Chapter 2

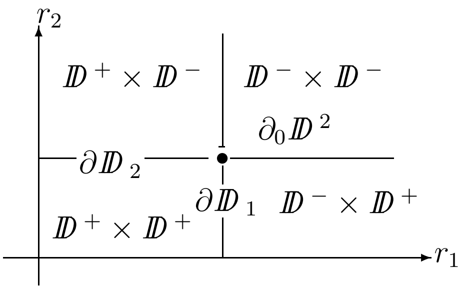
The Neumann Problem

2.1 The inhomogeneous pluriharmonic system

2.1.1 Preliminaries and Definition

The Neumann problem for the unit ball has been considered recently, see [2]. However for the polydisc which is another fundamental domain in $\mathbb{C}^n (n > 1)$ and as was first realized by C. Caratheodory topologically different from the unit ball, see [14], it remained open. The very first thing here is how to define the boundary function of the torus $\partial_0 \mathbb{D}^n (n > 1)$.

In the case of a unit bidisc, i.e. $\Omega = \mathbb{D}^2$ we know that $\partial \mathbb{D}_1 \cup \partial \mathbb{D}_2 \cup \partial_0 \mathbb{D}^2 =: (\partial \mathbb{D} \times \mathbb{D}) \cup (\mathbb{D} \times \partial \mathbb{D}) \cup (\partial_0 \mathbb{D}^2)$ is the whole boundary of $\mathbb{D} \times \mathbb{D}$ and $\mathbb{C}^2 \setminus \overline{\mathbb{D}^2} = (\mathbb{D}^- \times \mathbb{D}^+) \cup (\mathbb{D}^+ \times \mathbb{D}^-) \cup (\mathbb{D}^- \times \mathbb{D}^-)$ is the outside of $\mathbb{D} \times \mathbb{D}$ with respect to the whole boundary. The domains $\mathbb{D}^- \times \mathbb{D}^+$ and $\mathbb{D}^+ \times \mathbb{D}^-$ are the outside of $\mathbb{D}^+ \times \mathbb{D}^+$ related to $\partial \mathbb{D}_1$ and $\partial \mathbb{D}_2$. So the remaining part $\mathbb{D}^- \times \mathbb{D}^-$ may be considered as the outside of $\mathbb{D} \times \mathbb{D}$ related to the essential boundary $\partial_0 \mathbb{D}^2$.



Thus the essential boundary $\partial_0 \mathbb{D}^n (n > 1)$ divides $\mathbb{C}^n$ into 2^n parts rather than just in an inner and an outer domain as the whole boundary does or as in the case $n = 1$. For $\mathbb{D}^n = \times_{k=1}^n \mathbb{D}^+$, $\partial_0 \mathbb{D}^n = \times_{k=1}^n \partial \mathbb{D}$, let $out_{\partial_0 \mathbb{D}^n}(\mathbb{D}^n) := \times_{k=1}^n \mathbb{D}^- =: \mathbb{D}^{-n}$. This means that we have to consider $\mathbb{D}^{-n}$ as the outside of $\mathbb{D}^n$ relative to the essential boundary $\partial_0 \mathbb{D}^n$. The sense of this kind of devision of $\mathbb{C}^n$ related to $\partial_0 \mathbb{D}^n$ can be explained by further discussion of the Riemann problem related to $\partial_0 \mathbb{D}^n$, see [20]. Since analytic functions in $\mathbb{D}^n$ can be described completely by their values just on the essential boundary $\partial_0 \mathbb{D}^n$, see [27], we have to restrict our discussion to the essential boundary. Its boundary function ρ is defined by

$$\rho(z) = \sum_{j=1}^{n} (| z_j |^2 - 1) \quad , \qquad z = (z_1, \cdots, z_n) \in \mathbb{D}^n \cup \partial_0 \mathbb{D}^n \cup \mathbb{D}^{-n}. \tag{2.1}$$

Let Ω be a smooth domain in $\mathbb{C}^n$ and $f_{k\ell}(z) \in L_1(\overline{\Omega}) \cap C^1(\Omega)$ be given functions. Consider the following inhomogeneous system of n^2 independent equations

$$\frac{\partial^2 u}{\partial \overline{z}_k \partial z_\ell} = f_{k\ell}(z), \quad 1 \le k, \ell \le n \ , \tag{2.2}$$

with given right – hand sides, satisfying the condition

$$\frac{\partial f_{ks}}{\partial \overline{z}_\ell} - \frac{\partial f_{\ell s}}{\partial \overline{z}_k} = 0, \quad \frac{\partial f_{sk}}{\partial z_\ell} - \frac{\partial f_{s\ell}}{\partial z_k} = 0 \ , \qquad 1 \le s \le n \ . \tag{2.3}$$

The Problem N_1. Find a $C^1(\overline{\Omega})$ – solution of system (2.2), satisfying the Neumann condition

$$\frac{\partial u}{\partial \nu_\zeta} = \gamma_0(\zeta), \quad \zeta \in \partial\Omega \ , \tag{2.4}$$

where Ω is a smooth enough domain in $\mathbb{C}^n$ and $\partial u / \partial \nu_\zeta$ denotes the outward normal derivative of $u(z)$ at the point $\zeta \in \partial\Omega$.

By definition, see [2] page 172, it is known that

$$\frac{\partial u}{\partial \nu_\zeta} = \sum_{j=1}^{n} \Big(\rho_{\overline{z}_j} \frac{\partial u}{\partial z_j} + \rho_{z_j} \frac{\partial u}{\partial \overline{z}_j} \Big) \frac{2}{| \operatorname{grad}\rho |} \Big|_\zeta, \qquad \zeta \in \partial\Omega$$

where $\rho(z)$ is the boundary function of Ω defined on $\partial\Omega$ as follows.

$< i >$ $\rho(\zeta) = 0, \ | \operatorname{grad}\rho |_\zeta \ne 0, \quad \zeta \in \partial\Omega;$

$< ii >$ $\rho(z) < 0, \ z \in \Omega;$

$< iii >$ $\rho(z) > 0, \ z \in \text{out } \Omega = \mathbb{C}^n \setminus \overline{\Omega}.$

Hence from the representation (2.1) of the boundary function ρ for the polydisc $\partial_0 \mathbb{D}^n$ we have

$$\rho_{x_j} = 2x_j, \ \rho_{y_j} = 2y_j, \ \ | \operatorname{grad}\rho |^2_{\partial_0 \mathbb{D}^n} = 4 \sum_{j=1}^{n} (x_j^2 + y_j^2) \ |_{|z_j|=1} = 4n,$$

the Neumann condition (2.4) for the polydisc turns out to be

$$\sum_{j=1}^{n} \Big(z_j \frac{\partial u}{\partial z_j} + \overline{z}_j \frac{\partial u}{\partial \overline{z}_j} \Big) \Big|_\zeta = \gamma(\zeta), \qquad \zeta \in \partial_0 \mathbb{D}^n \ , \tag{2.5}$$

with $\gamma(\zeta) = \gamma_0(\zeta)\sqrt{n}$.

2.1.2 Specification and the Solution

The condition (2.5) can be written as

$$Re\, v(\zeta) = Re\, \gamma(\zeta),\ Re\, w(\zeta) = -Im \gamma(\zeta),\ \zeta \in \partial_0 \mathbb{D}^n, \tag{2.6}$$

with

$$v(z) = 2 \sum_{k=1}^{n} z_k (Re\, u)_{z_k},\ w(z) = 2i \sum_{k=1}^{n} z_k (Im\, u)_{z_k}\ .$$

According to (2.2) these functions satisfy the inhomogeneous Cauchy-Riemann systems

$$v_{\overline{z}_k} = \sum_{\ell=1}^{n} z_\ell (f_{k\ell} + \overline{f}_{\ell k}) =: F_k, \quad w_{\overline{z}_k} = i \sum_{\ell=1}^{n} z_\ell (f_{k\ell} - \overline{f}_{\ell k}) =: \Phi_k \tag{2.7}$$

in $\mathbb{D}^n$ with the right-hand sides because of (2.3) satisfying the compatibility conditions

$$F_{k\overline{z}_\ell} = F_{\ell \overline{z}_k}, \quad \Phi_{k\overline{z}_\ell} = \Phi_{\ell \overline{z}_k}, \qquad 1 \le k, \ell \le n.$$

Therefore if the compatibility conditions are satisfied then $v(z)$, $w(z)$ must be the solutions of the Schwarz problem (2.6) , (2.7) and this problem was solved, see [2] Chapter 5 and [3] . Thus under the solvability conditions

$$J - Re \sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n \\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}} \frac{(-1)^\nu}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} F_{k_1 \overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\lambda} \zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu}}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{1 - \overline{z}_{k_\tau} \zeta_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{1 - z_{k_\tau} \overline{\zeta}_{k_\tau}} \prod_{\tau=1}^{\nu} d\zeta_{k_\tau} d\eta_{k_\tau} = 0 \tag{2.8}$$

$$J^\star - Re \sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n \\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}} \frac{(-1)^\nu}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \Phi_{k_1 \overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\lambda} \zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu}}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{1 - \overline{z}_{k_\tau} \zeta_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{1 - z_{k_\tau} \overline{\zeta}_{k_\tau}} \prod_{\tau=1}^{\nu} d\zeta_{k_\tau} d\eta_{k_\tau} = 0 \tag{2.9}$$

we have the solutions

$$v(z) = \sum_{\nu=1}^{n} \sum_{1\le k_1<\cdots<k_\nu\le n} \frac{(-1)^\nu}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \Big\{ \overline{F_{k_1 \overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\nu}}}$$

$$\times \prod_{\tau=1}^{\nu} \frac{z_{k_\tau}}{1 - z_{k_\tau} \overline{\zeta}_{k_\tau}} - (-1)^\nu F_{k_1 \overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\nu}} \prod_{\tau=1}^{\nu} \frac{1}{\zeta_{k\tau} - z_{k_\tau}} \Big\} \prod_{\tau=1}^{\nu} d\xi d\eta_{k_\tau}$$

$$+ I + \sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n \\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}} \frac{(-1)^\lambda}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \overline{F_{k_1 \overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\lambda} \zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu}}}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{1 - z_{k_\tau}\overline{\zeta}_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{1}{\zeta_{k_\tau} - z_{k_\tau}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau} + iC_1 \qquad , \tag{2.10}$$

$$w(z) = \sum_{\nu=1}^{n} \sum_{1\le k_1<\cdots<k_\nu\le n} \frac{(-1)^\nu}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \Big\{ \overline{\Phi_{k_1\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\nu}}}$$

$$\times \prod_{\tau=1}^{\nu} \frac{z_{k_\tau}}{1 - z_{k_\tau}\overline{\zeta}_{k_\tau}} - (-1)^\nu \Phi_{k_1\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\nu}} \prod_{\tau=1}^{\nu} \frac{1}{\zeta_{k_\tau} - z_{k_\tau}} \Big\} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau}$$

$$+I^\star + \sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}} \frac{(-1)^\lambda}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \overline{\Phi_{k_1\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\lambda}\zeta_{k_{\lambda+1}}\cdots\zeta_{k_\nu}}}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{1 - z_{k_\tau}\overline{\zeta}_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{1}{\zeta_{k_\tau} - z_{k_\tau}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau} + iC_2 \tag{2.11}$$

where

$$F_{k_1\overline{z}_{k_2}\cdots\overline{z}_{k_\lambda}z_{k_{\lambda+1}}\cdots z_{k_\nu}} = \sum_{\alpha=\lambda+1}^{\nu} (f_{k_1k_\alpha} + \overline{f}_{k_\alpha k_1})_{\overline{z}_{k_2}\cdots\overline{z}_{k_\lambda}z_{k_{\lambda+1}}\cdots z_{k_{\alpha-1}}z_{k_{\alpha+1}}\cdots z_{k_\nu}}$$

$$+\sum_{\ell=1}^{n} z_\ell (f_{k_1\ell} + \overline{f}_{\ell k_1})_{\overline{z}_{k_2}\cdots\overline{z}_{k_\lambda}z_{k_{\lambda+1}}\cdots z_{k_\nu}} \qquad (\lambda < \nu) ,$$

$$F_{k_1\overline{z}_{k_2}\cdots\overline{z}_{k_\nu}} = \sum_{\ell=1}^{n} z_\ell (f_{k_1\ell} + \overline{f}_{\ell k_1})_{\overline{z}_{k_2}\cdots\overline{z}_{k_\nu}} \qquad (\lambda = \nu) ,$$

$$J = \frac{1}{(2\pi i)^n} \int_{\partial_0\mathbb{D}^n} Re\,\gamma(\zeta) \sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}} \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau} - z_{k\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \frac{d\zeta}{\zeta} ,$$

$$I = \frac{1}{(2\pi i)^n} \int_{\partial_0\mathbb{D}^n} Re\,\gamma(\zeta) \Big[2\frac{\zeta}{\zeta - z} - 1 \Big] \frac{d\zeta}{\zeta} ,$$

$$\Phi_{k_1\overline{z}_{k_2}\cdots\overline{z}_{k_\lambda}z_{k_{\lambda+1}}\cdots z_{k_\nu}} = i \sum_{\alpha=\lambda+1}^{\nu} (f_{k_1k_\alpha} - \overline{f}_{k_\alpha k_1})_{\overline{z}_{k_2}\cdots\overline{z}_{k_\lambda}z_{k_{\lambda+1}}\cdots z_{k_{\alpha-1}}z_{k_{\alpha+1}}\cdots z_{k_\nu}}$$

$$+ i\sum_{\ell=1}^{n} z_\ell (f_{k_1\ell} - \overline{f}_{\ell k_1})_{\overline{z}_{k_2}\cdots\overline{z}_{k_\lambda}z_{k_{\lambda+1}}\cdots z_{k_\nu}} \qquad (\lambda < \nu) ,$$

$$\Phi_{k_1\overline{z}_{k_2}\cdots\overline{z}_{k_\nu}} = i\sum_{\ell=1}^{n} z_\ell (f_{k_1\ell} - \overline{f}_{\ell k_1})_{\overline{z}_{k_2}\cdots\overline{z}_{k_\nu}} \qquad (\lambda = \nu) ,$$

$$J^\star = \frac{1}{(2\pi i)^n} \int_{\partial_0\mathbb{D}^n} \Big(-Im\,\gamma(\zeta) \Big) \sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}} \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau} - z_{k\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \frac{d\zeta}{\zeta} ,$$

$$I^\star = \frac{1}{(2\pi i)^n} \int_{\partial_0\mathbb{D}^n} \Big(-Im\,\gamma(\zeta) \Big) \Big[2\frac{\zeta}{\zeta - z} - 1 \Big] \frac{d\zeta}{\zeta} ,$$

and C_1, C_2, are arbitrary real constants. Since $v(0) = w(0) = 0$ follow by their definitions, from (2.10) and (2.11) we obtain

$$iC_1 + \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n} Re\,\gamma(\zeta)\frac{d\zeta}{\zeta} - \sum_{\ell,\nu=1}^{n}\sum_{1\le k_1<\cdots<k_\nu\le n}\frac{1}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}$$

$$\times\left[\zeta_\ell(f_{k_1\ell}+\overline{f}_{\ell k_1})_{\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\nu}}\right]\prod_{\tau=1}^{\nu}\frac{d\zeta_{k_\tau}d\eta_{k_\tau}}{\zeta_{k_\tau}} = 0 \tag{2.12}$$

$$iC_2 + \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\Big(-Im\gamma(\zeta)\Big)\frac{d\zeta}{\zeta} - i\sum_{\ell,\nu=1}^{n}\sum_{1\le k_1<\cdots<k_\nu\le n}\frac{1}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}$$

$$\times\left[\zeta_\ell(f_{k_1\ell}-\overline{f}_{\ell k_1})_{\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\nu}}\right]\prod_{\tau=1}^{\nu}\frac{d\zeta_{k_\tau}d\eta_{k_\tau}}{\zeta_{k_\tau}} = 0 \tag{2.13}$$

i.e.

