2 The Gauss theorem and Cauchy-Pompeiu representations

2.1 The Gauss theorem for the upper half plane

Let $D \subset \mathbb{C}$ be a regular domain D of the complex plane $\mathbb{C}$, that is a bounded domain with piecewise smooth boundary ∂D and let $w \in C^1(D;\mathbb{C}) \bigcap C(\overline{D},\mathbb{C})$. Then see [3], [28]

$$\frac{1}{2i}\int_{\partial D} w(z)dz = \int_D w_{\bar{z}}(z)dxdy,$$

$$-\frac{1}{2i}\int_{\partial D} w(z)d\bar{z} = \int_D w_z(z)dxdy.$$

Let $\mathbb{H} = \{z : 0 < Imz\}$ be the upper half plane and $w \in W^{1,1}(\mathbb{H};\mathbb{C}) \cap C(\overline{\mathbb{H}};\mathbb{C})$. For $\mathbb{H}_R = \{z : 0 < Imz, |z| < R\}$ the Gauss theorem is

$$\frac{1}{2i}\int_{\partial \mathbb{H}_R} w(z)dz = \int_{\mathbb{H}_R} w_{\bar{z}}(z)dxdy.$$

Assuming

$$\lim_{R\to+\infty} RM(R,w) = 0 \tag{2.1}$$

for

$$M(R,w) = \max_{\substack{|z|=R \\ 0\leqslant\varphi\leqslant\pi}} |w(z)| \quad (z = Re^{i\varphi})$$

leads to

$$\frac{1}{2i}\int_{-\infty}^{\infty} w(t)dt = \int_{\mathbb{H}} w_{\bar{z}}(z)dxdy.$$

The existence of the first integral follows from (2.1), that of the second one from $w \in W^{1,1}(\mathbb{H},\mathbb{C})$. Similarly

$$-\frac{1}{2i}\int_{-\infty}^{\infty} w(t)dt = \int_{\mathbb{H}} w_z(z)dxdy$$

follows.

2.2 Cauchy - Pompeiu representations for the upper half plane

Lemma 6 *Let $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$ for a regular domain D, then*

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta - z}, \tag{2.2}$$

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\bar\zeta}{\bar\zeta - z} - \frac{1}{\pi}\int_D w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\overline{\zeta - z}}}. \tag{2.3}$$

For a proof see [28], [3], [9].

Remark 3 Let $f \in L_1(D,\mathbb{C})$, then the Pompeiu operator T and its complex conjugate $\overline{T}$ are defined by

$$Tf(z) = -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{\zeta - z},$$

$$\overline{T}f(z) = -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{\overline{\overline{\zeta - z}}},$$

see [28],[3].

Theorem 2 *If $w : \mathbb{H} \to \mathbb{C}$ satisfies $|w(x)| \le C|x|^{-\varepsilon}$ for $|x| > k$ and $w_{\bar z} \in L_1(\mathbb{H};C)$, then*

$$w(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} w(t)\frac{dt}{t - z} - \frac{1}{\pi}\int_{0<Im\zeta} w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta - z}, \tag{2.4}$$

$$w(z) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} w(t)\frac{dt}{t - \bar z} - \frac{1}{\pi}\int_{0<Im\zeta} w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\overline{\zeta - z}}}, \tag{2.5}$$

where $z \in \mathbb{H}$.

Proof Applying the representation formula to the half disc $\mathbb{H}_r = [|z| < r, Imz > 0]$ gives

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{H}_r} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{\mathbb{H}_r} w_{\bar\zeta}(\zeta)\frac{d\zeta d\eta}{\zeta - z}$$

for $z \in \mathbb{H}_r$. Letting r tend to infinity

$$w(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}w(t)\frac{dt}{t-z}-\frac{1}{\pi}\int_{0<Im\zeta}w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta-z}$$

follows for $z\in\mathbb{H}$ because

$$\int_H w_{\bar\zeta}(\xi)\frac{d\xi d\eta}{\zeta-z}$$

exists and also as a Cauchy mean value

$$\oint_{-\infty}^{\infty}w(t)\frac{dt}{t-z}.$$

This is the case because

$$-\frac{1}{2\pi i}\int_0^{\pi}w(re^{i\phi})\frac{ire^{i\phi}d\phi}{re^{i\phi}-z}$$

tends to 0, if r tends to ∞ for $|x|\le|z|<r$ as with

$$M(r,w)=\max_{\substack{|z|=r\\0\leqslant\mathrm{Im}z}}|w(z)|\,,$$

$$\lim_{r\to\infty}M(r,w)=0.$$

The existence of

$$\int_{-\infty}^{\infty}w(t)\frac{dt}{t-z}$$

follows from the estimate for $2|z|<r$

$$\left|\int_r^{2k}w(t)\frac{dt}{t-z}\right|\leqslant\int_r^{2k}|w(t)|\frac{dt}{t-|z|}\leqslant C\int_r^{2k}\frac{1}{t^{\varepsilon+1}}\frac{t}{t-|z|}dt$$

$$\leqslant C\int_r^{2k}\frac{1}{t^{\varepsilon+1}}\frac{t}{t-\frac{r}{2}}dt\leqslant 2C\int_r^{2k}\frac{1}{t^{\varepsilon+1}}dt=-\frac{2C}{\varepsilon t^{\varepsilon}}|_r^{2k}<\frac{2C}{\varepsilon r^{\varepsilon}}.$$

Applying the Gauss theorem to the function $\frac{w(\zeta)}{\zeta-z}$, where $z\in\mathbb{H}$ is fixed and $\mathbb{H}_{R,\epsilon}=\mathbb{H}_R\backslash\{\zeta:|\zeta-z|\leqslant\varepsilon\}$ shows

$$\frac{1}{2\pi i}\int_{\partial\mathbb{H}_{R,\varepsilon}}\frac{w(\zeta)}{\zeta - z}d\zeta = \frac{1}{\pi}\int_{\mathbb{H}_{R,\varepsilon}}\frac{w_{\bar{z}}(\zeta)}{\zeta - z}d\xi d\eta.$$

Letting ε tends to 0, then

$$\frac{1}{2\pi i}\int_{\partial\mathbb{H}_R}\frac{w(\zeta)}{\zeta - z}d\zeta - w(z) = \frac{1}{\pi}\int_{\mathbb{H}_R}\frac{w_{\bar{\zeta}}(\zeta)}{\zeta - z}d\xi d\eta. \tag{2.6}$$

For $2|z| < R$

$$\left|\frac{1}{2\pi i}\int_0^\pi \frac{w(Re^{i\varphi})}{Re^{i\varphi} - z}Re^{i\varphi}d\varphi\right| \leqslant \frac{1}{2}\frac{M(R,w)}{R - |z|}R \leqslant M(R,w).$$

Assuming

$$\lim_{R\to+\infty} M(R,w) = 0$$

and as from $w \in W^{1,1}(\mathbb{H};\mathbb{C})$ the existence of

$$\frac{1}{\pi}\int_{\mathbb{H}} w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

follows, hence (2.4) holds and similarly (2.5).

Corollary 3 Under the assumption of Theorem 2

$$w(z) = -\frac{1}{\pi}\int_{0<Im\zeta}\frac{w(\zeta)}{(\bar{\zeta} - z)^2}d\xi d\eta - \frac{1}{\pi}\int_{0<Im\zeta}\partial_z G_1(z,\zeta)w_{\bar{\zeta}}(\zeta)d\xi d\eta. \tag{2.7}$$

Proof Introducing the harmonic Green function for $\mathbb{H}$

$$G_1(z,\zeta) = \log|\frac{\bar{\zeta} - z}{\zeta - z}|^2$$

for which

$$\partial_z G_1(z,\zeta) = \frac{1}{\zeta - z} - \frac{1}{\bar{\zeta} - z},$$

i.e.

$$\frac{1}{\zeta - z} = \partial_z G_1(z,\zeta) + \frac{1}{\bar{\zeta} - z},$$

then

$$\frac{1}{\pi}\int_{0<Im\zeta} w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\bar{\zeta}-z} = \frac{1}{\pi}\int_{0<Im\zeta}\{\partial_{\bar{\zeta}}(\frac{w(\zeta)}{\bar{\zeta}-z}) + \frac{w(\zeta)}{(\bar{\zeta}-z)^2}\}d\xi d\eta$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{w(t)}{t-z}dt + \frac{1}{\pi}\int_{0<Im\zeta} w(\zeta)\frac{d\xi d\eta}{(\bar{\zeta}-z)^2}.$$

Thus

$$w(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} w(t)\frac{dt}{t-z} - \frac{1}{\pi}\int_{0<Im\zeta}(\partial_z G_1(z,\zeta) + \frac{1}{\bar{\zeta}-z})w_{\bar{\zeta}}(\zeta)d\xi d\eta$$

$$= -\frac{1}{\pi}\int_{0<Im\zeta}\frac{w(\zeta)}{(\bar{\zeta}-z)^2}d\xi d\eta - \frac{1}{\pi}\int_{0<Im\zeta}\partial_z G_1(z,\zeta)w_{\bar{\zeta}}(\zeta)d\xi d\eta$$

$$= \varphi(z) + \psi(z).$$

This provides an orthogonal decomposition of w. The first term is obviously analytic in $\mathbb{H}$. To show the second term ψ being orthogonal to the set of analytic functions in $L_2(\mathbb{H};\mathbb{C})$ let ϕ be a function from this set, then

$$(\psi,\phi) = \int_{\mathbb{H}}\psi(z)\overline{\phi(z)}dxdy = -\frac{1}{\pi}\int_{\mathbb{H}}\int_{\mathbb{H}}\partial_z G_1(z,\zeta)\overline{\phi(z)}dxdy w_{\bar{\zeta}}(\zeta)d\xi d\eta,$$

$$(\partial_z G_1(\cdot,\zeta),\phi) = \int_{\mathbb{H}}\partial_z G_1(z,\zeta)\overline{\phi(z)}dxdy$$

$$= \int_H\{\partial_z[G_1(z,\zeta)\overline{\phi(z)}] - G_1(z,\zeta)\overline{\phi_{\bar{z}}(z)}\}dxdy$$

$$= -\frac{1}{2i}\int_{\partial\mathbb{H}} G_1(z,\zeta)\overline{\phi(z)}d\bar{z} = 0,$$

i.e.

$$(\partial_z G_1(\cdot,\zeta),\phi) = -(G_1(\cdot,\zeta),\partial_{\bar z}\phi) = 0.$$

Hence, $(\psi,\phi) = 0$.

2.3 Higher order Cauchy-Pompeiu representations and orthogonal decompositions

In order to find higher order Cauchy-Pompeiu representations the formulas (2.4) and (2.5) are iterated. One possibility for a second order representation is as follows. Consider

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{H}_R}\frac{w(\zeta)}{\zeta - z}d\zeta - \frac{1}{\pi}\int_{\mathbb{H}_R} w_{\bar{\tilde\zeta}}(\widetilde{\zeta})\frac{d\tilde\xi d\tilde\eta}{\tilde\zeta - z}$$

and

$$w_{\bar{\tilde\zeta}}(\tilde\zeta) = \frac{1}{2\pi i}\int_{\partial\mathbb{H}_R}\frac{w_{\bar\zeta}(\zeta)}{\zeta-\widetilde{\zeta}}d\zeta - \frac{1}{\pi}\int_{\mathbb{H}_R} w_{\bar\zeta\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta - \tilde\zeta},$$

so that

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{H}_R}\frac{w(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{\partial\mathbb{H}_R} w_{\bar\zeta}(\zeta)\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{d\tilde\xi d\tilde\eta}{(\zeta-\tilde\zeta)(\tilde\zeta - z)}d\zeta$$

$$+\frac{1}{\pi}\int_{\mathbb{H}_R} w_{\bar\zeta\bar\zeta}(\zeta)\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{d\tilde\xi d\tilde\eta}{(\zeta-\tilde\zeta)(\tilde\zeta - z)}d\xi d\eta.$$

Denoting

$$\psi(z,\zeta) = \frac{1}{\pi}\int_{\mathbb{H}_R}\frac{d\tilde\xi d\tilde\eta}{(\zeta-\tilde\zeta)(\tilde\zeta - z)}$$

and

$$\widetilde{\psi}(z,\zeta) = \frac{1}{2\pi i}\int_{\partial\mathbb{H}_R}\frac{\overline{\zeta - \widetilde{\zeta}}}{\zeta-\widetilde{\zeta}}\frac{d\widetilde{\zeta}}{\widetilde{\zeta} - z}$$

an application of (2.4) shows

$$\frac{\overline{\zeta - z}}{\zeta - z} = \frac{1}{2\pi i}\int_{\partial\mathbb{H}} \frac{\overline{\zeta - \widetilde{\zeta}}}{\zeta - \widetilde{\zeta}}\frac{d\widetilde{\zeta}}{\widetilde{\zeta} - z}$$

$$+ \frac{1}{\pi}\int_{\mathbb{H}_R} \frac{d\widetilde{\xi} d\widetilde{\eta}}{(\zeta - \widetilde{\zeta})(\widetilde{\zeta} - z)} = \widetilde{\psi}(z,\zeta) + \psi(z,\zeta) \qquad (2.8)$$

where

$$\partial_{\bar\zeta}\widetilde{\psi}(z,\zeta) = \frac{1}{2\pi i}\int_{\partial\mathbb{H}_R} \frac{d\widetilde{\zeta}}{(\zeta - \widetilde{\zeta})(\widetilde{\zeta} - z)} = \frac{1}{\zeta - z} - \frac{1}{\zeta - z} = 0.$$

Remark 4 The function $\frac{\overline{\zeta - z}}{\zeta - z}$ is weakly singular at $\zeta = z$ and satisfying the assumptions for the Gauss theorem only on $\mathbb{H} \setminus \{|\zeta - z| \le \zeta\}$. Applying the Gauss theorem there and letting ε tend to zero shows formula (2.4).

