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Elliptic curves over function fields of elliptic curves

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Contents

Introduction

Let k be a finite field. Let (C,O) be an elliptic curve over k and let $k(C)$ be the function field of C over k. The main objects of study of this dissertation are elliptic curves over $k(C)$. To begin, we wish to give the reader a small motivation for the problem we will consider.

Let $f(x,y)$ be a polynomial in two variables with $\mathbb{Q}$ coefficients and consider the algebraic curve given by the zero set of f, i.e.

$$C: \quad f(x,y) = 0.$$

Suppose moreover that C has a rational point. It was already known to the ancient Greeks that if C is a conic ($g(C) = 0$) with a rational point, then C always has infinitely many rational points and it was proven by G. Faltings [Fal86] that a curve C of genus $g(C) > 1$ never does. The most interesting case is $g(C) = 1$, i.e for C an elliptic curve. Here the rational points are not completely understood.

Let us briefly recall what elliptic curves are. An elliptic curve (E,O) over $\mathbb{Q}$ is the projective closure of a curve defined by an affine Weierstraß equation

$$E_{A,B}: \quad y^2 = x^3 + Ax + B \tag{0.1}$$

for some $A, B \in \mathbb{Q}$ satisfying $4A^3 + 27B^2 \neq 0$. These curves are the simplest examples of projective algebraic groups of positive dimension. The abelian group $E(\mathbb{Q})$ of rational points on E is finitely generated [Mor22] and hence

$$E(\mathbb{Q}) = \mathbb{Z}^r \oplus (\text{ torsion })$$

for some nonnegative integer r and some finite abelian group (torsion). The torsion subgroup is well understood [Maz78]. The number r measures the number of points needed to generate all rational points on the curve and it is called the (Mordell-Weil) rank of an elliptic curve. Hence the rank carries information about the elliptic curve and its rational points. Already in 1901 H. Poincare asked [Poi01] an important question

"What is the range of possibilities for the minimum number of generators of $E(\mathbb{Q})$?"

There are algorithms that compute the rank for given A and B of bounded size, but no general method is known.

There are however some results about how the rank behaves on average. Consider the set $\mathcal{E}$ consisting of elliptic curves $E_{A,B}$ given by the equation (0.1) and such that (0.1) is minimal (i.e. there is no $u \neq \pm 1$ such that u^6 divides A and u^4 divides B). We order these elliptic curves by their height, which we define as $H(E_{A,B}) = H(A;B) := \max\{4|A^3|, 27B^2\}$. Let $X \in \mathbb{R}$ and take $\mathcal{E}_{<X} := \{E_{AB} \in \mathcal{E} : \quad H(E_{AB}) < X\}$. Consider the average rank

$$\rho := \lim_{X\to\infty} \frac{\sum\limits_{E_{A,B}\in\mathcal{E}_{<X}} r(E_{A,B})}{\sum\limits_{E_{A,B}\in\mathcal{E}_{<X}} 1}, \tag{0.2}$$

(if the limit exists). It is conjectured (Goldfeld, Katz-Sarnak) that $\rho = \frac{1}{2}$. Recently M. Bhargava and A. Shankar [BS10] proved, that

$$\limsup_{X\to\infty} \frac{\sum_{E_{A,B}\in\mathcal{E}_{<X}} r(E_{A,B})}{\sum_{E_{A,B}\in\mathcal{E}_{<X}} 1} \leq 1.17$$

and that $\liminf_{X\to\infty}$ is non-zero (see [Poo12] for an overview about average ranks of elliptic curves over $\mathbb{Q}$).

It is natural to ask if one can obtain a result similar to the work of M. Bhargava and A. Shankar in the function field case. Let us first recall this set up. Let k be a finite field of characteristic $p \geq 5$. Let C be a smooth, geometrically connected, projective curve over k and let $k(C)$ be the function field of C over k. Let (E, O) be an elliptic curve over $k(C)$, i.e. a curve of genus 1 defined by an affine Weierstraß equation

