Chapter 2

General Properties of Fluid Membranes

2.1 Canham-Helfrich model

Closed lipid bilayers, such as red blood cells, can have an unswollen shape. At equilibrium, they minimize their free energy with respect to both volume and surface independently, thus yielding a vanishing surface tension. In this situation, one witnesses a 'flickering' phenomenon, i. e., the enhanced light scattering due to oscillations of the membrane, first observed in the 19th century [29].

In order to explain this effect, Canham [19] suggested that the behavior of red blood cell membranes is governed essentially by curvature energy. He proposed the curvature elastic energy of a membrane to have the form

$$E_{\mathrm{c}} = \tfrac{1}{2}\kappa \int \left(\frac{1}{R_1^2} + \frac{1}{R_2^2} \right) \mathrm{d}S, \tag{2.1}$$

where R_1 and R_2 are the principal curvature radii of the membrane surface (see Fig. 2.1), κ is its bending rigidity and the integral is over the total area of the membrane. The equilibrium configuration of a vesicle governed by this elastic energy reproduces the biconcave shape found in red blood cells, thus supporting Canham's model.

A more precise formulation of this model was given by Helfrich [20]. By introducing local Cartesian coordinates on the membrane, imposing that the z-axis be parallel to the surface normal vector at each point, he defined

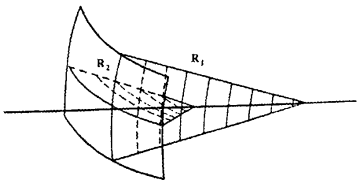


Figure 2.1: Principal curvature radii of a surface

the two principal curvatures as follows. Expressing the normal vector $\mathbf{n} = (n_x, n_y, n_z)$ as a function of the coordinates x and y, the principal curvatures at each point are given by the eigenvalues of the matrix

$$\bar{K} = \begin{pmatrix} \frac{\partial n_x}{\partial x} & \frac{\partial n_x}{\partial y} \\ \\ \frac{\partial n_y}{\partial x} & \frac{\partial n_y}{\partial y} \end{pmatrix}. \tag{2.2}$$

If the orientation of the local coordinate system is chosen in such a way that derivatives along its axes correspond to the largest and smallest curvatures of the membrane surface, mixed derivatives vanish, and the principal curvatures become

$$c_x = \frac{\partial n_x}{\partial x} \text{ and } c_y = \frac{\partial n_y}{\partial y}. \tag{2.3}$$

Since the membrane has rotational symmetry, only combinations of derivatives of $\mathbf{n}(x, y)$ which are independent of the orientation of the x- and y-axis may appear in the expression for the curvature elastic energy. They are

$$\frac{\partial n_x}{\partial x} + \frac{\partial n_y}{\partial y} \tag{2.4}$$

$$\left(\frac{\partial n_x}{\partial x} + \frac{\partial n_y}{\partial y}\right)^2 \tag{2.5}$$

$$\frac{\partial n_x}{\partial x}\frac{\partial n_y}{\partial y} - \frac{\partial n_x}{\partial y}\frac{\partial n_y}{\partial x}. \tag{2.6}$$

The curvature elastic energy density then becomes

$$\begin{aligned} e_c &= \tfrac{1}{2}\kappa \left(\frac{\partial n_x}{\partial x} + \frac{\partial n_y}{\partial y} - H_0\right)^2 + \bar{\kappa}\left(\frac{\partial n_x}{\partial x}\frac{\partial n_y}{\partial y} - \frac{\partial n_x}{\partial y}\frac{\partial n_y}{\partial x}\right) \\ &= \tfrac{1}{2}\kappa \left(\mathrm{Tr}\bar{K} - H_0\right)^2 + \bar{\kappa}\det \bar{K}, \end{aligned} \tag{2.7}$$

where the linear term (2.4) has been incorporated through the membrane spontaneous curvature H_0, and $\bar{\kappa}$ is its Gaussian rigidity.

The second term (2.5) and the integrand in Eq. (2.1) are equivalent, and correspond to the squared local mean curvature H^2 of the membrane. The third term (2.6) corresponds to the Gaussian curvature K, which is sensitive to topology changes.

2.2 A little differential geometry

Before proceeding in the study of the Canham-Helfrich model, it is useful to introduce some rudimentary differential geometry concepts to help us describe spatial configurations of a fluctuating surface.

A two-dimensional surface embedded in a three-dimensional Euclidean space can be mathematically represented by a parametric equation $\mathbf{X} = \mathbf{X}(\sigma_1, \sigma_2)$, where $\mathbf{X} = (x(\sigma_1, \sigma_2), y(\sigma_1, \sigma_2), z(\sigma_1, \sigma_2))$ is a three-dimensional vector function of the parameters $\bar{\sigma} = (\sigma_1, \sigma_2)$. In the following we shall assume that the vector function $\mathbf{X}$, referred to as the *parametrization* of the surface, is a smooth function of its parameters.

At each point, the tangent vectors $\mathbf{t}_i$ to the surface are given by the partial derivatives of the parametrization $\mathbf{X}$, that is, $\mathbf{t}_i \equiv \partial_i \mathbf{X}$, where ∂_i denotes the partial derivative with respect to the coordinate σ_i and $i = 1, 2$. If the surface does not self-intersect and does not have overhangs, the tangent vectors $\mathbf{t}_i$ are linearly independent, and generate the tangent plane to the surface at each point.

