Chapter 3

The Schwarz problem for analytic functions in torus related domains

3.1 Introduction

The Schwarz problem in polydiscs is considered in [2]. However about the analytic functions of the other torus related domains of $\mathbb{C}^n$ ($n > 1$) there is nothing known from the literature. Its study provides vital information for discussions of all kinds of boundary value problems related to the Shilov boundary.

The set of all complex-valued functions f on the unit circle of the complex plane $\mathbb{C}$ with absolutely convergent Fourier series

$$f(\zeta) = \sum_{-\infty}^{+\infty} a_k \zeta^k, \quad \zeta \in \partial\mathbb{D}\,, \quad \| f \|_W := \sum_{-\infty}^{+\infty} |a_k| < \infty$$

is called the *Wiener algebra* , see [18]. We denote the one dimensional Wiener algebra on $\partial\mathbb{D}$ as W^1 or $W(\partial\mathbb{D}\,;\mathbb{C}\,)$. By the Weierstrass theorem the Fourier series in the Wiener algebra are also uniformly convergent. Because of the independence of the variables on the Shilov boundary $\partial\mathbb{D}^n, n > 1$, one can easily get the version $W(\partial\mathbb{D}^n;\mathbb{C}\,)$ of the Wiener algebra :

$$W^n = \left\{ f \middle| f(z) = \sum_{-\infty}^{+\infty} a_\kappa \zeta^\kappa, \quad \zeta \in \partial_0\mathbb{D}^n, \quad \| f \|_{W^n} := \sum_{-\infty}^{+\infty} |a_\kappa| < \infty \right\}.$$

We focus our attention to the Wiener algebra as the function space under consideration. Just for the same reason as in [28], i.e., having difficulties with the resolution the result is restricted to the Wiener algebra. We were not able to get the same result in $C^\alpha, 0 < \alpha < 1$. However according to the Bernstein theorem $C^\alpha(\partial\mathbb{D}^n;\mathbb{C}\,)$ turns out to be a Wiener algebra for $\alpha > 1/2$ see [16]. Thus our discussion will be in the Wiener algebra.

In one variable case there are several equivalent definitions of analytic functions. Most important ones are via power series and Cauchy integral. In higher dimensional space, at least for the torus, most studies about analytic functions started from defining an analytic function by a Cauchy integral, see [7] and [17]. These considerations may be proper for some cases. However it faces major difficulties when the connection between the Riemann-Hilbert problem and the Riemann problem is considered even for the simplest cases. Taking these shortcomings into account we applied another approach - we have made power series as the starting point and derived the proper Cauchy kernel through a careful classification of the boundary values of holomorphic functions of the torus.

As the final part of the chapter we discuss the well-posed formulation of the Schwarz problems, i.e., the Schwarz problems with and without solvability conditions.

3.2 The Cauchy kernel, division of the boundary value

Let the real valued function γ belong to $W(\partial \mathbb{D}\,; \mathbb{C}\,)$. Then γ can be represented as

$$\gamma(\zeta) = \sum_{k=-\infty}^{+\infty} \alpha_k \zeta^k \ , \quad \alpha_k = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \gamma(\zeta) \zeta^{-k} \frac{d\zeta}{\zeta} \ , \ k \in \mathbb{Z} \ .$$

From $\alpha_k = \overline{\alpha_{-k}}$, $k \in \mathbb{Z}$, it follows that

$$\gamma(\zeta) = \sum_{k=1}^{+\infty} \alpha_k \zeta^k + \overline{\sum_{k=1}^{+\infty} \alpha_k \zeta^k} + \alpha_0 \ , \quad \zeta \in \partial \mathbb{D} \, .$$

This means that $\gamma(\zeta)$ is the boundary value of a harmonic function in $\mathbb{D}$ or equivalently the real part of a function which is analytic in $\mathbb{D}$. Moreover,

$$\overline{\sum_{k=1}^{+\infty} \alpha_k \zeta^k} \ , \quad \zeta \in \partial \mathbb{D}$$

is the boundary value of an anti-analytic function in $\mathbb{D}^+$ or in other words it is the boundary value of an analytic function in $\mathbb{D}^-$.

Thus $\gamma(\zeta)$ can be split into two parts: the boundary value of a function, which is analytic inside the domain $\mathbb{D}^+$; the boundary value of a function, which is analytic in the outer domain $\mathbb{D}^-$. In addition , each of them is the reflection of the other one with respect to $\partial \mathbb{D}$.

Let the real – valued γ belong to $W(\partial_0 \mathbb{D}^2; \mathbb{C}\,)$. Then

$$\gamma(\zeta_1, \zeta_2) = \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} a_{k_1,k_2} \zeta_1^{k_1} \zeta_2^{k_2} \ , \quad a_{k_1,k_2} = \frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \gamma(\zeta_1, \zeta_2) \zeta_1^{-k_1} \zeta_2^{-k_2} \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} \ ,$$

$$a_{-k_1,-k_2} = \overline{a_{k_1,k_2}} \ , \ k_1, k_2 \in \mathbb{Z} \ , \ (\zeta_1, \zeta_2) \in \partial_0 \mathbb{D}^2. \tag{3.1}$$

So for $(\zeta_1, \zeta_2) \in \partial_0 \mathbb{D}^2$ we have

$$\gamma(\zeta_1, \zeta_2) = \Big(\sum_{k_1=0}^{+\infty} + \sum_{k_1=-\infty}^{-1} \Big) \Big(\sum_{k_2=0}^{+\infty} + \sum_{k_2=-\infty}^{-1} \Big) a_{k_1,k_2} \zeta_1^{k_1} \zeta_2^{k_2}$$

$$= \sum_{k_1=0}^{+\infty}\sum_{k_2=0}^{+\infty} a_{k_1,k_2}\zeta_1^{k_1}\zeta_2^{k_2} + \sum_{k_2=1}^{+\infty} a_{0,-k_2}\zeta_2^{-k_2} + \sum_{k_1=1}^{+\infty}\sum_{k_2=1}^{+\infty} a_{k_1,-k_2}\zeta_1^{k_1}\zeta_2^{-k_2}$$

$$+ \sum_{k_1=1}^{+\infty} a_{-k_1,0}\zeta_1^{-k_1} + \sum_{k_1=1}^{+\infty}\sum_{k_2=1}^{+\infty} a_{-k_1,k_2}\zeta_1^{-k_1}\zeta_2^{k_2} + \sum_{k_1=1}^{+\infty}\sum_{k_2=1}^{+\infty} a_{-k_1,-k_2}\zeta_1^{-k_1}\zeta_2^{-k_2}$$

$$= \sum_{\substack{k_1=0\\ k_1+k_2>0}}^{+\infty}\sum_{k_2=0}^{+\infty} a_{k_1,k_2}\zeta_1^{k_1}\zeta_2^{k_2} + \overline{\sum_{\substack{k_1=0\\ k_1+k_2>0}}^{+\infty}\sum_{k_2=0}^{+\infty} a_{k_1,k_2}\zeta_1^{k_1}\zeta_2^{k_2}} + a_{0,0}$$

$$+ \sum_{k_1=1}^{+\infty}\sum_{k_2=1}^{+\infty} a_{k_1,-k_2}\zeta_1^{k_1}\zeta_2^{-k_2} + \overline{\sum_{k_1=1}^{+\infty}\sum_{k_2=1}^{+\infty} a_{k_1,-k_2}\zeta_1^{k_1}\zeta_2^{-k_2}}$$

$$= 2Re\Big\{ \sum_{\substack{k_1=0\\ k_1+k_2>0}}^{+\infty}\sum_{k_2=0}^{+\infty} a_{k_1,k_2}\zeta_1^{k_1}\zeta_2^{k_2}\Big\} + a_{0,0} + 2Re\Big\{\sum_{k_1=1}^{+\infty}\sum_{k_2=1}^{+\infty} a_{k_1,-k_2}\zeta_1^{k_1}\zeta_2^{-k_2}\Big\} .$$

Obviously, $\gamma(\zeta)$ can be split into the boundary values of two real pluriharmonic functions one in $\mathbb{D}^2 = \mathbb{D}^+ \times \mathbb{D}^+$ and one in $\mathbb{D}^+ \times \mathbb{D}^-$ respectively. This means that $\gamma(\zeta)$ consists of boundary values of four analytic functions in $\mathbb{D}^+ \times \mathbb{D}^+$, $\mathbb{D}^+ \times \mathbb{D}^-$, $\mathbb{D}^- \times \mathbb{D}^+$ and $\mathbb{D}^- \times \mathbb{D}^-$, respectively. For each corresponding pair of these functions the reflection principle holds and every one belongs to $W(\partial_0\mathbb{D}^2;\mathbb{C})$.

As we have seen a given real function $\gamma(\zeta)$ from $\partial_0\mathbb{D}^2$ is not always the real part of boundary values of a pluriharmonic function in $\mathbb{D}^2$. It is if and only if

$$Re\Big\{\sum_{k_1=1}^{+\infty}\sum_{k_2=1}^{+\infty} a_{k_1,-k_2}\zeta_1^{k_1}\zeta_2^{-k_2}\Big\} \;=\; 0 \;, \quad (\zeta_1,\zeta_2)\in\partial_0\mathbb{D}^2.$$

Just from this point the solvability conditions occur for the Schwarz problem for analytic functions in $\mathbb{D}^2$, see [2] page 244.

From the split boundary values which are uniformly and absolutely convergent, we have the respective analytic functions

$$\begin{aligned}\sum_{k_1=0}^{+\infty}\sum_{k_2=0}^{+\infty} a_{k_1,k_2}z_1^{k_1}z_2^{k_2} &= \frac{1}{(2\pi i)^2}\int_{\partial_0\mathbb{D}^2}\gamma(\zeta_1,\zeta_2)\sum_{k_1=0}^{+\infty}\sum_{k_2=0}^{+\infty}(z_1\overline{\zeta}_1)^{k_1}(z_2\overline{\zeta}_2)^{k_2}\frac{d\zeta_1}{\zeta_1}\frac{d\zeta_2}{\zeta_2}\\ &= \frac{1}{(2\pi i)^2}\int_{\partial_0\mathbb{D}^2}\gamma(\zeta)\Big[\frac{1}{1-z\overline{\zeta}}-1\Big]\frac{d\zeta}{\zeta} + a_{0,0}\\ &= \frac{1}{(2\pi i)^2}\int_{\partial_0\mathbb{D}^2}\gamma(\zeta)\frac{d\zeta}{\zeta - z}\\ &= =: \phi^{++}(z)\ ,\ z\in\mathbb{D}^2,\end{aligned}$$

similarly

$$\begin{aligned}\sum_{\substack{k_1=0\\k_1+k_2>0}}^{+\infty}\sum_{k_2=0}^{+\infty} a_{-k_1,-k_2} z_1^{-k_1} z_2^{-k_2} &= \frac{1}{(2\pi i)^2}\int_{\partial_0 \mathbb{D}^2}\gamma(\zeta)\Big[\frac{1}{1-\zeta z^{-1}}-1\Big]\frac{d\zeta}{\zeta}\\ &= \frac{1}{(2\pi i)^2}\int_{\partial_0 \mathbb{D}^2}\gamma(\zeta)\Big[\frac{z}{z-\zeta}-1\Big]\frac{d\zeta}{\zeta}\\ &=: \phi^{--}(z)\ , \quad z\in \mathbb{D}^-\times\mathbb{D}^-,\end{aligned}$$

$$\begin{aligned}\sum_{k_1=1}^{+\infty}\sum_{k_2=1}^{+\infty} a_{k_1,-k_2} z_1^{k_1} z_2^{-k_2} &= \frac{1}{(2\pi i)^2}\int_{\partial_0 \mathbb{D}^2}\gamma(\zeta)\frac{z_1\zeta_1^{-1}}{1-z_1\zeta_1^{-1}}\frac{z_2^{-1}\zeta_2}{1-z_2^{-1}\zeta_2}\frac{d\zeta}{\zeta}\\ &= \frac{-1}{(2\pi i)^2}\int_{\partial_0 \mathbb{D}^2}\gamma(\zeta)\frac{z_1}{\zeta_1-z_1}\frac{\zeta_2}{\zeta_2-z_2}\frac{d\zeta}{\zeta}\\ &=: \phi^{+-}(z)\ , \quad z\in \mathbb{D}^+\times\mathbb{D}^-,\end{aligned}$$

$$\begin{aligned}\sum_{k_1=1}^{+\infty}\sum_{k_2=1}^{+\infty} a_{-k_1,k_2} z_1^{-k_1} z_2^{k_2} &= \frac{1}{(2\pi i)^2}\int_{\partial_0 \mathbb{D}^2}\gamma(\zeta)\frac{z_1^{-1}\zeta_1}{1-z_1^{-1}\zeta_1}\frac{z_2\zeta_2^{-1}}{1-z_2\zeta_2^{-1}}\frac{d\zeta}{\zeta}\\ &= \frac{-1}{(2\pi i)^2}\int_{\partial_0 \mathbb{D}^2}\gamma(\zeta)\frac{\zeta_1}{\zeta_1-z_1}\frac{z_2}{\zeta_2-z_2}\frac{d\zeta}{\zeta}\\ &=: \phi^{-+}(z)\ , \quad z\in \mathbb{D}^-\times\mathbb{D}^+.\end{aligned}$$

Although one can describe these analytic functions in the two-dimensional case very easily it would not be very convenient to do the same in a higher dimensional space. One has to find a better way of description. Therefore we introduce the following notation.

3.3 Definition

Let $\chi = (\chi_1, \cdots, \chi_n)$ be a multi-sign, where

$$\chi_1,\cdots,\chi_n \in \{+,-\},\ 0\le \nu\le n\ ,1\le\sigma_1<\cdots<\sigma_\nu\le n\ ,$$

$$1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n\ ,\ \{\sigma_1,\cdots,\sigma_n\}=\{1,\cdots,n\}\ ,\chi_{\sigma_1}=-\ ,\cdots,\chi_{\sigma_\nu}=-\ ,$$

$$\chi_{\sigma_{\nu+1}}=+\ ,\cdots,\chi_{\sigma_n}=+\ ,\chi(\nu)=\chi_{\sigma_1\cdots\sigma_\nu}(\nu)\ ,$$

where ν gives the number of minus $(-)$ signs and the indices $\sigma_1,\cdots,\sigma_\nu$ show the position of these minus sign components. $\chi(\nu)$ obviously has $(n-\nu)$ plus $(+)$ sign components at the positions $\sigma_{\nu+1},\cdots,\sigma_n$. In addition $\chi(\nu)=\chi_{\sigma_1\cdots\sigma_\nu}(\nu)=-\chi_{\rho_1\cdots\rho_{n-\nu}}(n-\nu)=-\chi(n-\nu)$, for $0\le\nu\le n$ and $\{\rho_1,\cdots,\rho_{n-\nu}\}=\{1,\cdots,n\}\backslash\{\sigma_1\cdots\sigma_\nu\}=\{\sigma_{\nu+1}\cdots\sigma_n\}$, when treating $\chi(\nu)$ as a vector.

Actually $\chi_{\sigma_1\cdots\sigma_\nu}(\nu)$, $0\le\nu\le n$, can be understood as signs of vertices of the n dimensional cube $[-1,+1]^n$. In the case $n=2$ the signs $(+,+),(+,-),(-,+),(-,-)$ correspond to the signs of the vertices $(1,1),(1,-1),(-1,1),(-1,-1)$ of the unit square. Therefore we denote χ^* as the vertices of the $[-1,+1]^n$ cube, while χ represents the respective multi-sign.

