1 Higher order Green functions in the upper half plane

The Green function for the higher order Laplace operator was given in explicit form in [2] for the discs. They were used e.g. in [28], [29] to treat boundary value problems for polyharmonic functions in plane domains. In [4] this Green function was used to modify higher order Cauchy-Pompeiu representations, see e.g. [3], by replacing the higher order Cauchy kernel by proper derivatives of this related polyharmonic Green function.

The Green function for the upper half plane $\mathbb{H}$ of the complex plane $\mathbb{C}$ has the following properties. For any fixed $\zeta \in \mathbb{H}$ as a function of z

(1) $G(z,\zeta)$ is harmonic in $\mathbb{H} \setminus \{\zeta\}$,

(2) $G(z,\zeta) + \log|\zeta - z|$ is harmonic in $\mathbb{H}$,

(3) $\lim_{z\to t} G(z,\zeta) = 0$ for all $t \in \partial\mathbb{H}$.

Green functions exist for other domains than just for the upper half plane. The existence is related to the solvability of the Dirichlet problem for harmonic functions in the domain.

Twice the Green function of the half plane $\mathbb{H} = \{z : z \in \mathbb{C}, Imz > 0\}$ of the complex plane $\mathbb{C}$ is

$$G_1(z,\zeta) = \log\left|\frac{\bar{\zeta} - z}{\zeta - z}\right|^2,$$

where $z, \zeta \in \mathbb{H}, z \neq \zeta$. For convenience G_1 is used instead of G and G_1 is called the Green function of $\mathbb{H}$.

Definition 1 *Let for* $1 \leqslant n$ *and* $z, \zeta \in \mathbb{H}$, $z \neq \zeta$

$$\begin{aligned}(n-1)!^2 G_n(z,\zeta) &= |\zeta - z|^{2(n-1)} \log\left|\frac{\bar{\zeta} - z}{\zeta - z}\right|^2 \\ &+ \sum_{\nu=1}^{n-1} \frac{1}{\nu} |\zeta - z|^{2(n-1-\nu)} (\zeta - \bar{\zeta})^\nu (z - \bar{z})^\nu \quad (1.1)\end{aligned}$$

G_n is called the Green function of order n for $\mathbb{H}$.

Lemma 1 *The function G_n satisfies $G_n(z,\zeta) = G_n(\zeta, z)$ and*

$$
\begin{aligned}
\partial_z G_n(z,\zeta) &= -\frac{(\zeta-z)^{n-2}\overline{(\zeta-z)}^{n-1}}{(n-1)!(n-2)!}\log\left|\frac{\bar\zeta-z}{\zeta-z}\right|^2 \\
&- \frac{1}{(n-1)!(n-2)!}\sum_{\nu=1}^{n-2}\frac{1}{\nu}(\zeta-z)^{n-2-\nu} \\
&\times \overline{(\zeta-z)}^{n-1-\nu}(\zeta-\bar\zeta)^{\nu}(z-\bar z)^{\nu} \\
&- \frac{1}{(n-1)!^2}\frac{(\zeta-\bar\zeta)^{n-1}(z-\bar z)^{n-1}}{\bar\zeta-z},
\end{aligned}
\tag{1.2}
$$

$$
\begin{aligned}
\partial_z\partial_{\bar z} G_n(z,\zeta) &= G_{n-1}(z,\zeta) \\
&+ \frac{(\zeta-\bar\zeta)^{n-1}(z-\bar z)^{n-2}}{(n-1)!(n-2)!}\left(\frac{1}{\bar\zeta-z}-\frac{1}{\zeta-\bar z}\right)
\end{aligned}
\tag{1.3}
$$

for $z,\zeta\in\mathbb{H}$, $z\neq\zeta$ *and for* $0\leqslant\rho<n$

$$
\begin{aligned}
\partial_z^{\rho} G_n(z,\zeta) &= \frac{(-1)^{\rho}}{(n-1-\rho)!(n-1)!}(\zeta-z)^{n-\rho-1}\overline{(\zeta-z)}^{n-1}\log\left|\frac{\bar\zeta-z}{\zeta-z}\right|^2 \\
&+ \sum_{\nu=0}^{\rho-1}\binom{\rho}{\nu}(-1)^{\nu}\frac{(\zeta-z)^{n-1-\nu}(\overline{\zeta-z})^{n-1}}{(n-1-\nu)!(n-1)!}(\rho-\nu-1)! \\
&\times \left(\frac{1}{(\zeta-z)^{\rho-\nu}}-\frac{1}{(\bar\zeta-z)^{\rho-z}}\right)+\frac{1}{(n-1)!^2}\sum_{\nu=1}^{n-1}\frac{1}{\nu}(\zeta-\bar\zeta)^{\nu} \\
&\times \sum_{\sigma=max\{0,\rho-\nu\}}^{min\{\rho,n-1-\nu\}}\binom{\rho}{\sigma}\frac{(n-1-\nu)!}{(n-1-\nu-\sigma)!} \\
&\times (\zeta-z)^{n-1-\nu-\sigma}(\overline{\zeta-z})^{n-1-\nu} \\
&\times \frac{\nu!}{(\nu-\rho+\sigma)!}(z-\bar z)^{\nu-\rho+\sigma}
\end{aligned}
\tag{1.4}
$$