The infinitesimal Euclidean distance $\mathrm{d}\ell$ between two points on the surface with coordinates $\bar{\sigma}$ and $\bar{\sigma} + \mathrm{d}\bar{\sigma}$ is given by

$$\mathrm{d}\ell^2 = [\mathbf{X}(\bar{\sigma} + \mathrm{d}\bar{\sigma}) - \mathbf{X}(\bar{\sigma})]^2 = \mathbf{t}_i \cdot \mathbf{t}_j \mathrm{d}\sigma^i \mathrm{d}\sigma^j \equiv g_{ij} \mathrm{d}\sigma^i \mathrm{d}\sigma^j, \tag{2.8}$$

where

$$g_{ij} \equiv \mathbf{t}_i \cdot \mathbf{t}_j \tag{2.9}$$

is the *metric tensor* or *first fundamental form* on the surface, with Einstein's convention of summation over repeated indices.

The area of an infinitesimal element of surface can be approximated by the area of the parallelogram tangent to it, delimited by the tangent vectors $\mathbf{t}_i$ (see Fig. 2.2). It can be expressed as the absolute value of the cross product between the tangent vectors:

$$\mathrm{d}S = |\mathbf{t}_1 \times \mathbf{t}_2| \mathrm{d}\sigma^1 \mathrm{d}\sigma^2. \tag{2.10}$$

Remembering that

$$|\mathbf{t}_1 \times \mathbf{t}_2|^2 = \mathbf{t}_1^2 \mathbf{t}_2^2 - (\mathbf{t}_1 \cdot \mathbf{t}_2)^2, \tag{2.11}$$

the infinitesimal element of area on the surface can be rewritten as

$$\mathrm{d}S = \sqrt{g}\mathrm{d}\sigma^1 \mathrm{d}\sigma^2, \tag{2.12}$$

where $g = \det[g_{ij}]$.

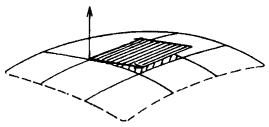


Figure 2.2: Infinitesimal element of area of a surface.

The unit normal vector to the surface is given by

$$\mathbf{N} = \frac{\mathbf{t}_1 \times \mathbf{t}_2}{|\mathbf{t}_1 \times \mathbf{t}_2|}, \tag{2.13}$$

and the surface is said to be *orientable* if it is possible to uniquely define a field of normal vectors on it. In the following we shall always consider orientable surfaces.

Each point of a surface is characterized by a local curvature tensor. The notion of curvature is more intuitive for a planar curve in two dimensions. In this case, the curvature tensor at each point is a scalar with magnitude equal to the inverse radius of the circle that locally follows the curve, as shown in Fig. 2.3.

By defining a field of normal vectors to the curve, thus specifying its unique orientation, one may define the sign of the curvature k at each point. If the curve rises towards its normal vector, the sign of k is positive, and if it falls away from the normal direction, the curvature is negative (see Fig. 2.4).

A similar analysis applies to a two-dimensional surface embedded in three dimensions. At each point on an oriented surface one may define a tangent plane. Planes normal to the tangent plane intersect the surface in a planar curve called a normal section. Each normal section has an associated curvature at the tangency point. The magnitude of the maximum and minimum

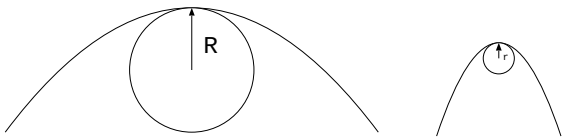


Figure 2.3: The circles tangent to the curves have a radius that is inversely proportional to the curvature at the tangency point.

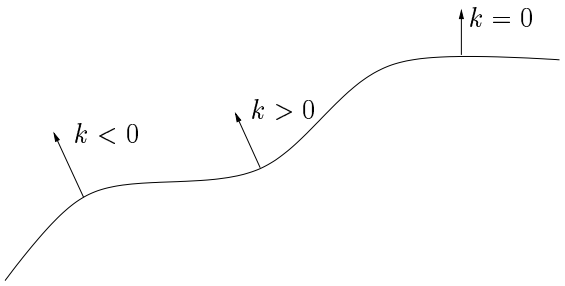


Figure 2.4: Sign of the curvature of a planar curve.

curvatures of the normal sections define the two principal curvatures k_1 and k_2 of the surface at each point. Their arithmetic mean defines the *mean curvature* $H = (k_1 + k_2)/2$ of the surface, and their product $K = k_1 k_2$ the *Gaussian curvature* at each point on the surface (see Fig. 2.5).

To formalize the notion of curvature of a surface one defines the *extrinsic curvature tensor* $\bar{K}_{ij}$ by

$$\partial_i \mathbf{N} = \bar{K}_{ij} \mathbf{t}_j, \tag{2.14}$$

also called *second fundamental form* on the surface. $\bar{K}_{ij}$ is a symmetric tensor, and it can be diagonalized at each point of the surface. Its eigenvectors correspond to the directions of the two principal curvatures on the tangent plane, and its eigenvalues to the principal curvatures. The mean curvature H is thus given by $H = \frac{1}{2}\mathrm{Tr}[\bar{K}_{ij}]$, and the Gaussian curvature by $K = \det[\bar{K}_{ij}]$.