Let the real–valued function φ belong to $W(\partial_0 \mathbb{D}^n;\mathbb{C})$. Then φ can be represented as

$$\varphi(\eta)=\sum_{\kappa\in\mathbb{Z}^n}\alpha_\kappa\eta^\kappa\ ,\quad \alpha_\kappa=\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\varphi(\zeta)\zeta^{-\kappa}\frac{d\zeta}{\zeta}\ ,\quad \overline{\alpha}_\kappa=\alpha_{-\kappa},\quad \kappa\in\mathbb{Z}^n. \tag{3.2}$$

This Fourier series is absolutely and uniformly convergent to $\varphi(\eta)$ because of $\varphi\in W(\partial_0 \mathbb{D}^n;\mathbb{C})$ and it can be split into 2^n parts:

$$\prod_{t=1}^{n}\Big(\sum_{k_t=0}^{+\infty}\zeta_t^{-k_t}\Big)\alpha_{-k_1,\cdots,-k_n}-\alpha_{0,\cdots,0}=\sum_{|\kappa|>0\ ,\ \kappa\in\mathbb{Z}^n_+}\alpha_{-\kappa}\zeta^{-\kappa}=:w^{\chi(n)}(\zeta)\ ,$$

$$\prod_{t=1}^{n}\Big(\sum_{k_t=1}^{+\infty}(\zeta_t^{\chi_t^*})^{k_t}+\delta_t^{\chi_t}\Big)\alpha_{\chi_1^*k_1,\cdots,\chi_n^*k_n}=:w^{\chi(\nu)}(\zeta),\ 0\le\nu<n\ ,\zeta\in\partial_0\mathbb{D}^n, \tag{3.3}$$

where

$$\delta_t^{\chi_t}=\frac{|\chi_t^*+\chi_{t^*+1}^*|}{2}\ ,\quad 1\le t\le n\ ,\quad t^*=t\ mod(n)\ .$$

Obviously, the $\delta_t^{\chi_t}$'s are 0 or 1 and about them there is an interesting fact

$$\delta_t^+ +\delta_t^- =\frac{|\chi_{t+1}^*+1|}{2}+\frac{|\chi_{t+1}^*-1|}{2}=1,\ 1\le t\le n-1\ ,$$

$$\delta_n^+ +\delta_n^- =\frac{|\chi_1^*+1|}{2}+\frac{|\chi_1^*-1|}{2}=1\ .$$

However this equality should not be used anywhere if we want to avoid any mistakes. Any representation which leads to the sum $\delta_t^+ +\delta_t^-$ is problematic. This can be shown very easily (applying the above equality to the left-hand side in Lemma 6 for n=2). Every $w^{\chi(\nu)}(\zeta)$ surely can be viewed as the boundary values of a holomorphic function $w^{\chi(\nu)}(z)$ in $\mathbb{D}^{\chi(\nu)}$. The correctness of this kind of partition (3.3) for any $w\in W(\partial_0\mathbb{D}^n;\mathbb{C})$ can be shown by the following lemmas.

Lemma 4 *Let* $a_t,b_t\in\mathbb{C}$, $1\le t\le n$. *Then*

$$\prod_{t=1}^{n}(a_t+b_t)=\sum_{t=0}^{n}\sum_{\substack{1\le\lambda_1<\cdots<\lambda_t\le n\\ 1\le\lambda_{t+1}<\cdots<\lambda_n\le n}}a_{\lambda_1}\cdots a_{\lambda_t}b_{\lambda_{t+1}}\cdots b_{\lambda_n}.$$

If $b_t=1\ ,1\le t\le n$, then the lemma becomes Lemma 5.2 in [2]. The proof of this lemma is trivial. So it is omitted. By this lemma, see [2], one can obtain

Lemma 5

$$\prod_{t=1}^{n}\Big(a_t+\overline{a_t}+1\Big)-\Big[\prod_{t=1}^{n}\Big(a_t+1\Big)+\prod_{t=1}^{n}\Big(\overline{a_t}+1\Big)-1\Big]$$

$$= \sum_{t=2}^{n} \sum_{\nu=1}^{t-1} \sum_{\substack{1\le\lambda_1<\cdots<\lambda_\nu\le n \\ 1\le\lambda_{\nu+1}<\cdots<\lambda_t\le n \\ cd\,\{\lambda_1,\ldots,\lambda_t\}=t}} \overline{a}_{\lambda_1}\cdots\overline{a}_{\lambda_\nu} a_{\lambda_{\nu+1}}\cdots a_{\lambda_t}$$

for $a_t \in \mathbb{C}$, $1 \le t \le n.$

Applying the last lemma we can obtain the following result.

Lemma 6

$$\prod_{t=1}^{n}\Big(a_t+\overline{a_t}+1\Big)+1=\sum_{\nu=0}^{n}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\prod_{t=1}^{n}\Big(a_t^{\chi_t}+\delta_t^{\chi}\Big)$$

$$=\sum_{\nu=0}^{n}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\prod_{t=1}^{\nu}\Big(\overline{a}_{\sigma_t}+\delta_{\sigma_t}^{-}\Big)\prod_{t=\nu+1}^{n}\Big(a_{\sigma_t}+\delta_{\sigma_t}^{+}\Big)$$

for $2 \le n,\ a_t \in \mathbb{C}$, $1 \le t \le n,$ *where* $a_\mu^- := \overline{a}_\mu$ *and*

$$cd\{\sigma_1,\cdots,\sigma_\nu,\sigma_{\nu+1},\cdots,\sigma_n\}=n\ ,\chi_{\sigma_1}=\cdots=\chi_{\sigma_\nu}=-\ ,\quad \chi_{\sigma_{\nu+1}}=\cdots=\chi_{\sigma_n}=+\ .$$

Proof 1. For $n = 2$

$$\sum_{\nu=1}^{n-1}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\prod_{t=1}^{n}\Big(a_t^{\chi_t}+\delta_t^{\chi}\Big)=\sum_{\substack{1\le\sigma_1\le 2\\1\le\sigma_2\le 2}}\Big(\overline{a}_{\sigma_1}+\delta_{\sigma_1}^{-}\Big)\Big(a_{\sigma_2}+\delta_{\sigma_2}^{+}\Big)=\sum_{\substack{1\le\sigma_1\le 2\\1\le\sigma_2\le 2}}\overline{a}_{\sigma_1}a_{\sigma_2}\ .$$

and

$$(a_1+\overline{a_1}+1)\,(a_2+\overline{a_2}+1)-(a_1+1)\,(a_2+1)-(\overline{a_1}+1)\,(\overline{a_2}+1)+1=a_1\overline{a_2}+\overline{a_1}a_2\ .$$

Therefore the formula holds in this case.

Next we check the case for $n = 3$.

$$\sum_{\nu=1}^{n-1}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\prod_{t=1}^{\nu}\Big(\overline{a}_{\sigma_t}+\delta_{\sigma_t}^{-}\Big)\prod_{t=\nu+1}^{n}\Big(a_{\sigma_t}+\delta_{\sigma_t}^{+}\Big)$$

$$=\sum_{\nu=1}^{2}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le 3\\1\le\sigma_{\nu+1}<\cdots<\sigma_3\le 3}}\prod_{t=1}^{\nu}\Big(\overline{a}_{\sigma_t}+\delta_{\sigma_t}^{-}\Big)\prod_{t=\nu+1}^{3}\Big(a_{\sigma_t}+\delta_{\sigma_t}^{+}\Big)$$

$$=\sum_{\substack{1\le\sigma_1\le 3\\1\le\sigma_2<\sigma_3\le 3}}\Big(\overline{a}_{\sigma_1}+\delta_{\sigma_1}^{-}\Big)\prod_{t=2}^{3}\Big(a_{\sigma_t}+\delta_{\sigma_t}^{+}\Big)+\sum_{\substack{1\le\sigma_1<\sigma_2\le 3\\1\le\sigma_3\le 3}}\prod_{t=1}^{2}\Big(\overline{a}_{\sigma_t}+\delta_{\sigma_t}^{-}\Big)\Big(a_{\sigma_3}+\delta_{\sigma_3}^{+}\Big)$$

$$= \overline{a}_1(a_2+1)a_3 + a_1\overline{a}_2(a_3+1) + (a_1+1)a_2\overline{a}_3 + (\overline{a}_1+1)\overline{a}_2 a_3 + \overline{a}_1 a_2(\overline{a}_3+1) + a_1(\overline{a}_2+1)\overline{a}_3$$

and

$$\sum_{t=2}^{3}\sum_{\nu=1}^{t-1}\sum_{\substack{1\le\lambda_1<\cdots<\lambda_\nu\le 3\\ 1\le\lambda_{\nu+1}<\cdots<\lambda_t\le 3}}\overline{a}_{\lambda_1}\cdots\overline{a}_{\lambda_\nu}a_{\lambda_{\nu+1}}\cdots a_{\lambda_t} = \sum_{\substack{1\le\lambda_1\le 3\\ 1\le\lambda_2\le 3}}\overline{a}_{\lambda_1}a_{\lambda_2} + \sum_{\substack{1\le\lambda_1\le 3\\ 1\le\lambda_2<\lambda_3\le 3}}\overline{a}_{\lambda_1}a_{\lambda_2}a_{\lambda_3} + \sum_{\substack{1\le\lambda_1<\lambda_2\le 3\\ 1\le\lambda_3\le 3}}\overline{a}_{\lambda_1}\overline{a}_{\lambda_2}a_{\lambda_3}$$

$$= \overline{a}_1a_2 + \overline{a}_1a_3 + \overline{a}_2a_1 + \overline{a}_2a_3 + \overline{a}_3a_1 + \overline{a}_3a_2 + \overline{a}_1a_2a_3$$
$$+\overline{a}_2a_1a_3 + \overline{a}_3a_1a_2 + \overline{a}_1\overline{a}_2a_3 + \overline{a}_1\overline{a}_3a_2 + \overline{a}_2\overline{a}_3a_1 \ .$$

So the Lemma holds in this case too. Now we look at the general case.

$$\sum_{\nu=0}^{n}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\prod_{t=1}^{\nu}\Big(\overline{a}_{\sigma_t}+\delta^-_{\sigma_t}\Big)\prod_{t=\nu+1}^{n}\Big(a_{\sigma_t}+\delta^+_{\sigma_t}\Big) = \prod_{t=1}^{n}\Big(a_t+1\Big)$$
$$+\prod_{t=1}^{n}\Big(\overline{a}_t+1\Big) + \sum_{\nu=1}^{n-1}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\prod_{t=1}^{\nu}\Big(\overline{a}_{\sigma_t}+\delta^-_{\sigma_t}\Big)\prod_{t=\nu+1}^{n}\Big(a_{\sigma_t}+\delta^+_{\sigma_t}\Big)\ .$$

Denote

$$I_p := \sum_{\nu=1}^{n-1}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\prod_{t=1}^{\nu}\Big(\overline{a}_{\sigma_t}+\delta^-_{\sigma_t}\Big)\prod_{t=\nu+1}^{n}\Big(a_{\sigma_t}+\delta^+_{\sigma_t}\Big)$$

and

$$I_g := \sum_{t=2}^{n}\sum_{\nu=1}^{t-1}\sum_{\substack{1\le\lambda_1<\cdots<\lambda_\nu\le n\\ 1\le\lambda_{\nu+1}<\cdots<\lambda_t\le n\\ cd\,\{\lambda_1,\ldots,\lambda_t\}=t}}\overline{a}_{\lambda_1}\cdots\overline{a}_{\lambda_\nu}a_{\lambda_{\nu+1}}\cdots a_{\lambda_t} = \sum_{t=2}^{n} I_g(t)\ .$$

Since we have shown that the lemma holds for $n = 2, 3$, one could automatically think about induction method to prove the lemma in general. For the proof by induction see [19]. One has to be very careful with applying the induction method here. This method does not lead to the intended result very easily. So we prefer a direct proof and show that $I_p = I_g$ holds. Applying Lemma 4 we have

$$I_p := \sum_{\nu=1}^{n-1}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\Big[\sum_{t=0}^{\nu}\sum_{\substack{1\le\alpha_1<\cdots<\alpha_t\le\nu\\ 1\le\alpha_{t+1}<\cdots<\alpha_\nu\le\nu}}\overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_t}}\delta^-_{\sigma_{\alpha_{t+1}}}\cdots\delta^-_{\sigma_{\alpha_\nu}}\Big]$$
$$\times\Big[\sum_{\ell=0}^{n-\nu}\sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{\nu+\ell}\le n\\ \nu+1\le\alpha_{\nu+\ell+1}<\cdots<\alpha_n\le n}}a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{\nu+\ell}}}\delta^+_{\sigma_{\alpha_{\nu+\ell+1}}}\cdots\delta^+_{\sigma_{\alpha_n}}\Big]$$

$$= \sum_{\nu=1}^{n-1} \sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}} \Big[\sum_{t=1}^{\nu} \sum_{\substack{1\le\alpha_1<\cdots<\alpha_t\le\nu\\ 1\le\alpha_{t+1}<\cdots<\alpha_\nu\le\nu}} \overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_t}}\delta^-_{\sigma_{\alpha_{t+1}}}\cdots\delta^-_{\sigma_{\alpha_\nu}} + \delta^-_{\sigma_1}\cdots\delta^-_{\sigma_\nu}\Big]$$

$$\times\Big[\sum_{\ell=1}^{n-\nu} \sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{\nu+\ell}\le n\\ \nu+1\le\alpha_{\nu+\ell+1}<\cdots<\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{\nu+\ell}}}\delta^+_{\sigma_{\alpha_{\nu+\ell+1}}}\cdots\delta^+_{\sigma_{\alpha_n}} + \delta^+_{\sigma_{\nu+1}}\cdots\delta^+_{\sigma_n}\Big].$$

Taking $1 \le \nu \le n-1$ into account we see that there exist always at least two integers $k \in \{\sigma_1, \cdots, \sigma_\nu\}$ and $h \in \{\sigma_{\nu+1}, \cdots, \sigma_n\}$ for which $\delta^-_k = 0$ and $\delta^+_h = 0$ respectively so that

$$\delta^-_{\sigma_1}\cdots\delta^-_{\sigma_\nu} = 0\,, \quad \delta^+_{\sigma_{\nu+1}}\cdots\delta^+_{\sigma_n} = 0\,.$$

Therefore

$$I_p = \sum_{\nu=1}^{n-1} \sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}} \Big[\sum_{t=1}^{\nu} \sum_{\substack{1\le\alpha_1<\cdots<\alpha_t\le\nu\\ 1\le\alpha_{t+1}<\cdots<\alpha_\nu\le\nu}} \overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_t}}\delta^-_{\sigma_{\alpha_{t+1}}}\cdots\delta^-_{\sigma_{\alpha_\nu}}\Big]$$

$$\times\Big[\sum_{\ell=1}^{n-\nu} \sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{\nu+\ell}\le n\\ \nu+1\le\alpha_{\nu+\ell+1}<\cdots<\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{\nu+\ell}}}\delta^+_{\sigma_{\alpha_{\nu+\ell+1}}}\cdots\delta^+_{\sigma_{\alpha_n}}\Big]$$

$$= \sum_{\nu=1}^{n-1} \sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}} \Big[\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu} + \sum_{\substack{1\le\alpha_1<\cdots<\alpha_{\nu-1}\le\nu\\ 1\le\alpha_\nu\le\nu}} \overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_{\nu-1}}}\delta^-_{\sigma_{\alpha_\nu}} + \cdots + \sum_{\substack{1\le\alpha_1\le\nu\\ 1\le\alpha_2<\cdots<\alpha_\nu\le\nu}} \overline{a}_{\sigma_{\alpha_1}}\delta^-_{\sigma_{\alpha_2}}\cdots\delta^-_{\sigma_{\alpha_\nu}}\Big]$$

$$\times\Big[a_{\sigma_{\nu+1}}\cdots a_{\sigma_n} + \sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{n-1}\le n\\ \nu+1\le\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{n-1}}}\delta^+_{\sigma_{\alpha_n}} + \cdots + \sum_{\substack{\nu+1\le\alpha_{\nu+1}\le n\\ \nu+1\le\alpha_{\nu+2}<\cdots<\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\delta^+_{\sigma_{\alpha_{\nu+2}}}\cdots\delta^+_{\sigma_{\alpha_n}}\Big]$$

$$= \sum_{\nu=1}^{n-1} \sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}} \Big[\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}a_{\sigma_{\nu+1}}\cdots a_{\sigma_n} + \Big(\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}\sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{n-1}\le n\\ \nu+1\le\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{n-1}}}\delta^+_{\sigma_{\alpha_n}}$$

$$+a_{\sigma_{\nu+1}}\cdots a_{\sigma_n} \sum_{\substack{1\le\alpha_1<\cdots<\alpha_{\nu-1}\le\nu\\ 1\le\alpha_\nu\le\nu}} \overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_{\nu-1}}}\delta^-_{\sigma_{\alpha_\nu}}\Big) + \Big(\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}\sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{n-2}\le n\\ \nu+1\le\alpha_{n-1}<\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{n-2}}}\delta^+_{\sigma_{\alpha_{n-1}}}\delta^+_{\sigma_{\alpha_n}}$$

$$+ \sum_{\substack{1\le\alpha_1<\cdots<\alpha_{\nu-1}\le\nu\\ 1\le\alpha_\nu\le\nu}} \overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_{\nu-1}}}\delta^-_{\sigma_{\alpha_\nu}} \sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{n-1}\le n\\ \nu+1\le\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{n-1}}}\delta^+_{\sigma_{\alpha_n}}$$

