

# Chapter 4

## The Riemann-Hilbert-Poincare Problem

### 4.1 Holomorphic Functions

#### 4.1.1 Introduction

Recently some results on holomorphic functions have been achieved both in the unit disc [1] and in polydiscs and in unit balls see [24] and [2]. On the basis of these results we study the Riemann-Hilbert-Poincare problem for holomorphic functions in polydiscs. This problem plays a significant role in solving the third boundary value problem for pluriholomorphic system and pluriharmonic system in polydiscs.

Let  $\mathbb{D}^n$  be the unit polydisc  $\{z \mid z = (z_1, \dots, z_n) \in \mathbb{C}^n, |z_k| < 1, 1 \leq k \leq n\}$  and  $\partial_0 \mathbb{D}^n$  its essential boundary  $\{z \mid z = (z_1, \dots, z_n) \in \mathbb{C}^n, |z_k| = 1, 1 \leq k \leq n\}$ .

We consider the Riemann-Hilbert-Poincare problem :

Find a function  $u(z)$ , which is holomorphic in  $\mathbb{D}^n$  and satisfies the boundary condition

$$Re \left[ \frac{\partial u}{\partial \nu_\zeta} + \alpha_0(\zeta)u \right] = \gamma_0(\zeta), \quad \zeta \in \partial_0 \mathbb{D}^n, \quad (4.1)$$

where  $\partial u / \partial \nu_\zeta$  denotes the outward normal derivative of  $u(z)$  at the point  $\zeta \in \partial_0 \mathbb{D}^n$  and  $\gamma_0, \alpha_0$  are given Hölder continuous functions on  $\partial_0 \mathbb{D}^n$ , further  $\alpha_0$  is boundary value of a function, which is holomorphic in  $\mathbb{D}^n$ .

By the fact that  $u(z)$  is holomorphic in  $\mathbb{D}^n$  and by the definition of boundary function in polydiscs [22] it is easy to see that the boundary condition (4.1) for the unit polydisc turns out to be

$$Re \left[ \sum_{j=1}^n \zeta_j \frac{\partial u}{\partial \zeta_j} + \alpha(\zeta)u \right] = \gamma(\zeta), \quad \zeta \in \partial_0 \mathbb{D}^n \quad (4.2)$$

with  $\gamma(\zeta) = \sqrt{n} \gamma_0(\zeta)$ ,  $\alpha(\zeta) = \sqrt{n} \alpha_0(\zeta)$ .

### 4.1.2 The problem

Since the left-hand side represents the real part of boundary values of a holomorphic function in  $\mathbb{D}^n$ , the right hand-side has to satisfy

$$\sum_{\nu=1}^n \sum_{\lambda=0}^{\nu-1} \sum_{\substack{1 \leq k_1 < \dots < k_\lambda \leq n \\ 1 \leq k_{\lambda+1} < \dots < k_\nu \leq n}} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{\bar{z}_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \frac{d\zeta}{\zeta} = 0, \quad z \in \mathbb{D}^n \quad (4.3)$$

see Lemma 2 in Section 2.2.1. Then by the Cauchy integral for equation (4.2) we get

$$\sum_{\ell=1}^n z_\ell \frac{\partial u(z)}{\partial z_\ell} + \alpha(z)u(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \left[ 2 \frac{1}{1 - z/\zeta} - 1 \right] \frac{d\zeta}{\zeta} + iC_0, \quad z \in \mathbb{D}^n, \quad (4.4)$$

where  $C_0$  is an arbitrary real constant. Let us denote

$$\frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \left[ 2 \frac{1}{1 - z/\zeta} - 1 \right] \frac{d\zeta}{\zeta} + iC_0 =: G_0(z) + iC_0, \quad z \in \mathbb{D}^n.$$

Then equation (4.4) becomes

$$\sum_{\ell=1}^n z_\ell \frac{\partial u(z)}{\partial z_\ell} + \alpha(z)u(z) = iC_0 + G_0(z), \quad z \in \mathbb{D}^n. \quad (4.5)$$

Clearly problem (4.5) is equivalent to problem (4.1). What we have to do is to determine the holomorphic function  $u$  and the real constant  $C_0$ .

Let the function  $\alpha$  be decomposed as

$$\alpha(z) = \alpha_0 + \sum_{i=1}^n z_i \alpha_i(z)$$

where  $\alpha_0$  is a complex constant and  $\alpha_i(z)$  is holomorphic in  $\mathbb{D}^n$ .

By the transformation

$$v(z) = u(z)e^{F(z)}, \quad \frac{\partial F(z)}{\partial z_k} = \alpha_k(z), \quad 1 \leq k \leq n,$$

where the function  $F$  has to be determined, problem (4.5) becomes

$$\sum_{\ell=1}^n z_\ell \frac{\partial v(z)}{\partial z_\ell} + \alpha_0 v(z) = (iC_0 + G_0(z))e^{F(z)}, \quad z \in \mathbb{D}^n. \quad (4.6)$$

In order to find the function  $F$  we have to solve the following system

$$F_{z_k} = \alpha_k(z), \quad 1 \leq k \leq n, \quad (4.7)$$

with the compatibility conditions

$$\frac{\partial \alpha_k}{\partial z_\ell} = \frac{\partial \alpha_\ell}{\partial z_k}, \quad 1 \leq k, \ell \leq n. \quad (4.8)$$

Suppose that the function  $F$  is holomorphic in  $\mathbb{D}^n$ . Then for  $n = 2$  we have

$$\begin{aligned} F(z) &= \sum_{|\kappa| \geq 0} \frac{z^\kappa}{\kappa!} \frac{\partial^\kappa F(0)}{\partial z^\kappa} = F(0) + \sum_{k_1=1}^{\infty} \frac{z_1^{k_1}}{k_1!} \frac{\partial^{k_1} F(0)}{\partial z^{k_1}} \\ &+ \sum_{k_2=1}^{\infty} \frac{z_2^{k_2}}{k_2!} \frac{\partial^{k_2} F(0)}{\partial z^{k_2}} + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \frac{z_1^{k_1} z_2^{k_2}}{k_1! k_2!} \frac{\partial^{k_1+k_2} F(0)}{\partial z^{k_1} \partial z^{k_2}} . \end{aligned}$$

Applying equation (4.7) we have a special solution to (4.7) and (4.8) as follows.

$$\begin{aligned} F_0(z) &= \sum_{k_1=0}^{\infty} \frac{z_1^{k_1+1}}{(k_1+1)!} \frac{\partial^{k_1} \alpha_1(0)}{\partial z^{k_1}} + \sum_{k_2=0}^{\infty} \frac{z_2^{k_2+1}}{(k_2+1)!} \frac{\partial^{k_2} \alpha_2(0)}{\partial z^{k_2}} \\ &+ \sum_{k_1=0}^{\infty} \sum_{k_2=1}^{\infty} \frac{z_1^{k_1+1} z_2^{k_2}}{(k_1+1)! k_2!} \frac{\partial^{k_1+k_2} \alpha_1(0)}{\partial z_1^{k_1} \partial z_2^{k_2}} \\ &= \int_0^{z_1} \alpha_1(\zeta_1, 0) d\zeta_1 + \int_0^{z_2} \alpha_2(0, \zeta_2) d\zeta_2 + \int_0^z \frac{\partial \alpha_1(\zeta)}{\partial \zeta_2} d\zeta \\ &= \int_0^{z_2} \alpha_2(0, \zeta_2) d\zeta_2 + \int_0^{z_1} \alpha_1(\zeta_1, z_2) d\zeta_1 . \end{aligned}$$

By the same way for  $n = 3$  we have

$$F_0(z) = \int_0^{z_3} \alpha_3(0, 0, \zeta_3) d\zeta_3 + \int_0^{z_2} \alpha_2(0, \zeta_2, z_3) d\zeta_2 + \int_0^{z_1} \alpha_1(\zeta_1, z_2, z_3) d\zeta_1 .$$

So for general  $n$  it is easy to prove that

$$F_0(z) = \sum_{k=1}^n \int_0^{z_k} \alpha_k(0, \dots, 0, \zeta_k, z_{k+1}, \dots, z_n) d\zeta_k , \quad z \in \mathbb{D}^n \quad (4.9)$$

is a special solution to equation (4.7) with compatibility condition (4.8).