If the two principal curvatures are positive, that is, both the mean curvature and the Gaussian curvature have a positive sign, all normal sections curve towards $\mathbf{N}$, as depicted in Fig 2.6. If both principal curvatures are

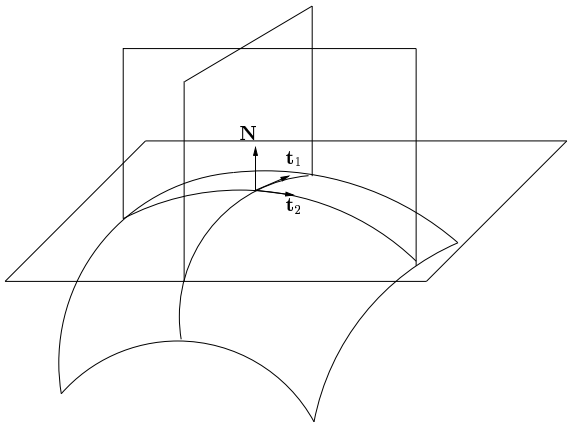


Figure 2.5: Principal curvatures of a surface.

negative, that is, $H < 0$ but $K > 0$, all normal sections curve away from $\mathbf{N}$. If the principal curvatures have opposite signs, that is, $K < 0$, one is at a *saddle point*, there are normal sections that curve both towards and away from $\mathbf{N}$.

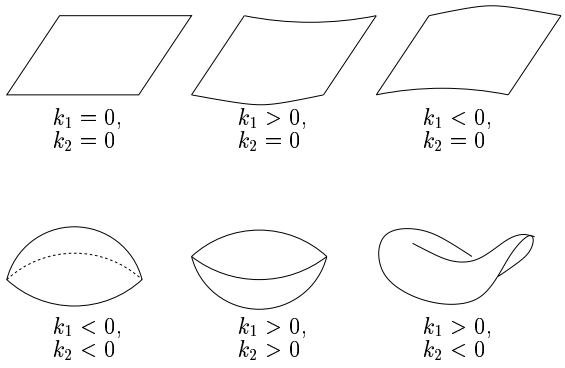


Figure 2.6: Different cases for the two principal curvatures.

An excellent discussion of the differential geometry concepts briefly reviewed here can be found in Refs. [30, 31].

2.3 Monge parametrization

We are now ready to discuss the properties of the Canham-Helfrich model in more detail. To investigate the role of thermal fluctuations, one must determine the partition function for a membrane governed by the energy

$$E_{\mathrm{c}} = \tfrac{1}{2} \int \mathrm{d}S \left(\kappa_0 H^2 + \bar{\kappa}_0 K^2 \right), \tag{2.15}$$

where the subscript 0 indicates that the bending and Gaussian rigidity are bare, microscopic parameters whose values will be modified by thermal fluctuations.

The partition function Z is given by the formal sum

$$Z = \sum_S \exp(-\beta E_{\mathrm{c}}) \tag{2.16}$$

over all possible spatial configurations S of the membrane. Here, and in the following, we define $\beta = 1/k_{\mathrm{B}}T$, where k_{B} is the Boltzmann constant and T is the temperature of the system.

To calculate the sum (2.16), we must specify each configuration S of the membrane by a parametrization of its surface. However, the same spatial configuration may be represented by several parametrizations, and the invariance of the properties of a surface with respect to changes in the parametrization is analogous to gauge invariance in electrodynamics. Consequently, the choice of a particular parametrization is equivalent to gauge fixing, and reparametrization invariance must be ensured via a Faddeev-Popov determinant.

Since, as discussed in the Introduction, fluid membranes under normal conditions are almost flat surfaces, a good gauge choice is the *Monge parametrization*, which is equivalent to specifying the height of the surface above a reference plane (see Fig. 2.7).

The points on the surface are then specified by a vector field $\mathbf{X}(\bar{\sigma}) = (\sigma_1, \sigma_2, \phi(\bar{\sigma}))$, where $\phi(\bar{\sigma})$ denotes the vertical displacement of the surface with respect to a base plane with Cartesian coordinates $\bar{\sigma} = (\sigma_1, \sigma_2)$.

We are going to concentrate on small fluctuations around a flat configuration, not allowing for topology changes. We may therefore disregard the Gaussian curvature term in the energy (2.15), since its integral over a surface with fixed topology amounts to a constant.

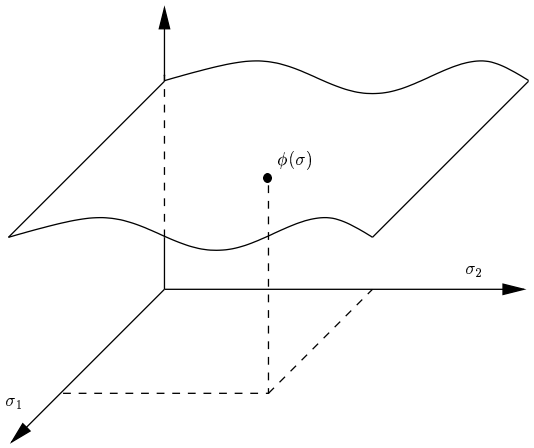


Figure 2.7: Monge parametrization.

Let us consider the energy

$$E_0 = \int \mathrm{d}^2\sigma \sqrt{g} \left(r_0 + \tfrac{1}{2}\kappa_0 H^2 \right), \tag{2.17}$$

where we have introduced a bare surface tension r_0. As discussed in the Introduction, the physical surface tension of a fluid membrane is vanishingly small. This applies to the *effective* surface tension, and there is no reason why the bare parameter should be set to zero *a priori*.