$$+a_{\sigma_{\nu+1}}\cdots a_{\sigma_n} \sum_{\substack{1\le\alpha_1<\cdots<\alpha_{\nu-2}\le\nu\\ 1\le\alpha_{\nu-1}<\alpha_\nu\le\nu}} \overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_{\nu-2}}}\delta^-_{\sigma_{\alpha_{\nu-1}}}\delta^-_{\sigma_{\alpha_\nu}}\Big) + \cdots +$$

$$\Big(\sum_{\substack{t+\ell=k \\ 1\le t\le\nu,\ 1\le\ell\le k-\nu}} \sum_{\substack{1\le\alpha_1<\cdots<\alpha_t\le\nu \\ 1\le\alpha_{t+1}<\cdots\alpha_\nu\le\nu}} \overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_t}}\delta^-_{\sigma_{\alpha_{t+1}}}\cdots\delta^-_{\sigma_{\alpha_\nu}} \sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{\nu+\ell}\le n \\ \nu+1\le\alpha_{\nu+\ell+1}<\cdots\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{\nu+\ell}}}\delta^+_{\sigma_{\alpha_{\nu+\ell+1}}}\cdots\delta^+_{\sigma_{\alpha_n}}\Big)$$

$$+\cdots+\sum_{\substack{1\le\alpha_1\le\nu \\ 1\le\alpha_2<\cdots\alpha_\nu\le\nu}}\overline{a}_{\sigma_{\alpha_1}}\delta^-_{\sigma_{\alpha_2}}\cdots\delta^-_{\sigma_{\alpha_\nu}}\sum_{\substack{\nu+1\le\alpha_{\nu+1}\le n \\ \nu+1\le\alpha_{\nu+2}<\cdots<\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\delta^+_{\sigma_{\alpha_{\nu+2}}}\cdots\delta^+_{\sigma_{\alpha_n}}\Big] = \sum_{k=2}^{n}\sum_{\nu=1}^{n-1}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n \\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}$$

$$\Big(\sum_{\substack{t+\ell=k \\ 1\le t\le\nu,\ 1\le\ell\le k-\nu}} \sum_{\substack{1\le\alpha_1<\cdots<\alpha_t\le\nu \\ 1\le\alpha_{t+1}<\cdots\alpha_\nu\le\nu}} \overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_t}}\delta^-_{\sigma_{\alpha_{t+1}}}\cdots\delta^-_{\sigma_{\alpha_\nu}} \sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{\nu+\ell}\le n \\ \nu+1\le\alpha_{\nu+\ell+1}<\cdots\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{\nu+\ell}}}\delta^+_{\sigma_{\alpha_{\nu+\ell+1}}}\cdots\delta^+_{\sigma_{\alpha_n}}\Big) =: \sum_{k=2}^{n} I_p(k).$$

Evidently

$$I_p(n) = \sum_{\nu=1}^{n-1}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n \\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}} \overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}a_{\sigma_{\nu+1}}\cdots a_{\sigma_n} = I_g(n)\ .$$

About the second term we have

$$I_p(n-1) = \sum_{\nu=1}^{n-1}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n \\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}} \Big[\ \overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}\sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{n-1}\le n \\ \nu+1\le\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{n-1}}}\delta^+_{\sigma_{\alpha_n}}$$

$$+a_{\sigma_{\nu+1}}\cdots a_{\sigma_n}\sum_{\substack{1\le\alpha_1<\cdots<\alpha_{\nu-1}\le\nu \\ 1\le\alpha_\nu\le\nu}} \overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_{\nu-1}}}\delta^-_{\sigma_{\alpha_\nu}}\Big]$$

By the definition of the δ's we know that

$$\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_{n-1}}\delta^+_{\sigma_n} = 0, \qquad a_{\sigma_2}\cdots a_{\sigma_n}\delta^-_{\sigma_1} = 0.$$

Therefore

$$I_p(n-1) = \sum_{\nu=1}^{n-2}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n \\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}} \overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}\sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{n-1}\le n \\ \nu+1\le\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{n-1}}}\delta^+_{\sigma_{\alpha_n}}$$

$$+\sum_{\nu=2}^{n-1}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n \\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}} a_{\sigma_{\nu+1}}\cdots a_{\sigma_n}\sum_{\substack{1\le\alpha_1<\cdots<\alpha_{\nu-1}\le\nu \\ 1\le\alpha_\nu\le\nu}} \overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_{\nu-1}}}\delta^-_{\sigma_{\alpha_\nu}}\ .$$

For an arbitrary given $\nu\in\{1,\cdots,n-2\}$ there are fixed number of sigmas, i.e. $\sigma_{\nu+1},\cdots,\sigma_n$ and they can arbitrarily have $n-\nu$ values from $\{1,\cdots,n\}$. As soon as α_n takes an arbitrary value among $\nu+1,\cdots,n$ then the rest $\alpha_{\nu+1},\cdots,\alpha_{n-1}$ are already fixed, because of their order. Assume that $\sigma_{\nu+1},\cdots,\sigma_n$ are fixed. Then due to that α_n varies from $\nu+1$ to n we have

$$\sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{n-1}\le n \\ \nu+1\le\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{n-1}}}\delta^+_{\sigma_{\alpha_n}} = a_{\sigma_{\nu+2}}\cdots a_{\sigma_n}\delta^+_{\sigma_{\nu+1}} + a_{\sigma_{\nu+1}}a_{\sigma_{\nu+3}}\cdots a_{\sigma_n}\delta^+_{\sigma_{\nu+2}}$$

$$+\cdots+a_{\sigma_{\nu+1}}\cdots a_{\sigma_{h-1}}a_{\sigma_{h+1}}\cdots a_{\sigma_n}\delta^+_{\sigma_h}+\cdots+a_{\sigma_{\nu+1}}\cdots a_{\sigma_{n-1}}\delta^+_{\sigma_n}$$

$$=\sum_{h=\nu+1}^{n}a_{\sigma_{\nu+1}}\cdots a_{\sigma_{h-1}}a_{\sigma_{h+1}}\cdots a_{\sigma_n}\delta^+_{\sigma_h}$$

i.e.,

$$I_p(n-1)=\sum_{\nu=1}^{n-2}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}\sum_{h=\nu+1}^{n}a_{\sigma_{\nu+1}}\cdots a_{\sigma_{h-1}}a_{\sigma_{h+1}}\cdots a_{\sigma_n}\delta^+_{\sigma_h}$$

$$+\sum_{\nu=2}^{n-1}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}a_{\sigma_{\nu+1}}\cdots a_{\sigma_n}\sum_{t=1}^{\nu}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_{t-1}}\overline{a}_{\sigma_{t+1}}\cdots\overline{a}_{\sigma_\nu}\delta^-_{\sigma_t}$$

If $\sigma_1,\cdots,\sigma_\nu$ are fixed, then $\{\sigma_{\nu+1},\cdots,\sigma_n\}=\{1,\cdots,n\}\backslash\{\sigma_1,\cdots,\sigma_\nu\}$ are also fixed. So for these fixed $\{\sigma_{\nu+1},\cdots,\sigma_n\}$ we have

$$\sum_{1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}\sum_{h=\nu+1}^{n}\delta^+_{\sigma_h}\prod_{\substack{\tau=\nu+1\\ \tau\ne h}}^{n}a_{\sigma_\tau}=\sum_{\substack{1\le\sigma'_{\nu+1}<\cdots<\sigma'_{n-1}\le n\ ,\ 1\le\sigma'_n\le n\\ \{\sigma'_{\nu+1},\cdots,\sigma'_{n-1},\sigma'_n\}=\{\sigma_{\nu+1},\cdots,\sigma_n\}}}\delta^+_{\sigma'_n}\prod_{\tau=\nu+1}^{n-1}a_{\sigma'_\tau}$$

and for the $\{\sigma_1,\cdots,\sigma_\nu\}$ similarly

$$\sum_{1\le\sigma_1<\cdots<\sigma_\nu\le n}\sum_{h=1}^{\nu}\delta^-_{\sigma_h}\prod_{\substack{\tau=1\\ \tau\ne h}}^{\nu}\overline{a}_{\sigma_\tau}=\sum_{\substack{1\le\sigma'_1<\cdots<\sigma'_{\nu-1}\le n\ ,\ 1\le\sigma'_\nu\le n\\ \{\sigma'_1,\cdots,\sigma'_{\nu-1},\sigma'_\nu\}=\{\sigma_1,\cdots,\sigma_\nu\}}}\delta^-_{\sigma'_\nu}\prod_{\tau=1}^{\nu-1}\overline{a}_{\sigma'_\tau}$$

Without loss of generality we assume that $\sigma_h=k\in\{1,\cdots,n\}$, $h\in\{\nu+1,\cdots,n\}$ and note $k^*=kmod(n)$. By the definition of the δ's it is clear that $\delta^+_{\sigma_h}$ is not zero if and only if $\sigma_{h^*+1}=\sigma^*_h+1$ i.e., $\sigma^*_h+1\in\{\sigma_{\nu+1},\cdots,\sigma_n\}$. This means

$$\sum_{\substack{1\le\sigma'_{\nu+1}<\cdots<\sigma'_{n-1}\le n\ ,\ 1\le k\le n\\ \{\sigma'_{\nu+1},\cdots,\sigma'_{n-1},k\}=\{\sigma_{\nu+1},\cdots,\sigma_n\}}}\delta^+_k\prod_{\tau=\nu+1}^{n-1}a_{\sigma'_\tau}=\sum_{\substack{1\le\sigma^0_{\nu+1}<\cdots<\sigma^0_{n-2}\le n\ ,\ 1\le k\le n\\ \{k,k^*+1,\sigma^0_{\nu+1},\cdots,\sigma^0_{n-2}\}=\{\sigma_{\nu+1},\cdots,\sigma_n\}}}\delta^+_k a_{k^*+1}\prod_{\tau=\nu+1}^{n-2}a_{\sigma^0_\tau}$$

$$=\sum_{\substack{1\le\sigma^0_{\nu+1}<\cdots<\sigma^0_{n-2}\le n\ ,\ 1\le k^*+1\le n\\ \{k,k^*+1,\sigma^0_{\nu+1},\cdots,\sigma^0_{n-2}\}=\{\sigma_{\nu+1},\cdots,\sigma_n\}}}a_{k^*+1}\prod_{\tau=\nu+1}^{n-2}a_{\sigma^0_\tau}=\sum_{\substack{1\le\sigma''_{\nu+1}<\cdots<\sigma''_{n-1}\le n\ ,\ k^*+1\in\{\sigma''_{\nu+1},\cdots,\sigma''_{n-1}\}\\ k\in\{1,\cdots,n\}\backslash\{\sigma_1,\cdots,\sigma_\nu,\sigma''_{\nu+1},\cdots,\sigma''_{n-1}\}}}\prod_{\tau=\nu+1}^{n-1}a_{\sigma''_\tau}$$

$$=\sum_{\substack{1\le\sigma_{\nu+1}<\cdots<\sigma_{n-1}\le n\ ,\ k^*+1\in\{\sigma_{\nu+1},\cdots,\sigma_{n-1}\}\\ k\in\{1,\cdots,n\}\backslash\{\sigma_1,\cdots,\sigma_\nu,\sigma_{\nu+1},\cdots,\sigma_{n-1}\}}}\prod_{\tau=\nu+1}^{n-1}a_{\sigma_\tau}$$

and

$$\sum_{\substack{1\le\sigma_1'<\cdots<\sigma_{\nu-1}'\le n\ ,\ 1\le\ell\le n\\ \{\sigma_1',\cdots,\sigma_{\nu-1}',\ell\}=\{\sigma_1,\cdots,\sigma_\nu\}}}\delta_\ell^-\prod_{\tau=1}^{\nu-1}\overline{a}_{\sigma_\tau'}=\sum_{\substack{1\le\sigma_1^0<\cdots<\sigma_{\nu-2}^0\le n\ ,\ 1\le\ell\le n\\ \{\sigma_1^0,\cdots,\sigma_{\nu-2}^0,\ell,\ell^*+1\}=\{\sigma_1,\cdots,\sigma_\nu\}}}\delta_\ell^-\overline{a}_{\ell^*+1}\prod_{\tau=1}^{\nu-2}\overline{a}_{\sigma_\tau^0}$$

$$=\sum_{\substack{1\le\sigma_1^0<\cdots<\sigma_{\nu-2}^0\le n\ ,\ 1\le\ell^*+1\le n\\ \{\sigma_1^0,\cdots,\sigma_{\nu-2}^0,\ell,\ell^*+1\}=\{\sigma_1,\cdots,\sigma_\nu\}}}\overline{a}_{\ell^*+1}\prod_{\tau=1}^{\nu-2}\overline{a}_{\sigma_\tau^0}=\sum_{\substack{1\le\sigma_1''<\cdots<\sigma_{\nu-1}''\le n\ ,\ \ell^*+1\in\{\sigma_1'',\cdots,\sigma_{\nu-1}''\}\\ \ell\in\{1,\cdots,n\}\setminus\{\sigma_1'',\cdots,\sigma_{\nu-1}'',\sigma_{\nu+1},\cdots,\sigma_n\}}}\prod_{\tau=1}^{\nu-1}\overline{a}_{\sigma_\tau''}\ .$$

Thus

$$I_p(n-1)=\sum_{\nu=1}^{n-2}\sum_{1\le\sigma_1<\cdots<\sigma_\nu\le n}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}\sum_{\substack{1\le\sigma_{\nu+1}<\cdots<\sigma_{n-1}\le n\ ,\ k^*+1\in\{\sigma_{\nu+1},\cdots,\sigma_{n-1}\}\\ k\in\{1,\cdots,n\}\setminus\{\sigma_1,\cdots,\sigma_\nu,\sigma_{\nu+1},\cdots,\sigma_{n-1}\}}}\prod_{\tau=\nu+1}^{n-1}a_{\sigma_\tau}$$

$$+\sum_{\nu=2}^{n-1}\sum_{1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}a_{\sigma_{\nu+1}}\cdots a_{\sigma_n}\sum_{\substack{1\le\sigma_1''<\cdots<\sigma_{\nu-1}''\le n\ ,\ \ell^*+1\in\{\sigma_1'',\cdots,\sigma_{\nu-1}''\}\\ \ell\in\{1,\cdots,n\}\setminus\{\sigma_1'',\cdots,\sigma_{\nu-1}'',\sigma_{\nu+1},\cdots,\sigma_n\}}}\prod_{\tau=1}^{\nu-1}\overline{a}_{\sigma_\tau''}$$

$$=\sum_{\nu=1}^{n-2}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\ ,\ 1\le\sigma_{\nu+1}<\cdots<\sigma_{n-1}\le n\\ k\in\{1,\cdots,n\}\setminus\{\sigma_1,\cdots,\sigma_\nu,\sigma_{\nu+1},\cdots,\sigma_{n-1}\}\ ,\ k^*+1\in\{\sigma_{\nu+1},\cdots,\sigma_{n-1}\}}}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}a_{\sigma_{\nu+1}}\cdots a_{\sigma_{n-1}}$$

$$+\sum_{\nu=1}^{n-2}\sum_{1\le\sigma_{\nu+1}<\cdots<\sigma_{n-1}\le n}a_{\sigma_{\nu+1}}\cdots a_{\sigma_{n-1}}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\ ,\ \ell^*+1\in\{\sigma_1,\cdots,\sigma_\nu\}\\ \ell\in\{1,\cdots,n\}\setminus\{\sigma_1,\cdots,\sigma_\nu,\sigma_{\nu+1},\cdots,\sigma_{n-1}\}}}\prod_{\tau=1}^{\nu}\overline{a}_{\sigma_\tau}$$

$$=\sum_{\nu=1}^{n-2}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\ ,\ 1\le\sigma_{\nu+1}<\cdots<\sigma_{n-1}\le n\\ k\in\{1,\cdots,n\}\setminus\{\sigma_1,\cdots,\sigma_\nu,\sigma_{\nu+1},\cdots,\sigma_{n-1}\}\ ,\ k^*+1\in\{\sigma_{\nu+1},\cdots,\sigma_{n-1}\}}}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}a_{\sigma_{\nu+1}}\cdots a_{\sigma_{n-1}}$$

$$+\sum_{\nu=1}^{n-2}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\ ,\ 1\le\sigma_{\nu+1}<\cdots<\sigma_{n-1}\le n\\ \ell\in\{1,\cdots,n\}\setminus\{\sigma_1,\cdots,\sigma_\nu,\sigma_{\nu+1},\cdots,\sigma_{n-1}\}\ ,\ \ell^*+1\in\{\sigma_1,\cdots,\sigma_\nu\}}}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}a_{\sigma_{\nu+1}}\cdots a_{\sigma_{n-1}}$$

$$=\sum_{\nu=1}^{n-2}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_{n-1}\le n}}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}a_{\sigma_{\nu+1}}\cdots a_{\sigma_{n-1}}=I_g(n-1).$$

