Chapter 5

Quantum Fluctuations of Fluid Membranes

In the previous chapter we verified that thermal fluctuations soften fluid membranes without destroying the covariance of the elastic energy (4.1). According to the results derived in Chapter 2, in particular Eq. (2.60), the softening of the membranes due to thermal fluctuations does not induce a phase transition: fluid membranes always appear crumpled at large length scales.

This is no longer true if one considers fluid membranes subject to thermal *and* quantum fluctuations[61]. While it may seem unnecessary to take quantum fluctuations into account when considering biomembranes, since they typically exist at high temperatures, a model incorporating the effect of quantum-fluctuations to the Canham-Helfrich energy may serve as a starting point for studying quantum interfaces, such as the Helium liquid-vapor interface at very low temperatures [62, 63]. In He^3-He^4 mixtures, the He^3 film on top of the He^4 bulk is a suitable candidate for the quantum membrane [63].

Moreover, quantum oscillations, analogous to the superconducting Josephson effect, have recently been detected in samples of superfluid He^3[64, 65]. The apparatus involved in these experiments consists of an inner cell, filled with He^3, contained in an outer cell, also filled with He^3. The two containers are separated by a rigid membrane glued to the bottom of the inner cell, and by a softer one attached to its top. The lower membrane contains an array of small apertures allowing for exchange of atoms between the cells, equivalent to the superconducting weak link. By manipulating this membrane, the pressure between the two systems can be kept at a fixed value, and the resulting mass current is determined by the displacement of the upper membrane from its original position. Josephson current oscillations at the weak link are then observed as oscillations of this membrane.

It is thus of interest to investigate whether the membranes themselves display quantum fluctuations, and to what extent such effects may be observable.

5.1 The model

The quantum-statistical partition function can be obtained from the quantum-mechanical one, that we encountered in Chapter 3 (see definition (3.4)) by continuing the time interval $t_b - t_a$ to the negative imaginary value [55]

$$t_b - t_a = -\frac{i\hbar}{k_{\rm B}T} = -i\hbar\beta. \tag{5.1}$$

This simple formal reason makes the quantum-mechanical partition function contain all information on the thermodynamic equilibrium properties of a quantum system.

For a quantum membrane described by a Lagrangian $\mathcal{L}$, the quantum-statistical partition function is a functional integral over all possible time-dependent surface configurations $\mathbf{X}(\bar{\sigma}, \tau)$:

$$Z = \int \mathcal{D}\mathbf{X} \exp(-S_0[\mathbf{X}]/\hbar), \tag{5.2}$$

with the Euclidean action

$$S_0 = \int_0^\beta \mathrm{d}\tau\, \mathrm{d}^2\sigma \sqrt{g}\mathcal{L}(\mathbf{X}, \dot{\mathbf{X}}), \tag{5.3}$$

and the Wick-rotated time $\tau = it/\hbar\beta$.

To obtain the Lagrangian for the quantum membrane we have to add a kinetic term to the classical Canham-Helfrich energy (2.17). For an incompressible membrane, the kinetic term reads, in Euclidean spacetime [66, 67],

$$\mathcal{T} = \frac{1}{2\nu_0} \int \mathrm{d}^2\sigma \sqrt{g}\, (\dot{\mathbf{X}} \cdot \mathbf{N})^2, \tag{5.4}$$

where $1/\nu_0$ is the bare mass density, and we allow only for the normal component of the velocity vector, since the tangential components correspond to the in-plane flow of molecules inside the membrane.

The resulting Euclidean action for the quantum membrane is

$$S_0 = \int_0^\beta \mathrm{d}\tau\, \mathrm{d}^2\sigma \sqrt{g} \left[\frac{1}{2\nu_0} (\dot{\mathbf{X}} \cdot \mathbf{N})^2 + r_0 + \frac{1}{2\alpha_0} H(\mathbf{X})^2 \right], \tag{5.5}$$

where $\alpha_0 \equiv 1/\kappa_0$ is the inverse bending rigidity of the membrane, and $H(\mathbf{X})$ is its mean curvature, as defined in Sec. 2.2. As discussed in Chapter 2, α_0 is proportional to the true expansion parameter

$$u_{\mathrm{th},0} \equiv k_{\mathrm{B}} T \alpha_0 \tag{5.6}$$

for perturbation theory, and we shall make use of this parameter from now on, in order to simplify the notation and the calculations.

We shall in this chapter study the quantum statistical mechanics of a membrane described by this action.

5.2 Perturbative calculation

5.2.1 Zero temperature

Let us first study the effect of quantum fluctuations alone, by setting the temperature equal to zero. We shall again work in the Monge parametrization, where, as described in Chapter 2, a point on the surface embedded in three-dimensional space is described by a displacement field $\phi(\bar{\sigma}, \tau)$ with respect to a reference plane $\bar{\sigma} = (\sigma_1, \sigma_2)$, such that

$$\mathbf{X}(\bar{\sigma}, \tau) = (\bar{\sigma}, \phi(\bar{\sigma}, \tau)) \,. \tag{5.7}$$

Inserting this parametrization in the action (5.5), and expanding the resulting expression up to fourth order in powers of the displacement field ϕ, we obtain

$$\begin{aligned} S_0 \;=\; \int \mathrm{d}\tau \mathrm{d}^2\sigma \Big\{ & \frac{1}{2} \left[\frac{1}{\nu_0} \dot{\phi}^2 + r_0 (\partial_i \phi)^2 + \frac{1}{\alpha_0} (\partial^2 \phi)^2 \right] - \frac{1}{4\nu_0} \dot{\phi}^2 (\partial_i \phi)^2 \\ & - \frac{r_0}{8} (\partial_i \phi)^2 (\partial_j \phi)^2 - \frac{1}{4\alpha_0} (\partial_i \phi)^2 (\partial^2 \phi)^2 && (5.8) \\ & - \frac{1}{\alpha_0} (\partial_i \phi)(\partial_j \phi)(\partial_i \partial_j \phi)(\partial^2 \phi) \Big\} \,. && (5.9) \end{aligned}$$

The displacement field describes the undulations of the surface. Its spectrum, $\omega^2 = c_s^2 q^2$, is gapless with $c_s = \sqrt{r_0 \nu_0}$ the velocity of the transversal waves.