Proof For $\rho=0$ obviously, (1.4) holds. Applying the Leibniz rule for $0\leqslant\rho\leqslant n-1$ by differentiating

$$G_n(z,\zeta) = \frac{1}{(n-1)!^2}|\zeta - z|^{2(n-1)} \log\left|\frac{\bar{\zeta}-z}{\zeta-z}\right|^2$$

$$+\frac{1}{(n-1)!^2}\sum_{\nu=1}^{n-1}\frac{1}{\nu}|\zeta - z|^{2(n-1-\nu)}(\zeta-\bar{\zeta})^\nu(z-\bar{z})^\nu,$$

one gets

$$\partial_z^{\rho+1}G_n(z,\zeta) = \frac{(-1)^{\rho+1}}{(n-2-\rho)!(n-1)!}(\zeta-z)^{n-\rho-2}\overline{(\zeta-z)}^{n-1}\log\left|\frac{\bar{\zeta}-z}{\zeta-z}\right|^2$$

$$+\frac{(-1)^\rho}{(n-1-\rho)!(n-1)!}(\zeta-z)^{n-\rho-1}\overline{(\zeta-z)^{n-1}}\left(\frac{1}{\zeta-z}-\frac{1}{\bar{\zeta}-z}\right)$$

$$+\sum_{\nu=0}^{\rho-1}\binom{\rho}{\nu}(-1)^{\nu+1}\frac{(\zeta-z)^{n-2-\nu}\overline{(\zeta-z)}^{n-1}}{(n-2-\nu)!(n-1)!}(\rho-\nu-1)!$$

$$\times\left(\frac{1}{(\zeta-z)^{\rho-\nu}}-\frac{1}{(\bar{\zeta}-z)^{\rho-\nu}}\right)+\sum_{\nu=0}^{\rho-1}\binom{\rho}{\nu}(-1)^\nu\frac{(\zeta-z)^{n-1-\nu}\overline{(\zeta-z)}^{n-1}}{(n-1-\nu)!(n-1)!}$$

$$\times(\rho-\nu)!\left(\frac{1}{(\zeta-z)^{\rho-\nu+1}}-\frac{1}{(\bar{\zeta}-z)^{\rho-\nu+1}}\right)$$

$$+\frac{1}{(n-1)!^2}\sum_{\nu=1}^{n-1}\frac{1}{\nu}(\zeta-\bar{\zeta})^\nu\sum_{\sigma=max\{0,\rho-\nu\}}^{min\{\rho,n-2-\nu\}}(-1)^{\sigma+1}\binom{\rho}{\sigma}\frac{(n-1-\nu)!}{(n-2-\nu-\sigma)!}$$

$$\times(\zeta-z)^{n-2-\nu-\sigma}\overline{(\zeta-z)}^{n-1-\nu}\frac{\nu!}{(\nu-\rho+\sigma)!}(z-\bar{z})^{\nu-\rho+\sigma}$$

$$+\frac{1}{(n-1)!^2}\sum_{\nu=1}^{n-1}\frac{1}{\nu}(\zeta-\bar{\zeta})^\nu\sum_{\sigma=max\{0,\rho-\nu+1\}}^{min\{\rho,n-1-\nu\}}(-1)^\sigma\binom{\rho}{\sigma}\frac{(n-1-\nu)!}{(n-1-\nu-\sigma)!}$$

$$\times(\zeta-z)^{n-1-\nu-\sigma}\overline{(\zeta-z)}^{n-1-\nu}\frac{\nu!}{(\nu-\rho-1-\sigma)!}(z-\bar{z})^{\nu-\rho-1-\sigma}$$

$$=\frac{(-1)^{\rho+1}}{(n-1-\rho-1)!(n-1)!}(\zeta-z)^{n-1-\rho-1}\overline{(\zeta-z)}^{n-1}\log\left|\frac{\bar{\zeta}-z}{\zeta-z}\right|^2$$

$$+\frac{(-1)^{\rho}}{(n-1-\rho)!(n-1)!}(\zeta-z)^{n-\rho-1}\overline{(\zeta-z)}^{n-1}\left(\frac{1}{\zeta-z}-\frac{1}{\bar{\zeta}-z}\right)$$

$$+\sum_{\nu=1}^{\rho}\binom{\rho}{\nu-1}(-1)^{\nu}\frac{(\zeta-z)^{n-1-\nu}\overline{(\zeta-z)}^{n-1}}{(n-1-\nu)!(n-1)!}(\rho-\nu)!$$

$$\times\left(\frac{1}{(\zeta-z)^{\rho-\nu+1}}-\frac{1}{(\bar{\zeta}-z)^{\rho-\nu+1}}\right)$$

$$+\sum_{\nu=0}^{\rho-1}\binom{\rho}{\nu}(-1)^{\nu}\frac{(\zeta-z)^{n-1-\nu}\overline{(\zeta-z)}^{n-1}}{(n-1-\nu)!(n-1)!}(\rho-\nu)!$$

$$\times\left(\frac{1}{(\zeta-z)^{\rho+2-\nu}}-\frac{1}{(\bar{\zeta}-z)^{\rho+1-\nu}}\right)$$