$$C_1 = Im\sum_{\ell,\nu=1}^{n}\sum_{1\le k_1<\cdots<k_\nu\le n}\frac{1}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}\left[\zeta_\ell(f_{k_1\ell}+\overline{f}_{\ell k_1})_{\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\nu}}\right]\prod_{\tau=1}^{\nu}\frac{d\zeta_{k_\tau}d\eta_{k_\tau}}{\zeta_{k_\tau}}\,,$$

$$C_2 = Re\sum_{\ell,\nu=1}^{n}\sum_{1\le k_1<\cdots<k_\nu\le n}\frac{1}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}\left[\zeta_\ell(f_{k_1\ell}-\overline{f}_{\ell k_1})_{\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\nu}}\right]\prod_{\tau=1}^{\nu}\frac{d\zeta_{k_\tau}d\eta_{k_\tau}}{\zeta_{k_\tau}}\,,$$

$$\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\gamma(\zeta)\frac{d\zeta}{\zeta} = \sum_{\ell,\nu=1}^{n}\sum_{1\le k_1<\cdots<k_\nu\le n}\frac{1}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}$$

$$\times\left[\zeta_\ell f_{k_1\ell\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\nu}}\prod_{\tau=1}^{\nu}\frac{d\zeta_{k_\tau}d\eta_{k_\tau}}{\zeta_{k_\tau}} + \overline{\zeta}_\ell f_{\ell k_1\zeta_{k_2}\cdots\zeta_{k_\nu}}\prod_{\tau=1}^{\nu}\frac{d\zeta_{k_\tau}d\eta_{k_\tau}}{\overline{\zeta}_{k_\tau}}\right]. \tag{2.14}$$

Again by the definitions of $v(z)$ and $w(z)$ we get

$$\begin{cases} \displaystyle\sum_{k=1}^{n} z_k u_{z_k}(z) = \frac{1}{2}(v(z) - iw(z)) = G(z,\overline{z})\,, \\ \displaystyle\sum_{k=1}^{n} \overline{z_k} u_{\overline{z}_k}(z) = \frac{1}{2}(\overline{v(z)} - i\overline{w(z)}) = H(z,\overline{z}) \end{cases} \tag{2.15}$$

Since

$$\sum_{k=1}^{n}\overline{z_k}G_{\overline{z}_k}(z,\overline{z}) = \sum_{k,\ell=1}\overline{z}_k z_\ell f_{k\ell}(z) = \sum_{k=1}^{n} z_k H_{z_k}(z,\overline{z})$$

the system (2.15) is compatible. Clearly, the homogeneous system

$$\sum_{k=1}^{n} z_k u_{z_k} = 0 \quad , \quad \sum_{k=1}^{n}\overline{z}_k u_{\overline{z}_k} = 0 \quad , \tag{2.16}$$

corresponding to (2.15) means that its C^1-solution is holomorphic as well as antiholomorphic in $\mathbb{D}^n$ and therefore is a constant. So the general solution to (2.15) is a sum of a particular solution to (2.15) and a constant. A particular solution to system (2.15) is given by

$$u(z) = \int_0^1 G(sz, s\overline{z})\frac{ds}{s} + \int_0^1 H(sz, s\overline{z})\frac{ds}{s} \, . \tag{2.17}$$

Next we simplify the conditions such as (2.8), (2.9) and the solution (2.17). Actually the conditions (2.8) and (2.9) are equivalent to

$$\sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}}\Big\{\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma(\zeta)\prod_{\tau=1}^{\lambda}\frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau}-z_{k\tau}}}\prod_{\tau=\lambda+1}^{\nu}\frac{z_{k\tau}}{\zeta_{k_\tau}-z_{k_\tau}}\frac{d\zeta}{\zeta}$$

$$-\frac{(-1)^\nu}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}\Big[\sum_{\alpha=\lambda+1}^{\nu} f_{k_1k_\alpha\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\lambda}\zeta_{k_{\lambda+1}}\cdots\zeta_{k_{\alpha-1}}\zeta_{k_{\alpha+1}}\cdots\zeta_{k_\nu}} + \sum_{\ell=1}^{n}\zeta_\ell f_{k_1\ell\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\lambda}\zeta_{k_{\lambda+1}}\cdots\zeta_{k_\nu}}\Big]$$

$$\times\prod_{\tau=1}^{\lambda}\frac{\overline{z}_{k_\tau}}{1-\overline{z}_{k_\tau}\zeta_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta}_{k_\tau}}\prod_{\tau=1}^{\nu}d\zeta_{k_\tau}d\eta_{k_\tau}$$

$$-\frac{(-1)^\nu}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}\Big[\sum_{\alpha=\lambda+1}^{\nu} f_{k_\alpha k_1\zeta_{k_2}\cdots\zeta_{k_\lambda}\overline{\zeta}_{k_{\lambda+1}}\cdots\overline{\zeta}_{k_{\alpha-1}}\overline{\zeta}_{k_{\alpha+1}}\cdots\overline{\zeta}_{k_\nu}} + \sum_{\ell=1}^{n}\overline{\zeta}_\ell f_{\ell k_1\zeta_{k_2}\cdots\zeta_{k_\lambda}\overline{\zeta}_{k_{\lambda+1}}\cdots\overline{\zeta}_{k_\nu}}\Big]$$

$$\times\prod_{\tau=1}^{\lambda}\frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta}_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{\overline{z}_{k_\tau}}{1-\overline{z}_{k_\tau}\zeta_{k_\tau}}\prod_{\tau=1}^{\nu}d\zeta_{k_\tau}d\eta_{k_\tau}\Big\} = 0 \, . \tag{2.18}$$

Substituting (2.10) and (2.11) into (2.15), the representation (2.17) gets the form

$$u(z) = \int_0^1\sum_{\ell,\nu=1}^{n}\sum_{1\le k_1<\cdots<k_\nu\le n}\frac{(-1)^\nu}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}\Big[\overline{\zeta}_\ell f_{\ell k_1\zeta_{k_2}\cdots\zeta_{k_\nu}}$$

$$\times\Big(\prod_{\tau=1}^{\nu}\frac{sz_{k_\tau}}{1-sz_{k_\tau}\overline{\zeta}_{k_\tau}} - (-1)^\nu\prod_{\tau=1}^{\nu}\frac{1}{\overline{\zeta_{k_\tau}-s z_{k_\tau}}}\Big)$$

$$+\zeta_\ell f_{k_1\ell\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\nu}}\Big(\prod_{\tau=1}^{\nu}\frac{s\overline{z}_{k_\tau}}{1-s\overline{z}_{k_\tau}\zeta_{k_\tau}} - (-1)^\nu\prod_{\tau=1}^{\nu}\frac{1}{\zeta_{k_\tau}-sz_{k_\tau}}\Big)\Big]\prod_{\tau=1}^{\nu}d\xi_{k_\tau}d\eta_{k_\tau}\frac{ds}{s}$$

$$+\int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma(\zeta)\Big[\frac{\zeta}{\zeta-sz}+\frac{\overline{\zeta}}{\overline{\zeta-sz}}-1\Big]\frac{d\zeta}{\zeta}\frac{ds}{s}+C$$

$$+\int_0^1\sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}}\frac{(-1)^\lambda}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}\Big\{$$

$$\times\Big[\sum_{\alpha=\lambda+1}^{\nu} f_{k_\alpha k_1\zeta_{k_2}\cdots\zeta_{k_\lambda}\overline{\zeta}_{k_{\lambda+1}}\cdots\overline{\zeta}_{k_{\alpha-1}}\overline{\zeta}_{k_{\alpha+1}}\cdots\overline{\zeta}_{k_\nu}} + \sum_{\ell=1}^{n}\overline{\zeta}_\ell f_{k_1\ell\zeta_{k_2}\cdots\zeta_{k_\lambda}\overline{\zeta}_{k_{\lambda+1}}\cdots\overline{\zeta}_{k_\nu}}\Big]$$

$$\times \prod_{\tau=1}^{\lambda} \frac{s z_{k_\tau}}{1 - s z_{k_\tau} \overline{\zeta}_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{1}{\zeta_{k_\tau} - s z_{k_\tau}}$$

$$+ \Big[\sum_{\alpha=\lambda+1}^{\nu} f_{k_1 k_\alpha \overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\lambda} \zeta_{k_{\lambda+1}} \cdots \zeta_{k_{\alpha-1}} \zeta_{k_{\alpha+1}} \cdots \zeta_{k_\nu}} + \sum_{\ell=1}^{n} \zeta_\ell f_{k_1 \ell \overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\lambda} \zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu}} \Big]$$

$$\times \prod_{\tau=1}^{\lambda} \frac{s \overline{z}_{k_\tau}}{1 - s \overline{z}_{k_\tau} \zeta_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{1}{\overline{\zeta}_{k_\tau} - s \overline{z}_{k_\tau}} \Big\} \prod_{\tau=1}^{\nu} d\zeta_{k_\tau} d\eta_{k_\tau} \frac{ds}{s} \,. \tag{2.19}$$

Theorem 6 *Let $f_{k\ell} \in C^1(\mathbb{D}^n) \cap L_1(\overline{\mathbb{D}^n})$ and satisfy the conditions (2.3). Then problem N_1 is solvable if and only if the condition (2.14) holds and the condition (2.18) holds for any $z \in \mathbb{D}^n$. The solution is given by (2.19). The corresponding homogeneous problem has no nontrivial solutions.*

Corollary 1 *Problem N_1 for pluriharmonic functions in $\mathbb{D}^n$ $(n \geq 1)$ is solvable if and only if*

$$\frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \frac{d\zeta}{\zeta} = 0 \tag{2.20}$$

holds and the condition

$$\sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{\substack{1 \leq k_1 < \cdots < k_\lambda \leq n \\ 1 \leq k_{\lambda+1} < \cdots < k_\nu \leq n}} \frac{1}{(2\pi i)^\nu} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau} - z_{k_\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \frac{d\zeta}{\zeta} = 0 \tag{2.21}$$

holds for any $z \in \mathbb{D}^n$. Then the solution to N_1 is

$$u(z) = \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \Big[\frac{\zeta}{\zeta - sz} + \frac{\overline{\zeta}}{\overline{\zeta - sz}} - 1 \Big] \frac{d\zeta}{\zeta} \frac{ds}{s} + C \,. \tag{2.22}$$

In this case (2.14) coincides with (2.20) and (2.18) with (2.21) and $C_1 = C_2 = 0$ in (2.14) because of $f_{k\ell}(z) = 0$. But from (2.10) and (2.11) it follows that

$$v(z) - iw(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \Big[2 \frac{\zeta}{\zeta - z} - 1 \Big] \frac{d\zeta}{\zeta}$$

$$\overline{v(z)} - i\overline{w(z)} = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \Big[2 \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 \Big] \frac{d\zeta}{\zeta}$$

So the equations (2.15) according to the representation $u(z) = \varphi(z) + \overline{\psi(z)}$ and together with condition (2.20) turns out to be the following partial differential equations for the holomorphic functions φ and ψ in $\mathbb{D}^n$.

$$\sum_{k=1}^{n} z_k \varphi_{z_k} = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \Big[\frac{\zeta}{\zeta - z} - 1 \Big] \frac{d\zeta}{\zeta} \tag{2.23}$$

$$\sum_{k=1}^{n} z_k \psi_{z_k} = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{\gamma(\zeta)} \Big[\frac{\zeta}{\zeta - z} - 1 \Big] \frac{d\zeta}{\zeta}$$

Transforming the variables via

$$\omega_1 = z_1, \quad \omega_2 = z_1/z_2, \cdots, \omega_n = z_1/z_n$$

we obtain

$$\omega_1 \frac{\partial \varphi}{\partial \omega_1} = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \Big[\frac{\zeta_1}{\zeta_1 - \omega_1} \frac{\zeta_2}{\zeta_2 - \omega_1/\omega_2} \cdots \frac{\zeta_n}{\zeta_n - \omega_1/\omega_n} - 1 \Big] \frac{d\zeta}{\zeta} \,,$$

$$\omega_1 \frac{\partial \psi}{\partial \omega_1} = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{\gamma(\zeta)} \Big[\frac{\zeta_1}{\zeta_1 - \omega_1} \frac{\zeta_2}{\zeta_2 - \omega_1/\omega_2} \cdots \frac{\zeta_n}{\zeta_n - \omega_1/\omega_n} - 1 \Big] \frac{d\zeta}{\zeta} \,.$$

Integrating these equations we get

$$\varphi(\omega) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \int_0^{\omega_1} \Big[\frac{\zeta_1}{\zeta_1 - t} \prod_{\tau=2}^{n} \frac{\zeta_\tau}{\zeta_\tau - t/\omega_\tau} - 1 \Big] \frac{dt}{t} \frac{d\zeta}{\zeta} + C_1^\circ \,,$$

$$\psi(\omega) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{\gamma(\zeta)} \int_0^{\omega_1} \Big[\frac{\zeta_1}{\zeta_1 - t} \prod_{\tau=2}^{n} \frac{\zeta_\tau}{\zeta_\tau - t/\omega_\tau} - 1 \Big] \frac{dt}{t} \frac{d\zeta}{\zeta} + C_2^\circ \,.$$

Transforming $0 \le s = t/\omega_1 \le 1$ and returning to the original variables gives

$$\varphi(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \int_0^1 \Big[\frac{1}{1 - sz\overline{\zeta}} - 1 \Big] \frac{ds}{s} \frac{d\zeta}{\zeta} + C_1^\circ \,, \tag{2.24}$$

$$\psi(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{\gamma(\zeta)} \int_0^1 \Big[\frac{1}{1 - sz\overline{\zeta}} - 1 \Big] \frac{ds}{s} \frac{d\zeta}{\zeta} + C_2^\circ \,.$$

It is also possible to solve (2.23) applying power series, i.e. let

$$\varphi(z) = \sum_k \alpha_k z^k,$$

where $k = (k_1, \cdots, k_n),\ k_1, \cdots, k_n \in N \cup \{0\}, \quad z \in \mathbb{D}^n$.

Keeping (2.20) in mind and representing both sides of (2.23) as series leads to

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (k_1 + \cdots + k_n) \alpha_{k_1 \cdots k_n} z_1^{k_1} \cdots z_n^{k_n}$$

$$= \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (z_1 \overline{\zeta}_1)^{k_1} \cdots (z_n \overline{\zeta}_n)^{k_n} \frac{d\zeta}{\zeta} \,.$$

By the Lebesgue theorem we know

$$\mid k \mid \alpha_k = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \overline{\zeta}^k \frac{d\zeta}{\zeta}, \quad 0 \le \mid k \mid .$$

This means

$$\alpha_k = \frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n} \gamma(\zeta)\frac{\overline{\zeta}^k}{\mid k\mid}\frac{d\zeta}{\zeta}, \quad \mid k \mid > 0.$$

i.e.

$$\varphi(z) = \sum_{|k|>0}\Big(\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n} \gamma(\zeta)\frac{\overline{\zeta}^k}{\mid k\mid}\frac{d\zeta}{\zeta}\Big)z^k + C_1^\circ$$

$$= \frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n} \gamma(\zeta)\sum_{|k|>0}\frac{(z\overline{\zeta})^k}{\mid k\mid}\frac{d\zeta}{\zeta} + C_1^\circ\,. \tag{2.25}$$

In the same way

$$\psi(z) = \frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n} \overline{\gamma(\zeta)}\sum_{|k|>0}\frac{(z\overline{\zeta})^k}{\mid k\mid}\frac{d\zeta}{\zeta} + C_2^\circ.$$

By the way, comparing (2.25) with (2.24) we get the sum of an interesting series :

$$\sum_{|k|>0}^{\infty}\frac{(z\overline{\zeta})^k}{|k|} = \int_0^1\Big[\frac{1}{1-sz\overline{\zeta}}-1\Big]\frac{ds}{s}, \quad for \quad z\in I\!D^n, \quad \zeta\in\partial_0 I\!D^n.$$

If we choose $\zeta = (1,\cdots,1)\in\partial_0 I\!D^n$ then it turns out to be

$$\sum_{\substack{k_1=0\\|k|>0}}^{\infty}\cdots\sum_{k_n=0}^{\infty}\frac{z_1^{k_1}\cdots z_n^{k_n}}{k_1+\cdots+k_n} = \int_0^1\Big[\frac{1}{1-sz_1}\cdots\frac{1}{1-sz_n}-1\Big]\frac{ds}{s}, \quad z\in I\!D^n.$$

By the induction method it is easy to get

$$\int_0^1\Big[\frac{1}{1-sz_1}\cdots\frac{1}{1-sz_n}-1\Big]\frac{ds}{s} = \sum_{\nu=1}^{n} z_\nu^{n-1}\prod_{\substack{\mu=1\\\mu\neq\nu}}^{\text{n}}\frac{1}{z_\nu-z_\mu}\int_0^1\frac{z_\nu}{1-sz_\nu}ds$$

$$= -\sum_{\nu=1}^{n} z_\nu^{n-1} log(1-z_\nu)\prod_{\substack{\mu=1\\\mu\neq\nu}}^{\text{n}}\frac{1}{z_\nu-z_\mu}\,.$$

2.2 The inhomogeneous pluriholomorphic system

2.2.1 Preliminaries and Definition

The Neumann problem for the inhomogeneous pluriholomorphic system in the unit ball was studied in [2] page 225. However, about the Neumann problem even for the homogeneous pluriholomorphic system in the unit polydisc nothing can be found in the literature , although a great deal of research has been done about the $\overline{\partial}-$ Neumann problem in polydiscs, see [6], [9], [25] etc.

