Chapter 4

Derivative Expansion for Fluid Membranes

After the discussion of the derivative expansion method in the previous chapter, we are now ready to apply it to the renormalization of fluid membranes.

Our starting point will be the modified Canham-Helfrich energy in the Monge parametrization (2.22), that is

$$E_0[\phi] = \int \mathrm{d}^2\sigma \sqrt{1+(\partial\phi_i)^2}\Big\{ r_0 + \tfrac{1}{2}\kappa_0 \Big[\frac{(\partial^2\phi)^2}{1+(\partial_i\phi)^2} - 2\frac{\partial_i\phi\partial_j\phi\partial_i\partial_j\phi\partial^2\phi}{[1+(\partial_k\phi)^2]^2} + \frac{(\partial_i\phi\partial_j\phi\partial_i\partial_j\phi)^2}{[1+(\partial\phi_k)^2]^3} \Big] \Big\}. \tag{4.1}$$

We will show that the ultraviolet divergent parts of the one-loop corrections induced by thermal fluctuations are of precisely the same form as in (4.1), and in particular, that the three terms in the curvature energy renormalize in the same way, resulting in an overall renormalization of κ_0 alone [58].

To apply the derivative expansion we write the partition function as a functional integral over the displacement field

$$Z = \int \mathrm{D}\phi \exp\left(-\beta E_0\right), \tag{4.2}$$

with each field configuration weighted with a Boltzmann factor.

In the one-loop approximation, the exponent in (4.2) may be expanded up to second order around a background configuration $\Phi(x)$ extremizing E_0.

A nontrivial background requires the presence of an extra source term. For brevity, this term will not be written down explicitly when setting $\delta E_0/\delta\Phi = 0$. The resulting integral is Gaussian and yields an effective energy

$$E_{\text{eff}}[\Phi] = E_0[\Phi] + E_1[\Phi] = E_0[\Phi] + \frac{1}{2\beta}\mathrm{Tr}\ln\left[\left.\frac{\delta^2(\beta E_0)}{\delta\phi(x)\delta\phi(y)}\right|_{\Phi}\right], \tag{4.3}$$

where the expression in square brackets corresponds to the matrix of second functional derivatives of E_0 and the trace Tr stands for the trace of this matrix, i.e., the integral $\int \mathrm{d}^2x$ over space, as well as the integral $\int \mathrm{d}^2k/(2\pi)^2$ over momentum [53].

The one-loop correction $E_1[\Phi]$ to the energy will now be calculated in a derivative expansion for an arbitrarily tilted, but nearly flat background configuration. A background configuration parallel to the reference surface is too trivial to generate a derivative expansion. The expansion has the general form

$$\begin{aligned} E_1[\Phi] = \int \mathrm{d}^2x\Big[\, & \mathcal{V}(V_k) + \mathcal{Z}^1(V_k)(\partial_i V_i)^2 + \mathcal{Z}^2_{ij}(V_k)\partial_i V_j \partial_m V_m \\ & + \mathcal{Z}^3_{ijmn}(V_k)\partial_i V_j \partial_m V_n + \cdots\Big], \end{aligned} \tag{4.4}$$

where we introduced the abbreviation $V_i = \partial_i\Phi$, while $\mathcal{V}$, $\mathcal{Z}^1$, $\mathcal{Z}^2_{ij}$, and $\mathcal{Z}^3_{ijmn}$ are functions of V_i to be determined. Following Ref. [53], we set $V_i(x) = \bar{V}_i + v_i(x)$, where $\bar{V}_i$ denotes the constant part of $V_i(x)$, and expand Eq. (4.4) in powers of $v_i(x)$ and its derivatives, to obtain

