Chapter 3

Derivative Expansion

The derivative expansion, first developed by C. Fraser [53], is a powerful method for calculating the one-loop effective Lagrangian of any given field theory. The resulting expansion of the effective Lagrangian in powers of the derivatives of the field is primarily useful when one is interested in low-energy or long-distance effects.

For ease of illustration, let us apply this method to a simple quantum mechanical problem.

3.1 Quantum corrections to effective action

Consider a particle of mass m moving in a one-dimensional potential $V(x)$. Its classical Lagrangian reads

$$\mathcal{L}(x, \dot{x}) = \tfrac{1}{2} m \dot{x}^2 - V(x), \tag{3.1}$$

where the dot indicates a time derivative. According to the path integral formulation of quantum mechanics [54], the probability amplitude of the particle initially at position x_a at time t_a to be found at position x_b at a later time t_b is given by the path integral

$$\langle x_a, t_a | x_b, t_b \rangle = \int \mathcal{D}x \exp(\frac{i}{\hbar} S[x]), \tag{3.2}$$

where $S[x]$ is the classical action

$$S[x] = \int_{t_a}^{t_b} \mathrm{d}t \; \mathcal{L}(x, \dot{x}), \tag{3.3}$$

and the path integral (3.2) runs over all paths with fixed end points at $x(t_a) = x_a$ and $x(t_b) = x_b$.

In the path integral formulation, the rules of quantum mechanics appear as a natural generalization of the rules of classical statistical mechanics [55]. In statistical mechanics, each volume in phase space is occupied with the Boltzmann probability. In the path integral formulation of quantum mechanics, each volume element in the *path phase space* is associated with a pure phase factor $\exp(iS[x]/\hbar)$. One may thus consider the quantum-mechanical partition function

$$Z_{\rm QM}(t_b, t_a)[X] \equiv \int_{x_a=x_b=0} \mathcal{D}x \exp(\frac{i}{\hbar}S[X+x]), \tag{3.4}$$

for the fluctuations x around some background orbit $X(t)$. The quantum-mechanical partition function can be used to define the effective action in the same way as the partition function in statistical mechanics is used to define the free energy of the model being studied. Accordingly, the effective action is given by

$$S_{\rm eff}[X] \equiv -i\hbar \ln Z_{\rm QM}[X]. \tag{3.5}$$

By calculating $Z_{\rm QM}$ we are able to introduce quantum corrections to the classical Lagrangian (3.1). However, the quantum-mechanical partition function cannot be calculated exactly for most potentials $V(x)$, and one must resort to some approximative method. In the so-called semi-classical approximation, $S_{\rm eff}$ is expanded around the classical action, and the quantum corrections are expressed as a series expansion in powers of $\hbar$, also referred to as loop expansion.

Let $x_{\rm cl}(t)$ be the classical path, solving the classical equation of motion

$$m\ddot{x}_{\rm cl} + V'(x_{\rm cl}) = 0. \tag{3.6}$$

Accordingly, the first functional derivative of the action (3.3) vanishes at $x_{\rm cl}$:

$$\left.\frac{\delta S[x]}{\delta x(t)}\right|_{x_{\rm cl}} = \left.\frac{\partial \mathcal{L}}{\partial x}\right|_{x_{\rm cl}} - \left.\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right|_{x_{\rm cl}} + \left.\frac{d^2}{dt^2}\frac{\partial \mathcal{L}}{\partial \ddot{x}}\right|_{x_{\rm cl}} + \cdots = 0. \tag{3.7}$$

Hence (3.3) has a functional Taylor series around the classical path starting as

$$S[x] = S[x_{\rm cl}] + \frac{1}{2}\int_{t,t'} \left.\frac{\delta^2 S[x]}{\delta x(t)\delta x(t')}\right|_{x_{\rm cl}} \bar{x}(t)\bar{x}(t') + \cdots, \tag{3.8}$$

where $\bar{x}(t) = x(t) - x_{\rm cl}(t)$ and the quantum-mechanical partition function reads, in the semi-classical approximation,

$$Z_{\rm QM}(t_b, t_a)[x_{\rm cl}] = e^{\frac{i}{\hbar}S[x_{\rm cl}]} \int_{\bar{x}_a=\bar{x}_b=0} \mathcal{D}\bar{x} \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} {\rm d}t\bar{x}(t)[m\omega^2 - V''(x_{\rm cl})]\bar{x}(t)\right\}, \tag{3.9}$$

where we have defined the time derivative operator $\hat{\omega} \equiv -i{\rm d}/{\rm d}t$. The path integral in (3.9) is Gaussian, and can be calculated analytically (for details see Ref. [55]), yielding an effective action

$$S_{\rm eff} \equiv S[x_{\rm cl}] + S_1[x_{\rm cl}], \tag{3.10}$$

with the one-loop quantum correction

$$S_1[x_{\rm cl}] = -\frac{i\hbar}{2}{\rm Tr}\ln[m\hat{\omega}^2 - V''(x_{\rm cl})]. \tag{3.11}$$

