |u|_{H^{1+s}(T)} \quad \forall T \in T_h \quad . \tag{3.3.1}$$

Combining this bound with Galerkin orthogonality shows an estimate in terms of piecewise contributions for $u \in H^{1+s}(\Omega_i)$:

$$|u - u_h|^2_{kH^1(\Omega)} \leq |u - I_N(u)|^2_{kH^1(\Omega)} = \sum_{l=1}^{n} k_i \, |u - I_N(u)|^2_{H^1(\Omega_i)} \preceq h^{2s} \sum_{l=1}^{n} k_i |u|^2_{H^{1+s}(\Omega_i)} \quad . \tag{3.3.2}$$

This bound shows that Finite Elements make in a certain sense use of piecewise regularity. As global regularity is restricted to $H^{3/2-\varepsilon}(\Omega)$, for any $\varepsilon > 0$, (see section 2.3.2), it follows, that for $s > 1/2$ the convergence is bounded by piecewise regularity and not by global regularity.
As the regularity parameter s, which depends on k, can be arbitrary small, the convergence rate can deteriorate.

Remark 3.1 *Take the singular function ψ_2 from example 2.2 on page 16 and choose $\lambda \in (0,1]$. Using approximation results from [47, p.265] [60] we see that on a simplex T, containing the singular point, the convergence of the solution u_h of problem (3.2.3) to the solution u of problem (3.2.2) could be arbitrarily bad*

$$c(u)h^{\lambda} \preceq |u - u_h|_{kH^1(T)} \preceq h^{\lambda-\varepsilon}C(u) \quad \forall\varepsilon > 0 \quad .$$

This is also true for polynomial Finite Elements of higher order [20]. In view of higher order convergence in regions where the solution is in H^2, the singularity will asymptotically dominate on uniform meshes leading to a global convergence rat of order $O(h^{\lambda-\varepsilon})$ for any $\varepsilon > 0$.

In order to enhance convergence for solutions with poor regularity, several techniques have been developed. One possibility is the enrichment of the Galerkin space with functions, that approximate the singular functions [56]. See also [20] for a modified procedure. Another approach consists in refining the mesh around the singularities.

3.4 Finite Element Method on adapted grids

Let us start with an initial mesh. By refinement of the initial mesh around a singular point we understand the construction of a triangulation with additional degrees of freedom around the singular point.
Here we differ between an approach, where the degree of the singularity should be known *a-priorily*, and a competing approach, where no a-priori knowledge is required and the mesh is refined on the basis of *a-posteriori* error estimators. The a-priori approach makes use of the fact that the form of the singular solution $r^{\lambda}s_{\lambda}(\varphi)$ is known. Here (r,φ) are polar coordinates with respect to a singular point.
A shape regular triangulation is constructed in such a way that the local mesh diameter behaves like $h \approx Hr^{1-\mu}, \mu < \lambda$, [47] [6]. Here H is a parameter and can be seen as global mesh size. Additionally, the simplex containing the singular point should have a diameter of order $H^{1/\mu}$. In order to construct such meshes, λ should be known or approximated [32]. Within this setting the error reduction proceeds with optimal order $O(H)$.
In 3D the parameter r is the distance to the singular edge and anisotropic simplices that are stretched in direction of the edge may be used. See [5] for the case of a constant diffusion coefficient $k(x) \equiv 1$.
The drawback of a-priori refinement is not only that λ should be known but also that there is no control of the refinement depth (the parameter H) in the vicinity of a singular point: It may be the case that in a particular situation the contribution of a singular function is zero or very small, when compared to the regular part or to other singularities. Then there is no need of refining the singularity.

Besides varying the local grid-size one can increase the degree of the polynomial shape functions. This leads to the p-version, developed for the case $k = 1$ [7] [8] [53]. The p-version has been extended to non-polynomial shape functions for interface problems in [48]. Here again the method suffers from the drawbacks of a-priori refinement and the convergence rate depends on the approximation of λ . But the method leads to a very high rate of convergence even in case of very low regularity.
An alternative *a-posteriori* approach is the construction of a new refined mesh by subdividing simplices on which the error is large into smaller ones. This corresponds to an introduction of new degrees of freedom. Since the error itself is not known, one uses a-posteriori error estimators η_T, which should reflect the behaviour of the error:

$$\eta_T \approx |u - u_h|_{kH^1(T)} \quad .$$

The a-posteriori error estimators can be calculated with low numerical effort on the basis of known data.
Such estimators have been derived for the Laplace equation. An overview is given in [57]. For the interface problem there are recent independent articles [21] [11] and [49]. These articles impose different restrictions on the structure of k. In [21] the case of essentially two subdomains is treated. In [11] a special criterion concerning the distribution of diffusion coefficients is used; see remarks 3.3 and 3.4. In the present chapter weaker assumptions on k are posed.
The reason for the restriction of the diffusion coefficient is that we will make use of interpolation operators which are robust in respective function spaces. In the next section we argue that the restriction imposed on k in the present work are necessary for the existence of such robust interpolation operators.

3.5 Interpolation operators

3.5.1 A non-robust interpolation operator

In this section we use the shorter terminus *coefficient* for the diffusion coefficient k_T.
For simplices T and for faces F we need estimates of the form

$$\begin{aligned} k_T^{1/2} \ \|u - I_L(u)\|_{L^2(T)} &\preceq h \ |u|_{kH^1(\tilde\omega_T)} \\ k_F^{1/2} \ \|u - I_L(u)\|_{L^2(F)} &\preceq h^{1/2} \, |u|_{kH^1(\tilde\omega_{T_F})} \quad , \end{aligned} \tag{3.5.1}$$

where $I_L : V \to V_h$ is an interpolation operator and $\tilde\omega_T, \tilde\omega_{T_F}$ consists of some neighbouring simplices of T and F. It is important that no additional factors depending on k do not enter into the bounds (3.5.1). An interpolation operator fulfilling the bounds (3.5.1) will be called *robust*.
The Clement interpolation operator I_C [16] [57] fulfills only the non-robust bounds

Lemma 3.1 *Let $\delta \le k \le \delta^{-1}$. For any $T \in \mathcal{T}_h$ and any $F \in \mathcal{F}_h$ the non-robust bounds*

$$\begin{aligned} k_T^{1/2} \ \|u - I_C(u)\|_{L^2(T)} &\preceq h \ \delta^{-1} \ |u|_{kH^1(\hat\omega_T)} \quad , \\ k_F^{1/2} \ \|u - I_C(u)\|_{L^2(F)} &\preceq h^{1/2} \ \delta^{-1} \ |u|_{kH^1(\hat\omega_F)} \quad , \end{aligned} \tag{3.5.2}$$

hold. Here $\hat{\omega}_T, \hat{\omega}_F$ denote the union of simplices from $\mathcal{T}_h$ that have a node with T resp. F in common. These bounds are optimal with respect to δ.

PROOF. The proof is an easy consequence of the properties of the Clement interpolation operator

$$\begin{aligned}\|u - I_C(u)\|_{L^2(T)} &\preceq h\ |u|_{H^1(\hat{\omega}_T)}\\ \|u - I_C(u)\|_{L^2(F)} &\preceq h^{1/2}\ |u|_{H^1(\hat{\omega}_F)} \quad .\end{aligned}$$

Denote by k_{min}, k_{max} the lower and upper bound of k on $\hat{\omega}_T$. It is easy to see that

$$|u|^2_{H^1(\tilde{\omega}_T)} \ \le \sum_{T' \subset \hat{\omega}_T} \frac{k_{T'}}{k_{min}}\ |u|^2_{H^1(T')} \ = \frac{1}{k_{min}}\ |u|^2_{kH^1(\hat{\omega}_T)} \ \le k_T^{-1} \frac{k_{max}}{k_{min}}\ |u|^2_{kH^1(\hat{\omega}_T)} \quad .$$

The relation $k_{max}/k_{min} \le \delta^{-2}$ finishes the proof of the first inequality. The second inequality is shown in the same way. ■

3.5.2 The quasi-monotonicity condition revisited

Robust interpolation operators have been derived under certain restrictions on the coefficient k [24] [13].
For some special configurations of diffusion coefficients it has been shown in [62] that there are *no robust interpolation operators*. In 2D regard a checkerboard like distribution of coefficients from $\{1, \varepsilon\}$. In 3D two cubes touch at a vertex or on an edge. Inside the cubes the diffusion is $k_1 = 1$ and in the remaining part of the domain $k_2 = \varepsilon$.
Following [24] we introduce the class of *quasi-monotone coefficients*. Within this class one can define robust interpolation operators [24]. For 2D the class of quasi-monotone coefficients was already defined in Definition 2.4 in chapter 2 and this class coincides with the one defined below.

Definition 3.3 *For a node $x \in \mathcal{N}_h(\bar{\Omega})$ we denote by $\tilde{T}_x$ a simplex from ω_x where the coefficient k_T achieves the maximum for $T \subset \omega_x$.*

Definition 3.4 *Choose a node $x \in \mathcal{N}_h(\bar{\Omega})$. The distribution of coefficients $k_T,\ \ T \subset \omega_x$ will be called* quasi-monotone with respect to a node x of a triangulation $\mathcal{T}_h$ and the part of the boundary $\Gamma_D \subset \partial\Omega$ *if the following conditions are fulfilled.*
For each simplex $T \subseteq \omega_x$ there exists a Lipschitz set $\tilde{\omega}_{T,x,qm}$ containing only simplices from ω_x such that

– *if $x \in \mathcal{N}_h(\bar{\Omega}/\bar{\Gamma}_D)$ then $T \cup \tilde{T}_x \subseteq \tilde{\omega}_{T,x,qm}$ and*

$$k_T \le k_{T'} \quad \forall T' \subseteq \tilde{\omega}_{T,x,qm} \quad .$$

– *if $x \in \mathcal{N}_h(\bar{\Gamma}_D)$ then $T \subseteq \tilde{\omega}_{T,x,qm}$, $meas_{d-1}(\partial\tilde{\omega}_{T,x,qm} \cap \Gamma_D) > 0$ and*

$$k_T \le k_{T'} \quad \forall T' \subseteq \tilde{\omega}_{T,x,qm} \quad .$$

Figure 3.1: The distribution of coefficients $k_T, T \subset \omega_x$ is quasi-monotone with respect to the node x on the left figure but not on the right one. The simplex T is colored dark and the set $\tilde{\omega}_{T,x,qm}$ is colored with different levels of grey in the left picture

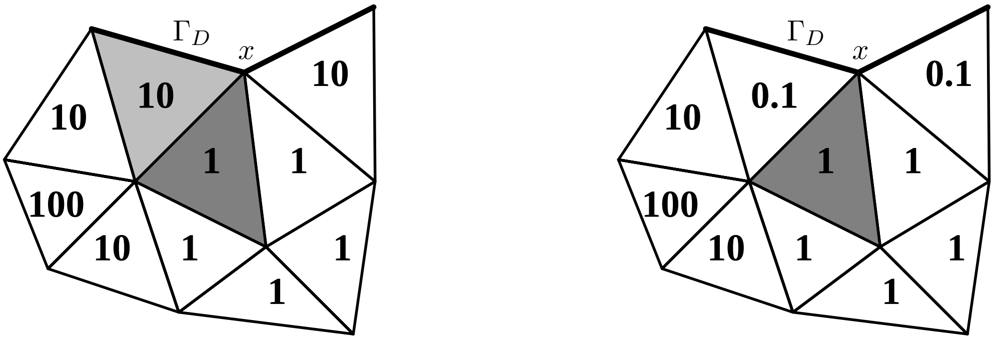

Figure 3.2: The distribution of coefficients $k_T, T \subset \omega_x$ is quasi-monotone with respect to the node $x \in \mathcal{N}_h(\Gamma_D)$ on the left figure but not on the right. The simplex T is colored dark and the set $\tilde{\omega}_{T,x,qm}$ is colored with different levels of grey in the left figure

One checks that the definition does not depend on the choice of $\tilde{T}_x$. Further it is important to note that this definition is in 2D also independent of the triangulation and describes a property of the diffusion coefficients.
If there is no danger of confusion, we will simply say that the distribution of coefficients is quasi-monotone with respect to a node x if the above definition is fulfilled for x.
We say that the distribution of coefficients $k_T, T \in \mathcal{T}_h$ is quasi-monotone if it is quasi-monotone with respect to all nodes of $\mathcal{N}_h$.
Here we use the definition of Lipschitz domains as given in [15]. It follows that the union of two simplices from $\mathcal{T}_h$ sharing a node is a Lipschitz domain if and only if they share a face.
We illustrate the quasi-monotonicity condition for an interior node (Figure 3.1) and for a node $x \in \Gamma_D$ (Figure 3.2).
Note that a checker board like distribution of the coefficients is not quasi-monotone.

Remark 3.2 *Definition 3.4 can be formulated in a more intuitive way. We give only the idea. For a node $x \in \mathcal{N}_h(\bar{\Omega}/\bar{\Gamma}_D)$ we demand that the trace of k on a small sphere B around x has* only one local maximum. *We say that a local maximum is attained on $B \cap \Omega_i$, if $k_i > k_j$ for all adjacent subdomains $\Omega_j : meas_{d-1}(\bar{\Omega}_i \cap \bar{\Omega}_j) > 0$ with node $x \in \Omega_j$. If $x \in \mathcal{N}_h(\bar{\Gamma}_D)$ each local maximum is adjacent to Γ_D.*

Remark 3.3 *Under the following restrictions any distribution of coefficients is quasi-monotone. Let $x \in \bar{\Omega}$ a point, denote by n the number of subdomain Ω_i to whose closure x belongs and by m the number of boundary types from $\bar{\Gamma}_D, \bar{\Gamma}_N$ to which x belongs. In other words set $m = 1$ if pure Dirichlet or pure Neumann boundary conditions are imposed on $x \in \partial\Omega$, set $m = 0$ if x is an interior point and otherwise $m = 2$. If in 2D n is restricted by*

$$n \leq 3 - m \quad ,$$

then the distribution of coefficients $k_T, T \in \omega_x$, is quasi-monotone for any values of k_T. A similar bound holds in the 3D-case. If

$$n \leq 2 - m$$

the distribution of the coefficients is quasi-monotone independent of the values of k_T.
One checks that these restrictions on n are sharp, *except for the case $m = 2$. To see this in case of an interior node x in 3D regard to cubes Ω_1, Ω_2, which touch only in x. The diffusion takes on the value 100 in the cubes and the value 1 in the remaining part Ω_3. Then $n = 3, m = 0$ and the distribution of the diffusion coefficients is not quasi-monotone.*

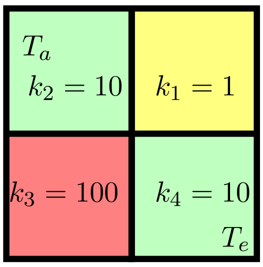

Figure 3.3: quasi-monotone diffusion coefficients which do not fulfill condition from [11]

The following restriction of the diffusion coefficient has been used in [11] to derive robust interpolation operators.