$$y^2 = x^3 + A_4x + A_6 \tag{0.3}$$

with $A_i \in k(C)$ and $\Delta = 4a_4^3 + 27A_6^2 \neq 0$. Néron [Ner] proved that the Mordell-Weil group $E(k(C))$ of $k(C)$-rational points on E is finitely generated. And therefore we can consider the (Mordell-Weil) rank $r(E(k(C)))$ of E over $k(C)$. Again the torsion subgroup is well understood, see [KM, Theorem 2.3.1]. To define the average rank we need the notion of a height of an elliptic curve over a function field. It is a little more complicated compared with the height over $\mathbb{Q}$. Every elliptic curve (E, O) over a function field $k(C)$ has its minimal model i.e. an elliptic surface $f : \mathcal{E} \to C$, where $\mathcal{E}$ is a unique smooth and proper surface over k admitting a flat and relatively minimal morphism f to C with generic fiber $E/k(C)$. We define the height $h(E, O)$ of an elliptic curve (E, O) as the degree of the line bundle $f_*\omega_{\mathcal{E}/C}$. The height is defined as $h(E, O) = \deg_C(f_*\omega_{\mathcal{E}/C})$. Now we can consider the average rank of elliptic curves over $k(C)$ of height d

$$AR_d = \frac{\sum_{[(E,O)];h(E)=d} r\left(E(k(C))\right)}{\sum_{[(E,O)];h(E)=d} 1}.$$

The closest result to the work of M. Bhargava and A. Shankar in the function field case is a theorem of of A. J. de Jong who proved that in the case $C = \mathbb{P}_k^1$ and $p \geq 7$ we have

$$\limsup_{q\to\infty} \limsup_{d\to\infty} AR_d \leq \frac{3}{2}. \tag{0.4}$$

Remark 0.0.1. We would like to mention that de Jong's work predates the work of M. Bhargava and A. Shankar.

The crucial part of the paper [deJ02] consists of showing that the following bound holds

$$\limsup_{d\to\infty} q^{-10d-1} \sum_{[(E,O)];h(E)=d} \frac{3^{r(E(k(C)))} - 1}{\#\mathrm{Aut}(E, O)} \leq \frac{3q^9}{(q^3-1)(q^2-1)^2(q-1)^2},$$

which he obtains by counting integral models of geometric objects representing elements of Mordell-Weil groups (In fact A. J. de Jong proves a stronger result, namely he bounds the average 3-Selmer rank, however as we are only interested in average ranks we do not recall it here). For an overview about average ranks over function fields see [Ulm04].

In this dissertation we treat the case when $C = (C, O)$ is an elliptic curve over k. The main result the author has obtained is the following bound

Theorem 4.5.10 *Let (C, O) be an elliptic curve over a finite field $k = \mathbb{F}_q$ with q elements and of characteristic $p > 7$. Let $\#C(\mathbb{F}_{q^n})$ denotes the number of $\mathbb{F}_{q^n}$-rational points of C. Then*

$$\limsup_{d\to\infty} q^{-10d+1}\left(\sum_{[(E,O)],h(E)=d} \frac{3^{r(E)}-1}{\#\mathrm{Aut}(E,O)}\right) \leq \frac{\#C(\mathbb{F}_q)A}{(q-1)^5(q+1)^2q}, \tag{0.5}$$

where the sum is taken over isomorphism classes of elliptic curves (E, O) over $\mathbb{F}_q(C)$ of height d and the number A is given by

$$A = 2\#C(\mathbb{F}_q)q^8 + \left(2\#C(\mathbb{F}_q)^2 + \#C(\mathbb{F}_{q^2})^2\right)q^7$$

$$+\left(\#C(\mathbb{F}_q)\#C(\mathbb{F}_{q^2}) + \#C(\mathbb{F}_{q^3})^3 + 1 - 2\#C(\mathbb{F}_q) + \#C(\mathbb{F}_{q^2})^2\right)q^6 + \left(-2 - 2\#C(\mathbb{F}_{q^2})^2 - 4\#C(\mathbb{F}_q)^2\right)q^5$$

$$+\left(-2\#C(\mathbb{F}_{q^3})^3 - 1 - 2\#C(\mathbb{F}_q) - 2\#C(\mathbb{F}_{q^2})^2 - 2\#C(\mathbb{F}_q)\#C(\mathbb{F}_{q^2})\right)q^4$$

$$+\left(\#C(\mathbb{F}_{q^2})^2 + 2\#C(\mathbb{F}_q)^2 + 4\right)q^3$$

$$+\left(\#C(\mathbb{F}_{q^3})^3 + \#C(\mathbb{F}_{q^2})^2 + \#C(\mathbb{F}_q)^2 - 1 + 2\#C(\mathbb{F}_q) + \#C(\mathbb{F}_q)\#C(\mathbb{F}_{q^2})\right)q^2 - 2q + 1$$

Remark 0.0.2. We would like to mention that as much as for the function field case of $\mathbb{P}^1_k$ (de Jong) there is a corresponding statement over rational numbers (Bhargava-Shankar). We do not have a corresponding bound on the number theory side.