The functional trace Tr in (3.11) contains a time integral $\int_{t_a}^{t_b} dt$ as well as a discrete sum over all eigenvalues ω_n of the operator $\hat{\omega}$ [53]. Since the summation over discrete eigenvalues introduces unnecessary complications in the calculation, we shall replace the sum $\sum_n f(\omega_n)$ with an integral $\int {\rm d}\omega/2\pi\ f(\omega)$. This approximation should not affect the results as long as $|t_b - t_a| \gg \sqrt{m/V''(x_{\rm cl}(t))}$ for all t.

If the classical path $x_{\rm cl}(t)$ is taken to be a constant, the determination of the functional trace in (3.11) is straightforward, and the quantum correction to the classical Lagrangian has no explicit time-dependence. We are interested in calculating corrections that are explicitly time-dependent, that is, we look for corrections proportional to $\dot{x}_{\rm cl}(t)$, $\ddot{x}_{\rm cl}(t)$, etc. The *time derivative expansion* of the correction to the effective action has the general form

$$S_1[x_{\rm cl}] = \int_{t_a}^{t_b} {\rm d}t \left[-\mathcal{V}(x_{\rm cl}(t)) + \tfrac{1}{2}\mathcal{Z}(x_{\rm cl}(t))\dot{x}_{\rm cl}^2 + \mathcal{Z}_2(x_{\rm cl}(t))\ddot{x}_{\rm cl}^2 + \cdots\right]. \tag{3.12}$$

The idea behind Fraser's derivative expansion [53] is to set $x_{\rm cl}(t)$ in (3.11) and (3.12) equal to $x_0 + \tilde{x}(t)$, where x_0 is a constant, and expand both (3.11) and (3.12) in powers of $\tilde{x}$ and its derivatives. By comparing the result of both expansions, the coefficients $\mathcal{V}(x)$, $\mathcal{Z}(x)$, etc., may be extracted.

To illustrate the method, let us calculate the first two coefficients in (3.12). Expanding it about x_0 up to terms of order $\tilde{x}^2$, one obtains

$$S_1[x_{\rm cl}] = \int_{t_a}^{t_b} {\rm d}t \left[-\mathcal{V}(x_0) - \mathcal{V}'(x_0)\tilde{x} - \tfrac{1}{2}\mathcal{V}''(x_0)\tilde{x}^2 + \tfrac{1}{2}\mathcal{Z}(x_0)\dot{\tilde{x}}^2 + \cdots\right]. \tag{3.13}$$

The corresponding expansion of (3.11) yields

$$S_1 = -\frac{i\hbar}{2}\mathrm{Tr}\ln[m\hat{\omega}^2 - V''(x_0)] - \frac{i\hbar}{2}\mathrm{Tr}\ln\left[1 - \frac{1}{m\hat{\omega}^2 - V''(x_0)}\Lambda(\tilde{x})\right], \quad (3.14)$$

where

$$\Lambda(\tilde{x}) = V'''(x_0)\tilde{x} + \tfrac{1}{2}V''''(x_0)\tilde{x}^2. \quad (3.15)$$

The first term in (3.14) is just the time-independent effective potential, and can be immediately calculated. The trace can be converted into a simple integral over time and over the eigenvalues ω_n of $\hat{\omega}$, since $V''(x_0)$ is time-independent. Using the integral formula

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{\omega^p}{[m\omega^2 - V''(x_0)]^q} = \frac{i^{-(p+1)}(-1)^{-q}[1+(-1)^p]\Gamma(q-\frac{p+1}{2})\Gamma(\frac{p+1}{2})}{4\pi[V''(x_0)]^{q-\frac{p+1}{2}}m^{\frac{p+1}{2}}\Gamma[q]} \quad (3.16)$$

for $p \to 0, q \to 0$, we obtain

$$\mathcal{V}(x_0) = \frac{\hbar}{2}\sqrt{\frac{V''(x_0)}{m}}. \quad (3.17)$$

The logarithm in the second term of (3.14) is also expanded up to terms of order $\tilde{x}^2$:

$$\begin{aligned} S_1[x_{\rm cl}] - S_1[x_0] &= \frac{i\hbar}{2}\mathrm{Tr}\left[\frac{1}{m\hat{\omega}^2 - V''(x_0)}\Lambda(\tilde{x})\right] \\ &+ \frac{i\hbar}{2}\mathrm{Tr}\left[\frac{1}{m\hat{\omega}^2 - V''(x_0)}\Lambda(\tilde{x})\frac{1}{m\hat{\omega}^2 - V''(x_0)}\Lambda(\tilde{x})\right]. \end{aligned} \quad (3.18)$$