### 4.1.3 The homogeneous problem

We suppose  $\gamma(\zeta) = 0$ . Then we consider problem (4.6) with  $G_0(z) = 0$ .

**Lemma 9** *Let  $v(z)$  be holomorphic in  $\mathbb{D}^n$  and satisfy*

$$\sum_{\ell=1}^n z_\ell \frac{\partial v(z)}{\partial z_\ell} + \alpha_0 v(z) = 0 , \quad z \in \partial_0 \mathbb{D}^n . \quad (4.10)$$

*If  $v(z)$  is single-valued and is not identically zero then  $\alpha_0$  must be an integer and must be non positive . If  $\alpha_0 > 0$  then  $v(z)$  must be identically zero.*

**Proof** Let  $v(z)$  be holomorphic in  $\Omega = \mathbb{D}^n$  and satisfy equation (4.10). Applying Green's theorem we get

$$\begin{aligned} \sum_{j=1}^n \int_{\Omega} \left( |v_{\bar{z}_j}|^2 + |v_{z_j}|^2 \right) d\Omega_z &= \sum_{j=1}^n \int_{\Omega} \left[ \frac{\partial}{\partial \bar{z}_j} (v \bar{v}_{z_j}) + \frac{\partial}{\partial z_j} (v \bar{v}_{\bar{z}_j}) \right] d\Omega_z \\ &= \sum_{j=1}^n \int_{\partial\Omega} v(\zeta) \left[ \overline{\left( \frac{\partial v(\zeta)}{\partial \bar{\zeta}_j} \right)} \rho_{\bar{\zeta}_j}(\zeta) + \overline{\left( \frac{\partial v(\zeta)}{\partial \zeta_j} \right)} \rho_{\zeta_j}(\zeta) \right] \frac{ds_{\zeta}}{|\text{grad}\rho(\zeta)|} \\ &= \sum_{j=1}^n \int_{\partial\Omega} v(\zeta) \left[ \overline{\left( \frac{\partial v(\zeta)}{\partial \zeta_j} \right)} \rho_{\zeta_j}(\zeta) \right] \frac{ds_{\zeta}}{|\text{grad}\rho(\zeta)|} \\ &= \int_{\partial_0 \mathbb{D}^n} v(\zeta) \left[ \sum_{j=1}^n \zeta_j \frac{\partial v(\zeta)}{\partial \zeta_j} \right] \frac{ds_{\zeta}}{\sqrt{n}} = -\frac{\bar{\alpha}_0}{\sqrt{n}} \int_{\partial_0 \mathbb{D}^n} |v(\zeta)|^2 ds_{\zeta} \end{aligned}$$

This means that if  $v(z)$  is not identically zero then  $\alpha_0$  must be non positive. Clearly if  $\alpha_0 > 0$  then  $v(z)$  must be identically zero.

Suppose that  $\alpha_0$  is not a negative integer and the holomorphic function  $v(z)$  which satisfies (4.10) has the form

$$v(z) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \gamma_{k_1 \dots k_n} z_1^{k_1} \cdots z_n^{k_n} =: \sum_{|\kappa| \geq 0} \gamma_{\kappa} z^{\kappa}, \quad |\kappa| = \sum_{j=1}^n k_j.$$

Then by equation (4.10) we have

$$\sum_{|\kappa| \geq 0} \gamma_{\kappa} (|\kappa| + \alpha_0) z^{\kappa} = 0, \quad z \in \partial_0 \mathbb{D}^n.$$

Thus if  $v(z)$  is single-valued and not identically zero on  $\partial_0 \mathbb{D}^n$ , then for some non negative integer  $|\kappa|$  it holds that  $\alpha_0 + |\kappa| = 0$ , i.e.,  $\alpha_0$  is a non positive integer.

**Lemma 10** *Let  $\alpha_0$  be a non-positive integer and  $v(z)$  satisfy (4.10), then*

$$v(z) = \sum_{|\kappa| = -\alpha_0} C_{\kappa} z^{\kappa}$$

where  $C_{\kappa}$  and  $|\kappa| = -\alpha_0$  are arbitrary complex constants.

**Lemma 11** *If  $\alpha_0 + |\kappa| \neq 0$  for all  $|\kappa| \in \mathbb{Z}^+$ , then*

$$v(z) = \sum_{|\kappa| \geq 0} \frac{i C_0 f_{\kappa}}{\kappa + \alpha_0} z^{\kappa}$$

is a solution of

$$\sum_{\ell=1}^n z_{\ell} \frac{\partial v(z)}{\partial z_{\ell}} + \alpha_0 v(z) = i C_0 e^{F_0(z)}, \quad z \in \partial_0 \mathbb{D}^n, \quad (4.11)$$

where

$$e^{F_0(z)} = \sum_{|\kappa| \geq 0} f_{\kappa} z^{\kappa} \quad (4.12)$$

and  $F_0(z)$  is defined as in (4.9).

The proof is trivial.

**Lemma 12** *Let  $-\alpha_0 = k_0 \in \mathbb{Z}^+$ . Assume that  $v$  is holomorphic in  $\mathbb{D}^n$  satisfying*

$$\sum_{\ell=1}^n z_\ell \frac{\partial v(z)}{\partial z_\ell} + \alpha_0 v(z) = \sum_{|\kappa|=k_0} C_\kappa^* z^\kappa$$

where  $C_\kappa^*$ ,  $|\kappa| = k_0$ , are arbitrary complex constants. Then  $C_\kappa^* = 0$ ,  $|\kappa| = k_0$ .

**Proof** Let

$$v(z) = \sum_{|\kappa| \geq 0} \gamma_\kappa z^\kappa.$$

From the assumption we have

$$\sum_{\ell=1}^n z_\ell \frac{\partial v(z)}{\partial z_\ell} + \alpha_0 v(z) = \sum_{|\kappa| \geq 0} (|\kappa| \gamma_\kappa - k_0 \gamma_\kappa) z^\kappa = \sum_{|\kappa|=k_0} C_\kappa^* z^\kappa.$$

Thus  $C_\kappa^* = 0$ ,  $|\kappa| = k_0$ .

From Lemmas 9-12 we obtain the following result.

**Theorem 14** *Let  $\alpha(z)$  be the boundary of a holomorphic function in  $\mathbb{D}^n$  and*

$$\alpha(z) = \alpha_0 + \sum_{\ell=1}^n z_\ell \alpha_\ell, \quad \frac{\partial \alpha_k}{\partial z_\ell} = \frac{\partial \alpha_\ell}{\partial z_k}, \quad 1 \leq k, \ell \leq n.$$

Then we have

1) *If  $\alpha_0 + k \neq 0$  for all  $k \in \mathbb{Z}^+$ , the homogeneous problem (4.1) has a nontrivial solution*

$$u(z) = iC_0 e^{-F_0(z)} \sum_{|\kappa| \geq 0} \frac{f_\kappa}{|\kappa| + \alpha_0} z^\kappa,$$

where  $F_0(z)$  is defined as in (4.9) and  $C_0$  is an arbitrary real constant.

2) *If  $-\alpha_0 \in \mathbb{Z}^+$  then*

a) *when  $f_\kappa|_{|\kappa|=-\alpha_0} \neq 0$ , then the homogeneous problem (4.1) has the solution*

$$u(z) = e^{-F_0(z)} \sum_{|\kappa|=-\alpha_0} C_\kappa z^\kappa$$

where  $C_\kappa$ ,  $|\kappa| = -\alpha_0$ , are arbitrary complex constants.

b) *when  $f_\kappa|_{|\kappa|=-\alpha_0} = 0$ , then*

$$u(z) = e^{-F_0(z)} \left[ \sum_{|\kappa|=-\alpha_0} C_\kappa z^\kappa + iC_0 \left( \sum_{\substack{|\kappa| \geq 0 \\ (|\kappa| + \alpha_0 \neq 0)}} \frac{f_\kappa}{|\kappa| + \alpha_0} z^\kappa \right) \right]$$

where  $C_0$  is an arbitrary real constant and  $C_\kappa$ ,  $|\kappa| = -\alpha_0$ , are arbitrary complex constants.

**Remark:**

1) Under the assumption on  $\alpha$  from the definition of  $\{f_\kappa\}$  in (4.12) the series appearing in Theorem 14 are seen to converge.