To prove the theorem we apply the construction of A. J. de Jong from [deJ02]. We first define a set $\mathcal{A}_{C,d}$, whose elements are elliptic families. We prove that it is finite and bound its number of elements. Then we show that $\mathcal{A}_{C,d}$ has a subset whose elements correspond to elements of Mordell-Weil groups.

Observe that the polynomial A in the variable q has degree 8 (as well as the dominator of (0.5)) and leading coefficient $2\#C(\mathbb{F}_q)$. The main difference between the rational function field case and the case of the function field of an elliptic curve is that in our case, the number of the $\mathbb{F}_q$-rational points of the elliptic curve contributes to the limit. This is due to the fact that we can identify an elliptic curve with its dual and therefore the number of line bundles of a given degree is equal to the number of its k−rational points.

Furthermore, the author believes that the bound

$$\limsup_{q\to\infty}\limsup_{d\to\infty} AR_d \leq \#C(\mathbb{F}_q) \tag{0.6}$$

holds. One should be able to write and solve a recurrency similar to [deJ02, Proposition 4.12] and get a lower bound for the number of weighted number of isomorphism classes of elliptic curves over $k(C)$ of a given height d. This bound should be $\#C(\mathbb{F}_q)q^{10d-1}F(q)$ where $F(q)$ is some rational function in q and then by a similar argument as in the proof of [deJ02, Corollary 1.3] (0.6) should hold. As the group of k-rational points of C can be large, this bound would not be comparable to any of the bounds of M. Bhargava and A. Shankar or A. J. de Jong, but no other precise information in this case is known.

Let us give a few words about the structure of the thesis. In Chapter 1.3 we introduce basic facts about vector bundles on elliptic curves, which we need especially in Chapter 3. In Chapter 2 we give necessary definitions and prove important results, about elliptic curves over function fields of elliptic curves and give an upper bound the weighted number of their isomorphism classes. Chapters 3 and 4 are the ingredients to prove the main result. In particular Sections 3.1, 3.2 of Chapter 3 and Chapter 4 are theoretical and Sections 3.2.1, 3.2.2, 3.2.3 and 3.3 are computational.

Conventions and notations

(i) We denote by k a finite field with $q = p^e$ elements of characteristic $p \geq 5$ and we work in the category of schemes of finite type over k. For a finite set X by $\#X$ we denote, the number of elements in X.

(ii) A curve is a geometrically integral variety over k of dimension 1, similarly a surface is a geometrically integral variety of dimension 2.

(iii) An elliptic curve defined over a field K is a couple (C, O), where C is a smooth projective curve of genus 1 defined over K and O is a fixed K-rational point of C. By $\mathrm{Aut}(E, O)$ we denote the group of automorphisms of the elliptic curve E, that preserve the rational point O.

(iv) An elliptic surface is a morphism $f : X \to C$ of a smooth projective surface to a smooth projective curve C whose generic fibre X_η is a smooth curve of genus 1 over $\kappa(\eta)$. An isomorphism of elliptic surfaces $f_1 : X_1 \to C$ and $f_2 : X_2 \to C$ is an isomorphism of surfaces $\phi : X_1 \to X_2$ such that it commutes with maps to C, $f_1 = f_2 \circ \phi$. If $f : X \to C$ admits a section σ (a map $\sigma : C \to X$ such that $f \circ \sigma = \mathrm{id}_C$), then by $\mathrm{Aut}(f : X \to C, \sigma)$ we denote the group of automorphisms $\phi : X \to X$ over C, that preserve the given section σ, $\phi \circ \sigma = \sigma$.

(v) Let X be a smooth surface. A (-1)-curve in X is a closed sub scheme $C \subset X$ such that $C \cong \mathbb{P}^1_{k'}$ for some field extension $k \subset k'$ and such that $\mathcal{N}_C X \cong \mathcal{O}_C(-1)$.