$$+\frac{1}{(n-1)!^2}\sum_{\nu=1}^{n-1}\frac{1}{\nu}(\zeta-\bar{\zeta})^{\nu}\sum_{\sigma=max\{1,\rho-\nu+1\}}^{min\{\rho+1,n-1-\nu\}}(-1)^{\sigma}\binom{\rho}{\sigma-1}\frac{(n-1-\nu)!}{(n-1-\nu-\sigma)!}$$

$$\times(\zeta-z)^{n-1-\nu-\sigma}\overline{(\zeta-z)}^{n-1-\nu}\frac{\nu!}{(\nu-\rho+\sigma-1)!}(z-\bar{z})^{\nu-\rho+\sigma-1}$$

$$+\frac{1}{(n-1)!^2}\sum_{\nu=1}^{n-1}\frac{1}{\nu}(\zeta-\bar{\zeta})^{\nu}\sum_{\sigma=max\{0,\rho-\nu+1\}}^{min\{\rho,n-1-\nu\}}(-1)^{\sigma}\binom{\rho}{\sigma}\frac{(n-1-\nu)!}{(n-1-\nu-\sigma)!}$$

$$\times(\zeta-z)^{n-1-\nu-\sigma}\overline{(\zeta-z)}^{n-1-\nu}\frac{\nu!}{(\nu-\rho-1-\sigma)!}(z-\bar{z})^{\nu+\rho-1-\sigma}$$

$$= \frac{(-1)^{\rho+1}}{(n-1-\rho-1)!(n-1)!}(\zeta - z)^{n-1-\rho-1}\overline{(\zeta - z)}^{n-1} \log \left| \frac{\bar{\zeta} - z}{\zeta - z} \right|^2$$

$$+ \sum_{\nu=0}^{\rho} (-1)^\nu \binom{\rho+1}{\nu} \frac{(\zeta - z)^{n-1-\nu}\overline{(\zeta - z)}^{n-1}}{(n-1-\nu)!(n-1)!}(\rho - \nu)!$$

$$\times \left(\frac{1}{(\zeta - z)^{\rho+1-\nu}} - \frac{1}{(\bar{\zeta} - z)^{\rho+1-\nu}} \right) + \frac{1}{(n-1)!^2} \sum_{\nu=1}^{n-1} \frac{1}{\nu}(\zeta - \bar{\zeta})^\nu$$

$$\times \sum_{\sigma=max\{0,\rho-\nu+1\}}^{min\{\rho+1,n-1-\nu\}} (-1)^\sigma \binom{\rho+1}{\sigma} \frac{(n-1-\nu)!}{(n-1-\nu-\sigma)!}(\zeta - z)^{n-1-\nu-\sigma}$$

$$\times \overline{(\zeta - z)}^{n-1-\nu} \frac{\nu!}{(\nu - \rho - 1 - \sigma)!}(z - \bar{z})^{\nu-\rho-1-\sigma},$$

This is (1.4) for $\rho + 1$ rather than for ρ. Thus (1.4) is proved by induction. Applying (1.4) for $\rho = n - 1$ gives

$$\partial_z^{n-1} G_n(z, \zeta) = \frac{(-1)^{n-1}}{(n-1)!}\overline{(\zeta - z)}^{n-1} \log |\frac{\bar{\zeta} - z}{\zeta - z}|^2 + \sum_{\nu=0}^{n-2} \binom{n-1}{\nu}$$

$$\times (-1)^\nu \frac{(\zeta - z)^{n-1-\nu}\overline{(\zeta - z)}^{n-1}}{(n-1-\nu)!(n-1)!}(n-2-\nu)!$$

$$\times \left(\frac{1}{(\zeta - z)^{n-1-\nu}} - \frac{1}{(\bar{\zeta} - z)^{n-1-\nu}} \right)$$

$$+ \frac{1}{(n-1)!^2} \sum_{\nu=1}^{n-1} \frac{1}{\nu}(\zeta - \bar{\zeta})^\nu (-1)^{n-1-\nu} \binom{n-1}{n-1-\nu} \frac{(n-1-\nu)!\nu!}{\nu!}$$

$$\times \overline{(\zeta - z)}^{n-1-\nu} = \frac{(-1)^{n-1}}{(n-1)!}\overline{(\zeta - z)}^{n-1} \log \left| \frac{\bar{\zeta} - z}{\zeta - z} \right|^2$$

$$+\sum_{\nu=0}^{n-2}(-1)^\nu\binom{n-1}{\nu}\frac{(\zeta-z)^{n-1-\nu}}{n-1-\nu}\frac{\overline{(\zeta-z)}^{n-1}}{(n-1)!}$$

$$\times\left(\frac{1}{(\zeta-z)^{n-1-\nu}}-\frac{1}{(\bar\zeta-z)^{n-1-\nu}}\right)+\frac{1}{(n-1)!}\sum_{\nu=1}^{n-1}\frac{(-1)^{n-1-\nu}}{\nu}(\zeta-\bar\zeta)^\nu$$