And about the third term

$$I_p(n-2)=\sum_{\nu=1}^{n-1}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\Big(\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}\sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{n-2}\le n\\ \nu+1\le\alpha_{n-1}<\alpha_n\le n}}a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{n-2}}}\delta^+_{\sigma_{\alpha_{n-1}}}\delta^+_{\sigma_{\alpha_n}}$$

$$+\sum_{\substack{1\le\alpha_1<\cdots<\alpha_{\nu-1}\le\nu\\ 1\le\alpha_\nu\le\nu}} \overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_{\nu-1}}}\delta^-_{\sigma_{\alpha_\nu}} \sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{n-1}\le n\\ \nu+1\le\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{n-1}}}\delta^+_{\sigma_{\alpha_n}}$$

$$+a_{\sigma_{\nu+1}}\cdots a_{\sigma_n}\sum_{\substack{1\le\alpha_1<\cdots<\alpha_{\nu-2}\le\nu\\ 1\le\alpha_{\nu-1}<\alpha_\nu\le\nu}} \overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_{\nu-2}}}\delta^-_{\sigma_{\alpha_{\nu-1}}}\delta^-_{\sigma_{\alpha_\nu}}\Big)$$

Keeping $\{\ \alpha_{n-1}\ ,\ \alpha_n \mid n\le\alpha_{n-1}<\alpha_n\le n\ \}=\emptyset$ in mind we have

$$I_p(n-2)=\sum_{\nu=1}^{n-2}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}} \overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}\sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{n-2}\le n\\ \nu+1\le\alpha_{n-1}<\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{n-2}}}\delta^+_{\sigma_{\alpha_{n-1}}}\delta^+_{\sigma_{\alpha_n}}$$

$$+\sum_{\nu=1}^{n-1}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\sum_{\substack{1\le\alpha_1<\cdots<\alpha_{\nu-1}\le\nu\\ 1\le\alpha_\nu\le\nu}} \overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_{\nu-1}}}\delta^-_{\sigma_{\alpha_\nu}}\sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{n-1}\le n\\ \nu+1\le\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{n-1}}}\delta^+_{\sigma_{\alpha_n}}$$

$$+\sum_{\nu=1}^{n-1}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}} a_{\sigma_{\nu+1}}\cdots a_{\sigma_n}\sum_{\substack{1\le\alpha_1<\cdots<\alpha_{\nu-2}\le\nu\\ 1\le\alpha_{\nu-1}<\alpha_\nu\le\nu}} \overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_{\nu-2}}}\delta^-_{\sigma_{\alpha_{\nu-1}}}\delta^-_{\sigma_{\alpha_\nu}}\ .$$

Again by applying the definition of δ and paying attention to

$$\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_{n-2}}\delta^+_{\sigma_{n-1}}\delta^+_{\sigma_n}=0\ ,\qquad \delta^-_{\sigma_1}a_{\sigma_{\alpha_2}}\cdots a_{\sigma_{\alpha_{n-1}}}\delta^+_{\sigma_{\alpha_n}}=0\ ,$$

$$\overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_{n-2}}}\delta^-_{\sigma_{\alpha_{n-1}}}\delta^+_{\sigma_n}=0\ ,\qquad \delta^-_{\sigma_1}\delta^-_{\sigma_2}a_{\sigma_3}\cdots a_{\sigma_n}=0$$

and that the first term of the third sum does not exist we get

$$I_p(n-2)=\sum_{\nu=1}^{n-3}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}} \overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}\sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{n-2}\le n\\ \nu+1\le\alpha_{n-1}<\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{n-2}}}\delta^+_{\sigma_{\alpha_{n-1}}}\delta^+_{\sigma_{\alpha_n}}$$

$$+\sum_{\nu=2}^{n-2}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\sum_{\substack{1\le\alpha_1<\cdots<\alpha_{\nu-1}\le\nu\\ 1\le\alpha_\nu\le\nu}} \overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_{\nu-1}}}\delta^-_{\sigma_{\alpha_\nu}}\sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{n-1}\le n\\ \nu+1\le\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{n-1}}}\delta^+_{\sigma_{\alpha_n}}$$

$$+\sum_{\nu=3}^{n-1}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}} a_{\sigma_{\nu+1}}\cdots a_{\sigma_n}\sum_{\substack{1\le\alpha_1<\cdots<\alpha_{\nu-2}\le\nu\\ 1\le\alpha_{\nu-1}<\alpha_\nu\le\nu}} \overline{a}_{\sigma_{\alpha_1}}\cdots\overline{a}_{\sigma_{\alpha_{\nu-2}}}\delta^-_{\sigma_{\alpha_{\nu-1}}}\delta^-_{\sigma_{\alpha_\nu}}\ .$$

Due to the fact that for fixed $\{\sigma_{\nu+1},\cdots,\sigma_n\}$

$$\sum_{\substack{\nu+1\le\alpha_{\nu+1}<\cdots<\alpha_{n-2}\le n\\ \nu+1\le\alpha_{n-1}<\alpha_n\le n}} a_{\sigma_{\alpha_{\nu+1}}}\cdots a_{\sigma_{\alpha_{n-2}}}\delta^+_{\sigma_{\alpha_{n-1}}}\delta^+_{\sigma_{\alpha_n}}=\delta^+_{\sigma_n}(\delta^+_{\sigma_{n-1}}\prod_{\substack{\tau=\nu+1\\ \tau\ne n-1}}^{n-1}a_{\sigma_\tau}+\delta^+_{\sigma_{n-2}}\prod_{\substack{\tau=\nu+1\\ \tau\ne n-2}}^{n-1}a_{\sigma_\tau}$$

$$+\cdots+\delta^+_{\sigma_{n-k}}\prod_{\substack{\tau=\nu+1\\ \tau\neq n-k}}^{n-1}a_{\sigma_\tau}+\cdots+\delta^+_{\sigma_{\nu+2}}\prod_{\substack{\tau=\nu+1\\ \tau\neq\nu+2}}^{n-1}a_{\sigma_\tau}+\delta^+_{\sigma_{\nu+1}}\prod_{\substack{\tau=\nu+1\\ \tau\neq\nu+1}}^{n-1}a_{\sigma_\tau})$$

$$+\delta^+_{\sigma_{n-1}}(\delta^+_{\sigma_{n-2}}\prod_{\substack{\tau=\nu+1\\ \tau\notin\{n-1,n-2\}}}^{n}a_{\sigma_\tau}+\delta^+_{\sigma_{n-3}}\prod_{\substack{\tau=\nu+1\\ \tau\notin\{n-1,n-3\}}}^{n}a_{\sigma_\tau}+\cdots+\delta^+_{\sigma_{n-k}}\prod_{\substack{\tau=\nu+1\\ \tau\notin\{n-1,n-k\}}}^{n}a_{\sigma_\tau}+\cdots+\delta^+_{\sigma_{\nu+2}}\prod_{\substack{\tau=\nu+1\\ \tau\notin\{n-1,\nu+2\}}}^{n}a_{\sigma_\tau}$$

$$+\delta^+_{\sigma_{\nu+1}}\prod_{\substack{\tau=\nu+1\\ \tau\notin\{n-1,\nu+1\}}}^{n}a_{\sigma_\tau})+\cdots+\delta^+_{\sigma_{k+1}}(\delta^+_{\sigma_k}\prod_{\substack{\tau=\nu+1\\ \tau\notin\{k+1,k\}}}^{n}a_{\sigma_\tau}+\delta^+_{\sigma_{k-1}}\prod_{\substack{\tau=\nu+1\\ \tau\notin\{k+1,k-1\}}}^{n}a_{\sigma_\tau}+\cdots+\delta^+_{\sigma_\ell}\prod_{\substack{\tau=\nu+1\\ \tau\notin\{k+1,k-\ell\}}}^{n}a_{\sigma_\tau}+\cdots+$$

$$\delta^+_{\sigma_{\nu+2}}\prod_{\substack{\tau=\nu+1\\ \tau\notin\{k+1,\nu+2\}}}^{n}a_{\sigma_\tau}+\delta^+_{\sigma_{\nu+1}}\prod_{\substack{\tau=\nu+1\\ \tau\notin\{k+1,\nu+1\}}}^{n}a_{\sigma_\tau})+\cdots+\delta^+_{\sigma_{\nu+3}}(\delta^+_{\sigma_{\nu+2}}\prod_{\substack{\tau=\nu+1\\ \tau\notin\{\nu+3,\nu+2\}}}^{n}a_{\sigma_\tau}+\delta^+_{\sigma_{\nu+1}}\prod_{\substack{\tau=\nu+1\\ \tau\notin\{\nu+3,\nu+1\}}}^{n}a_{\sigma_\tau})$$

$$+\delta^+_{\sigma_{\nu+2}}\delta^+_{\sigma_{\nu+1}}\prod_{\substack{\tau=\nu+1\\ \tau\notin\{\nu+2,\nu+1\}}}^{n}a_{\sigma_\tau}=\sum_{1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}\ \sum_{\nu+1\le h_1<h_2\le n}\delta^+_{\sigma_{h_1}}\delta^+_{\sigma_{h_2}}\prod_{\substack{\tau=\nu+1\\ \tau\notin\{h_1\,,\,h_2\}}}^{n}a_{\sigma_\tau}$$

we have

$$I_p(n-2)=\sum_{\nu=1}^{n-3}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}\sum_{\nu+1\le h_1<h_2\le n}\delta^+_{\sigma_{h_1}}\delta^+_{\sigma_{h_2}}\prod_{\substack{\tau=\nu+1\\ \tau\notin\{h_1\,,\,h_2\}}}^{n}a_{\sigma_\tau}$$

$$+\sum_{\nu=2}^{n-2}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\ \sum_{1\le h_1\le\nu}\delta^-_{\sigma_{h_1}}\prod_{\substack{\tau=1\\ \tau\neq h_1}}^{\nu}\overline{a}_{\sigma_\tau}\sum_{\nu+1\le h_2\le n}\delta^+_{\sigma_{h_2}}\prod_{\substack{\tau=\nu+1\\ \tau\neq h_2}}^{n}a_{\sigma_\tau}$$

$$+\sum_{\nu=3}^{n-1}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}a_{\sigma_{\nu+1}}\cdots a_{\sigma_n}\sum_{1\le h_1<h_2\le\nu}\delta^-_{\sigma_{h_1}}\delta^-_{\sigma_{h_2}}\prod_{\substack{\tau=1\\ \tau\notin\{h_1\,,\,h_2\}}}^{\nu}\overline{a}_{\sigma_\tau}$$

$$=\sum_{\nu=1}^{n-3}\sum_{1\le\sigma_1<\cdots<\sigma_\nu\le n}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}\sum_{\substack{1\le\sigma_{\nu+1}<\cdots<\sigma_{n-2}\le n\\ 1\le\sigma_{n-1}<\sigma_n\le n}}\delta^+_{\sigma_{n-1}}\delta^+_{\sigma_n}\prod_{\tau=\nu+1}^{n-2}a_{\sigma_\tau}$$

$$+\sum_{\nu=2}^{n-2}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_{\nu-1}\le n\\ 1\le\sigma_\nu\le n}}\delta^-_{\sigma_\nu}\prod_{\tau=1}^{\nu-1}\overline{a}_{\sigma_\tau}\sum_{\substack{1\le\sigma_{\nu+1}<\cdots<\sigma_{n-1}\le n\\ 1\le\sigma_n\le n}}\delta^+_{\sigma_n}\prod_{\tau=\nu+1}^{n-1}a_{\sigma_\tau}$$

$$+\sum_{\nu=3}^{n-1}\sum_{1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n} a_{\sigma_{\nu+1}}\cdots a_{\sigma_n}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_{\nu-2}\le n\\ 1\le\sigma_{\nu-1}<\sigma_\nu\le n}}\delta^-_{\sigma_{\nu-1}}\delta^-_{\sigma_\nu}\prod_{\tau=1}^{\nu-2}\overline{a}_{\sigma_\tau}\ .$$

The second part is obvious from the discussion about $I_p(n-1)$. It can be simplified as

$$\sum_{\nu=1}^{n-3}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\ ,\ 1\le\sigma_{\nu+1}<\cdots<\sigma_{n-2}\le n\\ \{\sigma_1,\cdots,\sigma_\nu,\sigma_{\nu+1},\cdots,\sigma_{n-2}\}=\{1,\cdots,n\}\backslash\{k_1,k_2\},\ k_1^*+1\in\{\sigma_1,\cdots,\sigma_\nu\},\ k_2^*+1\in\{\sigma_{\nu+1},\cdots,\sigma_{n-2}\}}}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}a_{\sigma_{\nu+1}}\cdots a_{\sigma_{n-2}}\ .$$

For the first part we have

$$\sum_{\nu=1}^{n-3}\sum_{1\le\sigma_1<\cdots<\sigma_\nu\le n}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}\Bigg(\sum_{\substack{1\le\sigma_{\nu+1}<\cdots<\sigma_{n-2}\le n\\ 1\le k_1<k_2\le n\ ,\ k_2-k_1=1}}\delta^+_{k_2-1}\delta^+_{k_2}+\sum_{\substack{1\le\sigma_{\nu+1}<\cdots<\sigma_{n-2}\le n\\ 1\le k_1<k_2\le n\ ,\ 1<k_2-k_1}}\delta^+_{k_1}\delta^+_{k_2}\Bigg)\prod_{\tau=\nu+1}^{n-2}a_{\sigma_\tau}$$

$$=\sum_{\nu=1}^{n-3}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\ ,\ 1\le\sigma_{\nu+1}<\cdots<\sigma_{n-2}\le n\\ \{\sigma_1,\cdots,\sigma_\nu,\sigma_{\nu+1},\cdots,\sigma_{n-2}\}=\{1,\cdots,n\}\backslash\{k_2-1,\ k_2\}\ ,\ k_2^*+1\in\{\sigma_{\nu+1},\cdots,\sigma_{n-2}\}}}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}a_{\sigma_\nu}\cdots a_{\sigma_{n-2}}$$

$$+\sum_{\nu=1}^{n-3}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\ ,\ 1\le\sigma_{\nu+1}<\cdots<\sigma_{n-2}\le n\\ \{\sigma_1,\cdots,\sigma_\nu,\sigma_{\nu+1},\cdots,\sigma_{n-2}\}=\{1,\cdots,n\}\backslash\{k_1,\ k_2\},\ \{k_1+1,\ k_2^*+1\big|1<k_2-k_1\}\subset\{\sigma_{\nu+1},\cdots,\sigma_{n-2}\}\ ,}}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}a_{\sigma_\nu}\cdots a_{\sigma_{n-2}}$$

$$=\sum_{\nu=1}^{n-3}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\ ,\ 1\le\sigma_{\nu+1}<\cdots<\sigma_{n-2}\le n\\ \{\sigma_1,\cdots,\sigma_\nu,\sigma_{\nu+1},\cdots,\sigma_{n-2}\}=\{1,\cdots,n\}\backslash\{k_1,\ k_2\big|k_1<k_2\}\ ,\ \{k_1+1,\ k_2^*+1\}\backslash\{k_1,\ k_2\}\subset\{\sigma_{\nu+1},\cdots,\sigma_{n-2}\}}}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}a_{\sigma_\nu}\cdots a_{\sigma_{n-2}}\ .$$

Similarly for the third term it is easy to get

$$\sum_{\nu=3}^{n-1}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_{\nu-2}\le n\ ,\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n\\ \{\sigma_1,\cdots,\sigma_{\nu-2},\sigma_{\nu+1},\cdots,\sigma_n\}=\{1,\cdots,n\}\backslash\{k_1,\ k_2\big|k_1<k_2\}\ ,\ \{k_1+1,\ k_2^*+1\}\backslash\{k_1,\ k_2\}\subset\{\sigma_1,\cdots,\sigma_{\nu-2}\}}}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_{\nu-2}}a_{\sigma_{\nu+1}}\cdots a_{\sigma_n}$$

$$=\sum_{\nu=1}^{n-3}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\ ,\ 1\le\sigma_{\nu+1}<\cdots<\sigma_{n-2}\le n\\ \{\sigma_1,\cdots,\sigma_\nu,\sigma_{\nu+1},\cdots,\sigma_{n-2}\}=\{1,\cdots,n\}\backslash\{k_1,\ k_2\big|k_1<k_2\}\ ,\ \{k_1+1,\ k_2^*+1\}\backslash\{k_1,\ k_2\}\subset\{\sigma_1,\cdots,\sigma_\nu\}}}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}a_{\sigma_{\nu+1}}\cdots a_{\sigma_{n-2}}.$$