(vi) An elliptic surface $f : X \to C$ is called relatively minimal if there are no (-1)-curves contained in the fibers of f. This is equivalent to saying, that every birational morphism of elliptic surfaces $\varphi : X_1 \to X_2$ over C is an isomorphism.

Chapter 1

Vector bundles on elliptic curves

This chapter contains basic facts about vector bundles on elliptic curves, which we use in later chapters, especially in Chapter 3. Section 1.1 includes general informations about vector bundles on curves, we give definitions of a semi-stable vector bundle, globally generated vector bundle, and their properties. We recall Riemann-Roch Theorem and vanishing of cohomology of semi-stable vector bundles. In Section 1.2 we show that on an elliptic curve over a finite field tensor product of semi-stable vector bundles is semi-stable. This is a well known fact, see [Sun99], [IMP03] however we include our proof here as it is elementary and uses only geometry of an elliptic curve. The idea of the proof comes from Georg Hein. In Section 1.3 we recall some facts about torsion sheaves on elliptic curves. The reader familiar with those notions may skip this chapter. We would like to point out, that in this chapter k is a field of an arbitrary positive characteristic p (not necessarily $p \geq 5$).

1.1 Vector bundles on curves

Let k be a finite field of characteristic $p > 0$. Let C be a smooth projective curve of genus g over k. If E is a vector bundle on C, we define the slope $\mu(E)$ of E as $\mu(E) = {}^{\deg(E)}/_{\mathrm{rk}(E)}$. Moreover we define the (semi)stability condition as follows.

Definition 1.1.1. *A vector bundle E on C is stable, if for all proper subbundles $E' \subsetneq E$ we have the inequality $\mu(E') < \mu(E)$. A vector bundle E on C is semi-stable, if for all subbundles $E' \subseteq E$ we have the inequality $\mu(E') \leq \mu(E)$.*

If E is not semi-stable, we say E is unstable.

Remark 1.1.2. In general one defines (semi)stability condition for subsheaves, however since we are dealing with vector bundles on smooth projective curves, instead of subsheaves we are allowed to restrict ourselves only to subbundles, see page 73 of [LeP].

Suppose now that K is an extension field of k. Let E be a vector bundle on C. Form the base extension

$$\begin{tikzcd} v^*E \arrow[r] \arrow[d] & E \arrow[d] \\ C_K \arrow[r, "v"] \arrow[d, "g"] & C \arrow[d, "f"] \\ \mathrm{Spec}\,(K) \arrow[r, "u"] & \mathrm{Spec}\,(k) \end{tikzcd} \tag{1.1}$$

The degree and rank are preserved under the base extension $k \to K$ [Lan75, page 97], hence is the slope $\mu(v^*E) = \mu(E)$.

Definition 1.1.3. *A vector bundle E on C is geometrically stable, if for any base field extension $C_K \to C$ the pull back $E \otimes_k K$ is stable. A vector bundle E on C is geometrically semi-stable, if for any base field extension $C_K \to C$ the pull back $E \otimes_k K$ is semi-stable.*

Proposition 1.1.4. *[Lan75, Proposition 3] Let E be a vector bundle on C. Then*

$$E \text{ is semi-stable} \iff E \text{ is geometrically semi-stable}$$

However there exist vector bundles that are stable but not geometrically stable (see: [ArEl92]).

We use the convention from [HL, Notation 1.2.5]. If in a statement both (semi)stable and $(\leq)$ appear, then the statement stands for two, one with stable and strict inequality and the other with semi-stable and the non-strict inequality.

One can consider quotients $E \twoheadrightarrow E'$ to obtain an equivalent statement of (semi)stability.

Lemma 1.1.5. *[Hei10, Proposition 2.6] A vector bundle on C is (semi)stable if and only if for all quotients $E \to E'$ we have*

$$\mu(E)(\leq)\mu(E').$$

We also recall the Riemann-Roch theorem and Serre duality for curves.

Theorem 1.1.6. *For a vector bundle E on C we have*

$$h^0(C, E) - h^1(C, E) = \deg(E) - \mathrm{rk}(E)(g - 1).$$

The $h^1(C, E)$–dimensional vector space $H^1(C, E)$ is dual to $H^0(C, \omega_C \otimes E^\vee)$.

