4 Boundary value problems for the inhomogeneous Cauchy-Riemann equation in the upper half plane

The Pompeiu operator

$$Tf(z) = -\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{\zeta - z},$$

where $f \in L_1(\mathbb{H};\mathbb{C})$, produces a particular weak solution to the inhomogeneous Cauchy-Riemann equation $w_{\bar{z}} = f$ in $\mathbb{H}$. Therefore $\varphi = w - Tf$ satisfies $\varphi_{\bar{z}} = 0$ i.e. is analytic in $\mathbb{H}$ as soon as w is a solution to $w_{\bar{z}} = f$, see [28]. In this way boundary value problems for the inhomogeneous Cauchy-Riemann equation is reduced to some for analytic functions.
At first again the Schwarz boundary value problem is studied.

4.1 Schwarz problem for the inhomogeneous Cauchy-Riemann equation in the upper half plane

Theorem 17 *Let $f \in L_{p,2}(\mathbb{H};\mathbb{C})$, $2 < p$, $\gamma \in C(\mathbb{R};\mathbb{R})$, $c \in \mathbb{R}$ such that γ is bounded on $\mathbb{R}$. Then the Schwarz problem $w_{\bar{z}} = f$ in $\mathbb{H}$, $\mathrm{Re}w = \gamma$ on $\mathbb{R}$, $\mathrm{Im}w(i) = c$ is uniquely solvable in the weak sense. The solution is*

$$\begin{aligned} w(z) &= ic + \frac{1}{\pi i}\int_{-\infty}^{\infty}\gamma(t)\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)dt \\ &- \frac{1}{\pi}\int_{\mathbb{H}}\{(f(\zeta)\left(\frac{1}{\zeta - z} - \frac{\zeta}{\zeta^2+1}\right) \\ &- \overline{f(\zeta)}\left(\frac{1}{\bar{\zeta} - z} - \frac{\bar{\zeta}}{\bar{\zeta}^2+1}\right)\}d\xi d\eta. \end{aligned} \tag{4.1}$$

Proof The function $\varphi = w - Tf + ic_0$ satisfies $\varphi_{\bar{z}} = 0$ in $\mathbb{H}$, $\mathrm{Re}\varphi = \gamma - \mathrm{Re}Tf$ on $\mathbb{R}$. Here $c_0 \in \mathbb{R}$ has to be determined by $\mathrm{Im}w(i) = c$. By the Schwarz integral formula, see Section 3.2,

$$\varphi(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty}(\gamma(t) - \mathrm{Re}Tf(t))\frac{dt}{t-z} + ic_0$$

From

$$\frac{1}{\pi i}\int_{-\infty}^{\infty} Tf(t)\frac{dt}{t-z} = -\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{1}{\pi}\int_{\mathbb{H}} f(\widetilde{\zeta})\frac{d\widetilde{\xi}d\widetilde{\eta}}{\widetilde{\zeta}-t}\frac{dt}{t-z}$$

$$= \frac{1}{\pi}\int_{\mathbb{H}} f(\widetilde{\zeta})\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{dt}{(t-\widetilde{\zeta})(t-z)}d\widetilde{\xi}d\widetilde{\eta} = 0,$$

$$\frac{1}{\pi i}\int_{-\infty}^{\infty} \overline{Tf(t)}\frac{dt}{t-z} = -\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{1}{\pi}\int_{\mathbb{H}} \overline{f(\widetilde{\zeta})}\frac{d\widetilde{\xi}d\widetilde{\eta}}{\overline{\widetilde{\zeta}}-t}\frac{dt}{t-z}$$

$$= \frac{1}{\pi}\int_{\mathbb{H}} \overline{f(\widetilde{\zeta})}\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{dt}{(t-\overline{\widetilde{\zeta}})(t-z)}d\widetilde{\xi}d\widetilde{\eta} = \frac{2}{\pi}\int_{\mathbb{H}} \overline{f(\widetilde{\zeta})}\frac{d\widetilde{\xi}d\widetilde{\eta}}{z-\overline{\widetilde{\zeta}}},$$

then

$$w(z) = \varphi(z) + Tf(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty}\gamma(t)\frac{dt}{t-z}$$