Remark 3.4 *Suppose that the coefficients $k_{T'}, T' \subset \omega_x$, are distributed in such a way that for any two simplices $T_b, T_e \subset \omega_x$ there are simplices $T_i \subset \omega_x, i = 1,..,n$, where $T_1 = T_a, T_n = T_b$ and $meas_{d-1}(T_i \cap T_{i+1}) > 0$ for $i = 1,..,n-1$. Suppose further that the sequence $k_{T_1}, k_{T_2}, .., k_{T_{n-1}}, k_{T_n}$ is monotone.*
Then it is not hard to see that the distribution of coefficients $k_{T'}, T' \subset \omega_x$ is quasi-monotone with respect to a node x if the space dimension is 2 or if x does not belong to Γ_D. If in 3D a node belongs to Γ_D this condition is not sufficient *to define robust interpolation operators. To*

see this, regard the 3D example of a cube Ω_1 *touching the Dirichlet boundary* $\Gamma_D = \partial\Omega$ *at one vertex only. Define* Ω_2 *to be the remaining part of the domain and set* $k_1 := 100, k_2 := 1$ *as proposed in [62].*
We show that the above condition is stronger than quasi-monotonicity *in the 2D case or if* $x \in \mathcal{N}_h(\bar{\Omega}/\bar{\Gamma}_D)$. *Define in 2D four coefficients which are numbered clockwise and which take on the value* $k_1 = 1, k_2 = 10, k_3 = 100, k_4 = 10$ *(Figure 3.5.2). Taking* $T_b = T_4, T_b = T_e$ *the above condition is not fulfilled but the distribution of coefficients* k_i *is quasi-monotone. In 3D proceed similarly.*

Remark 3.5 *Using the counter examples from Xu [62] one notices that* quasi-monotonically distributed coefficients are the largest class of coefficients for which robust interpolation operators exists. *This is easily seen from remark 3.2. The counter examples in [62] exploit the fact that there is more than one local maximum of the diffusion coefficient when restricted to a small sphere around a vertex in the sense of remark 3.2.*

We say that a triangulation $\mathcal{T}_{h'}$ is a refined triangulation of $\mathcal{T}_h$ if for the according Finite Element spaces holds $V_h \subset V_{h'}$. In 2D quasi-monotonicity is preserved during refinement of a triangulation.

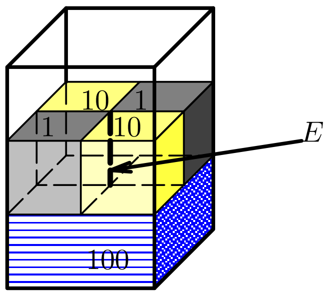

Figure 3.4: three horizontal layers, in the middle layer there is a checkerboard-like pattern of diffusion coefficients $1, 10$ around the edge E; for a new node on E quasi-monotonicity will be violated

In 3D this is not true. For example regard the case of a domain $\Omega = (-1,1) \times (-1,1) \times (-1,2)$ subdived into three horizontal layers of equal size as shown in Figure 3.5.2. Each layer is again subdivided into four similar cubes. In the bottom and upper layer the coefficient is 100. For the four cubes sharing the edge E given by the points $(0,0,0)$ and $(0,0,1)$ the coefficients 1 and 10 are distributed alternately. Given Dirichlet boundary conditions on $\partial\Omega$ and given a triangulation with nodes on vertices of the 12 cubes, the distribution of coefficients is quasi-monotone. But if in the course of refinement a new node is created on the edge E, the quasi-monotonicity condition is violated for this node.
To assure quasi-monotonicity also for triangulations obtained by refinement we introduce the following

Definition 3.5 *Let $d = 3$. Choose an edge $E \in \mathcal{E}_h$. The distribution of the coefficients $k_{T'}, T' \subset \omega_E$ will be called* quasi-monotone with respect to the edge E of a triangulation $\mathcal{T}_h$ and the part of the boundary $\Gamma_D \subset \partial\Omega$ *if following conditions are fulfilled:*
Denote by $\tilde{T}_E$ a simplex from ω_E where k_T achieves the maximum in ω_E.
For each simplex $T \subset \omega_E$ there exist a Lipschitz set $\tilde{\omega}_{T,E,qm}$ containing only simplices from ω_E, such that

– *if $E \in \mathcal{E}_h(\bar{\Omega}/\bar{\Gamma}_D)$ then $T \cup \tilde{T}_E \subseteq \tilde{\omega}_{T,E,qm}$ and*

$$k_T \leq k_{T'} \quad \forall T' \subseteq \tilde{\omega}_{T,E,qm} \quad .$$

– *if $E \in \mathcal{E}_h(\bar{\Omega}/\bar{\Gamma}_D)$ then $T \subseteq \tilde{\omega}_{T,E,qm}$, $meas_2(\partial\tilde{\omega}_{T,E,qm} \cap \Gamma_D) > 0$ and*

$$k_T \leq k_{T'} \quad \forall T' \subseteq \tilde{\omega}_{T,E,qm} \quad .$$

We say that the distribution of coefficients $k_T, T \in \mathcal{T}_h$ is quasi-monotone with respect to edges of $\mathcal{T}_h$ if the above condition holds for all edges $E \in \mathcal{E}_h$.
To illustrate this condition in 3D, denote by G_E a 1-dimensional sphere perpendicular to E, with center on E and contained in ω_E. This definition states that for interior edges E the coefficient function has only one local maximum on G_E.

Remark 3.6 *If in 3D the distribution of coefficients $k_T, T \in \mathcal{T}_h$, is quasi-monotone and additionally quasi-monotone with respect to edges of $\mathcal{T}_h$, then the distribution of coefficients $k_T, T \in \mathcal{T}_{h'}$, is quasi-monotone for any refined triangulation $\mathcal{T}_{h'}$.*

3.5.3 A robust interpolation operator

In the case of a mass term occuring in the elliptic equation one may need an interpolation operator which is continuous in $L^2(\Omega)$. Such stability does not hold in the case of the interpolation operator defined in [24]. A further difference is that we allow also for mixed boundary conditions. Our interpolation operator differs from the one presented in [24] only for nodes on the boundary $\partial\Omega$. The same operator was proposed independently in [11].
Let the distribution of coefficients $k_T, T \in \mathcal{T}_h$ be quasi-monotone. For a simplex T define a set containing some neighbouring simplices of T

$$\tilde{\omega}_T := \bigcup_{x \in \mathcal{N}_h(T)} \tilde{\omega}_{T,x,qm} \quad .$$

For a face F define $\tilde{\omega}_{T_F}$ by substituting in the above definition T with T_F.
The quasi-monotonicity condition implies then

$$k_T \leq k_{T'} \quad \forall T' \subset \tilde{\omega}_T \quad .$$

Remember that $\tilde{T}_x$ is defined in definition (3.3). The interpolation operator is defined by

$$I_L u := \sum_{x_i \in \mathcal{N}_h(\Omega \cup \Gamma_N)} \lambda_i p_{x_i} \quad , \quad \text{where } p_{x_i} := \frac{1}{|\tilde{T}_{x_i}|} \int_{\tilde{T}_{x_i}} u \quad , \quad x_i \in \mathcal{N}_h(\Omega \cup \Gamma_N)$$

and λ_i are the finite element shape functions of V_h. Hence $I_L : V \to V_h$.
For convenience define $p_x := 0$ for nodal points $x \in \Gamma_D$ so that in fact $I_L u := \sum_{x_i} \lambda_i p_{x_i}$ where the sum is taken over $\mathcal{N}_h(\bar{\Omega})$. Alternatively I_L can be defined by substituting in the definition of p_{x_i} the term $\tilde{T}_{x_i}$ by the union of simplices where k takes the maximum in ω_x.
We need a scaled version of a standard trace inequality. It states that

Lemma 3.2 *Let $\Omega_0 \subset R^d, d = 2,3$, be a domain with diameter h and Lipschitz boundary. Let F be a subset of Ω_0 with positive measure. Then for $v \in H^1(\Omega_0)$*

$$\|v\|^2_{L^2(F)} \preceq h^{-1} \|v\|^2_{L^2(\Omega_0)} + h\, |v|^2_{H^1(\Omega_0)} \quad .$$

PROOF. This is a refined version of a standard trace inequality [29] for domains with diameter $O(1)$. The constant in the bound depends on the Lipschitz constant of Ω_0. ■

Here we state the main result in this section.

Lemma 3.3 *Let $d = 2,3$. Let $u \in V$. Choose a simplex $T \in \mathcal{T}_h$ and a face $F \in \mathcal{F}_h$. Let the distribution of coefficients $k_{T'}, T' \subset \omega_x$ be quasi-monotone with respect to all nodes x of T and T_F. Then the following bounds hold for any $v \in V$*

$$\|I_L(v)\|_{L^2(T)} \preceq \|v\|_{L^2(\tilde{\omega}_T)} \tag{3.5.3}$$

$$\|v - I_L(v)\|_{L^2(T)} \preceq h\, |v|_{H^1(\tilde{\omega}_T)} \preceq k_T^{-1/2} h\, |v|_{kH^1(\tilde{\omega}_T)} \tag{3.5.4}$$

$$|v - I_L(v)|_{H^1(T)} \preceq |v|_{H^1(\tilde{\omega}_T)} \preceq k_T^{-1/2}\, |v|_{H^1(\tilde{\omega}_T)} \tag{3.5.5}$$

$$k_F^{1/2}\, \|v - I_L(v)\|_{L^2(F)} \preceq h^{1/2}\, |v|_{kH^1(\tilde{\omega}_{T_F})} \quad . \tag{3.5.6}$$

PROOF. The proof is similar to that in [24]. As the definition of I_L contains only integrals on simplices but no integrals on faces it is possible to bound the L^2-norm of I_L in terms of the L^2-norm.
Choose a simplex T and number its nodes with $x_i, i = 0, .., d$. Let $x \in \mathcal{N}_h(\Omega \cup \Gamma_N)$. Note that p_x can be written as $P_x(v)$ where P_x is the L^2-orthogonal projection on constant functions in $L^2(\tilde{T}_x)$. Exploiting the property of this projection it yields for nodes $x_i \in \mathcal{N}_h(T/\Gamma_D)$ and any $c \in R$

$$\begin{aligned} \|p_{x_i} - c\|^2_{L^2(T)} &\preceq \|p_{x_i} - c\|^2_{L^2(\tilde{T}_{x_i})} \\ &= \|P_{x_i}(v - c)\|^2_{L^2(\tilde{T}_{x_i})} \leq \|v - c\|^2_{L^2(\tilde{T}_{x_i})} \quad . \end{aligned} \tag{3.5.7}$$

Further, we use the decompositions

$$v = \sum_{i=0}^{d} \lambda_i v \quad \text{and} \quad I_L(v) = \sum_{i=0}^{d} \lambda_i p_{x_i} \quad . \tag{3.5.8}$$

We conclude from (3.5.7) with $c = 0$ and (3.5.8) that

$$\|I_L(v)\|^2_{L^2(T)} \approx \sum_{i=0}^{d} \|p_{x_i}\|^2_{L^2(T)} \preceq \sum_{i=0}^{d} \|v\|^2_{L^2(\tilde{T}_{x_i})} \preceq \|v\|^2_{L^2(\tilde{\omega}_T)} \quad .$$

This shows (3.5.3). Now let us prove (3.5.4). From (3.5.8) we obtain

$$\|v - I_L(v)\|^2_{L^2(T)} \;\leq \sum_{i=0}^{d} \|\lambda_i(v - p_{x_i})\|^2_{L^2(T)} \;\leq \sum_{i=0}^{d} \|v - p_{x_i}\|^2_{L^2(T)} \quad . \tag{3.5.9}$$

Inequality (3.5.7) applied to nodes $x_i \in \mathcal{N}_h(T/\Gamma_D)$ yields for any $c \in R$

$$\begin{aligned} \|v - p_{x_i}\|^2_{L^2(T)} &\leq \|v - c\|^2_{L^2(T)} + \|p_{x_i} - c\|^2_{L^2(T)} \\ &\preceq \|v - c\|^2_{L^2(T)} + \|v - c\|^2_{L^2(\tilde{T}_{x_i})} \quad . \end{aligned}$$

Recall that from definition $T \cup \tilde{T}_x \subset \tilde{\omega}_{T,x,qm} \subset \tilde{\omega}_T$. We use the last inequality and apply the Poincaré inequality [4] to the Lipschitz set $\tilde{\omega}_{T,x_i,qm}$ and arbitrary c to obtain

$$\|v - p_{x_i}\|^2_{L^2(T)} \;\preceq\; \|v - c\|^2_{L^2(\tilde{\omega}_{T,x_i,qm})} \;\preceq h^2\, |v|^2_{H^1(\tilde{\omega}_{T,x_i,qm})} \quad . \tag{3.5.10}$$

For nodes x_i from T lying on the Dirichlet boundary we use $p_{x_i} = 0$ and the fact that v vanishes on $\partial\tilde{\omega}_{T,x,qm} \cap \Gamma_D$. The Poincaré-Friedrichs inequality [4] yields

$$\|v - p_{x_i}\|^2_{L^2(T)} \;\leq\; \|v\|^2_{L^2(\tilde{\omega}_{T,x_i,qm})} \;\preceq h^2\, |v|^2_{H^1(\tilde{\omega}_{T,x_i,qm})} \quad . \tag{3.5.11}$$

Collecting inequalities (3.5.10) and (3.5.11), together with (3.5.9), shows

$$\|v - I_L(v)\|^2_{L^2(T)} \;\preceq h^2\, |v|^2_{H^1(\tilde{\omega}_T)} \quad . \tag{3.5.12}$$

It remains to use the quasi-monotonicity condition which states

$$k_T \leq k_{T'} \quad \forall T' \subset \tilde{\omega}_T \quad . \tag{3.5.13}$$