Lemma 1.1.7. *Let E be a vector bundle on C. For any subbundle $F \subseteq E$*

$$\mu(F) \leq h^0(C, E) + (g - 1). \tag{1.2}$$

Proof. If F is a subbundle of E, then

$$H^0(C, F) \subseteq H^0(C, E).$$

Moreover by Riemann-Roch we have

$$h^0(C, F) = \deg(F) - \mathrm{rk}(F)(g - 1) + h^1(C, F) \geq \deg(F) - \mathrm{rk}(F)(g - 1).$$

Hence

$$\mu(F) \leq h^0(C, E) + (g - 1).$$

□

An unstable vector bundle contains subbundles, that destabilize it, among those we distinguish the unique one of the maximal slope.

Definition/Proposition 1.1.8. [HL, Lemma 1.3.5] Assume that E is not semi-stable. There exists a subbundle $E_1 \subset E$ of the maximal slope among (1.2) and the maximal possible rank. It is unique and semi-stable. The vector bundle E_1 is called the maximal destabilizing subbundle of E.

One of the most important facts in the theory of vector bundles is that, every vector bundle can be uniquely classified by semi-stable bundles.

Theorem 1.1.9. *[LeP, Proposition 5.4.2] Each vector bundle E on C has a unique filtration*

$$0 = E_0 \subsetneq E_1 \subset E_2 \subset \quad \dots \quad \subset E_m = E \tag{1.3}$$

such that all the factors

$$E_i/E_{i-1}$$

are semi-stable bundles with

$$\mu(E_1) > \mu({}^{E_2}/_{E_1}) > \quad \dots \quad > \mu({}^{E}/_{E_{m-1}}).$$

The filtration (1.3) is called the Harder-Narasimhan filtration of E.

The basic properties of semi-stable vector bundles are collected in the following proposition.

Proposition 1.1.10. *[Hei10, Proposition 2.6] Let E and F be two vector bundles on a smooth projective curve C of genus g over k. Then*

i) $\mu(E \otimes F) = \mu(E) + \mu(F)$

ii) E is (semi)stable $\Longleftrightarrow$ $E^\vee$ *is (semi)stable.*

iii) If E is semi-stable of slope $\mu(E) < 0$, *then* $h^0(C, E) = 0$.

iv) If M is a line bundle on C, then E is (semi)stable $\Longleftrightarrow$ $E \otimes M$ *is (semi)stable.*

v) If E is semi-stable and $\mu(E) > 2g - 2$, *then* $h^1(C, E) = 0$.

Definition 1.1.11. *A vector bundle E is generated by its global sections at a point* $P \in C$ *(or globally generated at P) if the images of the global sections of E (i.e., elements of* $\Gamma(C, E)$*) in the stalk* E_P *generate that stalk as a* $\mathcal{O}_{C,P}$*-module.*

Definition 1.1.12. *A vector bundle E is generated by its global sections (or globally generated) if it is generated by its global sections at each point* $P \in C$.

This is equivalent to the surjectivity of the map

$$\Gamma(C, E) \otimes_k \mathcal{O}_C \to E,$$

and to the fact that E is a quotient

$$\mathcal{O}_C^{\oplus M} \to E \to 0$$

where M is a positive integer.

Proposition 1.1.13. *Consider the diagram (1.1). Let E be a vector bundle on C.*

$$E \text{ is globally generated} \iff v^* E \quad \text{is globally generated.}$$

Proof. Assume, that E is globally generated. Since $\Gamma(C,E)\otimes_k \mathcal{O}_C = f^* f_* E$, the vector bundle E is globally generated if and only if the adjunction morphism

$$f^* f_* E \xrightarrow{\phi} E \tag{1.4}$$

is surjective. The morphism v is faithfully flat, therfore (1.4) is surjective if and only if the map

$$v^* f^* f_* E \xrightarrow{v^*\phi} v^* E \tag{1.5}$$

is surjective. As u is flat, from [Har, Proposition 9.3] we have

$$v^* f^* f_*(E) = (f\circ v)^* f_*(E) = (u\circ g)^* f_*(E) = g^* u^* f_*(E) = g^* g_* v^*(E).$$

Hence the map (1.4) is surjective if and only if the map

$$g^* g_* v^*(E) \xrightarrow{\phi} v^*(E)$$

is surjective, however this is the condition that $v^*(E)$ needs to satisfy to be globally genereted . This finishes the proof of the proposition. □

1.2 Semistability of tensor product on elliptic curves

Let (C,O) be an elliptic curve over k. In this section, using elementary methods, we prove that the tensor product of semi-stable bundles on C is semi-stable. We also show, that the symmetric powers of a semi-stable vector bundle on C are semi-stable. The idea of the proof comes from Georg Hein.