The first term in the above expression can again be calculated using formula (3.16),

$$\frac{i\hbar}{2}\mathrm{Tr}\left[\frac{1}{m\hat{\omega}^2 - V''(x_0)}\Lambda(\tilde{x})\right] = \int_{t_a}^{t_b}\mathrm{d}t\,\frac{\hbar}{4}\left[-\frac{V'''(x_0)}{\sqrt{mV''(x_0)}}\tilde{x} - \frac{1}{2}\frac{V''''(x_0)}{\sqrt{mV''(x_0)}}\tilde{x}^2\right]. \quad (3.19)$$

By comparing the linear term in $\tilde{x}$ with that in (3.13) we identify

$$\mathcal{V}'(x_0) = \frac{\hbar}{4}\frac{V'''(x_0)}{\sqrt{mV''(x_0)}}. \quad (3.20)$$

To calculate the quadratic terms in (3.18), we must first move all operators $\hat{\omega}$ to the left, and all functions of t to the right. Then we can perform the traces independently. To do this, we use the commutator

$$[f(t), \hat{\omega}] = i\dot{f}(t), \tag{3.21}$$

where $f(t)$ is a smooth function of t. Thus we have

$$f(t)\hat{\omega}g(t) = [\hat{\omega} + i\partial_t]\, f(t)g(t), \tag{3.22}$$

for arbitrary functions $f(t)$ and $g(t)$ with the convention that the time derivative operator ∂_t acts *only* on the first term to its right. By repeatedly applying the above identity, the second term in (3.18) may be expanded in powers of time derivatives of $\tilde{x}$, and up to $\mathcal{O}(\dot{\tilde{x}})^2$, it gives

$$\begin{aligned}
&\mathrm{Tr}\left[\frac{1}{m\hat{\omega}^2 - V''(x_0)}\Lambda(\tilde{x})\frac{1}{m\hat{\omega}^2 - V''(x_0)}\Lambda(\tilde{x})\right] = \\
&\mathrm{Tr}\left[\frac{1}{[m\omega^2 - V''(x_0)]^2}\Lambda(\tilde{x})\Lambda(\tilde{x})\right] \\
&+\mathrm{Tr}\left[\frac{1}{[m\omega^2 - V''(x_0)]^3}\left(-2im\omega\partial_t + m\partial_t^2\right)\Lambda(\tilde{x})\Lambda(\tilde{x})\right] \\
&+\mathrm{Tr}\left[\frac{1}{[m\omega^2 - V''(x_0)]^4}\left(-2im\omega\partial_t + m\partial_t^2\right)^2\Lambda(\tilde{x})\Lambda(\tilde{x})\right].
\end{aligned} \tag{3.23}$$

In each line, the derivative operators ∂_t act only on the first $\Lambda(\tilde{x})$. After carrying out all integrations, we are left with

$$\begin{aligned}
&\frac{i\hbar}{2}\mathrm{Tr}\left[\frac{1}{m\hat{\omega}^2 - V''(x_0)}\Lambda(\tilde{x})\frac{1}{m\hat{\omega}^2 - V''(x_0)}\Lambda(\tilde{x})\right] = \\
&\qquad\int_{t_a}^{t_b} \mathrm{d}t\frac{\hbar}{16}\left[\frac{[V'''(x_0)]^2}{m^{1/2}[V''(x_0)]^{3/2}}\tilde{x}^2 + \frac{1}{4}\frac{[V'''(x_0)]^2 m^{1/2}}{[V''(x_0)]^{5/2}}\dot{\tilde{x}}^2\right].
\end{aligned} \tag{3.24}$$

Adding the second term of (3.19) to the coefficient of the $\tilde{x}^2$ term in the above expression one obtains $-\mathcal{V}''(x_0)/2$, as expected. From the second term in (3.24), proportional to $\dot{\tilde{x}}^2$, we extract an x_0-dependent contribution to the kinetic energy (3.12):

$$\mathcal{Z}(x_0) = \frac{\hbar}{32}\frac{[V'''(x_0)]^2 m^{1/2}}{[V''(x_0)]^{5/2}}. \tag{3.25}$$