$$+\frac{1}{\pi}\int_{\mathbb{H}}\overline{f(\zeta)}\frac{d\xi d\eta}{\overline{\zeta}-z} + ic_0 - \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

$$= \frac{1}{\pi i}\int_{-\infty}^{\infty}\gamma(t)\frac{dt}{t-z} - \frac{1}{\pi}\int_{\mathbb{H}}\left(\frac{f(\zeta)}{\zeta-z} - \frac{\overline{f(\zeta)}}{\overline{\zeta}-z}\right)d\xi d\eta + ic_0$$

follows. From

$$c = \mathrm{Im} w(i) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\gamma(t)\frac{tdt}{t^2+1}$$

$$-\frac{1}{\pi}\int_{\mathbb{H}}\mathrm{Im}\left(\frac{f(\zeta)}{\zeta-i} - \frac{\overline{f(\zeta)}}{\overline{\zeta}-i}\right)d\xi d\eta + c_0$$

it follows

$$c_0 = c + \frac{1}{\pi}\int_{-\infty}^{\infty}\gamma(t)\frac{tdt}{t^2+1}$$

$$+\frac{1}{2\pi i}\int_{\mathbb{H}}\left(\frac{f(\zeta)}{\zeta-i} - \frac{\overline{f(\zeta)}}{\overline{\zeta}+i} - \frac{\overline{f(\zeta)}}{\overline{\zeta}-i} + \frac{f(\zeta)}{\zeta+i}\right)d\xi d\eta$$

$$= c + \frac{1}{\pi}\int_{-\infty}^{\infty}\gamma(t)\frac{tdt}{t^2+1} + \frac{1}{\pi i}\int_{\mathbb{H}}\left(\frac{\zeta f(\zeta)}{\zeta^2+1} - \frac{\bar{\zeta}\overline{f(\zeta)}}{\bar{\zeta}^2+1}\right)d\xi d\eta.$$

Therefore

$$w(z) = ic + \frac{1}{\pi i}\int_{-\infty}^{\infty}\gamma(t)\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)dt$$
$$-\frac{1}{\pi}\int_{\mathbb{H}}\left(f(\zeta)\left(\frac{1}{\zeta-z} - \frac{\zeta}{\zeta^2+1}\right) - \overline{f(\zeta)}\left(\frac{1}{\bar{\zeta}-z} - \frac{\bar{\zeta}}{\bar{\zeta}^2+1}\right)\right)d\xi d\eta.$$

For checking this to be the solution observe

$$w_{\bar{z}}(z) = \partial_{\bar{z}}(Tf(z)) - \partial_{\bar{z}}\left(\overline{-\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{\zeta-\bar{z}}}\right)$$
$$= f(z) - \overline{\left(\partial_z\{-\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{\zeta-\bar{z}}\}\right)} = f(z)$$

and for $z = t \in \mathbb{R}$

$$\lim_{z\to t}\mathrm{Re}w(t) = \lim_{z\to t}\mathrm{Re}\frac{1}{\pi i}\int_{-\infty}^{\infty}\gamma(\tau)\frac{d\tau}{\tau - z} - \mathrm{Re}\frac{1}{\pi}\int_{\mathbb{H}}\left(\frac{f(\zeta)}{\zeta - t} - \frac{\overline{f(\zeta)}}{\overline{\zeta - t}}\right)d\xi d\eta$$
$$= \gamma(t)$$

and $\mathrm{Im}w(i) = c$.

4.2 Dirichlet problem for the inhomogeneous Cauchy-Riemann equation

Theorem 18 *The Dirichlet problem $w_{\bar{z}} = f$ in $\mathbb{H}$, $w = \gamma$ on $\mathbb{R}$, where $f \in L_{p,2}(\mathbb{H};\mathbb{C})$ for $2 < p$ and $\gamma \in L_2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$ is uniquely solvable in the weak sense by*

$$w(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma(t)\frac{dt}{t-z} - \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{\zeta - z} \qquad (4.2)$$

if and only if for $z \in \mathbb{H}$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma(t)\frac{dt}{t-\bar{z}} - \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{\zeta-\bar{z}} = 0. \qquad (4.3)$$