Let f_{kl} , γ be given functions with $f_{kl\overline{z}_j} \in L_1(\overline{I\!D^n}) \cap C(\overline{I\!D^n}), \gamma \in C(\partial_0 I\!D^n)$. We denote by $Z\!\!Z^+$ the set of non-negative integers, i.e., $Z\!\!Z^+ = \{0\} \cup I\!N$ where $I\!N$ is the set of positive integers. Consider the following inhomogeneous system of $n(n+1)/2$ independent equations

$$\frac{\partial^2 u}{\partial \overline{z}_k \partial \overline{z}_\ell} = f_{k\ell}(z) , \quad 1 \leq k , \ \ell \leq n, \tag{2.26}$$

with given right – hand sides satisfying the condition

$$f_{k\ell}(z) = f_{\ell k}(z) , \qquad \frac{\partial f_{k\ell}}{\partial \overline{z}_s} - \frac{\partial f_{ks}}{\partial \overline{z}_\ell} = 0 , \qquad 1 \leq s \leq n . \tag{2.27}$$

Problem N_2 . Find a $C^1(\overline{I\!D^n})$ – solution of system (2.26), satisfying the Neumann condition

$$\frac{\partial u}{\partial \nu_\zeta} = \gamma_0(\zeta) , \qquad \zeta \in \partial_0 I\!D^n , \tag{2.28}$$

where $\partial u / \partial \nu_\zeta$ denotes the outward normal derivative of $u(z)$ at the point $\zeta \in \partial_0 I\!D^n$.

By the definition it is known that, see the previous section or [22], the Neumann condition (2.28) for the unit polydisc turns out to be

$$\sum_{j=1}^{n} \Big(z_j \frac{\partial u}{\partial z_j} + \overline{z}_j \frac{\partial u}{\partial \overline{z}_j} \Big) \Big|_\zeta = \gamma(\zeta) , \qquad \zeta \in \partial_0 I\!D^n , \tag{2.29}$$

with $\gamma(\zeta) = \gamma_0(\zeta)\sqrt{n}/2$.

It is also known that the general solution to (2.26) is representable as

$$u(z) = \phi_0(z) + \ < \ \phi(z) \ , \ z \ > \ + u_0(z) \tag{2.30}$$

where $\phi(z) = \Big(\phi_1(z), \cdots, \phi_n(z) \Big)$ and every $\phi_k(z), k = 0, 1, \cdots, n$ is an arbitrary function, analytic in $I\!D^n$; $u_0(z)$ is a special solution to (2.26) given by

$$u_0 = \sum_{\mu=1}^{n} (-1)^{\mu+1} \sum_{\substack{1 \leq \ell_1 \leq n \\ 1 \leq \ell_2, \cdots, \ell_\mu \leq n}} T_{\ell_\mu} \cdots T_{\ell_2} T_{\ell_1}^2 f_{\ell_1 \ell_1 \overline{\zeta}_{\ell_2} \cdots \overline{\zeta}_{\ell_\mu}}$$

$$+ \sum_{\nu=2}^{n} (-1)^{\nu} \sum_{1 \leq \ell_1 < \cdots < \ell_\nu \leq n} T_{\ell_\nu} \cdots T_{\ell_1} f_{\ell_1 \ell_2 \overline{\zeta}_{\ell_3} \cdots \overline{\zeta}_{\ell_\nu}} \tag{2.31}$$

see the previous chapter or [21].

It is well known that for any given real – valued continuous function γ on $\partial I\!D$ there exists an analytic function w in $I\!D$, the real part of which has the boundary values γ on $\partial I\!D$, $\mathrm{Re}\, w = \gamma$. A solution can be given by the Schwarz integral $S\gamma$ which is the complex counterpart of the Poisson integral $P\gamma$. Hence γ turns out to be the boundary values of a harmonic function in $I\!D$. For two complex variables in order that a given real – valued function on the distinguished boundary $\partial_0 I\!D^2$ of the unit bidisc $I\!D^2$ is the boundary value function of the real part of an analytic function in $I\!D^2$ it has to belong to the space $\partial Ph_{I\!D^2}$ of boundary values of pluriharmonic functions in $I\!D^2$. It is known that not every function defined on $\partial_0 \mathbb{D}^2$ is in $\partial Ph_{\mathbb{D}^2}$, see [2] page 243. However, for our discussion we need to look at the problem a little bit further. Let the real– valued function γ on $\partial_0 \mathbb{D}^2$ be representable by a Fourier series

$$\gamma(z_1,z_2) = \sum_{i,k=-\infty}^{+\infty} a_{ik} z_1^i z_2^k \,, \quad a_{ik} = \frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \gamma(\zeta_1,\zeta_2) \overline{\zeta}_1^i \overline{\zeta}_2^k \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2}$$

$$a_{-i,-k} = \overline{a_{ik}} \,, \; i,k \in \mathbb{Z} \,, \; (z_1,z_2) \in \partial_0 \mathbb{D}^2. \tag{2.32}$$

Thus for the given γ we have two real pluriharmonic function in $\mathbb{C}^2$: one in $\mathbb{D}^{++} := \mathbb{D}^2$ $(\mathbb{D}^{--} = \{ \, z | z = (z_1,z_2) \, , \; |z_1| > 1 \, , \; |z_2| > 1 \, \})$, i.e.,

$$\sum_{i,k=0}^{+\infty} \Big\{ a_{ik} z_1^i z_2^k + a_{-i,-k} \overline{z}_1^i \overline{z}_2^k \Big\} - a_{0,0} \tag{2.33}$$

and one in $\mathbb{D}^{+-} = \{ \, z | z = (z_1,z_2) \, , \; |z_1| < 1 \, , \; |z_2| > 1 \, \} \; (\mathbb{D}^{-+})$,

$$\sum_{i,k=1}^{+\infty} \Big\{ a_{i,-k} z_1^i z_2^{-k} + a_{-i,k} z_1^{-i} z_2^k \Big\} \tag{2.34}$$

Clearly, if $\gamma \in \partial Ph_{\mathbb{D}^2}$, then obviously $a_{-i,k} = a_{i,-k} = 0$ for $i,k \in \mathbb{N}$,i.e.,

$$a_{i,-k} = \frac{-1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \gamma(\zeta_1,\zeta_2) \overline{\zeta}_1^i \zeta_2^k \frac{d\overline{\zeta}_1}{\overline{\zeta}_1} \frac{d\zeta_2}{\zeta_2} = 0 \,, \; i,k \in \mathbb{N} \,, \tag{2.35}$$

or equivalently

$$\frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \gamma(\zeta_1,\zeta_2) \frac{z_1 \overline{\zeta}_1}{1 - z_1 \overline{\zeta}_1} \frac{\overline{z}_2 \zeta_2}{1 - \overline{z}_2 \zeta_2} \frac{d\overline{\zeta}_1}{\overline{\zeta}_1} \frac{d\zeta_2}{\zeta_2} = 0 \,, \; (z_1,z_2) \in \mathbb{D}^2 \,. \tag{2.36}$$

If $\gamma \in \partial Ph_{\mathbb{D}^{+-}}$, then $a_{i,k} = a_{-i,-k} = 0$ for $i,k \in \mathbb{Z}^+$. This means γ satisfies

$$a_{i,k} = \frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \gamma(\zeta_1,\zeta_2) \overline{\zeta}_1^i \overline{\zeta}_2^k \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} = 0 \,, \; i,k \in \mathbb{Z}^+ \,, \tag{2.37}$$

equivalently

$$\frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \gamma(\zeta_1,\zeta_2) \frac{1}{1 - z_1 \overline{\zeta}_1} \frac{1}{1 - z_2 \overline{\zeta}_2} \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} = 0 \,, \; (z_1,z_2) \in \mathbb{D}^2 \,. \tag{2.38}$$

Evidently, it is easy to see that $\partial Ph_{\mathbb{D}^2} = \partial Ph_{\mathbb{D}^{--}}$, $\partial Ph_{\mathbb{D}^{+-}} = \partial Ph_{\mathbb{D}^{-+}}$.

Further, if γ belongs to $\partial H_{\mathbb{D}^2}$ the space of boundary values of functions, holomorphic in $\mathbb{D}^2$, then γ satisfies the condition (2.36) and

$$a_{-i,-k} = \frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \gamma(\zeta_1,\zeta_2) \zeta_1^i \zeta_2^k \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} = 0 \,, \; i,k \in \mathbb{Z}^+ \,, i+k \neq 0, \tag{2.39}$$

as well, i.e.,

$$\frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \gamma(\zeta_1,\zeta_2) \Big[\frac{1}{1 - \overline{z}_1 \zeta_1} \frac{1}{1 - \overline{z}_2 \zeta_2} - 1 \Big] \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} = 0 \,, \; (z_1,z_2) \in \mathbb{D}^2 \,, \tag{2.40}$$

equivalently

$$\frac{1}{(2\pi i)^2}\int_{\partial_0 \mathbb{D}^2}\gamma(\zeta_1,\zeta_2)\Big[\frac{\overline{z}_1\zeta_1}{1-\overline{z}_1\zeta_1}+\frac{\overline{z}_2\zeta_2}{1-\overline{z}_2\zeta_2}-\frac{\overline{z}_1\zeta_1}{1-\overline{z}_1\zeta_1}\frac{\overline{z}_2\zeta_2}{1-\overline{z}_2\zeta_2}\Big]\frac{d\zeta_1}{\zeta_1}\frac{d\zeta_2}{\zeta_2}=0\,. \tag{2.41}$$

For our further discussion we need Theorem 5.1 from [2]:

Theorem 7 *Let* γ *be real–valued continuous on* $\partial_0\mathbb{D}^n$ *satisfying*

$$\sum_{k=1}^{n-1}\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma(\zeta)\Big[\Big(\prod_{\nu=1}^{k}\frac{\zeta_\nu}{\zeta_\nu-z_\nu}-1\Big)\frac{\overline{z}_{k+1}}{\overline{\zeta_{k+1}-z_{k+1}}}+\Big(\prod_{\nu=1}^{k}\frac{\overline{\zeta}_\nu}{\overline{\zeta_\nu-z_\nu}}-1\Big)\frac{z_{k+1}}{\zeta_{k+1}-z_{k+1}}\Big]$$
$$\times\prod_{\nu=k+2}^{n}\Big(\frac{\zeta_\nu}{\zeta_\nu-z_\nu}+\frac{\overline{\zeta}_\nu}{\overline{\zeta_\nu-z_\nu}}-1\Big)\frac{d\zeta}{\zeta}\;=\;0$$

or equivalently $\gamma\in\partial Ph_{\mathbb{D}^n}$ *. Then for any real* c

$$\varphi(z):=\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma(\zeta)\Big[2\frac{\zeta}{\zeta-z}-1\Big]\frac{d\zeta}{\zeta}+ic$$

is analytic in $\mathbb{D}^n$ *satisfying* $Re\,\varphi(\zeta)=\gamma(\zeta)$ *on* $\partial_0\mathbb{D}^n$ *.*

On the basis of this theorem and from our discussion above we can get some conclusion about the boundary values of holomorphic functions in polydiscs.

Lemma 2 *Let* γ *be real–valued continuous on* $\partial_0\mathbb{D}^n$ *satisfying* $\gamma\in\partial H_{\mathbb{D}^n}$ *:*

$$\sum_{\nu=1}^{n}\sum_{\lambda=0}^{\nu-1}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}}\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma(\zeta)\prod_{\tau=1}^{\lambda}\frac{z_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau}-z_{k_\tau}}}\frac{d\zeta}{\zeta}\;=\;0\,. \tag{2.42}$$

Then

$$\phi(z):=\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma(\zeta)\frac{d\zeta}{\zeta-z} \tag{2.43}$$

is analytic in $\mathbb{D}^n$ *satisfying* $\phi(\zeta)=\gamma(\zeta)$ *on* $\partial_0\mathbb{D}^n$ *.*

2.2.2 The classical problem

From (2.30) it follows that

$$u_{\overline{z}_k}=\phi_k(z)+u_{0\overline{z}_k}\,,\quad u_{z_k}=\frac{\partial\phi_0}{\partial z_k}+\sum_{\mu=1}^{n}\overline{z}_\mu\frac{\partial\phi_\mu}{\partial z_k}+\frac{\partial u_0}{\partial z_k}\,. \tag{2.44}$$

where

$$u_{0\overline{z}_k}=\sum_{\nu=1}^{n}(-1)^{\nu+1}\sum_{1\le k_1<\cdots<k_\nu\le n}T_{k_\nu}\cdots T_{k_1}f_{k_1k\,\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\nu}}\,,\quad 1\le k\le n\,.$$

Substituting these expressions into (2.29), we obtain an equality for $\zeta \in \partial_0 \mathbb{D}^n$,

$$\sum_{k=1}^{n} \overline{\zeta}_k \Big(\phi_k(\zeta) + \sum_{j=1}^{n} \zeta_j \frac{\partial \phi_k}{\partial \zeta_j} + \frac{\zeta_k}{n} \sum_{j=1}^{n} \zeta_j \frac{\partial \phi_0}{\partial \zeta_j} \Big) = \sum_{k=1}^{n} \overline{\zeta}_k \Big(\frac{\zeta_k}{n} \gamma(\zeta) - \frac{\partial u_0}{\partial \overline{\zeta}_k} - \frac{\zeta_k}{n} \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \Big) . \quad (2.45)$$

Evidently this equality is satisfied if

$$\phi_k(\zeta) + \sum_{j=1}^{n} \zeta_j \frac{\partial \phi_k}{\partial \zeta_j} + \frac{\zeta_k}{n} \sum_{j=1}^{n} \zeta_j \frac{\partial \phi_0}{\partial \zeta_j} = \frac{\zeta_k}{n} \Big[\gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \Big] - \frac{\partial u_0}{\partial \overline{\zeta}_k} \quad (2.46)$$

hold for any $\zeta \in \partial_0 \mathbb{D}^n$ and $k \in \{1, 2, \cdots, n\}$. Since the left hand–side represents the boundary values of a holomorphic function in $\mathbb{D}^n$, the right hand–side does too. Thus according to Lemma.1, the problem is solvable if and only if the following conditions are satisfied,

$$\sum_{\nu=1}^{n} \sum_{\lambda=0}^{\nu-1} \sum_{\substack{1 \le k_1 < \cdots < k_\lambda \le n \\ 1 \le k_{\lambda+1} < \cdots < k_\nu \le n}} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Big\{ \frac{< \zeta, z >}{n} \Big[\gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \Big] - < grad_{\overline{\zeta}} u_0, z > \Big\}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau} - z_{k_\tau}}} \frac{d\zeta}{\zeta} = 0 \quad , \quad z \in \mathbb{D}^n . \quad (2.47)$$

Then

$$\phi_k(z) + \sum_{j=1}^{n} z_j \frac{\partial \phi_k}{\partial z_j} + \frac{z_k}{n} \sum_{j=1}^{n} z_j \frac{\partial \phi_0}{\partial z_j}$$

$$= \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Big\{ \frac{\zeta_k}{n} \Big[\gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \Big] - \frac{\partial u_0}{\partial \overline{\zeta}_k} \Big\} \frac{d\zeta}{\zeta - z} \; , \quad z \in \mathbb{D}^n \quad (2.48)$$

is analytic in $\mathbb{D}^n$ and satisfies condition (2.47).