An example for the values of  $f_\kappa$  :

If  $\alpha_k(z) = 1$  , then

$$e^{F_0(z)} = e^{z_1 + \dots + z_n} = f^{(1)}(z_1) \dots f^{(n)}(z_n) , \quad f^{(i)}(z_i) = e^{z_i} ,$$

$$f_\kappa = f_{k_1}^{(1)} \dots f_{k_n}^{(n)} , \quad \kappa = (k_1, \dots, k_n) , \quad f_{k_i}^{(i)} = \frac{1}{k_i!} .$$

If  $\alpha_k(z) = 2z_k$  , then

$$e^{F_0(z)} = e^{z_1^2 + \dots + z_n^2} = f^{(1)}(z_1) \dots f^{(n)}(z_n) , \quad f^{(i)}(z_i) = e^{z_i^2} ,$$

$$f_\kappa = f_{k_1}^{(1)} \dots f_{k_n}^{(n)} , \quad \kappa = (k_1, \dots, k_n) , \quad f_{k_i}^{(i)} = \frac{1}{(2t_i)!} , \quad k_i = 2t_i , t_i \in \mathbb{Z}^+ .$$

But terms  $f_\kappa$  for odd indices like  $k_i = 2t_i + 1$  ,  $t_i \in \mathbb{Z}^+$  do not appear, i.e.,  $f_{k_i}^{(i)} = 0$  for  $k_i = 2t_i + 1$  ,  $t_i \in \mathbb{Z}^+$  .

2) The conditions  $f_\kappa \Big|_{|\kappa|=-\alpha_0} = 0$  , are a finite number of conditions .

**4.1.4 The inhomogeneous problem**

Let

$$G_0(z)e^{F_0(z)} = \sum_{|\kappa| \geq 0} G_\kappa z^\kappa .$$

By finding a special solution of equation (4.6) we have :

**Theorem 15** *Suppose that the condition (4.3) is satisfied. Let  $\alpha(z)$  be the boundary value of a holomorphic function and*

$$\alpha(z) = \alpha_0 + \sum_{\ell=1}^n z_\ell \alpha_\ell , \quad \frac{\partial \alpha_k}{\partial z_\ell} = \frac{\partial \alpha_\ell}{\partial z_k} , \quad 1 \leq k, \ell \leq n .$$

1) when  $\alpha_0 + k \neq 0$  for all  $k \in \mathbb{Z}^+$ , problem (4.1) is solvable and its solutions are

$$u(z) = e^{-F_0(z)} \sum_{|\kappa| \geq 0} \frac{G_\kappa + iC_0 f_\kappa}{|\kappa| + \alpha_0} z^\kappa$$

where  $C_0$  is an arbitrary real number.

2) when  $\alpha_0 + k_0 = 0$  for some  $k_0 \in \mathbb{Z}^+$ , problem (4.1) is solvable if and only if there is a real constant  $C_0^*$  such that

$$\left( G_\kappa + iC_0^* f_\kappa \right) \Big|_{|\kappa|=k_0} = 0 , \quad \kappa = (k_1, \dots, k_n) . \quad (4.13)$$

If this condition is satisfied, then the solution can be given by

$$u(z) = e^{-F_0(z)} \left[ \sum_{|\kappa|=k_0} C_\kappa z^\kappa + \sum_{\substack{|\kappa| \geq 0 \\ (|\kappa| + \alpha_0 \neq 0)}} \frac{G_\kappa + iC_0^* f_\kappa}{|\kappa| + \alpha_0} z^\kappa \right]$$

where  $C_\kappa$ ,  $|\kappa| = k_0$ , are arbitrary complex numbers and  $F_0(z)$  is defined by (4.9).

**Remark:**

1. Clearly (4.13) are finitely many conditions.
2. The interesting case  $\alpha(z) = P(\bar{z})$  with  $\deg(P) = m$  could be considered in a similar way.

## 4.2 Anti-polynomial with non-integer free term coefficient

### 4.2.1 Introduction

Recently some results on holomorphic functions have been achieved both in the unit disc [1] and in polydiscs [24] and in the unit balls [2]. The work [1] has obtained quite an impressive result reducing the Riemann-Hilbert-Poincare problem for holomorphic functions to a Fuchsian type differential equation. However their anti-polynomial case was too special. In this sense further study is necessary. In this paper we consider general anti-polynomial case which in [1] remained open. This problem plays a significant role in solving the third boundary value problem for the pluriharmonic system in polydiscs. Also its Hele-Shaw applications is a motivating aspect of the problem, see [26].

Let  $\mathcal{D}$  be the unit disc  $\{z \mid z \in \mathcal{C}, |z| < 1\}$  and  $\partial\mathcal{D}$  its boundary  $\{z \mid z \in \mathcal{C}, |z| = 1\}$ . We consider the Riemann-Hilbert-Poincare problem :

Find a function  $u(z)$ , which is holomorphic in  $\mathcal{D}$  and satisfies the boundary condition

$$Re \left[ \frac{\partial u}{\partial \nu_\zeta} + \alpha(\zeta)u \right] = \gamma(\zeta), \quad \zeta \in \partial\mathcal{D}, \quad (4.14)$$

where  $\partial u / \partial \nu_\zeta$  denotes the outward normal derivative of  $u(z)$  at the point  $\zeta \in \partial\mathcal{D}$  and  $\gamma$ ,  $\alpha$  are given Hölder continuous functions on  $\partial\mathcal{D}$ .

The problem (4.14) is open for general coefficients. In some cases it is generally assumed that the norm of  $\alpha(z)$  is small enough, see [8] and [30]. However smallness of  $|\alpha(z)|$  does not tell if  $\alpha(z)$  is analytic or anti-analytic or even a mixture of these. Depending on these cases the number of solutions can be very different. Only for special coefficients, e.g., for  $\alpha(z) = a\bar{z}^k + b$ , problem (4.14) has been considered [1] and it is shown that the number of solutions depends on the coefficient  $\alpha(z)$ .

In this paper we consider a case of  $\alpha(z)$  being a general anti-polynomial for problem (4.14) and we show that even in this case the number of solutions still depends on the coefficient  $\alpha(z)$

but not on the smallness of  $|\alpha(z)|$  and not on the smoothness of  $\alpha(z)$ . It is interesting that solving this simple looking problem gives an impression that a simple looking problem not always has a simple answer.

For a holomorphic function  $u$  it is clear that

$$\frac{\partial u}{\partial \nu_\zeta} = \zeta \frac{du}{d\zeta} \quad \text{on } \partial \mathbb{D}$$

Thus the problem (4.14) for the unit disc turns out to be

$$\operatorname{Re} \left[ \zeta \frac{du}{d\zeta} + \alpha(\zeta)u \right] = \gamma(\zeta), \quad \zeta \in \partial \mathbb{D}. \quad (4.15)$$

From (4.15) we obtain an inhomogeneous Fuchsian type ordinary differential equation in the complex domain. We need to determine the holomorphic solution  $u(z)$  and some constants from the Riemann-Hilbert-Poincaré problem at the same time. Solvability conditions are also required to be taken into account. Therefore the main purpose of the paper remains on the exact number of solutions and the solvability conditions as these are the major originators for difficulties.

Applying the Fourier method we are able to calculate the exact number of solutions and the solvability conditions for problem (4.14). Seemingly the Fourier method is very compromising for both one- and several dimensional problems (in higher dimensional space at least for problems in polydiscs), see [19].

## 4.2.2 The problem

Problem (4.15) is a Schwarz problem. Thus a solution for (4.15) is given by the Cauchy integral

$$zu'(z) + \alpha(z)u(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \gamma(\zeta) \left[ 2 \frac{1}{1 - z/\zeta} - 1 \right] \frac{d\zeta}{\zeta} + i\gamma_0, \quad z \in \mathbb{D}, \quad (4.16)$$

where  $\gamma_0$  is an arbitrary real constant. Let us denote

$$\frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \left[ 2 \frac{1}{1 - z/\zeta} - 1 \right] \frac{d\zeta}{\zeta} + i\gamma_0 =: G(z) = \sum_{k=0}^{\infty} g_k z^k, \quad z \in \mathbb{D}^n.$$

Then equation (4.16) becomes

$$zu'(z) + \alpha(z)u(z) = G(z), \quad z \in \mathbb{D}. \quad (4.17)$$

Clearly problem (4.17) is equivalent to problem (4.14). What we have to do is to determine the holomorphic function  $u$  and the real constant  $\gamma_0$ .