$$\begin{aligned} E_1[\bar{V}_k + v_k] = \int \mathrm{d}^2x\Big[\, & \mathcal{V}(\bar{V}_k) + \frac{\partial\mathcal{V}(\bar{V}_k)}{\partial\bar{V}_i}v_i + \frac{1}{2}\frac{\partial^2\mathcal{V}(\bar{V}_k)}{\partial\bar{V}_i\partial\bar{V}_j}v_i v_j + \mathcal{Z}^1(\bar{V}_k)(\partial_i v_i)^2 \\ & + \mathcal{Z}^2_{ij}(\bar{V}_k)\partial_i v_j \partial_m v_m + \mathcal{Z}^3_{ijmn}(\bar{V}_k)\partial_i v_j \partial_m v_n + \cdots\Big], \end{aligned} \tag{4.5}$$

with space-independent $\mathcal{V}(\bar{V}_k)$ and $\mathcal{Z}(\bar{V}_k)$'s. These functions will now be extracted from the expansion of the Tr ln in (4.3) up to quadratic terms in v_i and $\partial_i v_j$, in the same way as discussed in the previous chapter.

The functional derivatives in (4.3) are again calculated using the Euler-Lagrange formula

$$\frac{\delta F[\phi]}{\delta\phi(x)} = \frac{\partial f}{\partial\phi} - \partial_i\frac{\partial f}{\partial(\partial_i\phi)} + \partial_i\partial_j\frac{\partial f}{\partial(\partial_i\partial_j\phi)} + \cdots, \tag{4.6}$$

with $F[\phi] = \int \mathrm{d}^2x f(\phi, \partial_i\phi, \partial_i\partial_j\phi, \ldots)$. To keep track of the many terms appearing in the resulting expression we have used the algebraic computer program `FORM` [59]. A detailed description of the `FORM` programs can be found in Appendix A.

We consider first the renormalization of the surface tension. Since the energy density $\sqrt{1+V^2}$ does not contain derivatives of V_i, we may set $v_i(x)$ to zero and consider $E_1[\bar{V}_k]$ only,

$$\beta E_1[\bar{V}_k] = -\tfrac{1}{4}\mathrm{Tr}\ln(1+\bar{V}^2) + \tfrac{1}{2}\mathrm{Tr}\ln[G^{-1}(p)], \tag{4.7}$$

where $p_i = -i\partial_i$. Here, $G^{-1}(p)$ denotes the inverse propagator:

$$G^{-1}(p) = (r_0 + \kappa_0 p^2)p^2 - (r_0 + 2\kappa_0 p^2)(\bar{U}\cdot p)^2 + \kappa_0(\bar{U}\cdot p)^4 \tag{4.8}$$

where $\bar{U}_i$ is the constant vector

$$\bar{U}_i = \frac{\bar{V}_i}{\sqrt{1+\bar{V}^2}}. \tag{4.9}$$

In dimensional regularization, the first term on the right-hand side of (4.7) is zero. To evaluate the remaining Tr ln, we apply a standard trick and first differentiate (4.7) with respect to r_0 to obtain

$$\frac{\partial E_1[\bar{V}_k]}{\partial r_0} = \frac{k_\mathrm{B}T}{2}\mathrm{Tr}\left[\frac{p^2 - (\bar{U}\cdot p)^2}{(r_0+\kappa_0 p^2)p^2 - (r_0+2\kappa_0 p^2)(\bar{U}\cdot p)^2 + \kappa_0(\bar{U}\cdot p)^4}\right]. \tag{4.10}$$

Since the integrand contains no space-dependence, the spatial part of the trace in (4.10) yields a volume factor $L^D = \int \mathrm{d}^D x$, and we are left with the momentum integral

$$\frac{\partial E_1[\bar{V}_k]}{\partial r_0} = L^D\frac{k_\mathrm{B}T}{2}\int\frac{\mathrm{d}^D k}{(2\pi)^D}\left[\frac{k^2 - (\bar{U}\cdot k)^2}{(r_0+\kappa_0 k^2)k^2 - (r_0+2\kappa_0 k^2)(\bar{U}\cdot k)^2 + \kappa_0(\bar{U}\cdot k)^4}\right]. \tag{4.11}$$