To derive the solution of problem N_2 we apply the Cauchy formula to (2.46) and by taking into account

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}} Tf(\zeta) \frac{d\zeta}{\zeta - z} = 0$$

i.e.,

$$\frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} u_{0\overline{z}_k} \frac{d\zeta}{\zeta - z} = 0$$

we get the following partial differential equations for $z \in \mathbb{D}^n$,

$$\phi_k(z) + \sum_{j=1}^{n} z_j \frac{\partial \phi_k}{\partial z_j} = -\frac{z_k}{n} \sum_{j=1}^{n} z_j \frac{\partial \phi_0}{\partial z_j} + \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{\zeta_k}{n} \Big[\gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \Big] \frac{d\zeta}{\zeta - z}. \quad (2.49)$$

By the transformation

$$\omega_1 = z_1 \; , \; \omega_2 = \frac{z_1}{z_2} \; , \cdots \; , \; \omega_n = \frac{z_1}{z_n} \; ,$$

we obtain for (2.49) the equations

$$\omega_1 \frac{\partial \phi_1}{\partial \omega_1} + \phi_1 = -\frac{\omega_1^2}{n} \frac{\partial \phi_0}{\partial \omega_1} + \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{\zeta_1}{n} \Big[\gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \Big]$$

$$\times \frac{d\zeta_1}{\zeta_1-\omega_1}\frac{d\zeta_2}{\zeta_2-\omega_1/\omega_2}\cdots\frac{d\zeta_n}{\zeta_n-\omega_1/\omega_n}\ ,$$

$$\omega_1\frac{\partial\phi_k}{\partial\omega_1}+\phi_k=-\frac{\omega_1^2}{n\omega_k}\frac{\partial\phi_0}{\partial\omega_1}+\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\frac{\zeta_k}{n}\Big[\gamma(\zeta)-\sum_{j=1}^n\zeta_j\frac{\partial u_0}{\partial\zeta_j}\Big]$$

$$\times \frac{d\zeta_1}{\zeta_1-\omega_1}\frac{d\zeta_2}{\zeta_2-\omega_1/\omega_2}\cdots\frac{d\zeta_n}{\zeta_n-\omega_1/\omega_n}\ ,\ k=2,\cdots,n.$$

Integrating these equations we have

$$\omega_1\phi_1=-\int_0^{\omega_1}\frac{t^2}{n}\frac{\partial\phi_0}{\partial t}dt+\int_0^{\omega_1}\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\frac{\zeta_1}{n}\Big[\gamma(\zeta)-\sum_{j=1}^n\zeta_j\frac{\partial u_0}{\partial\zeta_j}\Big]$$

$$\times \frac{d\zeta_1}{\zeta_1-t}\frac{d\zeta_2}{\zeta_2-t/\omega_2}\cdots\frac{d\zeta_n}{\zeta_n-t/\omega_n}dt+C_1$$

$$\omega_1\phi_k=-\int_0^{\omega_1}\frac{t^2}{n\omega_k}\frac{\partial\phi_0}{\partial t}dt+\int_0^{\omega_1}\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\frac{\zeta_k}{n}\Big[\gamma(\zeta)-\sum_{j=1}^n\zeta_j\frac{\partial u_0}{\partial\zeta_j}\Big]$$

$$\times \frac{d\zeta_1}{\zeta_1-t}\frac{d\zeta_2}{\zeta_2-t/\omega_2}\cdots\frac{d\zeta_n}{\zeta_n-t/\omega_n}dt+C_k\ ,\ k=2,\cdots,n.$$

Substituting $\omega_1=0$ on both sides we see that $C_k=0, k=1,2,\cdots,n$. Returning to the original variables we have

$$z_1\phi_1(z)=\int_0^1\frac{z_1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\frac{\zeta_1}{n}\Big[\gamma(\zeta)-\sum_{j=1}^n\zeta_j\frac{\partial u_0}{\partial\zeta_j}\Big]\frac{d\zeta}{\zeta-sz}ds$$

$$-z_1\int_0^1\frac{sz_1}{n}\sum_{j=1}^n(sz_j)\frac{\partial\phi_0(sz)}{\partial(sz_j)}ds$$

$$z_1\phi_k(z)=\int_0^1\frac{z_1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\frac{\zeta_k}{n}\Big[\gamma(\zeta)-\sum_{j=1}^n\zeta_j\frac{\partial u_0}{\partial\zeta_j}\Big]\frac{d\zeta}{\zeta-sz}ds$$

$$-z_1\int_0^1\frac{sz_k}{n}\sum_{j=1}^n(sz_j)\frac{\partial\phi_0(sz)}{\partial(sz_j)}ds\quad ,\ k=2,\cdots,n,$$

i.e., for $k=1,\cdots,n$, we have

$$\phi_k(z)=\int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\frac{\zeta_k}{n}\Big[\gamma(\zeta)-\sum_{j=1}^n\zeta_j\frac{\partial u_0}{\partial\zeta_j}\Big]\frac{d\zeta}{\zeta-sz}ds-\int_0^1\frac{s^2z_k}{n}d\phi_0(sz). \tag{2.50}$$

Hence the representation (2.30) gets the form

$$u(z)=\int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\frac{<\zeta,z>}{n}\Big[\gamma(\zeta)-\sum_{j=1}^n\zeta_j\frac{\partial u_0}{\partial\zeta_j}\Big]\frac{d\zeta}{\zeta-sz}ds+u_0(z)$$

$$+\phi_0(z) - \int_0^1 \frac{<sz, sz>}{n} d\phi_0(sz) \, . \tag{2.51}$$

If we take

$$\phi_0(z) = \sum_{|\kappa| \geq 0} a_\kappa z^\kappa \, , \quad z \in I\!D^n \, ,$$

then

$$\phi_0(z) - \int_0^1 \frac{<sz, sz>}{n} d\phi_0(sz) = \sum_{|\kappa| \geq 0} a_\kappa z^\kappa$$

$$- \int_0^1 \frac{s^2|z|^2}{n} \Big[\sum_{j=1}^n \Big(\sum_{|\kappa| \geq 1} a_\kappa \kappa_j (sz)^\kappa / sz_j \Big) z_j ds \Big]$$

$$= \sum_{|\kappa| \geq 0} a_\kappa z^\kappa - \int_0^1 \frac{s|z|^2}{n} \sum_{|\kappa| \geq 1} a_\kappa |\kappa| (sz)^\kappa ds$$

$$= a_0 + \sum_{|\kappa| \geq 1} a_\kappa \Big[1 - \frac{|\kappa||z|^2}{n(|\kappa|+2)} \Big] z^\kappa \, .$$

Thus

$$u(z) = \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial_0 I\!D^n} \frac{<\zeta, z>}{n} \Big[\gamma(\zeta) - \sum_{j=1}^n \zeta_j \frac{\partial u_0}{\partial \zeta_j} \Big] \frac{d\zeta}{\zeta - sz} ds + u_0(z)$$

$$+ \sum_{|\kappa| \geq 0} a_\kappa \Big[1 - \frac{|\kappa||z|^2}{n(|\kappa|+2)} \Big] z^\kappa \, , \quad z \in I\!D^n \, . \tag{2.52}$$

Theorem 8 *Problem N_2 is solvable if and only if its right–hand sides satisfy condition (2.47) on $\partial_0 I\!D^n$. The general solution can be given by (2.52). The corresponding homogeneous problem has infinitely many linearly independent non–trivial solutions,*

$$\Big[1 - \frac{|\kappa||z|^2}{n(|\kappa|+2)} \Big] z^\kappa \, , \; |\kappa| > 0, \quad z \in I\!D^n \, .$$

The problem N_2 is not well posed.

2.2.3 The modified problem

Since the solution (2.52) includes a free analytic function, clearly to get a fixed solution only a Schwarz problem is needed to be solved. So we introduce an additional boundary condition.

Problem N_2^* Find a $C^1(\overline{I\!D^n})$ solution to system (2.26) satisfying the Neumann condition (2.28) and

$$Re\, u(\zeta) = \gamma^*(\zeta) \, , \quad \zeta \in \partial_0 I\!D^n \, . \tag{2.53}$$

We call this problem the modified Neumann problem for system (2.26).

Let $f_{k\ell}=0$ in (2.26). Then the solvability condition (2.47) takes the form

$$\sum_{\nu=1}^{n}\sum_{\lambda=0}^{\nu-1}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}}\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{<\zeta,z>}{n}\gamma(\zeta)\prod_{\tau=1}^{\lambda}\frac{z_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}$$
$$\times\prod_{\tau=\lambda+1}^{\nu}\frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau}-z_{k_\tau}}}\frac{d\zeta}{\zeta}=0\quad,\quad\zeta\in\partial_0\mathbb{D}^n\,,\quad z\in\mathbb{D}^n\cup\partial_0\mathbb{D}^n \tag{2.54}$$

and it means that every $\zeta_k\gamma(\zeta)$ on $\partial_0\mathbb{D}^n$ ($k=1,\cdots,n$) belongs to $\partial H_{\mathbb{D}^n}$. Actually it is evident that $\gamma(\zeta)\in\partial H_{\mathbb{D}^n}$. Note $\zeta_1\gamma(\zeta)=\varphi_1(\zeta)\,,\varphi_1(\zeta)\in\partial H_{\mathbb{D}^n}$, then $\gamma(\zeta)=\overline{\zeta_1}\varphi_1(\zeta)$. If $\gamma(\zeta)\notin\partial H_{\mathbb{D}^n}$, then $\zeta_2\overline{\zeta_1}\varphi_1(\zeta)\notin\partial H_{\mathbb{D}^n}$. But by the condition above $\zeta_2\gamma(\zeta)\in\partial H_{\mathbb{D}^n}$. This is a contradiction. Hence condition (2.54) becomes

$$\sum_{\nu=1}^{n}\sum_{\lambda=0}^{\nu-1}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}}\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma(\zeta)\prod_{\tau=1}^{\lambda}\frac{z_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau}-z_{k_\tau}}}\frac{d\zeta}{\zeta}=0. \tag{2.55}$$

Substituting (2.53) into (2.52) shows

$$\sum_{|\kappa|\ge0}\frac{\overline{a}_\kappa\overline{\zeta}^\kappa+a_\kappa\zeta^\kappa}{2+|\kappa|}=\gamma^*(\zeta)-Re\int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{<\eta,\zeta>}{n}\gamma(\eta)\frac{d\eta}{\eta-s\zeta}ds=:2\Gamma(\zeta),$$

i.e.,

$$Re\sum_{|\kappa|\ge0}\frac{a_\kappa\zeta^\kappa}{2+|\kappa|}=\Gamma(\zeta)\,,\quad\zeta\in\partial_0\mathbb{D}^n\,. \tag{2.56}$$

Due to the character of the left–hand side of (2.56), the right–hand side $\Gamma(\zeta)$ on $\partial_0\mathbb{D}^n$ is also the boundary values of a function, pluriharmonic in $\mathbb{D}^n$. This means the given function $\Gamma(\zeta)$ on $\partial_0\mathbb{D}^n$must satisfy the condition,

$$\sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}}\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\Gamma(\zeta)\prod_{\tau=1}^{\lambda}\frac{z_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau}-z_{k_\tau}}}\frac{d\zeta}{\zeta}=0 \tag{2.57}$$

In fact due to $\gamma(\zeta)\in\partial H_{\mathbb{D}^n}$, it follows that

$$2\Gamma(\zeta)=\gamma^*(\zeta)-Re\int_0^1\frac{<s\zeta,\zeta>}{n}\gamma(s\zeta)ds=\gamma^*(\zeta)-Re\int_0^1 s\gamma(s\zeta)ds\,.$$

Hence

$$Re\int_0^1 s\gamma(s\zeta)ds\in\partial Ph_{\mathbb{D}^n}$$

the condition (2.57) implies that $\gamma^*(\zeta)\in\partial Ph_{\mathbb{D}^n}$,i.e.,

$$\sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}}\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma^*(\zeta)\prod_{\tau=1}^{\lambda}\frac{z_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau}-z_{k_\tau}}}\frac{d\zeta}{\zeta}=0. \tag{2.58}$$

So if condition (2.58) is satisfied, then the Schwarz problem (2.56) is solvable and the solution is given by

$$\sum_{|\kappa|\geq 0}\frac{a_\kappa z^\kappa}{|\kappa|+2}=\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\Gamma(\zeta)\Big[2\frac{\zeta}{\zeta-z}-1\Big]\frac{d\zeta}{\zeta}+iC^0$$

$$=\sum_{|\kappa|>0}\frac{2}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\Gamma(\zeta)(z\overline{\zeta})^\kappa\frac{d\zeta}{\zeta}+\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\Gamma(\zeta)\frac{d\zeta}{\zeta}+iC^0$$

with an arbitrary real constant C^0, is analytic in $I\!D^n$ and satisfies equation (2.56), see [2] page 251. One can see that

$$a_0=\frac{2}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\Gamma(\zeta)\frac{d\zeta}{\zeta}+i2C^0\ ,\quad a_\kappa=\frac{2(2+|\kappa|)}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\Gamma(\zeta)\overline{\zeta}^\kappa\frac{d\zeta}{\zeta}\ ,\quad |\kappa|>0\ . \tag{2.59}$$

Hence if conditions (2.55) and (2.58) are satisfied, i.e., if $\gamma(\zeta)\in\ \partial H_{I\!D^n}$ and $\gamma^*(\zeta)\in\partial Ph_{I\!D^n}$, then problem N_2^* with $f_{k\ell}=0$ is solvable and the solution is given by

$$u(z)=\sum_{|\kappa|\geq 0}a_\kappa\Big[1-\frac{|\kappa||z|^2}{n(|\kappa|+2)}\Big]z^\kappa+\int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\frac{<\zeta,z>}{n}\gamma(\zeta)\frac{d\zeta}{\zeta-sz}ds\ ,\ z\in I\!D^n\ . \tag{2.60}$$

But from

$$\sum_{|\kappa|\geq 0}a_\kappa\Big[1-\frac{|\kappa||z|^2}{n(|\kappa|+2)}\Big]z^\kappa=\sum_{|\kappa|>0}\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}2\Gamma(\zeta)\Big[2+|\kappa|\frac{n-|z|^2}{n}\Big](z\overline{\zeta})^\kappa\frac{d\zeta}{\zeta}+a_0$$

$$=a_0+\frac{2}{(2\pi i)^n}\int_{\partial_0 I\!D^n}2\Gamma(\zeta)\Big[\frac{1}{1-z\overline{\zeta}}-1\Big]\frac{d\zeta}{\zeta}+\sum_{|\kappa|>0}\frac{n-|z|^2}{n(2\pi i)^n}\int_{\partial_0 I\!D^n}2\Gamma(\zeta)|\kappa|(z\overline{\zeta})^\kappa\frac{d\zeta}{\zeta}$$

$$=a_0+\frac{2}{(2\pi i)^n}\int_{\partial_0 I\!D^n}2\Gamma(\zeta)\Big[\frac{1}{1-z\overline{\zeta}}-1\Big]\frac{d\zeta}{\zeta}+\sum_{|\kappa|>0}\frac{n-|z|^2}{n(2\pi i)^n}\int_{\partial_0 I\!D^n}2\Gamma(\zeta)\frac{\partial}{\partial t}(tz\overline{\zeta})^\kappa\Big|_{t=1}\frac{d\zeta}{\zeta}$$

$$=a_0+\frac{n-|z|^2}{n}\frac{\partial}{\partial t}\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}2\Gamma(\zeta)\Big[\frac{1}{1-tz\overline{\zeta}}-1\Big]\frac{d\zeta}{\zeta}\Big|_{t=1}$$
$$+\frac{2}{(2\pi i)^n}\int_{\partial_0 I\!D^n}2\Gamma(\zeta)\Big[\frac{1}{1-z\overline{\zeta}}-1\Big]\frac{d\zeta}{\zeta}$$

we get

$$u(z)=iC_0+\frac{n-|z|^2}{n}\frac{\partial}{\partial t}\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}2\Gamma(\zeta)\Big[\frac{1}{1-tz\overline{\zeta}}-1\Big]\frac{d\zeta}{\zeta}\Big|_{t=1}$$
$$+\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}2\Gamma(\zeta)\Big[\frac{2}{1-z\overline{\zeta}}-1\Big]\frac{d\zeta}{\zeta} \tag{2.61}$$

where C_0 is an arbitrary real constant. Next we make some simplifications. Let

$$I_1=\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}2\Gamma(\zeta)\Big[\frac{2}{1-z\overline{\zeta}}-1\Big]\frac{d\zeta}{\zeta}\ ,\quad z\in I\!D^n,$$

then

$$I_1 = \frac{-1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\Big\{Re\int_0^1 \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\frac{<\eta,\zeta>}{n}\gamma(\eta)\frac{d\eta}{\eta - s\zeta}ds\Big\}\Big[\frac{2}{1-z\overline{\zeta}}-1\Big]\frac{d\zeta}{\zeta}$$

$$+\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\gamma^*(\zeta)\Big[\frac{2}{1-z\overline{\zeta}}-1\Big]\frac{d\zeta}{\zeta} =: -I_{1a} - I_{1b} + I_{1c}$$

where

$$2I_{1a} = \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\int_0^1 \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\frac{<\eta,\zeta>}{n}\gamma(\eta)\frac{d\eta}{\eta - s\zeta}ds\Big[\frac{2}{1-z\overline{\zeta}}-1\Big]\frac{d\zeta}{\zeta}\ .$$