We assume that

$$\alpha(\zeta) = \sum_{i=0}^m \alpha_i \bar{\zeta}^i = \sum_{i=0}^m \frac{\alpha_i}{\zeta^i}, \quad \zeta \in \partial \mathbb{D},$$

where  $\alpha_i$ ,  $i = 0, 1, \dots, m$ , are complex constants.



From (4.17) and (4.14) we have

$$\operatorname{Re}[zu'(z) + \alpha(z)u(z)] = \operatorname{Re}G(z) \quad \text{on} \quad \partial\mathbb{D} . \quad (4.18)$$

Since for the function  $u$  holomorphic in  $\mathbb{D}$ , the function  $zu'(z) + \alpha(z)u(z)$  is holomorphic in  $\mathbb{D} \setminus \{0\}$  and has a pole of order at most  $m$  at 0. Therefore from (4.18) we have, see [29]

$$zu' + \left(\alpha_0 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2} + \cdots + \frac{\alpha_m}{z^m}\right)u = \sum_{k=1}^m \left(\frac{a_k}{z^k} - \bar{a}_k z^k\right) + ic_0 + G(z) , \quad z \in \mathbb{D} , \quad (4.19)$$

where  $a_i$ ,  $i = 1, \dots, m$ , are arbitrary complex constants and  $c_0$  is an arbitrary real number.

We are going to find the holomorphic function  $u$  and the coefficients  $a_1, \dots, a_m$ .

We consider now the case that  $G(z)$  is a polynomial with  $\deg(G) \leq m - 1$ .

### 4.2.3 The inhomogeneous problem with $\deg(G) \leq m - 1$

Let

$$u(z) = \sum_{k=0}^{\infty} u_k z^k , \quad z \in \mathbb{D} \quad \text{and} \quad \sum_{k=0}^{\infty} |u_k| < \infty .$$

Then from (4.19) we have

$$\sum_{k=1}^{\infty} k u_k z^k + \left(\alpha_0 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2} + \cdots + \frac{\alpha_m}{z^m}\right) \sum_{k=0}^{\infty} u_k z^k = \sum_{k=1}^m \left(\frac{a_k}{z^k} - \bar{a}_k z^k\right) + ic_0 + \sum_{k=0}^{m-1} g_k z^k$$

or equivalently

$$\begin{aligned} & \sum_{k=1}^{\infty} k u_k z^{k+m} + \left(\alpha_0 z^m + \alpha_1 z^{m-1} + \cdots + \alpha_{m-1} z^1 + \alpha_m\right) \sum_{k=0}^{\infty} u_k z^k \\ &= \sum_{k=1}^m \left(a_k z^{m-k} - \bar{a}_k z^{k+m}\right) + ic_0 z^m + \sum_{k=0}^{m-1} g_k z^{k+m} , \quad z \in \mathbb{D} . \end{aligned} \quad (4.20)$$

Thus by comparing the coefficients of  $z^k$  on both sides we get a system of algebraic equations

$$\alpha_m u_0 = a_m \quad \text{for} \quad z^0 , \quad (4.21)$$

$$\sum_{t=k+1}^m \alpha_t u_{t-k} = a_k - \alpha_k u_0 \quad \text{for} \quad z^{m-k} , \quad k = m-1, \dots, 2, 1, \quad (4.22)$$

$$\alpha_1 u_1 + \cdots + \alpha_{m-1} u_{m-1} + \alpha_m u_m = ic_0 + g_0 - \alpha_0 u_0 \quad \text{for} \quad z^m , \quad (4.23)$$

$$\begin{aligned} & (k - m + \alpha_0) u_{k-m} + \alpha_1 u_{k-m+1} + \alpha_2 u_{k-m+2} + \cdots + \alpha_{m-1} u_{k-1} + \alpha_m u_k \\ &= g_{k-m} - \bar{a}_{k-m} , \quad \text{for} \quad z^k , \quad k = m+1, \dots, 2m-1, \end{aligned} \quad (4.24)$$

$$(\alpha_0 + m) u_m + \alpha_1 u_{m+1} + \cdots + \alpha_{m-1} u_{2m-1} + \alpha_m u_{2m} = -\bar{a}_m \quad \text{for} \quad z^{2m} , \quad (4.25)$$

$$(k + \alpha_0) u_k + \alpha_1 u_{k+1} + \alpha_2 u_{k+2} + \cdots + \alpha_{m-1} u_{m+k-1} + \alpha_m u_{m+k} = 0$$

$$\text{for } z^k, \quad k \geq m + 1. \quad (4.26)$$

For convenience we denote  $ic_0 + g_0 =: a_0$ . Equation (4.21)-(4.23) can be written in matrix form as follows.

$$\begin{pmatrix} \alpha_m & 0 & 0 & \cdots & 0 & 0 \\ \alpha_{m-1} & \alpha_m & 0 & \cdots & 0 & 0 \\ \alpha_{m-2} & \alpha_{m-1} & \alpha_m & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_m & 0 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{m-1} & \alpha_m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{m-1} \\ u_m \end{pmatrix} = \begin{pmatrix} a_{m-1} \\ a_{m-2} \\ a_{m-3} \\ \vdots \\ a_1 \\ a_0 \end{pmatrix} - u_0 \begin{pmatrix} \alpha_{m-1} \\ \alpha_{m-2} \\ \alpha_{m-3} \\ \vdots \\ \alpha_1 \\ \alpha_0 \end{pmatrix} \quad (4.27)$$

If we denote the left-side  $m \times m$  matrix as  $A(m, 1)$  then its determinant  $|A(m, 1)| = \alpha_m^m$ . If  $\alpha_m^m \neq 0$  then  $A^{-1}(m, 1)$  exists. Suppose, without loss of generality,

$$A^{-1}(m, 1) = \begin{pmatrix} \beta_{11} & 0 & 0 & \cdots & 0 & 0 \\ \beta_{21} & \beta_{22} & 0 & \cdots & 0 & 0 \\ \beta_{31} & \beta_{32} & \beta_{33} & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \beta_{(m-1)1} & \beta_{(m-1)2} & \beta_{(m-1)3} & \cdots & \beta_{(m-1)(m-1)} & 0 \\ \beta_{m1} & \beta_{m2} & \beta_{m3} & \cdots & \beta_{m(m-1)} & \beta_{mm} \end{pmatrix}.$$

Interestingly

$$\beta_{(i+k)i} = (-1)^k \alpha_m^{-k-1} \begin{vmatrix} \alpha_{m-1} & \alpha_m & 0 & \cdots & 0 & 0 & 0 \\ \alpha_{m-2} & \alpha_{m-1} & \alpha_m & \cdots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_{m-k+2} & \alpha_{m-k+3} & \alpha_{m-k+4} & \cdots & \alpha_{m-1} & \alpha_m & 0 \\ \alpha_{m-k+1} & \alpha_{m-k+2} & \alpha_{m-k+3} & \cdots & \alpha_{m-2} & \alpha_{m-1} & \alpha_m \\ \alpha_{m-k} & \alpha_{m-k+1} & \alpha_{m-k+2} & \cdots & \alpha_{m-3} & \alpha_{m-2} & \alpha_{m-1} \end{vmatrix},$$

$$k = 2, \dots, m-1, \quad i = 1, 2, \dots, m-k, \quad \beta_{(i+1)i} = -\alpha_{m-1} \alpha_m^{-2}; \quad \beta_{jj} = \alpha_m^{-1}, \quad j = 1, \dots, m.$$

If  $\alpha_m^{m+1} \neq 0$  then for equation (4.27) we get the following solution.