Being interested only in the ultraviolet divergent terms, we obtain, in dimensional regularization

$$\frac{\partial E_1[\bar{V}_k]}{\partial r_0} = \frac{k_\mathrm{B}T}{4\pi\kappa_0}\frac{1}{\epsilon}\int \mathrm{d}^2x\sqrt{1+\bar{V}^2}. \tag{4.12}$$

After integrating again with respect to r_0 and comparing the result with (4.4) we find (up to an irrelevant additive constant)

$$\mathcal{V}(V_k) = k_{\rm B}T \frac{r_0}{4\pi\kappa_0}\frac{1}{\epsilon}\sqrt{1+V^2} = k_{\rm B}T \frac{r_0}{4\pi\kappa_0}\frac{1}{\epsilon}\sqrt{1+(\partial\Phi)^2}, \qquad (4.13)$$

where we replaced $\bar{V}_k$ with the full background field $V_k(x)$, to obtain the first term in (4.4). Note that this one-loop correction is precisely of the same form as the surface term contained in the original energy expression (4.1). This term can consequently be combined with the original one by introducing the renormalized tension

$$r_{\rm eff} = r_0 + \frac{k_{\rm B}T}{4\pi}\frac{r_0}{\kappa_0}\frac{1}{\epsilon}. \qquad (4.14)$$

This result, corresponding to $\alpha' = 1$ in (2.39), is in agreement with Refs. [41, 47] and with our discussion in Chapter 2. This proves the covariance, at least to one-loop order, of *all* terms in the expansion of the surface energy since the full expression has been maintained.

We continue to investigate the renormalization of the bending rigidity. Since the three terms involved contain derivatives of the background field V_i, we now have to employ the derivative expansion. As a first step, we expand the logarithm in (4.3) as:

$$\begin{aligned}\beta E_1[\bar{V}_k + v_k(x)] - \beta E_1[\bar{V}_k] &= \tfrac{1}{2}{\rm Tr}\ln[1+G(p)\Lambda(x,p)] \\ &= \tfrac{1}{2}{\rm Tr}[G(p)\Lambda(p,x)] \\ &- \tfrac{1}{4}{\rm Tr}[G(p)\Lambda(x,p)G(p)\Lambda(x,p)] + \cdots, \qquad (4.15)\end{aligned}$$

where $G(p)$ is the propagator defined in (4.8) and $\Lambda(x,p)$ contains the x-dependent terms obtained from functionally differentiating E_0 twice, setting $\partial_i\Phi(x) = V_i(x) = \bar{V}_i + v_i(x)$ and expanding up to second order in v_i and $\partial_i v_j$.

The first term in (4.15) can be calculated in a similar fashion as $\mathcal{V}(V_k)$. In the second term, all momentum operators have to be moved to the left [53], by repeatedly applying the identity

$$f(x)p_i g(x) = (p_i + i\partial_i)f(x)g(x), \qquad (4.16)$$

where $f(x)$ and $g(x)$ are arbitrary functions and the derivative ∂_i acts *only* on the next object to the right, as discussed in Chapter 3.

The typical momentum integrals showing up at one-loop order are of the form

$$I_{m,n} = \int \frac{{\rm d}^D k}{(2\pi)^D} k^m G^n(k) \sim \int {\rm d}k \begin{cases} k^{m+D-1-2n} & \text{infrared} \\ k^{m+D-1-4n} & \text{ultraviolet} \end{cases}, \qquad (4.17)$$

with $m, n > 0$. They diverge in the infrared when $m+D-1-2n \leq -1$, and in the ultraviolet when $m+D-1-4n \geq -1$. For $D=2$ these conditions become $m-2n \leq -2$, $m-4n \geq -2$, respectively, and the two types of divergences are seen to be separated by a wedge of finite integrals in the (m,n)-plane starting at $(-2,0)$, as depicted in Fig. 4.1. The explicit calculation of these