In the Monge parametrization, the metric tensor is given by

$$g_{ij} = \delta_{ij} + \partial_i \phi \partial_j \phi, \tag{2.18}$$

and so

$$g \equiv \det[g_{ij}] = 1 + (\partial_i \phi)^2. \tag{2.19}$$

The unit normal to the membrane surface is

$$\mathbf{N} = \frac{1}{\sqrt{1 + (\partial_i \phi)^2}} (\partial_1 \phi, \partial_2 \phi, -1), \tag{2.20}$$

and the extrinsic curvature tensor is

$$K_{ij} = \partial_i \left[\frac{\partial_j \phi}{\sqrt{1 + (\partial_k \phi)^2}} \right]. \tag{2.21}$$

The modified Canham-Helfrich energy (2.17) becomes, in the Monge parametrization,

$$E_0[\phi] = \int \mathrm{d}^2\sigma \sqrt{1+(\partial\phi_i)^2} \Bigg\{ r_0 + \tfrac{1}{2}\kappa_0 \Bigg[\frac{(\partial^2\phi)^2}{1+(\partial_i\phi)^2} - 2\frac{\partial_i\phi\partial_j\phi\partial_i\partial_j\phi\partial^2\phi}{[1+(\partial_k\phi)^2]^2} + \frac{(\partial_i\phi\partial_j\phi\partial_i\partial_j\phi)^2}{[1+(\partial\phi_k)^2]^3} \Bigg] \Bigg\}. \tag{2.22}$$

The partition function (2.16) is now given by the functional integral

$$Z = \int \mathcal{D}\phi \exp\left(-\beta E_0[\phi]\right), \tag{2.23}$$

where the measure $\mathcal{D}\phi$ is defined as follows. One discretizes the reference plane by introducing a square lattice of points (σ_i, σ_j) separated by a distance a, and defines a function $\phi_{ij} \equiv \phi(\sigma_i, \sigma_j)$ at each point. The integration measure $\mathcal{D}\phi$ is the defined as the limit

$$\mathcal{D}\phi = \lim_{a\to 0} \prod_{ij} \mathrm{d}\phi_{ij}. \tag{2.24}$$

Note that we chose the Monge parametrization in an arbitrary way, without worrying that reparametrization invariance be explicitly ensured. Before proceeding, let us deal with the problem of gauge fixing in the proper way, that is, by calculating the Faddeev-Popov determinant.

2.4 Faddeev-Popov determinant

Before choosing a special parametrization, the partition function should read

$$Z = \int \mathcal{D}\mathbf{X} \exp(-\beta E_0[\mathbf{X}]), \tag{2.25}$$

where the measure $\mathcal{D}\mathbf{X}$ is defined on the space of all membrane configurations. This measure should satisfy two conditions:

- *reparametrization invariance*
  The weight of a configuration should not depend on its parametrization.
- *locality*
  There are no long distance correlations on the surface, that is, deformations of the surface at distant points must be treated independently.

As discussed in Sec. 2.3, many configurations $\mathbf{X}(\bar{\sigma})$ are equivalent under a change of coordinates, and describe the same membrane. To avoid multiple counting in the partition function while respecting the above mentioned conditions, we use the same approach as in the functional quantization of gauge theories:

- We choose a measure which is local and reparametrization invariant, namely, Fujikawa's measure [32, 33, 34, 35, 36]

$$\mathcal{D}\mathbf{X} = \prod_{\bar{\sigma}} \prod_{\mu=1}^{3} \left[\mathrm{d}X^{\mu} [g(\bar{\sigma})]^{1/4} \right]. \tag{2.26}$$

- We pick up a set of inequivalent configurations ("gauge slice") by a gauge fixing condition. In the Monge parametrization, one fixes a reference surface $\mathbf{X}_0(\bar{\sigma})$ and requires that all other configurations $\mathbf{X}(\bar{\sigma})$ be of the form

$$\mathbf{X}(\bar{\sigma}) = \mathbf{X_0}(\bar{\sigma}) + \mathbf{x}(\bar{\sigma}), \tag{2.27}$$

  with the condition that $\mathbf{x}(\bar{\sigma})$ be orthogonal to the reference configuration $\mathbf{X}_0(\bar{\sigma})$. The gauge fixing condition is then implemented by introducing the two constraints $F_i(\bar{\sigma}) = 0$, with

$$F_i(\bar{\sigma}) = \mathbf{x}(\bar{\sigma}) \cdot \partial_i \mathbf{X}_0(\bar{\sigma}). \tag{2.28}$$

- The functional integral is written as an integral over the gauge slice, and each configuration in the gauge slice is weighted by the volume of the set of all configurations physically equivalent to it :

$$Z = \int \mathcal{D}\mathbf{X} \prod_{\bar{\sigma},i} \delta(F_i(\bar{\sigma})) \Delta_F \exp(-\beta E_0[\mathbf{X}]). \tag{2.29}$$

  The Faddeev-Popov determinant Δ_F is obtained by determining how the constraints $F_i(\bar{\sigma})$ are affected by an infinitesimal change of coordinates $\sigma^i \to \sigma^i + \varepsilon^i(\bar{\sigma})$:

$$\begin{aligned} \delta F_i(\bar{\sigma}) &= \delta \left[\mathbf{X}(\bar{\sigma}) - \mathbf{X}_0(\bar{\sigma}) \right] \partial_i \mathbf{X}_0(\bar{\sigma}) = \delta \mathbf{X}(\bar{\sigma}) \partial_i \mathbf{X}_0(\bar{\sigma}) \\ &= \varepsilon^j \partial_j \mathbf{X}(\bar{\sigma}) \partial_i \mathbf{X}_0(\bar{\sigma}) \\ &= \varepsilon^j \left[\partial_j \mathbf{X}_0(\bar{\sigma}) + \partial_j \mathbf{x}(\bar{\sigma}) \right] \partial_i \mathbf{X}_0(\bar{\sigma}) \\ &= \varepsilon^j \left[g^0_{ij}(\bar{\sigma}) + \partial_j \mathbf{x}(\bar{\sigma}) \partial_i \mathbf{X}_0(\bar{\sigma}) \right], \end{aligned}$$