Let us start with the following two lemmas.

Lemma 1.2.1. *Let F be a semi-stable vector bundle on C, and L a line bundle on C. If* $\deg(L) \geq \lfloor \mu(F) \rfloor + 2$, *then there exists an embedding*

$$F \hookrightarrow L^{\oplus M},$$

for some positive integer M.

Proof. The condition for the degree of L and semi-stability of F imply that the vector bundle $F^\vee \otimes L$ is semi-stable of slope

$$\begin{aligned}\mu(F^\vee\otimes L) &= \deg(L) - \mu(F)\\ &> \deg(L) - \lfloor \mu(F) \rfloor - 1 \geq 1,\end{aligned}$$

therefore $\mu(F^\vee \otimes L) > 1$. We claim that $F^\vee \otimes L$ is globally generated: By Proposition 1.1.13 it is enough to prove it for $v^*(F^\vee\otimes L)$ over C_K, where K is an algebraic closure of k. By Proposition 1.1.4 for any $P \in C_K$, the vector bundle $v^*(F^\vee \otimes L)(-P)$ is semi-stable of slope $\mu(v^*(F^\vee\otimes L)(-P)) = \mu(F^\vee\otimes L) - 1 > 0$, therefore

$$H^1(C_K, v^*(F^\vee \otimes L)(-P)) = 0 \;\; \text{(by Proposition 1.1.10)}.$$

Thus the long exact cohomology sequence coming from the exact sequence

$$0 \to v^*(F^\vee\otimes L)(-P) \to v^*(F^\vee\otimes L) \to v^*(F^\vee \otimes L)\otimes k(P) \to 0.$$

gives a surjection

$$H^0(C_K, v^*(F^\vee\otimes L)) \to H^0(C_K, v^*(F^\vee\otimes L)\otimes k(P)).$$

Therefore by Nakayama's lemma $v^*(F^\vee \otimes L)$ is globally generated at every $P \in C_K$, and therefore $v^*(F^\vee \otimes L)$ is globally generated. Hence we proved the claim. Since $F^\vee \otimes L$ is globally generated, we have a surjective map

$$\mathcal{O}_C^{\oplus M} \to F^\vee \otimes L.$$

Twisting with $L^\vee$ and taking dual, give the desired injective map. □

Lemma 1.2.2. *Let F, F' be two vector bundles on C. For any subbundle G of $F \otimes F'$ we have the estimate*

$$\mu(G) - \mu(F) - \mu(F') \leq 4.$$

Proof. Let L, L' be line bundles on C with $\deg L = \lfloor \mu(F) \rfloor + 2$ and $\deg L' = \lfloor \mu(F') \rfloor + 2$. By Lemma 1.2.1 we get embeddings

$$F \hookrightarrow L^{\oplus M}$$

and

$$F' \hookrightarrow L'^{\oplus M'}.$$

Therefore every subbundle G we can ambed into the direct sum of line bundles

$$G \hookrightarrow F \otimes F' \hookrightarrow (L \otimes L')^{\oplus M \cdot M'}.$$

The vector bundle $(L \otimes L')^{\oplus M \cdot M'}$ is a direct sum of line bundles of the same degree, hence is semi-stable of slope

$$\deg(L) + \deg(L') = \lfloor \mu(F) \rfloor + \lfloor \mu(F') \rfloor + 4.$$

Thus

$$\mu(G) - \mu(F) - \mu(F') \leq \mu(G) - \lfloor \mu(F) \rfloor - \lfloor \mu(F') \rfloor \leq 4.$$

and we obtain the result. □

In the proof of the below theorem we use Lemma 3.2.1 and Lemma 3.2.2 from [HL]. In this reference the authors assume characteristic 0, however if we assume additionally that f is separable, then the proof works as well in finite characteristic.