Proof Obviously (4.2) provides a weak solution to the equation $w_{\bar{z}} = f$.
1)The formula (4.3) is sufficient. Let $t_0 \in \mathbb{R}$, then

$$w(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \gamma(t)(\frac{1}{t-z} - \frac{1}{t-\bar{z}})dt$$

$$-\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{\zeta - z} + \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{\zeta - \bar{z}}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \gamma(t)\frac{y}{|t-z|^2}dt - \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)(\frac{1}{\zeta - z} - \frac{1}{\zeta - \bar{z}})d\xi d\eta.$$

The last formula tends to $\gamma(t_0)$ if z tends to $t_0 \in \mathbb{R}$.
2)Formula (4.3) is necessary. Consider for $z \in \mathbb{H}$ the function $\varphi = w - Tf$ and observe, see [28], $|z|^\alpha Tf(z)$ is bounded for $\alpha = (p-2)/p$. Then $\varphi_{\bar{z}} = 0$ and $\varphi = \gamma - Tf$ on $\mathbb{R}$. For $z \in \mathbb{H}$ then by (3.14)

$$\varphi(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} [\gamma(t) - Tf(t)]\frac{dt}{t-z}$$

and according to (3.15) the solvability condition is

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} [\gamma(t) - Tf(t)]\frac{dt}{t-\bar{z}} = 0$$

for $z \in \mathbb{H}$. From

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} Tf(t)\frac{dt}{t-z} = -\frac{1}{\pi}\int_{\mathbb{H}} \frac{f(\zeta)}{\zeta - z}\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\frac{1}{t-\zeta} - \frac{1}{t-z})d\xi d\eta = 0$$

and from

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} Tf(t)\frac{dt}{t-\bar{z}} = -\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dt}{(\zeta - t)(t-\bar{z})}d\xi d\eta$$

$$= \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{\zeta - \bar{z}}$$

it follows

$$w(z) = \varphi(z) + Tf(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma(t)\frac{dt}{t-z} - \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

if and only if

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma(t)\frac{dt}{t-\bar{z}} - \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{\zeta - \bar{z}} = 0.$$

3) *Verification*
Obviously from (4.1) $w_{\bar{z}} = f$ follows in $\mathbb{H}$. Moreover, for $t_0 \in \mathbb{R}$

$$\lim_{z\to t_0} w(z) = \lim[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma(t)\frac{dt}{t-z} - \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{\zeta - z}]$$

$$= \lim_{z\to t_0}[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma(t)(\frac{1}{t-z} - \frac{1}{t-\bar{z}})dt - \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)(\frac{1}{\zeta - z}) - \frac{1}{\zeta-\bar{z}})d\xi d\eta]$$

$$= \lim_{z\to t_0}\frac{1}{\pi}\int_{-\infty}^{\infty}\gamma(t)\frac{y}{|t-z|^2}dt = \gamma(t_0).$$

4.3 Neumann problem for the inhomogeneous Cauchy-Riemann equation

Theorem 19 *The Neumann problem $w_{\bar{z}} = f$ in $\mathbb{H}$, $\partial_y w = i\gamma$ on $\mathbb{R}$, $w(i) = c$ is uniquely solvable for $f \in L_{p,2}(\mathbb{H};\mathbb{C}) \cap C^1(\overline{\mathbb{H}};\mathbb{C}) \cap L_2(\mathbb{R};\mathbb{C})$, $\gamma \in L_2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$, $c \in \mathbb{C}$ is uniquely solvable if and only if for $z \in \mathbb{H}$*

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma(t) + f(t))\frac{dt}{t-\bar{z}} - \frac{1}{\pi}\int_{\mathbb{H}} f_\zeta(\zeta)\frac{d\xi d\eta}{\zeta - \bar{z}} = 0. \qquad (4.4)$$

The solution then is

$$
\begin{aligned}
w(z) &= c + \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma(t)+2f(t))\log\frac{t-i}{t-z}dt \\
&- \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{z-i}{(\zeta-z)(\zeta-i)}d\xi d\eta. \qquad (4.5)
\end{aligned}
$$

Proof Representing $w = \varphi + Tf$ with analytic φ this function φ satisfies in $\mathbb{H}$

$$\varphi' = w_z - w_{\bar z} + f - \Pi f$$

where

$$\Pi f(z) = -\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{(\zeta-z)^2}.$$