In the one-loop approximation, the exponent in (5.2) may be expanded up to second order around a background configuration $\Phi(\bar{\sigma}, \tau)$ extremizing S_0. The resulting integral is Gaussian and yields the effective action

$$S_{\text{eff}}[\Phi] = S_0[\Phi] + S_1[\Phi] = S_0[\Phi] + \frac{\hbar}{2} \text{Tr} \ln \left[\left. \frac{\delta^2 S_0}{\delta\phi(\bar{\sigma}, \tau)\delta\phi(\bar{\sigma}', \tau')} \right|_{\Phi} \right], \qquad (5.10)$$

where the expression in square brackets is a functional matrix given by the second functional derivative of S_0, and Tr denotes the functional trace, i.e., the integral $\int \mathrm{d}\tau\, \mathrm{d}^2\sigma$, as well as the integral $\int \mathrm{d}\omega\, \mathrm{d}^2 q/(2\pi)^3$ over the (angular) frequency ω and the wavevector $\mathbf{q}$.

Using the derivative-expansion method explained in Chapter 3 [53], we expand the one-loop correction S_1 in Eq. (5.10) in powers of the derivatives of the field $\phi(\bar{\sigma}, \tau)$, as in Chapter 4[58]. To obtain the renormalization of the parameters r_0, α_0 and ν_0, it suffices to keep only the first three terms of the expansion:

$$S_1 = \frac{\hbar}{2} \int \mathrm{d}\tau\, \mathrm{d}^2\sigma \left[I_1 \dot{\phi}^2 + I_2 (\partial_i \phi)^2 + I_3 (\partial^2 \phi)^2 + \ldots \right], \qquad (5.11)$$

with

$$I_1 = \frac{1}{2\nu_0} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{q^2}{\omega^2/\nu_0 + r_0 q^2 + q^4/\alpha_0} \qquad (5.12)$$

$$I_2 = \int \frac{\mathrm{d}\omega}{2\pi} \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{\frac{1}{2}\omega^2/\nu_0 - r_0 q^2 - \frac{3}{2} q^4/\alpha_0}{\omega^2/\nu_0 + r_0 q^2 + q^4/\alpha_0} \qquad (5.13)$$

$$I_3 = -\frac{3}{2} \frac{1}{\alpha_0} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{q^2}{\omega^2/\nu_0 + r_0 q^2 + q^4/\alpha_0}. \qquad (5.14)$$

After the integrals over the loop energy ω have been carried out, the resulting momentum integrals in Eqs. (5.12)–(5.14) diverge in the ultraviolet. To regularize them, we introduce a wavenumber cutoff Λ. Contributions proportional to positive powers of Λ are irrelevant to the renormalization group, and will be ignored. In dimensional regularization, where the number D of space dimensions of the membrane is analytically continued to be less than two, $D = 2 - \epsilon$, these powerlike divergences never appear in the first place. Only logarithmic divergences arise as poles in $1/\epsilon$, as discussed in Chapter

2. We now adopt regularization with a cutoff, since, as we shall see later, it proves useful when deriving the renormalization group equations for coupling constants with nonzero dimension. Substituting the results of the integration into Eq. (5.11), we obtain the effective action

$$S_{\text{eff}} = \frac{1}{2}\int \mathrm{d}\tau\, \mathrm{d}^2\sigma \left[\frac{1}{\nu}\dot{\phi}^2 + r(\partial_a\phi)^2 + \frac{1}{\alpha}(\partial^2\phi)^2 + \dots\right], \tag{5.15}$$

with the renormalized inverse mass density ν, surface tension r, and inverse rigidity α

$$\frac{1}{\nu} = \frac{1}{\nu_0}\left[1 + \frac{1}{16\pi}u_{\text{qm},0}\ln\left(\Lambda\xi_{\text{qm},0}\right)\right], \tag{5.16}$$

$$r = r_0, \tag{5.17}$$

$$\frac{1}{\alpha} = \frac{1}{\alpha_0}\left[1 + \frac{3}{16\pi}u_{\text{qm},0}\ln\left(\Lambda\xi_{\text{qm},0}\right)\right], \tag{5.18}$$

where the dimensionless parameter

$$u_{\text{qm},0} = \hbar r_0 \alpha_0^{3/2}\nu_0^{1/2} \tag{5.19}$$

is the (bare) expansion parameter in the quantum regime. Here, as in the following, we drop the subscript eff of renormalized quantities; to simplify the notation they will carry no subscript. Note that the surface tension is not renormalized by quantum fluctuations at this order. The parameter $\xi_{\text{qm},0} = 2/\sqrt{r_0\alpha_0}$ in the argument of the logarithm in Eqs. (5.16) and (5.18) defines a characteristic length scale of the problem. It sets the scale at which the tension and stiffness terms in the action (5.15) become equally important. At larger scales, the second term in the expansion (5.15) becomes more important and the undulations are dominated by tension, while at smaller scales, the third term dominates and the undulations are controlled by stiffness.

To obtain the renormalization flow, we apply Wilson's procedure (explained in Appendix C)[68]. Integrating out a momentum shell $\Lambda/s < q < \Lambda$, rescaling the coupling constants $\hat{\nu} \to s^{\epsilon-3}\nu$, $\hat{r} \to s^{-(\epsilon-3)}r$, $\hat{\alpha} \to s^{\epsilon-1}\alpha$, we arrive at

$$\beta_\nu(\nu, r, \alpha) = s\frac{\partial\hat{\nu}}{\partial s}\bigg|_{s=1} = (\epsilon-3)\nu - \frac{1}{16\pi}u_{\text{qm}}\nu, \tag{5.20}$$

$$\beta_r(\nu, r, \alpha) = s\frac{\partial\hat{r}}{\partial s}\bigg|_{s=1} = -(\epsilon-3)r, \tag{5.21}$$

$$\beta_\alpha(\nu, r, \alpha) = s\frac{\partial \hat{\alpha}}{\partial s}\bigg|_{s=1} = (\epsilon - 1)\alpha - \frac{3}{16\pi}u_{\rm qm}\alpha, \tag{5.22}$$

where $\epsilon = 2 - D$ is assumed to be small and will be set to zero at the end of the calculation. The coefficients of the first terms on the right-hand sides denote the scaling dimension of the scaling fields (see Appendix C). For small ϵ, the scaling fields ν and α are irrelevant, while r is relevant. Criticality is obtained by setting the relevant fields to zero, i.e., $r = 0$ in this case. Starting somewhere on the critical surface $r = 0$, the system flows towards the trivial fixed point $\nu = \alpha = 0$. This guarantees the stiffness of the membrane at large scales. There is no crumpling transition at the absolute zero of temperature; the membrane is always flat, where the normal vectors to its surface are strongly correlated. More specifically, the correlation function between the normal vectors to the surface behaves at large scales as

$$\langle \partial_i \mathbf{X}(\bar{\sigma}, \tau) \cdot \partial_j \mathbf{X}(\bar{\sigma}', \tau)\rangle \sim \frac{\delta_{ij}}{|\sigma - \sigma'|^3}. \tag{5.23}$$

Details of the calculation of this correlation function can be found in Appendix D.1. This algebraic fall-off implies the absence of a persistence length which would define the length scale above which the normals become uncorrelated and the surface becomes crumpled.