hence

$$\Delta_F = \prod_{\bar{\sigma}} \det \left[g^0_{ij}(\bar{\sigma}) + \partial_j \mathbf{x}(\bar{\sigma}) \partial_i \mathbf{X}_0(\bar{\sigma}) \right], \tag{2.30}$$

and since in the Monge parametrization $g^0_{ij}(\bar{\sigma}) = \delta_{ij}$ and $\partial_j \mathbf{x}(\bar{\sigma}) \cdot \partial_i \mathbf{X}_0(\bar{\sigma}) = 0$, the Faddeev-Popov determinant is simply equal to one [22].

As we have seen, the Monge parametrization has the advantage that there are no extra "Faddeev-Popov ghosts" in the partition function. We may now take the partition function (2.23) with the elastic energy (2.22) as the starting point of our calculations.

2.5 Harmonic approximation

The harmonic approximation to the modified Canham-Helfrich energy (2.22), obtained by expanding the integrand in that expression up to second order in ϕ, is

$$E_{\text{harmonic}}[\phi] = r_0 L^2 + \tfrac{1}{2} \int \mathrm{d}^2\sigma \left[r_0 (\partial_i \phi)^2 + \kappa_0 (\partial^2 \phi)^2 \right], \tag{2.31}$$

where L is the lateral size of the reference plane.

If the curvature term is dropped, E_{harmonic} becomes identical to the elastic Hamiltonian of the xy-model. Since in this model phase fluctuations become divergent at and below its lower critical dimension $D_{\text{l}} = 2$, one may ask if this is also the case in our membrane model. Let us calculate the *positional* correlation function $\langle \phi^2 \rangle$. Passing to Fourier space, we have

$$\langle \phi^2 \rangle = k_{\text{B}} T \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{1}{r_0 q^2 + \kappa_0 q^4} = \frac{k_{\text{B}} T}{2\pi r_0} \ln \left(\sqrt{\frac{r_0}{\kappa_0}} L \right). \tag{2.32}$$

The height fluctuations diverge with the lateral size of the surface, implying that the average position of the membrane becomes less well defined as its size increases: there is no *long-range positional order*. On the other hand, the direction of the surface normal remains well defined. The deviation of the normal vector from its average direction is, from Eq. (2.20),

$$\langle |\delta \mathbf{N}(\bar{\sigma})|^2 \rangle \approx \langle (\partial_i \phi)^2 \rangle = k_{\text{B}} T \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{q^2}{r_0 q^2 + \kappa_0 q^4} = \frac{k_{\text{B}} T}{2\pi \kappa_0} \ln \left(\sqrt{\frac{\kappa_0}{r_0}} \Lambda \right), \tag{2.33}$$

where Λ is an ultraviolet cutoff proportional to the inverse size of a lipid molecule. Thus, normal fluctuations remain finite as the system size increases: the surface has *long-range orientational order*.

If the surface tension is set equal to zero, so that the curvature term in $E_{\rm harmonic}$ alone determines the shape of the membrane, positional and orientational fluctuations are much more violent. In this case

$$\langle \phi^2 \rangle = k_{\rm B}T \int \frac{{\rm d}^2 q}{(2\pi)^2} \frac{1}{\kappa_0 q^4} = \frac{k_{\rm B}T}{4\pi\kappa_0} L^2, \tag{2.34}$$

and

$$\langle |\delta \mathbf{N}(\bar{\sigma})|^2 \rangle = k_{\rm B}T \int \frac{{\rm d}^2 q}{(2\pi)^2} \frac{q^2}{r_0 q^2 + \kappa_0 q^4} = \frac{k_{\rm B}T}{2\pi\kappa_0} \ln\left(L\Lambda\right), \tag{2.35}$$

so that there is neither long-range positional order nor long-range orientational order. This means that, for lateral sizes larger than the orientational *persistence length*

$$\xi_{\rm p} = \Lambda^{-1} \exp\left(\frac{2\pi\kappa_0}{k_{\rm B}T}\right), \tag{2.36}$$

at which $\langle |\delta \mathbf{N}(\bar{\sigma})|^2 \rangle$ becomes of order unity, the assumption of an almost flat membrane, with no overhangs, breaks down. $\xi_{\rm p}$ is the de Gennes-Taupin [25] persistence length for fluctuating surfaces in the harmonic approximation.

2.6 Anharmonic terms and renormalization

From our discussion of the harmonic approximation of the elastic energy of a membrane we see that thermally excited fluctuations play a crucial role in the determination of the spatial configuration of such a system. It was first shown by Helfrich [37] that thermal fluctuations are not only controlled by the bending rigidity, but also reduce it for long-wavelength undulations.

The fact that short-wavelength fluctuations modify the long-wavelength bending rigidity is a fundamentally nonlinear effect, since in a harmonic system modes of different wavelengths are decoupled. In addition, as we shall see, thermal fluctuations induce spontaneous generation of a surface tension, and that is the reason why we include a bare surface tension in our calculations from the beginning.