Representing, see [28],

$$\Pi f(z) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} f(t)\frac{d\tau}{\tau - z} - \frac{1}{\pi}\int_{\mathbb{H}} f_\zeta(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

we see for $t \in \mathbb{R}$

$$
\begin{aligned}
(\Pi f)^+(t) = \lim_{\substack{z\to t\\ z\in\mathbb{H}}} \Pi f(z) &= -\frac{1}{2}f(\tau) - \frac{1}{2\pi i}\oint_{-\infty}^{\infty} f(\tau)\frac{d\tau}{\tau - t} \\
&- \frac{1}{\pi}\int_{\mathbb{H}} f_\zeta(\zeta)\frac{d\xi d\eta}{\zeta - t}. \qquad (4.6)
\end{aligned}
$$

Thus the boundary values of φ' on $\mathbb{R}$ are

$$\varphi' = \gamma + f - (\Pi f)^+ = \gamma + \frac{3}{2}f + \frac{1}{2\pi i}\oint_{-\infty}^{\infty} f(\tau)\frac{d\tau}{\tau - t} + \frac{1}{\pi}\int_{\mathbb{H}} f_\zeta(\zeta)\frac{d\xi d\eta}{\zeta - t}.$$

Applying the result of Theorem 19 the solution to this Dirichlet problem is

$$
\begin{aligned}
\varphi'(z) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}[\gamma(t) + \frac{3}{2}f(t) \\
&+ \frac{1}{2\pi i}\oint_{-\infty}^{\infty} f(\tau)\frac{d\tau}{\tau - t} + \frac{1}{\pi}\int_{\mathbb{H}} f_\zeta(\zeta)\frac{d\xi d\eta}{\zeta - t}]\frac{dt}{t - z} \qquad (4.7)
\end{aligned}
$$

if and only if

$$\begin{aligned}\frac{1}{2\pi i}\int_{-\infty}^{\infty}[\gamma(t) &+ \frac{3}{2}f(t)+\frac{1}{2\pi i}\oint_{-\infty}^{\infty}f(\tau)\frac{d\tau}{\tau-t}\\ &+ \frac{1}{\pi}\int_{\mathbb{H}}f_{\zeta}(\zeta)\frac{d\xi d\eta}{\zeta-t}]\frac{dt}{t-\bar{z}}=0. \end{aligned}\tag{4.8}$$

Because for $z\in\mathbb{H}$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{\pi i}\oint_{-\infty}^{\infty}f(\tau)\frac{d\tau}{\tau-t}\frac{dt}{t-z}=\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(\tau)\frac{d\tau}{\tau-z},$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{\pi}\int_{\mathbb{H}}f_{\zeta}(\zeta)\frac{d\xi d\eta}{\zeta-t}\frac{dt}{t-z}=0,$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{\pi i}\oint_{-\infty}^{\infty}f(\tau)\frac{d\tau}{\tau-t}\frac{dt}{t-\bar{z}}=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(\tau)\frac{d\tau}{\tau-\bar{z}},$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{\pi}\int_{\mathbb{H}}f_{\zeta}(\zeta)\frac{d\xi d\eta}{\zeta-t}\frac{dt}{t-\bar{z}}=-\frac{1}{\pi}\int_{\mathbb{H}}f_{\zeta}(\zeta)\frac{d\xi d\eta}{\zeta-\bar{z}}=Tf_{\zeta}(\bar{z})$$

then

$$\varphi^{'}(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}[\gamma(t)+2f(t)]\frac{dt}{t-z}\tag{4.9}$$

if and only if

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}[\gamma(t)+f(t)]\frac{dt}{t-\bar{z}}=\frac{1}{\pi}\int_{\mathbb{H}}f_{\zeta}(\zeta)\frac{d\xi d\eta}{\zeta-\bar{z}}\tag{4.10}$$

which is (4.5).
Integrating (4.9) leads to

$$\varphi(z)=\varphi(i)+\frac{1}{2\pi i}\int_{-\infty}^{\infty}[\gamma(t)+2f(t)]\log\frac{t-i}{t-z}dt$$

so that

$$w(z)=c+\frac{1}{2\pi i}\int_{-\infty}^{\infty}[\gamma(t)+2f(t)]\log\frac{t-i}{t-z}dt$$

$$-\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{z-i}{(\zeta-i)(\zeta-z)}d\xi d\eta$$

which is (4.5).
At last (4.5) is verified under (4.4) to be a solution. Obviously, $w_{\bar z} = f$ is satisfied in $\mathbb{H}$. Differentiation gives

$$w_z(z) - w_{\bar z}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}[\gamma(t) + 2f(t)]\frac{dt}{t-z} + \Pi f(z) - f(z).$$

