Appendix C

The renormalization group

C.1 The renormalization group transformation

The idea behind the renormalization group [68] is concerned with the basic physics of phase transitions, namely the long wavelength fluctuations that are the cause of singularities at the critical point.

For a D-dimensional physical system near a critical point, the number of degrees of freedom interacting with each other is of the order of ξ^D, where ξ is the correlation length. At the critical point, this number diverges, since there the correlation length is infinite.

The renormalization group method consists in systematically reducing the number of interacting degrees of freedom by integrating over short-wavelength fluctuations. Suppose we start with a physical system on a lattice of lattice spacing a, then the minimum wavelength for fluctuations is of order a. If we integrate over fluctuations having wavelengths $a < \lambda < sa$, where $s > 1$ is called the dilatation factor, this makes no difference to the behavior of correlation functions $G(\mathbf{x})$ for $x > sa$: integration over short-wavelength fluctuations assigns to the original system a corresponding one with the same long-distance behavior. The transformation from one to the other is called a renormalization group transformation (RGT). It can be iterated by integrating over fluctuations having wavelengths $sa < \lambda < s^2 a$, etc., thus establishing a whole sequence of corresponding physical systems, all with the same long-distance behavior.

After the first RGT, the effective lattice spacing of the transformed system

is *sa*. Accordingly, one associates with each RGT a dilatation of the unit of length by a factor s, thus making it possible to compare the original and the transformed systems in a lattice common to both.

The dilatation of the unit of length links two correlation functions of arguments x and x/s, respectively. This is not equivalent to simple dimensional analysis, since the integration over short-wavelength fluctuations changes (renormalizes) the system parameters.

C.2   Critical surface and fixed points

To give a more precise formulation to the concepts we wish to discuss in this section, let us consider a physical system which, near the critical point, is described by the generalized Ginzburg-Landau Hamiltonian

$$\mathcal{H}_0 = \int \mathrm{d}^D x \left[\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}r_0\phi^2 + \tfrac{1}{4!}u_0\phi^4 + \tfrac{1}{6!}v_0\phi^6 + \cdots \right] \qquad (C.1)$$

The coupling constants define a *parameter space* for the system under study, we shall write them collectively as

$$\mu = \{r_0, u_0, v_0, \cdots\}. \qquad (C.2)$$

Each possible state of the system is described by a point in parameter space, and an RGT R_s maps a point μ of this space into another such point μ', such that

$$\mu' = R_s\mu = \{r_0', u_0', v_0', \cdots\}. \qquad (C.3)$$

The renormalization group strategy consists in iterating R_s many times, and one can thus define the nth iterate $R_{s^n} = R_s \ldots R_s$. Suppose we are at the critical point, where the correlation length ξ diverges. After applying an RGT, the transformed correlation length $\xi' = \xi/s$ remains infinite, that is, the system is still critical. The locus of points in parameter space that correspond to systems at the critical point ($\xi = \infty$) is called the *critical surface* S_∞. An RGT applied to a point in S_∞ always transforms it on a point likewise in S_∞. On the other hand, if we start from a point far from the critical surface, each RGT will move it further away, since each operation divides the correlation length by s. Under iterations of the RGT each point in parameter space describes a trajectory, and the set of all trajectories is called the *renormalization flow.*

Although the behavior of the points in S_∞ arising from successive iterations of an RGT is in principle arbitrary, the physically interesting case is that where there is a *fixed point*, characterized by a set μ^* of parameters such that

$$\mu^* = R_s \mu^*. \tag{C.4}$$

In general, the position of the fixed point is related to the values of the parameters at a phase transition (if, for example, one of the parameters is the temperature, the value of the coordinate of the fixed point on the temperature axis in parameter space will correspond to the critical temperature), and it depends on the particular form chosen for the RGT. On the other hand, universal physical quantities, such as critical exponents, do not depend on the explicit form of the RGT.