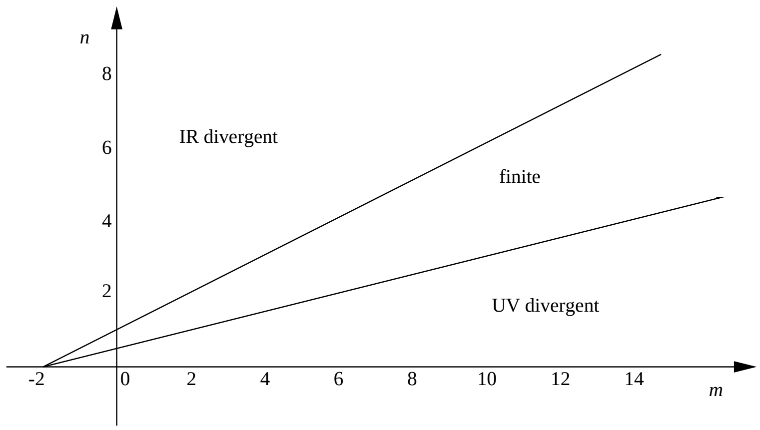

Figure 4.1: Behavior of the one-loop integrals $I_{m,n}$ in Eq. (4.17).

momentum integrals can be found in Appendix B.

After a lengthy calculation, involving of the order of 10^4 terms, done with help of a program written in `FORM` [59], we obtained the divergent terms to second order in derivatives of the field v_i (for details see Appendix A):

$$\begin{aligned}
&\beta(E_1[\bar{V}_k + v_k] - E_1[\bar{V}_k]) = \\
&\int \mathrm{d}^2x \Bigg\{ \frac{r_0}{4\pi\kappa_0}\frac{1}{\epsilon}\left[\frac{\bar{V}_i}{(1+\bar{V}^2)^{1/2}} v_i + \frac{1}{2}\left(\frac{\delta_{ij}}{(1+\bar{V}^2)^{1/2}} + \frac{\bar{V}_i\bar{V}_j}{(1+\bar{V}^2)^{3/2}}\right) v_i v_j\right] \\
&-\frac{3}{8\pi}\frac{1}{\epsilon}\left[\frac{1}{(1+\bar{V}^2)^{1/2}}(\partial_i v_i)^2 - 2\frac{\bar{V}_i\bar{V}_j}{(1+\bar{V}^2)^{3/2}}\partial_i v_j \partial_m v_m + \frac{\bar{V}_i\bar{V}_j\bar{V}_m\bar{V}_n}{(1+\bar{V}^2)^{5/2}}\partial_i v_j \partial_m v_n\right] \\
&-\frac{1}{4\pi}\frac{1}{\epsilon_{\mathrm{ir}}}\left[\frac{\bar{V}_i\bar{V}_j}{(1+\bar{V}^2)^{3/2}}\partial_i v_j \partial_m v_m - \frac{\bar{V}_i\bar{V}_j\bar{V}_n\bar{V}_m}{(1+\bar{V}^2)^{5/2}}\partial_i v_j \partial_m v_n\right]\Bigg\}. \qquad (4.18)
\end{aligned}$$

In deriving this expression we also encountered infrared divergences. These are regularized in the same scheme as used to regularize the ultraviolet divergences. To distinguish the two we gave epsilon an index ir in case of an infrared divergence. We leave the discussion of the infrared divergences to the next paragraph, and first analyze the ultraviolet ones. Comparing (4.18) to (4.5) with $\mathcal{V}(\bar{V}_k)$ given by (4.13), we see that the terms proportional to r_0 precisely correspond to the first two terms at the right-hand side of (4.5), as it should be. Moreover, we conclude that the $\mathcal{Z}$-functions in (4.5) are given by