To investigate this further, let us calculate the Hausdorff dimension $d_{\rm H}$ of the membrane. It can be defined by the relation between its mean surface area $\langle A\rangle$, where

$$A = \int {\rm d}^2\sigma\sqrt{g}, \tag{5.24}$$

and the frame, or projected area $A_0 = \int {\rm d}^2\sigma$. This relation is

$$\langle A\rangle \sim A_0^{d_{\rm H}/2}, \tag{5.25}$$

so that the Hausdorff dimension is given by

$$d_{\rm H} = 2\frac{\partial \ln\langle A\rangle}{\partial \ln A_0}. \tag{5.26}$$

Since the frame area A_0 scales with the cutoff Λ as

$$A_0 = \int {\rm d}^2\sigma \sim \Lambda^{-2}, \tag{5.27}$$

Eq. (5.26) can also be written in the form

$$d_{\mathrm{H}} = -\frac{\partial \ln\langle A\rangle}{\partial \ln \Lambda}. \tag{5.28}$$

At the one-loop level, the mean surface area is

$$\begin{aligned}\langle A\rangle &= \left\langle \int \mathrm{d}^2\sigma \left[1 + \frac{1}{2}(\partial_i \phi)^2 + \ldots\right]\right\rangle \\ &= A_0\left(1 + \frac{\hbar}{2}\int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{q^2}{\omega^2/\nu_0 + r_0 q^2 + q^4/\alpha_0}\right),\end{aligned} \tag{5.29}$$

so that we obtain from relation (5.28) the Hausdorff dimension

$$d_{\mathrm{H}} = 2 + \frac{1}{16\pi} u_{\mathrm{qm}}. \tag{5.30}$$

For large membranes, $u_{\mathrm{qm}} \to 0$, implying a Hausdorff dimension $d_{\mathrm{H}} = 2$. Expressed in group theoretic terms, the $\mathrm{SO}(d)$ rotational symmetry of d-dimensional space is spontaneously broken to its $\mathrm{SO}(d-2) \times \mathrm{SO}(2)$ subgroup. According to the Mermin-Wagner theorem [69], long-range order is destroyed in a two-dimensional system at any finite temperature. Since we are studying quantum fluctuations, where the membrane has an extra (time) dimension, this spontaneous symmetry breaking does not violate the Mermin-Wagner theorem. The algebraic decay found in Eq. (5.23) is an immediate consequence of this symmetry breaking and identifies the two resulting Goldstone modes.

5.2.2 Finite temperature

Let us now investigate how the high-temperature regime, dominated by thermal fluctuations, where the membrane is found to be always crumpled, goes over into the low-temperature regime dominated by quantum fluctuations, where the membrane remains flat.

To investigate this temperature dependence, we adopt the imaginary-time approach to thermal field theory [55]. It can be derived from the corresponding Euclidean quantum theory at zero temperature by restricting the Euclidean time to the finite interval $0 \leq \tau \leq \hbar/k_{\mathrm{B}}T$, and substituting

$$\int \frac{\mathrm{d}\omega}{2\pi} g(\omega) \to \frac{k_{\mathrm{B}}T}{\hbar}\sum_n g(\omega_n), \tag{5.31}$$

where g is an arbitrary function, and ω_n denote the Matsubara frequencies,

$$\omega_n = 2\pi n k_{\rm B} T/\hbar, \qquad n = 0, \pm 1, \pm 2 \cdots. \tag{5.32}$$

Using this substitution in the zero-temperature integrals (5.12)–(5.14), as well as the formula [70]

$$\frac{k_{\rm B}T}{\hbar} \sum_n \frac{1}{\omega_n^2 + a^2} = \frac{1}{2a} \coth\left(\frac{\hbar a}{2 k_{\rm B} T}\right) \tag{5.33}$$

to carry out the sum over the Matsubara frequencies, we arrive at

$$\frac{1}{\nu} = \frac{1}{\nu_0}\left[1 - \frac{1}{8\pi} u_{\rm qm,0} F_1(T, \Lambda')\right], \tag{5.34}$$

$$r = r_0 \left[1 - \frac{1}{2\pi} u_{\rm qm,0} F_2(T, \Lambda')\right], \tag{5.35}$$

$$\frac{1}{\alpha} = \frac{1}{\alpha_0}\left[1 - \frac{3}{8\pi} u_{\rm qm,0} F_1(T, \Lambda')\right], \tag{5.36}$$

with

$$F_1(T, \Lambda') = \int^{\Lambda'} {\rm d}q' \frac{q'^2 \coth(\frac{1}{2}\gamma_0 q' \sqrt{1+q'^2})}{\sqrt{1+q'^2}}, \tag{5.37}$$

$$F_2(T, \Lambda') = \int^{\Lambda'} {\rm d}q' \frac{(\frac{3}{4}q'^2 + q'^4) \coth(\frac{1}{2}\gamma_0 q' \sqrt{1+q'^2})}{\sqrt{1+q'^2}}. \tag{5.38}$$

Here, we rescaled the integration variable $q' = q\xi_{\rm qm,0}/2$, and therefore $\Lambda' = \Lambda \xi_{\rm qm,0}/2$; and introduced $\gamma_0 = \hbar r_0 \alpha_0^{1/2} \nu_0^{1/2} / k_{\rm B} T$. This dimensionless parameter can be expressed as the ratio of the expansion parameters in the quantum and classical regime,

$$\gamma_0 = u_{\rm qm,0}/u_{\rm th,0}. \tag{5.39}$$

The integrals in Eqs. (5.38) and (5.37) diverge in the ultraviolet as $\Lambda' \to \infty$. As before, we disregard powerlike divergences, and consider only the logarithmically diverging terms. We thus arrive at:

$$\frac{1}{\nu} = \frac{1}{\nu_0}\left[1 - \frac{1}{4\pi}\left(u_{\rm th,0} - \frac{1}{4} u_{\rm qm,0}\right) \ln(\Lambda')\right], \tag{5.40}$$

$$r = r_0 \left[1 + \frac{1}{4\pi} u_{\rm th,0} \ln(\Lambda')\right], \tag{5.41}$$

$$\frac{1}{\alpha} = \frac{1}{\alpha_0}\left[1 - \frac{3}{4\pi}\left(u_{\rm th,0} - \frac{1}{4} u_{\rm qm,0}\right) \ln(\Lambda')\right]. \tag{5.42}$$

One may check that as the temperature T and, consequently, $u_{\mathrm{th},0} \propto T$ tend to infinity, Eqs. (5.41) and (5.42) reproduce, up to finite terms, the high-temperature results (4.14) and (4.23). On the other hand, in the limit $u_{\mathrm{th},0} \propto T \to 0$, we recover the zero-temperature results (5.16)–(5.18).