As

$$\frac{1}{\pi}\int_{\mathbb{H}_R} w_{\bar\zeta\bar\zeta}(\zeta)\widetilde{\psi}(z,\zeta)d\xi d\eta = \frac{1}{\pi}\int_{\mathbb{H}_R} \{\partial_{\bar\zeta}(w_{\bar\zeta}(\zeta)\widetilde{\psi}(z,\zeta)) - w_{\bar\zeta}(\zeta)\partial_{\bar\zeta}\widetilde{\psi}(z,\zeta)\}d\xi d\eta$$

$$= \frac{1}{2\pi i}\int_{\partial\mathbb{H}_R} w_{\bar\zeta}(\zeta)\widetilde{\psi}(z,\zeta)d\zeta$$

then

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{H}_R} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\partial\mathbb{H}_R} w_{\bar\zeta}(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\zeta$$

$$+\frac{1}{\pi}\int_{\mathbb{H}_R} w_{\bar\zeta\bar\zeta}(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\xi d\eta$$

follows. From the estimates

$$\left|\frac{1}{2\pi}\int_0^\pi w(Re^{i\varphi})\frac{Re^{i\varphi}\partial\varphi}{Re^{i\varphi} - z}\right| \leqslant 2M(R,w),$$

where $|z| < \frac{R}{2}$ and

$$\left|\frac{1}{2\pi i}\int_0^\pi w_{\bar\zeta}(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\zeta\right| \leqslant RM(w_{\bar z}, R)$$

the following result is seen.

Theorem 3 *Let $w \in W^{2,1}(\mathbb{H}, \mathbb{C})$ and*

$$\lim_{R\to+\infty} M(w, R) = 0$$

and

$$\lim_{R\to+\infty} RM(w_{\bar z}, R) = 0,$$

where $w_{\bar z} \in L_1(\mathbb{R}, \mathbb{C})$. Then

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{w(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{-\infty}^{\infty} w_{\bar\zeta}(\zeta)\frac{\zeta - \bar z}{\zeta - z}d\zeta \\ &+ \frac{1}{\pi}\int_{\mathbb{H}} w_{\bar\zeta\bar\zeta}(z)\frac{\overline{\zeta - z}}{\zeta - z}d\xi d\eta, \end{aligned} \tag{2.9}$$

where $z \in \mathbb{H}$.

The representation (2.5) can be transformed into an orthogonal decomposition of w. To this end consider the biharmonic Green function

$$G_2(z, \zeta) = |\zeta - z|^2 \log\left|\frac{\bar\zeta - z}{\zeta - z}\right|^2 + (z - \bar z)(\zeta - \bar\zeta)$$

satisfying

$$G_{2z}(z, \zeta) = -\overline{(\zeta - z)}\log\left|\frac{\bar\zeta - z}{\zeta - z}\right|^2 + \overline{(\zeta - z)}\frac{\bar\zeta - \zeta}{\bar\zeta - z} + (\zeta - \bar\zeta),$$

$$G_{2zz}(z, \zeta) = -\overline{(\zeta - z)}\frac{\bar\zeta - \zeta}{(\zeta - z)(\bar\zeta - z)} + \overline{(\zeta - z)}\frac{\bar\zeta - \zeta}{(\bar\zeta - z)^2}$$

$$= \frac{\overline{(\zeta - z)(\tilde{\zeta} - \zeta)}}{(\zeta - z)(\tilde{\zeta} - z)^2}(\zeta - z - \bar{\zeta} + z) = -\frac{\overline{(\zeta - z)}(\bar{\zeta} - \zeta)^2}{(\zeta - z)(\bar{\zeta} - z)^2}$$

$$= -\frac{\overline{\zeta - z}}{\zeta - z} - \frac{\overline{(\zeta - z)}(\bar{\zeta} - \zeta) - \overline{(\zeta - z)}(\bar{\zeta} - z)}{(\zeta - z)(\bar{\zeta} - z)} + \frac{\overline{(\zeta - z)}(\bar{\zeta} - \zeta)}{(\bar{\zeta} - z)^2}$$

$$= -\frac{\overline{\zeta - z}}{\zeta - z} + \frac{\overline{\zeta - z}}{\bar{\zeta} - z} + \frac{\overline{(\zeta - z)}(\bar{\zeta} - \zeta)}{(\bar{\zeta} - z)^2}.$$

Then

$$-\frac{\overline{\zeta - z}}{\zeta - z} = G_{2zz}(z, \zeta) + \frac{\overline{(\zeta - z)}(\bar{\zeta} - \zeta)^2}{(\zeta - z)(\bar{\zeta} - z)^2} - \frac{\overline{\zeta - z}}{\zeta - z}.$$

Corollary 4 Under the assumptions of Theorem 3

$$\begin{aligned} w(z) &= \frac{1}{\pi}\int_{\mathbb{H}} w(\zeta)\{4\frac{\bar{\zeta} - \zeta}{(\zeta - z)(\bar{\zeta} - z)^2} + 2\frac{\overline{\zeta - z}}{(\zeta - z)(\bar{\zeta} - z)^2} \\ &- 4\frac{\overline{(\zeta - z)}(\bar{\zeta} - \zeta)}{(\zeta - z)(\bar{\zeta} - z)^3} - 4\frac{(\bar{\zeta} - \zeta)^2}{(\zeta - z)(\bar{\zeta} - z)^3} \\ &+ 6\frac{\overline{(\zeta - z)}(\bar{\zeta} - \zeta)^2}{(\zeta - z)(\bar{\zeta} - z)^4}\}d\xi d\eta \\ &+ \frac{1}{\pi}\int_{\mathbb{H}} G_{2zz}(z, \zeta) w_{\bar{\zeta}\bar{\zeta}}(\zeta) d\xi d\eta. \end{aligned} \tag{2.10}$$

Proof Consider

$$\frac{1}{\pi}\int_{\mathbb{H}} w_{\bar{\zeta}\bar{\zeta}}(\zeta)\{\frac{\overline{(\zeta - z)}(\bar{\zeta} - \zeta)^2}{(\zeta - z)(\bar{\zeta} - z)^2} - \frac{\overline{\zeta - z}}{\zeta - z}\}d\xi d\eta$$

$$= \frac{1}{\pi}\int_{\mathbb{H}} \partial_{\bar{\zeta}}\{w_{\bar{\zeta}}(\zeta)\{\frac{\overline{(\zeta - z)}(\bar{\zeta} - \zeta)^2}{(\zeta - z)(\bar{\zeta} - z)^2} - \frac{\overline{\zeta - z}}{\zeta - z}\}\}$$

$$-w_{\bar{\zeta}}(\zeta)\{\frac{(\bar{\zeta} - \zeta)^2}{(\zeta - z)(\bar{\zeta} - z)^2} + \frac{2\overline{(\zeta - z)}(\bar{\zeta} - \zeta)}{(\zeta - z)(\bar{\zeta} - z)^2}$$

$$-2\frac{(\overline{\zeta - z})(\bar{\zeta} - \zeta)^2}{(\zeta - z)(\bar{\zeta} - z)^3} - \frac{1}{\zeta - z}\}d\xi d\eta$$

$$= \frac{1}{2\pi i}\int_{\partial\mathbb{H}} w_{\bar{\zeta}}(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\zeta$$

$$-\frac{1}{\pi}\int_{\mathbb{H}} \partial_{\bar{\zeta}}\{w(\zeta)\{\frac{(\bar{\zeta} - \zeta)^2}{(\zeta - z)(\bar{\zeta} - z)^2} + \frac{2(\overline{\zeta - z})(\bar{\zeta} - \zeta)}{(\zeta - z)(\bar{\zeta} - z)^2}$$

$$-2\frac{(\overline{\zeta - z})(\bar{\zeta} - \zeta)^2}{(\zeta - z)(\bar{\zeta} - z)^3} - \frac{1}{\zeta - z}\}$$

$$-w(\zeta)\{\frac{2(\bar{\zeta} - \zeta)}{(\zeta - z)(\bar{\zeta} - z)^2} - 2\frac{(\bar{\zeta} - \zeta)^2}{(\zeta - z)(\bar{\zeta} - z)^3} + 2\frac{(\bar{\zeta} - \zeta)}{(\zeta - z)(\bar{\zeta} - z)^2}$$

$$+2\frac{(\overline{\zeta - z})}{(\zeta - z)(\bar{\zeta} - z)^2} - 4\frac{(\overline{\zeta - z})(\bar{\zeta} - \zeta)}{(\zeta - z)(\bar{\zeta} - z)^3} - 2\frac{(\bar{\zeta} - \zeta)^2}{(\zeta - z)(\bar{\zeta} - z)^3}$$

$$-4\frac{(\overline{\zeta - z})(\bar{\zeta} - \zeta)}{(\zeta - z)(\bar{\zeta} - z)} + 6\frac{(\overline{\zeta - z})(\bar{\zeta} - \zeta)^2}{(\zeta - z)(\bar{\zeta} - z)^4}\}\}d\xi d\eta$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} w_{\bar{\zeta}}(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{w(\zeta)}{\zeta - z}d\zeta$$

$$+\frac{1}{\pi}\int_{\mathbb{H}} w(\zeta)\{4\frac{\bar{\zeta} - \zeta}{(\zeta - z)(\bar{\zeta} - z)^2} + 2\frac{\overline{\zeta - z}}{(\zeta - z)(\bar{\zeta} - z)^2}$$

$$-4\frac{(\overline{\zeta - z})(\bar{\zeta} - \zeta)}{(\zeta - z)(\bar{\zeta} - z)^3} - 4\frac{(\bar{\zeta} - \zeta)^2}{(\zeta - z)(\bar{\zeta} - z)^3} + 6\frac{(\overline{\zeta - z})(\bar{\zeta} - \zeta)^2}{(\zeta - z)(\bar{\zeta} - z)^4}\}.$$

This means (2.10).
Denoting the first term by φ the second by ψ obviously φ is a bianalytic function and ψ is orthogonal to the set of bianalytic functions in $L_2(\mathbb{H};\mathbb{C})$. To show the latter it is enough to show $(G_{2zzz}, \phi) = 0$ for any bianalytic ϕ in $L_2(\mathbb{H};\mathbb{C})$. But because G_2 and G_{2z} vanish for $z \in \mathbb{R}$ this implies

$$(G_{2zz}, \phi) = (G_2, \phi_{\bar{z}\bar{z}}) = 0.$$

Theorem 4 *Let for* $w \in W^{k,1}(\mathbb{H}, \mathbb{C})$, $\lim_{R\to+\infty} R^\nu M(\partial_{\bar{z}}^\nu w, R) = 0$, *where* $0 \leqslant \nu \leqslant k-1$. *And also* $\bar{z}^{k-2}\partial_{\bar{z}}^k w \in L^1(\mathbb{H}, \mathbb{C})$ *when* $k \geq 2$. *Then,*

$$\begin{aligned} w(z) &= \sum_{\nu=0}^{k-1} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\nu!} \frac{\overline{(z-\zeta)}^\nu}{\zeta - z} \partial_{\bar{\zeta}}^\nu w(\zeta) d\zeta \\ &- \frac{1}{\pi} \int_{\mathbb{H}} \frac{1}{(k-1)!} \frac{\overline{(z-\zeta)}^{k-1}}{\zeta - z} \partial_{\bar{\zeta}}^k w(\zeta) d\xi d\eta, \end{aligned} \tag{2.11}$$

where $z \in \mathbb{H}$.