The resulting effective action reads

$$S_{\rm eff}[x] = \int_{t_a}^{t_b} {\rm d}t \quad \left[\tfrac{1}{2} m_{\rm eff}\dot{x}^2 - V_{\rm eff}(x)\right], \tag{3.26}$$

where

$$m_{\rm eff} = m + \frac{\hbar}{32}\frac{[V'''(x)]^2 m^{1/2}}{[V''(x)]^{5/2}} \tag{3.27}$$

and

$$V_{\rm eff}(x) = V(x) + \frac{\hbar}{2}\sqrt{\frac{V''(x)}{m}} \tag{3.28}$$

are the effective mass and potential, respectively. We have replaced x_0 with x, since the functional dependence of the coefficients $\mathcal{V}$ and $\mathcal{Z}$ on x_0 and x is the same.

We see that the quantum corrections to the classical Lagrangian not only generate an effective potential, but also introduce an effective coordinate-dependent mass, leading to the modified equation of motion

$$m_{\rm eff}(x)\ddot{x} + \tfrac{1}{2} m'_{\rm eff}(x)\dot{x}^2 + V'_{\rm eff}(x) = 0. \tag{3.29}$$

3.2 Comparison with graphical method

It is useful to compare the derivative expansion method with the calculation using Feynman graphs. To do that, we first expand the potential $V(x)$ about an extremum x_0, and write the classical Lagrangian as

$$\mathcal{L}(x,\dot{x}) = \tfrac{1}{2} m\dot{\bar{x}}^2 - \tfrac{1}{2} V''(x_0)\bar{x}^2 - \tfrac{1}{3!} V'''(x_0)\bar{x}^3 + \cdots, \tag{3.30}$$

where now $\bar{x}(t) = x(t) - x_0$ and we dropped the constant term $V(x_0)$. To one loop order, we need to expand only up to terms of order $\bar{x}^3$.

Expanding the effective action about x_0, one obtains:

$$S_{\rm eff}[x] = S[x_0] + \tfrac{1}{2}\int_{t_a}^{t_b} {\rm d}t{\rm d}t' \left.\frac{\delta^2 S_{\rm eff}}{\delta x(t)\delta x(t')}\right|_{x_0} \bar{x}(t)\bar{x}(t') + \cdots. \tag{3.31}$$

From the Fourier transform $\Gamma^{(2)}(\omega)$ of the second derivative term

$$\Gamma^{(2)}(t,t') \equiv \left.\frac{\delta^2 S_{\rm eff}}{\delta x(t)\delta x(t')}\right|_{x_0} \tag{3.32}$$

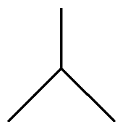
the effective mass can be extracted:

$$\Gamma^{(2)}(\omega) = -i\hbar(m_{\text{eff}}\omega^2 + \cdots). \tag{3.33}$$

$\Gamma^{(2)}(\omega)$ is calculated in the perturbation scheme, where the free propagator, read off from the Lagrangian (3.30), is

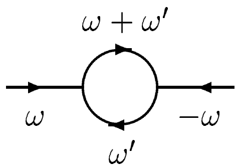
$$G_0(\omega) = \frac{1}{m\omega^2 - V''(x_0)} \tag{3.34}$$

and there is one vertex, corresponding to the cubic term in (3.30):



$$V'''(x_0). \tag{3.35}$$

The only one-loop diagram contributing to $\Gamma^{(2)}(\omega)$ is



$$\frac{[V'''(x_0)]^2}{2}\int \frac{\mathrm{d}\omega'}{2\pi}\frac{1}{m\omega'^2 - V''(x_0)}\frac{1}{m(\omega+\omega')^2 - V''(x_0)}. \tag{3.36}$$

After carrying out the integration in (3.36), we obtain

$$\Gamma^{(2)}(\omega) = \frac{\hbar}{32}\frac{[V'''(x_0)]^2 m^{1/2}}{[V''(x_0)]^{5/2}}\omega^2 + \cdots. \tag{3.37}$$

Comparing the above expression with (3.33), one readily identifies the correction to the effective mass, in agreement with the one obtained by the derivative expansion in (3.27).

3.3 Coordinate-dependent mass

Now we allow the mass to be coordinate-dependent from the outset [56]. This is necessary to describe a larger variety of interesting physical systems, for instance compound nuclei, where the collective Hamiltonian, commonly derived from a microscopic description via a quantized adiabatic time-dependent Hartree-Fock theory (ATDHF)[57], contains coordinate-dependent collective mass parameters.