The second term we can split into two parts $\{k_1,\ k_2|k_1<k_2\}\ :\ \{k_1+1,\ k_2^*+1\}$ and $\{k_1,\ k_2|k_1>k_2\}\ :\ \{k_1^*+1,\ k_2+1\}$. Each belongs to two different zones of the plane. The first and the third term represents another two parts. Thus we have four zones of the plane complete, i.e.,

$$I_p(n-2)=\sum_{\nu=1}^{n-3}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\ ,\ 1\le\sigma_{\nu+1}<\cdots<\sigma_{n-2}\le n\\ \{k_1,\ k_2\}=\{1,\cdots,n\}\backslash\{\sigma_1,\cdots,\sigma_\nu,\sigma_{\nu+1},\cdots,\sigma_{n-2}\}\ ,\ \{k_1^*+1,\ k_2^*+1\}\backslash\{k_1,\ k_2\}\subset\{\sigma_1,\cdots,\sigma_\nu,\sigma_{\nu+1},\cdots,\sigma_{n-2}\}}}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}a_{\sigma_{\nu+1}}\cdots a_{\sigma_{n-2}}$$

$$=\sum_{\nu=1}^{n-3}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\1\le\sigma_{\nu+1}<\cdots<\sigma_{n-2}\le n}}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_\nu}a_{\sigma_{\nu+1}}\cdots a_{\sigma_{n-2}}=I_g(n-2).$$

For $I_p(k)$ $(2\le k\le n)$ it is not difficult to see that the equalities

$$\{\alpha_1,\cdots,\alpha_t,\alpha_{t+1},\cdots,\alpha_\nu\Big|1\le\alpha_1<\cdots<\alpha_t\le\nu,1\le\alpha_{t+1}<\cdots<\alpha_\nu\le\nu\}$$

$$=\{1,\cdots,\nu\}\ ,\ \{\sigma_{\alpha_1},\cdots,\sigma_{\alpha_t},\sigma_{\alpha_{t+1}},\cdots,\sigma_{\alpha_\nu}\}=\{\sigma_1,\cdots,\sigma_\nu\Big|1\le\sigma_1<\cdots<\sigma_\nu\le n\}$$

actually are nothing more than

$$\{\sigma_1,\cdots,\sigma_t\Big|1\le\sigma_1<\cdots<\sigma_t\le n\ \}\cup\{\sigma_{t+1},\cdots,\sigma_\nu\Big|1\le\sigma_{t+1}<\cdots<\sigma_\nu\le n\ \}$$

$$=\{\sigma_1,\cdots,\sigma_\nu\Big|1\le\sigma_1<\cdots<\sigma_\nu\le n\}.$$

Since the same holds for the other part of $I_p(k)$, it can be written as

$$I_p(k)=\sum_{\nu=1}^{n-1}\sum_{\substack{t+\ell=k\\1\le t\le\nu,\ 1\le\ell\le k-\nu}}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_t\le n\\1\le\sigma_{t+1}<\cdots<\sigma_\nu\le n}}\prod_{\tau=1}^{t}\overline{a}_{\sigma_\tau}\prod_{\tau=t+1}^{\nu}\delta^-_{\sigma_\tau}\sum_{\substack{1\le\sigma_{\nu+1}<\cdots<\sigma_{\nu+\ell}\le n\\1\le\sigma_{\nu+\ell+1}<\cdots<\sigma_n\le n}}\prod_{\tau=\nu+1}^{\nu+\ell}a_{\sigma_\tau}\prod_{\tau=\nu+\ell+1}^{n}\delta^+_{\sigma_\tau}$$

As it is stated by the notation of $\delta_k^{\chi_k}$ it is obvious that $\delta_{k_1}^{\chi_{k_1}}=1$ if and only if $\chi_{k_1}=\chi_{k_1+1}$. If $\delta^-_{k_1}=1$ then $\chi_{k_1+1}=-$, i.e., $k_1+1\in\{\sigma_1,\cdots,\sigma_\nu\}$, but at this stage it is not yet clear if $k_1+1\in\{\sigma_1,\cdots,\sigma_t\}$ or $k_1+1\in\{\sigma_{t+1},\cdots,\sigma_\nu\}$. To know this we look at whether there is $\delta^-_{k_2}=1$ with $k_2=k_1^*+1$. If it is the case then $\{k_1,\ k_1+1\}\subset\{\sigma_{t+1},\cdots,\sigma_\nu\}$, else $k_1+1\in\{\sigma_1,\cdots,\sigma_t\}$ but $\{k_1,\ k_2\}\subset\{\sigma_{t+1},\cdots,\sigma_\nu\}$. Anyway for k_1 it is sure that $\{k_1^*+1\}\backslash\{k_2\}\subset\{\sigma_1,\cdots,\sigma_t\}$ where $k_1,k_2\in\{1,\cdots,n\}$. Similarly $\delta^+_{k_1}=1$ can be considered. Thus

$$I_p(k)=\sum_{\nu=1}^{n-1}\sum_{\substack{t+\ell=k\\1\le t\le\nu,\ 1\le\ell\le k-\nu}}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_t\le n\ ,\ 1\le\sigma_{\nu+1}<\cdots<\sigma_{\nu+\ell}\le n\\1\le\sigma_{t+1}<\cdots<\sigma_\nu\le n\ ,\ 1\le\sigma_{\nu+\ell+1}<\cdots<\sigma_n\le n\\\{\sigma_{t+1}+1,\cdots,\sigma^*_\nu+1\}\backslash\{\sigma_{t+1},\cdots,\sigma_\nu\}\subset\{\sigma_1,\cdots,\sigma_t\}\\\{\sigma_{\nu+\ell+1}+1,\cdots,\sigma^*_n+1\}\backslash\{\sigma_{\nu+\ell+1},\cdots,\sigma_n\}\subset\{\sigma_{\nu+1},\cdots,\sigma_{\nu+\ell}\}}}\overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_t}a_{\sigma_{\nu+1}}\cdots a_{\sigma_{\nu+\ell}}$$

$$=\sum_{\nu=1}^{n-1}\sum_{\substack{t+\ell=k\\1\le t\le\nu,\ 1\le\ell\le k-\nu}}\sum_{\substack{1\le\lambda_1<\cdots<\lambda_t\le n\ ,\ 1\le\lambda_{t+1}<\cdots<\lambda_k\le n\\\{h_1+1,\cdots,\ h_{\nu-t-1}+1,\ h^*_{\nu-t}+1\}\backslash\{h_1,\cdots,h_{\nu-t}\}\subset\{\lambda_1,\cdots,\lambda_t\}\\\{h_{\nu-t+1}+1,\cdots,h_{n-k-1}+1,h^*_{n-k}+1\}\backslash\{h_{\nu-t+1},\ \cdots,\ h_{n-k}\}\subset\{\lambda_{t+1},\cdots,\lambda_k\}\\1\le h_1<\cdots<h_{\nu-t}\le n\ ,\ 1\le h_{\nu-t+1}<\cdots<h_{n-k}\le n}}\overline{a}_{\lambda_1}\cdots\overline{a}_{\lambda_t}a_{\lambda_{t+1}}\cdots a_{\lambda_k}$$

Observing the set of indices about δ s it is not difficult to see that we have two different elements δ^- and δ^+ to fulfill $n-k$ positions: the positions $\{h_1,\cdots,h_{\nu-t}\}$ with δ^- s and the positions $\{h_{\nu-t+1},\ \cdots,\ h_{n-k}\}$ with δ^+ s. Due to $1\le\nu\le n-1$ and $1\le t\le k-1$ it is clear that $\nu-t\in\{0,1,\cdots,n-k\}$. For $\nu-t=0$ from the set of indices we have $\{h_1+1,\cdots,h_{n-k-1}+1,h^*_{n-k}+1\}\backslash\{h_1,\ \cdots,\ h_{n-k}\}\in\{\lambda_{t+1},\cdots,\lambda_k\}$, i.e. there is no δ^- at all and that is only $C^0_{n-k}=1$

possible case. For $\nu - t = 1$ then there is one δ^- which position can be different with possibility C^1_{n-k}. So for $\{0, 1, \cdots, n-k\}$ we have altogether $C^0_{n-k} + C^1_{n-k} + \cdots + C^{n-k}_{n-k} = 2^{n-k}$ different cases. Each one illustrates exactly one block of a $\mathbb{C}^{n-k}$ dimensional space, i.e. the set covers the whole $\mathbb{C}^{n-k}$ space fully. Now it is more clear that every element of $I_p(k)$ comes exactly from one block of the $\mathbb{C}^{n-k}$ space and it contains all the 2^{n-k} different blocks. In another word, the restrictions caused by the set finally vanish completely. Thus we get

$$I_p(k) = \sum_{\nu=1}^{n-1} \sum_{\substack{t+\ell=k \\ 1\le t\le\nu,\ 1\le\ell\le k-\nu}} \sum_{\substack{1\le\lambda_1<\cdots<\lambda_t\le n\ ,\ 1\le\lambda_{t+1}<\cdots<\lambda_k\le n \\ \{h_1,\cdots,h_{\nu-t},h_{\nu-t+1},\ \cdots,\ h_{n-k}\}=\{1,\cdots,n\}\setminus\{\lambda_1,\cdots,\lambda_t,\lambda_{t+1},\cdots,\lambda_k\} \\ \{h_1+1,\cdots,\ h_{\nu-t-1}+1,\ h^*_{\nu-t}+1,h_{\nu-t+1}+1,\cdots,h_{n-k-1}+1,h^*_{n-k}+1\}\setminus\{h_1,\cdots,h_{\nu-t},h_{\nu-t+1},\ \cdots,\ h_{n-k}\} \\ \subset\{\lambda_1,\cdots,\lambda_t,\lambda_{t+1},\cdots,\lambda_k\}}} \overline{a}_{\lambda_1}\cdots\overline{a}_{\lambda_t}a_{\lambda_{t+1}}\cdots a_{\lambda_k}$$

$$= \sum_{\nu=1}^{n-1} \sum_{\substack{t+\ell=k \\ 1\le t\le\nu,\ 1\le\ell\le k-\nu}} \sum_{\substack{1\le\sigma_1<\cdots<\sigma_t\le n \\ 1\le\sigma_{t+1}<\cdots<\sigma_k\le n}} \overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_t}a_{\sigma_{t+1}}\cdots a_{\sigma_k}\ .$$

Paying attention to the fact that $t+\ell = k \le n$ we have actually $1 \le t \le k-1$, $1 \le \ell \le k-1$, and $t + \ell = k$. Therefore the expression can be simplified as

$$I_p(k) = \sum_{t=1}^{k-1} \sum_{\substack{1\le\sigma_1<\cdots<\sigma_t\le n \\ 1\le\sigma_{t+1}<\cdots<\sigma_k\le n}} \overline{a}_{\sigma_1}\cdots\overline{a}_{\sigma_t}a_{\sigma_{t+1}}\cdots a_{\sigma_k}\ .$$

This means $I_p(k) = I_g(k)$ and the proof is completed.

Applying Lemma 6 for

$$a_t^{\chi_t} = \sum_{k_t=1}^{\infty} \alpha_{\chi_t^* k_t} \zeta_t^{\chi_t^* k_t}\ , \quad \zeta_t \in \partial\mathbb{D}_t\ , \quad \sum_{k_t=1}^{\infty} |\alpha_{\chi_t^* k_t}| < \infty\ , \quad 1 \le t \le n\ ,$$

we see that

$$\begin{aligned}
&\prod_{t=1}^{n}\Big(\sum_{k_t=1}^{\infty}\zeta_t^{k_t} + \sum_{k_t=1}^{\infty}\overline{\zeta}_t^{k_t} + 1\Big)\alpha_{\chi_1^*k_1,\cdots,\chi_n^*k_n} \\
&-\Big[\prod_{t=1}^{n}\Big(\sum_{k_t=1}^{\infty}\zeta_t^{k_t}+1\Big) + \prod_{t=1}^{n}\Big(\sum_{k_t=1}^{\infty}\overline{\zeta}_t^{k_t}+1\Big) - 1\Big]\alpha_{\chi_1^*k_1,\cdots,\chi_n^*k_n} \\
= &\sum_{\nu=1}^{n-1} \sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n \\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}} \prod_{t=1}^{\nu}\Big(\sum_{k_{\sigma_t}=1}^{\infty}\overline{\zeta}_{\sigma_t}^{k_{\sigma_t}} + \delta^-_{\sigma_t}\Big) \prod_{t=\nu+1}^{n}\Big(\sum_{k_{\sigma_t}=1}^{\infty}\zeta_{\sigma_t}^{k_{\sigma_t}} + \delta^+_{\sigma_t}\Big)\alpha_{\chi_1^*k_1,\cdots,\chi_n^*k_n} \\
=: &\sum_{\nu=1}^{n-1} \sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n \\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}} \prod_{t=1}^{n}\Big(\sum_{k_t=1}^{\infty}\zeta_t^{\chi_t^*k_t} + \delta_t^{\chi_t}\Big)\alpha_{\chi_1^*k_1,\cdots,\chi_n^*k_n}\ , \quad z \in \mathbb{D}^n\ , \quad \zeta \in \partial_0\mathbb{D}^n,
\end{aligned}$$

where

$$\prod_{t=1}^{n}\Big(\sum_{k_t=1}^{\infty}\zeta_t^{k_t}+\sum_{k_t=1}^{\infty}\overline{\zeta}_t^{k_t}+1\Big)\alpha_{\chi_1^*k_1,\cdots,\chi_n^*k_n}=\sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty}\alpha_{\chi_1^*k_1,\cdots,\chi_n^*k_n}\zeta_1^{\chi_1^*k_1}\cdots\zeta_n^{\chi_n^*k_n}\ .$$

Therefore by this devision of boundary values, an arbitrary analytic function $w^{\chi(\nu)}(z)$ in $I\!D^{\chi(\nu)}$, without loss of generality, possesses the form

$$\prod_{t=1}^{n}\Big(\sum_{k_t=1}^{+\infty}(z_t^{k_t})^{\chi_t^*}+\delta_t^{\chi_t}\Big)\alpha_{\chi_1^*k_1,\cdots,\chi_n^*k_n}=:w^{\chi(\nu)}(z),\quad 0\le\nu<n\ ,\quad z\in I\!D^{\chi(\nu)};$$

$$\Big[\prod_{t=1}^{n}\Big(\sum_{k_t=1}^{+\infty}z_t^{-k_t}+1\Big)-1\Big]\alpha_{\chi_1^*k_1,\cdots,\chi_n^*k_n}=:w^{\chi(n)}(z)=w^{-\chi(0)}(z)\ ,\quad z\in(I\!D^-)^n\ ,$$

and they converge absolutely and uniformly even on $\partial_0 I\!D^n$. Obviously by means of (3.2) the above analytic functions can be written as

$$w^{\chi(\nu)}(z)=\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\varphi(\zeta,z)C^{\chi(\nu)}(z,\zeta)\frac{d\zeta}{\zeta}\tag{3.4}$$

where

$$C^{\chi(\nu)}(z,\zeta)=\begin{cases}\prod_{k=1}^{n}\Big[\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}}+\delta_k^{\chi_k}\Big] & ,\quad 0\le\nu\le n-1\ ,\ z\in I\!D^{\chi(\nu)}\\ \Big[\frac{z}{z-\zeta}-1\Big] & ,\quad z\in(I\!D^-)^n\\ \frac{\zeta}{\zeta-z} & ,\quad z\in I\!D^n\ .\end{cases}$$

We call (3.4) the torus related Cauchy integral. For the function $w^{\chi(\nu)}(z)$, $0<\nu<n$, defined by (3.4) there exist at least two integers $h\neq k, h,k\in\{1,\cdots,n\}$, such that

$$w^{\chi(\nu)}(z)\Big|_{z_h=\infty}=0\ ,\ h\in\{\sigma_1,\cdots,\sigma_\nu\}\ ;\quad w^{\chi(\nu)}(z)\Big|_{z_k=0}=0\ ,\ k\in\{\sigma_{\nu+1},\cdots,\sigma_n\}\ .\tag{3.5}$$

About $w^{\chi(0)}(z)$ and $w^{\chi(n)}(z)$ there is nothing to say, except $w^{\chi(n)}(\infty)=0$. For the other part: in the two dimensional case we do not have $\delta_t^{\chi_t}\neq 0$; but from three dimension on we have at least n times $\delta_t^{\chi_t}=1$ and this means at least n times $w^{\chi(\nu)}(z)|_{z_t=\infty}\neq 0$. In this sense (3.5) is only a partial phenomena from three dimension on. Solving the problem in 2 dimensional space does not mean that it always is the same as solving the problem really in n dimensional space.