$$\times\overline{(\zeta-z)}^{n-1-\nu}=\frac{(-1)^{n-1}}{(n-1)!}\overline{(\zeta-z)}^{n-1}\log\left|\frac{\bar\zeta-z}{\zeta-z}\right|^2$$

$$+\frac{\overline{(\zeta-z)}^{n-1}}{(n-1)!}\sum_{\nu=0}^{n-2}\frac{(-1)^\nu}{n-1-\nu}\binom{n-1}{\nu}\left(1-\left(\frac{\zeta-z}{\bar\zeta-z}\right)^{n-1-\nu}\right)$$

$$+\frac{\overline{(\zeta-z)}^{n-1}}{(n-1)!}\sum_{\nu=1}^{n-1}\frac{(-1)^{n-1-\nu}}{\nu}\left(\frac{\zeta-\bar\zeta}{\overline{\bar\zeta-z}}\right)^\nu.$$

Corollary 1 For any $n\in\mathbb{N}$

$$\partial_z^n G_n(z,\zeta)=\frac{-1}{(n-1)!}\left(\frac{\zeta-\bar\zeta}{\bar\zeta-z}\right)^n\frac{\overline{(\zeta-z)}^{n-1}}{\zeta-z}. \tag{1.5}$$

Proof Differentiating the preceding formula shows

$$\partial_z^n G_n(z,\zeta)=\frac{(-1)^{n-1}}{(n-1)!}\overline{(\zeta-z)}^{n-1}\left(\frac{1}{\zeta-z}-\frac{1}{\bar\zeta-z}\right)$$

$$+\sum_{\nu=0}^{n-2}(-1)^{\nu+1}\binom{n-1}{\nu}(\zeta-z)^{n-2-\nu}\frac{\overline{(\zeta-z)}^{n-1}}{(n-1)!}$$

$$\times\left(\frac{1}{(\zeta-z)^{n-1-\nu}}-\frac{1}{(\bar\zeta-z)^{n-1-\nu}}\right)+\sum_{\nu=0}^{n-2}(-1)^\nu\binom{n-1}{\nu}$$

$$\times(\zeta-z)^{n-1-\nu}\frac{\overline{(\zeta-z)}^{n-1}}{(n-1)!}\left(\frac{1}{(\zeta-z)^{n-\nu}}-\frac{1}{(\bar\zeta-z)^{n-\nu}}\right)$$

$$= \sum_{\nu=0}^{n-1} (-1)^\nu \binom{n}{\nu} (\zeta - z)^{n-1-\nu} \frac{\overline{(\zeta - z)}^{n-1}}{(n-1)!} \left(\frac{1}{(\zeta - z)^{n-\nu}} - \frac{1}{(\bar{\zeta} - z)^{n-\nu}} \right)$$

$$= \frac{\overline{(\zeta - z)}^{n-1}}{(n-1)!} \sum_{\nu=0}^{n-1} (-1)^\nu \binom{n}{\nu} \left(1 - (\frac{\zeta - z}{\bar{\zeta} - z})^{n-\nu} \right) \frac{1}{\zeta - z}$$

$$= \frac{\overline{(\zeta - z)}^{n-1}}{(n-1)!(\zeta - z)} [\sum_{\nu=0}^{n-1} (-1)^\nu \binom{n}{\nu} - \sum_{\nu=0}^{n-1} (-1)^\nu \binom{n}{\nu} \left(\frac{\zeta - z}{\bar{\zeta} - z} \right)^{n-\nu}]$$

$$= \frac{\overline{(\zeta - z)}^{n-1}}{(n-1)!} \frac{1}{\zeta - z} [(1-1)^n - (-1)^n - \left(-1 + \frac{\zeta - z}{\bar{\zeta} - z} \right)^n + (-1)^n]$$

$$= \frac{-1}{(n-1)!} \frac{(\overline{\zeta - z})^{n-1}}{(\zeta - z)} \left(\frac{\zeta - z}{\bar{\zeta} - z} - 1 \right)^n = \frac{-1}{(n-1)!} \frac{(\zeta - \bar{\zeta})^n}{|\zeta - z|^2} \left(\frac{\overline{(\zeta - z)}}{\bar{\zeta} - z} \right)^n$$

$$= \frac{-1}{(n-1)!} \left(\frac{\zeta - \bar{\zeta}}{\bar{\zeta} - z} \right)^n \frac{\overline{(\zeta - z)}^{n-1}}{\zeta - z}.$$

Lemma 2 *For* $0 \leqslant \rho < n$, $z = \bar{z}$, $\partial_z^\rho G_n(z, \zeta) = 0$.