$$\begin{aligned}
\mathcal{Z}^1(\bar{V}_k) &= -\frac{3k_\mathrm{B}T}{8\pi}\frac{1}{\epsilon}\frac{1}{(1+\bar{V}^2)^{1/2}}, &(4.19)\\
\mathcal{Z}^2_{ij}(\bar{V}_k) &= \frac{3k_\mathrm{B}T}{4\pi}\frac{1}{\epsilon}\frac{\bar{V}_i\bar{V}_j}{(1+\bar{V}^2)^{3/2}}, &(4.20)\\
\mathcal{Z}^3_{ijmn}(\bar{V}_k) &= -\frac{3k_\mathrm{B}T}{8\pi}\frac{1}{\epsilon}\frac{\bar{V}_i\bar{V}_j\bar{V}_m\bar{V}_n}{(1+\bar{V}^2)^{5/2}}. &(4.21)
\end{aligned}$$

By replacing the constant $\bar{V}_k$ with the full background field $V_i(x) = \partial_i\Phi(x)$, we obtain for the UV-divergent parts of the expansion (4.4) the explicit form

$$E_1[\Phi] = \frac{k_\mathrm{B}T}{4\pi\epsilon\kappa_0}\int \mathrm{d}^2x\sqrt{1+(\partial\Phi)^2}\left\{r_0 - \frac{3\kappa_0}{2}\left[\frac{(\partial^2\Phi)^2}{1+(\partial\Phi)^2} - 2\frac{\partial_i\Phi\partial_j\Phi\partial_i\partial_j\Phi\partial^2\Phi}{[1+(\partial\Phi)^2]^2} + \frac{(\partial_i\Phi\partial_j\Phi\partial_i\partial_j\Phi)^2}{[1+(\partial\Phi)^2]^3}\right]\right\}. \quad (4.22)$$

We see that the thermally generated terms at the one-loop level are precisely of the same form as those present in the original energy expression (4.1). In addition, the relative weights of the curvature terms produced by the fluctuations are the same as those found there. They can therefore be combined with the original terms by introducing the renormalized rigidity

$$\kappa_\mathrm{eff} = \kappa_0 - \frac{3k_\mathrm{B}T}{4\pi}\frac{1}{\epsilon}, \quad (4.23)$$

whose value is in agreement with [40, 41, 42, 43, 44].

As seen in (4.18), the one-loop corrections seem to have introduced infrared divergences in the theory. A closer inspection reveals that the infrared-divergent contributions all stem from the surface energy term in (4.1), so that it suffices to analyze the one-loop corrections to the truncated energy

$$E_0' = r_0\int \mathrm{d}^2x\sqrt{1+(\partial\phi)^2}. \quad (4.24)$$

Infrared divergences in this model have previously been studied in [60], where they were shown to disappear for an infinitely small dimension D of the membrane to all orders in D. In our calculation the problem arises for $D = 2 - \epsilon$. When calculating the effective action, we expand (4.24) around the background field Φ extremizing E_0', i.e.,

$$\left.\frac{\delta E_0'}{\delta \phi}\right|_{\Phi} = 0, \tag{4.25}$$

which reads explicitly

$$\frac{\partial^2 \Phi}{[1 + (\partial \Phi)^2]^{1/2}} - \frac{\partial_n \Phi \partial_m \Phi \partial_m \partial_n \Phi}{[1 + (\partial \Phi)^2]^{3/2}} = 0. \tag{4.26}$$

The presence of the implicitly assumed sources turns this equation into a nontrivial one. Rewriting $\partial_i \Phi(x) = \bar{V}_i + v_i(x)$, expanding to linear order in v_i, and substituting the resulting expression in (4.18), we see the infrared divergences to vanish for a two-dimensional membrane.

In conclusion, we have demonstrated that all logarithmically divergent one-loop corrections induced by thermal fluctuations are precisely of the same form as in the original energy (4.1), so that they can be removed by a renormalization of the surface tension and bending rigidity.