Naturally, for the torus related analytic functions it is necessary to satisfy condition (3.5). This property holds at least for one pair of components of the variable but not for every pair. In the case $n=2k+1$, $k\in I\!N$, there exists at least one component for which the function fails to satisfy condition (3.5). In the case $n=2k$, $k\in I\!N$ there exists one and only one pair of 4^k-2 analytic functions totally satisfying condition (3.5). Regarding the other 4^k-4

analytic functions, for every analytic function there exist at least two pairs of components of the variable and for one pair the function satisfies the condition (3.5), while for the second pair it does not, see **Appendix** .

Interestingly

$$\overline{\prod_{t=1}^{n}\left(a_t^{\chi_t}+\delta_t^{\chi_t}\right)}=\prod_{t=1}^{n}\left(a_t^{-\chi_t}+\delta_t^{\chi_t}\right)$$

so if $\varphi(\eta)$ is real and $\varphi(0)=0$, then

$$\overline{w^{\chi(\nu)}(\zeta)}:=\overline{\prod_{t=1}^{n}\Big(\sum_{k_t=1}^{+\infty}(\zeta_t^{k_t})^{\chi_t^*}+\delta_t^{\chi_t}\Big)\alpha_{\chi_1^*k_1,\cdots,\chi_n^*k_n}}$$

$$=\prod_{t=1}^{n}\Big(\sum_{k_t=1}^{+\infty}(\zeta_t^{k_t})^{-\chi_t^*}+\delta_t^{\chi_t}\Big)\alpha_{-\chi_1^*k_1,\cdots,-\chi_n^*k_n}=w^{-\chi(\nu)}(\zeta)\ ,\quad \zeta\in\partial_0\mathbb{D}^n,\ 0\le\nu\le n$$

holds and $w^{\chi(\nu)}(\zeta)$ can be seen as the reflection of $w^{-\chi(\nu)}(\zeta)$ with respect to $\partial_0\mathbb{D}^n$. This property of boundary values will be quite useful in our further discussion.

It is known that if ϕ^+ , ϕ^- are boundary values of analytic functions in $\mathbb{D}^+$, $\mathbb{D}^-$ respectively and are continuous on $\partial\mathbb{D}$, then

$$(a)\qquad \frac{1}{(2\pi i)^n}\int_{\partial\mathbb{D}}\phi^+(\zeta)\frac{\zeta/z}{1-\zeta/z}\frac{d\zeta}{\zeta}\ =\ 0\ ,\quad z\in\mathbb{D}^-\ ,$$

$$(b)\qquad \frac{1}{(2\pi i)^n}\int_{\partial\mathbb{D}}\phi^-(\zeta)\frac{z/\zeta}{1-z/\zeta}\frac{d\zeta}{\zeta}\ =\ 0\ ,\quad z\in\mathbb{D}^+\ .$$

Repeatedly applying these two formulas leads to the following result which will be useful in the sequel.

Lemma 7 *Let* $\phi^{\chi(\nu)}(\zeta)$ *be boundary values of a function, holomorphic in* $\mathbb{D}^{\chi(\nu)}$ *and continuous on* $\mathbb{D}^{\chi(\nu)}\cup\partial_0\mathbb{D}^n$ *. Then*

$$\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\phi^{\chi(\nu)}(\zeta)C^{\chi^0(\mu)}(z,\zeta)\frac{d\zeta}{\zeta}=0,\quad z\notin\partial_0\mathbb{D}^n,\ \chi(\nu)\ne\chi^0(\mu),\quad 0\le\nu,\mu\le n,\qquad(3.6)$$

where $C^{\chi^0(\mu)}(\zeta,z)$ *is defined as in (3.4) and* $\phi^{\chi(\nu)}(\zeta)$ *is defined as in (3.3) .*

Proof If $\nu=0,\ \mu=n$, then by (a) for $z\in(\mathbb{D}^-)^n$ it is clear that

$$\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\phi^{\chi(0)}(\zeta)\Big[\frac{z}{z-\zeta}-1\Big]\frac{d\zeta}{\zeta}=\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\phi^{\chi(0)}(\zeta)\Big[\frac{1}{1-\zeta/z}-1\Big]\frac{d\zeta}{\zeta}=0\ .$$

If $\nu=n,\ \mu=0,$ then by (b) and (3.5) for $z\in\mathbb{D}^n$ we have

$$\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\phi^{-\chi(0)}(\zeta)\frac{\zeta}{\zeta-z}\frac{d\zeta}{\zeta}=\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\phi^{-\chi(0)}(\zeta)\frac{1}{1-z\zeta^{-1}}\frac{d\zeta}{\zeta}=\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}$$

$$\times\phi^{-\chi(0)}(\zeta)\Big[\Big(\frac{1}{1-z/\zeta}-1\Big)+1\Big]\frac{d\zeta}{\zeta}=\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\phi^{-\chi(0)}(\zeta)\frac{d\zeta}{\zeta}=\phi^{-\chi(0)}(\infty)=0\ .$$

In all other cases if $\chi(\nu) \neq \chi^0(\mu)$ then there exists at least an index t, $t \in \{1, \cdots, n\}$ such that $\chi_t \neq \chi_t^0$, i.e., $\chi_t = -\chi_t^0$. This means we have to deal with

$$\frac{1}{(2\pi i)} \int_{\partial \mathbb{D}_t} \phi^{\chi_t}(\zeta_t, .) \Big[\delta_t^{\chi_t^0} + \frac{(z_t\zeta_t^{-1})^{\chi_t^0}}{1-(z_t\zeta_t^{-1})^{\chi_t^0}}\Big] \frac{d\zeta_t}{\zeta_t} = \frac{1}{(2\pi i)} \int_{\partial \mathbb{D}_t} \phi^{-\chi_t^0}(\zeta_t, .) \Big[\delta_t^{\chi_t^0} + \frac{(z_t\zeta_t^{-1})^{\chi_t^0}}{1-(z_t\zeta_t^{-1})^{\chi_t^0}}\Big] \frac{d\zeta_t}{\zeta_t}$$

$$= \frac{1}{(2\pi i)} \int_{\partial \mathbb{D}_t} \phi^{-\chi_t^0}(\zeta_t, .) \delta_t^{\chi_t^0} \frac{d\zeta_t}{\zeta_t} = \frac{\phi(.)}{(2\pi i)} \int_{\partial \mathbb{D}_t} \delta_t^{-\chi_t^0} \delta_t^{\chi_t^0} \frac{d\zeta_t}{\zeta_t} = 0 ,$$

where

$$\delta_t^{-\chi_t^0} \delta_t^{\chi_t^0} = \frac{|-\chi_t^0 + \chi_{t+1}^0|}{2} \frac{|\chi_t^0 + \chi_{t+1}^0|}{2} = 0 , \text{ for } \chi_t^0, \ \chi_{t+1}^0 \in \{+1, -1\} .$$

and (a) or (b) were applied.

Thus (3.6) is always true in the sense of (3.3) and (3.4).

The sense of the Lemma is that by classifying the boundary values and defining the kernel as in (3.3) and (3.4) for a given function on $\partial_0 \mathbb{D}^n$, for a kernel of the domain $\mathbb{D}^{\chi(\nu)}$ only the domain $\mathbb{D}^{\chi(\nu)}$ related part of the given function which is boundary value of a function, holomorphic in $\mathbb{D}^{\chi(\nu)}$, produces a nontrivial result with the kernel of the domain $\mathbb{D}^{\chi(\nu)}$.

3.4 The Schwarz problem for polydiscs

The Schwarz problem for analytic functions in polydiscs was considered in [2] Chapter 5. Therefore we need to consider the Schwarz problem only for analytic functions in the other torus related domains.

Lemma 8 *Let $\phi^{\chi(\nu)}$ be holomorphic in $\mathbb{D}^{\chi(\nu)}$ and continuous on $\mathbb{D}^{\chi(\nu)} \cup \partial_0 \mathbb{D}^n$. Then*

$$\phi^{\chi(0)}(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Big(\mathrm{Re}\phi^{\chi(0)}(\zeta)\Big) \Big[2\frac{1}{1-z/\zeta} - 1\Big] \frac{d\zeta}{\zeta} + iIm\phi^{\chi(0)}(0) , \ z \in \mathbb{D}^n , \tag{3.7}$$

$$\mathrm{Re}\phi^{\chi(0)}(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Big(\mathrm{Re}\phi^{\chi(0)}(\zeta)\Big) \prod_{k=1}^n \mathrm{Re}\frac{\zeta_k + z_k}{\zeta_k - z_k} \frac{d\zeta}{\zeta} , \ z \in \mathbb{D}^n , \tag{3.8}$$

$$\phi^{\chi(n)}(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Big(2\mathrm{Re}\phi^{\chi(n)}(\zeta)\Big) \Big[\frac{1}{1-\zeta/z} - 1\Big] \frac{d\zeta}{\zeta}, \quad z \in (\mathbb{D}^-)^n , \tag{3.9}$$

$$\mathrm{Re}\phi^{\chi(n)}(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Big(\mathrm{Re}\phi^{\chi(n)}(\zeta)\Big) \Big[\prod_{k=1}^n \mathrm{Re}\frac{z_k + \zeta_k}{z_k - \zeta_k} - 1\Big] \frac{d\zeta}{\zeta} , \quad z \in (\mathbb{D}^-)^n , \tag{3.10}$$

$$\phi^{\chi(\nu)}(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Big(2\mathrm{Re}\phi^{\chi(\nu)}(\zeta)\Big) \prod_{k=1}^n \Big[\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}} + \delta_k^{\chi_k}\Big] \frac{d\zeta}{\zeta} ,$$
$$0 < \nu < n , \quad z \in \mathbb{D}^{\chi(\nu)} , \tag{3.11}$$

$$\mathrm{Re}\phi^{\chi(\nu)}(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Big(\mathrm{Re}\phi^{\chi(\nu)}(\zeta)\Big) \prod_{k=1}^n \Big[2\mathrm{Re}\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}} + \delta_k^{\chi_k}\Big] \frac{d\zeta}{\zeta} ,$$
$$0 < \nu < n , \quad z \in \mathbb{D}^{\chi(\nu)} . \tag{3.12}$$

The first part of the lemma is proved in [2] page 244. The second and the third part has to be checked and it can be done by a similar method.

Proof From the Cauchy formula

$$\phi^{\chi(n)}(z) = \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n} \phi^{\chi(n)}(\zeta)\left[\frac{1}{1-\zeta/z}-1\right]\frac{d\zeta}{\zeta}$$

$$= \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n} \left(\phi^{\chi(n)}(\zeta)+\overline{\phi^{\chi(n)}(\zeta)}\right)\left[\frac{1}{1-\zeta/z}-1\right]\frac{d\zeta}{\zeta}$$

$$\overline{-\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n} \phi^{\chi(n)}(\zeta)\left[\frac{1}{1-\overline{\zeta}/\overline{z}}-1\right]\frac{d\zeta}{\zeta}}$$

$$= \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n} \left(2\mathrm{Re}\phi^{\chi(n)}(\zeta)\right)\left[\frac{1}{1-\zeta/z}-1\right]\frac{d\zeta}{\zeta}, \quad z\in \mathbb{D}^{-n} ,$$

follows. Passing to the real part leads to

$$\mathrm{Re}\phi^{\chi(n)}(z) = \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n} \left(\mathrm{Re}\phi^{\chi(n)}(\zeta)\right)\left[\frac{z}{z-\zeta}+\frac{\overline{z}}{\overline{\overline{z-\zeta}}}-2\right]\frac{d\zeta}{\zeta} , \quad z\in (\mathbb{D}^-)^n . \tag{3.13}$$

To show this to be the same as (3.10) we apply induction.

Let $n=2$. Applying $\mathrm{Re}[(z_k+\zeta_k)/(z_k-\zeta_k)] = z_k/(z_k-\zeta_k)+\overline{z}_k/(\overline{z}_k-\overline{\zeta}_k)-1$,

$$\left(\frac{z_1}{z_1-\zeta_1}+\frac{\overline{z}_1}{\overline{z}_1-\overline{\zeta}_1}-1\right)\left(\frac{z_2}{z_2-\zeta_2}+\frac{\overline{z}_2}{\overline{z}_2-\overline{\zeta}_2}-1\right)-1$$

$$= \frac{z_1}{z_1-\zeta_1}\frac{z_2}{z_2-\zeta_2}+\frac{\overline{z}_1}{\overline{z}_1-\overline{\zeta}_1}\frac{\overline{z}_2}{\overline{z}_2-\overline{\zeta}_2}-2+\frac{\zeta_1}{z_1-\zeta_1}\frac{\overline{\zeta}_2}{\overline{z}_2-\overline{\zeta}_2}+\frac{\overline{\zeta}_1}{\overline{z}_1-\overline{\zeta}_1}\frac{\zeta_2}{z_2-\zeta_2}$$

and by the Cauchy formula

$$\frac{1}{(2\pi i)^2}\int_{\partial_0 \mathbb{D}^2} \phi^{--}(\zeta)\frac{\zeta_1}{z_1-\zeta_1}\frac{\overline{\zeta}_2}{\overline{z}_2-\overline{\zeta}_2}\frac{d\zeta}{\zeta}$$

$$= \frac{1}{(2\pi i)^2}\int_{\partial_0 \mathbb{D}^2} \phi^{--}(\zeta)\frac{\zeta_1}{z_1-\zeta_1}\frac{\overline{\zeta}_2/\overline{z}_2}{1-\overline{\zeta}_2/\overline{z}_2}\frac{d\zeta_1}{\zeta_1}\frac{d\zeta_2}{\zeta_2} = 0 ,$$

$$\frac{1}{(2\pi i)^2}\int_{\partial_0 \mathbb{D}^2} \overline{\phi^{--}(\zeta)}\frac{\zeta_1}{z_1-\zeta_1}\frac{\overline{\zeta}_2}{\overline{z}_2-\overline{\zeta}_2}\frac{d\zeta}{\zeta}$$

$$= \overline{\frac{1}{(2\pi i)^2}\int_{\partial_0 \mathbb{D}^2} \phi^{--}(\zeta)\frac{\overline{\zeta}_1/\overline{z}_1}{1-\overline{\zeta}_1/\overline{z}_1}\frac{\zeta_2}{z_2-\zeta_2}\frac{d\zeta_1}{\zeta_1}\frac{d\zeta_2}{\zeta_2}} = 0$$

etc. (3.10) is seen to coincide with (3.13) for $n=2$. The remaining part is similar to the first case in the proof of (3.7) and (3.8).

Again by the Cauchy formula

$$\phi^{\chi(\nu)}(z) = \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n} \phi^{\chi(\nu)}(\zeta)\prod_{k=1}^{n}\left[\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}}+\delta_k^{\chi_k}\right]\frac{d\zeta}{\zeta}$$

$$= \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\Big(\phi^{\chi(\nu)}(\zeta)+\overline{\phi^{\chi(\nu)}(\zeta)}\Big)\prod_{k=1}^{n}\Big[\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}}+\delta_k^{\chi_k}\Big]\frac{d\zeta}{\zeta}$$

$$-\overline{\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\phi^{\chi(\nu)}(\zeta)\prod_{k=1}^{n}\Big[\frac{(\overline{z}_k\zeta_k)^{\chi_k^*}}{1-(\overline{z}_k\zeta_k)^{\chi_k^*}}+\delta_k^{\chi_k}\Big]\frac{d\zeta}{\zeta}}$$

$$= \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\Big(2\mathrm{Re}\phi^{\chi(\nu)}(\zeta)\Big)\prod_{k=1}^{n}\Big[\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}}+\delta_k^{\chi_k}\Big]\frac{d\zeta}{\zeta}\,,\quad 0<\nu<n\,,\quad z\in\mathbb{D}^{\chi(\nu)},$$

follows. The second term above vanishes because for ν $(0<\nu<n)$ there exists an integer k $(1\le k\le n)$ such that $\delta_k^{\chi_k}=0$ and both the function and the kernel are analytic in $\zeta_k^{\chi_k^*}$ and therefore the integral vanishes.