By changing the order of integration, we have

$$2I_{1a} = \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\int_0^1 \gamma(\eta)\Big\{\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\frac{<\eta,\zeta>}{n}\frac{1}{1-s\zeta\overline{\eta}}\Big[\frac{2}{1-z\overline{\zeta}}-1\Big]\frac{d\zeta}{\zeta}\ \Big\}ds\frac{d\eta}{\eta}\ ,$$

but

$$\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\frac{<\eta,\zeta>}{n}\frac{1}{1-s\zeta\overline{\eta}}\Big[\frac{2}{1-z\overline{\zeta}}-1\Big]\frac{d\zeta}{\zeta}$$

$$= \sum_{k=1}^n \frac{1}{n(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\frac{\eta_k\overline{\zeta}_k}{1-s\zeta_k\overline{\eta}_k}\prod_{\substack{\tau=1\\ \tau\neq k}}^n \frac{1}{1-s\zeta_\tau\overline{\eta}_\tau}\frac{2d\zeta}{\zeta - z} - \overline{\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\frac{<\zeta,\eta>}{n}\frac{d\zeta}{\zeta - s\eta}}$$

$$= \sum_{k=1}^n \frac{2}{n(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\eta_k\Big[\overline{\zeta}_k + \frac{s\overline{\eta}_k}{1-s\zeta_k\overline{\eta}_k}\Big]\prod_{\substack{\tau=1\\ \tau\neq k}}^n \frac{1}{1-s\zeta_\tau\overline{\eta}_\tau}\frac{1}{1-z\overline{\zeta}}\frac{d\zeta}{\zeta} - \frac{\overline{<s\eta,\eta>}}{n}$$

$$= \sum_{k=1}^n \frac{2}{n}\eta_k\frac{s\overline{\eta}_k}{1-sz_k\overline{\eta}_k}\prod_{\substack{\tau=1\\ \tau\neq k}}^n \frac{1}{1-sz_\tau\overline{\eta}_\tau} - s = s\Big[\frac{2}{1-sz\overline{\eta}}-1\Big]\ ,$$

leads to

$$2I_{1a} = \int_0^1 \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\gamma(\eta)\Big[2\frac{\eta}{\eta - sz}-1\Big]\frac{d\eta}{\eta}\ sds$$

The second part of I_1 which has to be simplified is

$$2I_{1b} = \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\int_0^1 \overline{\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\frac{<\eta,\zeta>}{n}\gamma(\eta)\frac{d\eta}{\eta - s\zeta}}ds\Big[\frac{2}{1-z\overline{\zeta}}-1\Big]\frac{d\zeta}{\zeta}\ .$$

By changing the order of the integrals

$$2I_{1b} = \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\int_0^1 \overline{\gamma(\eta)}\Big\{\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\frac{<\zeta,\eta>}{n}\frac{1}{1-s\eta\overline{\zeta}}\Big[\frac{2}{1-z\overline{\zeta}}-1\Big]\frac{d\zeta}{\zeta}\Big\}ds\frac{d\eta}{\eta}$$

and from

$$\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\frac{<\zeta,\eta>}{n}\frac{1}{1-s\eta\overline{\zeta}}\Big[\frac{2}{1-z\overline{\zeta}}-1\Big]\frac{d\zeta}{\zeta} = \sum_{k=1}^n \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\frac{\zeta_k\overline{\eta}_k}{n}$$

$$\times\Big[1+s\eta_k\overline{\zeta}_k+\frac{(s\eta_k\overline{\zeta}_k)^2}{1-s\eta_k\overline{\zeta}_k}\Big]\prod_{\substack{\tau=1\\ \tau\neq k}}^{n}\frac{1}{1-s\eta_\tau\overline{\zeta}_\tau}\Big\{2\Big[1+\frac{z_k\overline{\zeta}_k}{1-z_k\overline{\zeta}_k}\Big]\prod_{\substack{\tau=1\\ \tau\neq k}}^{n}\frac{1}{1-sz_\tau\overline{\zeta}_\tau}-1\Big\}\frac{d\zeta}{\zeta}$$

$$=\sum_{k=1}^{n}\frac{1}{2\pi i}\int_{\partial D_k}\frac{1}{n}\Big[\zeta_k\overline{\eta}_k+s+s\frac{s\eta_k\overline{\zeta}_k}{1-s\eta_k\overline{\zeta}_k}\Big]\Big\{2\Big[1+\frac{z_k\overline{\zeta}_k}{1-z_k\overline{\zeta}_k}\Big]-1\Big\}\frac{d\zeta_k}{\zeta_k}$$

$$=\sum_{k=1}^{n}\frac{1}{2\pi i}\int_{\partial D_k}\frac{1}{n}\Big[\zeta_k\overline{\eta}_k+s+s\frac{s\eta_k\overline{\zeta}_k}{1-s\eta_k\overline{\zeta}_k}\Big]\Big[1+\frac{2z_k\overline{\zeta}_k}{1-z_k\overline{\zeta}_k}\Big]\frac{d\zeta_k}{\zeta_k}=\sum_{k=1}^{n}\frac{1}{n}\Big[s+2z_k\overline{\eta}_k\Big]$$

we have

$$2I_{1b}=\frac{1}{(2\pi i)^n}\int_{\partial_0 D^n}\int_0^1\overline{\gamma(\eta)}\Big[2\frac{<z,\eta>}{n}+s\Big]ds\frac{d\eta}{\eta}$$

$$=\frac{2}{(2\pi i)^n}\int_{\partial_0 D^n}\frac{<z,\eta>}{n}\overline{\gamma(\eta)}\frac{d\eta}{\eta}+\frac{1}{2(2\pi i)^n}\int_{\partial_0 D^n}\overline{\gamma(\eta)}\frac{d\eta}{\eta}\,.$$

Thus we have got I_1 calculated as

$$I_1=\frac{1}{(2\pi i)^n}\int_{\partial_0 D^n}\gamma^*(\zeta)\Big[2\frac{\zeta}{\zeta-z}-1\Big]\frac{d\zeta}{\zeta}-\frac{1}{2}\int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0 D^n}\gamma(\zeta)\Big[2\frac{\zeta}{\zeta-sz}-1\Big]\frac{d\zeta}{\zeta}sds$$
$$-\frac{1}{(2\pi i)^n}\int_{\partial_0 D^n}\frac{<z,\zeta>}{n}\overline{\gamma(\zeta)}\frac{d\zeta}{\zeta}-\frac{1}{4(2\pi i)^n}\int_{\partial_0 D^n}\overline{\gamma(\zeta)}\frac{d\zeta}{\zeta}\,.$$

Let

$$I_2:=\frac{1}{(2\pi i)^n}\int_{\partial_0 D^n}2\Gamma(\zeta)\Big[\frac{1}{1-t\overline{z\zeta}}-1\Big]\frac{d\zeta}{\zeta}\,.$$

Similar to I_1 it is easy to get

$$I_2=\frac{1}{(2\pi i)^n}\int_{\partial_0 D^n}\gamma^*(\zeta)\Big[\frac{\zeta}{\zeta-tz}-1\Big]\frac{d\zeta}{\zeta}-\frac{1}{2}\int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0 D^n}\gamma(\zeta)\Big[\frac{\zeta}{\zeta-stz}-1\Big]\frac{d\zeta}{\zeta}sds$$
$$-\frac{1}{2(2\pi i)^n}\int_{\partial_0 D^n}\frac{<tz,\zeta>}{n}\overline{\gamma(\zeta)}\frac{d\zeta}{\zeta}$$

So we have

$$u(z)=iC_0+\frac{n-|z|^2}{n}\frac{\partial}{\partial t}\Big\{\frac{1}{(2\pi i)^n}\int_{\partial_0 D^n}\gamma^*(\zeta)\Big[\frac{\zeta}{\zeta-tz}-1\Big]\frac{d\zeta}{\zeta}$$
$$-\frac{1}{2}\int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0 D^n}\gamma(\zeta)\Big[\frac{\zeta}{\zeta-stz}-1\Big]\frac{d\zeta}{\zeta}sds-\frac{1}{2(2\pi i)^n}\int_{\partial_0 D^n}\frac{<tz,\zeta>}{n}\overline{\gamma(\zeta)}\frac{d\zeta}{\zeta}\Big\}\Big|_{t=1}$$
$$+\frac{1}{(2\pi i)^n}\int_{\partial_0 D^n}\gamma^*(\zeta)\Big[2\frac{\zeta}{\zeta-z}-1\Big]\frac{d\zeta}{\zeta}-\frac{1}{2}\int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0 D^n}\gamma(\zeta)\Big[2\frac{\zeta}{\zeta-sz}-1\Big]\frac{d\zeta}{\zeta}sds$$
$$-\frac{1}{(2\pi i)^n}\int_{\partial_0 D^n}\frac{<z,\zeta>}{n}\overline{\gamma(\zeta)}\frac{d\zeta}{\zeta}-\frac{1}{4(2\pi i)^n}\int_{\partial_0 D^n}\overline{\gamma(\zeta)}\frac{d\zeta}{\zeta}\,, \tag{2.62}$$

where C_0 is an arbitrary real constant.

Lemma 3 *The modified Neumann problem N_2^* for pluriholomorphic functions in D^n is uniquely solvable if and only if the condition (2.55) and (2.58) are satisfied,i.e., $\gamma\in\ \partial H_{D^n}$ and $\gamma^*\in\partial Ph_{D^n}$. The solution which is unique up to an arbitrary real constant, is given by (2.62). The problem is well – posed.*

Next we clarify the solution and the solvability conditions of the modified problem N_2^* for the inhomogeneous system (2.26). By substituting the condition (2.53) into the representation (2.52) we have ,

$$\sum_{|\kappa|\ \geq 0}\left(\overline{a}_\kappa\overline{\zeta}^\kappa + a_\kappa\zeta^\kappa\right)\frac{1}{|\kappa|+2} = \gamma^*(\zeta) - Re\ u_0(\zeta)$$

$$-Re\int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\frac{<\eta,\zeta>}{n}\Big[\gamma(\eta) - \sum_{j=1}^n \eta_j\frac{\partial u_0}{\partial\eta_j}\Big]\frac{d\eta}{\eta - s\zeta}ds =: 2F(\zeta)\ ,\ \zeta\in\partial_0 I\!D^n\ ,$$

i.e.,

$$Re\sum_{|\kappa|\ \geq 0}\frac{a_\kappa\zeta^\kappa}{|\kappa|+2}\ =\ F(\zeta)\ ,\quad \zeta\in\partial_0 I\!D^n\ . \tag{2.63}$$

This means again $F\in\partial Ph_{I\!D^n}$, because the left-hand side belongs to $\partial Ph_{I\!D^n}$, i.e.,

$$\sum_{\nu=2}^n\sum_{\lambda=1}^{\nu-1}\sum_{\substack{1\leq k_1<\cdots<k_\lambda\leq n\\ 1\leq k_{\lambda+1}<\cdots<k_\nu\leq n}}\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}F(\zeta)\prod_{\tau=1}^{\lambda}\frac{z_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{\overline{z}_{k_\tau}}{\overline{\zeta}_{k_\tau}-\overline{z}_{k_\tau}}\frac{d\zeta}{\zeta}\ =\ 0\quad ,$$

$$z\in I\!D^n\ . \tag{2.64}$$

Then the Schwarz problem (2.63) is solvable and the solution can be given by

$$\sum_{|\kappa|\ \geq\ 0}\frac{a_\kappa z^\kappa}{|\kappa|+2} = \frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}F(\zeta)\Big[2\frac{\zeta}{\zeta - z} - 1\Big]\frac{d\zeta}{\zeta} + iC^1$$

and from it one can derive that

$$a_0 = \frac{2}{(2\pi i)^n}\int_{\partial_0 I\!D^n}F(\zeta)\frac{d\zeta}{\zeta} + i2C^1\ ,\quad a_\kappa = \frac{2(2+|\kappa|)}{(2\pi i)^n}\int_{\partial_0 I\!D^n}F(\zeta)\overline{\zeta}^\kappa\frac{d\zeta}{\zeta}\ ,\quad |\kappa|\ >0\ ,$$

where C^1 is an arbitrary real constant. Substituting them into (2.52) we get

$$u(z) = \int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\frac{<\zeta,z>}{n}\Big[\gamma(\zeta) - \sum_{j=1}^n\zeta_j\frac{\partial u_0}{\partial\zeta_j}\Big]\frac{d\zeta}{\zeta - sz}ds + u_0(z)$$

$$+\sum_{|\kappa|\ \geq 0}\frac{2+|\kappa|}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\Big[1 - \frac{|\kappa||z|^2}{n(|\kappa|+2)}\Big]F(\zeta)(z\overline{\zeta})^\kappa\frac{d\zeta}{\zeta} + i2C^1\ ,\quad z\in I\!D^n\ .$$

Similarly to the case of the pluriholomorphic system we obtain

$$u(z) = \int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\frac{<\zeta,z>}{n}\Big[\gamma(\zeta) - \sum_{j=1}^n\zeta_j\frac{\partial u_0}{\partial\zeta_j}\Big]\frac{d\zeta}{\zeta - sz}ds + u_0(z) + iC^*$$

$$+\frac{\partial}{\partial t}\frac{n-|z|^2}{n(2\pi i)^n}\int_{\partial_0 I\!D^n}F(\zeta)\Big[\frac{1}{1-tz\overline{\zeta}} - 1\Big]\frac{d\zeta}{\zeta}\Big|_{t=1} + \frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}F(\zeta)\Big[\frac{2}{1-z\overline{\zeta}} - 1\Big]\frac{d\zeta}{\zeta} \tag{2.65}$$

where C^* is an arbitrary real constant.

Theorem 9 *The modified Neumann problem N_2^* for the inhomogeneous pluriholomorphic system (2.26) in $I\!D^n$ is solvable if and only if the conditions (2.47) and (2.64) are satisfied. The solution which is unique up to an arbitrary real constant, is given by (2.65). The problem is well – posed.*

2.2.4 A simple application

Find the sums

$$\sum_{|k|>0} \frac{x^k}{|k|} , \qquad \sum_{|k|>0} |k| x^k , \quad |x_1| < 1, \cdots , |x_n| < 1 .$$

By the above method we get

$$\sum_{|k|>0} \frac{x_1^{k_1} \cdots x_n^{k_n}}{k_1 + \cdots + k_n} = \int_0^1 \Big(\frac{1}{1 - sx_1} \cdots \frac{1}{1 - sx_n} - 1 \Big) \frac{ds}{s} ,$$

$$\sum_{|k|>0} (k_1 + \cdots + k_n)(x_1^{k_1} \cdots x_n^{k_n}) = \frac{\partial}{\partial s} \Big(\frac{1}{1 - sx_1} \cdots \frac{1}{1 - sx_n} - 1 \Big) \Big|_{s=1} .$$

2.3 The inhomogeneous Cauchy-Riemann system

2.3.1 Preliminaries and Definition

From the literature it is known that about the $\overline{\partial}$ – Neumann problem a great deal of research has been done, not only in polydiscs and in the unit ball, see [6], [9] and [25] etc., but also in general domains, see [15] and [12]. However, about the Neumann problem even for the homogeneous holomorphic system in the unit polydisc nothing can be found in the literature, while similar problem is solved in the case of the unit ball, see [3].

Let f_k , γ be given functions with $f_{k\overline{z}_j} \in C(\overline{\mathbb{D}^n}), \gamma \in C(\partial_0 \mathbb{D}^n)$. Consider the following Cauchy-Riemann system of n independent equations

$$\frac{\partial u}{\partial \overline{z}_k} = f_k(z) , \quad 1 \le k \le n , \tag{2.66}$$

with given right – hand sides, satisfying the condition

$$\frac{\partial f_k}{\partial \overline{z}_\ell} - \frac{\partial f_\ell}{\partial \overline{z}_k} = 0 , \qquad 1 \le \ell, k \le n . \tag{2.67}$$

Problem N_3 . Find a $C^1(\overline{\mathbb{D}^n})$ – solution of system (2.66), satisfying the Neumann condition

$$\frac{\partial u}{\partial \nu_\zeta} = \gamma_0(\zeta) , \qquad \zeta \in \partial_0 \mathbb{D}^n , \tag{2.68}$$

where $\partial u / \partial \nu_\zeta$ denotes the outward normal derivative of $u(z)$ at the point $\zeta \in \partial_0 \mathbb{D}^n$.

By definition it is known that, see the previous sections or [22], the Neumann condition (2.68) for the unit polydisc turns out to be

$$\sum_{j=1}^n \Big(z_j \frac{\partial u}{\partial z_j} + \overline{z}_j \frac{\partial u}{\partial \overline{z}_j} \Big) \Big|_\zeta = \gamma(\zeta) , \qquad \zeta \in \partial_0 \mathbb{D}^n , \tag{2.69}$$

with $\gamma(\zeta) = \gamma_0(\zeta)\sqrt{n}$.

