Boundary Value Problems of Complex Variables

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Alip Mohammed

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Betreuer:	Prof. Dr. Heinrich Begehr
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Abstract

Until now there has been still a great deal of activity in the study of integral representations for holomorphic functions of one and several complex variables in terms of their boundary values. These problems are of interest both because of their theoretical importance and the implications for applications, such as quantum field theory. In contrast with the one variable case, studies on holomorphic functions of several complex variables encounter mainly two kinds of difficulties: the geometrical complexity of general higher dimensional domains and the analytic complexity of complex integrability conditions from overdetermination. Many new tools are developed and many new insights are achieved.

The aim of the PhD project is finding integral representations for elliptic equations and overdetermined elliptic systems as well as well-posed formulation of the classical problems in higher dimensional space regarding the torus.

Introduction

This PhD project concerns with all three kinds of main boundary value problems for polydiscs as well as for the other torus related domains. Some plane case problems are also included as being step-stones to higher dimensional problems. But higher dimensional cases remain as our main interest. The project consists of four chapters.

In Chapter 1 the classical Dirichlet problem for the inhomogeneous pluriholomorphic system in a polydisc is studied. To get a unique solution the boundary condition is modified and its solvability conditions as well as the unique solution are given explicitly. Further applying integral properties a necessary and sufficient condition for the boundary values of a holomorphic function in polydiscs is given. But it is extendable to any torus related domains. This will be shown in Chapter 3 .

In Chapter 2 Section 2.1 the Neumann problem for the inhomogeneous pluriharmonic system in a polydisc is considered. The definition of the torus (Shilov boundary) related boundary function for a polydisc, the solvability conditions and the explicit solution which is unique up to an arbitrary constant are given.

In Chapter 2 Section 2.2 the classical Neumann problem for the inhomogeneous pluriholomorphic system in a polydisc is considered. Its solvability conditions and its solution are given. It is shown that the problem is not well posed. To fix the solution the boundary condition is modified. For the modified problem the solvability conditions and the solution which is unique up to an arbitrary constant are explicitly given.

In Chapter 2 Section 2.3 the Neumann problem for the inhomogeneous Cauchy–Riemann system in a polydisc is considered. The solvability conditions and the explicit solution which is unique up to an arbitrary real constant are given. Applying Fourier series method a necessary and sufficient condition for the boundary values of a holomorphic function in polydiscs ($\mathbb{D}^n$) is given.

In Chapter 3 boundary values of functions, analytic in torus related domains, are classified in the Wiener algebra. The general integral representation formulas of these functions , the solvability conditions and the solutions of the corresponding Schwarz problems are given explicitly. A necessary and sufficient condition for the boundary values of a holomorphic function for any torus related domain ($\mathbb{D}^{\chi(\nu)}$) is given. At the end, well-posed formulations of the torus related problems are considered.

In Chapter 4 section 4.1 the Riemann-Hilbert-Poincare problem for holomorphic functions with holomorphic coefficients in polydiscs is considered by the Fourier series method. The solvability conditions and the explicit solutions are given.

In Chapter 4 section 4.2 the Riemann-Hilbert-Poincare problem for holomorphic functions in the unit disc with anti-polynomial with non-integer free term coefficient is considered by the Fourier method. The explicit solutions and the exact number of linearly independent solutions of the homogenous problem are calculated. The solvability conditions are given. Through our discussion we see that these features do not depend on the the smoothness of its coefficients on the boundary and not on the smallness of its coefficients but rather on the interior analytic continuation property of the coefficients.