Proof If $z = \bar{z}$ and $0 \leqslant \rho < n$ then from (1.4)

$$\partial_z^\rho G_n(z, \zeta) = \sum_{\nu=0}^{\rho-1} (-1)^\nu \binom{\rho}{\nu} \frac{(\zeta - z)^{(n-1-\rho)}}{(n-1-\nu)!} \frac{\overline{(\zeta - z)}^{n-1}}{(n-1)!} (\rho - \nu - 1)!$$

$$\times \left(1 - (\frac{\zeta - z}{\bar{\zeta} - z})^{\rho-\nu} \right) + \frac{1}{(n-1)!^2} \sum_{\nu=1}^{\rho} \frac{1}{\nu} (\zeta - \bar{\zeta})^\nu (-1)^{\rho-\nu} \binom{\rho}{\rho - \nu}$$

$$\times \frac{(n-1-\nu)!}{(n-1-\rho)!} \nu! (\zeta - z)^{n-1-\rho} \overline{(\zeta - z)}^{n-1-\nu}$$

$$= \sum_{\nu=1}^{\rho}(-1)^{\rho-\nu}\binom{\rho}{\nu}\frac{(\nu-1)!}{(n-1-\rho+\nu)!}\frac{(\zeta-z)^{n-1-\rho}\overline{(\zeta-z)}^{n-1}}{(n-1)!}$$

$$\times\left(1-(\frac{\zeta-z}{\bar\zeta-z})^{\nu}\right)+\frac{1}{(n-1)!^2}\sum_{\nu=1}^{\rho}(-1)^{\rho-\nu}\binom{\rho}{\nu}(\nu-1)!$$

$$\times\binom{n-1-\nu}{\rho-\nu}(\rho-\nu)!(\zeta-z)^{n-1-\rho}\overline{(\zeta-z)}^{n-1-\nu}(\zeta-\bar\zeta)^{\nu}$$

$$=\frac{1}{(n-1)!^2}\sum_{\nu=1}^{\rho}(-1)^{\rho-\nu}\binom{\rho}{\nu}(\nu-1)!\binom{n-1}{\rho-\nu}(\rho-\nu)!$$

$$\times(\zeta-z)^{n-1-\rho}\overline{(\zeta-z)}^{n-1}\sum_{\mu}^{\nu}\binom{\nu}{\mu}\left(\frac{\zeta-\bar\zeta}{\bar\zeta-z}\right)^{\mu}$$

$$+\frac{1}{(n-1)!^2}\sum_{\nu=1}^{\rho}(-1)^{\rho-\nu}\binom{\rho}{\nu}(\nu-1)!\binom{n-1-\nu}{\rho-\nu}$$

$$\times(\rho-\nu)!(\zeta-z)^{n-1-\rho}\overline{(\zeta-z)}^{n-1-\nu}(\zeta-\bar\zeta)^{\nu}$$

$$=\frac{1}{(n-1)!^2}\sum_{\mu=1}^{\rho}\sum_{\nu=\mu}^{\rho}(-1)^{\rho-\nu-1}\frac{\rho!(\nu-1)!}{(\nu-\mu)!\mu!}\binom{n-1}{\rho-\nu}$$

$$\times(\zeta-z)^{n-1-\rho}(\bar\zeta-z)^{n-1-\mu}(\zeta-\bar\zeta)^{\mu}+\frac{1}{(n-1)!^2}$$

$$\times\sum_{\mu=1}^{\rho}(-1)^{\rho-\mu}\frac{\rho!}{\mu}\binom{n-1-\mu}{\rho-\mu}(\zeta-z)^{n-1-\rho}\overline{(\zeta-z)}^{n-1-\mu}(\zeta-\bar\zeta)^{\mu}$$

follows. From

$$\sum_{\nu=\mu}^{\rho}(-1)^{\nu-\mu}\binom{\nu-1}{\mu-1}\binom{n-1}{\rho-\nu}=\binom{n-1-\nu}{\rho-\mu}$$

follows

$$\partial_z^\rho G_n(z,\zeta) = -\frac{1}{(n-1)!^2}\sum_{\mu=1}^{\rho}(-1)^{\rho-1}\frac{\rho!}{\mu}\binom{n-1-\mu}{\rho-\mu}$$

$$\times(\zeta-z)^{n-1-\rho}(\bar\zeta-z)^{n-1-\mu}(\zeta-\bar\zeta)^{\mu}+\frac{1}{(n-1)!^2}\sum_{\mu=1}^{\rho}(-1)^{\rho-\mu}$$

$$\times\frac{\rho!}{\mu}\binom{n-1-\mu}{\rho-\mu}(\zeta-z)^{n-1-\rho}(\bar\zeta-z)^{n-1-\nu}(\zeta-\bar\zeta)^{\mu}=0$$

Lemma 3 *For* $1 \leqslant \rho < n$

$$\begin{aligned}(\partial_z\partial_{\bar z})^\rho G_n(z,\zeta) &= G_{n-\rho}(z,\zeta)+\sum_{\mu=0}^{\rho-1}\frac{(\zeta-\bar\zeta)^{n-\mu-1}}{(n-\mu-2)!(n-\mu-1)!} \\ &\times (\partial_z\partial_{\bar z})^{\rho-\mu}\tilde G_{n-\mu}(z,\zeta), \qquad (1.6)\end{aligned}$$

where $\tilde G_n(z,\zeta)$ *is some function satisfying*

$$\begin{aligned}\partial_z\partial_{\bar z}\tilde G_n(z,\zeta) &= (z-\bar z)^{n-2}\left(\frac{1}{\bar\zeta-z}-\frac{1}{\zeta-\bar z}\right) \\ &= (z-\bar z)^{n-2}g_1(z,\zeta). \qquad (1.7)\end{aligned}$$