2.3.2 The problem

It is known that the general solution to (2.66) is representable as

$$u(z) = \phi(z) + u_0(z) \tag{2.70}$$

where $\phi(z)$ is an arbitrary function analytic in $\mathbb{D}^n$ and

$$u_0(z) = \sum_{\mu=1}^{n}(-1)^{\mu+1} \sum_{1\le k_1<\cdots<k_\mu\le n} T_{k_\mu}\cdots T_{k_1} f_{k_1\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\mu}}\,, \quad z\in\mathbb{D}^n\,, \tag{2.71}$$

see Chapter 1 Theorem 1 or [2] Theorem 5.2 . Substituting (2.70) into (2.69) we have

$$\sum_{\alpha=1}^{n}\zeta_\alpha\phi_{\zeta_\alpha} = \gamma(\zeta) - \sum_{\alpha=1}^{n}(\overline{\zeta}_\alpha f_\alpha + \zeta_\alpha u_{0\zeta_\alpha})\,, \quad \zeta\in\partial_0\mathbb{D}^n\,. \tag{2.72}$$

Since the left–hand side represents the boundary values of an analytic function in $\mathbb{D}^n$, the right hand–side has to satisfy

$$\sum_{\nu=1}^{n}\sum_{\lambda=0}^{\nu-1}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}} \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\left[\gamma(\zeta)-\sum_{\alpha=1}^{n}\Big(\overline{\zeta}_\alpha f_\alpha+\zeta_\alpha u_{0\zeta_\alpha}\Big)\right]$$

$$\times\prod_{\tau=1}^{\lambda}\frac{z_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau}-z_{k_\tau}}}\frac{d\zeta}{\zeta} = 0\,, \quad z\in\mathbb{D}^n, \tag{2.73}$$

see Lemma 1 in the previous section or [23]. Then by the Cauchy integral for equation (2.72) we get

$$\sum_{\alpha=1}^{n} z_\alpha\frac{\partial\phi}{\partial z_\alpha} = \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\left[\gamma(\zeta)-\sum_{\alpha=1}^{n}\Big(\overline{\zeta}_\alpha f_\alpha+\zeta_\alpha u_{0\zeta_\alpha}\Big)\right]\frac{d\zeta}{\zeta-z}\,. \tag{2.74}$$

Applying

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\overline{\zeta}^\kappa Tf(\zeta)\frac{d\zeta}{\zeta-z} = 0, \quad f\in L_1(\overline{\mathbb{D}}), \quad 0\le\kappa, \quad z\in\mathbb{D}\,, \tag{2.75}$$

see [2] page 305, from (2.71) and (2.74) it is obvious that

$$\sum_{\alpha=1}^{n} z_\alpha\frac{\partial\phi}{\partial z_\alpha} = \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\left[\gamma(\zeta)-\sum_{\alpha=1}^{n}\Big(\overline{\zeta}_\alpha f_\alpha+\zeta_\alpha \Pi_\alpha f_\alpha\Big)\right]\frac{d\zeta}{\zeta-z} \tag{2.76}$$

where, $\Pi_j : C^\alpha(\overline{\mathbb{D}}) \to C^\alpha(\overline{\mathbb{D}})$, see [29],

$$\Pi_j f_j = -\frac{1}{\pi}\int_{\mathbb{D}_j} f_j(\zeta)\frac{d\zeta_j d\eta_j}{(\zeta_j - z_j)^2}\,, \quad z_j\in\overline{\mathbb{D}}_j$$

and

$$\Pi f = Tf_\zeta - \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\zeta)\frac{d\overline{\zeta}}{\zeta-z}\,, \quad f\in C^1(\mathbb{D})\cap C(\overline{\mathbb{D}}),$$

see again [2] page 282.

Since the left–hand side of (2.74) vanishes for $z = 0 \in \mathbb{D}^n$ so the right–hand side has to vanish, i.e.

$$\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n} \gamma(\zeta)\frac{d\zeta}{\zeta} = \sum_{\alpha=1}^{n}\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\Big(\overline{\zeta}_\alpha f_\alpha + \zeta_\alpha \Pi_\alpha f_\alpha\Big)\frac{d\zeta}{\zeta}\,. \tag{2.77}$$

Transforming the variables via

$$u_1 = z_1, \quad u_2 = z_1/z_2, \cdots, \quad u_n = z_1/z_n\,,$$

integrating the transformed equation and returning to the original variables one obtains

$$\phi(z) = \int_0^1 \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\left[\gamma(\zeta) - \sum_{\alpha=1}^{n}\Big(\overline{\zeta}_\alpha f_\alpha + \zeta_\alpha \Pi_\alpha f_\alpha\Big)\right]\frac{d\zeta}{\zeta - sz}\frac{ds}{s} + C \tag{2.78}$$

where C is an arbitrary complex number.

Taking into account that

$$\frac{1}{2\pi i}\int_{\partial \mathbb{D}} \zeta^\kappa Tf(\zeta)\frac{d\zeta}{\zeta - z} = \frac{1}{\pi}\int_{\mathbb{D}} f(\zeta)\frac{\zeta^\kappa - z^\kappa}{\zeta - z}d\xi d\eta,$$

$$f \in L_1(\overline{\mathbb{D}}), \quad 0 < \kappa\,, \quad z \in \mathbb{D}\,, \tag{2.79}$$

see [2] page 305, and

$$\frac{1}{2\pi i}\int_{\partial \mathbb{D}} f(\zeta)\frac{d\overline{\zeta}}{\zeta - z} = \frac{1}{\pi}\int_{\mathbb{D}} \overline{\zeta} f(\zeta)\frac{d\overline{\zeta}}{1 - z\overline{\zeta}} = -\frac{1}{\pi}\int_{\mathbb{D}} \overline{\zeta} f_\zeta(\zeta)\frac{d\xi d\eta}{1 - z\overline{\zeta}}\,,$$

$$f \in C^1(\mathbb{D}) \cap C(\overline{\mathbb{D}})$$

it is easy to derive

$$\Pi f = Tf_\zeta - \frac{1}{2\pi i}\int_{\partial \mathbb{D}} f(\zeta)\frac{d\overline{\zeta}}{\zeta - z} = Tf_\zeta + \frac{1}{\pi}\int_{\mathbb{D}} \overline{\zeta} f_\zeta(\zeta)\frac{d\xi d\eta}{1 - z\overline{\zeta}}, \quad z \in \mathbb{D}\,,$$

and further with $\eta = \xi + i\tau$

$$\frac{1}{2\pi i}\int_{\partial \mathbb{D}} \zeta \Pi f\frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{\partial \mathbb{D}} \zeta\left[Tf_\zeta + \frac{1}{\pi}\int_{\mathbb{D}} \overline{\eta} f_\eta(\eta)\frac{d\xi d\tau}{1 - \zeta\overline{\eta}}\right]\frac{d\zeta}{\zeta - z}$$

$$= \frac{1}{\pi}\int_{\mathbb{D}} f_\eta(\eta)d\xi d\tau + \frac{1}{\pi}\int_{\mathbb{D}} f_\eta(\eta)\left[\frac{1}{2\pi i}\int_{\partial \mathbb{D}} \frac{\zeta\overline{\eta}}{1 - \zeta\overline{\eta}}\frac{d\zeta}{\zeta - z}d\xi d\tau\right]$$

$$= \frac{1}{\pi}\int_{\mathbb{D}}\left[f_\eta(\eta) + f_\eta(\eta)\frac{z\overline{\eta}}{1 - z\overline{\eta}}\right]d\xi d\tau = \frac{1}{\pi}\int_{\mathbb{D}} f_\eta(\eta)\frac{d\xi d\tau}{1 - z\overline{\eta}}\,.$$

By the Green's theorem it follows that

$$\frac{1}{2\pi i}\int_{\partial \mathbb{D}} \overline{\zeta} f(\zeta)\frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{\partial \mathbb{D}} f(\zeta)\frac{1}{\zeta - z}\frac{d\zeta}{\zeta}$$

$$= -\frac{1}{2\pi i}\int_{\partial \mathbb{D}} f(\zeta)\frac{d\overline{\zeta}}{1-z\overline{\zeta}} = \frac{1}{\pi}\int_{\mathbb{D}} f_\zeta(\zeta)\frac{d\xi d\tau}{1-z\overline{\zeta}} .$$

Therefore

$$\frac{1}{2\pi i}\int_{\partial \mathbb{D}} \left(\overline{\zeta} f + \zeta \Pi f\right)\frac{d\zeta}{\zeta - z} = \frac{2}{\pi}\int_{\mathbb{D}} f_\zeta(\zeta)\frac{d\xi d\eta}{1-z\overline{\zeta}} . \tag{2.80}$$

Thus (2.78) takes the form

$$\phi(z) = \int_0^1 \left[\frac{1}{(2\pi i)^n}\int_{\partial \mathbb{D}^n} \gamma(\zeta)\frac{d\zeta}{\zeta - sz} - \sum_{j=1}^n \frac{1}{(2\pi i)^{n-1}}\int_{\partial \mathbb{D}^{n-1}} \frac{2}{\pi}\int_{\mathbb{D}_j} f_{j\zeta_j}\frac{d\xi_j d\eta_j}{1-sz_j\overline{\zeta}_j}\right.$$

$$\left.\times \prod_{\substack{k=1\\k\neq j}} \frac{d\zeta_k}{\zeta_k - sz_k}\right]\frac{ds}{s} + C .$$

Further, by the Pompeiu formula for $z_k \in \mathbb{D}_k$, $1 \le k \le n$, $k \neq j$ one derives

$$\Phi(z) = \int_0^1 \left[\frac{1}{(2\pi i)^n}\int_{\partial \mathbb{D}^n} \gamma(\zeta)\frac{d\zeta}{\zeta - sz}\right.$$

$$-\sum_{\mu=1}^n \sum_{\substack{1\le\lambda_1\le n\\1\le\lambda_2<\cdots<\lambda_\mu\le n}} \frac{2}{\pi^\mu}\int_{\mathbb{D}_{\lambda_1}}\int_{\mathbb{D}_{\lambda_2}}\cdots\int_{\mathbb{D}_{\lambda_\mu}} f_{\lambda_1\zeta_{\lambda_1}\overline{\zeta}_{\lambda_2}\cdots\overline{\zeta}_{\lambda_\mu}}$$

$$\left.\times \frac{1}{1-sz_{\mu_1}\overline{\zeta}_{\lambda_1}}\prod_{\tau=2}^{\mu}\frac{1}{\zeta_{\lambda_\tau} - sz_{\lambda_\tau}}\prod_{\tau=1}^{\mu} d\xi_{\lambda_\tau} d\eta_{\lambda_\tau}\right]\frac{ds}{s} + C . \tag{2.81}$$

So the solution to problem N_3 can be given by

$$u(z) = -\sum_{\nu=1}^n \sum_{1\le k_1<\cdots<k_\nu} \frac{1}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}} f_{k_1\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\nu}}\prod_{\tau=1}^{\nu}\frac{d\xi_{k_\tau} d\eta_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}}$$

$$+\int_0^1 \left[\frac{1}{(2\pi i)^n}\int_{\partial \mathbb{D}^n} \gamma(\zeta)\frac{d\zeta}{\zeta - sz} - \sum_{\mu=1}^n \sum_{\substack{1\le k_1\le n\\1\le k_2<\cdots<k_\mu\le n}} \frac{2}{\pi^\mu}\int_{\mathbb{D}_{k_1}}\int_{\mathbb{D}_{k_2}}\cdots\int_{\mathbb{D}_{k_\mu}} f_{k_1\zeta_{k_1}\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\mu}}\right.$$

$$\left.\times \frac{1}{1-sz_k\overline{\zeta}_{k_1}}\prod_{\tau=2}^{\mu}\frac{1}{\zeta_{k_\tau} - sz_{k_\tau}}\prod_{\tau=1}^{\mu} d\xi_{k_\tau} d\eta_{k_\tau}\right]\frac{ds}{s} + C . \tag{2.82}$$

Also making some simplifications one can get a more explicit form of (2.73). We rewrite condition (2.73) as

$$\sum_{\nu=1}^n \sum_{\lambda=0}^{\nu-1} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\1\le k_{\lambda+1}<\cdots<k_\nu\le n}} \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n} \gamma(\zeta)\prod_{\tau=1}^{\lambda}\frac{z_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{\overline{\zeta}_{k_\tau}}{\overline{\zeta_{k_\tau} - z_{k_\tau}}}\frac{d\zeta}{\zeta}$$

$$= \sum_{\alpha,\,\nu=1}^n \sum_{\lambda=0}^{\nu-1} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\1\le k_{\lambda+1}<\cdots<k_\nu\le n}} \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n} \left(\overline{\zeta}_\alpha f_\alpha + \zeta_\alpha u_{0\zeta_\alpha}\right)$$

$$\times \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau} - z_{k_\tau}}} \frac{d\zeta}{\zeta} =: I \ . \tag{2.83}$$

Applying

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}} f \frac{z}{\zeta - z} \frac{d\zeta}{\zeta} = \frac{1}{\pi} \int_{\mathbb{D}} f_\zeta \frac{z}{1 - z\overline{\zeta}} d\xi d\eta \ , \quad z \in \mathbb{D} \, ,$$

and

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}} f \frac{\overline{z}}{\overline{\zeta - z}} \frac{d\zeta}{\zeta} = \frac{1}{\pi} \int_{\mathbb{D}} f_{\overline{\zeta}} \frac{\overline{z}}{1 - \overline{z}\zeta} d\xi d\eta \ , \quad z \in \mathbb{D} \, ,$$

we have

$$I = \sum_{\alpha,\, \nu=1}^{n} \sum_{\lambda=0}^{\nu-1} \sum_{\substack{1 \le k_1 < \cdots < k_\lambda \le n \\ 1 \le k_{\lambda+1} < \cdots < k_\nu \le n \\ 1 \le k_{\nu+1} < \cdots < k_n \le n}} \frac{1}{(2\pi i)^{n-\nu}} \int_{\partial_0 \mathbb{D}^{\, n-\nu}} \frac{1}{\pi^\nu} \int_{\mathbb{D}_{\, k_1}} \cdots \int_{\mathbb{D}_{\, k_\nu}}$$

$$\times \Big(\overline{\zeta}_\alpha f_\alpha + \zeta_\alpha u_{0\zeta_\alpha} \Big)_{\zeta_{k_1} \cdots \zeta_{k_\lambda} \overline{\zeta}_{k_{\lambda+1}} \cdots \overline{\zeta}_{k_\nu}} \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau} d\xi_{k_\tau} d\eta_{k_\tau}}{1 - z_{k_\tau} \overline{\zeta}_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{\overline{z}_{k_\tau} d\xi_{k_\tau} d\eta_{k_\tau}}{1 - \overline{z}_{k_\tau} \zeta_{k_\tau}} \prod_{\tau=\nu+1}^{n} \frac{d\zeta_{k_\tau}}{\zeta_{k_\tau}}$$

$$= \sum_{\alpha,\, \nu=1}^{n} \sum_{\lambda=0}^{\nu-1} \sum_{\substack{1 \le k_1 < \cdots < k_\lambda \le n \\ 1 \le k_{\lambda+1} < \cdots < k_\nu \le n \\ 1 \le k_{\nu+1} < \cdots < k_n \le n}} \frac{1}{\pi^\nu (2\pi i)^{n-\nu}} \int_{\mathbb{D}_{\, k_1}} \cdots \int_{\mathbb{D}_{\, k_\nu}} \int_{\partial_0 \mathbb{D}^{\, n-\nu}}$$

$$\times \left[\Big(\overline{\zeta}_\alpha f_\alpha \Big)_{\zeta_{k_1} \cdots \zeta_{k_\lambda} \overline{\zeta}_{k_{\lambda+1}} \cdots \overline{\zeta}_{k_\nu}} + \Big(\zeta_\alpha f_{k_\nu \, \zeta_\alpha} \Big)_{\zeta_{k_1} \cdots \zeta_{k_\lambda} \overline{\zeta}_{k_{\lambda+1}} \cdots \overline{\zeta}_{k_{\nu-1}}} \right]$$

$$\times \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{1 - z_{k_\tau} \overline{\zeta}_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{\overline{z}_{k_\tau}}{1 - \overline{z}_{k_\tau} \zeta_{k_\tau}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau} \prod_{\tau=\nu+1}^{n} \frac{d\zeta_{k_\tau}}{\zeta_{k_\tau}}$$

Let

$$\sum_{\nu=1}^{n} \sum_{\lambda=0}^{\nu-1} \sum_{\substack{1 \le k_1 < \cdots < k_\lambda \le n \\ 1 \le k_{\lambda+1} < \cdots < k_\nu \le n}} F_{\zeta_{k_1} \cdots \zeta_{k_\lambda} \overline{\zeta}_{k_{\lambda+1}} \cdots \overline{\zeta}_{k_\nu}} \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{1 - z_{k_\tau} \overline{\zeta}_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{\overline{z}_{k_\tau}}{1 - \overline{z}_{k_\tau} \zeta_{k_\tau}} = J.$$