Taking the real part gives

$$\mathrm{Re}\phi^{\chi(\nu)}(\zeta) = \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\Big(\mathrm{Re}\phi^{\chi(\nu)}(\zeta)\Big)\Big[\prod_{k=1}^{n}\Big(\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}}+\delta_k^{\chi_k}\Big)$$

$$+\prod_{k=1}^{n}\Big(\frac{(\overline{z}_k\zeta_k)^{\chi_k^*}}{1-(\overline{z}_k\zeta_k)^{\chi_k^*}}+\delta_k^{\chi_k}\Big)\Big]\frac{d\zeta}{\zeta}\,,\quad z\in\mathbb{D}^{\chi(\nu)}. \tag{3.14}$$

In order to show this to be identical with (3.12) we consider

$$I := \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\Big(\phi^{\chi(\nu)}(\zeta)+\overline{\phi^{\chi(\nu)}(\zeta)}\Big)\prod_{k=1}^{n}\Big[\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}}+\frac{(\overline{z}_k\zeta_k)^{\chi_k^*}}{1-(\overline{z}_k\zeta_k)^{\chi_k^*}}+\delta_k^{\chi_k}\Big]\frac{d\zeta}{\zeta}$$

Applying Lemma 7 we have actually

$$I = \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\phi^{\chi(\nu)}(\zeta)\prod_{k=1}^{n}\Big[\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}}+\delta_k^{\chi_k}\Big]\frac{d\zeta}{\zeta}$$

$$+\overline{\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\phi^{\chi(\nu)}(\zeta)\prod_{k=1}^{n}\Big[\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}}+\delta_k^{\chi_k}\Big]\frac{d\zeta}{\zeta}}\,,\quad z\in\mathbb{D}^{\chi(\nu)}$$

and because of (a) or (b) we get

$$I = \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\phi^{\chi(\nu)}(\zeta)\prod_{k=1}^{n}\Big[\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}}+\frac{(\overline{z}_k\zeta_k)^{\chi_k^*}}{1-(\overline{z}_k\zeta_k)^{\chi_k^*}}+\delta_k^{\chi_k}\Big]\frac{d\zeta}{\zeta}$$

$$+\overline{\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\phi^{\chi(\nu)}(\zeta)\prod_{k=1}^{n}\Big[\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}}+\frac{(\overline{z}_k\zeta_k)^{\chi_k^*}}{1-(\overline{z}_k\zeta_k)^{\chi_k^*}}+\delta_k^{\chi_k}\Big]\frac{d\zeta}{\zeta}}$$

$$= \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\phi^{\chi(\nu)}(\zeta)\prod_{k=1}^{n}\Big[2\mathrm{Re}\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}}+\delta_k^{\chi_k}\Big]\frac{d\zeta}{\zeta}$$

$$+\overline{\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\phi^{\chi(\nu)}(\zeta)\prod_{k=1}^{n}\Big[2\mathrm{Re}\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}}+\delta_k^{\chi_k}\Big]\frac{d\zeta}{\zeta}}$$

$$= \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \left[2\mathrm{Re}\phi^{\chi(\nu)}(\zeta) \right] \prod_{k=1}^{n} \left[2\mathrm{Re} \frac{(z_k \zeta_k^{-1})^{\chi_k^*}}{1 - (z_k \zeta_k^{-1})^{\chi_k^*}} + \delta_k^{\chi_k} \right] \frac{d\zeta}{\zeta} , \quad z \in \mathbb{D}^{\chi(\nu)}$$

and this shows that (3.14) coincides with (3.12).

Theorem 11 *Let* γ *be real-valued and belong to* $W(\partial \mathbb{D}^n; \mathbb{C})$ *satisfying*

$$\sum_{\substack{\mu=0 \\ \mu \neq \nu, n-\nu}}^{n} \sum_{\substack{1 \le \sigma_1 < \cdots < \sigma_\mu \le n \\ 1 \le \sigma_{\mu+1} < \cdots < \sigma_n \le n}} \frac{1}{(2\pi i)^n} \int_{\partial \mathbb{D}^n} \gamma(\zeta) C^{\chi(\mu)}(z, \zeta) \frac{d\zeta}{\zeta} = 0 . \tag{3.15}$$

Then

$$\phi^{\chi(0)}(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \left[2 \frac{1}{1 - z/\zeta} - 1 \right] \frac{d\zeta}{\zeta} + iC_1 , \quad z \in \mathbb{D}^n , \tag{3.16}$$

$$\phi^{\chi(n)}(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\gamma(\zeta) \left[\frac{1}{1 - \zeta/z} - 1 \right] \frac{d\zeta}{\zeta}, \quad z \in (\mathbb{D}^-)^n , \tag{3.17}$$

$$\phi^{\chi_{\sigma_1 \cdots \sigma_\nu}(\nu)}(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\gamma(\zeta) \prod_{k=1}^{n} \left(\frac{(z_k \zeta_k^{-1})^{\chi_k^*}}{1 - (z_k \zeta_k^{-1})^{\chi_k^*}} + \delta_k^{\chi_k} \right) \frac{d\zeta}{\zeta} ,$$

$$z \in \mathbb{D}^{\chi_{\sigma_1 \cdots \sigma_\nu}(\nu)} , \ 0 < \nu < n, \tag{3.18}$$

are analytic functions in respective domains with arbitrary real C_1 *and satisfying*

$$\mathrm{Re}\phi^{\pm\chi_{\sigma_1 \cdots \sigma_\nu}(\nu)}(\zeta) = \gamma(\zeta), \quad \zeta \in \partial_0 \mathbb{D}^n , \tag{3.19}$$

where

$$\gamma(\zeta) = \sum_{\nu=0}^{n} \sum_{\substack{1 \le \sigma_1 < \cdots < \sigma_\nu \le n \\ 1 \le \sigma_{\nu+1} < \cdots < \sigma_n \le n}} \gamma^{\chi_{\sigma_1 \cdots \sigma_\nu}(\nu)}(\zeta) , \tag{3.20}$$

$$\gamma^{\chi_{\sigma_1 \cdots \sigma_\nu}(\nu)}(\zeta) = \prod_{t=1}^{n} \left(\sum_{k_t=1}^{+\infty} (\zeta_t^{\chi_t^*})^{k_t} + \delta_t^{\chi_k} \right) \alpha_{\chi_1^* k_1, \cdots, \chi_n^* k_n}, \ 0 \le \nu < n , \quad \zeta \in \partial_0 \mathbb{D}^n,$$

$$\gamma^{\chi(n)}(\zeta) = \sum_{|\kappa|>0 \, , \, \kappa \in \mathbb{Z}_+^n} \alpha_{-\kappa} \zeta^{-\kappa} = \left[\prod_{t=1}^{n} \left(\sum_{k_t=1}^{+\infty} \zeta_t^{-k_t} + 1 \right) - 1 \right] \alpha_{-k_1, \cdots, -k_n}$$

$$\alpha_{\kappa(\chi(\nu))} = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \zeta^{-\kappa(\chi(\nu))} \frac{d\zeta}{\zeta} , \quad \overline{\alpha_{\kappa(\chi(\nu))}} = \alpha_{-\kappa(\chi(\nu))} .$$

The condition (3.15) is not only sufficient but also necessary.

Due to $\gamma \in W(\partial \mathbb{D}^n; \mathbb{C})$, by applying Lemma 7 and Lemma 8, the sufficiency proof of the theorem is quite trivial.

Conversely suppose $\gamma \in W(\partial \mathbb{D}^n; \mathbb{C})$ and the function $\phi^{\chi_{\sigma_1 \cdots \sigma_\nu}(\nu)}(z)$ defined by (3.18) is analytic in $\mathbb{D}^{\chi_{\sigma_1 \cdots \sigma_\nu}(\nu)}$ and satisfies on $\partial_0 \mathbb{D}^n$ condition (3.19). But this can be true if and only if

$$\gamma^{\chi_{\sigma_1 \cdots \sigma_\nu}(\nu)}(\zeta) + \gamma^{-\chi_{\sigma_1 \cdots \sigma_\nu}(\nu)}(\zeta) = \gamma(\zeta) \, , \zeta \in \partial_0 \mathbb{D}^n .$$

By (3.20) this means condition (3.15) holds.

Remark In general $\mathrm{Re}\phi^{\chi(0)}(\zeta) = \gamma(\zeta)$ could not be true anymore for $n \geq 2$ unless

$$\sum_{\nu=1}^{n-1} \sum_{\substack{1\leq\sigma_1<\cdots<\sigma_\nu\leq n \\ 1\leq\sigma_{\nu+1}<\cdots<\sigma_n\leq n}} \gamma^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta) = 0\,, \tag{3.21}$$

or equivalently

$$\gamma^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta) = 0\,, \quad 0 < \nu < n.$$

This means for $z \in \mathbb{D}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}$ we have

$$\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \prod_{k=1}^{\nu}\Big(\frac{\zeta_{\sigma_k}/z_{\sigma_k}}{1-\zeta_{\sigma_k}/z_{\sigma_k}} + \delta^{\chi_k}_{\sigma_k}\Big) \prod_{k=\nu+1}^{n}\Big(\frac{z_{\sigma_k}/\zeta_{\sigma_k}}{1-z_{\sigma_k}/\zeta_{\sigma_k}} + \delta^{\chi_k}_{\sigma_k}\Big)\frac{d\zeta}{\zeta} = 0\,.$$

Thus (3.21) can be written as

$$\sum_{\nu=1}^{n-1} \sum_{\substack{1\leq\sigma_1<\cdots<\sigma_\nu\leq n \\ 1\leq\sigma_{\nu+1}<\cdots<\sigma_n\leq n}} \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \prod_{k=1}^{\nu}\Big(\frac{\zeta_{\sigma_k}/z_{\sigma_k}}{1-\zeta_{\sigma_k}/z_{\sigma_k}} + \delta^{\chi_k}_{\sigma_k}\Big)$$

$$\times \prod_{k=\nu+1}^{n}\Big(\frac{z_{\sigma_k}/\zeta_{\sigma_k}}{1-z_{\sigma_k}/\zeta_{\sigma_k}} + \delta^{\chi_k}_{\sigma_k}\Big)\frac{d\zeta}{\zeta} = 0\,, \quad z \in \mathbb{D}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}.$$

Applying Lemma 4, changing the summation indices and paying attention to the δ^{χ}_k $(0 < \nu < n)$ one can get it in another form as

$$\sum_{\mu=2}^{n}\sum_{\nu=1}^{\mu-1} \sum_{\substack{1\leq\sigma_1<\cdots<\sigma_\nu\leq n \\ 1\leq\sigma_{\nu+1}<\cdots<\sigma_\mu\leq n}} \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \prod_{k=1}^{\nu}\frac{\overline{z}_{\sigma_k}\zeta_{\sigma_k}}{1-\overline{z}_{\sigma_k}\zeta_{\sigma_k}} \prod_{k=\nu+1}^{\mu}\frac{z_{\sigma_k}\overline{\zeta}_{\sigma_k}}{1-z_{\sigma_k}\overline{\zeta}_{\sigma_k}}\frac{d\zeta}{\zeta} = 0\,, \ z \in \mathbb{D}^n$$

and this is just an equivalent form of Theorem 5.1 in [2].

3.5 Well-posed formulation of the Schwarz problem

3.5.1 Plane case

In the case of the unit disc the given real value on the unit circle is enough to determine the real part of one analytic function in the unit disc or in the outer domain of the unit disc. In another word, alone with the given real values on the circle, one can determine one analytic function in the unit disc and one other analytic function outside the disc. The given real values on the circle are sufficient to determine two analytic functions in domains which completely cover up the whole space. To determine an analytic function in or outside the unit disc it is necessary to have its real part on the unit circle. This means the given real value on the circle is necessary and sufficient to determine analytic functions in the two domains which completely cover up the whole space.

3.5.2 Higher dimensional space

From two dimensions on we have two possible interpretation of the unit disc on the plane: unit ball and unit polydisc. The boundary of the unit ball, the sphere divides the whole space only in two parts. In this sense the unit ball is not very different from the unit disc. But for the polydisc the situation is very different. The boundary of the unit polydisc can be defined in two different ways: the whole boundary or the characteristic boundary - torus. This makes the essential difference, see [14]. Since analytic functions in polydiscs can be described completely by their values just on the essential boundary $\partial_0 \mathbb{D}^n$, see [27], we restrict our discussion to the essential boundary - torus.

It is well known that all the torus related problems are accompanied always with some solvability conditions due to the fact that the given values on the torus have more components than necessary components for the problem under consideration. This phenomena appears because of that the torus has more domains then simply pure inner and pure outer domains, i.e., the torus has some domains which is neither a pure inner nor a pure outer domain. Without taking these domains into consideration it is not possible to cover up the whole space with domains of analytic functions by the given real values on the torus and that is why we have to always accept some solvability conditions which do not exist in one variable case. In this sense investigating the analytic functions in other torus related domains has major impact on all kinds of torus related problem solving. After having the properly established analytic functions for every domain of the torus it is obvious that concerning only one special domain of the torus is always accompanied with some solvability conditions. These solvability conditions are seen as natural phenomena for the torus. However this can be understood also as ill-posed formulation of the original problem - we have usually more information than we need and less equations than necessary. Taking into account that the original problem was established for half of the space by the given values on the circle(the other half can be obtained by the given value too), if we formulate the problem exactly for half of the torus domains(for half space) by the given values on the torus, then no any solvability conditions could appear. Now it is clear that if we want to get an analytic function for a very tiny part of the space defined by the torus(for one torus domain) by the given values on the torus, the other non relevant part of the given values has to vanish and so we have solvability conditions. If we consider more torus domains we would have less solvability conditions. If we consider the half or more of the torus domains then we have no any solvability conditions. Thus we can have a well-posed analog of the Schwarz problem for the torus which is originally well defined for the circle in the plane.

Before we give the well-posed or modified definition of the Schwarz problem, applying some notation in the third section of this chapter we define some sets.

Let $I^* = \{\chi^* \mid \chi^* = (\chi_1^*, \cdots, \chi_n^*)$ be the set of vertices of the $[-1,+1]^n$ cube$\}$. For every element $\chi_1^* \in I^*$ there is exactly one and only one element $\chi_2^* \in I^*$ so that $\chi_1^* = -\chi_2^*$. Denote $I_+^* = \{\chi^* \mid \chi^* = (+1, \chi_2^*, \cdots, \chi_n^*) \in I^*\}$. Clearly I_+^* contains exactly half of the elements of I^* and has no any reflected element.

Respectively we denote $I = \{\chi \mid \chi = (\chi_1, \cdots, \chi_n)$ sign of the vertices $\chi^* \in I^*\}$ and $I_+ = \{\chi \mid \chi = (+, \chi_2, \cdots, \chi_n) \in I\}$.

Now we give our modified well-posed definition of the Schwarz problem for the torus.

The Modified Problem Let γ be real-valued and belong to $W(\partial \mathbb{D}^n; \mathbb{C})$. Find a holomorphic function $\phi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)$ in $\mathbb{D}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}$ for $\chi_{\sigma_1\cdots\sigma_\nu}\in I_+$ so that

$$\sum_{\nu=0}^{[\frac{n}{2}]}\sum_{\substack{2\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\operatorname{Re}\phi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)=\gamma(\zeta),\quad \zeta\in\partial_0\mathbb{D}^n\,, \tag{3.22}$$

On the basis of the previous Theorem one can easily obtain the following conclusion.