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{m-1} \\ u_m \end{pmatrix} = A^{-1}(m, 1) \left( \begin{pmatrix} a_{m-1} \\ a_{m-2} \\ a_{m-3} \\ \vdots \\ a_1 \\ a_0 \end{pmatrix} - u_0 \begin{pmatrix} \alpha_{m-1} \\ \alpha_{m-2} \\ \alpha_{m-3} \\ \vdots \\ \alpha_1 \\ \alpha_0 \end{pmatrix} \right)$$

or

$$U(1, m) = A^{-1}(m, 1) [a(m-1, 0) - u_0 \alpha(m-1, 0)] \quad (4.28)$$

For equation (4.23)-(4.25) we have

$$\begin{pmatrix} g_1 - \bar{a}_1 \\ g_2 - \bar{a}_2 \\ g_3 - \bar{a}_3 \\ \dots \\ g_{m-1} - \bar{a}_{m-1} \\ -\bar{a}_m \end{pmatrix} = \begin{pmatrix} \alpha_0 + 1 & \alpha_1 & \alpha_2 & \dots & \alpha_{m-2} & \alpha_{m-1} \\ 0 & \alpha_0 + 2 & \alpha_1 & \dots & \alpha_{m-3} & \alpha_{m-2} \\ 0 & 0 & \alpha_0 + 3 & \dots & \alpha_{m-4} & \alpha_{m-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha_0 + m - 1 & \alpha_1 \\ 0 & 0 & 0 & \dots & 0 & \alpha_0 + m \end{pmatrix} \times \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{m-1} \\ u_m \end{pmatrix} = \begin{pmatrix} \alpha_m & 0 & 0 & \dots & 0 & 0 \\ \alpha_{m-1} & \alpha_m & 0 & \dots & 0 & 0 \\ \alpha_{m-2} & \alpha_{m-1} & \alpha_m & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_2 & \alpha_3 & \alpha_4 & \dots & \alpha_m & 0 \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{m-1} & \alpha_m \end{pmatrix} \begin{pmatrix} u_{m+1} \\ u_{m+2} \\ u_{m+3} \\ \vdots \\ u_{2m-1} \\ u_{2m} \end{pmatrix} \quad (4.29)$$

or briefly

$$G(1, m) - M(1, m)U(1, m) = A(m, 1)U(m + 1, 2m) . \quad (4.30)$$

Equation (4.26) is equivalent to the matrix equation

$$A(m, 1)U((\ell + 1)m + 1, (\ell + 2)m) = -M(\ell m + 1, (\ell + 1)m)U(\ell m + 1, (\ell + 1)m) , \ell \geq 1. \quad (4.31)$$

Denote  $C^\ell =: A^{-1}(m, 1)M(\ell m + 1, (\ell + 1)m) = (C_{ij}^\ell)$  ,  $\ell \geq 1$  . Then for equation (4.31) we get the solution

$$\begin{aligned} U((\ell + 1)m + 1, (\ell + 2)m) &= -C^\ell U(\ell m + 1, (\ell + 1)m) \\ &= \left[ (-1)^\ell \prod_{h=1}^{\ell} (C^{\ell-h+1}) \right] U(m + 1, 2m) , \ell \geq 1 . \end{aligned} \quad (4.32)$$

Now we look at the elements of  $C^\ell = (C_{ij}^\ell)$  in detail. It is easy to see that

$$\begin{aligned} C_{ij}^\ell &= \beta_{i1}\alpha_{j-1} + \beta_{i2}\alpha_{j-2} + \dots + \beta_{i(j-1)}\alpha_1 + \beta_{ij}(\alpha_0 + \ell m + j) \\ &= \sum_{t=1}^{j-1} \beta_{it}\alpha_{j-t} + \beta_{ij}(\alpha_0 + \ell m + j) , \text{ for } j \leq i , \\ C_{ij}^\ell &= \beta_{i1}\alpha_{j-1} + \beta_{i2}\alpha_{j-2} + \dots + \beta_{ii}\alpha_{j-i} = \sum_{t=1}^i \beta_{it}\alpha_{j-t} , \text{ for } j > i . \end{aligned}$$

One can see that above the diagonal of the matrix  $C^\ell$  there is no  $C_{ij}^\ell$  which includes  $\alpha_0 + \ell m$  terms and that below and on the diagonal every  $C_{ij}^\ell$  includes only one  $\alpha_0 + \ell m + j$  term.

**Case:**  $\alpha_0 + k \neq 0$  for all  $k \geq m + 1$ .

We suppose  $\alpha_0 + k \neq 0$  for all  $k \geq m + 1$ . Denote  $C^\ell C^{\ell-1} =: C^{(\ell, \ell-1)} = (C_{ij}^{(\ell, \ell-1)})$ . We calculate some elements of  $C^{(\ell, \ell-1)}$  and look at the terms which includes second order terms of  $\ell$ .

$$C_{11}^{(\ell, \ell-1)} = \sum_{k=1}^m C_{1k}^\ell C_{k1}^{\ell-1} = \left[ \beta_{11}(\alpha_0 + \ell m + 1) + \sum_{k=1}^{m-1} \alpha_k \beta_{(k+1)1} \right] \beta_{11}(\alpha_0 + (\ell - 1)m + 1) .$$

Among  $C_{11}^{(\ell, \ell-1)}$ ,  $C_{12}^{(\ell, \ell-1)}$ ,  $\dots$ ,  $C_{1m}^{(\ell, \ell-1)}$  the only element which includes second order term of  $\ell$  is  $C_{11}^{(\ell, \ell-1)}$ . The others can only include first order  $\ell$  terms due to the fact that the upper triangle of the matrices has no any  $\ell$  term both in  $C^\ell$  and in  $C^{\ell-1}$ . So for  $C^{(\ell, \ell-1)}$  second order terms of  $\ell$  do not appear in the upper triangle. They occur only in the lower triangle including the diagonal. The second order term of  $C_{11}^{(\ell, \ell-1)}$  comes from  $C_{11}^\ell C_{11}^{\ell-1}$ . Thus it is easy to see that  $C_{11}^{(\ell, \ell-1, \dots, 2, 1)}$  includes its highest order term

$$C_{11}^\ell C_{11}^{\ell-1} \dots C_{11}^2 C_{11}^1 = \beta_{11}^\ell (\alpha_0 + \ell m + 1)(\alpha_0 + (\ell - 1)m + 1) \dots (\alpha_0 + 2m + 1)(\alpha_0 + m + 1)$$

and that if  $u_{m+1} \neq 0$  then

$$\begin{aligned} u_{\ell m+1} &= C_{11}^{(\ell, \ell-1, \dots, 2, 1)} u_{m+1} + C_{12}^{(\ell, \ell-1, \dots, 2, 1)} u_{m+2} + \dots + C_{1m}^{(\ell, \ell-1, \dots, 2, 1)} u_{2m} \\ &= C_{11}^{(\ell, \ell-1, \dots, 2, 1)} u_{m+1} \rightarrow \infty \quad \text{for } \ell \rightarrow \infty \quad \text{if } u_{m+1} \neq 0. \end{aligned}$$

This means  $u_{m+1} = 0$ .

$$\begin{aligned} C_{21}^{(\ell, \ell-1)} &= \sum_{k=1}^m C_{2k}^\ell C_{k1}^{\ell-1} = \left[ \beta_{21} \left( \beta_{11}(\alpha_0 + \ell m + 1) + \beta_{22}(\alpha_0 + \ell m + 2) \right) \right. \\ &\quad \left. + \beta_{21}^2 \alpha_1 + \sum_{k=3}^m (\beta_{21} \alpha_{k-1} + \beta_{22} \alpha_{k-2}) \beta_{31} \right] (\alpha_0 + (\ell - 1)m + 1) \\ C_{22}^{(\ell, \ell-1)} &= \sum_{k=1}^m C_{2k}^\ell C_{k2}^{\ell-1} = \beta_{22}^2 (\alpha_0 + \ell m + 2)(\alpha_0 + (\ell - 1)m + 2) \\ &\quad + \beta_{21} \alpha_1 \left[ \beta_{11}(\alpha_0 + \ell m + 1) + \beta_{22}(\alpha_0 + (\ell - 1)m + 2) + \beta_{22}(\alpha_0 + \ell m + 2) \right] \\ &\quad + \sum_{k=3}^m (\beta_{21} \alpha_{k-1} + \beta_{22} \alpha_{k-2}) (\beta_{k1} \alpha_1 + \beta_{k2} (\alpha_0 + (\ell - 1)m + 1)). \end{aligned}$$