In order to explore the behavior of the membrane at large length scales, we compute the flow equations corresponding to the three parameters of the theory, as we did above. They are given by:

$$\beta_\nu(\nu, r, \alpha) = (\epsilon - 3)\nu + \frac{1}{4\pi}\left(u_{\mathrm{th}} - \frac{1}{4}u_{\mathrm{qm}}\right)\nu, \tag{5.43}$$

$$\beta_r(\nu, r, \alpha) = -(\epsilon - 3)r + \frac{1}{4\pi}u_{\mathrm{th}}r, \tag{5.44}$$

$$\beta_\alpha(\nu, r, \alpha) = (\epsilon - 1)\alpha + \frac{3}{4\pi}\left(u_{\mathrm{th}} - \frac{1}{4}u_{\mathrm{qm}}\right)\alpha. \tag{5.45}$$

This system of equations admits two possible fixed points (see Fig. 5.1), viz. the trivial one at $\nu = r = \alpha = 0$ which we already found above at $T = 0$, and a new one at $\nu = r = 0, \alpha = \alpha^*$, with

$$\alpha^* = \frac{4\pi}{3k_{\mathrm{B}}T}(1 - \epsilon). \tag{5.46}$$

Note that this fixed point exists even for a two-dimensional membrane ($\epsilon = 0$). The scaling field r is found to be relevant, while ν is found to be irrelevant with respect to both fixed points. Criticality is obtained by setting $r = 0$. The scaling field α behaves differently: it is irrelevant with respect to the trivial fixed point, but relevant with respect to the new one. The presence of the unstable fixed point implies the existence of a crumpling transition for a two-dimensional membrane at a critical temperature

$$T_{\mathrm{c}} = \frac{4\pi}{3k_{\mathrm{B}}}\frac{1}{\alpha^*}. \tag{5.47}$$

For $T < T_{\mathrm{c}}$, α flows away from α^* and towards the trivial fixed point $\alpha = 0$. The correlation between the normals to the surface is long-ranged, and the membrane remains flat. For $T > T_{\mathrm{c}}$, on the other hand, α flows away from α^* in the other direction, that is $\alpha \to \infty$. In this case, thermal fluctuations dominate. The correlation between the normals to the surface is short-ranged, and the membrane is found to be crumpled. As the temperature tends to infinity, the time dimension shrinks to a point, making the integration $\int \mathrm{d}\tau$ disappear

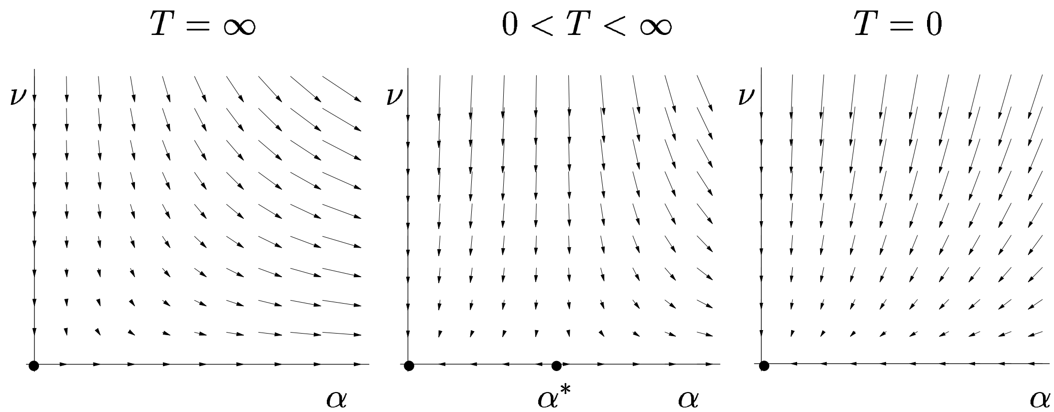

Figure 5.1: Flow diagrams in the (α, ν)-plane. As T becomes finite from above (middle panel), a nontrivial fixed point ($\alpha^* \neq 0$) starts to move to the right away from the origin, and disappears at infinity when T tends to zero (right panel).

from the action (5.5). This implies that the parameter ϵ in the flow equations (5.43)–(5.45) must be set equal to $\epsilon = \epsilon' + 1$ [71, 72], with $\epsilon' = 0$ for $D = 2$. We see that the fixed point (5.46) reduces in the limit $T \to \infty$ to the trivial one for a two-dimensional membrane. A similar nontrivial fixed point to the one in (5.46) was found in Refs. [40, 2] for a $(2 + |\epsilon'|)$-dimensional membrane ($\epsilon' < 0$) described by the classical Canham-Helfrich model embedded in $3+|\epsilon'|$ dimensions, as discussed in Chapter 2. In our case, due to the presence of the extra time dimension, the nontrivial fixed point (5.46) exists also in two dimensions.

5.3 Large-d calculation

Let us now analyze the behavior of the quantum membrane for very large dimension d of the embedding space at all temperatures, and make use of the fact that the model is exactly solvable in this limit. We will calculate all of its relevant properties, in particular its order parameter and phase diagram [73]. This is not possible in the perturbative approach, since, by its nature, the perturbative expansion we used in the previous section breaks down as the membrane crumples.