Subtracting (4.4) shows

$$w_z(z) - w_{\bar z}(z) = \frac{1}{\pi}\int_{-\infty}^{\infty}[\gamma(t) + f(t)]\frac{ydt}{|t-z|^2} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} f(t)\frac{dt}{t-z}$$

$$-Tf_\zeta(\bar z) + \Pi f(z) - f(z)$$

so that by letting z tend to $t \in \mathbb{R}$

$$w_z^+(t) - w_{\bar z}^+(t) = \gamma(t) + f(t) + \frac{1}{2}f(t) + \frac{1}{2\pi i}\oint_{-\infty}^{\infty} f(\tau)\frac{d\tau}{\tau - t}$$

$$-Tf_\zeta(t) + (\Pi f)^+(t) - f(t) = \gamma(t).$$

Here the Plemelj-Sokhotzki formula

$$\lim_{\substack{z\to t\\ z\in\mathbb{H}}}\frac{1}{2\pi i}\int_{-\infty}^{\infty} f(\tau)\frac{d\tau}{\tau - z} = \frac{1}{2}f(t) + \frac{1}{2\pi i}\oint_{-\infty}^{\infty} f(\tau)\frac{d\tau}{\tau - t}$$

and (4.6) is used.

4.4 Robin problem for the inhomogeneous Cauchy-Riemann equation

Here a particular case is studies, see [13].
A special form of the Robin problem for the inhomogeneous Cauchy-Riemann equation in the upper half plane is

$$w_{\bar{z}} = f \quad \text{in} \quad \mathbb{H}$$

$$w + \partial_\nu w = \gamma \quad \text{on} \quad \partial\mathbb{H}.$$

Again $\partial_\nu = -\partial_y$ on $\partial\mathbb{H}$.

Theorem 20 *The Robin problem for the inhomogeneous Cauchy-Riemann equation $w_{\bar{z}} = f$ in $\mathbb{H}$, $w + \partial_\nu w = \gamma$ on $\mathbb{R}$,*

$$w(0) + \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\zeta}{\zeta} = c$$

is uniquely solvable for given $f \in L_{p,2}(\mathbb{H};\mathbb{C}) \cap C^1(\overline{\mathbb{H}};\mathbb{C}) \cap L_2(\mathbb{R};\mathbb{C})$ for $2 < p$, $\gamma \in L_2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$ if and only if for $z \in \mathbb{H}$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} (i\gamma(t) + f(t))\,\frac{dt}{t - \bar{z}} + iTf(\bar{z}) + Tf_\zeta(\bar{z}) = 0. \qquad (4.11)$$

The solution then is given by

$$\begin{aligned} w(z) &= ce^{-iz} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} (\gamma(t) - 2if(t))\,e^{i(t-z)}\int_{-t}^{z-t}\frac{e^{i\zeta}}{i\zeta}d\zeta dt \\ &- \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{\zeta - z}. \end{aligned} \qquad (4.12)$$

Proof Consider $w = \varphi + Tf$ where φ is analytic. Then

$$\partial_\nu w = -\partial_y w = -i(\partial_z - \partial_{\bar{z}})w = -i\varphi' - i\Pi f + if$$

so that

$$w + \partial_\nu w = \varphi - i\varphi' + if + Tf - i\Pi f.$$

Observing on $\mathbb{R}$

$$(\Pi f)^+(t) = -\frac{1}{2}f(t) - \frac{1}{2\pi i}\oint_{-\infty}^{\infty} f(\tau)\frac{d\tau}{\tau - t} + Tf_\zeta(t)$$

therefore the analytic function $\varphi - i\varphi'$ satisfies on $\mathbb{R}$

$$\varphi - i\varphi' = \gamma - if - Tf + i(\Pi f)^+$$

i.e.