Replacing the summation index ν by $\mu = \nu - \lambda$ we have

$$J = \sum_{\lambda=0}^{n-1} \sum_{\mu=1}^{n-\lambda} \sum_{\substack{1 \le k_1 < \cdots < k_\lambda \le n \\ 1 \le k_{\lambda+1} < \cdots < k_{\lambda+\mu} \le n}} F_{\zeta_{k_1} \cdots \zeta_{k_\lambda} \overline{\zeta}_{k_{\lambda+1}} \cdots \overline{\zeta}_{k_{\lambda+\mu}}} \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{1 - z_{k_\tau} \overline{\zeta}_{k_\tau}} \prod_{\tau=\lambda+1}^{\lambda+\mu} \frac{\overline{z}_{k_\tau}}{1 - \overline{z}_{k_\tau} \zeta_{k_\tau}} \ .$$

Denote $\{k_{\lambda+1}, \cdots, k_{\lambda+\mu}\} = \{h_1, \cdots, h_\mu\}$, $\{k_1, \cdots, k_\lambda\} = \{h_{\mu+1}, \cdots, h_{\mu+\lambda}\}$, where

$$1 \le h_1 < \cdots < h_\mu \le n \ , 1 \le h_{\mu+1} < \cdots < h_{\mu+\lambda} \le n \ .$$

Thus J can be written as

$$J = \sum_{\lambda=0}^{n-1} \sum_{\mu=1}^{n-\lambda} \sum_{\substack{1 \le h_1 < \cdots < h_\mu \le n \\ 1 \le h_{\mu+1} < \cdots < h_{\mu+\lambda} \le n}} F_{\overline{\zeta}_{h_1} \cdots \overline{\zeta}_{h_\mu} \zeta_{h_{\mu+1}} \cdots \zeta_{h_{\mu+\lambda}}} \prod_{\tau=1}^{\mu} \frac{\overline{z}_{h_\tau}}{1 - \overline{z}_{h_\tau} \zeta_{h_\tau}} \prod_{\tau=\mu+1}^{\mu+\lambda} \frac{z_{h_\tau}}{1 - z_{h_\tau} \overline{\zeta}_{h_\tau}} \ .$$

Again replacing the summation index λ by $\nu = \mu + \lambda$ we get

$$J = \sum_{\nu=1}^{n} \sum_{\mu=1}^{\nu} \sum_{\substack{1 \le h_1 < \cdots < h_\mu \le n \\ 1 \le h_{\mu+1} < \cdots < h_\nu \le n}} F_{\overline{\zeta}_{h_1} \cdots \overline{\zeta}_{h_\mu} \zeta_{h_{\mu+1}} \cdots \zeta_{h_\nu}} \prod_{\tau=1}^{\mu} \frac{\overline{z}_{h_\tau}}{1 - \overline{z}_{h_\tau} \zeta_{h_\tau}} \prod_{\tau=\mu+1}^{\nu} \frac{z_{h_\tau}}{1 - z_{h_\tau} \overline{\zeta}_{h_\tau}}.$$

By virtue of

$$\frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} f(\zeta) \frac{d\zeta}{\zeta} = f(z) + \sum_{k=1}^{n} \sum_{\mu=0}^{k} \sum_{\substack{1 \le \nu_1 < \cdots < \nu_\mu \le n \\ 1 \le \nu_{\mu+1} < \cdots < \nu_k \le n}} \frac{(-1)^{k-\mu}}{\pi^k} \int_{\mathbb{D}_{\nu_1}} \cdots \int_{\mathbb{D}_{\nu_k}}$$

$$\times f_{\overline{\zeta}_{\nu_1} \cdots \overline{\zeta}_{\nu_\mu} \zeta_{\nu_{\mu+1}} \cdots \zeta_{\nu_k}} \prod_{\tau=1}^{\mu} \frac{1}{\zeta_{\nu_\tau} - z_{\nu_\tau}} \prod_{\tau=\mu+1}^{k} \frac{z_{\nu_\tau}}{1 - z_{\nu_\tau} \overline{\zeta}_{\nu_\tau}} \prod_{\tau=1}^{k} d\xi_{k_\tau} d\eta_{k_\tau}, \ f \in C^1(\overline{\mathbb{D}^n}), \tag{2.84}$$

see [2] page 262, we obtain

$$I = \sum_{\alpha,\, \nu=1}^{n} \sum_{\lambda=1}^{\nu} \sum_{\substack{1 \le k_1 < \cdots < k_\lambda \le n \\ 1 \le k_{\lambda+1} < \cdots < k_\nu \le n}} \frac{1}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \Bigg\{$$

$$\times \left[\left(\overline{\zeta}_\alpha f_\alpha \right)_{\overline{\zeta}_{k_1} \cdots \overline{\zeta}_{k_\lambda} \zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu}} + \left(\zeta_\alpha f_{k_\lambda\, \zeta_\alpha} \right)_{\overline{\zeta}_{k_1} \cdots \overline{\zeta}_{k_{\lambda-1}} \zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu}} \right]$$

$$+ \sum_{\ell=1}^{n-\nu} \sum_{\mu=0}^{\ell} \sum_{\substack{1 \le \sigma_1 < \cdots < \sigma_\mu \le n-\nu \\ 1 \le \sigma_{\mu+1} < \cdots < \sigma_\ell \le n-\nu \\ \{\sigma_1, \cdots, \sigma_\ell\} = \{1, \cdots, \ell\}}} \frac{(-1)^{\ell-\mu}}{\pi^\ell} \int_{\mathbb{D}_{k_{\nu+\sigma_1}}} \cdots \int_{\mathbb{D}_{k_{\nu+\sigma_\ell}}}$$

$$\times \left[\left(\overline{\zeta}_\alpha f_\alpha \right)_{\overline{\zeta}_{k_1} \cdots \overline{\zeta}_{k_\lambda}} + \left(\zeta_\alpha f_{k_\lambda\, \zeta_\alpha} \right)_{\overline{\zeta}_{k_1} \cdots \overline{\zeta}_{k_{\lambda-1}}} \right]_{\zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu} \overline{\zeta}_{k_{\nu+\sigma_1}} \cdots \overline{\zeta}_{k_{\nu+\sigma_\mu}} \zeta_{k_{\nu+\sigma_{\mu+1}}} \cdots \zeta_{k_{\nu+\sigma_\ell}}}$$

$$\times \prod_{\tau=1}^{\mu} \frac{1}{\zeta_{k_{\nu+\sigma_\tau}} - z_{k_{\nu+\sigma_\tau}}} \prod_{\tau=\mu+1}^{\ell} \frac{z_{k_{\nu+\sigma_\tau}}}{1 - z_{k_{\nu+\sigma_\tau}} \overline{\zeta}_{k_{\nu+\sigma_\tau}}} \prod_{\tau=1}^{\ell} d\xi_{k_{\nu+\sigma_\tau}} d\eta_{k_{\nu+\sigma_\tau}} \Bigg\}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{1 - \overline{z}_{k_\tau} \zeta_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{1 - z_{k_\tau} \overline{\zeta}_{k_\tau}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau}$$

$$= \sum_{\alpha,\, \nu=1}^{n} \sum_{\lambda=1}^{\nu} \sum_{\substack{1 \le k_1 < \cdots < k_\lambda \le n \\ 1 \le k_{\lambda+1} < \cdots < k_\nu \le n}} \frac{1}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \Bigg\{$$

$$\times \left[\left(\overline{\zeta}_\alpha f_\alpha \right)_{\overline{\zeta}_{k_1} \cdots \overline{\zeta}_{k_\lambda} \zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu}} + \left(\zeta_\alpha f_{k_\lambda\, \zeta_\alpha} \right)_{\overline{\zeta}_{k_1} \cdots \overline{\zeta}_{k_{\lambda-1}} \zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu}} \right]$$

$$+ \sum_{\ell=1}^{n-\nu} \sum_{\substack{1 \le \sigma_1 < \cdots < \sigma_\ell \le n-\nu \\ \{\sigma_1, \cdots, \sigma_\ell\} = \{1, \cdots, \ell\}}} \frac{(-1)^\ell}{\pi^\ell} \int_{\mathbb{D}_{k_{\nu+\sigma_1}}} \cdots \int_{\mathbb{D}_{k_{\nu+\sigma_\ell}}}$$

$$\times\left[\left(\overline{\zeta}_\alpha f_\alpha\right)_{\overline{\zeta}_{k_1}\cdots\overline{\zeta}_{k_\lambda}}+\left(\zeta_\alpha f_{k_\lambda\,\zeta_\alpha}\right)_{\overline{\zeta}_{k_1}\cdots\overline{\zeta}_{k_{\lambda-1}}}\right]_{\zeta_{k_{\lambda+1}}\cdots\zeta_{k_\nu}\zeta_{k_{\nu+\sigma_1}}\cdots\zeta_{k_{\nu+\sigma_\ell}}}$$

$$\times\prod_{\tau=1}^{\ell}\frac{z_{k_{\nu+\sigma_\tau}}}{1-z_{k_{\nu+\sigma_\tau}}\overline{\zeta}_{k_{\nu+\sigma_\tau}}}\prod_{\tau=1}^{\ell}d\xi_{k_{\nu+\sigma\tau}}d\eta_{k_{\nu+\sigma\tau}}$$

$$+\sum_{\ell=1}^{n-\nu}\sum_{\mu=1}^{\ell}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\mu\le n-\nu\\ 1\le\sigma_{\mu+1}<\cdots<\sigma_\ell\le n-\nu\\ \{\sigma_1,\cdots,\sigma_\ell\}=\{1,\cdots,\ell\}}}\frac{(-1)^{\ell-\mu}}{\pi^\ell}\int_{\mathbb{D}_{k_{\nu+\sigma_1}}}\cdots\int_{\mathbb{D}_{k_{\nu+\sigma_\ell}}}$$

$$\times\left[\left(\overline{\zeta}_\alpha f_\alpha\right)_{\overline{\zeta}_{k_1}\cdots\overline{\zeta}_{k_\lambda}}+\left(\zeta_\alpha f_{k_\lambda\,\zeta_\alpha}\right)_{\overline{\zeta}_{k_1}\cdots\overline{\zeta}_{k_{\lambda-1}}}\right]_{\zeta_{k_{\lambda+1}}\cdots\zeta_{k_\nu}\overline{\zeta}_{k_{\nu+\sigma_1}}\cdots\overline{\zeta}_{k_{\nu+\sigma_\mu}}\zeta_{k_{\nu+\sigma_{\mu+1}}}\cdots\zeta_{k_{\nu+\sigma_\ell}}}$$

$$\times\prod_{\tau=1}^{\mu}\frac{1}{\zeta_{k_{\nu+\sigma_\tau}}-z_{k_{\nu+\sigma_\tau}}}\prod_{\tau=\mu+1}^{\ell}\frac{z_{k_{\nu+\sigma_\tau}}}{1-z_{k_{\nu+\sigma_\tau}}\overline{\zeta}_{k_{\nu+\sigma_\tau}}}\prod_{\tau=1}^{\ell}d\xi_{k_{\nu+\sigma\tau}}d\eta_{k_{\nu+\sigma\tau}}\Bigg\}$$

$$\times\prod_{\tau=1}^{\lambda}\frac{\overline{z}_{k_\tau}}{1-\overline{z}_{k_\tau}\zeta_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta}_{k_\tau}}\prod_{\tau=1}^{\nu}d\xi_{k_\tau}d\eta_{k_\tau}$$

Replacing the summation over ℓ by one over $\gamma := \nu+\ell$ and changing the order of summations give for the last two terms

$$\sum_{\alpha=1}^{n}\sum_{\gamma=2}^{n}\sum_{\nu=1}^{\gamma-1}\sum_{\lambda=1}^{\nu}\sum_{\substack{1\le h_1<\cdots<h_\lambda\le n,\ 1\le h_{\lambda+1}<\cdots<h_\nu\le n\ ,\{k_1,\cdots,k_\lambda\}=\{h_1,\cdots,h_\lambda\}\\ 1\le h_{\nu+1}<\cdots<h_\gamma\le n\ ,\{k_{\lambda+1},\cdots,k_\nu,k_{\nu+\sigma_1},\cdots,k_{\nu+\sigma_{\gamma-\nu}}\}=\{h_{\lambda+1},\cdots,h_\nu,h_{\nu+1},\cdots,h_\gamma\}}}\frac{(-1)^{\gamma-\nu}}{\pi^\gamma}$$

$$\times\int_{\mathbb{D}_{h_1}}\cdots\int_{\mathbb{D}_{h_\gamma}}\left[\left(\overline{\zeta}_\alpha f_\alpha\right)_{\overline{\zeta}_{h_1}\cdots\overline{\zeta}_{h_\lambda}}+\left(\zeta_\alpha f_{h_\lambda\,\zeta_\alpha}\right)_{\overline{\zeta}_{h_1}\cdots\overline{\zeta}_{h_{\lambda-1}}}\right]_{\zeta_{h_{\lambda+1}}\cdots\zeta_{h_\gamma}}$$

$$\times\prod_{\tau=1}^{\lambda}\frac{\overline{z}_{h_\tau}}{1-\overline{z}_{h_\tau}\zeta_{h_\tau}}\prod_{\tau=\lambda+1}^{\gamma}\frac{z_{h_\tau}}{1-z_{h_\tau}\overline{\zeta}_{h_\tau}}\prod_{\tau=1}^{\gamma}d\xi_{h_\tau}d\eta_{h_\tau}$$

$$+\sum_{\alpha=1}^{n}\sum_{\gamma=2}^{n}\sum_{\nu=1}^{\gamma-1}\sum_{\lambda=1}^{\nu}\sum_{\mu=1}^{\gamma-\nu}\sum_{\substack{1\le h_1<\cdots<h_\lambda\le n,\ 1\le h_{\lambda+1}<\cdots<h_\nu\le n\ ,\\ 1\le h_{\nu+1}<\cdots<h_{\nu+\mu}\le n\ ,1\le h_{\nu+\mu+1}<\cdots<h_\gamma\le n\ ,\\ \{k_{\lambda+1},\cdots,k_\nu,k_{\nu+\sigma_{\mu+1}},\cdots,k_{\nu+\sigma_{\gamma-\nu}}\}=\{h_{\lambda+\mu+1},\cdots,h_{\nu+\mu},h_{\nu+\mu+1},\cdots,h_\gamma\}\\ \{k_1,\cdots,k_\lambda,k_{\nu+\sigma_1},\cdots,k_{\nu+\sigma_\mu}\}=\{h_1,\cdots,h_\lambda,h_{\lambda+1},\cdots,h_{\lambda+\mu}\}}}\frac{(-1)^{\gamma-\nu-\mu}}{\pi^\gamma}$$

$$\times\int_{\mathbb{D}_{h_1}}\cdots\int_{\mathbb{D}_{h_\gamma}}\left[\left(\overline{\zeta}_\alpha f_\alpha\right)_{\overline{\zeta}_{h_1}}+\left(\zeta_\alpha f_{h_1\,\zeta_\alpha}\right)\right]_{\overline{\zeta}_{h_2}\cdots\overline{\zeta}_{h_\lambda}\overline{\zeta}_{h_{\lambda+1}}\cdots\overline{\zeta}_{h_{\lambda+\mu}}\zeta_{h_{\lambda+1+\mu}}\cdots\zeta_{h_{\nu+\mu}}\zeta_{h_{\nu+\mu+1}}\cdots\zeta_{h_\gamma}}$$

$$\times\prod_{\tau=1}^{\lambda}\frac{\overline{z}_{h_\tau}}{1-\overline{z}_{h_\tau}\zeta_{h_\tau}}\prod_{\tau=\lambda+1}^{\lambda+\mu}\frac{1}{\zeta_{h_\tau}-z_{h_\tau}}\prod_{\tau=\nu+\mu+1}^{\gamma}\frac{z_{h_\tau}}{1-z_{h_\tau}\overline{\zeta}_{h_\tau}}\prod_{\tau=1}^{\gamma}d\xi_{h_\tau}d\eta_{h_\tau}\ .$$

Since

$$\sum_{\{k_{\lambda+1},\cdots,k_\nu,k_{\nu+\sigma_1},\cdots,k_{\nu+\sigma_{\gamma-\nu}}\}=\{h_{\lambda+1},\cdots,h_\nu,h_{\nu+1},\cdots,h_\gamma\}}1\ =\ \binom{\boldsymbol{\gamma-\lambda}}{\boldsymbol{\nu-\lambda}},$$

for some a_λ with $1 \le \lambda \le \nu$ we have

$$\sum_{\nu=1}^{\gamma-1}\sum_{\lambda=1}^{\nu}(-1)^{\gamma-\nu}\binom{\gamma-\lambda}{\nu-\lambda}a_\lambda = \sum_{\lambda=1}^{\gamma-1}\sum_{\nu=\lambda}^{\gamma-1}(-1)^{\gamma-\nu}\binom{\gamma-\lambda}{\nu-\lambda}a_\lambda$$

$$= \sum_{\lambda=1}^{\gamma-1}\sum_{\nu=0}^{\gamma-\lambda-1}(-1)^{\gamma-\lambda-\nu}\binom{\gamma-\lambda}{\nu}a_\lambda = \sum_{\lambda=1}^{\gamma-1}\left[\Big(1-1\Big)^{\gamma-\lambda} - \binom{\gamma-\lambda}{\gamma-\lambda}\Big(-1\Big)^{0}\right]a_\lambda = -\sum_{\lambda=1}^{\gamma-1}a_\lambda \,.$$