Proof If $\rho = 1$ then

$$(n-1)!^2\partial_z G_n(z,\zeta) = (-n-1)(\zeta-z)^{n-2}\overline{(\zeta-z)}^{n-1}\log\left|\frac{\bar\zeta-z}{\zeta-z}\right|^2$$

$$-(n-1)\sum_{\nu=1}^{n-2}\frac{1}{\nu}(\zeta-z)^{n-2-\nu}(\zeta-\bar z)^{n-1-\nu}(z-\bar z)^{\nu}(\zeta-\bar\zeta)^{\nu}-\frac{(z-\bar z)^{n-1}(\zeta-\bar\zeta)^{n-1}}{\bar\zeta-z},$$

$$\partial_z\partial_{\bar z}G_n(z,\zeta) = (n-1)^2G_{n-1}(z,\zeta)+(n-1)(z-\bar z)^{n-2}(\zeta-\bar\zeta)^{n-1}$$

$$\times\left(\frac{1}{\bar\zeta - z} - \frac{1}{\zeta - \bar z}\right) = (n-1)^2 G_{n-1}(z,\zeta) + (n-1)(\zeta-\bar\zeta)^{n-1}\partial_z\partial_{\bar z}\tilde G_n(z,\zeta).$$

By an induction argument from (1.6) follows

$$(\partial_z\partial_{\bar z})^{\rho+1}G_n(z,\zeta) = \partial_z\partial_{\bar z}G_{n-\rho}(z,\zeta)$$

$$+\sum_{\mu=0}^{\rho-1}\frac{(\zeta-\bar\zeta)^{n-\mu-1}}{(n-\mu-2)!(n-\nu-1)!}(\partial_z\partial_{\bar z})^{\rho-\mu+1}\tilde G_{n-\mu}(z,\zeta)$$

$$= [(n-\rho-1)^2 G_{n-\rho-1}(z,\zeta) + (n-\rho-1)(\zeta-\bar\zeta)^{n-\rho-1}\partial_z\partial_{\bar z}\tilde G_{n-\rho}(z,\zeta)]$$

$$+\sum_{\mu=0}^{\rho-1}\frac{(\zeta-\bar\zeta)^{n-\mu-1}}{(n-\mu-2)!(n-\mu-1)!}(\partial_z\partial_{\bar z})^{\rho-\mu+1}\tilde G_{n-\mu}(z,\zeta)$$

$$= G_{n-\rho-1}(z,\zeta) + \frac{(\zeta-\bar\zeta)^{n-\rho-1}}{(n-\rho-2)!(n-\rho-1)!}\partial_z\partial_{\bar z}\tilde G_{n-\rho}(z,\zeta)$$

$$+\sum_{\mu=0}^{\rho-1}\frac{(\zeta-\bar\zeta)^{n-\mu-1}}{(n-\mu-2)!(n-\mu-1)!}(\partial_z\partial_{\bar z})^{\rho-\mu+1}\tilde G_{n-\mu}(z,\zeta)$$

$$= G_{n-\rho-1}(z,\zeta) + \sum_{\mu=0}^{\rho}\frac{(\zeta-\bar\zeta)^{n-\mu-1}}{(n-\mu-2)!(n-\mu-1)!}(\partial_z\partial_{\bar z})^{n-\mu+1}\tilde G_{n-\mu}(z,\zeta)$$

This is (1.6) for $\rho+1$ rather than for ρ.

Lemma 4 *For* $2 \leqslant 2\rho \leqslant n-2$

$$\begin{aligned}(\partial_z\partial_{\bar z})^{\rho}\tilde G_n(z,\zeta) &= (-1)^{\rho+1}\sum_{\mu=\rho+1}^{2\rho}\frac{(n-2)!}{(n-\mu)!}(z-\bar z)^{n-\mu} \\ &\times \frac{(\rho-1)!}{(\mu-\rho-1)!}g_{1+2\rho-\mu}(z,\zeta),\end{aligned} \tag{1.8}$$

where

$$g_\alpha(z,\zeta) = \frac{1}{(\bar{\zeta}-z)^\alpha} + \frac{(-1)^\alpha}{(\zeta-\bar{z})^\alpha} \tag{1.9}$$

for all $\alpha \in \mathbb{N}$.

Proof For $\rho = 1$ we have

$$\partial_z\partial_{\bar{z}}\tilde{G}_n(z,\zeta) = (z-\bar{z})^{n-2} g_1(z,\zeta),$$

then differentiating (1.8)

$$\begin{aligned}
\partial_z(\partial_z\partial_{\bar{z}})^\rho \tilde{G}_n(z,\zeta) &= (-1)^{\rho+1}\sum_{\mu=\rho+1}^{2\rho}[\frac{(n-2)!}{(n-\mu-1)!}(z-\bar{z})^{n-\mu-1} \\
&\times \frac{(\rho-1)!}{(\mu-\rho-1)!} g_{1+2\rho-\mu}(z,\zeta) \\
&+ \frac{(n-2)!}{(n-\mu)!}(1+2\rho-\mu)(z-\bar{z})^{n-\mu} \\
&\times \frac{(\rho-1)!}{(\mu-\rho-1)}\frac{1}{(\bar{\zeta}-z)^{2+2\rho-\mu}}]
\end{aligned}$$