With this bound we prove

$$|v|^2_{H^1(\tilde{\omega}_T)} \;= \sum_{T' \subset \tilde{\omega}_T} k_{T'}^{-1}\, |v|^2_{kH^1(T')} \;\leq k_T^{-1}\, |v|^2_{kH^1(\tilde{\omega}_T)} \tag{3.5.14}$$

which yields due to (3.5.12) assertion (3.5.4).
For showing (3.5.5) we use as before (3.5.8) to conclude

$$|v - I_L(v)|^2_{H^1(T)} \;\leq \sum_{i=0}^{d} |\lambda_i(v - p_{x_i})|^2_{H^1(T)} \quad .$$

The properties of the shape functions λ_i imply then

$$\begin{aligned} |\lambda_i(v - p_{x_i})|^2_{H^1(T)} &\preceq \|(\nabla\lambda_i)(v - p_{x_i})\|^2_{L^2(T)} + \|\lambda_i \nabla(v - p_{x_i})\|^2_{L^2(T)} \\ &\preceq h^{-2}\, \|v - p_{x_i}\|^2_{L^2(T)} + |v|^2_{H^1(T)} \quad . \end{aligned} \tag{3.5.15}$$

We combine again (3.5.10) and (3.5.11) to bound

$$h^{-2} \sum_{i=0}^{d} \|v - p_{x_i}\|^2_{L^2(T)} \;\preceq\; |v|^2_{H^1(\tilde{\omega}_T)} \quad . \tag{3.5.16}$$

From the last three inequalities and (3.5.13) follows now (3.5.5).
The trace inequality from Lemma 3.2 and inequalities (3.5.4) , (3.5.5) show

$$\begin{aligned} \|v - I_L(v)\|^2_{L^2(F)} &\preceq h^{-1} \|v - I_L(v)\|^2_{L^2(T_F)} + h\, |v - I_L(v)|^2_{H^1(T_F)} \\ &\preceq h\, |v|^2_{H^1(T_F)} \quad . \end{aligned}$$

Multiplication with $k_{T_F} \approx k_F$ proves due to (3.5.13) with $\tilde{\omega}_T$ substituted by $\tilde{\omega}_{T_F}$ the last assertion (3.5.6). ■

3.5.4 Does regularity imply interpolation properties ?

In this section we set $d = 2$. We saw that the property of quasi-monotonicity implies on the one hand regularity $H^{5/4}$ (section 2.5.4) and on the other hand approximation properties of Finite Elements in norms depending on the coefficient k (see inequality (3.5.5)). It is interesting whether there is a connection between regularity $H^{5/4}$ and these approximation properties. This open question will be addressed in the following.
In classical interpolation results, as for instance for the Lagrange interpolation operator, regularity $H^{d/2+\varepsilon}$ (for some $\varepsilon > 0$) is needed [15] and the stronger semi -norm $|\cdot|_{H^{d/2+\varepsilon}}$ *will appear* on the right hand side of the interpolation inequality . This is a difference to our approach since we want to use only the weaker norm $\|k^{1/2}\nabla \cdot\|_{L^2(\Omega)}$.
Fix an interior singular point x_0 and denote by B a ball centered at x_0 and contained in ω_x and with diameter approximately h. Suppose that B contains no other singular point. Our aim is to have an interpolation result of the analogon of inequality 3.5.4 in following form

$$\|k^{1/2}(v - v_h)\|_{L^2(B)} \leq c\, \lambda^{-1/2}\, h\, \|k^{1/2}\, \nabla v\|_{L^2(B)} \quad . \tag{3.5.17}$$

for functions $v \in V$ of the special form $v := u - u_h$ where u, u_h, are solutions of problems (3.2.2),(3.2.3) and $u \in H^{1+s}(\Omega_i)$ for any $0 < s < \lambda$. Here $v_h \in V_h$ is an approximation of v and c does not depend on the data k nor on v, u or u_h. Comparison with inequality 3.5.4 shows an additional factor $\lambda^{-1/2}$ that depends on the regularity of v and thus on k. This factor is necessary since it has been shown in [Xu] that inequalities like (3.5.17) do not hold for arbitrary $v \in V$ and $v_h \in V_h$ without additional factors depending on v.
The bound (3.5.17) is stronger than (3.5.2) since we conclude from Theorem 2.16 that $\delta \preceq \lambda$ and hence $\lambda^{-1/2} \preceq \delta^{-1/2} < \delta^{-1}$. Setting for a moment $u_h = 0$, $v_h = 0$ and $v = \psi_2$, where ψ_2 is defined in example 2.2, we see that the factor $\lambda^{-1/2}$ is necessary. The same is true for functions v_h with a small norm with respect to v. We have to pay for decreasing regularity with a higher factor.
We do not know if an estimate like (3.5.17) holds but we want to give some ideas how to check it, if it holds. Defining $v_h(x_0) = v(x_0)$ and $v_h(x_i) = p_{x_i}$ for points different from x_0 and using techniques from the proof of Lemma 3.3 inequality (3.5.17) may be shown, if we were able to show

$$\|k^{1/2}(u - u_h)\|_{L^2(B)} \preceq \lambda^{-1/2} h\, \|k^{1/2}\nabla(u - u_h)\|_{L^2(B)} \quad . \tag{3.5.18}$$

Suppose that $f|_B = 0$. This implies that u is piecewise harmonic on $B \cap \Omega_i$. We now use techniques based on decomposition into orthogonal bases, see for instance [56].

In this subsection we use the notation $\lambda_i\ ,\ i = 1, 2, ..$ for the eigenvalues λ_i^2 of the Sturm-Liouville eigenvalue problem (2.4.1), (2.4.2). This shall not lead to confusion with the similar notation for the Finite Element shape functions. Denote by $s_{\lambda_i}, i = 1, 2, ..$ the according eigenfunctions. Therefore, functions $s_{\lambda_i}, s_{\lambda_j}, i \neq j$ are orthogonal in the scalar products induced by the norm $\|k^{1/2}(\cdot)\|_{L^2(\partial B)}$ and the norm $\|k^{1/2}(\nabla\cdot)\|_{L^2(\partial B)}$. Using the expansion $u|_{\partial B} = \sum_i a_i s_{\lambda_i}, a_i \in R$ and the fact that $r^{\lambda_i} s_{\lambda_i}$ is piecewise harmonic and satisfies the interface conditions we can expand u in the sequence [56]

$$u = \sum_i a_i r^{\lambda_i} s_{\lambda_i} \quad . \tag{3.5.19}$$

Regularity of u is restricted by regularity of functions $r^{\lambda_i} s_{\lambda_i}$ and we observe that it holds piecewise H^{1+s}-regularity for any $s < \lambda$ where $\lambda = \min_i \{\lambda_i\}$.
Orthogonality of the functions s_{λ_i} implies orthogonality of functions the $r^{\lambda_i} s_{\lambda_i}$ in the scalar products defined by the norms $\|k^{1/2}(\cdot)\|_{L^2(B)}$ and $\|k^{1/2}\nabla(\cdot)\|_{L^2(B)}$. Further we use $\|k^{1/2}(r^{\lambda_i} s_{\lambda_i})\|_{L^2(B)} \approx \lambda_i^{-1/2}\, h\, \|k^{1/2}(\nabla r^{\lambda_i} s_{\lambda_i})\|_{L^2(B)}$. We may prove now

$$\|k^{1/2}\, u\|_{L^2(B)} \preceq \lambda^{-1/2}\, h\, \|k^{1/2}\, \nabla u\|_{L^2(B)} \tag{3.5.20}$$

by expanding u in the orthogonal basis given in (3.5.19), applying Parseval's identity and showing appropriate inequalities for each of the functions $r^{\lambda_i} s_{\lambda_i}$.
Using standard Finite Element techniques it is not hard to prove the estimate

$$\|k^{1/2}\, u_h\|_{L^2(B)} \preceq \lambda^{-1/2} h\, \|k^{1/2}\, \nabla u_h\|_{L^2(B)} \quad . \tag{3.5.21}$$

If one wants to combine (3.5.20), (3.5.21) to show (3.5.18) one would need a sharpened Cauchy-Schwarz inequality between the Spaces V and V_h equipped with the norm $\|k^{1/2}\nabla \cdot\|_{L^2(B)}$ (or an additional condition for the norm $\|k^{1/2} \cdot\|_{L^2(B)}$). We do not know how to derive such an inequality or whether it holds at all.
Another attempt in proving the relation (3.5.17) could be to proceed as in the proof of the Poincaré inequality which relies on compact embeddings $L^2 \subset\subset H^1$. Therefore, it suffices to check whether the function $v = 1$ defined on $(0, 1)$ is *not contained* in the closure of functions $r^{\lambda_i}, 0 < \lambda_0 \leq \lambda_i, i = 1, 2, ..$ defined on the interval $(0, 1)$ in the $H^{1/2}$-semi-norm and *does belong* to the closure of that functions in $L^2(0, 1)$. Note that r^{λ_i} are the traces of functions $r^{\lambda_i}\, s_{\lambda_i}(\varphi)$.
But this attempt fails as one can show that the function v belongs to the closure in $H^{1/2}(0, 1)$ *iff* it belongs to the closure in $L^2(0, 1)$. The proof of this fact is not straightforward. It uses ideas of the proof of Müntz theorem [19, p. 174]. Instead of using in the proof the Hilbert space $H^{1/2}(0, 1)$ we can use the space $H^1(B_0)$ where B_0 is the unit ball. The definition of the functions r^{λ_i} remains the same.

3.6 Residual based error estimators

3.6.1 Theoretical basis for a-posteriori error estimators

We will use the interpolation operator defined in section 3.5. If the diffusion coefficients $k_T, T \in \mathcal{T}_h$, are distributed quasi-monotonically, we can exploit the approxima-

tion estimates (3.5.1). See remarks 3.3, 3.4 in section 3.5 for sufficient conditions for the quasi-monotonicity.
If the distribution of diffusion coefficients is not quasi-monotone with respect to some nodes, then there are no robust interpolation operators satisfying equations (3.5.1). In this case we have to admit a constant δ from relation (3.2.1).
We define the residual $r(u) \in V^\star$ for the solution u of (3.2.2) and u_h of equation (3.2.3):

$$r(u)(v) := \int_\Omega k\nabla\left(u - u_h\right)\nabla v \;\text{ for } v \in V \quad .$$

The following decomposition of the residual will be used in the derivation of an upper bound for the error

Lemma 3.4 *For any $v \in V$ and any $v_h \in V_h$ we have the following representation of the residual:*

$$\int_\Omega k\nabla\left(u - u_h\right)\nabla v \;=\sum_{T\in\mathcal{T}_h}\int_T f\left(v - v_h\right) \;+\sum_{F\in\mathcal{F}_h}\int_F j_F\left(v - v_h\right) \quad . \tag{3.6.1}$$

PROOF. Recall that we suppose $g_D = g_{D,h}$, that means that problem (3.2.2) and (3.2.3) fulfill the same Dirichlet conditions. Integration by parts allows for splitting the residual into local contributions. We use Galerkin orthogonality together with the fact that Δu_h vanishes for linear functions. Note that in the definition of j_F we have included the Neumann boundary data. ■

3.6.2 A residual based error estimator

Extending an estimator from [57] we define a residual based estimator η_R. The global estimator η_R consists of the sum of local estimators $\eta_{R,T}$.

Definition 3.6

$$\eta_R^2 := \sum_{T\in\mathcal{T}_h}\eta_{R,T}^2 \quad .$$

$$\eta_{R,T}^2 := \frac{h^2}{k_T}\,\|f_h\|_{L^2(T)}^2 \;+\sum_{F\subset\partial T/\Gamma_D}\frac{h}{k_F}\,\|j_F\|_{L^2(F)}^2 \quad .$$

The next theorem is the main result in this section. We show the estimator to be reliable and efficient. A robust upper bounds holds in the case of quasi-monotone diffusion coefficients. Otherwise additional constants enter in the upper bound for the estimator, see remark 3.7.

Theorem 3.5 *Let $d = 2,3$ and $g_D = g_{D,h}$ and let g_N be piecewise constant on faces $F \in \mathcal{F}_h(\Gamma_N)$.*
If the distribution of the diffusion coefficients $k_T, T \in \mathcal{T}_h$, is quasi-monotone, then for the solution u of equation (3.2.2) and u_h of equation (3.2.3) it holds that the estimator η_R is globally reliable, that is

$$|u - u_h|_{kH^1(\Omega)}^2 \;\preceq \sum_{T\in\mathcal{T}_h}\eta_{R,T}^2 \;+\sum_{T\in\mathcal{T}_h}\frac{h^2}{k_T}\,\|f - f_h\|_{L^2(T)}^2 \quad . \tag{3.6.2}$$

Without any assumptions about the distribution of the diffusion coefficients the estimator η_R is locally efficient, that is for any simplex $T \in \mathcal{T}_h$

$$\eta_{R,T}^2 \preceq |u-u_h|^2_{kH^1(\omega_T)} + \sum_{T' \subset \omega_T} \frac{h^2}{k_{T'}} \|f-f_h\|^2_{L^2(T')} ,$$

where ω_T contains all simplices sharing a face with T. The constants in these bounds neither depend on the diffusion k nor on other data, but only on the shape regularity parameter of $\mathcal{T}_h$.

In the 2D-case it suffices to demand in Theorem 3.5 quasi-monotonicity for the initial grid. But remember that the quasi-monotonicity is preserved during refinement in 3D only under an additional condition given in Definition 3.5.

Corollary 3.1 *Let $d = 3$ and let an initial triangulation $\mathcal{T}_{h_0}$ be given. The distribution of diffusion coefficients $k_T, T \in \mathcal{T}_{h_0}$, is quasi-monotone and additionally the distribution of diffusion coefficients $k_T, T \in \mathcal{T}_{h_0}$, is quasi-monotone with respect to edges of $\mathcal{E}_{h_0}$. Then for each triangulation $\mathcal{T}_h$ obtained by refining $\mathcal{T}_{h_0}$ (that means for the corresponding Finite Element spaces holds $V_h \subset V_{h_0}$) the error estimator η_R is reliable*

$$|u-u_h|^2_{kH^1(\Omega)} \preceq \sum_{T\in\mathcal{T}_h} \eta_{R,T}^2 + \sum_{T\in\mathcal{T}_h} \frac{h^2}{k_T} \|f-f_h\|^2_{L^2(T)} .$$

Remark 3.7 *With the bounds $\delta \le k(x) \le \delta^{-1}$ the non robust upper bound*

$$|u-u_h|^2_{kH^1(\Omega)} \preceq \delta^{-2} \left(\sum_{T\in\mathcal{T}_h} \eta_{R,T}^2 + \sum_{T\in\mathcal{T}_h} \frac{h^2}{k_T} \|f-f_h\|^2_{L^2(T)} \right) \tag{3.6.3}$$

holds. For a proof proceed as in [57] for the case $k = 1$.