3.6 The Schwarz problem without solvability conditions

Theorem 12 *Let* γ *be real-valued and belong to* $W(\partial\mathbb{D}^n;\mathbb{C})$ *. Then*

$$\phi^{\chi(0)}(z)=\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma(\zeta)\left[2\frac{1}{1-z/\zeta}-1\right]\frac{d\zeta}{\zeta}+iC_1\,,\quad z\in\mathbb{D}^n\,,$$

$$\phi^{\chi(n)}(z)=\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}2\gamma(\zeta)\left[\frac{1}{1-\zeta/z}-1\right]\frac{d\zeta}{\zeta},\quad z\in(\mathbb{D}^-)^n\,, \tag{3.23}$$

$$\phi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(z)=\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}2\gamma(\zeta)\prod_{k=1}^{n}\Big(\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}}+\delta_k^{\chi_k}\Big)\frac{d\zeta}{\zeta}\,,$$

$$z\in\mathbb{D}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}\,,\ 0<\nu<n, \tag{3.24}$$

are analytic functions in respective domains with arbitrary real C_1 *and satisfying*

$$\sum_{\nu=0}^{[\frac{n}{2}]}\sum_{\substack{2\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\operatorname{Re}\phi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)=\gamma(\zeta),\quad \zeta\in\partial_0\mathbb{D}^n\,, \tag{3.25}$$

$$\operatorname{Re}\phi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)=\gamma^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)+\gamma^{\chi_{\sigma_{\nu+1}\cdots\sigma_n}(\nu)}(\zeta) \tag{3.26}$$

where

$$\gamma(\zeta)=\sum_{\nu=0}^{n}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\gamma^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)\,,$$

$$\gamma^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)=\prod_{t=1}^{n}\Big(\sum_{k_t=1}^{+\infty}(\zeta_t^{\chi_t^*})^{k_t}+\delta_t^{\chi_k}\Big)\alpha_{\chi_1^*k_1,\cdots,\chi_n^*k_n},\ 0\le\nu<n\,,\quad \zeta\in\partial_0\mathbb{D}^n,$$

$$\gamma^{\chi(n)}(\zeta)=\sum_{|\kappa|>0\,,\,\kappa\in\mathbb{Z}_+^n}\alpha_{-\kappa}\zeta^{-\kappa}=\Big[\prod_{t=1}^{n}\Big(\sum_{k_t=1}^{+\infty}\zeta_t^{-k_t}+1\Big)-1\Big]\alpha_{-k_1,\cdots,-k_n}$$

$$\alpha_{\kappa(\chi(\nu))}=\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma(\zeta)\zeta^{-\kappa(\chi(\nu))}\frac{d\zeta}{\zeta}\,,\quad \overline{\alpha_{\kappa(\chi(\nu))}}=\alpha_{-\kappa(\chi(\nu))}\,.$$

Evidently the condition (3.25) is not a solvability condition and it is always satisfied.

3.7 A necessary and sufficient condition for the boundary values of holomorphic functions of the torus domains

In the previous chapters for holomorphic functions in polydiscs we have shown some equivalent methods to check the boundary values and the real part of the boundary values. After having fixed the structures of holomorphic functions in arbitrary torus domain, we may be interested to solve problems in this domain. Then surely we are confronted with the boundary values of the holomorphic functions in these torus domains. That is why we want to give a simple checking method.

So we take a closer look at the boundary values of holomorphic functions of the unit circle in the plane.

Let $\varphi \in W(\partial\mathbb{D}\,;\mathbb{C}\,)$, i.e. ,

$$\varphi(\zeta) = \sum_{-\infty}^{+\infty} \alpha_k \zeta^k, \quad \zeta \in \partial\mathbb{D}\,, \quad \sum_{-\infty}^{+\infty} |\alpha_k| \leq +\infty\,. \tag{3.27}$$

We denote

$$\phi(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \varphi(\zeta)\frac{d\zeta}{\zeta - z}\,, \quad \zeta \notin \partial\mathbb{D} \tag{3.28}$$

It is well known that the function ϕ is holomorphic in $\widehat{\mathbb{C}}\setminus\partial\mathbb{D}$ - where $\widehat{\mathbb{C}}$ is the Riemann sphere $\mathbb{C}\cup\{\infty\}$ - vanishing at ∞, see [5] .

Applying (3.27) to (3.28) for $z \in \mathbb{D}$ we have

$$\phi(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \varphi(\zeta)\frac{1}{1 - z\overline{\zeta}}\frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \Big[\sum_{-\infty}^{+\infty}\alpha_k\zeta^k\Big]\Big[\sum_{h=0}^{+\infty}(z\overline{\zeta})^h\Big]\frac{d\zeta}{\zeta} = \sum_{k=0}^{+\infty}\alpha_k z^k,\ z \in \mathbb{D}\,,$$

and for $z \in \mathbb{D}^-$,

$$\phi(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \varphi(\zeta)\frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \varphi(\zeta)\Big[-\frac{z^{-1}\zeta}{1 - z^{-1}\zeta}\Big]\frac{d\zeta}{\zeta}$$

$$= -\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \Big[\sum_{-\infty}^{+\infty}\alpha_k\zeta^k\Big]\Big[\sum_{h=1}^{+\infty}(z^{-1}\zeta)^h\Big]\frac{d\zeta}{\zeta} = -\sum_{k=1}^{+\infty}\alpha_{-k}z^{-k},\ z \in \mathbb{D}^-\,.$$

Due to $\varphi \in W(\partial\mathbb{D}\,;\mathbb{C}\,)$ it is clear that for $\eta \in \partial\mathbb{D}$ the series

$$\sum_{k=0}^{+\infty}\alpha_k\eta^k \quad \text{and} \quad \sum_{k=1}^{+\infty}\alpha_{-k}\eta^{-k}$$

are absolutely and uniformly convergent. Therefore

$$\phi^+(\eta) := \lim_{\substack{z\to\eta\\ z\in D^+}} \phi(z) = \sum_{k=0}^{+\infty}\alpha_k\eta^k \quad \text{and} \quad \phi^-(\eta) := \lim_{\substack{z\to\eta\\ z\in D^-}} \phi(z) = -\sum_{k=1}^{+\infty}\alpha_{-k}\eta^{-k}$$

and of course $\phi^+(\eta)$, $\phi^-(\eta) \in W(\partial\mathbb{D}\,;\mathbb{C}\,)$. Further

$$\overline{\phi^-}(\zeta) = -\sum_{k=1}^{+\infty} \overline{\alpha}_{-k}\zeta^k \ \in \partial\mathcal{H}(\mathbb{D}\,),\ \zeta \in \partial\mathbb{D}\ ,$$

i.e., $\overline{\phi^-}(\zeta)$ is the boundary values of a function, which is holomorphic in $\mathbb{D}$. This means that in order to know whether $\phi^-(\zeta) \in \partial\mathcal{H}(\mathbb{D}^-)$ it is enough to know if $\overline{\phi^-}(\zeta) \in \partial\mathcal{H}(\mathbb{D}\,)$. This idea can be applied to check boundary values of holomorphic functions in arbitrary torus domains. However we need to introduce a slightly modified version of complex conjugate.

Let $\varphi \in W(\partial_0\mathbb{D}^{\,n};\mathbb{C}\,)$ and $\phi^{\chi(\nu)}$ be a holomorphic function in $\mathbb{D}^{\,\chi(\nu)}$ which has the boundary values defined as in (3.3) for the given function φ .

We define *boundary partial conjugate* of $\phi^{\chi(\nu)}$ as following.

$$\mathcal{C}_\zeta\left[\phi^{\chi(n)}(\zeta)\right] := \overline{\left[\phi^{\chi(n)}(\zeta)\right]}_\zeta := \overline{\Big[\prod_{t=1}^{n}\Big(1+\sum_{k_t=1}^{+\infty}\zeta_t^{-k_t}\Big)}\overline{\alpha}_{-k_1,\cdots,-k_n} - \overline{\alpha}_{0,\cdots,0}\Big]$$

$$= \Big[\prod_{t=1}^{n}\Big(1+\sum_{k_t=1}^{+\infty}\zeta_t^{k_t}\Big)\overline{\alpha}_{-k_1,\cdots,-k_n} - \overline{\alpha}_{0,\cdots,0}\Big]\ ,\ \zeta \in \partial_0\mathbb{D}^{\,n} \tag{3.29}$$

$$\mathcal{C}_{\zeta_{\sigma_1}\cdots\,\zeta_{\sigma_\nu}}\left[\phi^{\chi(\nu)}(\zeta)\right] =: \overline{\left[\phi^{\chi(\nu)}(\zeta)\right]}_{\zeta_{\sigma_1}\cdots\,\zeta_{\sigma_\nu}}$$

$$:= \Big[\overline{\prod_{t=1}^{\nu}\Big(\delta^-_{\sigma_t}+\sum_{k_{\sigma_t}=1}^{+\infty}\zeta_{\sigma_t}^{-k_{\sigma_t}}\Big)}\prod_{t=\nu+1}^{k}\Big(\delta^+_{\sigma_t}+\sum_{k_{\sigma_t}=1}^{+\infty}\zeta_{\sigma_t}^{k_{\sigma_t}}\Big)\alpha_{\chi_1^*k_1,\cdots,\chi_n^*k_n}\Big]$$

$$= \Big[\prod_{t=1}^{n}\Big(\delta_t^{\chi_t}+\sum_{k_t=1}^{+\infty}\zeta_t^{k_t}\Big)\alpha_{\chi_1^*k_1,\cdots,\chi_n^*k_n}\Big],\ 0<\nu<n\ ,\ \zeta \in \partial_0\mathbb{D}^{\,n}. \tag{3.30}$$

We call $\mathcal{C}_{\zeta_{\sigma_1}\cdots\,\zeta_{\sigma_\nu}}\left[\phi^{\chi(\nu)}(\zeta)\right]$ *boundary partial conjugate* of $\phi^{\chi(\nu)}$. Obviously $\phi^{\chi(\nu)}(\zeta) \in \partial\mathcal{H}(\mathbb{D}^{\,\chi(\nu)})$, $0<\nu\le n$ is equivalent to $\mathcal{C}_{\zeta_{\sigma_1}\cdots\,\zeta_{\sigma_\nu}}\left[\phi^{\chi(\nu)}(\zeta)\right] \in \partial\mathcal{H}(\mathbb{D}^{\,n})$, $0<\nu\le n$ and $\mathcal{C}_{\zeta_{\sigma_1}\cdots\,\zeta_{\sigma_\nu}}\left[\mathcal{C}_{\zeta_{\sigma_1}\cdots\,\zeta_{\sigma_\nu}}\left[\phi^{\chi(\nu)}(\zeta)\right]\right] = \phi^{\chi(\nu)}$. Thus we have

Theorem 13 *Let* $\phi^{\chi(\nu)}$ *be an arbitrary function which is continuous in* $\mathbb{D}^{\,\chi(\nu)}$ *and* $\phi^{\chi(\nu)} \in W(\partial_0\mathbb{D}^{\,n},\mathbb{C}\,)$ *. Then* $\mathcal{C}_{\zeta_{\sigma_1}\cdots\,\zeta_{\sigma_\nu}}\left[\phi^{\chi(\nu)}(\zeta)\right] \in \partial\mathcal{H}(\mathbb{D}^{\,n})$ *is the necessary and sufficient condition for* $\phi^{\chi(\nu)}(\zeta) \in \partial\mathcal{H}(\mathbb{D}^{\,\chi(\nu)})$ *.*

In Section 2.3.3 a simple checking method for boundary values of holomorphic functions in polydisc is given. So by Theorem 13 we can check boundary values of holomorphic functions in any torus domains.

3.8 Appendix

In the three dimensional case from (3.4) it is easy to see that

$$w^{++-}(z) = \frac{1}{(2\pi i)^3}\int_{\partial_0 \mathbb{D}^3} \varphi(\zeta)\left[\frac{z_1\zeta_1^{-1}}{1-z_1\zeta_1^{-1}}+1\right]\frac{z_2\zeta_2^{-1}}{1-z_2\zeta_2^{-1}}\frac{z_3^{-1}\zeta_3}{1-z_3^{-1}\zeta_3}\frac{d\zeta}{\zeta}$$

$$w^{--+}(z) = \frac{1}{(2\pi i)^3}\int_{\partial_0 \mathbb{D}^3} \varphi(\zeta)\left[\frac{z_1^{-1}\zeta_1}{1-z_1^{-1}\zeta_1}+1\right]\frac{z_2^{-1}\zeta_2}{1-z_2^{-1}\zeta_2}\frac{z_3\zeta_3^{-1}}{1-z_3\zeta_3^{-1}}\frac{d\zeta}{\zeta}$$

and the others can be written in the same way. From them one can have

$$w^{--+}(z)|_{z_1=\infty} \neq 0\,, \quad w^{--+}(z)|_{z_2=\infty} = 0\,, \quad w^{+--}(z)|_{z_2=\infty} \neq 0\,,$$

$$w^{+--}(z)|_{z_3=\infty} \neq 0\,, \quad w^{-+-}(z)|_{z_3=\infty} \neq 0\,, \quad w^{-+-}(z)|_{z_1=\infty} = 0\,.$$

In the case of four dimension the most interesting ones are

$$w^{-+-+}(z) = \frac{1}{(2\pi i)^4}\int_{\partial_0 \mathbb{D}^4} \varphi(\zeta)\frac{z_1^{-1}\zeta_1}{1-z_1^{-1}\zeta_1}\frac{z_2\zeta_2^{-1}}{1-z_2\zeta_2^{-1}}\frac{z_3^{-1}\zeta_3}{1-z_3^{-1}\zeta_3}\frac{z_4\zeta_4^{-1}}{1-z_4\zeta_4^{-1}}\frac{d\zeta}{\zeta}$$

$$w^{+-+-}(z) = \frac{1}{(2\pi i)^4}\int_{\partial_0 \mathbb{D}^4} \varphi(\zeta)\frac{z_1\zeta_1^{-1}}{1-z_1\zeta_1^{-1}}\frac{z_2^{-1}\zeta_2}{1-z_2^{-1}\zeta_2}\frac{z_3\zeta_3^{-1}}{1-z_3\zeta_3^{-1}}\frac{z_4^{-1}\zeta_4}{1-z_4^{-1}\zeta_4}\frac{d\zeta}{\zeta}$$

$$w^{--++}(z) = \frac{1}{(2\pi i)^4}\int_{\partial_0 \mathbb{D}^4} \varphi(\zeta)\left[\frac{z_1^{-1}\zeta_1}{1-z_1^{-1}\zeta_1}+1\right]\frac{z_2\zeta_2^{-1}}{1-z_2\zeta_2^{-1}}\left[\frac{z_3^{-1}\zeta_3}{1-z_3^{-1}\zeta_3}+1\right]\frac{z_4\zeta_4^{-1}}{1-z_4\zeta_4^{-1}}\frac{d\zeta}{\zeta}$$

$$w^{++--}(z) = \frac{1}{(2\pi i)^4}\int_{\partial_0 \mathbb{D}^4} \varphi(\zeta)\left[\frac{z_1\zeta_1^{-1}}{1-z_1\zeta_1^{-1}}+1\right]\frac{z_2\zeta_2^{-1}}{1-z_2\zeta_2^{-1}}\left[\frac{z_3^{-1}\zeta_3}{1-z_3^{-1}\zeta_3}+1\right]\frac{z_4^{-1}\zeta_4}{1-z_4^{-1}\zeta_4}\frac{d\zeta}{\zeta}$$

$$w^{-+++}(z) = \frac{1}{(2\pi i)^4}\int_{\partial_0 \mathbb{D}^4} \varphi(\zeta)\frac{z_1^{-1}\zeta_1}{1-z_1^{-1}\zeta_1}\left[\frac{z_2\zeta_2^{-1}}{1-z_2\zeta_2^{-1}}+1\right]\left[\frac{z_3\zeta_3^{-1}}{1-z_3\zeta_3^{-1}}+1\right]\frac{z_4\zeta_4^{-1}}{1-z_4\zeta_4^{-1}}\frac{d\zeta}{\zeta}$$

$$w^{+---}(z) = \frac{1}{(2\pi i)^4}\int_{\partial_0 \mathbb{D}^4} \varphi(\zeta)\frac{z_1\zeta_1^{-1}}{1-z_1\zeta_1^{-1}}\left[\frac{z_2^{-1}\zeta_2}{1-z_2^{-1}\zeta_2}+1\right]\left[\frac{z_3^{-1}\zeta_3}{1-z_3^{-1}\zeta_3}+1\right]\frac{z_4^{-1}\zeta_4}{1-z_4^{-1}\zeta_4}\frac{d\zeta}{\zeta}$$

and the others can be similarly described. They have following property.

$$w^{-+-+}(z)|_{z_1=\infty} = 0\,, \quad w^{-+-+}(z)|_{z_3=\infty} = 0\,, \quad w^{++--}(z)|_{z_3=\infty} \neq 0$$

$$w^{+-+-}(z)|_{z_2=\infty} = 0\,, \quad w^{+-+-}(z)|_{z_4=\infty} = 0\,, \quad w^{++--}(z)|_{z_4=\infty} = 0$$

$$w^{-+++}(z)|_{z_1=\infty} = 0\,, \quad w^{+---}(z)|_{z_2=\infty} \neq 0\,, \quad w^{+---}(z)|_{z_3=\infty} \neq 0$$

$$w^{--++}(z)|_{z_1=\infty} \neq 0\,, \quad w^{--++}(z)|_{z_2=\infty} = 0\,, \quad w^{+---}(z)|_{z_4=\infty} = 0$$

Only $w^{-+-+}(z)$, $w^{+-+-}(z)$ satisfies the condition (3.5) totally. But $w^{--++}(z)$, $w^{++--}(z)$, $w^{+---}(z)$, $w^{-+++}(z)$ fulfill only half of the condition (3.5).