5.3.1 Simplified model for large d

The surface describing the membrane is parametrized by a vector field $\mathbf{X}(\bar{\sigma},\tau)$ in the d-dimensional embedding space. In this parametrization, the Canham-Helfrich energy (2.17) reads

$$E_0 = \int \mathrm{d}^2\sigma\sqrt{g}\left[r_0 + \frac{1}{2\alpha_0}(\Delta\mathbf{X})^2\right], \tag{5.48}$$

where

$$g_{ij} = \partial_i\mathbf{X}\cdot\partial_j\mathbf{X} \tag{5.49}$$

is the metric induced by the embedding, and $\Delta = g^{-1/2}\partial_i g^{ij} g^{1/2}\partial_j$ is the Laplace-Beltrami operator. As in the previous section, we add to the Hamiltonian (5.48) a kinetic term to account for quantum fluctuations:

$$\mathcal{T} = \frac{1}{2\nu_0}\int \mathrm{d}^2\sigma\sqrt{g}\,\dot{\mathbf{X}}^2, \tag{5.50}$$

where $1/\nu_0$ is the bare mass density[1]. The Euclidean action describing the quantum membrane is thus

$$S_0 = \int \mathrm{d}\tau\,\mathrm{d}^2\sigma\sqrt{g}\left[\frac{1}{2\nu_0}\dot{\mathbf{X}}^2 + r_0 + \frac{1}{2\alpha_0}(\Delta\mathbf{X})^2\right]. \tag{5.51}$$

5.3.2 Auxiliary field variable approximation

For large d it is useful to consider g_{ij} as an independent field [35], and impose relation (5.49) with help of a Lagrange multiplier λ_{ij}. We consider the case where the classical action will have an extremum around an almost flat configuration. In the d-dimensional generalization of the Monge parametrization of an almost flat surface, the metric tensor becomes $g_{ij} = \delta_{ij} + \partial_i\mathbf{X}\cdot\partial_j\mathbf{X}$. The partition function for the membrane can then be written as

$$Z = \int \mathcal{D}g\,\mathcal{D}\lambda\,\mathcal{D}\mathbf{X}\,\mathrm{e}^{-S_0/\hbar}, \tag{5.52}$$

[1] The kinetic term $\dot{\mathbf{X}}^2$ has the unphysical feature that it is not invariant under time-dependent reparametrizations. It should therefore be replaced by the normal gradient energy $(\mathbf{N}\cdot\dot{\mathbf{X}})^2$, as discussed in the previous section [66, 67]. However, in the limit of large d, which we are going to investigate, the difference between the two kinetic terms will be negligible.

with the Euclidean action

$$S_0 = \int \mathrm{d}\tau\, \mathrm{d}^2\sigma \sqrt{g} \left\{ \frac{1}{2\nu_0}\dot{\mathbf{X}}^2 + r_0 + \frac{1}{2\alpha_0}\left[(\Delta\mathbf{X})^2 + \lambda^{ij}(\delta_{ij} + \partial_i\mathbf{X}\cdot\partial_j\mathbf{X} - g_{ij})\right] \right. \\ \left. - \frac{c_0}{4}\lambda_{ii}^2 \right\}. \tag{5.53}$$

We have added a term proportional to λ_{ii}^2, with the proportionality constant c_0 being the in-plane compressibility of the membrane. This is necessary to absorb infinities, and the renormalized compressibility c will be set equal to zero at the end to describe an incompressible planar fluid. Note that the functional integration over the Lagrange multipliers λ_{ij} in (5.52) has to be performed along the imaginary axis for convergence.

The functional integral over all possible surface configurations $\mathbf{X}(\bar{\sigma},\tau)$ in (5.52) is Gaussian, and can be immediately carried out, yielding an effective action $S_{\text{eff}} = \tilde{S}_0 + S_1$, with

$$\tilde{S}_0 = \int \mathrm{d}\tau\, \mathrm{d}^2\sigma \sqrt{g} \left[r_0 + \frac{\lambda^{ij}}{2\alpha_0}(\delta_{ij} - g_{ij}) - \frac{c_0}{4}\lambda_{ii}^2 \right], \tag{5.54}$$

and

$$S_1 = \frac{\hbar}{2} d\, \mathrm{Tr}\ln\left[-\partial_0^2 + \frac{\nu_0}{\alpha_0}(\Delta^2 - \partial_i\lambda^{ij}\partial_j) \right]. \tag{5.55}$$

For large d, the partition function (5.52) is dominated by the saddle point of the effective action with respect to the metric g_{ij} and the Lagrange multiplier λ^{ij}, and we are left with a mean-field theory in these fields. For very large membranes, translational invariance allows us to assume that this saddle point is symmetric and homogeneous [75, 76, 77, 78], such that $g_{ij} = \varrho_0\delta_{ij}$, $\lambda^{ij} = \lambda_0 g^{ij} = \lambda_0/\varrho_0\, \delta^{ij}$, with constant ϱ_0 and λ_0. There, the functional trace in (5.55) becomes an integral $\int \mathrm{d}\tau\, \mathrm{d}^2\sigma \int \mathrm{d}\omega\, \mathrm{d}^2 q/(2\pi)^3$ over the $(2+1)$-dimensional phase space, after replacing $\partial_0^2 \to -\omega^2$ and $g^{ij}\partial_i\partial_j \to -\mathbf{q}^2$.

5.3.3 Zero temperature

At zero temperature, the phase space integral in (5.55) yields

$$S_1 = \frac{\hbar}{2} d \int \mathrm{d}\tau \mathrm{d}^2\sigma \varrho_0 \sqrt{\frac{\nu_0}{\alpha_0}} \left\{ \frac{\Lambda^4}{8\pi} + \frac{\lambda_0}{8\pi}\Lambda^2 + \frac{\lambda_0^2}{64\pi}\left[1 - 2\ln\left(\frac{4\Lambda^2}{\lambda_0}\right)\right] \right\}, \tag{5.56}$$

where the ultraviolet divergences of the integral have again been regularized by a wavevector cutoff Λ. The first term in (5.56) is a constant and renormalizes the surface tension to

$$r = r_0 + \frac{\hbar d}{16\pi}\sqrt{\frac{\nu_0}{\alpha_0}}\Lambda^4. \tag{5.57}$$

The quadratically divergent term renormalizes the bending rigidity in the second term of (5.54). The logarithmically divergent term proportional to λ_0^2 modifies the in-plane compressibility to

$$c = c_0 + \frac{\hbar d}{32\pi}\sqrt{\frac{\nu_0}{\alpha_0}}\ln\left(4e^{-1/2}\frac{\Lambda^2}{\mu^2}\right), \tag{5.58}$$

where μ is a renormalization scale. We now set r and c equal to zero, to describe a critical incompressible membrane. The renormalized effective action is then

$$S_{\text{eff}} = \int \mathrm{d}\tau\, \mathrm{d}^2\sigma\, \varrho\lambda \left\{\frac{1}{\alpha}\left(\frac{1}{\varrho} - 1\right) + \frac{1}{\sqrt{\alpha\alpha_{\text{c}}}} + \frac{a}{\sqrt{\alpha}}\lambda\left[\ln\left(\frac{\lambda}{\bar{\lambda}}\right) - \frac{1}{2}\right]\right\}, \tag{5.59}$$

where we have defined the critical bending rigidity $1/\alpha_{\text{c}} \equiv \hbar^2 d^2 \nu_0 \Lambda^4/256\pi^2$ and the constants $a \equiv \hbar d \nu^{1/2}/(64\pi)$, $\bar{\lambda} \equiv \mu^2 e^{-1/2}$. From the second-derivative matrix of S_{eff} with respect to ϱ and λ we find that the stability of the saddle point is guaranteed only for $\lambda < \bar{\lambda}$. Note that the integration over λ_{ij} in (5.52) along the imaginary axis requires a maximum of (5.59) with respect to λ for stability.