$$\varphi' + i\varphi = i\gamma + f - iTf - (\Pi f)^+$$

which is

$$\varphi'(t) + i\varphi(t) = i\gamma(t) + \frac{3}{2}f(t) + \frac{1}{2\pi i}\oint_{-\infty}^{\infty} f(\tau)\frac{d\tau}{\tau - t} - iTf(t) - Tf_\zeta(t) = \hat{\gamma}(t).$$

The solution to this Dirichlet problem is

$$\varphi'(z) + i\varphi(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \hat{\gamma}(t)\frac{dt}{t - z}$$

if and only if for $z \in \mathbb{H}$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \hat{\gamma}(t)\frac{dt}{t - \bar{z}} = 0.$$

From

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{\pi i}\oint_{-\infty}^{\infty} f(\tau)\frac{d\tau}{\tau - t}\frac{dt}{t - z} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} f(\tau)\frac{d\tau}{\tau - z},$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{\pi i}\oint_{-\infty}^{\infty} f(\tau)\frac{d\tau}{\tau - t}\frac{dt}{t - \bar{z}} = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} f(\tau)\frac{d\tau}{\tau - \bar{z}},$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{\zeta - t}\frac{dt}{t - \bar{z}} = Tf(\bar{z}),$$

it follows

$$\varphi'(z) + i\varphi(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \left(i\gamma(t) + 2f(t)\right)\frac{dt}{t - z} = \widetilde{\gamma}(z) \qquad (4.13)$$

if and only if in $\mathbb{H}$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}(i\gamma(t)+f(t))\frac{dt}{t-\bar{z}}+iTf(\bar{z})+Tf_{\zeta}(\bar{z})=0.$$

This last condition is (4.11). Solving the ordinary differential equation (4.13) leads to

$$\begin{aligned}\varphi(z) &= e^{-iz}\{\varphi(0)+\int_0^z \widetilde{\gamma}(\zeta)e^{i\zeta}d\zeta\} \\ &= \varphi(0)e^{-iz}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}(i\gamma(t)+2f(t))\int_0^z \frac{e^{i(\zeta-z)}}{t-\zeta}d\zeta dt \\ &= \varphi(0)e^{-iz}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma(t)-2if(t))e^{i(t-z)}\int_{-t}^{z-t}\frac{e^{i\zeta}}{i\zeta}d\zeta dt.\end{aligned}$$

Thus

$$w(z)=(w(0)-Tf(0))e^{-iz}$$

$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma(t)-2if(t))e^{i(t-z)}\int_{-t}^{z-t}\frac{e^{i\zeta}}{i\zeta}d\zeta dt+Tf(z).$$

This is (4.12).

Next (4.12) is verified to be a solution to the Robin problem provided the solvability condition (4.11) holds. Obviously (4.12) is a solution to the inhomogeneous Cauchy-Riemann equation. Differentiating (4.12) shows

$$\begin{aligned}\partial_y w(z) &= i(\partial_z-\partial_{\bar{z}})w(z) \\ &= ce^{-iz}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma(t)-2if(t))e^{i(t-z)}\int_{-t}^{z-t}\frac{e^{i\zeta}}{i\zeta}d\zeta dt \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma(t)-2if(t))\frac{dt}{z-t}+i\Pi f(z)-if(z)\end{aligned}$$

so that

$$w(z)-\partial_y w(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma(t)-2if(t))\frac{dt}{t-z}+if(z)-i\Pi f(z)+Tf(z).$$

Multiplying (4.11) by i and adding this to the last equation shows

$$\begin{aligned} w(z) - \partial_y w(z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(\gamma(t) - if(t))\frac{ydt}{|t-z|^2} - \frac{i}{2\pi i}\int_{-\infty}^{\infty} f(t)\frac{dt}{t-z} \\ &+ if(z) - i\Pi f(z) + Tf(z) - Tf(\bar{z}) + iTf_\zeta(\bar{z}). \end{aligned}$$

Thus on $\mathbb{R}$

$$\begin{aligned} (w + \partial_\nu w)(t) &= \gamma(t) - if(t) - \frac{i}{2}f(t) - \frac{i}{2\pi i}\oint_{-\infty}^{\infty} f(\tau)\frac{d\tau}{\tau - t} \\ &+ if(t) - i(\Pi f)^+(t) + iTf_\zeta(t) = \gamma(t). \end{aligned}$$