Proof If $k = 1$, then

$$w(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\zeta - z} w(\zeta) d\zeta - \frac{1}{\pi} \int_{\mathbb{H}} \frac{1}{\zeta - z} \partial_{\bar{\zeta}} w(\zeta) d\xi d\eta.$$

Let again $\mathbb{H}_R = \mathbb{H} \bigcap \{|z| < R\}$. Then assuming for $z \in \mathbb{H}_R$

$$\begin{aligned} w(z) &= \sum_{\nu=0}^{k-1} \frac{1}{2\pi i} \int_{\partial\mathbb{H}_R} \frac{1}{\nu!} \frac{\overline{(z-\zeta)}^\nu}{\zeta - z} \partial_{\bar{\zeta}}^\nu w(\zeta) d\zeta \\ &- \frac{1}{\pi} \int_{\mathbb{H}_R} \frac{1}{(k-1)!} \frac{\overline{(z-\zeta)}^{k-1}}{\zeta - z} \partial_{\bar{\zeta}}^k w(\zeta) d\xi d\eta \end{aligned} \tag{2.12}$$

to hold, this formula will be proved for $k+1$ instead of k. An application of (2.2) gives for $\zeta \in \mathbb{H}_R$

$$\partial_{\bar{\zeta}}^k w(\zeta) = \frac{1}{2\pi i} \int_{\partial\mathbb{H}_R} \frac{1}{\tilde{\zeta} - \zeta} \partial_{\bar{\tilde{\zeta}}}^k w(\tilde{\zeta}) d\tilde{\zeta} - \frac{1}{\pi} \int_{\mathbb{H}_R} \frac{1}{\tilde{\zeta} - \zeta} \partial_{\bar{\tilde{\zeta}}}^{k+1} w(\tilde{\zeta}) d\tilde{\xi} d\tilde{\eta}.$$

Inserting this in the preceding formula shows

$$
\begin{aligned}
w(z) &= \sum_{\nu=0}^{k-1} \frac{1}{2\pi i}\int_{\partial\mathbb{H}_R} \frac{1}{\nu!}\frac{\overline{(z-\zeta)}}{\zeta - z}\partial_{\bar{\zeta}}^{\nu} w(\zeta)d\zeta \\
&- \frac{1}{2\pi i}\int_{\partial\mathbb{H}_R} \partial_{\widetilde{\zeta}}^{k} w(\widetilde{\zeta})\frac{1}{\pi}\int_{\mathbb{H}_R} \frac{1}{(k-1)!}\frac{\overline{(z-\zeta)}^{k-1}}{\zeta - z} \\
&\times \frac{1}{\widetilde{\zeta}-\zeta}d\xi d\eta d\widetilde{\zeta} + \frac{1}{\pi}\int_{\mathbb{H}_R} \partial_{\widetilde{\zeta}}^{k+1} w(\widetilde{\zeta}) \\
&\times \frac{1}{\pi}\int_{\mathbb{H}_R} \frac{1}{(k-1)!}\frac{\overline{(z-\zeta)}^{k-1}}{\zeta - z}\frac{1}{\widetilde{\zeta}-\zeta}d\xi d\eta d\widetilde{\xi} d\widetilde{\eta}. \qquad (2.13)
\end{aligned}
$$

Let

$$
\varphi(z,\widetilde{\zeta}) = \frac{1}{\pi}\int_{\mathbb{H}_R} \frac{1}{(k-1)!}\frac{\overline{(z-\zeta)}^{k-1}}{\zeta - z}\frac{1}{\widetilde{\zeta}-\zeta}d\xi d\eta.
$$

Then, similarly to (2.8)

$$
\begin{aligned}
\frac{1}{k!}\frac{\overline{(z-\widetilde{\zeta})}^{k}}{\tilde{\zeta}-z} &= \frac{1}{2\pi i}\int_{\partial\mathbb{H}_R} \frac{1}{k!}\frac{\overline{(z-\zeta)}^{k}}{\zeta - z}\frac{d\zeta}{\zeta-\widetilde{\zeta}} \\
&+ \frac{1}{\pi}\int_{\mathbb{H}_R} \frac{1}{(k-1)!}\frac{\overline{(z-\zeta)}^{k-1}}{\zeta - z}\frac{d\xi d\eta}{\zeta-\widetilde{\zeta}} \\
&= \widetilde{\psi}(z,\widetilde{\zeta}) - \varphi(z,\widetilde{\zeta}).
\end{aligned}
$$

As obviously $\partial_{\widetilde{\zeta}}\widetilde{\psi}(z,\widetilde{\zeta}) = 0$, then

$$
\frac{1}{\pi}\int_{\mathbb{H}_R} \partial_{\widetilde{\zeta}}^{k} w(\widetilde{\zeta})\widetilde{\psi}(z,\widetilde{\zeta})d\widetilde{\xi} d\widetilde{\eta} = \frac{1}{\pi}\int_{\mathbb{H}_R} \partial_{\widetilde{\zeta}}\{\partial_{\widetilde{\zeta}}^{k-1} w(\widetilde{\zeta})\widetilde{\psi}(z,\widetilde{\zeta})\}d\widetilde{\xi} d\widetilde{\eta}
$$

$$
= \frac{1}{2\pi i}\int_{\partial\mathbb{H}_R} \partial_{\widetilde{\zeta}}^{k-1} w(\widetilde{\zeta})\widetilde{\psi}(z,\widetilde{\zeta})d\widetilde{\zeta}.
$$

Adding

$$\frac{1}{2\pi i}\int_{\partial\mathbb{H}_R}\partial_{\bar{\widetilde{\zeta}}}^{k-1}w(\widetilde{\zeta})\widetilde{\psi}(z,\widetilde{\zeta})d\widetilde{\zeta}-\frac{1}{\pi}\int_{\mathbb{H}_R}\partial_{\bar{\widetilde{\zeta}}}^{k}w(\widetilde{\zeta})\widetilde{\psi}(z,\widetilde{\zeta})d\widetilde{\zeta}=0$$

to (2.13) shows

$$\begin{aligned} w(z) &= \sum_{\nu=0}^{k}\frac{1}{2\pi i}\int_{\partial\mathbb{H}_R}\frac{1}{\nu!}\frac{(\overline{z-\zeta})^{\nu}}{\zeta-z}\partial_{\bar{\zeta}}^{\nu}w(\zeta)d\zeta \\ &+ \frac{1}{2\pi i}\int_{\partial\mathbb{H}_R}\frac{1}{k!}\frac{(\overline{z-\zeta})^{k}}{\zeta-z}\partial_{\bar{\zeta}}^{k}w(\zeta)d\zeta \\ &- \frac{1}{\pi}\int_{\mathbb{H}_R}\frac{1}{k!}\frac{(\overline{z-\zeta})^{k}}{\zeta-z}\partial_{\bar{\zeta}}^{k+1}w(\zeta)d\xi d\eta \end{aligned}$$

which is (2.11) for $k+1$.
From the estimate

$$\left|\frac{1}{2\pi i}\int_{|\zeta|=R,0<Im\zeta}\frac{1}{\nu!}\frac{\overline{(z-\zeta)}^{\nu}}{\zeta-z}\partial_{\bar{\zeta}}^{\nu}w(\zeta)d\zeta\right|\leq\frac{1}{\nu!}(R+|z|)^{\nu-1}M(R,\partial_{\bar{z}}^{\nu}w)R$$

it follows that this tends to zero for $\mathbb{R}$ tending to ∞. As also

$$\int_{\mathbb{H}}\frac{1}{(k-1)!}\frac{\overline{(z-\zeta)}^{k-1}}{\zeta-z}\partial_{\bar{\zeta}}^{k}w(\zeta)d\xi d\eta$$

exists by the respective assumption (2.11) follows from (2.12).

Theorem 5 *Let* $w\in W^{k,1}(\mathbb{H},\mathbb{C})$ *satisfy* $\lim_{R\to+\infty}R^{\nu}M(R,\partial_{\bar{z}}^{\nu}w)=0$, *where* $0\leqslant\nu\leqslant k-1$ *and* $\bar{z}^{k-2}\partial_{\bar{z}}^{k}w\in L^{1}(\mathbb{H};\mathbb{C})$. *Then,*

$$\begin{aligned} w(z) &= -\sum_{\nu=0}^{k-1}\frac{1}{\pi}\int_{\mathbb{H}}\frac{k}{\nu!}\frac{(\bar{\zeta}-\zeta)^{k-1}(\overline{z-\zeta})^{\nu}}{(\bar{\zeta}-z)^{k+1}}\partial_{\bar{\zeta}}^{\nu}w(\zeta)d\xi d\eta \\ &- \frac{1}{\pi}\int_{\mathbb{H}}\frac{1}{(k-1)!^{2}}\partial_{\zeta}^{k}G_{k}(z,\zeta)\partial_{\bar{\zeta}}^{k}w(\zeta)d\xi d\eta \end{aligned}$$

Proof. From the above calculation, see Chapter 1

$$\partial_z^k G_k(z,\zeta) = (k-1)!\frac{(\overline{z-\zeta})^{k-1}}{(\bar{\zeta}-z)^k}\frac{(\bar{\zeta}-\zeta)^k}{\zeta - z}.$$

Applying the Gauss theorem we have

$$\frac{1}{\pi}\int_{\mathbb{H}_R}\left(\frac{1}{(k-1)!}\frac{(\overline{z-\zeta})^{k-1}}{\zeta-z} - \frac{1}{(k-1)!}\frac{(\overline{z-\zeta})^{k-1}}{(\bar{\zeta}-z)^k}\frac{(\bar{\zeta}-\zeta)^k}{\zeta-z}\right)\partial_{\bar{\zeta}}^k w(\zeta)d\xi d\eta$$

$$= \frac{1}{\pi}\int_{\mathbb{H}_R}\frac{1}{(k-1)!}\frac{(\overline{z-\zeta})^{k-1}}{\zeta-z}\left(1-\frac{(\bar{\zeta}-\zeta)^k}{(\bar{\zeta}-z)^k}\right)\partial_{\bar{\zeta}}^k w(\zeta)d\xi d\eta$$

$$= \sum_{\nu=0}^{k-1}\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{1}{(k-1)!}\frac{(\overline{z-\zeta})^{k-1}}{\zeta-z}\frac{(\bar{\zeta}-\zeta)^\nu}{(\bar{\zeta}-z)^\nu}\left(1-\frac{\bar{\zeta}-\zeta}{\bar{\zeta}-z}\right)\partial_{\bar{\zeta}}^k w(\zeta)d\xi d\eta$$

$$= \sum_{\nu=0}^{k-1}\frac{1}{\pi}\int_{\mathbb{H}_R}\{\partial_{\bar{\zeta}}\left(\frac{1}{(k-1)!}\frac{(\overline{z-\zeta})^{k-1}(\bar{\zeta}-\zeta)^\nu}{(\bar{\zeta}-z)^{\nu+1}}\partial_{\bar{\zeta}}^{k-1}w(\zeta)\right)$$

$$-\{-\frac{(\overline{z-\zeta})^{k-2}(\bar{\zeta}-\zeta)^\nu}{(k-2)!(\bar{\zeta}-z)^{\nu+1}} + \frac{\nu(\overline{z-\zeta})^{k-1}(\bar{\zeta}-\zeta)^{\nu-1}}{(k-1)!(\bar{\zeta}-z)^{\nu+1}}$$

$$-\frac{(\nu+1)(\overline{z-\zeta})^{k-1}(\bar{\zeta}-\zeta)^\nu}{(k-1)!(\bar{\zeta}-z)^{\nu+2}}\}\partial_{\bar{\zeta}}^{k-1}w(\zeta)\}d\xi d\eta$$

$$= \sum_{\nu=0}^{k-1}\frac{1}{2\pi i}\int_{\partial\mathbb{H}_R}\frac{1}{(k-1)!}\frac{(\overline{z-\zeta})^{k-1}(\bar{\zeta}-\zeta)^\nu}{(\bar{\zeta}-z)^{\nu+1}}\partial_{\bar{\zeta}}^{k-1}w(\zeta)d\zeta$$

$$+\sum_{\nu=0}^{k-1}\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{(\overline{z-\zeta})^{k-2}(\bar{\zeta}-\zeta)^\nu}{(k-2)!(\bar{\zeta}-z)^{\nu+1}}\partial_{\tilde{\zeta}}^{k-1}w(\zeta)d\xi d\eta$$

$$-\sum_{\nu=0}^{k-2}\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{(\nu+1)(\overline{z-\zeta})^{k-1}(\bar{\zeta}-\zeta)^\nu}{(k-1)!(\zeta-z)^{\nu+2}}\partial_{\bar{\zeta}}^{k-1}w(\zeta)d\xi d\eta$$

$$+\sum_{\nu=0}^{k-1}\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{(\nu+1)(\overline{z-\zeta})^{k-1}(\bar\zeta-\zeta)^{\nu}}{(k-1)!(\bar\zeta-z)^{\nu+2}}\partial_{\bar\zeta}^{k-1}w(\zeta)d\xi d\eta$$

$$=\frac{1}{2\pi i}\int_{-R}^{R}\frac{1}{(k-1)!}\frac{(\overline{z-\zeta})^{k-1}}{(\bar\zeta-z)}\partial_{\bar\zeta}^{k-1}w(\zeta)d\zeta$$

$$+\sum_{\nu=0}^{k-1}\frac{1}{2\pi i}\int_{|\zeta|=R,0<Im\zeta}\frac{1}{(k-1)!}\frac{(\overline{z-\zeta})^{k-1}(\bar\zeta-\zeta)^{\nu}}{(\bar\zeta-z)^{\nu+1}}\partial_{\bar\zeta}^{k-1}w(\zeta)d\zeta$$

$$+\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{k(\overline{z-\zeta})^{k-1}(\bar\zeta-\zeta)^{k-1}}{(k-1)!(\bar\zeta-z)^{k+1}}\partial_{\bar\zeta}^{k-1}w(\zeta)d\xi d\eta$$

$$+\sum_{\nu=0}^{k-1}\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{(\overline{z-\zeta})^{k-2}(\bar\zeta-\zeta)^{\nu}}{(k-2)!(\bar\zeta-z)^{\nu+1}}\partial_{\bar\zeta}^{k-1}w(\zeta)d\xi d\eta.$$