The extremization of (5.59) with respect to ϱ leads to two different solutions for the saddle point, namely

$$\lambda = 0 \quad \text{or} \quad \lambda\left[\ln\left(\frac{\lambda}{\bar{\lambda}}\right) - \frac{1}{2}\right] = \frac{1}{a}\left(\frac{1}{\alpha^{1/2}} - \frac{1}{\alpha_{\text{c}}^{1/2}}\right). \tag{5.60}$$

These describe two different phases existing at zero temperature. For $\alpha < \alpha_{\text{c}}$, $\lambda = 0$ is the only possible solution for the saddle point. This solution corresponds to the flat phase, as we shall verify below. For $\alpha > \alpha_{\text{c}}$, on the other hand, there exists a solution of the second equation in (5.60) for nonzero λ. This solution corresponds to the crumpled phase. The behavior of the effective action (5.59) is shown in Fig. 5.2. As α approaches the critical point from below, i. e. the membrane softens, λ becomes nonzero (see Fig. 5.3), and the surface crumples.

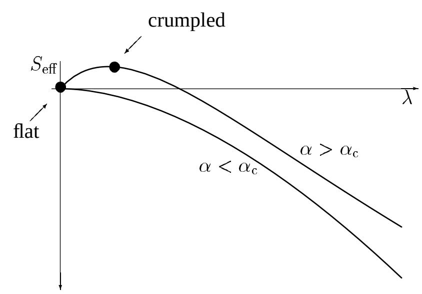

Figure 5.2: Effective action at $T = 0$ in units of a. The physical solution of the saddle point equations (5.60) lies at the maxima.

Note that, in the crumpled phase, the effective action reduces to

$$S_{\rm eff} = \int {\rm d}\tau {\rm d}^2\sigma \ \frac{\lambda}{\alpha}, \tag{5.61}$$

thus providing us with the physical interpretation of the Lagrange multiplier λ : it is proportional to the effective, spontaneusly generated surface tension. Also, by definition,

$$\rho = \frac{\langle A \rangle}{A_0}, \tag{5.62}$$

i. e., the parameter ρ corresponds to the ratio of the mean total area of the surface to its projected area.

To determine the saddle point solution for ϱ we extremize (5.59) with respect to λ. In the flat phase, where $\lambda = 0$, we obtain

$$\varrho_-^{-1} = 1 - \left(\frac{\alpha}{\alpha_{\rm c}} \right)^{1/2}, \tag{5.63}$$

showing that the total area of the membrane increases as α approaches $\alpha_{\rm c}$ from below, with a crumpling transition at $\alpha_{\rm c}$. In the crumpled phase, ϱ is given by

$$\varrho_+^{-1} = \left(\frac{\alpha}{\alpha_{\rm c}} \right)^{1/2} - 1 - a\sqrt{\alpha}\lambda, \tag{5.64}$$

with nonzero λ. As α approaches $\alpha_{\rm c}$ from above, λ tends to zero, and ϱ goes again to infinity. The positivity of ϱ and the stability of the saddle point imply that there is an upper bound for the bending rigidity, given by

$1/\alpha_{\rm max}^{1/2} = 1/\alpha_{\rm c}^{1/2} - a\bar{\lambda}$ below which an incompressible membrane becomes unstable. The behavior of λ in the two phases is shown in Fig. 5.3. The behavior of ϱ^{-1} is shown in Fig. 5.4.

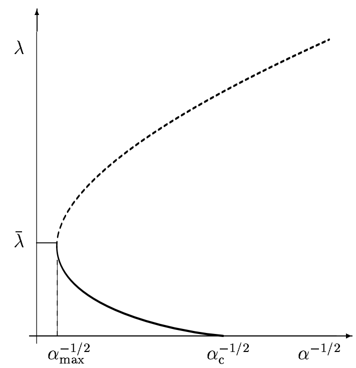

Figure 5.3: Physical branch of the solution of Eq. (5.60) for λ as a function of the stiffness α^{-1}. The dashed curve indicates the unstable extremum of the action.

5.3.4 Finite temperature

At finite temperature, the phase space integral in (5.55) involves a sum over the Matsubara frequencies [55] $\omega_n = 2\pi n k_{\rm B}T/\hbar$, $n = 0, \pm 1, \pm 2\cdots$, as in Sec. 5.2.2. For small λ_0, a series expansion (see Appendix D.2) leads to

$$
\begin{aligned}
S_1 = \int {\rm d}\tau\, {\rm d}^2\sigma\, \varrho_0 \Bigg\{ & \frac{\hbar d}{16\pi}\sqrt{\frac{\nu_0}{\alpha_0}}\Lambda^4 - \frac{\hbar d \pi}{24}\sqrt{\frac{\alpha_0}{\nu_0}}\left(\frac{k_{\rm B}T}{\hbar}\right)^2 + \frac{\hbar d}{16\pi}\lambda_0\sqrt{\frac{\nu_0}{\alpha_0}}\Lambda^2 \\
& + \lambda_0 \frac{d k_{\rm B}T}{8\pi} \ln\left(L^2 \frac{k_{\rm B}T}{\hbar}\sqrt{\frac{\alpha_0}{\nu_0}}\right) \\
& + a\frac{\lambda_0^2}{\sqrt{\alpha_0}}\left[3 - 2\gamma + 2\ln\left(\frac{\lambda_0}{8\pi}\frac{L^2}{\Lambda^2}\frac{k_{\rm B}T}{\hbar}\sqrt{\frac{\alpha_0}{\nu_0}}\right)\right] \\
& + \frac{\hbar d}{2}\sqrt{\pi}\sum_{m=3}^{\infty}\frac{(-1)^{m+1}\lambda_0^m}{m\, 2^{2m}\pi^m}\left(\frac{\hbar}{k_{\rm B}T}\right)^{m-2}\left(\frac{\nu_0}{\alpha_0}\right)^{\frac{m-1}{2}} \\
& \qquad \times \frac{\Gamma(\frac{m-1}{2})}{\Gamma(\frac{m}{2})}\zeta(m-1)\Bigg\}. \qquad (5.65)
\end{aligned}
$$