Remark 3.8 *In [11] similar error estimates were derived independently for the class of diffusion coefficients, defined in remark 3.4, which is smaller than the class of quasi-monotone diffusion coefficients. See also the independent article [21], where the case of essentially two subdomains was considered.*

PROOF of Theorem 3.5. The techniques used in the proof are essentially those of [57]. Define $v = u - u_h$, where u and u_h are solutions of (3.2.2) and (3.2.3) and observe that v vanishes on Γ_D. Let $v_h := I_L(v)$. Reliability will be shown using the representation of the residual in Lemma 3.4 and Lemma 3.3 from section 3.5 for bounding the terms $v - v_h$. By Lemma 3.4 we split the residual into contributions $\int_T f\,(v-v_h)$ and $\int_F j_F\,(v-v_h)$. We use Lemma 3.3 from section 3.5 to bound

$$\int_T f(v-v_h) \le \|f\|_{L^2(T)} \; \|v-v_h\|_{L^2(T)} \preceq \frac{h}{k_T^{1/2}} \|f\|_{L^2(T)} \; |v|_{kH^1(\tilde{\omega}_T)} . \tag{3.6.4}$$

In a second step we estimate the contributions from the terms j_F. Using the approximation inequality (3.5.6) from Lemma 3.3 from section 3.5 we have the local estimate

$$\begin{aligned} \int_F j_F(v-v_h) &\le \|j_F\|_{L^2(F)} \; \|v-v_h\|_{L^2(F)} \\ &\preceq \left(\frac{h}{k_F}\right)^{1/2} \|j_F\|_{L^2(F)} \; |v|_{kH^1(\tilde{\omega}_{T_F})} . \end{aligned} \tag{3.6.5}$$

Now we use (3.6.1) together with the bounds (3.6.4), (3.6.5) and apply the Cauchy-Schwarz inequality. Further we make use of the fact that each simplex T is covered by finite number of $\tilde{\omega}_T$ or $\tilde{\omega}_{T_F}$ and obtain

$$\int_\Omega k\nabla (u-u_h)\nabla v = |v|^2_{kH^1(\Omega)} \preceq |v|_{kH^1(\Omega)} \cdot \left\{ \sum_{T\in\mathcal{T}_h} \left(\frac{h^2}{k_T} \|f\|^2_{L^2(T)} + \sum_{F\subset\partial T/\Gamma_D} \frac{h}{k_F} \|j_F\|^2_{L^2(F)} \right) \right\}^{1/2} .$$

Cancelation and the triangle inequality $\|f\|_{L^2(T)} \le \|f_h\|_{L^2(T)} + \|f-f_h\|_{L^2(T)}$ finish the proof of (3.6.2).

The proof of the lower bound goes in two steps. First we estimate the element residual. We denote by ϕ_T an element bubble function vanishing outside T as defined in section 3.2.2.

Note that since f_h is a constant on each simplex, the local equivalences

$$|f_h\phi_T|_{H^1(T)} \approx \|f_h\|_{L^2(T)}\, h^{-1} \;,\quad \|f_h\phi_T\|_{L^2(T)} \approx \|f_h\|_{L^2(T)}$$

follow directly from the relation $|\phi_T|_{H^1(T)} \approx h^{-1}\,\|1\|_{L^2(T)}$ and $\|\phi_T\|_{L^2(T)} \approx \|1\|_{L^2(T)}$.

Using $v = f_h\phi_T$ as a test function and seting $v_h = 0$ in (3.6.1) the Cauchy-Schwarz inequality and the local equivalences imply

$$\begin{aligned}
\|f_h\|^2_{L^2(T)} &\approx \int_T f_h\,(f_h\phi_T) \\
&= \int_T k\,\nabla(u-u_h)\;\nabla(f_h\phi_T) - \int_T (f-f_h)\,(f_h\phi_T) \\
&\le \left(\left(\frac{h}{k_T^{1/2}}\right)^{-1} |u-u_h|_{kH^1(T)} + \|f-f_h\|_{L^2(T)} \right) \|f_h\|_{L^2(T)} .
\end{aligned}$$

Simplifying the last expression we get an upper bound

$$\frac{h}{k_T^{1/2}}\,\|f_h\|_{L^2(T)} \preceq |u-u_h|_{kH^1(T)} + \frac{h}{k_T^{1/2}}\,\|f-f_h\|_{L^2(T)} . \tag{3.6.6}$$

To complete the estimate from below we need to estimate the jump term. We will exploit local equivalences of the type

$$\|j_F\phi_F\|_{L^2(T)} \approx h^{1/2}\,\|j_F\|_{L^2(F)} \;,\quad |j_F\phi_F|_{H^1(T)} \approx h^{-1/2}\,\|j_F\|_{L^2(F)} .$$

These equivalences follow from $\|\phi_F\|_{L^2(T)} \approx h^{1/2}\,\|1\|_{L^2(F)}$ and $|\phi_F|_{H^1(T)} \approx h^{-1/2}\,\|1\|_{L^2(F)}$.

We insert $v = j_F\phi_F$ as a test function and let $v_h = 0$ in (3.6.1) and obtain together with the above local equivalences

$$\begin{aligned}
\|j_F\|^2_{L^2(F)} &\approx \int_F j_F\,(j_F\phi_F) \\
&= -\int_{\omega_F} f_h\,(j_F\phi_F)\; - \int_{\omega_F} (f-f_h)\,(j_F\phi_F)\; + \int_{\omega_F} k\,\nabla\,(u-u_h)\,\nabla\,(j_F\phi_F) \\
&\preceq \sum_{T\subset\omega_F} \Big\{ \left(\|f_h\|_{L^2(T)} \;+\; \|f-f_h\|_{L^2(T)} \right) \|j_F\|_{L^2(F)}\; h^{1/2} \\
&\quad + k_T\; |u-u_h|_{H^1(T)} \quad \|j_F\|_{L^2(F)}\; h^{-1/2} \Big\} \quad .
\end{aligned}$$

Cancelation and insertion of (3.6.6) into the last right hand side gives

$$\begin{aligned}
h^{1/2}\; \|j_F\|_{L^2(F)} \;&\preceq \sum_{T\subset\omega_F} \left\{ h\; \|f-f_h\|_{L^2(T)} \;+ k_T^{1/2}\; |u-u_h|_{kH^1(T)} \right\} \\
&\le h\; \|f-f_h\|_{L^2(\omega_F)} \;+ k_F^{1/2}\; |u-u_h|_{kH^1(\omega_F)} \quad .
\end{aligned} \tag{3.6.7}$$

Combing (3.6.6) and (3.6.7) we obtain the upper bound for $\eta_{R,T}$

$$\eta^2_{R,T} \;\preceq\; |u-u_h|^2_{kH^1(\omega_T)} \;+ \sum_{T'\subset\omega_T} \frac{h^2}{k_{T'}}\; \|f-f_h\|^2_{L^2(T')} \quad .$$

■

3.6.3 Error estimators based on local problems

In the literature one finds estimators based on the solution of local problems [57]. Let the domain Ω be covered by patches of simplices. On each patch one defines a Galerkin space of bubble functions and solves within this space a local analogon of the continuous problem. The energy norm of the resulting solution is taken as an error estimator. The local problems can be classified into Dirichlet and Neumann problems.

Error estimators based on local Dirichlet problems

Here we present an approach with Dirichlet boundary conditions on a patch consisting of two neighbouring simplices. For a face $F \in \mathcal{F}_h$ let T and T' be simplices sharing the face F. If $F \in \mathcal{F}_h(\Gamma_N)$ regard only one simplex T with face F. The Galerkin space consists of two element bubble functions $\phi_T, \phi_{T'}$ vanishing outside T resp. T' and one bubble function aligned with the face ϕ_F vanishing outside ω_F. Let V_D be the space spanned by three bubble functions $\phi_T, \phi_{T'}, \phi_F$. Such bubble functions are defined in subsection 3.2.2.
We seek $v_D \in V_D$ fulfilling:

$$\int_{\omega_F} k\nabla v_D \nabla\phi = \int_{\omega_F} f_h\phi \;+ \int_{F\cap\Gamma_N} g_N\phi \;- \int_{\omega_F} k\nabla u_h\nabla\phi \quad \forall\phi\in V_D \quad . \tag{3.6.8}$$

Definition 3.7 *For a face $F \in \mathcal{F}_h$ we define*

$$\eta_{D,F} := |v_D|_{kH^1(\omega_F)} \quad .$$

Similar estimators have been proposed in the context of hierarchical bases in [12], see also section 3.7.1.
We show that the residual based estimator η_R and the estimator η_D are equivalent.

Theorem 3.6 *Let $d = 2, 3$. For each face $F \in \mathcal{F}_h$ and neighbouring simplices T, T' we have*

$$\eta_{D,F} \preceq \eta_{R,T} + \eta_{R,T'} \text{ and } \eta_{R,T} \preceq \sum_{F \subset \partial T / \Gamma_D} \eta_{D,F} \quad .$$

PROOF. The proof uses techniques from [57] developed for the case $k = 1$ adapted to the case of piecewise constant diffusion coefficients as done in the proof of Theorem 3.5. ∎

Error estimators based on a local Neumann problems

It may be desirable to construct estimators based on local problems with just one simplex as support for the bubble functions. Then one has to impose Neumann boundary conditions. For varying coefficients it is not straightforward what Neumann boundary conditions to impose in order to keep the resulting estimator robust.
We propose an estimator with one bubble function ϕ_F per face F of T and one ϕ_T for the simplex as done in the estimator η_N in [57]. The Galerkin space spanned by these functions is denoted by V_N.
We seek $v_N \in V_N$ satisfying:

$$\int_T k_T \nabla v_N \nabla \phi = \int_T f_h \phi - \sum_{F \subset \partial T / \Gamma_D} \left(\frac{k_T}{k_F} \right)^{1/2} \int_F j_F \, \phi \quad \forall \phi \in V_N \quad . \tag{3.6.9}$$

The estimator η_N will then be defined as:

Definition 3.8

$$\eta_{N,T} := |v_N|_{kH^1(T)} \quad .$$

Again we can show an equivalence.

Theorem 3.7 *The estimators $\eta_{R,T}$ and $\eta_{N,T}$ are locally equivalent:*

$$\eta^2_{N,T} \approx \eta^2_{R,T} \quad .$$

PROOF. The proof is done as the proof of Theorem 3.6. ∎

3.7 Other estimators

3.7.1 Estimators based on hierarchical bases

In distinction from residual based error estimators, where one needs interpolation results for the derivation of upper bounds for the error, there is an alternative approach based on hierarchical bases, see [12] [9]. Here the upper bound is shown by the so-called *saturation assumption*.
The analysis in [12] was done for the case $k \approx 1$ but carries over directly to the case of discontinuous diffusion coefficient.
Let $V_h \subset Q \subset V$ where Q is for example the space of piecewise quadratic Finite Elements

$$Q = V_h \oplus \mathcal{V} \quad .$$

Here $\mathcal{V}$ is called the hierarchical extension.
We define u_Q as the solution of the variational problem with Galerkin space Q: $u_Q - g_{D,h} \in Q$ and

$$\int_\Omega k \nabla u_Q \nabla v_Q \;= \int_\Omega f v_Q + \int_{\partial\Omega} g_N v_Q \quad \forall v_Q \in Q \quad . \tag{3.7.1}$$

We define the saturation assumption.

Definition 3.9 *We say that the saturation assumption is fulfilled if there exists a number* $\beta \in (0,1)$ *such that for the solutions* u, u_h, u_Q *of problems (3.2.2) (3.2.3) (3.7.1) holds*

$$|u - u_Q|_{kH^1(\Omega)} \;\le \beta\, |u - u_h|_{kH^1(\Omega)}$$

and β *does not depend on the data* $f, g_{D,h}, g_N$ *or* k*.*

The hierarchical extension $\mathcal{V}$ is spanned by face bubble functions ϕ_F for faces $F \in \mathcal{F}_h$.
We define the local error estimator for a face $F \in \mathcal{F}_h$

$$\eta^2_{H,F} \;= \left(\int_{\omega_F} f\ \phi_F \;- \int_F j_F\ \phi_F \right)^2 \; |\phi_F|^{-2}_{kH^1(\omega_F)}$$

and the global error estimator

$$\eta_H^2 := \sum_{F \in \mathcal{F}_h} \eta^2_{H,F} \quad .$$

Following [12] one can show

Lemma 3.8 *For solutions* u, u_h *of problems (3.2.2) (3.2.3) holds the lower bound*

$$\eta_H^2 \preceq \; |u - u_h|^2_{kH^1(\Omega)} \quad .$$

The saturation assumption is equivalent to the upper bound

$$|u - u_h|^2_{kH^1(\Omega)} \;\preceq \eta_H^2 \quad .$$

Similar estimators can be constructed for different spaces $Q \supset V_h$.
The problem is to prove the saturation assumption. For the case of a constant diffusion coefficient k this has been done recently using a-posteriori error analysis techniques [23]. There it is shown that a small ratio between the oscillation $\sum_T \|f - f_h\|^2_{L^2(T)}$ and $\|f\|^2_{L^2(\Omega)}$ implies the saturation assumption. However, for the case of a varying k until now there is no result showing the saturation assumption on the basis of a-priori data f. If one is solely interested in the saturation assumption, one can use reliability of the adjoint error estimator to prove the saturation assumption [12]. This can be done if the space $\mathcal{V}$ contains for each simplex $T \in \mathcal{T}_h$ a shape function with support contained in T as φ_T [12].
See [21] for connections between so called *a-posteriori saturation assumption* that drives the quality of the approximation of the data and the refinement strategy and between the above defined saturation assumption.