Now we calculate the last term in the last formula. It is equal to

$$\sum_{\nu=0}^{k-1}\frac{1}{\pi}\int_{\mathbb{H}_R}\{\partial_{\bar\zeta}\{\frac{(\overline{z-\zeta})^{k-2}(\bar\zeta-\zeta)^{\nu}}{(k-2)!(\bar\zeta-z)^{\nu+1}}\partial_{\bar\zeta}^{k-2}w(\zeta)\}$$

$$-\{-\frac{(\overline{z-\zeta})^{k-3}(\bar\zeta-\zeta)^{\nu}}{(k-3)!(\bar\zeta-z)^{\nu+1}}+\frac{\nu(\overline{z-\zeta})^{k-2}(\bar\zeta-\zeta)^{\nu-1}}{(k-2)!(\bar\zeta-z)^{\nu+1}}$$

$$-\frac{(\nu+1)(\overline{z-\zeta})^{k-2}(\bar\zeta-z)^{\nu}}{(k-2)!(\bar\zeta-z)^{\nu+2}}\}\partial_{\bar\zeta}^{k-2}w(\zeta)\}d\xi d\eta$$

$$=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{(\bar z-\zeta)^{k-2}}{(k-2)!(\bar\zeta-z)}\partial_{\bar\zeta}^{k-2}w(\zeta)d\zeta$$

$$+\sum_{\nu=0}^{k-1}\frac{1}{2\pi i}\int_{|\zeta|=R,0<Im\zeta}\frac{(\overline{z-\zeta})^{k-2}(\bar\zeta-\zeta)^{\nu}}{(k-2)!(\bar\zeta-z)^{\nu+1}}\partial_{\bar\zeta}^{k-2}w(\zeta)d\zeta$$

$$+\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{k(\overline{z-\zeta})^{k-2}(\bar\zeta-\zeta)^{k-1}}{(k-2)!(\bar\zeta-z)^{k+1}}\partial_{\bar\zeta}^{k-2}w(\zeta)d\xi d\eta$$

$$+\sum_{\nu=0}^{k-1}\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{(\overline{z-\zeta})^{k-3}(\bar\zeta-\zeta)^{\nu}}{(k-3)!(\bar\zeta-z)^{\nu+1}}\partial_{\bar\zeta}^{k-2}w(\zeta)d\xi d\eta.$$

Continuing using induction at last this becomes

$$\sum_{\nu=0}^{k-1}\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{(\bar\zeta-\zeta)^{\nu}}{(\bar\zeta-z)^{\nu+1}}\partial_{\bar\zeta}w(\zeta)d\xi d\eta=\sum_{\nu=0}^{k-1}\frac{1}{\pi}\int_{\mathbb{H}_R}\{\partial_{\bar\zeta}\left(\frac{(\bar\zeta-\zeta)^{\nu}}{(\bar\zeta-z)^{\nu+1}}w(\zeta)\right)$$

$$-\left(\frac{\nu(\bar\zeta-\zeta)^{\nu-1}}{(\bar\zeta-z)^{\nu+1}}-\frac{(\nu+1)(\bar\zeta-\zeta)^{\nu}}{(\bar\zeta-z)^{\nu+2}}\right)w(\zeta)\}d\xi d\eta$$

$$=\sum_{\nu=0}^{k-1}\frac{1}{2\pi i}\int_{\partial\mathbb{H}_R}\frac{(\bar\zeta-\zeta)^{\nu}}{(\bar\zeta-z)^{\nu+1}}w(\zeta)d\zeta$$

$$-\sum_{\nu=0}^{k-2}\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{(\nu+1)(\bar\zeta-\zeta)^{\nu}}{(\bar\zeta-z)^{\nu+2}}w(\zeta)d\xi d\eta$$

$$+\sum_{\nu=0}^{k-1}\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{(\nu+1)(\bar\zeta-\zeta)^{\nu}}{(\bar\zeta-z)^{\nu+2}}w(\zeta)d\xi d\eta$$

$$=\frac{1}{2\pi i}\int_{-R}^{R}\frac{w(\zeta)}{\bar\zeta-z}d\zeta+\sum_{\nu=0}^{k-1}\frac{1}{2\pi i}\int_{|\zeta|=R,0<Im\zeta}\frac{(\bar\zeta-\zeta)^{\nu}}{(\bar\zeta-z)^{\nu+1}}w(\zeta)d\zeta$$

$$+\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{k(\bar\zeta-\zeta)^{k-1}}{(\bar\zeta-z)^{k+1}}w(\zeta)d\xi d\eta.$$

Hence,

$$\frac{1}{\pi}\int_{\mathbb{H}_R}\left(\frac{1}{(k-1)!}\frac{(\overline{z-\zeta})^{k-1}}{\zeta-z}-\frac{1}{(k-1)!}\frac{(\overline{z-\zeta})^{k-1}(\bar\zeta-\zeta)^{k}}{(\bar\zeta-z)^{k}(\zeta-z)}\right)\partial_{\bar\zeta}^{k}w(\zeta)d\xi d\eta$$

$$=\sum_{\nu=0}^{k-1}\frac{1}{2\pi i}\int_{-R}^{R}\frac{1}{\nu!}\frac{(\overline{z-\zeta})^{\nu}}{\bar\zeta-z}\partial_{\bar\zeta}^{\nu}w(\zeta)d\zeta$$

$$+\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}\frac{1}{2\pi i}\int_{\substack{|\zeta|=R\\ Im\zeta>0}}\frac{(\overline{z-\zeta})^{\mu}}{\mu!}\frac{(\bar\zeta-\zeta)^{\nu}}{(\bar\zeta-z)^{\nu+1}}\partial_{\bar\zeta}^{\mu}w(\zeta)d\zeta$$

$$+\sum_{\mu=0}^{k-1}\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{k(\overline{z-\zeta})^{\mu}}{\mu!}\frac{(\bar\zeta-\zeta)^{k-1}}{(\bar\zeta-z)^{k+1}}\partial_{\bar\zeta}^{\mu}w(\zeta)d\xi d\eta.$$

Thus we have

$$-\sum_{\nu=0}^{k-1}\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{k}{\nu!}\frac{(\bar\zeta-\zeta)^{k-1}(\overline{z-\zeta})^{\nu}}{(\bar\zeta-z)^{k+1}}\partial_{\bar\zeta}^{\nu}w(\zeta)d\xi d\eta$$

$$-\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{1}{(k-1)!^2}\partial_{\zeta}^{k}G_k(z,\zeta)\partial_{\bar\zeta}^{k}w(\zeta)d\xi d\eta$$

$$=-\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{1}{(k-1)!}\frac{(\overline{z-\zeta})^{k-1}}{\zeta-z}\partial_{\bar\zeta}^{k}w(\zeta)d\xi d\eta$$

$$+\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}\frac{1}{2\pi i}\int_{\substack{|\zeta|=R\\ Im\zeta>0}}\frac{(\overline{z-\zeta})^{\mu}}{\mu!}\frac{(\bar\zeta-\zeta)^{\nu}}{(\bar\zeta-z)^{\nu+1}}\partial_{\bar\zeta}^{\mu}w(\zeta)d\zeta$$

$$+\sum_{\nu=0}^{k-1}\frac{1}{2\pi i}\int_{-R}^{R}\frac{1}{\nu!}\frac{(\overline{z-\zeta})^{\nu}}{\bar\zeta-z}\partial_{\bar\zeta}^{\nu}w(\zeta)d\zeta$$

$$=-\frac{1}{\pi}\int_{\mathbb{H}_R}\frac{1}{(k-1)!}\frac{(\overline{z-\zeta})^{k-1}}{\zeta-z}\partial_{\bar\zeta}^{k}w(\zeta)d\xi d\eta$$

$$+\sum_{\nu=0}^{k-1}\frac{1}{2\pi i}\int_{-R}^{R}\frac{1}{\nu!}\frac{(\overline{z-\zeta})^{\nu}}{\zeta-z}\partial_{\bar\zeta}^{\nu}w(\zeta)d\zeta+o(1).$$

The right-hand side of the last formula tends to $w(z)$ as R tends to infinity, see Theorem 4. This proves Theorem 5.

Corollary 5 Any $w\in W^{k,1}(\mathbb{H};\mathbb{C})$ satisfying $\bar z^{k-2}\partial_{\bar z}^{k}w\in L^1(\mathbb{H};\mathbb{C})$ and $\lim_{R\to\infty}R^{\nu}M(R,\partial_{\bar z}^{\nu}w)=0$ for $0\le\nu\le k-1$ is projected by

$$-\sum_{\nu=0}^{k-1}\frac{1}{\pi}\int_{\mathbb{H}}\frac{k}{\nu!}\frac{(\overline{z-\zeta})^{\nu}(\bar\zeta-\zeta)^{k-1}}{(\bar\zeta-z)^{k+1}}\partial_{\bar\zeta}^{\nu}w(\zeta)d\xi d\eta \tag{2.14}$$

into the set of polyanalytic functions in $\mathbb{H}$ of order at most k, $\mathcal{O}_{2,k}(\mathbb{H})$, and by

$$-\frac{1}{\pi}\int_{\mathbb{H}}\frac{1}{(k-1)!^2}\partial_z^k G_k(z,\zeta)\partial_{\bar\zeta}^k w(\zeta)d\xi d\eta \tag{2.15}$$

into the subset of $L_2(\mathbb{H};\mathbb{C})$ functions orthogonal to $\mathcal{O}_{2,k}(\mathbb{H})$.

Proof As (2.12) obviously is a polynomial in $\bar z$ of degree at most $k-1$ with analytic coefficients it remains to prove the second statement. This then follows from the boundary behaviour of $\partial_z^\nu G_k(z,\zeta)$ for $0 \le \nu \le k-1$ from what

$$(\partial_z^k G_k(z,\zeta),\phi(z)) = (-1)^k(G_k(z,\zeta)\partial_{\bar z}^k\phi(z)) = 0$$

follows for any $\phi \in \mathcal{O}_{2,k}(\mathbb{H})$.

Lemma 7 *Any* $w \in C^{2n}(\mathbb{H};\mathbb{C})\cap C^{2n-1}(\overline{\mathbb{H}};\mathbb{C})$ *for which* $|z|^{2\nu+\delta}(\partial_z\partial_{\bar z})^\nu w$ *for* $0 \leqslant \nu \leqslant n$, $|z|^{2\nu+1+\delta}\partial_z^\nu\partial_{\bar z}^{\nu+1}w$, $|z|^{2\nu+1+\delta}\partial_z^{\nu+1}\partial_{\bar z}^\nu w$ *for* $0 \leqslant \nu \leqslant n-1$ *are bounded in* $\mathbb{H}$ *satisfies*

$$\begin{aligned}
&\frac{2}{\pi}\int_{\mathbb{H}} G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^n w(\zeta)d\xi d\eta \\
&= \sum_{\nu=0}^{k}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{(\partial_\zeta\partial_{\bar\zeta})^\nu G_n(z,\zeta)[\partial_\zeta^{n-\nu}\partial_{\bar\zeta}^{n-\nu-1}w(\zeta)d\zeta \\
&-\partial_\zeta^{n-\nu-1}\partial_{\bar\zeta}^{n-\nu}w(\zeta)d\bar\zeta] - (\partial_\zeta\partial_{\bar\zeta})^{n-\nu-1}w(\zeta) \\
&\times[\partial_\zeta^{\nu+1}\partial_{\bar\zeta}^{\nu}G_n(z,\zeta)d\zeta - \partial_\zeta^{\nu}\partial_{\bar\zeta}^{\nu+1}G_n(z,\zeta)d\bar\zeta]\} \\
&+\frac{2}{\pi}\int_{\mathbb{H}}(\partial_\zeta\partial_{\bar\zeta})^{k+1}G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^{n-k-1}w(\zeta)d\xi d\eta
\end{aligned} \tag{2.16}$$

Proof
1) If $k=0$, then

$$\frac{2}{\pi}\int_{\mathbb{H}} G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^n w(\zeta)d\xi d\eta = \frac{1}{\pi}\int_{\mathbb{H}}\{\partial_{\bar\zeta}[G_n(z,\zeta)\partial_\zeta^n\partial_{\bar\zeta}^{n-1}w(\zeta)]$$

$$+\partial_\zeta[G_n(z,\zeta)\partial_\zeta^{n-1}\partial_{\bar\zeta}^{n}w(\zeta)] - \partial_{\bar\zeta}G_n(z,\zeta)\partial_\zeta^n\partial_{\bar\zeta}^{n-1}w(\zeta)$$

$$-\partial_\zeta G_n(z,\zeta)\partial_\zeta^{n-1}\partial_{\bar\zeta}^n w(\zeta)\}d\xi d\eta = \frac{1}{\pi}\int_{\mathbb{H}}\{\partial_{\bar\zeta}[G_n(z,\zeta)\partial_\zeta^n\partial_{\bar\zeta}^{n-1}w(\zeta)$$

$$-\partial_\zeta G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^{n-1}w(\zeta)] + \partial_\zeta[G_n(z,\zeta)\partial_\zeta^{n-1}\partial_{\bar\zeta}^n w(\zeta)$$

$$-\partial_{\bar\zeta}G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^{n-1}w(\zeta)] + 2\partial_\zeta\partial_{\bar\zeta}G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^{n-1}w(\zeta)\}d\xi d\eta$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\{G_n(z,\zeta)[\partial_\zeta^n\partial_{\bar\zeta}^{n-1}w(\zeta)d\zeta - \partial_\zeta^{n-1}\partial_{\bar\zeta}^n w(\zeta)d\bar\zeta]$$

$$-(\partial_\zeta\partial_{\bar\zeta})^{n-1}w(\zeta)[\partial_\zeta G_n(z,\zeta)d\zeta - \partial_{\bar\zeta}G_n(z,\zeta)d\bar\zeta]\}$$

$$+\frac{2}{\pi}\int_{\mathbb{H}}(\partial_\zeta\partial_{\bar\zeta})G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^{n-1}w(\zeta)d\xi d\eta.$$

2) Assume (2.16) holds for k. Then using

$$\frac{2}{\pi}\int_{\mathbb{H}}(\partial_\zeta\partial_{\bar\zeta})^{k+1}G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^{n-k-1}w(\zeta)d\xi d\eta$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\partial_\zeta\partial_{\bar\zeta})^{k+1}G_n(z,\zeta)[\partial_\zeta^{n-k-1}\partial_{\bar\zeta}^{n-k-2}w(\zeta)d\zeta$$

$$-\partial_\zeta^{n-k-2}\partial_{\bar\zeta}^{n-k-1}w(\zeta)d\bar\zeta] - (\partial_\zeta\partial_{\bar\zeta})^{n-k-2}w(\zeta)[\partial_\zeta^{k+2}\partial_{\bar\zeta}^{k+1}$$

$$\times G_n(z,\zeta)d\zeta - \partial_\zeta^{k+1}\partial_{\bar\zeta}^{k+2}G_n(z,\zeta)d\bar\zeta] + \frac{2}{\pi}\int_{\mathbb{H}}(\partial_\zeta\partial_{\bar\zeta})^{k+2}$$

$$\times G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^{n-k-2}w(\zeta)d\xi d\eta$$

(2.16) follows for $k+1$.