Explicitly, thermal fluctuations soften the bending rigidity at large length scales, reducing it from the bare value κ_0 as follows

$$\kappa_{\rm eff} = \kappa_0 - \alpha \frac{k_{\rm B}T}{4\pi} \ln(\Lambda L). \tag{2.37}$$

Various authors derived different values for α, first $\alpha = 1$ [37, 38, 39] was obtained, later $\alpha = 3$ [40, 41, 42, 43]. The second result has also been found in computer simulations [44]. For either value of α, the rigidity disappears at length scales larger than the persistence length

$$\xi_{\rm p} = \Lambda^{-1} \exp\left(\frac{1}{\alpha}\frac{4\pi}{k_{\rm B}T}\kappa_0\right), \tag{2.38}$$

beyond which the normal vectors of the surface become uncorrelated — the surface looks crumpled. More recent calculations [45] suggest the value $\alpha = -1$, implying a stiffening instead of a softening of the bending rigidity. This new result was argued to arise from the use of another integration measure which respects the incompressible-fluid nature of the membrane from the outset. This is, however, in contrast to previous studies of in-plane fluid [43] and elastic effects [46] which did not show any change in the value $\alpha = 3$ (they only enter at the two-loop level [22]).

The effect of thermal fluctuations on the surface tension has also been investigated by several authors. The results can be summarized by the formula

$$r_{\rm eff} = r_0 + \alpha' \frac{k_{\rm B}T}{4\pi}\frac{r_0}{\kappa_0} \ln(\Lambda L), \tag{2.39}$$

with the value $\alpha' = 1$ found in [41, 47] and $\alpha' = 3$ in [40, 42]. In Ref. [48], an attempt was made to reconcile the differences. An almost planar surface without overhangs was considered in the Monge parametrization. The modified Canham-Helfrich energy Eq. (2.22) was expanded to fourth order in ϕ. The relative weights of the resulting terms are fixed by their covariant origin. The authors encountered considerable problems in showing that this remains true after including the thermal fluctuations.

To illustrate the difficulties encountered when the first anharmonic terms are added to the elastic energy (2.31), let us summarize the perturbation and renormalization group calculations leading to the results described above. The modified Canham-Helfrich energy (2.22) is now expanded up to fourth order in ϕ, and we consider the energy

$$E_{\rm anharmonic} = E_{\rm harmonic} + E_{\rm int}, \tag{2.40}$$

where

$$E_{\rm int} = \int {\rm d}^2\sigma \Big[-\tfrac{1}{8} r_0 (\partial_i \phi)^2 (\partial_j \phi)^2 - \tfrac{1}{4} \kappa_0 (\partial_i \phi)^2 (\partial^2 \phi)^2 - \kappa_0 (\partial_i \phi)(\partial_j \phi)(\partial_i \partial_j \phi)(\partial^2 \phi) \Big]. \tag{2.41}$$

$E_{\rm int}$ may be regarded as a perturbation around the harmonic energy $E_{\rm harmonic}$, and we calculate physical quantities as a series in the expansion parameter $k_{\rm B}T/\kappa_0$. We are considering the situation where the ratio $\kappa_0/k_{\rm B}T$ is large: the persistence length $\xi_{\rm p}$ (see Eq. 2.38) is much larger than the membrane size, so that the membrane can be considered to be almost flat.

The partition function is written as

$$Z[h] = \int \mathcal{D}\phi \exp \left[-\beta (E_{\rm harmonic} + E_{\rm int}) + \beta \int {\rm d}^2\sigma \phi(\bar{\sigma}) h(\bar{\sigma}) \right], \tag{2.42}$$

where the last term in the exponent serves to generate correlation functions, and the field h shall be set equal to zero at the end of the calculations. The effective potential reads

$$\Gamma[\Phi] \equiv -\frac{1}{\beta} \ln Z + \int {\rm d}^2\sigma \Phi(\bar{\sigma}) h(\bar{\sigma}), \tag{2.43}$$

where

$$\Phi(\bar{\sigma}) \equiv \langle \phi(\bar{\sigma}) \rangle = \frac{1}{\beta Z[0]} \left. \frac{\delta Z[h]}{\delta h(\bar{\sigma})} \right|_{h=0}. \tag{2.44}$$

The effective energy can be expanded around $\Phi = 0$:

$$\Gamma[\Phi] = \Gamma[0] + \tfrac{1}{2} \int {\rm d}^2\bar{\sigma} {\rm d}^2\bar{\sigma}' \left. \frac{\delta^2 \Gamma}{\delta\Phi(\bar{\sigma}) \delta\Phi(\bar{\sigma}')} \right|_{\Phi=0} \Phi(\bar{\sigma}) \Phi(\bar{\sigma}') + \cdots \tag{2.45}$$

The Fourier transform $\Gamma^{(2)}(q)$ of the second derivative term

$$\Gamma^{(2)}(\bar{\sigma}, \bar{\sigma}') \equiv \left. \frac{\delta^2 \Gamma}{\delta\Phi(\bar{\sigma}) \delta\Phi(\bar{\sigma}')} \right|_{\Phi=0} \tag{2.46}$$

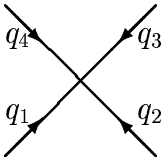
is used to define the effective surface tension $r_{\rm eff}$ and bending rigidity $\kappa_{\rm eff}$ as follows

$$\Gamma^{(2)}(q) = k_{\rm B}T (r_{\rm eff} q^2 + \kappa_{\rm eff} q^4 + \cdots). \tag{2.47}$$