and

$$(\partial_z\partial_{\bar{z}})^{\rho+1}\tilde{G}_n(z,\zeta) = (-1)^\rho \sum_{\mu=\rho+1}^{2\rho}[\frac{(n-2)!}{(n-\mu-1)!}(z-\bar{z})^{n-\mu-2}\frac{(\rho-1)!}{(\mu-\rho-1)!}$$

$$\times g_{1+2\rho-\mu}(z,\zeta) + \frac{(n-2)!}{(n-\mu-1)!}(z-\bar{z})^{n-\mu-1}\frac{(\rho-1)!}{(\mu-\rho-1)!}$$

$$\times(1+2\rho-\mu)\frac{(-1)^{2\rho-\mu}}{(\zeta-\bar{z})^{2+2\rho-\mu}} + \frac{(n-2)!}{(n-\mu-1)!}(z-\bar{z})^{n-\mu-1}$$

$$\times\frac{(\rho-1)!}{(\mu-\rho-1)!}(1+2\rho-\mu)\frac{1}{(\bar{\zeta}-z)^{2+2\rho-\mu}}]$$

$$= (-1)^\rho \sum_{\mu=\rho+1}^{2\rho} [\frac{(n-2)!}{(n-\mu-2)!}(z-\bar{z})^{n-\mu-2}\frac{(\rho-1)!}{(\mu-\rho-1)!}g_{1+2\rho-\mu}(z,\zeta)$$

$$+\frac{(n-2)!}{(n-\mu-1)!}(z-\bar{z})^{n-\mu-1}\frac{(\rho-1)!}{(\mu-\rho-1)!}(1+2\rho-\mu)g_{2+2\rho-\mu}(z,\zeta)]$$

$$= (-1)^\rho \sum_{\mu=\rho+3}^{2\rho+2} [\frac{(n-2)!}{(n-\mu)!}(z-\bar{z})^{n-\mu}\frac{(\rho-1)!}{(\mu-\rho-3)!}g_{3+2\rho-\mu}(z,\zeta)$$

$$+(-1)^\rho \sum_{\mu=\rho+2}^{2\rho+1} \frac{(n-2)!}{(n-\mu)!}(z-\bar{z})^{n-\mu}\frac{(\rho-1)!}{(\mu-\rho-2)!}(2+2\rho-\mu)g_{3+2\rho-\mu}(z,\zeta)]$$

$$= (-1)^\rho[\frac{(n-2)!}{(n-\rho-2)!}(z-\bar{z})^{n-\rho-2}(\rho-1)!\rho g_{\rho+1}(z,\zeta)$$

$$+\sum_{\mu=\rho+3}^{2\rho+1} \frac{(n-2)!}{(n-\mu)!}(z-\bar{z})^{n-\mu}\frac{(\rho-1)!}{(\mu-\rho-2)!}[\mu-\rho-2+2+2\rho-\mu]$$

$$\times g_{3+2\rho-\mu}(z,\zeta)+\frac{(n-2)!}{(n-2\rho-2)!}(z-\bar{z})^{n-2\rho-2}\frac{(\rho-1)!}{(\rho-1)!}g_1(z,\zeta)]$$

$$= (-1)^\rho[\frac{(n-2)!}{(n-\rho-2)!}(z-\bar{z})^{n-\rho-2}\frac{\rho!}{\mu!}g_{\rho+1}(z,\zeta)+\sum_{\mu=\rho+3}^{2\rho+1}\frac{(n-2)!}{(n-\mu)!}$$

$$\times(z-\bar{z})^{n-\mu}\frac{\rho!}{(\mu-\rho-2)!}g_{3+2\rho-\mu}(z,\zeta)$$

$$+\frac{(n-2)!}{(n-2\rho-2)!}(z-\bar{z})^{n-2\rho-2}\frac{\rho!}{\rho!}g_1(z,\zeta)]$$

$$= (-1)^{\rho+2} \sum_{\mu=\rho+2}^{2\rho+2} \frac{(n-2)!}{(n-\mu)!}(z-\bar{z})^{n-\mu} \frac{\rho!}{(\mu-\rho-2)!} g_{3+2\rho-\mu}(z,\zeta)$$

follows. This reproduces (1.8) with $\rho+1$ replacing ρ.

Theorem 1 *For* $0 \leq \rho \leq [\frac{n-1}{2}]$

$$\begin{aligned}(\partial_z\partial_{\bar{z}})^\rho G_n(z,\zeta) &= G_{n-\rho}(z,\zeta) \\ &+ \sum_{\nu=0}^{\rho-1}\sum_{\mu=0}^{\rho-1-\nu}(-1)^{\rho-\mu-1}\frac{(\rho-\mu-1)!}{(n-\nu-\rho-1)!(n-1-\mu)!\nu!} \\ &\times (\zeta-\bar{\zeta})^{n-\mu-1}(z-\bar{z})^{n-\rho-1-\nu}g_{\rho-\mu-\nu}(z,\zeta) \qquad (1.10)\end{aligned}$$

Remark 1 $(\partial_z\partial_{\bar{z}})^\rho G_n(z,\zeta) = 0$ on $z=\bar{z}$ for $2\rho < n$ i.e. $0 \leqslant \rho \leqslant [\frac{n-1}{2}]$ because $0 < n-\rho-1-\nu$ i.e. $\nu < n-\rho-1$, as $\nu \leqslant \rho-1 < n-\rho-1$.