Corollary 6 Under the preceding assumptions

$$\frac{2}{\pi}\int_{\mathbb{H}} G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^n w(\zeta)d\xi d\eta$$

$$=\sum_{\nu=0}^{n-2}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{(\partial_\zeta\partial_{\bar\zeta})^\nu G_n(z,\zeta)[\partial_\zeta^{n-\nu}\partial_{\bar\zeta}^{n-\nu-1}w(\zeta)d\zeta$$

$$-\partial_\zeta^{n-\nu-1}\partial_{\bar\zeta}^{n-\nu}w(\zeta)d\bar\zeta]-(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-1}w(\zeta)$$

$$\times[\partial_\zeta^{\nu+1}\partial_{\bar\zeta}^{\nu}G_n(z,\zeta)d\zeta-\partial_\zeta^{\nu}\partial_{\bar\zeta}^{\nu+1}G_n(z,\zeta)d\bar\zeta]\}$$

$$+\frac{2}{\pi}\int_{\mathbb{H}}(\partial_\zeta\partial_{\bar\zeta})^{n-1}G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})w(\zeta)d\xi d\eta$$

Lemma 8 *Any* $w\in C^2(\mathbb{H};\mathbb{C})\cap C^1(\overline{\mathbb{H}};\mathbb{C})$ *for which* $|z|^{2+\delta}w_{z\bar z}$, $|z|^{1+\delta}w_z$, $|z|^{1+\delta}w_{\bar z}$ *are bounded in* $\mathbb{H}$ *can be represented as*

$$\frac{2}{\pi}\int_{\mathbb{H}}(\partial_\zeta\partial_{\bar\zeta})^{n-1}G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})w(\zeta)d\xi d\eta$$

$$=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{(\partial_\zeta\partial_{\bar\zeta})^{n-1}G_n(z,\zeta)[\partial_\zeta w(\zeta)d\zeta-\partial_{\bar\zeta}w(\zeta)d\bar\zeta]$$

$$+w(\zeta)[\partial_\zeta^{n}\partial_{\bar\zeta}^{n-1}G_n(z,\zeta)d\zeta-\partial_\zeta^{n-1}\partial_{\bar\zeta}^{n}G_n(z,\zeta)d\bar\zeta]\}-2w(z).$$

Proof

$$\frac{2}{\pi}\int_{\mathbb{H}}(\partial_\zeta\partial_{\bar\zeta})^{n-1}G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})w(\zeta)d\xi d\eta$$

$$=\frac{1}{\pi}\int_{\mathbb{H}}\{\partial_{\bar\zeta}[(\partial_\zeta\partial_{\bar\zeta})^{n-1}G_n(z,\zeta)\partial_\zeta w(\zeta)]+\partial_\zeta[(\partial_\zeta\partial_{\bar\zeta})^{n-1}$$

$$\times G_n(z,\zeta)\partial_{\bar\zeta}w(\zeta)]-\partial_\zeta^{n-1}\partial_{\bar\zeta}^{n}G_n(z,\zeta)\partial_\zeta w(\zeta)-\partial_\zeta^{n}\partial_{\bar\zeta}^{n-1}$$

$$\times G_n(z,\zeta)\partial_{\bar\zeta}w(\zeta)\}d\xi d\eta=\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\partial_\zeta\partial_{\bar\zeta})^{n-1}G_n(z,\zeta)$$

$$\times[\partial_\zeta w(\zeta)d\zeta - \partial_{\bar\zeta} w(\zeta)d\bar\zeta] - \frac{1}{\pi}\int_{\mathbb{H}} \{(-1)^n \left(\frac{z-\bar z}{\bar z - \zeta}\right)^n$$

$$\times \frac{1}{z-\zeta}\partial_{\bar\zeta} w(\zeta) + (-1)^n \left(\frac{\bar z - z}{z - \bar\zeta}\right)^n \frac{1}{\overline{\overline{z-\zeta}}}\partial_\zeta w(\zeta)\} d\xi d\eta,$$

where

$$\frac{1}{\pi}\int_{\mathbb{H}} (-1)^n \left(\frac{z-\bar z}{\bar z - \zeta}\right)^n \frac{1}{z-\zeta}\partial_{\bar\zeta} w(\zeta) d\xi d\eta = w(z)$$

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty} (-1)^n \left(\frac{z-\bar z}{\bar z - \zeta}\right)^n \frac{w(\zeta)}{\zeta - z} d\zeta$$

and

$$\frac{1}{\pi}\int_{\mathbb{H}} (-1)^n \left(\frac{\bar z - z}{z - \bar\zeta}\right)^n \frac{1}{\overline{\overline{z-\zeta}}}\partial_\zeta w(\zeta) d\xi d\eta = w(z)$$

$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty} (-1)^n \left(\frac{\bar z - z}{z - \bar\zeta}\right)^n \frac{\overline{\overline{w(\zeta)}}}{\overline{\overline{\zeta - z}}} d\bar\zeta.$$

Theorem 6 *Any* $w \in C^{2n}(\mathbb{H};\mathbb{C}) \cap C^{2n-1}(\overline{\mathbb{H}};\mathbb{C})$ *for which* $|z|^{2\nu+\delta}(\partial_z\partial_{\bar z})^\nu w$ *for* $0 \leqslant \nu \leqslant n$, $|z|^{2\nu+1+\delta}\partial_z^\nu \partial_{\bar z}^{\nu+1} w$, $|z|^{2\nu+1+\delta}\partial_z^{\nu+1}\partial_{\bar z}^\nu w$ *for* $0 \leqslant \nu \leqslant n-1$ *are bounded in* $\mathbb{H}$ *can be represented as*

$$w(z) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} [\partial_\zeta^n \partial_{\bar\zeta}^{n-1} G_n(z,t) - \partial_\zeta^{n-1}\partial_{\bar\zeta}^n G_n(z,t)]$$

$$\times w(t)dt - \sum_{\nu=1}^{[\frac{n-1}{2}]} \frac{1}{2\pi i}\int_{-\infty}^{\infty} [\partial_\zeta^{n-\nu}\partial_{\bar\zeta}^{n-\nu-1} G_n(z,t)$$

$$-\partial_\zeta^{n-\nu-1}\partial_{\bar\zeta}^{n-\nu} G_n(z,t)](\partial_\zeta\partial_{\bar\zeta})^\nu w(t)dt$$

$$+\sum_{\nu=0}^{[\frac{n}{2}]-1} \frac{1}{2\pi i}\int_{-\infty}^{\infty} (\partial_\zeta\partial_{\bar\zeta})^{n-\nu-1} G_n(z,t)[\partial_\zeta^{\nu+1}\partial_{\bar\zeta}^\nu w(t)$$

$$-\partial_\zeta^\nu \partial_{\bar\zeta}^{\nu+1} w(t)]dt - \frac{1}{\pi}\int_{\mathbb{H}} G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^n w(\zeta) d\xi d\eta.$$

Proof From Lemma 8, Corollary 6 and formula (1.5) follows

$$w(z)=\frac{1}{4\pi i}\int_{-\infty}^{\infty}[\left(\frac{\bar{z}-z}{\bar{z}-t}\right)^n\frac{1}{t-z}-\left(\frac{z-\bar{z}}{z-t}\right)^n\frac{1}{t-\bar{z}}]w(t)dt$$

$$+\frac{1}{4\pi i}\int_{-\infty}^{\infty}(\partial_\zeta\partial_{\bar{\zeta}})^{n-1}G_n(z,t)[\partial_\zeta w(t)-\partial_{\bar{\zeta}}w(t)]dt$$

$$+\sum_{\nu=0}^{n-2}\frac{1}{4\pi i}\int_{-\infty}^{\infty}(\partial_\zeta\partial_{\bar{\zeta}})^{\nu}G_n(z,t)[\partial_\zeta^{n-\nu}\partial_{\bar{\zeta}}^{n-\nu-1}w(t)-\partial_\zeta^{n-\nu-1}\partial_{\bar{\zeta}}^{n-\nu}$$

$$\times w(t)]dt-\sum_{\nu=0}^{n-2}\frac{1}{4\pi i}\int_{-\infty}^{\infty}(\partial_\zeta\partial_{\bar{\zeta}})^{n-\nu-1}w(t)[\partial_\zeta^{\nu+1}\partial_{\bar{\zeta}}^{\nu}G_n(z,t)$$

$$-\partial_\zeta^{\nu}\partial_{\bar{\zeta}}^{\nu+1}G_n(z,t)]dt-\frac{1}{\pi}\int_{\mathbb{H}}G_n(z,\zeta)(\partial_\zeta\partial_{\bar{\zeta}})^n w(\zeta)d\xi d\eta.$$

From Remark 1 and Corollary 2 follows $(\partial_\zeta\partial_{\bar{\zeta}})^\rho G_n(z,\zeta)=0$ for $2\rho\leqslant n-1$ on $\zeta=\bar{\zeta}$ and $\partial_\zeta^{\rho+1}\partial_{\bar{\zeta}}^{\rho}G_n(z,\zeta)=0$, $\partial_\zeta^{\rho}\partial_{\bar{\zeta}}^{\rho+1}G_n(z,\zeta)=0$ for $2\rho\leqslant n-2$ on $\zeta=\bar{\zeta}$. Hence,

$$w(z)=\frac{1}{4\pi i}\int_{-\infty}^{\infty}[\left(\frac{\bar{z}-z}{\bar{z}-t}\right)^n\frac{1}{t-z}-\left(\frac{z-\bar{z}}{z-t}\right)^n\frac{1}{t-\bar{z}}]w(t)dt$$

$$-\sum_{\nu=0}^{n-2}\frac{1}{4\pi i}\int_{-\infty}^{\infty}[\partial_\zeta^{\nu+1}\partial_{\bar{\zeta}}^{\nu}G_n(z,t)-\partial_\zeta^{\nu}\partial_{\bar{\zeta}}^{\nu+1}G_n(z,t)](\partial_\zeta\partial_{\bar{\zeta}})^{n-\nu-1}$$

$$\times w(t)dt+\sum_{\nu=0}^{n-1}\frac{1}{4\pi i}\int_{-\infty}^{\infty}(\partial_\zeta\partial_{\bar{\zeta}})^{\nu}G_n(z,t)[\partial_\zeta^{n-\nu}\partial_{\bar{\zeta}}^{n-\nu-1}w(t)$$

$$-\partial_\zeta^{n-\nu-1}\partial_{\bar{\zeta}}^{n-\nu}w(t)]dt-\frac{1}{\pi}\int_{\mathbb{H}}G_n(z,\zeta)(\partial_\zeta\partial_{\bar{\zeta}})^n w(\zeta)d\xi d\eta$$

$$=\frac{1}{4\pi i}\int_{-\infty}^{\infty}[\left(\frac{\bar{z}-z}{\bar{z}-t}\right)^n\frac{1}{t-z}-\left(\frac{z-\bar{z}}{z-t}\right)^n\frac{1}{t-\bar{z}}]w(t)dt$$

$$-\sum_{\nu=1}^{n-1}\frac{1}{4\pi i}\int_{-\infty}^{\infty}[\partial_\zeta^{n-\nu}\partial_{\bar\zeta}^{n-\nu-1}G_n(z,t)-\partial_\zeta^{n-\nu-1}\partial_{\bar\zeta}^{n-\nu}G_n(z,t)](\partial_\zeta\partial_{\bar\zeta})^\nu$$

$$\times w(t)dt+\sum_{\nu=0}^{n-1}\frac{1}{4\pi i}\int_{-\infty}^{\infty}(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-1}G_n(z,t)[\partial_\zeta^{\nu+1}\partial_{\bar\zeta}^{\nu}w(t)$$

$$-\partial_\zeta^{\nu}\partial_{\bar\zeta}^{\nu+1}w(t)]dt-\frac{1}{\pi}\int_{\mathbb{H}}G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^n w(\zeta)d\xi d\eta$$

$$=\frac{1}{4\pi i}\int_{-\infty}^{\infty}[\left(\frac{\bar z-z}{\bar z-t}\right)^n\frac{1}{t-z}-\left(\frac{z-\bar z}{z-t}\right)^n\frac{1}{t-\bar z}]w(t)dt$$

$$-\sum_{\nu=1}^{[\frac{n-1}{2}]}\frac{1}{4\pi i}\int_{-\infty}^{\infty}[\partial_\zeta^{n-\nu}\partial_{\bar\zeta}^{n-\nu-1}G_n(z,t)-\partial_\zeta^{n-\nu-1}\partial_{\bar\zeta}^{n-\nu}G_n(z,t)](\partial_\zeta\partial_{\bar\zeta})^\nu$$

$$\times w(t)dt+\sum_{\nu=0}^{[\frac{n}{2}]-1}\frac{1}{4\pi i}\int_{-\infty}^{\infty}(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-1}G_n(z,t)[\partial_\zeta^{\nu+1}\partial_{\bar\zeta}^{\nu}w(t)$$

$$-\partial_\zeta^{\nu}\partial_{\bar\zeta}^{\nu+1}w(t)]dt-\frac{1}{\pi}\int_{\mathbb{H}}G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^n w(\zeta)d\xi d\eta.$$