3.7.2 A Zienkiewicz-Zhu like estimator

Besides residual based and hierarchical based error estimators there is another approach based on averaging techniques originating from Zienkiewicz and Zhu [63] [57]. This approach is popular among engineers and is validated for the case of a constant diffusion coefficient. In this section we want to provide an argumentation that it may be not possible to adapt this kind of estimators to the general case of discontinuous diffusion coefficients in a robust way. We do this in the light that there are applications using averaging techniques for mesh adaptation which fail to provide reasonable refined meshes in case of discontinuous diffusion coefficients.
For the case of a constant diffusion coefficient these estimators are based on a reference solution obtained by an averaging procedure of the numerical solution. Let $G_{u_h} \in (V_h)^d$ defined by

$$G_{u_h}(x) := \sum_{T \subset \omega_x} \frac{meas_d(T)}{meas_d(\omega_x)} \, k_T \nabla u_h|_T \quad , \quad x \in \mathcal{N}_h \quad . \tag{3.7.2}$$

An error indicator can then be defined as

$$\eta^2_{ZZ,T} := \frac{1}{k_T} \, \|G_{u_h} - k_T \nabla u\|^2_{L^2(T)} \quad . \tag{3.7.3}$$

In case of $k \equiv 1$ we obtain the well-known ZZ-estimator. But in case of a discontinuous diffusion coefficient $\eta_{ZZ,T}$ cannot serve as indicator. This is clear if one observes that the derivatives of the numerical solution tangential to the interface are continuous whereas the derivatives normal to the interface are not continuous but close to be continuous when weighted with k. Thus in a proper averaging procedure the tangential and the normal part should be weighted differently. This may be hard to realize near singular points x, that means on a part of the interface Γ_0 which is not a straight line. The direction normal to the interface is not constant in $\Gamma_0 \cap \omega_x$ and this will cause serious difficulties for properly defining the normal and tangential derivatives.

3.8 Extension to more general problems

In this section we extend the estimators η_R, η_D to diffusion problems with a mass term and boundary conditions of Dirichlet-, Neumann- or Cauchy-type in polygonal (polyhedral) domains $\Omega \subset R^d, d = 2, 3$. As the mass term can become very big when compared to the diffusion coefficient k these problem comprise the so-called singularly perturbed reaction-diffusion problems.
Let a function $g_D \in H^{1/2}(\Gamma_D)$ be given that corresponds to the Dirichlet boundary data and could be extended to $g_D \in H^1(\Omega)$. Further the function $g_N \in L^2(\Gamma_N)$ corresponds to Neumann boundary data and functions $g_C \in L^2(\Gamma_C), \gamma \in L^2(\Gamma_C), \gamma > 0$ to Cauchy boundary data. Let $m \in L^\infty(\Omega), 0 < m$ be given. As in section 3.2 we define the space

$$V := \left\{ v \in H^1(\Omega) : v|_{\Gamma_D} = 0 \right\} \quad .$$

Let use define the energy scalar product

$$a(\, u, v \,) := \int_\Omega k \,\nabla u \,\nabla v \,+ \int_\Omega m \, u \, v \,+ \int_{\Gamma_C} \gamma \, u \, v \quad .$$

As before we do not explicitly denote that integration is done over the space variable x. The variational form of the problem under consideration is as follows: we seek u, satisfying $u \in g_D + V$ and

$$a(\, u, v \,) \;= \int_\Omega f \, v \,+ \int_{\Gamma_C} \gamma \, g_C \, v \,+ \int_{\Gamma_N} g_N \, v \quad \forall v \in V \quad . \tag{3.8.1}$$

We introduce the energy norm:

$$|\!|\!| v |\!|\!|^2_{\Omega'} := \; |k^{1/2} \, v|^2_{H^1(\Omega')} \;\; + \; \|m^{1/2} \, v\|^2_{L^2(\Omega')} \;\; + \; \|\gamma^{1/2} \, v\|^2_{L^2(\partial\Omega' \cap \Gamma_C)} \quad ,$$

and a norm that coincides with the energy norm in the case $\Gamma_C = \emptyset$

$$\|v\|^2_{b,\Omega'} \; := \; |k^{1/2} \, v|^2_{H^1(\Omega')} \;\; + \; \|m^{1/2} \, v\|^2_{L^2(\Omega')} \quad ,$$

for measurable subdomains $\Omega' \subset \Omega$. Denote by $b(\, \cdot, \cdot \,)_{\Omega'}$ the scalar product for the norm $\| \cdot \|_{b,\Omega'}$.
For simplicity we assume $meas_{d-1}(\Gamma_D) > 0$. Using Riesz's theorem one can show that there exists a unique solution of problem (3.8.1).
Let $V_h \subset V$ be an Finite Element space with continuous and piecewise linear functions. Let $g_{D,h} \in V_h$ be an approximation of g_D. Then the solution of the discrete problem u_h satisfies $u_h \in g_{D,h} + V_h$ and

$$a(\, u_h, v_h \,) = \int_\Omega f \, v_h \,+ \int_{\Gamma_C} \gamma \, g_C \, v_h \,+ \int_{\Gamma_N} g_N \, v_h \quad , \quad \forall v_h \in V_h \quad . \tag{3.8.2}$$

We assume that the diffusion coefficient k and the mass term m are piecewise constant on simplices of $\mathcal{T}_h$. Further we assume that functions $g_D = g_{D,h}$ and that g_N, g_C and γ are piecewise constant on faces of $\partial\Omega$. Otherwise an additional a-priori error occurs which is due to the approximation of the boundary data.

A-posteriori error estimators were derived for the case $k = 1$, $\Gamma_C = \emptyset$ in [59] and for the case $k = 1, m = 0$ in [37]. Similar error estimators have been derived in the case of a constant diffusion coefficient and pure Dirichlet boundary conditions in [2]. We derive a-posteriori error estimators for the case of discontinuous diffusion coefficients and Cauchy boundary conditions by adapting ideas from [59] [37]. We assume that the mass term m is piecewise constant on simplices from $\mathcal{T}_h$ and does not vary too much from simplex to simplex: For any neighbouring simplices T, T' with $T \cap T' \neq \emptyset$ it holds

$$m_T \approx m_{T'} \quad ,$$

where $m_T, m_{T'}$ denotes the value of m on the simplices T, T'. In the following we will skip the index and simply write m.

3.8.1 Notation

We will frequently make use of the following terms defined for a simplex T resp. a face F

$$\alpha_T := \min\left\{\frac{h}{k_T^{1/2}}, m^{-1/2}\right\} \quad , \quad \alpha_F := \frac{1}{k_F^{1/4}} \min\left\{\frac{h^{1/2}}{k_F^{1/4}}, m^{-1/4}\right\} \quad .$$

For each face $F \in \mathcal{F}_h$ we need a special shape function adapted to the energy norm and with support in ω_F . This function was introduced in [59]. We shortly sketch its definition. Denote by $\widehat{T}$ the reference simplex with vertices given by the unit vectors $e_i, i = 1, .., d$ and the point $e_{d+1} := 0 \in R^d$. Let $\widehat{F}$ be the face opposite to the vertex e_d. For a given number $0 < \delta \leq 1$ we introduce an affine transformation $\Psi_\delta : (x_1, .., x_{d-1}, x_d) \mapsto (x_1, .., x_{d-1}, \delta\, x_d)$ and denote the barycentric coordinates of the image $\widehat{T}_\delta := \Psi(\widehat{T}) \subset \widehat{T}$ by $\widehat{\lambda}_{i,\delta}, i = 1, .., d+1$. Now define the shape function on the reference simplex $\widehat{T}$ by:

$$\widehat{\phi}|_{\widehat{F},\delta} := d^d\, \widehat{\lambda}_{d+1,\delta}\ \Pi_{i=1,..,d-1} \widehat{\lambda}_{i,\delta}$$

with support in $\widehat{T}_\delta$. Let us choose a simplex $T \subset \omega_F$. By the affine transformation $G : \widehat{T} \to T$ mapping $\widehat{F}$ onto F we define $\phi_{F,\delta}(x) := \widehat{\phi}|_{\widehat{F},\delta}(G^{-1}(x))$. The function $\phi_{F,\delta}$ will be defined on the other simplex sharing the face F in the same way. That means that $\phi_{F,\delta}$ and ϕ_F coincide on the face F and in the case $\delta = 1$ they coincide everywhere. For $\delta < 1$ the function $\phi_{F,\delta}$ will have smaller support. With an appropriate value of $\delta = \alpha_F^2\, k_F\, h^{-1}$ and for $k = 1$ the function $\phi_{F,\delta}$ can be viewed to be a function which energy $\|\cdot\|_\Omega$ is close to the extension from F to ω_F with minimal energy $\|\cdot\|_\Omega$ [2].
Further we extend the definition of the jump term j_F to the case of boundary conditions of third kind. For a face F denote by T, T' the simplices from ω_F.

$$J_F := \begin{cases} \frac{1}{2}\left(k_T \frac{\partial u_h}{\partial n_T} + k_{T'} \frac{\partial u_h}{\partial n_{T'}}\right) & \text{if } F \subset \Omega \\ g_N - k_T \frac{\partial u_h}{\partial n} & \text{if } F \subset \Gamma_N \\ \gamma_F (g_C - u_h) - k_T \frac{\partial u_h}{\partial n} & \text{if } F \subset \Gamma_C \end{cases} \quad .$$

In the following we denote as usual the L^2-scalar product on a subdomain $\Omega' \subseteq \Omega$ by $(\cdot,\cdot)_{\Omega'}$. If $\Omega' = \Omega$, we write $(\cdot,\cdot)$.

3.8.2 Stability of the interpolation operator in mixed norms

In this section we assume that $\Gamma_C = \emptyset$. The aim of this subsection is to show that the interpolation operator $I_L : V \to V_h$ has the following interpolation properties with respect to the energy norm defined in section 3.8

$$\|u - I_C(u)\|_{L^2(T)} \preceq \alpha_T \ \|u\|_{b,\tilde{\omega}_T} \tag{3.8.3}$$

$$\|u - I_C(u)\|_{L^2(F)} \preceq \alpha_F \ \|u\|_{b,\tilde{\omega}_{T_F}} \tag{3.8.4}$$

under the restriction to quasi-monotone diffusion coefficients. Here $\tilde{\omega}_T$ and $\tilde{\omega}_{T_F}$ contains some neighbours of T or F. For a definition of $\tilde{\omega}_T$ or $\tilde{\omega}_{T_F}$ see section 3.5.3. Similar estimates have been shown for the case $k = 1$ [59]. For the proof we use a trace inequality [59].

Lemma 3.9 *Let $T \subset R^d, d = 2, 3$ be a simplex with diameter h. Let $F \subset \partial T$ a face of T. Then for $u \in H^1(T)$ it holds*

$$\|u\|^2_{L^2(F)} \preceq h^{-1} \|u\|^2_{L^2(T)} + \|u\|_{L^2(T)} \ |u|_{H^1(T)} \quad .$$

PROOF. See [59]. ■

Here we state the interpolation results:

Lemma 3.10 *Let $\Gamma_C = \emptyset$ and let the distribution of weights $k_T, T \in \mathcal{T}_h$ be quasi-monotone. For each simplex $T \in \mathcal{T}_h$ and for each face $F \in \mathcal{F}_h$ the bounds (3.8.3), (3.8.4) hold.*

PROOF. For the proof we exploit Lemma 3.9 and Lemma 3.3. A combination of (3.5.3), (3.5.4) yields

$$\|u - I_L(u)\|^2_{L^2(T)} \le m^{-1} \|u\|^2_{b,\tilde{\omega}_T}$$
$$\|u - I_L(u)\|^2_{L^2(T)} \le \frac{h^2}{k_T} \|u\|^2_{b,\tilde{\omega}_T}$$

proves (3.8.3). Choose a face F and the neighboring simplex T_F. Through inequalities (3.8.3), (3.5.5) and Lemma 3.9 and we establish the bound

$$\begin{aligned}
\|u - I_L(u)\|^2_{L^2(F)} &\preceq h^{-1} \|u - I_L(u)\|^2_{L^2(T_F)} \\
&\quad + \|u - I_L(u)\|_{L^2(T_F)} \ |u - I_L(u)|_{H^1(T_F)} \\
&\preceq h^{-1} \alpha^2_{T_F} \ \|u\|^2_{b,\tilde{\omega}_{T_F}} + \alpha_{T_F} \ \|u\|_{b,\tilde{\omega}_{T_F}} \frac{1}{k^{1/2}_{T_F}} |u|_{kH^1(\tilde{\omega}_{T_F})} \\
&= \left(h^{-1} \alpha^2_{T_F} + \alpha_{T_F} \frac{1}{k^{1/2}_{T_F}} \right) \|u\|^2_{b,\tilde{\omega}_{T_F}} \quad .
\end{aligned}$$

It remains to show

$$h^{-1} \alpha^2_{T_F} + \alpha_{T_F} \frac{1}{k^{1/2}_{T_F}} \approx \alpha^2_F \quad . \tag{3.8.5}$$

This is done by observing

$$h^{-1}\min\left\{\frac{h^2}{k_{T_F}}, m^{-1}\right\} + \min\left\{\frac{h}{k_{T_F}^{1/2}}, m^{-1/2}\right\}\frac{1}{k_{T_F}^{1/2}}$$
$$\approx \frac{1}{k_{T_F}^{1/2}}\min\left\{\frac{h}{k_{T_F}^{1/2}}, m^{-1/2}\right\}\left(\left(\frac{h}{k_{T_F}^{1/2}}\right)^{-1}\min\left\{\frac{h}{k_{T_F}^{1/2}}, m^{-1/2}\right\}+1\right)$$
$$\approx \frac{1}{k_{T_F}^{1/2}}\min\left\{\frac{h}{k_{T_F}^{1/2}}, m^{-1/2}\right\} \quad .$$

■

3.8.3 Stability of a modified interpolation operator

Now we allow for Cauchy boundary conditions. In this section we define the interpolation operator $I_C : V \to V_h$ which is a modification of the operator I_L. The modification is done in order to take into account the Cauchy boundary conditions. The new operator I_C fulfills the following interpolation properties with respect to the energy norm (where now the Cauchy boundary data are included)

$$\|u - I_C(u)\|_{L^2(T)} \preceq \alpha_T \, |||u|||_{\tilde{\omega}_T} \quad \forall u \in V \tag{3.8.6}$$
$$\|u - I_C(u)\|_{L^2(F)} \preceq \min\left\{ \alpha_F \,, \gamma_F^{-1/2}\right\} |||u|||_{\tilde{\omega}_{T_F}} \quad \forall u \in V \quad . \tag{3.8.7}$$

The last inequality remains valid for a face $F \not\subseteq \bar{\Gamma}_C$ if we define formally $\gamma_F = 0$ and $\gamma_F^{-1/2} = +\infty$. Defining the new operator we follow an idea of [37], where the problem of a globally constant diffusion coefficient was regarded. We define two subsets of boundary faces:

$$\Gamma_0 := \Gamma_D \cup \Gamma_{C0}, \text{ where } \Gamma_{C0} := \left\{F \in \mathcal{F}_h(\bar{\Gamma}_C) : \; \alpha_F{}^{-2} \le \gamma_F\right\} \quad .$$