In the element  $C_{23}^{(\ell, \ell-1)} = \sum_{k=1}^m C_{2k}^\ell C_{k3}^{\ell-1}$  the terms of  $\ell$  cannot meet each other and therefore here can only the first order terms of  $\ell$  occur. This is true also for  $C_{24}^{(\ell, \ell-1)}$ ,  $\dots$ ,  $C_{2m}^{(\ell, \ell-1)}$ . Thus again it is not difficult to see that  $C_{21}^{(\ell, \ell-1, \dots, 2, 1)}$  has the same growth-rate of  $\ell$  as  $C_{11}^{(\ell, \ell-1, \dots, 2, 1)}$  does. Now supposing  $u_{m+2} \neq 0$  and taking into account that  $u_{m+1} = 0$  we have

$$\begin{aligned} u_{\ell m+2} &= C_{21}^{(\ell, \ell-1, \dots, 2, 1)} u_{m+1} + C_{22}^{(\ell, \ell-1, \dots, 2, 1)} u_{m+2} + \dots + C_{2m}^{(\ell, \ell-1, \dots, 2, 1)} u_{2m} \\ &= C_{21}^{(\ell, \ell-1, \dots, 2, 1)} u_{m+1} + C_{22}^{(\ell, \ell-1, \dots, 2, 1)} u_{m+2} = C_{22}^{(\ell, \ell-1, \dots, 2, 1)} u_{m+2} \rightarrow \infty \quad \text{for } \ell \rightarrow \infty. \end{aligned}$$

This means  $u_{m+2} = 0$ . By a similar way we obtain  $u_{m+3} = 0$ ,  $\dots$ ,  $u_{2m} = 0$ . Clearly  $C^{(\ell, \ell-1, \dots, 1)}$  has diagonal dominance for large enough  $\ell$ .

**Lemma 13** *Suppose that*

$$\alpha_0 + k \neq 0 \quad \text{for all } k \geq m + 1. \quad (4.33)$$

*Then*

$$u_{m+k} = 0 \quad \text{for all } k \geq 1. \quad (4.34)$$

Now we discuss condition (4.30) in two cases.

By the necessary condition (4.34) and (4.29) we have

$$-\bar{a}_m - (\alpha_0 + m)u_m = 0 \tag{4.35}$$

and

$$\begin{aligned} & \begin{pmatrix} \alpha_0 + 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{m-3} & \alpha_{m-2} \\ 0 & \alpha_0 + 2 & \alpha_1 & \cdots & \alpha_{m-4} & \alpha_{m-3} \\ 0 & 0 & \alpha_0 + 3 & \cdots & \alpha_{m-5} & \alpha_{m-4} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & \alpha_0 + m - 2 & \alpha_1 \\ 0 & 0 & 0 & \cdots & 0 & \alpha_0 + m - 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m-2} \\ u_{m-1} \end{pmatrix} \\ &= \begin{pmatrix} g_1 - \bar{a}_1 \\ g_2 - \bar{a}_2 \\ \dots \\ g_{m-2} - \bar{a}_{m-2} \\ g_{m-1} - \bar{a}_{m-1} \end{pmatrix} - u_m \begin{pmatrix} \alpha_{m-1} \\ \alpha_{m-2} \\ \dots \\ \alpha_2 \\ \alpha_1 \end{pmatrix} \end{aligned}$$

or briefly

$$M(1, m - 1)U(1, m - 1) = g(1, m - 1) - \bar{a}(1, m - 1) - u_m\alpha(m - 1, 1) . \tag{4.36}$$

We transform equation (4.27) as

$$A(m, 2)U(1, m - 1) = a(m - 1, 1) - u_0\alpha(m - 1, 1) , \tag{4.37}$$

$$\alpha_m u_m = a_0 - \alpha_0 u_0 - (\alpha_1 u_1 + \cdots + \alpha_{m-1} u_{m-1}) . \tag{4.38}$$

**Case.I :**  $\alpha_0 + k \neq 0$  for all  $k \geq m + 1$  and  $\alpha_0 + m = 0$

If  $\alpha_0 + m = 0$  then from (4.35) it is clear that  $a_m = 0$  and therefore  $u_0 = 0$  . In this case, taking into account that  $u_0 = 0$  due to equation (4.21) and substituting  $u_m$  from (4.38) in equation (4.36) we have

$$\begin{aligned} & \left[ \begin{pmatrix} \alpha_0 + 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{m-3} & \alpha_{m-2} \\ 0 & \alpha_0 + 2 & \alpha_1 & \cdots & \alpha_{m-4} & \alpha_{m-3} \\ 0 & 0 & \alpha_0 + 3 & \cdots & \alpha_{m-5} & \alpha_{m-4} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & \alpha_0 + m - 2 & \alpha_1 \\ 0 & 0 & 0 & \cdots & 0 & \alpha_0 + m - 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m-2} \\ u_{m-1} \end{pmatrix} \right] \\ & - \tau_0 \begin{pmatrix} \alpha_1 \alpha_{m-1} & \alpha_2 \alpha_{m-1} & \cdots & \alpha_{m-2} \alpha_{m-1} & \alpha_{m-1} \alpha_{m-1} \\ \alpha_1 \alpha_{m-2} & \alpha_2 \alpha_{m-2} & \cdots & \alpha_{m-2} \alpha_{m-2} & \alpha_{m-1} \alpha_{m-2} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_1 \alpha_2 & \alpha_2 \alpha_2 & \cdots & \alpha_{m-2} \alpha_2 & \alpha_{m-1} \alpha_2 \\ \alpha_1 \alpha_1 & \alpha_2 \alpha_1 & \cdots & \alpha_{m-2} \alpha_1 & \alpha_{m-1} \alpha_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m-2} \\ u_{m-1} \end{pmatrix} \\ &= \begin{pmatrix} g_1 - \bar{a}_1 \\ g_2 - \bar{a}_2 \\ \dots \\ g_{m-2} - \bar{a}_{m-2} \\ g_{m-1} - \bar{a}_{m-1} \end{pmatrix} - a_0 \tau_0 \begin{pmatrix} \alpha_{m-1} \\ \alpha_{m-2} \\ \dots \\ \alpha_2 \\ \alpha_1 \end{pmatrix} , \quad \tau_0 = \frac{1}{\alpha_m} \end{aligned}$$

or in matrix form

$$M_{\tau_0}^0(1, m-1)U(1, m-1) = g(1, m-1) - \bar{a}(1, m-1) - a_0\tau_0\alpha(m-1, 1), \quad (4.39)$$

where

$$M_{\tau_0}^0(1, m-1) = M(1, m-1) - \frac{1}{\alpha_m}\alpha(m-1, 1)\alpha(1, m-1)^T.$$

Let us denote  $M_{\tau_0}^0(1, m-1)A^{-1}(m, 2) =: C^0(1, m-1)$ , where

$$C^0(1, m-1) = \begin{pmatrix} C_{11}^0 & C_{12}^0 & \cdots & C_{1(m-1)}^0 \\ C_{21}^0 & C_{22}^0 & \cdots & C_{2(m-1)}^0 \\ \dots & \dots & \dots & \dots \\ C_{(m-2)1}^0 & C_{(m-2)2}^0 & \cdots & C_{(m-2)(m-1)}^0 \\ C_{(m-1)1}^0 & C_{(m-1)2}^0 & \cdots & C_{(m-1)(m-1)}^0 \end{pmatrix}.$$

Substituting  $U(1, m-1)$  from (4.37) in equation (4.39) we have the equation

$$C^0(1, m-1)a(m-1, 1) + \bar{a}(1, m-1) = g(1, m-1) - a_0\tau_0\alpha(m-1, 1). \quad (4.40)$$

Rotating the stones  $180^\circ$  (for the matrices it is better two times  $90^\circ$  rotation in the same direction) and taking conjugate we get

$$\bar{C}^0(m-1, 1)\bar{a}(1, m-1) + a(m-1, 1) = \bar{g}(m-1, 1) - \bar{a}_0\bar{\tau}_0\bar{\alpha}(1, m-1) \quad (4.41)$$

where

$$C^0(m-1, 1) = \begin{pmatrix} C_{(m-1)(m-1)}^0 & C_{(m-1)(m-2)}^0 & \cdots & C_{(m-1)1}^0 \\ C_{(m-2)(m-1)}^0 & C_{(m-2)(m-2)}^0 & \cdots & C_{(m-2)1}^0 \\ \dots & \dots & \dots & \dots \\ C_{2(m-1)}^0 & C_{2(m-2)}^0 & \cdots & C_{21}^0 \\ C_{1(m-1)}^0 & C_{1(m-2)}^0 & \cdots & C_{11}^0 \end{pmatrix}. \quad (4.42)$$