Theorem 7 *Any* $w\in C^{2n}(\mathbb{H};\mathbb{C})\cap C^{2n-1}(\overline{\mathbb{H}};\mathbb{C})$ *for which* $|z|^{2\nu+\delta}(\partial_z\partial_{\bar z})^\nu w$ *for* $0\leqslant\nu\leqslant n$, $|z|^{2\nu+1+\delta}\partial_z^\nu\partial_{\bar z}^{\nu+1}w$, $|z|^{2\nu+1+\delta}\partial_z^{\nu+1}\partial_{\bar z}^{\nu}w$ *for* $0\leqslant\nu\leqslant n-1$ *are bounded in* $\mathbb{H}$ *can be represented as*

$$\begin{aligned}w(z)= & -\sum_{\nu=0}^{[\frac{n-1}{2}]}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\partial_\zeta^{n-\nu}\partial_{\bar\zeta}^{n-\nu-1}G_n(z,t)(\partial_\zeta\partial_{\bar\zeta})^\nu w(t)dt\\ & -\sum_{\nu=0}^{[\frac{n}{2}]-1}\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-1}G_n(z,t)\partial_\zeta^{\nu}\partial_{\bar\zeta}^{\nu+1}w(t)dt\\ & -\frac{1}{\pi}\int_{\mathbb{H}}G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^n w(\zeta)d\xi d\eta \qquad (2.17)\end{aligned}$$

and

$$
\begin{aligned}
w(z) &= \sum_{\nu=0}^{[\frac{n-1}{2}]} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \partial_\zeta^{n-\nu-1} \partial_{\bar\zeta}^{n-\nu} G_n(z,t)(\partial_\zeta \partial_{\bar\zeta})^\nu w(t) dt \\
&+ \sum_{\nu=0}^{[\frac{n}{2}]-1} \frac{1}{2\pi i} \int_{-\infty}^{\infty} (\partial_\zeta \partial_{\bar\zeta})^{n-\nu-1} G_n(z,t) \partial_\zeta^{\nu+1} \partial_{\bar\zeta}^{\nu} w(t) dt \\
&- \frac{1}{\pi} \int_{\mathbb{H}} G_n(z,\zeta)(\partial_\zeta \partial_{\bar\zeta})^n w(\zeta) d\xi d\eta. \qquad (2.18)
\end{aligned}
$$

Proof For formula (2.17) observe

$$
\frac{1}{\pi} \int_{\mathbb{H}} G_n(z,\zeta)(\partial_\zeta \partial_{\bar\zeta})^n w(\zeta) d\xi d\eta = \frac{1}{\pi} \int_{\mathbb{H}} \{\partial_\zeta [G_n(z,\zeta) \partial_\zeta^{n-1} \partial_{\bar\zeta}^{n} w(\zeta)]
$$

$$
-\partial_{\bar\zeta}[\partial_\zeta G_n(z,\zeta)(\partial_\zeta \partial_{\bar\zeta})^{n-1} w(\zeta)] + \partial_\zeta \partial_{\bar\zeta} G_n(z,\zeta)(\partial_\zeta \partial_{\bar\zeta})^{n-1} w(\zeta)\} d\xi d\eta
$$

$$
= \frac{1}{\pi} \int_{\mathbb{H}} \{\sum_{\nu=0}^{n-2} [\partial_\zeta[(\partial_\zeta \partial_{\bar\zeta})^\nu G_n(z,\zeta) \partial_\zeta^{n-\nu-1} \partial_{\bar\zeta}^{n-\nu} w(\zeta)] - \partial_{\bar\zeta}[\partial_\zeta^{\nu+1} \partial_{\bar\zeta}^{\nu}
$$

$$
\times G_n(z,\zeta)(\partial_\zeta \partial_{\bar\zeta})^{n-\nu-1} w(\zeta)]] + (\partial_\zeta \partial_{\bar\zeta})^{n-1} G_n(z,\zeta) \partial_\zeta \partial_{\bar\zeta} w(\zeta)\} d\xi d\eta,
$$

where the last two terms are

$$
\frac{1}{\pi} \int_{\mathbb{H}} (\partial_\zeta \partial_{\bar\zeta})^{n-1} G_n(z,\zeta) \partial_\zeta \partial_{\bar\zeta} w(\zeta) d\xi d\eta = \frac{1}{\pi} \int_{\mathbb{H}} \{\partial_\zeta [(\partial_\zeta \partial_{\bar\zeta})^{n-1}
$$

$$
\times G_n(z,\zeta) \partial_{\bar\zeta} w(\zeta)] - \partial_\zeta^{n} \partial_{\bar\zeta}^{n-1} G_n(z,\zeta) \partial_{\bar\zeta} w(\zeta)\} d\xi d\eta
$$

and

$$
\frac{1}{\pi} \int_{\mathbb{H}} \partial_\zeta^{n} \partial_{\bar\zeta}^{n-1} G_n(z,\zeta) \partial_{\bar\zeta} w(\zeta) d\xi d\eta = w(z)
$$

$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\partial_\zeta^n\partial_{\bar\zeta}^{n-1}G_n(z,\zeta))w(\zeta)dt.$$

By formula (1.5) we have

$$\partial_\zeta^n\partial_{\bar\zeta}^{n-1}G_n(z,\zeta)=(-1)^n\left(\frac{z-\bar z}{\bar z-\zeta}\right)^n\frac{1}{z-\zeta}=\left(\frac{z-\bar z}{\zeta-\bar z}\right)^n\frac{1}{z-\zeta}$$

$$=-\left(\frac{z-\bar z}{\zeta-\bar z}\right)^n\frac{1}{\zeta-z}.$$

Thus

$$w(z)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\partial_\zeta^n\partial_{\bar\zeta}^{n-1}G_n(z,t)w(t)dt-\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\partial_\zeta\partial_{\bar\zeta})^{n-1}$$

$$\times G_n(z,t)\partial_{\bar\zeta}w(t)dt-\sum_{\nu=0}^{n-2}\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\partial_\zeta\partial_{\bar\zeta})^\nu G_n(z,t)\partial_\zeta^{n-\nu-1}\partial_{\bar\zeta}^{n-\nu}$$

$$\times w(t)dt+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\partial_\zeta^{\nu+1}\partial_{\bar\zeta}^{\nu}G_n(z,t)(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-1}w(t)dt$$

$$-\frac{1}{\pi}\int_{\mathbb{H}}G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^n w(\zeta)\}d\xi d\eta$$

$$=-\sum_{\nu=0}^{n-1}\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\partial_\zeta\partial_{\bar\zeta})^\nu G_n(z,t)\partial_\zeta^{n-\nu-1}\partial_{\bar\zeta}^{n-\nu}w(t)dt$$

$$-\sum_{\nu=0}^{n-1}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\partial_\zeta^{\nu+1}\partial_{\bar\zeta}^{\nu}G_n(z,t)(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-1}w(t)dt$$

$$-\frac{1}{\pi}\int_{\mathbb{H}}G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^n w(\zeta)d\xi d\eta$$

$$=-\sum_{\nu=0}^{[\frac{n-1}{2}]}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\partial_\zeta^{n-\nu}\partial_{\bar\zeta}^{n-\nu-1}G_n(z,t)(\partial_\zeta\partial_{\bar\zeta})^\nu w(t)dt$$

$$-\sum_{\nu=0}^{[\frac{n}{2}]-1}\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-1}G_n(z,t)\partial_\zeta^\nu\partial_{\bar\zeta}^{\nu+1}w(t)dt$$

$$-\frac{1}{\pi}\int_{\mathbb{H}}G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^n w(\zeta)d\xi d\eta.$$

This is the representation (2.17). Formula (2.18) can be similarly proved.

2.4 The Dirichlet and the Neumann problem for the Poisson equation

The Neumann boundary value problem is investigated for some complex model equations up to fourth order including the inhomogeneous Cauchy-Riemann, [17], the Poisson and bi-Poisson and n-Poisson equation for the unit disc, see [14], [15].
The Neumann problem is well studied for harmonic functions and solved under certain conditions via the Neumann function, sometimes also called Green function of second kind, see [22]. For the half plane $\mathbb{H}$ this function is up to some constant factor.

$$N_1(z,\zeta)=\log\left|\frac{(\zeta-z)(\bar\zeta-z)}{(z+i)^2(\zeta+i)^2}\right|^2 \tag{2.19}$$

Remark 5 The Neumann function has the properties
1) $\partial_z\partial_{\bar z}N_1(z,\zeta)=0$ in $\mathbb{H}\setminus\{\zeta\}$
2) $N_1(z,\zeta)-2\log|\zeta-z|$ is harmonic in $\mathbb{H}$ for all $\zeta\in\mathbb{H}$
3) $\partial_y N_1(z,\zeta)=-\frac{4}{1+x^2}$ for $y=0$, $\zeta\in\mathbb{H}$
4) $N_1(i,\zeta)=\log\frac{1}{16}|\frac{\zeta-i}{\zeta+i}|^2$, $\zeta\neq i,\zeta\in\mathbb{H}$.
They can be checked by direct calculations.

From

$$\partial_\eta N_1(z,\zeta)=\frac{2(\eta-y)}{|\zeta-z|^2}+\frac{2(\eta+y)}{|\bar\zeta-z|^2}-4\frac{\eta+1}{|\zeta+i|^2}$$

for $\eta=0$ then

$$\partial_\eta N_1(z,\zeta)=-\frac{4}{1+\xi^2}$$

follows. Thus on $\eta = 0$ the outward normal derivative is

$$\partial_{\nu_\zeta} N_1(z,\zeta) = \frac{4}{1+\xi^2}.$$

Theorem 8 *Any* $w \in C^2(\mathbb{H};\mathbb{C}) \cap C^1(\bar{\mathbb{H}};\mathbb{C})$ *for which* $z^\delta w(z)$, $z^{1+\delta} w_z(z)$, $z^{1+\delta} w_{\bar z}(z)$, $z^{2+\delta} w_{z\bar z}(z)$ *for some* $0 < \delta$ *are bounded in* $\mathbb{H}$ *can be represented as*

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(t)}{t-z}dt + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \log|t-z|^2 \partial_{\bar\zeta} w(t)dt \\ &+ \frac{1}{\pi}\int_{\mathbb{H}} \log|\zeta - z|^2 \partial_\zeta \partial_{\bar\zeta} w(\zeta) d\xi d\eta \end{aligned} \tag{2.20}$$

and as

$$\begin{aligned} w(z) &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(t)}{t-\bar z}dt - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \log|t-z|^2 \partial_{\zeta} w(t)dt \\ &+ \frac{1}{\pi}\int_{\mathbb{H}} \log|\zeta - z|^2 \partial_\zeta \partial_{\bar\zeta} w(\zeta) d\xi d\eta. \end{aligned} \tag{2.21}$$

Proof To prove formula (2.20) let us take the point $z \in \mathbb{H}_R$ and denote $\mathbb{H}_{R,\varepsilon} : \mathbb{H}_R \setminus \{|\zeta - z| < \varepsilon\}$ for $\varepsilon > 0$ small enough. Then the Gauss theorem shows

$$\frac{1}{\pi}\int_{\mathbb{H}_{R,\varepsilon}(z)} \log|\zeta - z|^2 \partial_\zeta \partial_{\bar\zeta} w(\zeta) d\xi d\eta$$

$$= \frac{1}{\pi}\int_{\mathbb{H}_{R,\varepsilon}(z)} \{\partial_\zeta \log|\zeta - z|^2 \partial_{\bar\zeta} w(\zeta) - \partial_{\bar\zeta}(\frac{1}{\zeta - z} w(\zeta))\} d\xi d\eta$$

$$= -\frac{1}{2\pi i}\int_{\partial\mathbb{H}_{R,\varepsilon}(z)} \{\log|\zeta - z|^2 \partial_{\bar\zeta} w(\zeta) d\bar\zeta + \frac{1}{\zeta - z} w(\zeta) d\zeta\}$$

If R tends to $+\infty$, then the right-hand side of the last formula tends to

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty} \{\log|\zeta - z|^2 \partial_{\bar\zeta} w(\zeta) d\bar\zeta + \frac{w(\zeta)}{\zeta - z} d\zeta\}$$

$$+\frac{1}{2\pi i}\int_{|\zeta-z|=\varepsilon}\{\log|\zeta-z|^2\partial_{\bar\zeta}w(\zeta)d\bar\zeta+\frac{w(\zeta)}{\zeta-z}d\zeta\}.$$

Observing

$$\lim_{R\to+\infty}\int_{|\zeta|=R,0<Im\zeta}\{\log|\zeta-z|^2\partial_{\bar\zeta}w(\zeta)d\bar\zeta+\frac{w(\zeta)}{\zeta-z}d\zeta\}=0,$$

because $\lim_{R\to+\infty}R^{1+\delta}\log M(R,\partial_{\bar z}w)=0$ and $\lim_{R\to\infty}M(R,w)=0$, and the convergence of the integrals along the real line these terms become

$$-\frac{1}{2\pi i}\int_{\mathbb{R}}\{\log|\zeta-z|^2\partial_{\bar\zeta}w(\zeta)d\bar\zeta+\frac{w(\zeta)}{\zeta-z}d\zeta\}$$

$$+\frac{1}{2\pi i}\int_{|\zeta-z|=\varepsilon}\{\log|\zeta-z|^2\partial_{\bar\zeta}w(\zeta)d\bar\zeta+\frac{w(\zeta)}{\zeta-z}d\zeta\}.$$