Proof From (1.6) and (1.8)

$$(\partial_z\partial_{\bar{z}})^\rho G_n(z,\zeta) - G_{n-\rho}(z,\zeta) = \sum_{\mu=0}^{\rho-1}\frac{(\zeta-\bar{\zeta})^{n-\mu-1}}{(n-\mu-2)!(n-\mu-1)!}$$

$$\times(-1)^{\rho-\mu-1}\sum_{\nu=\rho-\mu+1}^{2(\rho-\mu)}\frac{(n-\mu-2)!}{(n-\mu-\nu)!}(z-\bar{z})^{n-\mu-\nu}\frac{(\rho-\mu-1)!}{(\nu-\rho+\mu-1)!}$$

$$\times g_{1+2\rho-2\mu-\nu}(z,\zeta) = \sum_{\mu=0}^{\rho-1}\sum_{\nu=0}^{\rho-1-\mu}(-1)^{\rho-1-\mu}\frac{(\rho-\mu-1)!}{(n-\mu-1)!(n-\rho-\nu-1)!\nu!}$$

$$\times(\zeta-\bar{\zeta})^{n-\mu-1}(z-\bar{z})^{n-\rho-\nu-1}g_{\rho-\mu-\nu}(z,\zeta)$$

$$= \sum_{\mu=0}^{\rho-1}\sum_{\nu=\mu}^{\rho-1}(-1)^{\rho-1-\mu}\frac{(\rho-\mu-1)!}{(n-1-\mu)!(n-\rho-1-\nu+\mu)!(\nu-\mu)!}$$

$$\times(\zeta-\bar{\zeta})^{n-\mu-1}(z-\bar{z})^{n-\rho-\nu+\mu-1}g_{\rho-\nu}(z,\zeta)$$

$$=\sum_{\nu=0}^{\rho-1}\sum_{\mu=0}^{\nu}(-1)^{\rho-1+\mu-\nu}\frac{(\rho-\nu+\mu-1)!}{(n-1-\nu+\mu)!(n-\rho-1-\mu)!\mu!}$$

$$\times(\zeta-\bar{\zeta})^{n-\nu+\mu-1}(z-\bar{z})^{n-\rho-1-\mu}g_{\rho-\nu}(z,\zeta)$$

Lemma 5 *For* $0\leqslant\sigma,\rho,\ \sigma+2\rho\leqslant n-1$

$$\begin{aligned}
\partial_z^\sigma(\partial_z\partial_{\bar{z}})^\rho G_n(z,\zeta) &= \frac{1}{(n-1-\rho)!^2}\partial_z^\sigma G_{n-\rho}(z,\zeta)\\
&+ \sum_{\nu=0}^{\rho-1}\sum_{\mu=0}^{\rho-1-\nu}(-1)^{\rho-\mu-1}\frac{(\rho-\mu-1)!(\zeta-\bar{\zeta})^{n-\mu-1}}{(n-\rho-1-\nu)!(n-1-\mu)!\nu!}\\
&\times [\frac{(n-\rho-1-\nu)!}{(n-\rho-1-\nu-\sigma)!}(z-\bar{z})^{n-\rho-1-\nu-\sigma}g_{\rho-\mu-1}(z,\zeta)\\
&+ \sum_{\lambda=0}^{min\{\sigma-1,n-\rho-1-\nu\}}\frac{(n-\rho-1-\nu)!}{(n-\rho-1-\nu-\lambda)!}\\
&\times \frac{(\rho-\mu-\nu+\sigma-\lambda-1)!}{(\rho-\mu-\nu-1)!}\\
&\times (z-\bar{z})^{n-\rho-1-\nu-\lambda}\frac{1}{(\bar{\zeta}-z)^{\rho-\mu-\nu+\sigma-\lambda}}]
\end{aligned}\qquad(1.11)$$

Proof Formula (1.11) follows from (1.10) by differentiation.

Corollary 2 If $\sigma+2\rho\leqslant n-1$, then

$$\partial_z^\sigma(\partial_z\partial_{\bar{z}})^\rho G_n(z,\zeta)=0$$

for $z=\bar{z}$.

Proof Because $\sigma\leqslant n-\rho-1$ then $\partial_z^\sigma G_{n-\rho}(z,\zeta)=0$ for $z=\bar{z}$. From $\sigma+2\rho\leqslant n-1$ if follows $\sigma<n-2\rho\leqslant n-\rho-1-\nu$ for $0\leqslant\nu\leqslant\rho-1$ i.e. $0<n-\rho-1-\nu-\lambda$ for $0\leqslant\lambda\leqslant\sigma<n-\rho-1-\nu$. Hence setting $z=\bar{z}$ in (11) and applying Lemma 2 the result follows.

Remark 2 The result of Corollary 2 can be reformulated: if $\mu + \nu \leqslant n - 1$, then

$$\partial_z^\mu \partial_{\bar{z}}^\nu G_n(z, \zeta) = 0$$

for $z = \bar{z}$.