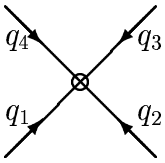
$\Gamma^{(2)}(q)$ is calculated in the perturbation scheme, where the free propagator is given by

$$\text{———} \qquad G_0(q) = \frac{1}{r_0 q^2 + \kappa_0 q^4}. \tag{2.48}$$

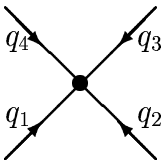
and there are three types of vertices corresponding to the three terms in E_{int}:



$$r_0[(q_1 \cdot q_2)(q_3 \cdot q_4) + 2 \text{ permutations}], \tag{2.49}$$

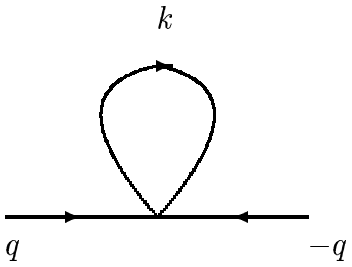


$$-\kappa_0[(q_1 \cdot q_2) q_3^2 q_4^2 + 5 \text{ permutations}], \tag{2.50}$$

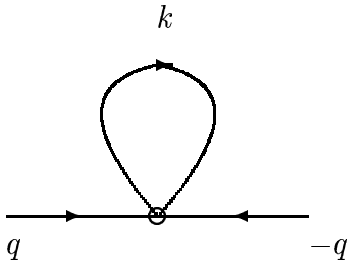


$$-2\kappa_0[q_1^2 (q_2 \cdot q_4)(q_3 \cdot q_4) + 11 \text{ permutations}]. \tag{2.51}$$

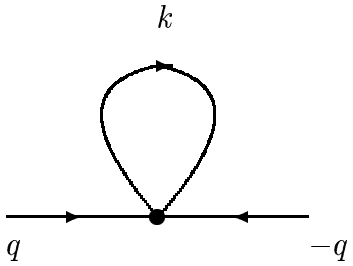
To one-loop order, that is, to first order in the expansion parameter $k_{\text{B}}T/\kappa_0$, the following diagrams contribute to $\Gamma^{(2)}(q)$:



$$\frac{r_0}{2} \int \frac{d^2k}{(2\pi)^2} \frac{q^2 k^2 + 2(q \cdot k)^2}{r_0 k^2 + \kappa_0 k^4},$$



$$\frac{\kappa_0}{2}\int \frac{d^2k}{(2\pi)^2}\frac{q^4k^2+q^2k^4}{r_0k^2+\kappa_0k^4},$$



$$2\kappa_0\int \frac{d^2k}{(2\pi)^2}\frac{k^2(q\cdot k)^2+q^2(q\cdot k)^2}{r_0k^2+\kappa_0k^4}. \quad (2.52)$$

Since all these diagrams are ultraviolet divergent for a membrane of $D = 2$ dimensions, we must introduce some regularization scheme before calculating the integrals. Following Ref. [2] we adopt dimensional regularization to handle the divergent integrals. The great advantage of dimensional regularization over regularization with a momentum cutoff Λ is that terms diverging with a strictly positive power of Λ are suppressed. Only logarithmic divergences show up as poles in ϵ, where $\epsilon = 2 - D$. The connection between the two types of regularization is, at one-loop order,

$$\frac{1}{\epsilon} \to \ln(\Lambda L); \quad (2.53)$$

with the lateral size L of the membrane representing the relevant long-distance scale. The rationale for using dimensional regularization is that contributions to the effective potential with strictly positive powers of the ultraviolet cutoff are connected to $\delta^{(2)}(\bar{\sigma} = 0)$. These highly local terms are uninteresting at large length scales [49, 50].

After evaluating $\Gamma^{(2)}(q)$ and using relation (2.47), we arrive at

$$r_{\rm eff} = r_0 \left(1 + \frac{k_{\rm B}T}{4\pi\kappa_0}\frac{1}{\epsilon}\right), \tag{2.54}$$

$$\kappa_{\rm eff} = \kappa_0 \left(1 - \frac{3k_{\rm B}T}{4\pi\kappa_0}\frac{1}{\epsilon}\right). \tag{2.55}$$

This result for $r_{\rm eff}$, corresponding to $\alpha' = 1$ in (2.39), is in agreement with Refs. [41, 47, 48], but disagrees with Refs. [40, 42] where the value $\alpha' = 3$ was obtained. To understand the differences, we note that in these last two references, the energy (2.40) with $r_0 = 0$ was used instead. That is, the renormalization of the surface tension calculated by these authors was generated solely by the curvature terms. However, the surface term also contributes. In fact, it generates a contribution with $\alpha' = -2$, which, together with the contribution obtained from the curvature terms, results in the value $\alpha' = 1$. The result for $\kappa_{\rm eff}$ reproduces the one from Refs. [40, 41, 42, 43, 44]. The discrepancy between this result with $\alpha = 3$ and the result $\alpha = 1$ from Refs. [37, 38] is due to the fact that in these References the authors treated the fluctuations in a Hartree-like approximation, ignoring the contribution from terms other than harmonic in (2.40). In Ref. [39] an attempt was made to justify the result $\alpha = 1$ based on the in-plane incompressibility of the membrane. This argument has been invalidated in Ref. [51].