Observing the boundedness of $z^{2+\varepsilon}w_{z\bar z}(z)$ and the existence of

$$\int_{|\zeta-z|<\varepsilon}\log|\zeta-z|^2\partial_\zeta\partial_{\bar\zeta}w(\zeta)d\xi d\eta$$

it follows

$$\lim_{\substack{R\to\infty\\ \varepsilon\to0}}\int_{\mathbb{H}_{R,\varepsilon}}\log|\zeta-z|^2\partial_\zeta\partial_{\bar\zeta}w(\zeta)d\xi d\eta=\int_{\mathbb{H}}\log|\zeta-z|^2\partial_\zeta\partial_{\bar\zeta}w(\zeta)d\xi d\eta.$$

Now

$$\frac{1}{2\pi i}\int_{|\zeta-z|=\varepsilon}\{\log|\zeta-z|^2\partial_{\bar\zeta}w(\zeta)d\bar\zeta+\frac{w(\zeta)}{\zeta-z}d\zeta\}$$

$$=\frac{1}{2\pi i}\int_0^{2\pi}\{2\log|\varepsilon e^{i\varphi}|\partial_{\bar\zeta}w(z+\varepsilon e^{i\varphi})(-i\varepsilon e^{-i\varphi})$$

$$+\frac{w(z+\varepsilon e^{i\varphi})}{\varepsilon e^{i\varphi}}i\varepsilon e^{i\varphi}\}d\varphi.$$

If ε tends to zero, then the last formula tends to $w(z)$. Thus (2.20) follows. Similarly (2.21) can be shown.
Also

$$\frac{1}{\pi}\int_{\mathbb{H}_R}\log|\bar{\zeta}-z|^2\partial_\zeta\partial_{\bar{\zeta}}w(\zeta)d\xi d\eta$$
$$=\frac{1}{\pi}\int_{\mathbb{H}_R}\{\partial_\zeta\left(\log|\bar{\zeta}-z|^2\partial_{\bar{\zeta}}w(\zeta)\right)-\partial_{\bar{\zeta}}\left(\frac{w(\zeta)}{\zeta-\bar{z}}\right)\}d\xi d\eta$$
$$=-\frac{1}{2\pi i}\int_{\partial\mathbb{H}_R}\{\log|\bar{\zeta}-z|^2\partial_{\bar{\zeta}}w(\zeta)d\bar{\zeta}+\frac{w(\zeta)}{\zeta-\bar{z}}d\zeta\}$$

follows. If R tends to $+\infty$, then the right-hand side of the last formula tends to

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{\log|t-z|^2\partial_{\bar{\zeta}}w(t)dt+\frac{w(t)}{t-\bar{z}}dt\}.$$

Thus for $z\in\mathbb{H}$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{\frac{w(t)}{t-\bar{z}}+\log|t-z|^2\partial_{\bar{\zeta}}w(t)\}dt+\frac{1}{\pi}\int_{\mathbb{H}}\log|\bar{\zeta}-z|^2\partial_\zeta\partial_{\bar{\zeta}}w(\zeta)d\xi d\eta=0 \tag{2.22}$$

follows. Similarly

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{\frac{w(t)}{t-z}+\log|t-z|^2\partial_\zeta w(t)\}dt$$
$$+\frac{1}{\pi}\int_{\mathbb{H}}\log|\bar{\zeta}-z|^2\partial_\zeta\partial_{\bar{\zeta}}w(\zeta)d\xi d\eta=0 \tag{2.23}$$

follows.
The representation formulas (2.20) and (2.21) can be modified according to the Dirichlet problem and the Neumann problem.

Theorem 9 *Under the assumptions in Theorem 8 the function w can be represented as*

$$w(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} w(t)\frac{y}{|t-z|^2}dt + \frac{1}{\pi}\int_{\mathbb{H}} G_1(z,\zeta)\partial_\zeta\partial_{\bar\zeta}w(\zeta)d\xi d\eta \quad (2.24)$$

where $G_1(z,\zeta)$ is the Green function for $\mathbb{H}$.

Proof Subtracting formula (2.22) from (2.20) proves

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} w(t)\left(\frac{1}{t-z} - \frac{1}{t-\bar z}\right)dt \\ &- \frac{1}{\pi}\int_{\mathbb{H}} \log|\frac{\bar\zeta - z}{\zeta - z}|^2 \partial_\zeta\partial_{\bar\zeta}w(t)d\xi d\eta \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} w(t)\frac{y}{|t-z|^2}dt \\ &+ \frac{1}{\pi}\int_{\mathbb{H}} G_1(z,\zeta)\partial_\zeta\partial_{\bar\zeta}w(t)d\xi d\eta. \end{aligned}$$

Formula (2.24) provides the solution to the Dirichlet problem for the Poisson equation.

Theorem 10 *The Dirichlet problem $w_{z\bar z} = f$ in $\mathbb{H}$, $w = \gamma$ on $\mathbb{R}$ is uniquely solvable for $f \in L_1(\mathbb{H};\mathbb{C})$, $\gamma \in C(\mathbb{R};\mathbb{C})$, γ bounded. The solution is given as*

$$w(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \gamma(t)\frac{y}{|t-z|^2}dt + \frac{1}{\pi}\int_{\mathbb{H}} G_1(z,\zeta)f(\zeta)d\xi d\eta. \quad (2.25)$$

Proof Obviously (2.25) is a solution as the first integral is the Poisson integral, which is harmonic, (for details see Section 3.1) and the second integral vanishing on the boundary $\mathbb{R}$ is a particular solution to the Poisson equation. The solution is unique because any harmonic function in $\mathbb{H}$ continuous in $\overline{\mathbb{H}}$, vanishing at infinity and with vanishing boundary values on $\mathbb{R}$ vanishes identically. This follows from the maximum principle for harmonic functions.
For another alteration of the representation (2.20) consider

$$\frac{1}{\pi}\int_{\mathbb{H}_R} \partial_\zeta\partial_{\bar\zeta}w(\zeta)d\xi d\eta = \frac{1}{2\pi i}\int_{\mathbb{H}_R} \partial_\zeta w(\zeta)d\zeta$$

leading to

$$-\frac{1}{\pi}\int_{\mathbb{H}}\log|z+i|^2\partial_\zeta\partial_{\bar\zeta}w(\zeta)d\xi d\eta-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\log|z+i|^2\partial_{\bar\zeta}w(t)dt=0 \tag{2.26}$$

and similarly

$$-\frac{1}{\pi}\int_{\mathbb{H}}\log|z+i|^2\partial_\zeta\partial_{\bar\zeta}w(\zeta)d\xi d\eta+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\log|z+i|^2\partial_{\zeta}w(t)dt=0. \tag{2.27}$$

Also from

$$\frac{1}{\pi}\int_{\mathbb{H}_R}\log|\zeta+i|^2\partial_\zeta\partial_{\bar\zeta}w(\zeta)d\xi d\eta=\frac{1}{\pi}\int_{\mathbb{H}_R}\{\partial_\zeta\left(\log|\zeta+i|^2\partial_{\bar\zeta}w(\zeta)\right)$$

$$-\partial_{\bar\zeta}\left(\frac{w(\zeta)}{\zeta+i}\right)\}d\xi d\eta=-\frac{1}{2\pi i}\int_{\partial\mathbb{H}_R}\{\log|\zeta+i|^2\partial_{\bar\zeta}w(\zeta)d\bar\zeta+\frac{w(\zeta)}{\zeta+i}d\zeta\}$$

we have

$$\begin{aligned}&-\frac{1}{\pi}\int_{\mathbb{H}}\log|\zeta+i|^2\partial_\zeta\partial_{\bar\zeta}w(\zeta)d\xi d\eta\\&-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{\log|t+i|^2\partial_{\bar\zeta}w(t)dt+\frac{w(t)}{t+i}\}dt=0\end{aligned} \tag{2.28}$$

and by symmetry

$$\begin{aligned}&-\frac{1}{\pi}\int_{\mathbb{H}}\log|\zeta+i|^2\partial_\zeta\partial_{\bar\zeta}w(\zeta)d\xi d\eta\\&+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{\log|t+i|^2\partial_{\zeta}w(t)+\frac{w(t)}{t-i}\}dt=0.\end{aligned} \tag{2.29}$$

Adding (2.20),(2.21),(2.22),(2.23) and two times of (2.26), (2.27), (2.28), (2.29) shows

$$w(z)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{w(\zeta)}{|t-i|^2}dt$$

$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}(w_{\bar\zeta}(t)-w_\zeta(t))\log\left|\frac{(t-z)^2}{(t+i)^2(z+i)^2}\right|^2 dt$$

$$+\frac{1}{\pi}\int_{\mathbb{H}}w_{\zeta\bar\zeta}(\zeta)\log\left|\frac{(\zeta-z)(\bar\zeta-z)}{(\zeta+i)^2(z+i)^2}\right|^2 d\xi d\eta.$$

Observing

$$\partial_\eta=\frac{\partial\zeta}{\partial\eta}\partial_\zeta+\frac{\partial\bar\zeta}{\partial\eta}\partial_{\bar\zeta}=i\partial_\zeta-i\partial_{\bar\zeta}=i(\partial_\zeta-\partial_{\bar\zeta}),$$

this is

$$\begin{aligned} w(z) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{w(t)}{1+t^2}dt+\frac{1}{4\pi}\int_{-\infty}^{\infty}\partial_\eta w(t)\log\left|\frac{(t-z)^2}{(t+i)^2(z+i)^2}\right|^2 dt \\ &+ \frac{1}{\pi}\int_{\mathbb{H}}w_{\zeta\bar\zeta}(\zeta)\log\left|\frac{(\zeta-z)(\bar\zeta-z)}{(\zeta+i)^2(z+i)^2}\right|^2 d\xi d\eta. \end{aligned} \tag{2.30}$$

Thus the following result is proved.

Theorem 11 *Any $w\in C^2(\mathbb{H};\mathbb{C})\cap C^1(\bar{\mathbb{H}};\mathbb{C})$ for which $z^\delta w(z)$, $z^{1+\delta}w_y(z)$, $z^{2+\delta}w_{z\bar z}(z)$ are bounded for some $0<\delta$ can be represented by (2.30).*

Lemma 9 *For $z\in\mathbb{H}$*

$$\arctan z=-\frac{1}{4\pi i}\int_{-\infty}^{\infty}N_1(z,t)\frac{dt}{1+t^2}.$$

Proof For $z\in\mathbb{H}$

$$\partial_y\arctan z=\frac{i}{1+z^2},$$

$$\partial_z\partial_{\bar z}\arctan z=\partial_{\bar z}\frac{1}{1+z^2}=0.$$

Applying (2.30) shows

$$\arctan z=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\arctan t}{1+t^2}dt+\frac{i}{4\pi}\int_{-\infty}^{\infty}N_1(z,t)\frac{dt}{1+t^2}.$$

Thus the lemma follows because

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\arctan t}{1+t^2}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} d(\arctan t)^2 = 0.$$

Remark 6 The derivative ∂_y is the derivative in the inner normal direction on $\mathbb{R}$ with respect to $\mathbb{H}$.

Remark 7 A more concise proof of formula (2.30) is based on

$$\frac{1}{\pi}\int_{\mathbb{H}_{R,\varepsilon}} N_1(z,\zeta)w_{\zeta\bar{\zeta}}(\zeta)d\xi d\eta = \frac{1}{2\pi}\int_{\mathbb{H}_{R,\varepsilon}} 2N_1(z,\zeta)w_{\zeta\bar{\zeta}}d\xi d\eta$$

$$= \frac{1}{2\pi}\int_{\mathbb{H}_{R,\varepsilon}} \left(\partial_{\bar{\zeta}}(N_1 w_\zeta) + \partial_\zeta(N_1 w_{\bar{\zeta}}) - \partial_{\bar{\zeta}}N_1 w_\zeta - \partial_\zeta N_1 w_{\bar{\zeta}}\right) d\xi d\eta$$

$$= \frac{1}{2\pi}\int_{\mathbb{H}_{R,\varepsilon}} (\partial_{\bar{\zeta}}(N_1 w_\zeta) + \partial_\zeta(N_1 w_{\bar{\zeta}}) - \partial_\zeta\{\partial_{\bar{\zeta}}N_1 w\}$$

$$-\partial_{\bar{\zeta}}\{\partial_\zeta N_1 w\} + 2\partial_\zeta\partial_{\bar{\zeta}}N_1 w)d\xi d\eta$$

$$= \frac{1}{4\pi i}\int_{\partial\mathbb{H}_{R,\varepsilon}} (\{N_1(z,\zeta)w_\zeta(\zeta) - \partial_\zeta N_1(z,\zeta)w(\zeta)\}d\zeta$$

$$-\{N_1(z,\zeta)w_{\bar{\zeta}}(\zeta) - \partial_{\bar{\zeta}}N_1(z,\zeta)w(\zeta)\}d\bar{\zeta})$$

$$= \frac{1}{4\pi i}\int_{\partial\mathbb{H}_{R,\varepsilon}} (N_1(z,\zeta)(w_\zeta(\zeta)d\zeta - w_{\bar{\zeta}}(\zeta)d\bar{\zeta})$$

$$-w(\zeta)\{\partial_\zeta N_1(z,\zeta)d\zeta - \partial_{\bar{\zeta}}N_1(z,\zeta)d\bar{\zeta}\}).$$