The interpolation properties (3.8.6) and (3.8.7) hold if the distribution of coefficients $k_{T'}, \;\; T' \subset \omega_x$ is quasi-monotone with respect to any node x of a triangulation $\mathcal{T}_h$ and the part of the boundary $\Gamma_0 \subset \partial\Omega$.
The new operator I_C is defined by its values in the nodal points

$$I_C(v)(x_i) = \begin{cases} 0 & \text{if } x_i \in \mathcal{N}_h(\Gamma_0) \\ p_{x_i} & \text{otherwise} \end{cases} \quad .$$

For a definition of p_{x_i} see section 3.5.3. We see that I_C and I_L coincides in all nodal points but those on Γ_{C0}. We need the following Poincaré inequality:

Lemma 3.11 *Let $\Omega_0 \subset R^d, d = 2,3$, be a Lipschitz domain with diameter h. Let $F \subset \partial\Omega_0, meas_{d-1}(F) \approx h$. Then for $u \in H^1(\Omega_0)$*

$$\|u\|^2_{L^2(\Omega_0)} \preceq h^2 \, |u|^2_{H^1(\Omega_0)} + h \, \|u\|^2_{L^2(F)} \quad .$$

PROOF. This is a refined version of a standard Poincaré inequality [4] for domains with diameter $O(1)$. The constant in the inequality depends on the Lipschitz constant of Ω_0. ■

Lemma 3.12 *Let the distribution of coefficients $k_{T'}$, $T' \subset \omega_x$ be quasi-monotone with respect to any node x of the triangulation $\mathcal{T}_h$ and the part of the boundary $\Gamma_0 \subset \partial\Omega$. Then for each simplex $T \in \mathcal{T}_h$ and for each face $F \in \mathcal{F}_h$ the bounds (3.8.6), (3.8.7) hold.*

PROOF. In the proof we exploit elements from the proofs of Lemma 3.3 and Lemma 3.10. Choose a simplex $T \in \mathcal{T}_h$ and denote its vertices by $x_i, i = 0, .., d$. Clearly $u - I_C(u) = \sum_{i=0}^d \lambda_i(u - p_{x_i})$. We set $p_{x_i} = 0$, if there is a face $F \subset \Gamma_0$ with node x_i. We show as in the proof of Lemma 3.3 that

$$\|I_C(u)\|_{L^2(T)} \preceq \|u\|_{L^2(T)} \quad .$$

Accordingly

$$\|I_C(u)\|^2_{L^2(T)} = m^{-1} \|m^{1/2}u\|^2_{L^2(T)} \le m^{-1}|\!|\!|u|\!|\!|^2_T \quad . \tag{3.8.8}$$

Now we want to show

$$\|u - I_C(u)\|^2_{L^2(T)} \preceq \left(\frac{h^2}{k_T} + \frac{h}{k_T^{1/2}}\,\alpha_T\right) |\!|\!|u|\!|\!|^2_{\tilde\omega_T} \quad . \tag{3.8.9}$$

As in the proof of Lemma 3.3 we use $\|u - I_C(u)\|^2_{L^2(T)} \preceq \sum_{i=0}^d \|u - p_{x_i}\|^2_{L^2(T)}$. Thus in order to show (3.8.9) it suffices to prove

$$\|u - p_{x_i}\|^2_{L^2(T)} \preceq \left(\frac{h^2}{k_T} + \frac{h}{k_T^{1/2}}\,\alpha_T\right) |\!|\!|u|\!|\!|^2_{\tilde\omega_T} \quad . \tag{3.8.10}$$

For nodes $x_i \in \mathcal{N}_h(T/\Gamma_{C0})$ we show as in Lemma 3.3

$$\|u - p_{x_i}\|^2_{L^2(T)} \preceq h^2\, |u|^2_{H^1(\tilde\omega_{T,x_i,qm})} \quad . \tag{3.8.11}$$

We use the quasi-monotonicity condition

$$k_T \le k_{T'} \;\;,\;\; T' \subset \tilde\omega_{T,x_i,qm} \quad . \tag{3.8.12}$$

to bound

$$\begin{aligned} |u|^2_{H^1(\tilde\omega_{T,x_i,qm})} &= \sum_{T' \subset \tilde\omega_{T,x_i,qm}} \frac{1}{k_{T'}}\, |u|^2_{kH^1(T')} \\ &\le \sum_{T' \subset \tilde\omega_{T,x_i,qm}} \frac{1}{k_T}\, |u|^2_{kH^1(T')} = \frac{1}{k_T}\, |u|^2_{kH^1(\tilde\omega_{T,x_i,qm})} \preceq \frac{1}{k_T} |\!|\!|u|\!|\!|^2_{\tilde\omega_{T,x_i,qm}} \quad . \end{aligned} \tag{3.8.13}$$

Hence we are done with nodes x_i in $\mathcal{N}_h(T/\Gamma_{C0})$.

If a node x_i belongs to $\mathcal{N}_h(T \cap \Gamma_{C0})$, there is a face $F \subset \Gamma_{C0} \cap \partial\tilde{\omega}_{T,x_i,qm}$ where $T_F \subset \tilde{\omega}_{T,x_i,qm}$ and we may use Lemma 3.11

$$(3.8.14) \quad \begin{aligned} \|u - p_{x_i}\|^2_{L^2(T)} &= \|u\|^2_{L^2(T)} \leq \|u\|^2_{L^2(\tilde{\omega}_{T,x_i,qm})} \\ &\preceq h^2 \, |u|^2_{H^1(\tilde{\omega}_{T,x_i,qm})} + h \, \|u\|^2_{L^2(F)} \quad . \end{aligned}$$

Using the relation $\gamma_F^{-1} \leq \alpha_F^2$ the last term of ineq. (3.8.14) is bounded by

$$(3.8.15) \qquad h \, \|u\|^2_{L^2(F)} = h \, \gamma_F^{-1} \, \|\gamma_F^{1/2} u\|^2_{L^2(F)} \leq h \, \alpha_F^2 \|\!|u|\!\|^2_{\tilde{\omega}_{T,x_i,qm}} \quad .$$

Using the relation $\alpha_F^2 \approx \frac{1}{k_F}\alpha_{T_F}$ and the quasi-monotonicity assumption (3.8.12) we show

$$(3.8.16) \qquad h \, \alpha_F^2 \approx \frac{h}{k_{T_F}^{1/2}} \alpha_{T_F} \leq \frac{h}{k_T^{1/2}} \alpha_T \quad .$$

Again we use the quasi-monotonicity condition to bound the term $|u|^2_{H^1(\tilde{\omega}_{T,x_i,qm})}$ in ineq. (3.8.14) like in ineq. (3.8.13).
With ineq. (3.8.13), (3.8.14), (3.8.15), (3.8.16), the definition of $\tilde{\omega}_T = \bigcup_{i=1}^{d+1} \tilde{\omega}_{T,x_i,qm}$ and help of ineq. (3.8.10) we are able to show ineq. (3.8.9).
Combining ineq. (3.8.8) and ineq. (3.8.9) we conclude

$$\begin{aligned} \|u - I_C(u)\|^2_{L^2(T)} &\preceq \min\left\{ m^{-1}, \frac{h^2}{k_T} + \frac{h}{k_T^{1/2}} \min\left\{ \frac{h}{k_T^{1/2}}, m^{-1/2} \right\} \right\} \|\!|u|\!\|^2_{\tilde{\omega}_T} \\ &\preceq \min\left\{ \frac{h^2}{k_T}, m^{-1} \right\} \|\!|u|\!\|^2_{\tilde{\omega}_T} \quad , \end{aligned}$$

what shows assertion (3.8.6).
For a fixed face F we want to derive assertion (3.8.7). We set $T = T_F$. We first bound $|u - I_C(u)|^2_{H^1(T)}$ in terms of $\|\!|u|\!\|^2_{\tilde{\omega}_{T_F}}$. We conclude as in Lemma 3.3 (ineq. (3.5.15) and (3.5.16)) that

$$|u - I_C(u)|^2_{H^1(T)} \preceq h^{-2} \sum_{i=0}^{d} \|u - p_{x_i}\|^2_{L^2(T)} + |u|^2_{H^1(T)} \quad .$$

With help of the bounds (3.8.10), (3.8.13) we see

$$(3.8.17) \qquad |u - I_C(u)|^2_{H^1(T)} \preceq \left[h^{-2} \left(\frac{h^2}{k_T} + \frac{h}{k^{1/2}} \, \alpha_T \right) + k_T^{-1} \right] \|\!|u|\!\|^2_{\tilde{\omega}_{T_F}} \quad .$$

Now using the bounds (3.8.6) and (3.8.17) we proceed as in the proof of ineq. (3.8.4) and apply Lemma 3.9. We reach at

$$\begin{aligned} \|u - I_C(u)\|^2_{L^2(F)} &\preceq \left(h^{-1} \, \alpha_T^2 + \alpha_T \left[\left(\frac{1}{k_T} + \frac{1}{h k_T^{1/2}} \, \alpha_T \right) + k_T^{-1} \right]^{1/2} \right) \|\!|u|\!\|^2_{\tilde{\omega}_{T_F}} \\ &\preceq \left(h^{-1} \, \alpha_T^2 + \alpha_T \, \frac{1}{k_T^{1/2}} + \left(\frac{1}{h k_{T_F}^{1/2}} \, \alpha_T \right)^{1/2} \alpha_T \right) \|\!|u|\!\|^2_{\tilde{\omega}_{T_F}} \quad . \end{aligned}$$

It remains to bound the factor in the right hand side of the last inequality by α_F . From inequality (3.8.5) we know that $h^{-1}\ \alpha_T^2\ +\ \alpha_T\ \frac{1}{k_T^{1/2}}$ is bounded by α_F^2 . The third addend is estimated by

$$\left(\frac{1}{hk_{T_F}^{1/2}}\ \alpha_{T_F}\ \right)^{1/2}\ \alpha_T\ \ = \frac{1}{k_{T_F}^{1/2}}\ \alpha_{T_F}\ \left(\frac{k_{T_F}^{1/4}}{h^{1/2}}\cdot\ \alpha_{T_F}\ ^{1/2}\right) \le\ \alpha_F^2\ \cdot 1\quad .$$

This shows

$$\|u - I_C(u)\|^2_{L^2(F)}\ \preceq\ \alpha_F^2\ \| u \|^2_{\tilde{\omega}_{T_F}} \quad . \tag{3.8.18}$$

Suppose $\gamma_F^{-1} \le\ \alpha_F^2$. Then $F \subset \Gamma_{C0}$ and $I_C(u)|_F = 0$. Hence it is easy to see that

$$\|u - I_C(u)\|^2_{L^2(F)}\ =\ \|u\|^2_{L^2(F)}\ = \gamma_F^{-1}\ \|\gamma_F^{1/2}u\|^2_{L^2(F)}\ \ \le \gamma_F^{-1} \| u \|^2_{\tilde{\omega}_{T_F}} \quad . \tag{3.8.19}$$

Combining the bounds (3.8.18) and (3.8.19) we have shown assertion (3.8.7).

3.8.4 Theoretical basis for the lower bound

In the following using ideas from [59] we define δ in such a way that $h\delta\ =\ k_F\ \alpha_F^2$. Observe that then

$$\delta = \frac{k_F}{h}\frac{1}{k_F^{1/2}}\min\left\{\frac{h}{k_F^{1/2}}, m^{-1/2}\right\} \le 1\ , \tag{3.8.20}$$

according to our assumptions.

We will use an extension operator $E_F : L^\infty(F) \to L^\infty(\omega_F)$ as defined in [57, p. 58]. It relies on constant extension onto the simplices $T, T' \subset \omega_F$ containing the face F.

For the proof of the lower bound of the error estimator defined in the next section we need a technical lemma

Lemma 3.13 *Let V_T be the finite dimensional space spanned by the linear functions on T and denote by V_F the space of functions which are linear on F.*
For any linear function $\varphi \in V_T$ holds

$$\alpha_T^2\ \ \|\phi_T\varphi\|^2_{b,T}\ \approx\ \|\phi_T\varphi\|^2_{L^2(T)}\ \approx\ \|\varphi\|^2_{L^2(T)}\ \approx (\ \varphi, \phi_T\varphi\ )_T \tag{3.8.21}$$

where the bounds are independent of φ.
The following estimates hold for any $0 < \delta \le 1$ and $\varphi \in V_F$ independent of δ and φ

$$\|\phi_{F,\delta}\varphi\|^2_{L^2(F)}\ \approx\ \|\varphi\|^2_{L^2(F)}\ \approx (\ \phi_{F,\delta}, \varphi\ )_F \tag{3.8.22}$$

$$\|\phi_{F,\delta}E_F(\varphi)\|^2_{L^2(T)}\ \approx h\delta\ \|\varphi\|^2_{L^2(F)} \tag{3.8.23}$$

$$\alpha_F^2\ \ \|\phi_{F,\delta}E_F(\varphi)\|^2_{b,\omega_F}\ \approx\ \|\phi_{F,\delta}\varphi\|^2_{L^2(F)} \quad . \tag{3.8.24}$$

PROOF. Note that $|\phi_T\varphi|_{H^1(T)}$ is a norm in the finite dimensional space V_T and since norms in finite dimensional spaces are equivalent we show after transformation to a reference element $\widehat{T}$ that $|\phi_T\varphi|_{H^1(T)} \approx h^{-1} \|\phi_T\varphi\|_{L^2(T)}$. Hence, it is easy to see that

(3.8.25)
$$\begin{aligned} \|\phi_T\varphi\|^2_{b,T} &= m\, \|\phi_T\varphi\|^2_{L^2(T)} + k_T\, |\phi_T\varphi|^2_{H^1(T)} \\ &\approx \left(\frac{k_T}{h^2} + m\right) \|\phi_T\varphi\|^2_{L^2(T)} \approx \left(\min\left\{\frac{h^2}{k_T}, m^{-1}\right\}\right)^{-1} \|\phi_T\varphi\|^2_{L^2(T)} \quad . \end{aligned}$$

Similarly, $(\ \varphi, \phi_T\varphi\ )_T \approx \ \|\varphi\|^2_{L^2(T)} \ \approx \ \|\phi_T\varphi\|^2_{L^2(T)}$ what proves (3.8.21).
Inequalities (3.8.22) follow from equivalence of norms on finite dimensional spaces.
Argumenting as in the proof [59, Lem. 3.4] we get

$$\|\phi_{F,\delta} E_F(\varphi)\|^2_{L^2(T)} \approx h\delta\, \|\phi_{F,\delta}\varphi\|^2_{L^2(F)} \quad ,$$

what proves (3.8.23).
Before we prove (3.8.24) observe first that $|\nabla\phi_{F,\delta}|_{l^d(L^\infty)} \approx 1/(\delta h)$. Performing transformation to a reference simplex and applying standard arguments (see the proof of [59, Lem. 3.4]) we arrive at

$$|\phi_{F,\delta} E_F(\varphi)|^2_{H^1(T)} \approx (h\delta)^{-1}\, \|\phi_{F,\delta}\varphi\|^2_{L^2(F)} \quad .$$

Straightforward calculation shows

$$\begin{aligned} k_F\,(h\delta)^{-1}\ \alpha_F^2 &= 1 \\ m\,(h\delta)\ \alpha_F^2 = m\,k_F\,\alpha_F^4 &= \min\left\{m\,h^2\,k_F^{-1}, m\,m^{-1}\right\} \le 1 \quad . \end{aligned}$$

Combining the last three inequalities with (3.8.23) we obtain

$$\begin{aligned} \|\phi_{F,\delta} E_F(\varphi)\|^2_{b,\omega_F} &= \sum_{T\in\omega_F} m\, \|\phi_{F,\delta E_F(\varphi)}\|^2_{L^2(T)} + \sum_{T\in\omega_F} k_T\, |\phi_{F,\delta} E_F(\varphi)|^2_{H^1(T)} \\ &\approx \left\{m\,h\,\delta + k_F\,(h\delta)^{-1}\right\} \|\phi_{F,\delta}\varphi\|^2_{L^2(F)} \approx \alpha_F^{-2}\ \|\phi_{F,\delta}\varphi\|^2_{L^2(F)} \quad . \end{aligned}$$

This finishes the proof of (3.8.24). ■

3.8.5 Residual based error estimator

We define the residual $r(u) \in V^\star$

$$r(u)(v) := a(\ u - u_h, v\ ) \quad \text{for } v \in V$$

where u and u_h are solutions of (3.8.1) and (3.8.2).
The following representation of the residual holds.