Now we have a system of equations (4.40)-(4.41) and if

$$C^0(1, m-1)\bar{C}^0(m-1, 1) \neq I_{m-1} \quad (4.43)$$

then system (4.40) - (4.41) is uniquely solvable. If condition (4.43) is not satisfied, then system (4.40) - (4.41) is solvable if only if

$$g(1, m-1) - C^0(1, m-1)\bar{g}(m-1, 1) = a_0\tau_0\alpha(m-1, 1) - C^0(1, m-1)\bar{a}_0\bar{\tau}_0\bar{\alpha}(1, m-1). \quad (4.44)$$

We have got for the solvability of problem (4.14)

**Theorem 16** *Let  $\alpha_0 + k \neq 0$  for  $k \geq m+1$  and  $\alpha_0 + m = 0$ . If (4.43) is not satisfied, (4.44) holds. Then problem (4.14) is solvable.*

**Case.II:**  $\alpha_0 + k \neq 0$  for all  $k \geq m+1$  and  $\alpha_0 + m \neq 0$

If  $\alpha_0 + m \neq 0$  then from equation (4.21), (4.35) and (4.38) we have

$$\frac{\alpha_0}{\alpha_m}a_m - \frac{\alpha_m}{\alpha_0 + m}\bar{a}_m = a_0 - \alpha(1, m-1)^T U(1, m-1). \quad (4.45)$$

Substituting  $u_0$  from (4.21) into (4.37) we get

$$A(m, 2)U(1, m-1) = a(m-1, 1) - \frac{a_m}{\alpha_m} \alpha(m-1, 1). \quad (4.46)$$

Since  $A^{-1}(m, 2)$  exists  $U(1, m-1)$  can be uniquely represented in terms of  $a(m, 1)$  and  $\alpha(m, 1)$ . Substituting  $U(1, m-1)$  from (4.46) into (4.45) we have

$$\theta_1 a_m - \theta_2 \bar{a}_m = f \quad (4.47)$$

where

$$f = a_0 - \tau_2, \quad \theta_1 = \frac{\alpha_0 - \tau_1}{\alpha_m}, \quad \theta_2 = \frac{\alpha_m}{\alpha_0 + m}, \quad \tau_1 = c_1(1, m-1)^T \alpha(m-1, 1),$$

$$\tau_2 = c_1(1, m-1)^T a(m-1, 1), \quad c_1(1, m-1)^T = \alpha(m-1, 1)^T A^{-1}(m, 2).$$

Adding equation (4.47) to its conjugate we get

$$\operatorname{Re}[(\theta_1 - \bar{\theta}_2)a_m] = \operatorname{Re}[g_0 - \tau_2]. \quad (4.48)$$

If

$$\theta_1 = \bar{\theta}_2 \quad (4.49)$$

then

$$\operatorname{Re}[g_0] = \operatorname{Re}[\tau_2] \quad (4.50)$$

has to hold and  $a_m$  is arbitrary. The real constant  $c_0$  is determined by (4.45).

**Case.II.1:**  $\alpha_0 + m \neq 0$  and  $|\alpha_m|^2 \neq (\alpha_0 - \tau_1)(\bar{\alpha}_0 + m)$

If (4.49) does not hold, then equation (4.48) can be solved for  $a_m$  in terms of  $a(1, m-1)$  and  $\alpha(0, m)$ . Thus solving (4.47) for  $a_m$  we have

$$a_m = c^* [\bar{\theta}_1 f + \theta_2 \bar{f}], \quad c^* = \frac{1}{|\theta_1^2| - |\theta_2^2|}. \quad (4.51)$$

Substituting  $U(1, m-1)$  from (4.46) and  $u_m$  from (4.35) into (4.36) we have

$$C(1, m-1)a(m-1, 1) + \bar{a}(1, m-1)$$

$$= g(1, m-1) + \left[ C(1, m-1) \frac{a_m}{\alpha_m} - I_{m-1} \frac{\bar{a}_m}{\alpha_0 + m} \right] \alpha(m-1, 1), \quad (4.52)$$

where  $C(1, m-1) = M(1, m-1)A^{-1}(m, 2)$ .

Substituting  $a_m$  from (4.51) into (4.52) we have

$$C_1(1, m-1)a(m-1, 1) + C_2(1, m-1)I_{m-1}^* \bar{a}(m-1, 1)$$

$$= g(1, m-1) + \beta_1(m-1, 1), \quad (4.53)$$

where  $I_{m-1}^* = \operatorname{rot}(2/\pi)I_{m-1}$  and

$$\beta_1(m-1, 1) = \left[ \left( \frac{\bar{\theta}_1}{\alpha_m} C(1, m-1) - \frac{\bar{\theta}_2}{\alpha_0 + m} I_{m-1} \right) a_0 \right]$$

$$\begin{aligned}
& + \left( \frac{\theta_2}{\alpha_m} C(1, m-1) - \frac{\theta_1}{\alpha_0 + m} I_{m-1} \right) \bar{a}_0 \Big] c^* \alpha(m-1, 1), \\
C_1(1, m-1) &= C(1, m-1) + B_1(1, m-1), \quad C_2(1, m-1) = I_{m-1} + B_2(1, m-1), \\
B_1(1, m-1) &= \left[ \frac{\bar{\theta}_1 c^*}{\alpha_m} C(1, m-1) - \frac{\bar{\theta}_2 c^*}{\alpha_0 + m} I_{m-1} \right] \alpha(m-1, 1) c_1(1, m-1)^T, \\
B_2(1, m-1) &= \left[ \frac{\theta_2 c^*}{\alpha_m} C(1, m-1) - \frac{\theta_1 c^*}{\alpha_0 + m} I_{m-1} \right] \alpha(m-1, 1) \bar{c}_1(1, m-1)^T.
\end{aligned}$$

If  $|C_1(1, m-1)| \neq 0$  then equation (4.53) can be transformed as

$$a(m-1, 1) + C_1^*(1, m-1) \bar{a}(m-1, 1) = f_1$$

where

$$\begin{aligned}
f_1 &= C_1^{-1}(1, m-1) \left[ g(1, m-1) + \beta_1(m-1, 1) \right], \\
C_1^*(1, m-1) &= C_1^{-1}(1, m-1) C_2(1, m-1) I_{m-1}^*.
\end{aligned}$$

In this case equation (4.53) is uniquely solvable for  $a(m-1, 1)$  if

$$C_1^*(1, m-1) \bar{C}_1^*(1, m-1) \neq I_{m-1}. \quad (4.54)$$

Otherwise it is solvable if  $f_1 = C_1^*(1, m-1) \bar{f}_1$ , i.e.,

$$g(1, m-1) - C_1^\#(1, m-1) \bar{g}(1, m-1) = C_1^\#(1, m-1) \bar{\beta}_1(m-1, 1) - \beta_1(m-1, 1) \quad (4.55)$$

where  $C_1^\#(1, m-1) = C_2(1, m-1) I_{m-1}^* \bar{C}_1^{-1}(1, m-1)$ .

If  $|C_1(1, m-1)| = 0$  but  $|C_2(1, m-1) I_{m-1}^*| \neq 0$  then equation (4.53) can be transformed as

$$\bar{a}(m-1, 1) + C_2^*(1, m-1) a(m-1, 1) = f_2,$$

where

$$\begin{aligned}
f_2 &= \left( C_2(1, m-1) I_{m-1}^* \right)^{-1} \left[ g(1, m-1) + \beta_1(m-1, 1) \right], \\
C_2^*(1, m-1) &= \left( C_2(1, m-1) I_{m-1}^* \right)^{-1} C_1(1, m-1).
\end{aligned}$$

In this case equation (4.53) is uniquely solvable for  $a(m-1, 1)$  if

$$\bar{C}_2^*(1, m-1) C_2^*(1, m-1) \neq I_{m-1}. \quad (4.56)$$

Otherwise it is solvable if  $f_2 = C_2^*(1, m-1) \bar{f}_2$ , i.e.,

$$g(1, m-1) - C_2^\#(1, m-1) \bar{g}(1, m-1) = C_2^\#(1, m-1) \bar{\beta}_1(m-1, 1) - \beta_1(m-1, 1), \quad (4.57)$$

where  $C_2^\#(1, m-1) = C_1(1, m-1) \overline{\left( C_2(1, m-1) I_{m-1}^* \right)^{-1}}$ .