Similarly from

$$\sum_{\{k_1,\cdots,k_\lambda,k_{\nu+\sigma_1},\cdots,k_{\nu+\sigma_\mu}\}=\{h_1,\cdots,h_{\mu+\lambda}\}} 1 = \binom{\mu+\lambda}{\lambda},$$

$$\sum_{\{k_{\lambda+1},\cdots,k_\nu,k_{\nu+\sigma_{\mu+1}},\cdots,k_{\nu+\sigma_{\gamma-\nu}}\}=\{h_{\lambda+\mu+1},\cdots,h_\gamma\}} 1 = \binom{\gamma-\mu-\lambda}{\nu-\lambda},$$

for some $a_{\lambda\mu}$ with $1 \le \lambda \le \nu$, $1 \le \mu \le \gamma-\nu$ it can be easily shown that

$$\sum_{\nu=1}^{\gamma-1}\sum_{\lambda=1}^{\nu}\sum_{\mu=1}^{\gamma-\nu}(-1)^{\gamma-\nu-\mu}\binom{\gamma-\mu-\lambda}{\nu-\lambda}\binom{\mu+\lambda}{\lambda}a_{\lambda\mu}$$

$$=\sum_{\lambda=1}^{\gamma-1}\sum_{\nu=\lambda}^{\gamma-1}\sum_{\mu=1}^{\gamma-\nu}(-1)^{\gamma-\nu-\mu}\binom{\gamma-\mu-\lambda}{\nu-\lambda}\binom{\mu+\lambda}{\lambda}a_{\lambda\mu}$$

$$=\sum_{\lambda=1}^{\gamma-1}\sum_{\mu=1}^{\gamma-\lambda}\sum_{\nu=\lambda}^{\gamma-\mu}(-1)^{\gamma-\nu-\mu}\binom{\gamma-\mu-\lambda}{\nu-\lambda}\binom{\mu+\lambda}{\lambda}a_{\lambda\mu}$$

$$=\sum_{\lambda=1}^{\gamma-1}\sum_{\mu=1}^{\gamma-\lambda}\sum_{\nu=0}^{\gamma-\mu-\lambda}(-1)^{\gamma-\nu-\lambda-\mu}\binom{\gamma-\mu-\lambda}{\nu}\binom{\mu+\lambda}{\lambda}a_{\lambda\mu}$$

$$=\sum_{\lambda=1}^{\gamma-1}\sum_{\mu=1}^{\gamma-\lambda}\left[(1-1)^{\gamma-\mu-\lambda}\right]\binom{\mu+\lambda}{\lambda}a_{\lambda\mu} \;=\; 0.$$

Finally we have simplified I as

$$I = \sum_{\alpha,\,\nu=1}^{n}\sum_{\lambda=1}^{\nu}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}}\frac{1}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}\left[\Big(\overline{\zeta}_\alpha f_\alpha\Big)_{\overline{\zeta}_{k_1}} + \Big(\zeta_\alpha f_{k_1\,\zeta_\alpha}\Big)\right]_{\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\lambda}\zeta_{k_{\lambda+1}}\cdots\zeta_{k_\nu}}$$

$$\times\prod_{\tau=1}^{\lambda}\frac{\overline{z}_{k_\tau}}{1-\overline{z}_{k_\tau}\zeta_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta}_{k_\tau}}\prod_{\tau=1}^{\nu}d\xi_{k_\tau}d\eta_{k_\tau}$$

$$-\sum_{\alpha=1}^{n}\sum_{\gamma=2}^{n}\sum_{\lambda=1}^{\gamma-1}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\gamma\le n}}\frac{1}{\pi^\gamma}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\gamma}}\left[\Big(\overline{\zeta}_\alpha f_\alpha\Big)_{\overline{\zeta}_{k_1}} + \Big(\zeta_\alpha f_{k_1\,\zeta_\alpha}\Big)\right]_{\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\lambda}\zeta_{k_{\lambda+1}}\cdots\zeta_{k_\gamma}}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{1-\overline{z}_{k_\tau}\zeta_{k_\tau}} \prod_{\tau=\lambda+1}^{\gamma} \frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta}_{k_\tau}} \prod_{\tau=1}^{\gamma} d\xi_{k_\tau} d\eta_{k_\tau}$$

$$= \sum_{\alpha,\,\nu=1}^{n} \sum_{1\le k_1<\cdots<k_\nu\le n} \frac{1}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \left[\left(\overline{\zeta}_\alpha f_\alpha\right)_{\overline{\zeta}_{k_1}} + \left(\zeta_\alpha f_{k_1\,\zeta_\alpha}\right)\right]_{\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\nu}}$$

$$\times \prod_{\tau=1}^{\nu} \frac{\overline{z}_{k_\tau}}{1-\overline{z}_{k_\tau}\zeta_{k_\tau}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau}\ .$$

Therefore condition (2.83) turns out to be

$$\sum_{\nu=1}^{n}\sum_{\lambda=0}^{\nu-1} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}} \frac{1}{(2\pi i)^n} \int_{\mathbb{D}^n} \gamma(\zeta) \prod_{\tau=1}^{\lambda} \frac{z_{\lambda_\tau}}{\zeta_{\lambda_\tau}-z_{\lambda_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{\overline{\zeta}_{\lambda_\tau}}{\overline{\zeta_{\lambda_\tau}-z_{\lambda_\tau}}} \frac{d\zeta}{\zeta} =$$

$$\sum_{\alpha,\,\nu=1}^{n} \sum_{1\le k_1<\cdots<k_\nu\le n} \frac{1}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \left[\left(\overline{\zeta}_\alpha f_\alpha\right)_{\overline{\zeta}_{k_1}} + \left(\zeta_\alpha f_{k_1\,\zeta_\alpha}\right)\right]_{\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\nu}} \prod_{\tau=1}^{\nu} \frac{\overline{z}_{k_\tau} d\xi_{k_\tau} d\eta_{k_\tau}}{1-\overline{z}_{k_\tau}\zeta_{k_\tau}}. \tag{2.85}$$

2.3.3 An alternative

In the following we give an alternative to the solvability condition (2.85). Let φ belong to the Wiener algebra:

$$W(\partial\mathbb{D}\,;\mathbb{C}\,) = \left\{ f \middle| f(z) = \sum_{-\infty}^{+\infty} \alpha_\kappa z^\kappa, \quad z\in\partial\mathbb{D}\,, \quad \|\, f\,\|_W := \sum_{-\infty}^{+\infty} |\alpha_\kappa| < \infty \right\},$$

see [18] and [28]. Then φ is representable as a Fourier series

$$\varphi(\zeta) = \sum_{k=0}^{\infty} \alpha_k \zeta^k + \sum_{k=1}^{\infty} \alpha_{-k}\zeta^{-k}$$

and the series converges absolutely and uniformly to $\varphi(\zeta)$, $\zeta\in\partial\mathbb{D}$. If $\alpha_{-k}=0$ ($\alpha_k = 0$) $k\in\mathbb{N}$ then $\varphi(\zeta)$ is the boundary value of a function, holomorphic in $\mathbb{D}$ ($\mathbb{D}^-$) , i.e., $\varphi\in BH(\mathbb{D}\,)(\varphi\in BH(\mathbb{D}^-))$.

So, if $\varphi\in W(\partial\mathbb{D}\,;\mathbb{C}\,)$ then condition $\varphi(\zeta)\in BH(\mathbb{D}\,)$ is equivalent to $\alpha_{-k}=0,\ k\in\mathbb{N}$, i.e.,

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \varphi(\zeta)\zeta^k \frac{d\zeta}{\zeta} = 0\ .$$

This leads to

$$\sum_{k=1}^{\infty} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \varphi(\zeta)(\overline{z}\zeta)^k \frac{d\zeta}{\zeta} = 0\,, \quad z\in\mathbb{D}\ ,$$

namely

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \varphi(\zeta)\frac{\overline{z}\zeta}{1-\overline{z}\zeta}\frac{d\zeta}{\zeta} = 0\,, \quad z\in\mathbb{D}\ .$$

Let $\varphi \in W(\partial \mathbb{D}^2, \mathbb{C})$ and $\varphi(\zeta) \in BH(D^2)$. Then

$$\alpha_{k_1 - k_2} = 0, \quad \alpha_{-k_1\ -k_2} = 0, \quad \alpha_{k_1 - k_2} = 0, \quad (k_1, k_2) \in \mathbb{Z}_+^2 \setminus (0,0)$$

where

$$\alpha_{k_1,k_2} = \frac{1}{(2\pi i)^2} \int_{\partial \mathbb{D}^2} \varphi(\zeta) \zeta_1^{-k_1} \zeta_1^{-k_2} \frac{d\zeta}{\zeta}\,, \quad (k_1, k_2) \in \mathbb{Z}^2$$

and it holds if and only if

$$\frac{1}{(2\pi i)^2} \int_{\partial \mathbb{D}^2} \varphi(\zeta) \left[\frac{1}{1 - \overline{z}\zeta} - 1 \right] \frac{d\zeta}{\zeta} = 0\,, \quad z \in \mathbb{D}^2\,.$$

This is the same as

$$\sum_{\nu=1}^{n} \sum_{\mu=0}^{\nu-1} \sum_{\substack{1 \le \lambda_1 < \cdots < \lambda_\mu \le n \\ 1 \le \lambda_{\mu+1} < \cdots < \lambda_\nu \le n}} \frac{1}{(2\pi i)^n} \int_{\partial \mathbb{D}^n} \varphi(\zeta) \prod_{\tau=1}^{\mu} \frac{z_{\lambda\tau}}{\zeta_{\lambda_\tau} - z_{\lambda_\tau}} \prod_{\tau=\mu+1}^{\nu} \frac{\overline{z}_{\lambda_\tau}}{\overline{\zeta_{\lambda_\tau} - z_{\lambda_\tau}}} \frac{d\zeta}{\zeta} = 0, \ n = 2.$$

II. In the case $\varphi \in W(\partial \mathbb{D}^n, \mathbb{C})$, it is easy to conclude from the Plemejl formula that $\varphi \in BH(\mathbb{D}^n)$ if and only if

$$\frac{1}{(2\pi i)^k} \int_{\partial_0 \mathbb{D}^k} \varphi(\eta) \frac{d\eta^*}{\eta^* - \zeta^*} = \frac{1}{2^k} \varphi(\zeta^{(k)}),$$

$$\eta = (\eta^*, \eta')\,,\ \zeta^{(k)} = (\zeta^*, \eta') \in \partial_0 \mathbb{D}^n\,;\ \zeta^*\,,\ \eta^* \in \partial_0 \mathbb{D}^k\,,\ k = 1, 2, \cdots, n-1, n. \tag{2.86}$$

Clearly from (2.86) it follows that

$$\frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \varphi(\eta) \frac{d\eta}{\eta - \zeta} = \frac{1}{2^n} \varphi(\zeta), \quad \zeta \in \partial_0 \mathbb{D}^n\,. \tag{2.87}$$

However, this is not a sufficient condition. For example take

$$\varphi(\zeta) = \zeta_1^{k_1} \zeta_2^{-k_2} \zeta_3^{-k_3},\ \zeta \in \partial_0 \mathbb{D}^3,\ k_1, k_2, k_3 \in \mathbb{N},$$

then by

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}} \zeta^k \frac{d\zeta}{\zeta - \eta} = \frac{1}{2} \eta^k\,,\ \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \zeta^{-h} \frac{d\zeta}{\zeta - \eta} = -\frac{1}{2} \eta^{-h}\,,\ k \in \mathbb{Z}_+\,,\ h \in \mathbb{N}\,,\ \eta \in \partial \mathbb{D}\,,$$

it is evident that

$$\frac{1}{(2\pi i)^3} \int_{\partial \mathbb{D}^3} \varphi(\zeta) \frac{d\zeta}{\zeta - \eta} = \frac{1}{2^3} \varphi(\eta), \quad \text{but} \quad \varphi(\zeta) = \zeta_1^{k_1} \zeta_2^{-k_2} \zeta_3^{-k_3} \notin BH(\mathbb{D}^3)\,.$$

And yet, together with (2.87) the condition

$$\frac{1}{(2\pi i)^k} \int_{\partial \mathbb{D}^k} \varphi(\zeta^*, \zeta') \frac{d\zeta^*}{\zeta^* - \eta^*} = \frac{1}{2^k} \varphi(\eta^*, \zeta'),$$

$$\zeta^* = (\zeta_1, \cdots, \zeta_k)\,,\ \eta^* = (\eta_1, \cdots, \eta_k)\ \in \partial \mathbb{D}^k;\ \zeta' \in \partial \mathbb{D}^{n-k}\ (k = 1, \cdots, n-1)\,, \tag{2.88}$$

without loss of generality, is a sufficient condition. In one variable case (2.88) vanishes automatically. Actually, it is evident that condition (2.87) together with (2.88) is equivalent to

$$\frac{1}{2\pi i}\int_{\partial \mathbb{D}_k} \varphi(\zeta)\frac{d\zeta_k}{\zeta_k-\eta_k} = \frac{1}{2}\varphi(\zeta)\Big|_{\zeta_k=\eta_k}, \quad \eta_k \in \partial\mathbb{D}_k,\ k=1,\cdots,n. \tag{2.89}$$

This is exactly the definition for a given function φ on $\partial\mathbb{D}^n$ to satisfy $\varphi \in BH(\mathbb{D}^n)$.

Thus the solvability condition (2.85) for N_3 just can be replaced by

$$\frac{1}{(2\pi i)^n}\int_{\partial\mathbb{D}^n}\Big[\gamma(\zeta)-\sum_{\alpha=1}^{n}(\overline{\zeta}_\alpha f_\alpha+\zeta_\alpha u_{0_{\zeta_\alpha}})\Big]\frac{d\zeta}{\zeta-\eta}$$

$$=\frac{1}{2^n}\Big[\gamma(\eta)-\sum_{\alpha=1}^{n}(\overline{\eta}_\alpha f_\alpha+\eta_\alpha u_{0_{\eta_\alpha}})\Big], \eta\in\partial\mathbb{D}^n, \tag{2.90}$$

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\Big[\gamma(\zeta)-\sum_{\alpha=1}^{n}(\overline{\zeta}_\alpha f_\alpha+\zeta_\alpha u_{0_{\zeta_\alpha}})\Big]\frac{d\zeta_k}{\zeta_k-\eta_k}$$

$$=\frac{1}{2}\Big[\gamma(\zeta)-\sum_{\alpha=1}^{n}(\overline{\zeta}_\alpha f_\alpha+\zeta_\alpha u_{0_{\zeta_\alpha}})\Big]\ |_{\zeta_k=\eta_k}, \quad k=1,\cdots,n-1\,. \tag{2.91}$$

Since

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} Tf\frac{d\zeta}{\zeta-z}=0$$

is valid even on the boundary $\partial\mathbb{D}$, (2.90) can be simplified as

$$\frac{1}{(2\pi i)^n}\int_{\partial\mathbb{D}^n}\Big[\gamma(\zeta)-\sum_{\alpha=1}^{n}(\overline{\zeta}_\alpha f_\alpha+\zeta_\alpha\Pi_\alpha f_\alpha)\Big]\frac{d\zeta}{\zeta-\eta}$$

$$=\frac{1}{2^n}\Big[\gamma(\eta)-\sum_{\alpha=1}^{n}(\overline{\eta}_\alpha f_\alpha+\eta_\alpha u_{0_{\eta_\alpha}})\Big], \eta\in\partial\mathbb{D}^n\,. \tag{2.92}$$

Thus the following result is proved.

Theorem 10 *Let $f_{k\overline{z}_j} \in L_1(\overline{\mathbb{D}^n})\cap C(\overline{\mathbb{D}^n})$ satisfy (2.66).*

$< a >$ The conditions (2.77) and (2.92) are necessary in order that problem N_3 is solvable. If the condition (2.91) is also satisfied then the problem is uniquely solvable with a normalising condition. The solution can be given by (2.82).

$< b >$ If the conditions (2.77) and (2.85) hold then problem N_3 is uniquely solvable up to an arbitrary constant. The solution is given by (2.82). The homogeneous problem has only the trivial solution. The problem is well – posed.