C.3 Linearization near a fixed point

Let $\mu = \{g_1, g_2, \cdots\}$ be a point in parameter space near the fixed point $\mu^* = \{g_i^*\}$, with

$$g_i = g_i^* + \delta g_i, \tag{C.5}$$

and let $\mu' = R_s \mu$ be its transform under an RGT. If μ and μ^* are close enough, the relation between δg_i and $\delta g_i'$ is approximately linear:

$$\delta g_i' \simeq T_{ij} \delta g_j, \tag{C.6}$$

where

$$T_{ij}(s) = \left. \frac{\partial g_i'}{\partial g_j} \right|_{\mu^*}. \tag{C.7}$$

Since by the definition of the RGT, $T(s_1)T(s_2) = T(s_1 s_2)$, for an infinitesimal dilatation parameter $s = 1 + \delta$, $T(s)$ must have the form

$$T(s) = \exp(\mathcal{T} \ln s), \tag{C.8}$$

with

$$\mathcal{T} = \left. \frac{\mathrm{d}T}{\mathrm{d}\ln s} \right|_{s=1}. \tag{C.9}$$

Let e^a be an eigenvector of $\mathcal{T}_{ij}$, corresponding to an eigenvalue y_a:

$$\mathcal{T}_{ij} e_j^a = y_a e_i^a. \tag{C.10}$$

Accordingly

$$T_{ij}e_j^a = s^{y_a}e_i^a. \tag{C.11}$$

Any point in parameter space has a representation in the basis of the eigen-vectors $\{e^a\}$:

$$\delta g_i = t_a e_i^a, \tag{C.12}$$

and, in the linear approximation

$$\delta g_i' = t_a s^{y_a} e_i^a = s^{y_a}\delta g_i. \tag{C.13}$$

The coefficients t_i are called *scaling fields*. According to (C.13), three cases are possible:

- $y_a > 0$: the scaling field increases under iterations of the RGT: t_i is called a *relevant* field.

- $y_a = 0$: the scaling field remains constant (in the linear approximation) under iterations of the RGT: t_i is called a *marginal* field.

- $y_a < 0$: the scaling field decreases under iterations of the RGT: t_i is called an *irrelevant* field.

We thus see that, to be at the critical point, one must set all relevant fields equal to zero: if they are nonzero, the RGT will move the point μ' further away from the fixed point. The most frequent case is when there is only one relevant field (usually corresponding to the temperature).

C.4   A simple example: the Ginzburg-Landau model

To illustrate the renormalization method, let us consider the Hamiltonian (C.1), and disregard the terms proportional to powers of ϕ higher than four. As we shall see, these terms are irrelevant for $D \geq 4$.

The partition function is given by

$$Z[\phi] = \int \mathcal{D}\phi e^{-\beta\mathcal{H}_0[\phi]}, \tag{C.14}$$

and we calculate it as an a expansion in powers of u_0, in the standard perturbative approach. The effective Hamiltonian is then, to second order in the expansion,

$$
\mathcal{H} = \int \mathrm{d}^D x \left\{ \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}\left[r_0 + \frac{k_\mathrm{B}T}{2}\int_{\Lambda/s}^{\Lambda} \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{u_0}{k^2 + r_0} \right]\phi^2 \right.
$$
$$
\left. + \tfrac{1}{4!}\left[u_0 - \frac{3k_\mathrm{B}T}{2}\int_{\Lambda/s}^{\Lambda} \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{u_0^2}{(k^2 + r_0)^2} \right]\phi^4 \right\}, \qquad \text{(C.15)}
$$

According to the renormalization group prescription, we integrate over fluctuations with wavelengths between a and sa, this is equivalent, in Fourier space, to integrating over wavenumbers between $\Lambda = 1/a$ and Λ/s.

The second step of the RGT consists in the dilatation of the unit of length by a factor s, $x \rightarrow x/s$; accordingly, in momentum space we have $k \rightarrow sk$. Due to the rescaling of the coordinates, the field variable ϕ transforms as $\phi \rightarrow s^{d_\phi}\phi$. In general, d_ϕ does not simply assume the value found by dimensional analysis, but it is influenced by fluctuations, and depends on the fixed point under consideration. However, to the order to which we are now working, fluctuations do not affect the gradient term in (C.15), and we may therefore set $d_\phi = D/2 - 1$, the *canonical dimension* of the field. The effective Hamiltonian is, after the dilatation,

$$
\mathcal{H} = \int \mathrm{d}^D x \left\{ \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}s^2\left[r_0 + \frac{k_\mathrm{B}T}{2}\int_{\Lambda/s}^{\Lambda} \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{u_0}{k^2 + r_0} \right]\phi^2 \right.
$$
$$
\left. + \tfrac{1}{4!}s^{4-D}\left[u_0 - \frac{3k_\mathrm{B}T}{2}\int_{\Lambda/s}^{\Lambda} \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{u_0^2}{(k^2 + r_0)^2} \right]\phi^4 \right\}.
$$
$$
\text{(C.16)}
$$

The transformation laws for the coupling constants are therefore

$$
r = s^2 r_0 \left(1 - \frac{k_\mathrm{B}T}{16\pi^2}u_0 \ln s \right), \qquad \text{(C.17)}
$$

$$
u = s^\epsilon u_0 \left(1 - \frac{3k_\mathrm{B}T}{16\pi^2}u_0 \ln s \right), \qquad \text{(C.18)}
$$

where we have carried out the integrals in $D = 4 - \epsilon$ dimensions, for small ϵ.