Let us take the classical action

$$S[x] = \int_{t_a}^{t_b} \mathrm{d}t \, \left[m(x)\dot{x}^2 - V(x)\right], \tag{3.38}$$

where the mass $m(x)$ is explicitly coordinate-dependent, as our starting point. As discussed in the previous section, the quantum correction to the classical action in the semi-classical approximation is now given by

$$\begin{aligned} S_1[x_{\rm cl}] = -\frac{i\hbar}{2}\mathrm{Tr}\ln\Big[& m(x_{\rm cl})\hat{\omega}^2 - V''(x_{\rm cl}) - im'(x_{\rm cl})\dot{x}_{\rm cl}\hat{\omega} \\ & -m'(x_{\rm cl})\ddot{x}_{\rm cl} - \tfrac{1}{2}m''(x_{\rm cl})\dot{x}_{\rm cl}^2\Big]. \end{aligned} \tag{3.39}$$

After replacing $x_{\rm cl}$ with $x_0 + \tilde{x}$ and expanding up to the second order in $\tilde{x}$ and its derivatives, we obtain

$$S_1 = -\frac{i\hbar}{2}\mathrm{Tr}\ln[G^{-1}(\hat{\omega}) + \Lambda(\tilde{x})], \tag{3.40}$$

where the inverse free propagator $G^{-1}(\hat{\omega})$ is given by

$$G^{-1}(\hat{\omega}) = m(x_0)\hat{\omega}^2 - V''(x_0), \tag{3.41}$$

and

$$\begin{aligned} \Lambda(\tilde{x}) &= \tilde{x}[m'(x_0)\hat{\omega}^2 - V'''(x_0)] + \tfrac{1}{2}\tilde{x}^2[m''(x_0)\hat{\omega}^2 - V''''(x_0)] - i\dot{\tilde{x}}m'(x_0)\hat{\omega} \\ &- i\tilde{x}\dot{\tilde{x}}m''(x_0)\hat{\omega} - \ddot{\tilde{x}}m'(x_0) - \tilde{x}\ddot{\tilde{x}}m''(x_0) - \tfrac{1}{2}\dot{\tilde{x}}^2 m''(x_0). \end{aligned} \tag{3.42}$$

Following the same steps as in the previous section, we expand the logarithm in (3.40) up to the second order in $\tilde{x}$:

$$S_1 = -\frac{i\hbar}{2}\mathrm{Tr}[G^{-1}(\hat{\omega})] - \frac{i\hbar}{2}\mathrm{Tr}[G(\hat{\omega})\Lambda(\tilde{x})] + \frac{i\hbar}{4}\mathrm{Tr}[G(\hat{\omega})\Lambda(\tilde{x})G(\hat{\omega})\Lambda(\tilde{x})]. \tag{3.43}$$

After carrying out the remaining steps of the derivative expansion, we obtain

$$S_1[x_{\rm cl}] = \int_{t_a}^{t_b} \mathrm{d}t\left[-\mathcal{V}(x_0) - \mathcal{V}'(x_0)\tilde{x} - \tfrac{1}{2}\mathcal{V}''(x_0)\tilde{x}^2 + \tfrac{1}{2}\mathcal{Z}(x_0)\dot{\tilde{x}}^2 + \cdots\right], \tag{3.44}$$

with

$$\mathcal{V}(x_0) = \frac{\hbar}{2}\sqrt{\frac{V''(x_0)}{m(x_0)}}, \tag{3.45}$$

as in the previous section, and

$$\begin{aligned}\mathcal{Z}(x_0) &= \frac{\hbar}{32}\frac{[V'''(x_0)]^2[m(x_0)]^{1/2}}{[V''(x_0)]^{5/2}} - \frac{5\hbar}{16}\frac{m'(x_0)V'''(x_0)}{[m(x_0)]^{1/2}[V''(x_0)]^{3/2}} \\ &- \frac{11\hbar}{32}\frac{[m'(x_0)]^2}{[m(x_0)]^{3/2}[V''(x_0)]^{1/2}} + \frac{\hbar}{4}\frac{m''(x_0)}{[m(x_0)]^{1/2}[V''(x_0)]^{1/2}}. \end{aligned} \tag{3.46}$$

The effective potential and mass to be introduced in the corrected equation of motion (3.29) are now given by

$$V_{\text{eff}} = V(x) + \mathcal{V}(x), \tag{3.47}$$

and

$$m_{\text{eff}} = m(x) + \tfrac{1}{2}\mathcal{Z}(x), \tag{3.48}$$

where we again replaced x_0 with the full x.