Lemma 3.14 *For any $v \in V$ and any $v_h \in V_h$ we have the following identity:*

$$
\begin{aligned}
r(u)(v) &= \int_\Omega k(\nabla u - \nabla u_h)v + \int_\Omega m(u-u_h)v + \int_{\Gamma_C} \gamma_F(u-u_h)v \\
&= \sum_{T\in\mathcal{T}_h} \int_T (f - mu_h)\, v + \sum_{F\in\mathcal{F}_h(\Omega)} \int_F j_F v \\
&+ \sum_{F\in\mathcal{F}_h(\Gamma_N)} \int_F (g_N - k\frac{\partial u_h}{\partial n})v + \sum_{F\in\mathcal{F}_h(\Gamma_C)} \int_F \left\{ \gamma_F(g_C - u_h) - k\frac{\partial u_h}{\partial n}\right\} v \\
&= \sum_{T\in\mathcal{T}_h} \int_T (f - mu_h)\, v + \sum_{F\in\mathcal{F}_h/\Gamma_D} \int_F J_F\, v \\
&= \sum_{T\in\mathcal{T}_h} \int_T (f - mu_h)\,(v-v_h) + \sum_{F\in\mathcal{F}_h/\Gamma_D} \int_F J_F(v-v_h)
\end{aligned}
\tag{3.8.26}
$$

Here J_F denotes the jump term defined in definition 3.2.

PROOF. The proof follows from Galerkin orthogonality and partial integration. ∎

We extend the residual based estimator in [59] to the class of problems (3.8.1) with discontinuous diffusion coefficients. For the case of $k = 1, m = 0$ and Cauchy boundary conditions a similar residual based a-posteriori error estimator has been proposed in [37].

Definition 3.10

$$
\begin{aligned}
\eta_{R,T}^2 := \alpha_T^2 \; \|f_h - m\,u_h\|^2_{L^2(T)} &+ \sum_{F\subset\mathcal{F}_h(\partial T/(\Gamma_D\cup\Gamma_C))} \alpha_F^2 \; \|J_F\|^2_{L^2(F)} \\
&+ \sum_{F\subset\mathcal{F}_h(\partial T\cap\Gamma_C)} \min\left\{ \alpha_F^2\, , \gamma_F^{-1}\right\} \; \|J_F\|^2_{L^2(F)} \quad .
\end{aligned}
$$

If $m = 0$ the definition of α_T and α_F has still a meaning setting formally $m^{-1} = \infty$. In this sense the error estimator $\eta_{R,T}$ defined in section 3.6.2 coincides we the new one defined above in the case $m = 0$ and $\Gamma_C = \emptyset$ and we agree to use the same symbol.
We define the global error estimator as the sum of the local contributions

$$
\eta_R^2 := \sum_{T\in\mathcal{T}_h} \eta_{R,T}^2 \quad . \tag{3.8.27}
$$

The following theorem states robust efficiency and reliability of the estimator η_R.

Theorem 3.15 *Set $d = 2, 3$ and let u, u_h be the solutions of problems (3.8.1), (3.8.2). For any simplex $T \in \mathcal{T}_h$, the estimator is locally efficient, that is*

$$
\eta_{R,T}^2 \preceq |||u - u_h|||^2_{\omega_T} + \sum_{T'\subset\omega_T} \alpha_T^2 \; \|f - f_h\|^2_{L^2(T')} \quad ,
$$

where ω_T contains all simplices sharing a face with T.

Let the distribution of coefficients $k_T, \quad T \subset \omega_x$ be quasi-monotone with respect to any node x of a triangulation $\mathcal{T}_h$ and the part of the boundary $\Gamma_0 \subset \partial\Omega$. Then the estimator η_R is globally reliable, that is

$$|||u - u_h|||_\Omega^2 \preceq \sum_{T\in\mathcal{T}_h} \eta_{R,T}^2 + \sum_{T\in\mathcal{T}_h} \alpha_T^2 \; \|f - f_h\|_{L^2(T)}^2 \quad .$$

The constants in these estimates depend only on the shape regularity and not on the parameters like k, m and boundary data.

PROOF. The proof goes essentially like that for the case $k = 1$ in [59] [37]. We choose $v = u - u_h \in V$ and set $v_h = I_C(v)$. From the representation of the residual (Lemma 3.14) and interpolation inequalities from Lemma 3.10 and Lemma 3.12 follows

$$\begin{aligned}
a(\, u - u_h, v\,) \;&= \sum_{T\in\mathcal{T}_h} \int_T (f - mu_h)\,(v - v_h) \;+ \sum_{F\in\mathcal{F}_h/\Gamma_D} \int_F J_F\,(v - v_h) \\
&\le \sum_{T\in\mathcal{T}_h} \|f - mu_h\|_{L^2(T)} \;\|v - v_h\|_{L^2(T)} \\
&\quad + \sum_{F\in\mathcal{F}_h/\Gamma_D} \|J_F\|_{L^2(F)} \;\|v - v_h\|_{L^2(F)} \\
&\le \sum_{T\in\mathcal{T}_h} \alpha_T \;|||v|||_{\tilde\omega_T} \;\|f - mu_h\|_{L^2(T)} \\
&\quad + \sum_{F\in\mathcal{F}_h/(\Gamma_D\cup\Gamma_C)} \alpha_F \;|||v|||_{\tilde\omega_{T_F}} \;\|J_F\|_{L^2(F)} \\
&\quad + \sum_{F\in\mathcal{F}_h\cap\Gamma_C} \min\{\,\alpha_F^2\,, \gamma_F^{-1}\}\; |||v|||_{\tilde\omega_{T_F}} \;\|J_F\|_{L^2(F)} \quad .
\end{aligned} \tag{3.8.28}$$

Now we apply the Cauchy-Schwarz inequality and make use of the fact that each simplex T is covered at most by finite number of $\tilde\omega_T$ or $\tilde\omega_{T_F}$ that depends on the shape regularity parameter of $\mathcal{T}_h$. At the end substitute f by the approximation f_h and use the triangle inequality

$$\|f - mu_h\|_{L^2(T)}^2 \;\preceq\; \|f_h - mu_h\|_{L^2(T)}^2 \;+\; \|f - f_h\|_{L^2(T)}^2 \quad .$$

This proves reliability of the error estimator η_R.
The proof of the efficiency goes in two steps. First estimate the element residual using the bubble function

$$w_T := (f_h - mu_h)\,\phi_T \quad .$$

In the identity

$$(\, f_h - mu_h, \varphi\,)_T \;= a(\, u - u_h, \varphi\,) + (\, f - f_h, \varphi\,)_T$$

we substitute φ by w_T and exploit inequality (3.8.21) to get

$$\begin{aligned}
\|f_h - mu_h\|_{L^2(T)}^2 \;&\approx (\, f_h - mu_h, w_T\,)_T \;= a(\, u - u_h, w_T\,) + (\, f - f_h, w_T\,)_T \\
&\le\; \|u - u_h\|_{b,T} \;\|w_T\|_{b,T} \;+\; \|f - f_h\|_{L^2(T)} \;\|w_T\|_{L^2(T)} \\
&\le\; \|f_h - mu_h\|_{L^2(T)}^2 \;\left\{\, \alpha_T^{-1} \;\|u - u_h\|_{b,T} \;+\; \|f - f_h\|_{L^2(T)}\,\right\} \quad .
\end{aligned}$$

After cancelation we arrive at the bound

$$\alpha_T^2 \ \|f_h - m u_h\|_{L^2(T)}^2 \ \preceq \ \|u - u_h\|_{b,T}^2 \ + \ \alpha_T^2 \ \|f - f_h\|_{L^2(T)}^2 \quad . \tag{3.8.29}$$

In a second step we estimate the jump term J_F. Let us first take an interior face F or a face from Γ_N. We exploit the following identity which follows from partial integration

$$(\ J_F, w_F\ ) = \ b(\ u - u_h, w_F\ ) \ - (\ f_h - m\, u_h, w_F\ )_{\omega_F} - (\ f - f_h, w_F\ )_{\omega_F} \quad . \tag{3.8.30}$$

Here $w_F := J_F \phi_{F,\delta}$ is a special trial function. Note that J_F is a constant. As $w_F \in V_F$ we can apply Lemma 3.13 with inequalities (3.8.22), (3.8.23) , (3.8.24).

$$\begin{aligned} \|J_F\|_{L^2(F)}^2 \ &\approx (\ J_F, w_F\ ) \\ &\leq \sum_{T \subset S_F} \Big\{ \ \|u - u_h\|_{b,T} \ \|w_F\|_{b,T} \\ &\quad + \big(\ \|f_h - m\, u_h\|_{L^2(T)} \ + \ \|f - f_h\|_{L^2(T)}\ \big) \ \|w_F\|_{L^2(T)} \ \Big\} \\ &\preceq \ \|J_F\|_{L^2(F)} \Big\{ \sum_{T \subset S_F} \ \alpha_F^{-1} \ \|u - u_h\|_{b,T} \\ &\quad + (h\delta)^{\frac{1}{2}} \ \big(\ \|f_h - m\, u_h\|_{L^2(T)} \ + \ \|f - f_h\|_{L^2(T)}\ \big) \ \Big\} \quad . \end{aligned} \tag{3.8.31}$$

Cancelation and application of equation (3.8.29) yields

$$\|J_F\|_{L^2(F)}^2 \ \preceq \sum_{T \subset S_F} \Big\{ \ \big(\ \alpha_F^{-2} \ + h\delta\ \alpha_T^{-2}\ \big) \ \|u - u_h\|_{b,T}^2 + h\delta\ \|f - f_h\|_{L^2(T)}^2 \ \Big\} \quad . \tag{3.8.32}$$

Multiplication (3.8.32) with α_F^2 and the fact $h\delta\ \alpha_F^2 \ = k_F \alpha_F^4 \approx \ \alpha_{T_F}^2 \ \leq \ \alpha_T^2$ imply

$$\alpha_F^2 \ \|J_F\|_{L^2(F)}^2 \ \preceq \ \|u - u_h\|_{b,T_F}^2 \ + \sum_{T \subset T_F} \ \alpha_T^2 \ \|f - f_h\|_{L^2(T)}^2 \tag{3.8.33}$$

an upper bound for the jump term. If $\Gamma_C = \emptyset$ we are done.
Otherwise take $F \subset \Gamma_C$. As before due to definition of J_F we get

$$\begin{aligned} (\ J_F, w_F\ ) = \ b(\ u - u_h, w_F\ ) + \int_F \gamma_F (u - u_h) w_F \\ - (\ f_h - m\, u_h, w_F\ ) - (\ f - f_h, w_F\ ) \quad . \end{aligned} \tag{3.8.34}$$

We proceed as in the case of $F \subset \mathcal{F}_h / \Gamma_C$ and set $w_F := E_F(J_F)\, \phi_{F,\delta} \in V_F$. Here E_F is the extension operator introduced in section 3.8.4. We bound the additional term in (3.8.34) by applying the Cauchy-Schwarz inequality and inequality (3.8.22)

$$\int_F \gamma_F (u - u_h) w_F \ \leq \gamma_F\ \|u - u_h\|_{L^2(F)} \ \|w_F\|_{L^2(F)} \ \approx \gamma_F^{1/2} |\!|\!| u - u_h |\!|\!|_{T_F} \ \|J_F\|_{L^2(F)} \quad .$$

Proceeding as before (compare with inequality (3.8.32)) we gain

$$\begin{aligned} \|J_F\|_{L^2(F)}^2 \ \preceq \sum_{T \subset S_F} \Big\{ \ \big(\ \alpha_F^{-2} \ + h\delta\ \alpha_T^{-2}\ \big) \ \|u - u_h\|_{b,T}^2 \\ + h\delta\ \|f - f_h\|_{L^2(T)}^2 \ + \gamma_F |\!|\!| u - u_h |\!|\!|_{T_F}^2 \ \Big\} \quad . \end{aligned} \tag{3.8.35}$$

respectively

$$\|J_F\|^2_{L^2(F)} \preceq \left(\alpha_F^{-2} + \gamma_F \right) \|u - u_h\|^2_{T_F} + \alpha_F^{-2} \sum_{T \subset T_F} \alpha^2_{T_F} \|f - f_h\|^2_{L^2(T)} \quad . \tag{3.8.36}$$

Observe

$$\alpha_F^{-2} + \gamma_F \approx \max\left\{ \alpha_F^{-2}, \gamma_F \right\} = \left\{ \min\left\{ \alpha_F^2, \gamma_F^{-1} \right\} \right\}^{-1} \quad . \tag{3.8.37}$$

Thus from inequality (3.8.36) follows

$$\min\left\{ \alpha_F^2, \gamma_F^{-1} \right\} \|J_F\|^2_{L^2(F)} \preceq \|u - u_h\|^2_{T_F} + \sum_{T \subset T_F} \alpha_T^2 \|f - f_h\|^2_{L^2(T)} \quad . \tag{3.8.38}$$

Here we used $\min\left\{ \alpha_F^2, \gamma_F^{-1} \right\} \alpha_F^{-2} \leq 1$ and $\alpha^2_{T_F} \leq \alpha_T^2$. Combining inequalities (3.8.29), (3.8.33) and (3.8.38) we finish the proof of local efficiency. ■

3.8.6 Error estimators based on local Dirichlet problems

We define an estimator based on solutions of local Dirichlet problems. The estimator is similar to that derived in section 3.6.3. For each face $F \in \mathcal{F}_h$ the estimator is based on a Dirichlet problem on ω_F.