As in the zero-temperature discussion, we absorb the logarithmic divergence by renormalizing the in-plane compressibility via Eq. (5.58), setting c equal to zero for incompressible membranes. The surface tension receives now a temperature dependent renormalization

$$r = r_0 + \frac{\hbar d}{16\pi}\sqrt{\frac{\nu_0}{\alpha_0}}\Lambda^4 - \frac{\hbar d\pi}{24}\sqrt{\frac{\alpha_0}{\nu_0}}\left(\frac{k_{\rm B}T}{\hbar}\right)^2 , \tag{5.66}$$

and r_0 is chosen to make $r = 0$ at all temperatures. Extremization of the renormalized combined effective action (5.54) and (5.65) with respect to ϱ leads again to two possible solutions for the saddle point, namely $\lambda = 0$ or $\lambda = \lambda_{\rm T}$, with

$$\begin{aligned}
&\lambda_{\rm T}\left[\ln\left(\frac{\lambda_{\rm T}}{\bar\lambda}\right) - \frac{1}{2}\right] + \lambda_{\rm T}\left[1 - \gamma + \ln\left(\frac{L^2}{8\pi}\frac{k_{\rm B}T}{\hbar}\sqrt{\frac{\alpha}{\nu}}\right)\right] \\
&+32\pi^{3/2}\sum_{m=3}^{\infty}\frac{(-1)^{m+1}\lambda_{\rm T}^{m-1}}{m\,2^{2m}\pi^m}\left(\frac{\hbar}{k_{\rm B}T}\right)^{m-2}\left(\frac{\nu}{\alpha}\right)^{\frac{m-2}{2}}\frac{\Gamma(\frac{m-1}{2})}{\Gamma(\frac{m}{2})}\zeta(m-1) \\
&= \frac{1}{a}\left[1 - \alpha^{1/2}\left(1 - \sqrt{\frac{\alpha_{\rm c}}{\alpha_{\rm T}}}\right)\right]\left(\frac{1}{\alpha^{1/2}} - \frac{1}{\alpha_{\rm T}^{1/2}}\right)
\end{aligned} \tag{5.67}$$

where

$$\frac{1}{{\alpha_{\rm T}}^{1/2}} \approx \frac{1}{\alpha_{\rm c}^{1/2}}\left[\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{dk_{\rm B}T}{8\pi}\alpha_{\rm c}\ln\left(L^2\frac{k_{\rm B}T}{\hbar}\sqrt{\frac{\alpha_{\rm c}}{\nu}}\right)}\right] \tag{5.68}$$

is the critical bending rigidity at finite temperature. Alternatively, we find the critical temperature at a fixed bending rigidity $1/\alpha$:

$$T_{\rm c}\ln\left(L^2\frac{k_{\rm B}T_{\rm c}}{\hbar}\sqrt{\frac{\alpha}{\nu}}\right) = \frac{8\pi}{dk_{\rm B}}\left(\frac{1}{\alpha} - \frac{1}{\sqrt{\alpha\alpha_{\rm c}}}\right), \tag{5.69}$$

in qualitative agreement with the perturbative critical temperature. For $\alpha < \alpha_{\rm T}$, Eq. (5.67) has no solution for $\lambda_{\rm T}$. In this case, the membrane is in the flat phase, the only available solution for the saddle point being $\lambda = 0$. For $\alpha > \alpha_{\rm T}$, $\lambda_{\rm T}$ is nonzero, and the membrane is crumpled.

Let us now examine the saddle point solutions for ϱ. In the crumpled phase, where $\lambda = \lambda_{\rm T}$ is nonzero, we may expand the effective action into a high-temperature series. The details of the calculation of the series expansion can be found in Appendix D.2. Extremization with respect to λ_T leads to

$$\begin{aligned}\varrho_+^{-1} = \quad & \left[\left(\frac{\alpha}{\alpha_{\mathrm{T}}}\right)^{1/2} - 1\right]\left[1 - \alpha^{1/2}\left(1 - \sqrt{\frac{\alpha_{\mathrm{c}}}{\alpha_{\mathrm{T}}}}\right)\right] - a\sqrt{\alpha}\lambda_{\mathrm{T}} \\ & -\hbar d\alpha\frac{\sqrt{\pi}}{2}\sum_{m=3}^{\infty}\frac{(-1)^{m+1}\lambda_{\mathrm{T}}^{m-1}}{2^{2m}\pi^m}\left(1-\frac{2}{m}\right)\left(\frac{\hbar}{k_{\mathrm{B}}T}\right)^{m-2}\left(\frac{\nu}{\alpha}\right)^{\frac{m-1}{2}} \\ & \times\frac{\Gamma(\frac{m-1}{2})}{\Gamma(\frac{m}{2})}\zeta(m-1). \end{aligned} \tag{5.70}$$

The positivity of ϱ and the stability of the saddle point again define an upper bound for the inverse bending rigidity, given by

$$\frac{1}{\sqrt{\alpha_{\max}^T}} = \frac{1}{\sqrt{\alpha_{\max}}}\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{T}{T_{\mathrm{stab}}}}\right) \tag{5.71}$$

with

$$k_{\mathrm{B}}T_{\mathrm{stab}} = \frac{4\pi}{d\alpha_{\max}\ln(16\pi/\hbar d\sqrt{\nu\alpha_{\mathrm{c}}}\bar{\lambda})}. \tag{5.72}$$

For temperatures lower than T_{stab} the effective action becomes unstable if the rigidity is lower than $1/\alpha_{\max}^{\mathrm{T}}$. Above T_{stab}, the membrane is stable at all rigidities. In the flat phase, the situation is more delicate. For $\lambda = 0$, ϱ can be calculated exactly, and we obtain

$$\varrho_-^{-1} = 1 - \alpha\frac{dk_{\mathrm{B}}T}{8\pi}\ln\left[\frac{\sinh\left(\frac{16\pi}{dk_{\mathrm{B}}T}\frac{1}{\sqrt{\alpha\alpha_{\mathrm{c}}}}\right)}{\frac{\hbar}{2k_{\mathrm{B}}T}\sqrt{\frac{\nu}{\alpha}}L^{-2}}\right], \tag{5.73}$$

with an infrared regulator L equal to the inverse lateral size of the membrane. For low temperatures, (5.73) may be approximated by

$$\varrho_-^{-1} \approx \left[1 - \left(\frac{\alpha}{\alpha_{\mathrm{T}}}\right)^{1/2}\right]\left[1 - \alpha^{1/2}\left(1 - \sqrt{\frac{\alpha_{\mathrm{c}}}{\alpha_{\mathrm{T}}}}\right)\right]. \tag{5.74}$$