Equations (C.17) and (C.18) can be transformed into differential equations, by considering an infinitesimal dilatation factor $s = 1 + \delta$. One obtains

$$
\frac{\mathrm{d}r(s)}{\mathrm{d}\ln s} = 2r(s) - \frac{k_\mathrm{B}T}{16\pi^2}u(s)r(s), \qquad \text{(C.19)}
$$

$$\frac{\mathrm{d}u(s)}{\mathrm{d}\ln s} \;=\; \epsilon u(s) - \frac{3k_\mathrm{B}T}{16\pi^2}u^2(s). \tag{C.20}$$

These equations are known as the *flow equations*, and they make it easy to calculate the position of the fixed points. The system of equations above allows two fixed points, namely $r = u = 0$ or $r = 0, u = u^*$, with $u^* = 16\epsilon\pi^2/3k_\mathrm{B}T$. Let us now analyze their stability.

C.4.1   The Gaussian fixed point

Let us start with the trivial, or *Gaussian* fixed point $r = u = 0$. We have

$$\mathcal{T} = \begin{pmatrix} 2 & 0 \\ 0 & \epsilon \end{pmatrix}, \tag{C.21}$$

with eigenvalues and eigenvectors given by

$$e^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y_1 = 2, \tag{C.22}$$

$$e^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad y_2 = \epsilon. \tag{C.23}$$

Thus r is a relevant field, and for $D > 4$ ($\epsilon < 0$), u is an irrelevant field. Provided that $D > 4$, all higher order terms are also irrelevant, the sixth-order term v, for example, transforms as

$$v = s^{6-2D}v_0, \tag{C.24}$$

being therefore irrelevant. The parameter r vanishes at the critical point, as expected. However, the model with $u = 0$ is not defined for $r < 0$, since there it becomes unstable.

For $D < 4$, u becomes relevant, and the Gaussian fixed point is no longer adequate to describe the transition. A *non-Gaussian* fixed point appears, whose properties we will investigate in the next section.

C.4.2   Non-Gaussian fixed point

For the fixed point $r = 0, u = u^*$, we have

$$\mathcal{T} = \begin{pmatrix} 2 - \frac{\epsilon}{3} & 0 \\ 0 & -\epsilon \end{pmatrix}. \tag{C.25}$$

Thus r is again a relevant field, an u is irrelevant for $D < 4$. Now the model is well defined both for $r > 0$ and for $r < 0$. At the phase transition, the relevant parameter r vanishes.

The flow diagrams for $D > 4$ and $D < 4$ are sketched in Fig. (C.1).
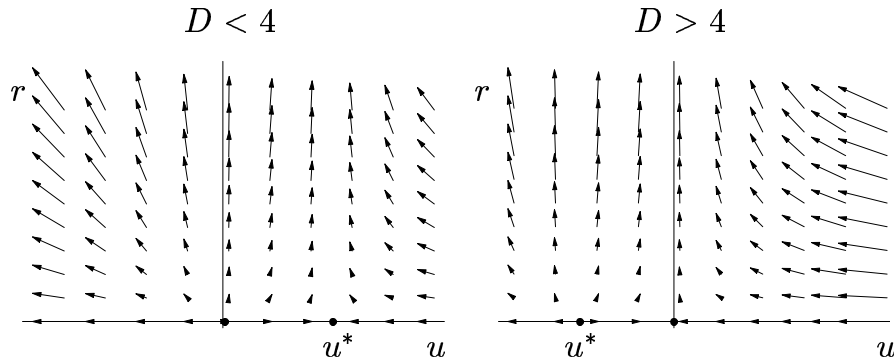

Figure C.1: Flow diagrams for $D < 4$ and $D > 4$. For $D > 4$ the fixed point at the negative u^* is unphysical, the transition is described by the Gaussian fixed point at the origin. For $D < 4$ the situation is reversed.