Denote by ϕ_T and $\phi_{F,\delta}$ the shape functions that are defined as before. The local Galerkin V_D will be:

$$V_D := \{\varphi\, \phi_{F,\delta} : \varphi \in V_F\} + \bigcup_{T \subseteq \omega_F} \{\varphi\, \phi_T : \varphi \in V_T\} \quad , \tag{3.8.39}$$

where the spaces V_T, V_F are defined in Lemma 3.13. If $F \not\subseteq \Gamma_C$ then V_F can be modified to contain only constant functions. In this case V_D will be spanned be seven (four if $F \subset \Gamma_N$) shape functions in the 2D case. In the 3D case the maximum dimension of V is 9.

We seek $v_D \in u_h|_{\omega_F} + V_D$ fulfilling

$$a(\, v_D, \varphi\,) = (\, f_h - m\, u_h, \varphi\,) - (\, J_F, \varphi\,)_F \quad , \quad \forall \varphi \in V_D \quad . \tag{3.8.40}$$

The solution v_D can be viewed as the solution of a Dirichlet problem with $v_D|_{\partial T_F} = u_h$ and with the Galerkin space V_D.

We define the estimator η_D in the following way:

Definition 3.11 *For a face $F \in \mathcal{F}_h$ we define*

$$\eta_{D,F} := \|v_D\|_{T_F} \quad .$$

We can then show that the residual based estimator η_R and the estimator η_D are equivalent.

Theorem 3.16 *For each face F and neighboring simplices T, T' it yields*

$$\eta_{D,F} \preceq \eta_{R,T} + \eta_{R,T'} \text{ and } \eta_{R,T} \preceq \sum_{F \subset \partial T} \eta_{D,F} \quad .$$

PROOF. For the proof of the upper bound for $\eta_{D,F}$ use v_D as a trial function in (3.8.40). Lemma 3.13 implies then

$$\begin{aligned} \|v_D\|_{S_F}^2 &\leq \sum_{T \subset S_F} \|v_D\|_{L^2(T)} \; \|f_h - m\, u_h\|_{L^2(T)} \; + \; \|v_D\|_{L^2(F)} \; \|J_F\|_{L^2(F)} \\ &\leq \sum_{T \subset S_F} \alpha_T^2 \, \|v_D\|_{S_F} \, \|f_h - m\, u_h\|_{L^2(T)} \; + \; \alpha_F^2 \, \|v_D\|_{S_F} \, \|J_F\|_{L^2(F)} \quad , \end{aligned}$$

what proves after cancelation the upper bound for $\eta_{D,F}$.
For the proof of the lower bound for $\eta_{D,F}$ we proceed as in the proof of Theorem 3.15. We define bubble functions

$$w_T := (f_h - m u_h)\, \phi_T \text{ and } w_F := E_F(J_F) \phi_{F,\delta} \quad .$$

Substitution of $\varphi = w_T$ in equation (3.8.40) shows as before

$$\alpha_T \;\; \|f_h - m\, u_h\|_{L^2(T)} \;\; \preceq \|v_D\|_T \quad . \tag{3.8.41}$$

Insertion of w_F in equation (3.8.40) and the bound (3.8.41) allows to establish the bound

$$\alpha_F^2 \;\; \|J_F\|_{L^2(F)}^2 \;\; \preceq \|v_D\|_{\omega_F} \quad .$$

This finishes the proof of lower bound. ■

3.9 Application to transient problems

There is a variety of articles about error estimators for transient problems. There are articles based on the dual problem which require H^2-regularity of the according stationary problem [26]. Due to the singularities of interface problems this higher regularity is not given and estimators of this type are not applicable. Another approach allows for lower and upper bounds of the error [58] within the abstract framework developed in [57].
In order to demonstrate the application of the a-posteriori techniques developed for the case of discontinuous diffusion coefficients we use a simple approach based on known energy techniques [46] [14]. The resulting error estimators combines errors which are due to time and space discretization and allow for an upper bound of the error.
In the following we apply the notation for the stationary problem given in section 3.2.1 and 3.8.1. For a positive time T we pose the transient problem in weak form and seek $u \in C^1([0,T], V)$

$$\left(\frac{\partial u}{\partial t}, v \right) + (k\, \nabla u, \nabla v\,) \; = (\; f, v\;) \quad \forall v \in V \quad , \quad \forall t \in (0,T) \tag{3.9.1}$$

with homogeneous boundary conditions of Dirichlet or Neumann type and an initial condition $u(\cdot,0) = u_0(\cdot) \in V$.
Problem (3.9.1) will be first discretized in time. At time-steps $t_0 = 0 < t_1 < ... < t_N = T$ we define Finite Element spaces V_h^n containing linear Finite Elements. The spaces $V_h^n, n = 0,..,N$ satisfy the conditions posed in section 3.2.2. That means, among other things, that the shape parameter of the underlying triangulation $\mathcal{T}_h^n$ is not too small and that $\mathcal{T}_h^n$ is aligned with the diffusion coefficients and parts of the boundary Γ_D and Γ_N. Let us define the time-step size $\tau_n := t_n - t_{n-1}, n = 1,..,N$. At each time-step $t_n, n = 1,..,N$ we solve the discrete elliptic problem: seek $u_h^n \in V_h^n$

$$\tau_n^{-1} \left(u_h^n, v_h^n \right) + \left(k \nabla u_h^n, \nabla v_h^n \right) = \left(f(t_n) + \tau_n^{-1} u_h^{n-1}, v_h^n \right) \quad \forall v_h^n \in V_h^n \quad . \tag{3.9.2}$$

Here u_h^0 is an approximation of u_0. A straightforward approach is to apply to each discrete problem the error estimator defined for the perturbed elliptic problems as discussed in section 3.8. Here the mass factor is $m := 1/\tau_n$. The error introduced by use of the term u_h^{n-1} instead of $u(\cdot, t_{n-1})$ in right hand side of (3.9.2) can be controlled by subsequently applying the estimator for the previous time-step as done in [2]. However, this approach does not make use of the parabolic nature of the problem.
Instead we define a-posteriori error estimators for $n = 1,..,N$ by

$$\eta_n^2 := \sum_{T \in \mathcal{T}_h} \frac{h^2}{k_T} \left\| \frac{u_h^n - u_h^{n-1}}{\tau_n} - f(t_n) \right\|_{L^2(T)}^2 + \left| u_h^n - u_h^{n-1} \right|_{kH^1(T)}^2 + \sum_{F \subset \partial T / \Gamma_D} \frac{h}{k_F} \left\| j_F^n \right\|_{L^2(F)}^2 \quad .$$

Here j_F^n denotes the jump of the flux of u_h^n normal to the face F. We denote the piecewise linear interpolant of the sequence $\{u_h^n\}, n = 0,..,N$ by

$$U(\cdot, t) := \frac{t_n - t}{\tau_n} u_h^{n-1}(\cdot) + \left(1 - \frac{t_n - t}{\tau_n} \right) u_h^n(\cdot) \quad .$$

With the above estimator we prove the upper bound for the error:

Theorem 3.17 *Let $d = 2, 3$. If the distribution of the diffusion coefficients $k_T, T \in \mathcal{T}_h^n$ is quasi-monotone for $n = 1,..,N$, the following upper bound holds for the solution u of equation (3.9.1) and u_h of equation (3.9.2):*

$$\sup_{t \in (0,T]} \|(u - U)(t)\|_{L^2(\Omega)}^2 + \int_0^T |(u - U)(t)|_{kH^1(\Omega)}^2 \, dt \preceq \|u(\cdot, 0) - u_h^0(\cdot)\|_{L^2(\Omega)}^2 + \sum_{n=1}^N \tau_n \, \eta_n^2 \quad . \tag{3.9.3}$$

In our analysis we assume that the terms $\left(u_h^{n-1}, v_h^n \right)$ from equation (3.9.2) are evaluated exactly for $u_h^{n-1} \in V_h^{n-1}$ and $v_h^n \in V_h^n$. A formal evaluation is not a problem, but if the calculation of the integral is done in course of mesh adaptation in a computer code, proper evaluation requires a suitable implementation. See for instance the software package ALBERT [52] where hierarchical bases have been used. If the exact evaluation of $\left(u_h^{n-1}, \varphi_h^n \right)$ is not possible, an additional error of the form $\|u_h^{n-1} - I_h^n(u_h^{n-1})\|_{H^{-1}(\Omega)}$ enters in the error estimator η_n and has to be controlled. Here $I_h^n : V \to V_h^n$ is an interpolation operator. Similarly to [14] one can regard additional errors which arise if the right hand side $f(t_n)$ in (3.9.1) has to be approximated.

PROOF. The proof uses standard energy techniques for linear parabolic problems. We refer to [46] for an application of these techniques to a more complicated problem.

The idea is to apply the residuum to the error $E := u - U$ and to make use of the discrete problem (3.9.2). First we calculate the residuum on the basis of equation (3.9.1).

$$\left(\frac{\partial E}{\partial t}, v \right) + (k \nabla E, \nabla v) = -\left(\frac{\partial U}{\partial t}, v \right) - (k \nabla U, \nabla v) + (f, v) \quad .$$

In a second step we set $v = E$ and substitute in the discrete problem (3.9.2) v_h^n by $I_L(E)$ where I_L is the interpolation operator I_L defined in section 3.5.3.

(3.9.4)

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t} \|E\|^2_{L^2(\Omega)} &+ |E|^2_{kH^1(\Omega)} \\ &= -\left(\frac{\partial U}{\partial t}, E \right) - (k \nabla U, \nabla E) + (f, E) \\ &= \left(\frac{\partial U}{\partial t} - f, I_L(E) - E \right) + (k \nabla u_h^n, \nabla(I_L(E) - E)) + (k \nabla(u_h^n - U), \nabla E) \\ &= I_1 + I_2 + I_3 \quad . \end{aligned}$$

The terms I_1, I_3 will be bounded using techniques already applied in the derivation of Theorem 3.5. We exploit interpolation estimates from Lemma 3.3 and the Cauchy-Schwarz inequality.

$$\begin{aligned} I_1 = \sum_{T \in \mathcal{T}_h} \left(\frac{\partial U}{\partial t} - f, I_L(E) - E \right)_T &\preceq \sum_{T \in \mathcal{T}_h} \frac{h}{k_T^{1/2}} \|\frac{\partial U}{\partial t} - f\|_{L^2(T)} \; |E|_{kH^1(T)} \\ &\le \left(\sum_{T \in \mathcal{T}_h} \left\{ \frac{h^2}{k_T} \|\frac{\partial}{\partial t} U - f\|^2_{L^2(T)} \right\} \right)^{1/2} |E|_{kH^1(\Omega)} \quad . \end{aligned}$$

The term I_2 is estimated using Gauss's theorem and the fact that $k \Delta u_h^n$ vanishes on the simplices $T \in \mathcal{T}_h^n$.

$$\begin{aligned} I_2 &= \sum_{T \in \mathcal{T}_h} (k \nabla u_h^n, \nabla(I_L(E) - E))_T = \sum_{F \in \mathcal{F}_h} (j_F^n, E)_F \\ &\preceq \sum_{F \in \mathcal{F}_h} \left(\frac{h}{k_F} \right)^{1/2} \|j_F^n\|_{L^2(F)} \; |E|_{kH^1(\tilde{\omega}_{T_F})} \le \left(\sum_{F \in \mathcal{F}_h} \frac{h}{k_F} \|j_F^n\|^2_{L^2(F)} \right)^{1/2} |E|_{kH^1(\Omega)} \quad . \end{aligned}$$

To bound the term I_3 we apply simply the Cauchy-Schwarz inequality

$$I_3 = (k\,\nabla(u_h^n - U), \nabla E\ ) \ \leq \left(\sum_{T\in\mathcal{T}_h} |u_h^n - U|^2_{kH^1(T)} \right)^{1/2} \ |E|_{kH^1(\Omega)} \quad .$$

The above estimates of I_1, I_2 and I_3 and inequality (3.9.4) yield together with Young's inequality which states $a\,b \leq \varepsilon a^2 + (4\varepsilon)^{-1} b^2$ for any real numbers a, b and $\varepsilon > 0$ the bound

(3.9.5)

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\ \|E\|^2_{L^2(\Omega)} \ + \ |E|^2_{kH^1(\Omega)} \\
&\preceq \sum_{T\in\mathcal{T}_h} \left(\frac{h^2}{k_T}\ \|\frac{\partial U}{\partial t} - f\|^2_{L^2(T)} \ + \ |u_h^n - U|^2_{kH^1(T)} \ + \sum_{F\subset\partial T/\Gamma_D} \frac{h}{k_F}\ \|j_F^n\|^2_{L^2(F)} \right) \\
&\leq \sum_{T\in\mathcal{T}_h} \left(\frac{h^2}{k_T}\ \|\frac{u_h^n - u_h^{n-1}}{\tau_n} - f\|^2_{L^2(T)} \ + \ |u_h^n - u_h^{n-1}|^2_{kH^1(T)} \ + \sum_{F\subset\partial T/\Gamma_D} \frac{h}{k_F}\ \|j_F^n\|^2_{L^2(F)} \right) \quad .
\end{aligned}$$

In the last step we exploited the identities $\frac{\partial U}{\partial t} = \frac{u_h^n - u_h^{n-1}}{\tau_n}$ and $u_h^n - U = (1 - \frac{t_n - t}{\tau_n})(u_h^n - u_h^{n-1})$. Integration of (3.9.5) over the time interval $[0, T]$ finishes the proof. ∎