Let us now derive the renormalization flow for the parameters $r_{\rm eff}$ and $\kappa_{\rm eff}$. By dimensional analysis, $r_{\rm eff}$ is a *relevant* parameter, equivalent to the mass in an usual field theory. Criticality is obtained by setting all relevant parameters equal to zero, which is equivalent to being at the critical temperature in an usual statistical field theory (remember that the critical behavior in the φ^4-theory is obtained by treating the massless theory, and the mass m is defined there as $m^2 \sim |T - T_{\rm c}|$, $T_{\rm c}$ being the critical temperature). We thus see that only a tensionless membrane can exhibit critical behavior. The expansion parameter $k_{\rm B}T/\kappa_0$, on the other hand, being dimensionless, is a *marginal* parameter, and its behavior at large length scales determines also the phase behavior of the membrane. To determine the beta-function corresponding to $\kappa_{\rm eff}$, we first rewrite Eq. (2.55) as

$$\kappa(\mu) = \mu^{\epsilon}\kappa_0 \left(1 - \frac{3k_{\rm B}T}{4\pi\kappa_0}\frac{1}{\epsilon}\right), \tag{2.56}$$

where we introduced the momentum scale μ, all other quantities being now

dimensionless. The beta-function is then given by

$$\beta(\kappa) \equiv \mu \left. \frac{\mathrm{d}\kappa}{\mathrm{d}\mu} \right|_{\kappa_0}, \tag{2.57}$$

where κ_0, kept constant, is $\kappa_0 = \mu^{-\epsilon}\kappa(\mu)Z_\kappa(\kappa,\epsilon)$, and the renormalization factor $Z_\kappa(\kappa,\epsilon)$ is given by the inverse of the term in brackets on the RHS of (2.56). By expanding Z_κ in powers of $k_\mathrm{B}T/\kappa$ and $1/\epsilon$:

$$Z_\kappa = 1 + A_\kappa \frac{k_\mathrm{B}T}{\kappa}\frac{1}{\epsilon} + \cdots, \tag{2.58}$$

one obtains, for the beta-function

$$\beta(\kappa) = \epsilon\kappa + A_\kappa k_\mathrm{B}T + \cdots. \tag{2.59}$$

From (2.56) and (2.57) one readily identifies the coefficient A_κ, so that

$$\beta(\kappa) = \kappa\left(\epsilon + \frac{3k_\mathrm{B}T}{4\pi\kappa}\right), \tag{2.60}$$

For $\epsilon = 0$, that is, for a two-dimensional membrane, the beta-function has no fixed point other than the trivial one (see Fig. 2.8), thus implying the absence of a phase transition. From Eq. (2.60) we see that the membrane is softened at large length scales. That means that, for scales larger than the persistence length ξ_p, the membrane is crumpled, according to the general discussion of the harmonic model.

Interestingly, for a hypothetical $(2+\epsilon)$-dimensional membrane, with $\epsilon > 0$, Eq. (2.60) implies the existence of a nontrivial fixed point (see Fig. 2.8) at

$$T_\mathrm{c} = \frac{4\pi\kappa}{3k_\mathrm{B}}|\epsilon|, \tag{2.61}$$

corresponding to a *crumpling transition*: for $T < T_\mathrm{c}$ the bending rigidity increases with the lateral size of the membrane, and it behaves as a flat object with $\xi_\mathrm{p} = \infty$. For $T > T_\mathrm{c}$, on the other hand, the membrane is crumpled, and ξ_p is finite [2].

Contrary to the case of the φ^4-theory, where higher order terms are *irrelevant* to the renormalization flow, *all* terms generated by the expansion of the curvature term in the modified Canham-Helfrich energy (2.22) are still

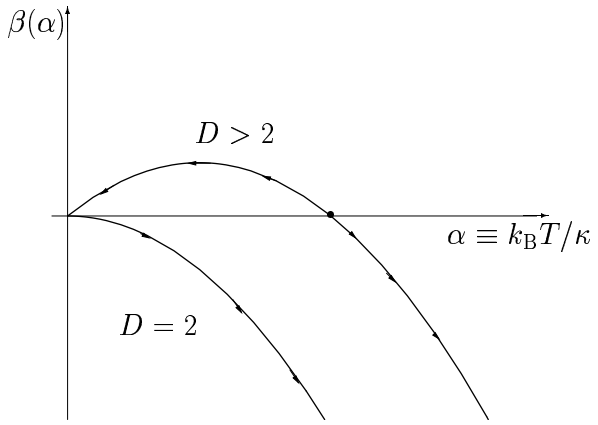


Figure 2.8: Beta-function for the expansion parameter $k_{\rm B}T/\kappa$ calculated to order ϵ. For $D = 2$ the theory is asymptotically free, corresponding to a crumpled membrane. For $D > 2$ a crumpling transition fixed point is found.

marginal, and thus contribute to the renormalization. The relative coefficients of the contributions generated by each of these terms is fixed by their geometrical nature. The same is true for terms related to the surface tension. As previously mentioned, the proof of the covariance of the theory with respect only to the terms contributing to the renormalization of the surface tension is extremely difficult [48]. An attempt to verify this covariance by calculating the contributions from the higher order terms in the expansion starting from the first anharmonic terms led to inconsistencies and showed that thermal fluctuations introduce also infrared divergences in the theory [52].

To solve the problems mentioned above, and to show the renormalizability of the theory at least at the one-loop order, a treatment of the full energy (2.22) is required. The most adequate way to do that is to use a *derivative expansion* [53], which allows one to explicitly keep track of the many terms resulting from the expansion of (2.22). Since the derivative expansion method, though conceptually simple, involves highly technical calculations when applied to membranes, we shall make a digression, and dedicate the next chapter to illustrating this, before proceeding in our analysis of the renormalizability of the Canham-Helfrich energy.