If  $|C_1(1, m-1)| = 0$  and  $|C_2(1, m-1)| = 0$  then equation (4.53) is solvable for  $a(m-1, 1)$  if

$$\text{rank}(C_1(1, m-1), C_2(1, m-1) I_{m-1}^*) = \text{rank}(C_1(1, m-1), C_2(1, m-1) I_{m-1}^*, f). \quad (4.58)$$

Thus for the solvability of problem (4.14) we have got



**Theorem 17** Let  $\alpha_0 + m \neq 0$  and  $|\alpha_m|^2 \neq (\alpha_0 - \tau_1)(\bar{\alpha}_0 + m)$ . If (4.1)-(4.4) are not satisfied, (4.58) holds. Then problem (4.14) is solvable.

**Case.II.2** :  $\alpha_0 + m \neq 0$  and  $|\alpha_m|^2 = (\alpha_0 - \tau_1)(\bar{\alpha}_0 + m)$

If condition (4.49) holds, then  $a_m$  is arbitrary and by (4.21) also  $u_0$  is arbitrary. In this case (4.52) becomes

$$C^*(1, m-1)a(1, m-1) + \bar{a}(1, m-1) = g(1, m-1) + \beta_2(m-1, 1) \quad (4.59)$$

where  $C^*(1, m-1) = C(1, m-1)I_{m-1}^*$  and

$$\beta_2(m-1, 1) = \left[ C(1, m-1) \frac{a_m}{\alpha_m} - I_{m-1} \frac{\bar{a}_m}{\alpha_0 + m} \right] \alpha(m-1, 1) .$$

Again conjugating (4.59) we get

$$\bar{C}^*(1, m-1)\bar{a}(1, m-1) + a(1, m-1) = \bar{g}(1, m-1) + \bar{\beta}_2(m-1, 1) \quad (4.60)$$

If we have

$$C^*(1, m-1)\bar{C}^*(m-1, 1) \neq I_{m-1} \quad (4.61)$$

then the system (4.59)-(4.60) is uniquely solvable.

If (4.61) is not satisfied, then it is solvable if only if

$$g(1, m-1) - C^*(1, m-1)\bar{g}(1, m-1) = C^*(1, m-1)\bar{\beta}_2(m-1, 1) - \beta_2(m-1, 1). \quad (4.62)$$

In this case we have for the solvability of problem (4.14)

**Theorem 18** Let  $\alpha_0 + k \neq 0$  for  $k \geq m+1$ ,  $\alpha_0 + m \neq 0$  and  $|\alpha_m|^2 = (\alpha_0 - \tau_1)(\bar{\alpha}_0 + m)$ . If (4.61) is not satisfied, (4.62) holds. Then problem (4.14) is solvable.

#### 4.2.4 Number of solutions of the homogenous problem

We calculate the number of solutions of the homogenous problem (4.14), which is  $G(z) = 0$  in (4.19). Respectively  $a_0 = ic_0$ .

$$zu' + \left( \alpha_0 + \frac{\alpha_1}{z^1} + \frac{\alpha_2}{z^2} + \cdots + \frac{\alpha_m}{z^m} \right) u = \sum_{k=1}^m \left( \frac{a_k}{z^k} - \bar{a}_k z^k \right) + ic_0, \quad z \in \mathbb{D}. \quad (4.63)$$

Depending on  $\alpha_0 + m$  we consider three possible cases.

**Case H.I** :  $\alpha_0 + m = 0$

In this case from (4.35) it is clear that  $a_m = 0$  and therefore  $u_0 = 0$ .

From (4.29) and Lemma 1 we have respective forms of (4.40)-(4.41) as

$$C^0(1, m-1)a(m-1, 1) + \bar{a}(1, m-1) = -\frac{a_0}{\alpha_m} \alpha(m-1, 1), \quad (4.64)$$

$$\overline{C}^0(m-1, 1)\overline{a}(1, m-1) + a(m-1, 1) = -\frac{\overline{a}_0}{\overline{\alpha}_m}\overline{\alpha}(1, m-1). \quad (4.65)$$

Now we have the system of equations (4.64)-(4.65) and if (4.43) is satisfied then it is uniquely solvable.

If condition (4.43) is not satisfied, then system (4.64)-(4.65) is solvable if only if

$$\frac{a_0}{\alpha_m}\alpha(m-1, 1) = C^0(1, m-1)\frac{\overline{a}_0}{\overline{\alpha}_m}\overline{\alpha}(1, m-1). \quad (4.66)$$

**Case H.II :**  $\alpha_0 + m \neq 0$  and  $|\alpha_m|^2 \neq (\alpha_0 - \tau_1)(\overline{\alpha}_0 + m)$

In this case  $a_m$  is given by (4.52) and we have the respective form of (4.53) as

$$C_1(1, m-1)a(m-1, 1) + C_2(1, m-1)I_{m-1}^*\overline{a}(m-1, 1) = \beta_1(m-1, 1). \quad (4.67)$$

Equation (4.67) is uniquely solvable for  $a(m-1, 1)$  if (4.54) is satisfied. Otherwise it is solvable if

$$\beta_1(m-1, 1) = C_1^\#(1, m-1)\overline{\beta}_1(m-1, 1). \quad (4.68)$$

Suppose both of (4.54) and (4.68) are not satisfied. Then equation (4.67) is uniquely solvable for  $a(m-1, 1)$  if (4.56) is satisfied. Otherwise it is solvable if

$$\beta_1(m-1, 1) = C_2^\#(1, m-1)\overline{\beta}_1(m-1, 1). \quad (4.69)$$

If none of (4.54), (4.68), (4.56) and (4.69) is satisfied, then equation (4.67) is solvable if (4.58) is satisfied.

**Case H.III :**  $\alpha_0 + m \neq 0$  and  $|\alpha_m|^2 = (\alpha_0 - \tau_1)(\overline{\alpha}_0 + m)$

In this case  $a_m$  is arbitrary and by (4.21) and (4.35) also  $u_0$  and  $u_m$  are arbitrary.

In this case corresponding form of (4.50) and (4.59) become

$$\operatorname{Re}\left[\alpha(m-1, 1)^T A^{-1}(m, 2)a(m-1, 1)\right] = 0, \quad (4.70)$$

$$C^*(1, m-1)a(1, m-1) + \overline{a}(1, m-1) = \beta_2(m-1, 1). \quad (4.71)$$

Equation (4.72) is uniquely solvable for  $a(1, m-1)$  if (4.61) is satisfied. Otherwise it is solvable if

$$\beta_2(m-1, 1) = C^*(1, m-1)\overline{\beta}_2(m-1, 1). \quad (4.72)$$

We have for the number of linearly independent solutions of the homogenous problem (4.14)

**Theorem 19** *Let  $\alpha_0 + k \neq 0$  for  $k \geq m+1$ . Then we have*

1. *If (4.43) is satisfied and  $\alpha_0 + m = 0$ . Then problem (4.14) has one nontrivial solution.*
2. *If  $\alpha_0 + m \neq 0$  and  $|\alpha_m|^2 \neq (\alpha_0 - \tau_1)(\overline{\alpha}_0 + m)$ . If one of (4.54) or (4.56) is satisfied, then problem (4.14) has one nontrivial solution.*
3. *If  $\alpha_0 + m \neq 0$  and  $|\alpha_m|^2 = (\alpha_0 - \tau_1)(\overline{\alpha}_0 + m)$ . If (4.61) is satisfied, then problem (4.14) has two independent nontrivial solutions over the field of real numbers.*

**Remark:** We here considered only the case  $\deg(G) \leq m-1$ . Since  $G(z)$  is reducible to a polynomial  $P(z)$  with  $\deg(P) \leq m-1$ , generality is not lost, see [1]. The case  $\alpha_0 + k_0 = 0$  for some  $k_0 \geq m+1$  will be considered in the next paper.