At high temperatures, however, the positivity of ϱ is not guaranteed. For fixed, but high temperatures, and for fixed membrane lateral size L, there is a characteristic value of the inverse bending rigidity defined by

$$\alpha^* = \frac{8\pi}{dk_{\mathrm{B}}T\ln\left(\frac{16\pi L^2}{\hbar d\sqrt{\nu\alpha_{\mathrm{c}}}}\right)}, \tag{5.75}$$

above which ϱ changes sign, and (5.73) is no longer applicable. Interestingly, for all L and at all temperatures T, the critical bending rigidity $1/\alpha_{\rm T}$ is larger than $1/\alpha^*$, so that the crumpling transition still occurs. The behavior of ϱ is depicted in Fig. 5.4.

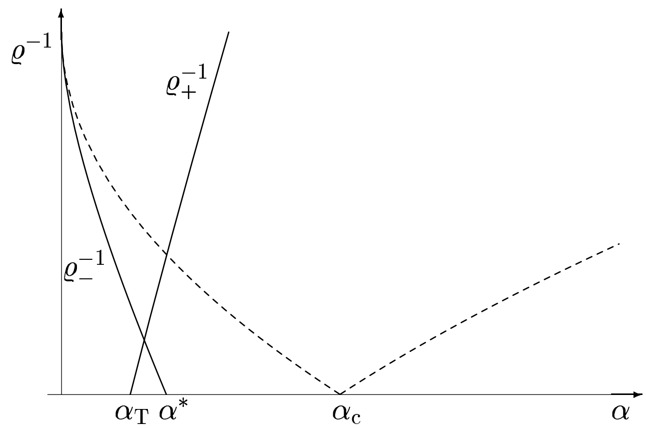

Figure 5.4: Behavior of ϱ^{-1} for fixed L^2. In the flat phase, ϱ is given by (5.73), and in the crumpled phase by (5.70). The transition happens at $\alpha_{\rm T}$, where the flat phase becomes crumpled. The dashed lines show the behavior of ϱ^{-1} at zero temperature.

Note that the existence of the characteristic inverse rigidity α^* reflects the existence of a persistence length. At fixed temperature, for $\alpha_{\rm T} < \alpha < \alpha^*$, the membrane is flat at scales smaller than $L_{\rm p} = \Lambda^{-1}\exp(4\pi/dk_{\rm B}T\alpha)$, and crumpled at larger scales. This agrees with the de Gennes-Taupin persistence length $\xi_{\rm p}$. As the projected area $A_0 = L^2$ of the membrane approaches infinity, the root $\alpha_{\rm T}$ of the branch ϱ_+^{-1} (see Fig. 5.4) goes to zero, and the branch ϱ_-^{-1} becomes unphysical. The phase diagram of the quantum membrane is plotted in Fig. 5.5. As the lateral size L of the membrane goes to infinity, the inverse critical bending rigidity $\alpha_{\rm T}$ goes to zero, and the crumpling transition is washed out. In the limit of infinite area, the ratio $\alpha^*/\alpha_{\rm T} = 1$. The membrane is crumpled at large scales, and flat at scales smaller than the persistence length. Its behavior can thus be described by the classical Canham-Helfrich model alone.

Let us finally characterize the two phases in terms of the correlation functions between the normal vectors to the surface of the membrane. For

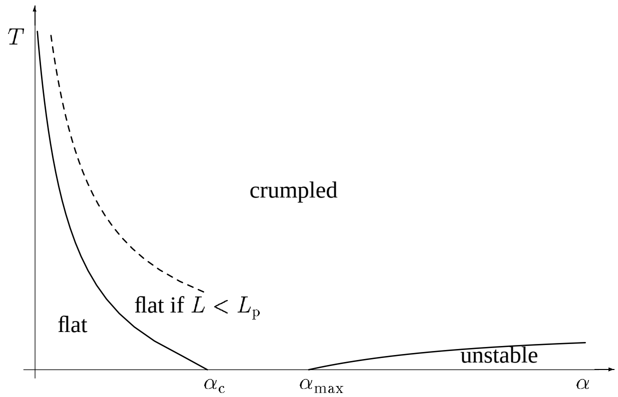

Figure 5.5: Phase diagram of the quantum membrane. At $T = 0$, there is a crumpling transition as the rigidity $1/\alpha$ falls below $1/\alpha_c$. For membranes of fixed lateral size L, the crumpling transition still takes place at higher temperatures. The critical inverse rigidity tends asymptotically to zero as the temperature goes to infinity. Below the dotted line the membrane is still flat if its lateral size is smaller than the persistence length. The unstable region disappears for nonzero in-plane compressibility.

the solution $\lambda = 0$, this correlation function coincides with the one we found perturbatively for the zero temperature case, namely

$$\langle \partial_a \mathbf{X}(\bar{\sigma}, \tau) \cdot \partial_b \mathbf{X}(\bar{\sigma}', \tau) \rangle \sim \frac{\delta_{ij}}{|\bar{\sigma} - \bar{\sigma}'|^3}. \tag{5.76}$$

This solution corresponds to the flat, low temperature phase, where the normal vectors are strongly correlated. Such behavior can incidentally also be obtained in the large-d limit at high temperatures by adding curvature terms of higher orders to the Canham-Helfrich Hamiltonian (2.17) to stabilize a negative bending rigidity $1/\alpha_0$ [79, 80, 81]. For nonzero $\lambda = \lambda_T$ the correlation function behaves as

$$\langle \partial_a \mathbf{X}(\bar{\sigma}, \tau) \cdot \partial_b \mathbf{X}(\bar{\sigma}', \tau) \rangle \sim \delta_{ab} \exp(-\sqrt{\lambda_T} |\bar{\sigma} - \bar{\sigma}'|). \tag{5.77}$$

In this case, the normals to the membranes are uncorrelated beyond a length scale $\lambda_T^{-1/2}$. The exponential decay of the correlation function shows that this solution corresponds to the crumpled phase. The length scale $\lambda_T^{-1/2}$ may also be identified as the persistence length ξ_p [2, 22, 78].