We observe

$$\frac{1}{4\pi i}\int_{\substack{|\zeta|=R\\ Im\zeta>0}}\{N_1(z,\zeta)(\zeta w_\zeta(\zeta)+\bar\zeta w_{\bar\zeta}(\zeta))$$

$$-w(\zeta)(\zeta\partial_\zeta N_1(z,\zeta)+\bar\zeta\partial_{\bar\zeta}N_1(z,\zeta))\}\frac{d\zeta}{\zeta}$$

$$=\frac{1}{4\pi i}\int_{\substack{|\zeta|=R\\ Im\zeta>0}}(N_1(z,\zeta)R\partial_R w(\zeta)-w(\zeta)R\partial_R N_1(z,\zeta))\frac{d\zeta}{\zeta}$$

$$=\frac{1}{4\pi i}\int_{\substack{|\zeta|=R\\ Im\zeta>0}}\{\log\left|\frac{(\zeta-z)(\zeta-\bar z)}{(z+i)^2(\zeta+i)^2}\right|^2 R\partial_R w(\zeta)$$

$$-w(\zeta)\{\zeta(\frac{1}{\zeta-z}+\frac{1}{\zeta-\bar z}-\frac{2}{\zeta+i})$$

$$+\bar\zeta(\frac{1}{\overline{\overline{\zeta}-z}}+\frac{1}{\bar\zeta-z}-\frac{2}{\bar\zeta-i})\}\}\frac{d\zeta}{\zeta}.$$

The last formula tends to 0 if R tends to ∞ assuming w is bounded and $R\partial_R w(Re^{i\varphi})$, where $0\leqslant\varphi\leqslant\pi$, tends to zero. As

$$\partial_\eta N_1(z,\zeta)=\frac{2(\eta+y)}{|\zeta-\bar z|^2}-\frac{4(\eta+1)}{|\zeta+i|^2}+\frac{2(\eta-y)}{|\bar\zeta-z|^2},$$

then

$$\partial_\eta N_1(z,\zeta)=\frac{2y}{|t-\bar z|^2}-\frac{4}{|t+i|^2}-\frac{2y}{|t-z|^2}=-\frac{4}{1+t^2},$$

when $\eta=0$ and $\xi=t$. If $\zeta=\xi+i\eta$ and $\eta\equiv 0$ then

$$\partial_\zeta d\zeta-\partial_{\bar\zeta}\partial\bar\zeta=\frac{1}{2}(\partial_\xi-i\partial_\eta)(d\xi+id\eta)-\frac{1}{2}(\partial_\xi+id\eta)(d\xi-id\eta)$$

$$=-i\partial_\eta d\xi.$$

Hence,

$$\frac{1}{4\pi i}\int_{-R}^{R}\{N_1(z,\zeta)(w_\zeta(\zeta)d\zeta - w_{\bar\zeta}(\zeta)d\bar\zeta)$$

$$-w(\zeta)(\partial_\zeta N_1(z,\zeta)d\zeta - \partial_{\bar\zeta}N_1(z,\zeta)d\bar\zeta)\}$$

$$= \frac{1}{4\pi i}\int_{-R}^{R}\{N_1(z,t)(-i)\partial_\eta w(t) - w(t)(-i)\partial_\eta N_1(z,t)\}dt$$

$$= -\frac{1}{4\pi}\int_{-R}^{R}\{N_1(z,t)\partial_\eta w(t) - w(t)\partial_\eta N_1(z,t)\}dt.$$

Moreover,

$$\frac{1}{4\pi i}\int_{|\zeta-z|=\varepsilon}\{N_1(z,\zeta)\left((\zeta-z)w_\zeta + (\overline{\zeta-z})w_{\bar\zeta}\right)$$

$$-w(\zeta)\left((\zeta-z)N_{1\zeta}(z,\zeta) + (\overline{\zeta-z})N_{1\bar\zeta}(z,\zeta)\right)\}\frac{d\zeta}{\zeta-z}$$

$$= \frac{\varepsilon i}{4\pi i}\int_0^{2\pi}\log\left|\frac{\varepsilon e^{i\varphi}(\varepsilon e^{-i\varphi}+\bar z - z)}{|z+i|^2|\varepsilon e^{i\varphi}+z+i|^2}\right|^2$$

$$\times\left(e^{i\varphi}w_\zeta(z+\varepsilon e^{i\varphi}) + e^{-i\varphi}w_{\bar\zeta}(z+\varepsilon e^{i\varphi})\right)d\varphi$$

$$-\frac{i}{4\pi i}\int_0^{2\pi}w(z+\varepsilon e^{i\varphi})\{\varepsilon e^{i\varphi}(\frac{1}{\varepsilon e^{i\varphi}} + \frac{1}{\varepsilon e^{i\varphi}+z-\bar z} - \frac{2}{\varepsilon e^{i\varphi}+z+i})$$

$$+\varepsilon e^{-i\varphi}(\frac{1}{\varepsilon e^{-iy}} + \frac{1}{\varepsilon e^{-iy}+\bar z - z} - \frac{2}{\varepsilon e^{-i\varphi}+\bar z - i})\}d\varphi.$$

The last formula tends to $-\frac{1}{2}2w(z) = -w(z)$ if ε tends to 0. Hence

$$\frac{1}{\pi}\int_{\mathbb{H}}N_1(z,\zeta)w_{\zeta\bar\zeta}(\zeta)d\xi d\eta = \lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\frac{1}{\pi}\int_{\mathbb{H}_{R,\varepsilon}}N_1(z,\zeta)w_{\zeta\bar\zeta}(\zeta)d\xi d\eta,$$

$$-\frac{1}{4\pi}\int_{\mathbb{R}}\{N_1(z,t)\partial_\eta w(t) - w(t)\partial_\eta N_1(z,t)\}dt + w(z)$$

$$= -\frac{1}{\pi}\int_{\mathbb{R}} \frac{w(t)}{1+t^2}dt - \frac{1}{4\pi}\int_{\mathbb{R}} N_1(z,t)\partial_\eta w(t)dt + w(z).$$

This is (2.30) which can also be written as

$$\begin{aligned} w(z) &= -\frac{1}{4\pi}\int_{-\infty}^{\infty} w(t)\partial_\eta N_1(z,t)dt + \frac{1}{4\pi}\int_{-\infty}^{\infty} \partial_\eta w(t) N_1(z,\zeta)dt \\ &+ \frac{1}{\pi}\int_{-\infty}^{\infty} \partial_\zeta \partial_{\bar\zeta} w(\zeta) N_1(z,\zeta)d\xi d\eta. \end{aligned} \tag{2.31}$$

Theorem 12 *The Neumann problem $w_{z\bar z} = f$ in $\mathbb{H}$, $\partial_\eta w(t) = \varphi(t)$ on $\mathbb{R}$, for $f \in L_1(\mathbb{H};\mathbb{C})$, $\varphi \in C(\mathbb{R};\mathbb{C})$ such that $t^{1+\delta}\varphi(t), z^{2+\delta}f(z)$ are bounded for some $0 < \delta$ under the normalization condition*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{w(t)}{1+t^2}dt - \frac{1}{\pi i}\int_{-\infty}^{\infty} \varphi(t)\arctan t dt - \frac{4}{\pi i}\int_{\mathbb{H}} f(\zeta)\arctan\zeta d\xi d\eta = C \tag{2.32}$$

for $C \in \mathbb{R}$, is solvable if and only if

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \varphi(t)dt + \frac{4}{\pi}\int_{\mathbb{H}} f(\zeta)d\xi d\eta = 0. \tag{2.33}$$

The unique solution is given as

$$w(z) = C + \frac{1}{4\pi}\int_{-\infty}^{\infty} \varphi(t)N_1(z,t)dt + \frac{1}{\pi}\int_{\mathbb{H}} N_1(z,\zeta)f(\zeta)d\xi d\eta. \tag{2.34}$$

Verification.
1)For $z = \bar z$, i.e. $z \in \mathbb{R}$, $y = 0$, we consider (2.34). Differentiating (2.34) we have, see Remark 5,

$$\partial_y w(z) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \varphi(t)\{\frac{2y}{|t-z|^2} + \frac{2y}{|t-\bar z|^2} - 2\frac{2(y+1)}{|z+i|^2}\}dt$$

$$+\frac{2}{\pi}\int_{\mathbb{H}} f(\zeta)\{\frac{y-\eta}{|\zeta-z|^2}+\frac{\eta+y}{|\zeta-\bar{z}|^2}-2\frac{y+1}{|z+i|^2}\}d\xi d\eta.$$

The right-hand side of the last formula tends to

$$\varphi(t_0)-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\varphi(t)}{|x+i|^2}dt-\frac{4}{\pi}\int_{\mathbb{H}}\frac{f(\zeta)}{|x+i|^2}d\xi d\eta$$

if z tends to $t_0 \in \mathbb{R}$ where $z \in \mathbb{H}$. Because of (2.33) this is $\varphi(t_0)$.
2)Differentiating (2.34) shows

$$w_z = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\varphi(t)\left(\frac{\bar{\zeta}-z+\zeta-z}{(\zeta-z)(\bar{\zeta}-z)}+\frac{2}{z+i}\right)dt$$

$$-\frac{1}{\pi}\int_{\mathbb{H}}\left(\frac{1}{\zeta-z}+\frac{1}{\bar{\zeta}-z}+\frac{2}{z+i}\right)f(\zeta)d\xi d\eta.$$

Differentiating again proves $w_{z\bar{z}} = f$ in $\mathbb{H}$.
3)By integration

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{w(t)}{1+t^2}dt = \frac{C}{\pi}\int_{-\infty}^{\infty}\frac{dt}{1+t^2}+\frac{1}{4\pi}\int_{-\infty}^{\infty}\varphi(\xi)\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{N_1(t,\xi)dt}{1+t^2}d\xi$$

$$+\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{N_1(t,\zeta)}{1+t^2}dt d\xi d\eta = C+\frac{1}{\pi i}\int_{-\infty}^{\infty}\varphi(t)\arctan t dt$$

$$+\frac{4}{\pi i}\int_{\mathbb{H}} f(\zeta)\arctan\zeta d\xi d\eta$$

i.e. (2.32) follows. Here Lemma 9 is used. Hence (2.34) is a solution under the condition (2.33).
Proof of Theorem 12
It remains to prove that if there is a solution then (2.33) is satisfied. But any solution of the Neumann problem has the form (2.30) by Theorem 11, i.e. using (2.32)

$$w(z) = C + \frac{1}{\pi i}\int_{-\infty}^{\infty}\varphi(t)\arctan t dt + \frac{4}{\pi i}\int_{\mathbb{H}} f(\zeta)\arctan\zeta d\xi d\eta$$

$$+\frac{1}{4\pi}\int_{-\infty}^{\infty}\varphi(t)N_1(z,t)dt + \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)N_1(z,\zeta)d\xi d\eta.$$

By differentiating

$$\begin{aligned}
\partial_y w(z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(t)\{2\frac{y}{|t-z|^2} - 2\frac{y+1}{|z+i|^2}\}dt \\
&+ \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\{2\frac{y-\eta}{|\zeta-z|^2} + 2\frac{y+\eta}{|\bar{\zeta}-z|^2} - 4\frac{y+1}{|z+i|^2}\}d\xi d\eta \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(t)\{\frac{y}{|t-z|^2} - \frac{1+y}{|z+i|^2}\}dt \\
&+ \frac{2}{\pi}\int_{\mathbb{H}} f(\zeta)\{\frac{y-\eta}{|\zeta-z|^2} + \frac{y+\eta}{|\bar{\zeta}-z|^2} - 2\frac{y+1}{|z+i|^2}\}d\xi d\eta \quad (2.35)
\end{aligned}$$

is seen. Then letting $z \in \mathbb{H}$ tend to $t_0 \in \mathbb{R}$

$$\lim_{z\to t_0}\partial_y w(z) = \lim_{z\to t_0}\{\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(t)\frac{y}{|t-z|^2}dt\} - \frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(t)\frac{dt}{1+t_0^2}$$

$$-\frac{4}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{|t_0+i|^2} = \varphi(t_0)$$

if and only if

$$\frac{1}{\pi}\int_{\mathbb{R}}\varphi(t)dt + \frac{4}{\pi}\int_{\mathbb{H}} f(\zeta)d\xi d\eta = 0.$$

Remark 8 On the basis of (2.30) the uniqueness of N_1 can be shown. Let $\widetilde{N_1}(z,\zeta)$, be another function with the properties (1)-(4) of $N_1(z,\zeta)$. Then $N(z,\zeta) = N_1(z,\zeta) - \widetilde{N_1}(z,\zeta)$ satisfies

1) $\partial_z\partial_{\bar{z}}N(z,\zeta) = 0$ in $\mathbb{H}\backslash\{\zeta\}$

2) $N(z,\zeta) = N_1(z,\zeta) - \log|\zeta - z|^2 - \widetilde{N_1}(z,\zeta) + \log|\zeta - z|^2$ is harmonic

in $\mathbb{H}$ for all $\zeta \in \mathbb{H}$
3)$\partial_y N(z,\zeta) = 0$ for $y = 0, \zeta \in \mathbb{H}$
4) $N(i,\zeta) = 0$.

Applying formula (2.30) to $N(z,\zeta)$, from 1),2) and 3) follows that $N(z,\zeta) = M(\zeta)$ is constant for $z \in \mathbb{H}$. Then using 4) $N(z,\zeta) = N(i,\zeta) = 0$.