Theorem 1.2.3. *Let F and F' be two vector bundles on C. If F and F' are semi-stable, then $F \otimes F'$ is semi-stable.*

Proof. Let ℓ be a prime number different from p. Consider the multiplication by ℓ map on C

$$[\ell] : C \to C$$
$$P \mapsto \ell P.$$

The morphism $[\ell]$ [KM, Theorem 2.3.1] is a degree ℓ^2 separable morphism of smooth curves, therefore

$$F \text{ is semi-stable } \Leftrightarrow [\ell]^* F \text{ is semi-stable (by Lemma 3.2.2 from [HL])}.$$

Suppose that $F \otimes F'$ is not semi-stable. Let G be the maximal destabilizing subbundle of $F \otimes F'$. This gives

$$\mu(G) - \mu(F) - \mu(F') > 0.$$

Moreover [HL, Lemma 3.2.2], the vector bundle $[\ell]^* G$ is the maximal destabilizing subbundle of $[\ell]^*(F \otimes F')$ with slope $\mu([\ell]^* G) = \ell^2 \mu(G)$ [HL, Lemma 3.2.1]. Therefore using Lemma 1.2.2 we get

$$0 < \ell^2(\mu(G) - \mu(F') - \mu(F)) \leq 4.$$

Taking ℓ large enough gives a contradiction and proves the result. □

Theorem 1.2.4. *Let E be a semi-stable vector bundle on C. Then for any n, the symmetric tensor $\mathrm{Sym}^n E$ of E is semi-stable.*

Proof. Let E be a semi-stable vector bundle on C of rank r. By [HL, page 65] the rank of $\mathrm{Sym}^n(E)$ is equal $\mathrm{rk}(\mathrm{Sym}^n E) = \binom{n+r-1}{r-1}$ and the determinant of $\mathrm{Sym}^n E$ is equal $\det(\mathrm{Sym}^n E) = \det E^{\otimes \binom{n+r-1}{r}}$. Therefore

$$\mu(\mathrm{Sym}^n E) = \deg(E)\frac{(r-1)!n!}{(n-1)!r!} = n\mu(E) = \mu(E^{\otimes n}).$$

Hence the map

$$E^{\otimes n} \to \mathrm{Sym}^n E$$

is a surjective map of vector bundles of the same slope. Assume $\mathrm{Sym}^n E$ is not semi-stable, then there exists a vector bundle G and a surjective map $\mathrm{Sym}^n E \twoheadrightarrow G$ with $\mu(\mathrm{Sym}^n E) < \mu(G)$. However this gives a surjective map $E^{\otimes n} \twoheadrightarrow G$, where $\mu(E^{\otimes n}) = \mu(\mathrm{Sym}^n E) > \mu(G)$ and this contradicts the semistability of $E^{\otimes n}$. □

1.3 Torsion sheaves on elliptic curves

References for the proposition below are [HP05, Proposition 4] and [Pol, Theorem 14.7, Remark on page 179].

Proposition 1.3.1. *Let $\mathcal{S}(r,d)$ be the set of all isomorphism classes of semi-stable bundles of rank r and degree d. There is an isomorphism between $\mathcal{S}(r,d)$ and $\mathrm{Torsion}_{\mathrm{length}=(r,d)}$*

$$\mathrm{FM}_{\mathcal{G}} : \mathrm{Torsion}_{\mathrm{length}=(r,d)} \to \mathcal{S}(r,d).$$

Moreover for semi-stable vector bundles F and F', such that $F = \mathrm{FM}_{\mathcal{G}}(T)$ and $F' = \mathrm{FM}_{\mathcal{G}}(T')$ we have

$$\begin{aligned} \mathrm{Ext}^{\mathrm{i}}(F,F') &= \mathrm{Ext}^{\mathrm{i}}(\mathrm{FM}_{\mathcal{G}}(T),\mathrm{FM}_{\mathcal{G}}(T')) \\ &= \mathrm{Ext}^{\mathrm{i}}(T,T'). \end{aligned}$$

Here $\mathrm{Torsion}_{\mathrm{length}=(r,d)}$ denotes the set of isomorphism classes of coherent torsion sheaves T on C of length (r,d). As T is a dimension zero sheaf on a curve, it can be therefore written as a finite direct sum of coherent skyscraper sheaves $k_{P_i}^{m_i} = \mathcal{O}_{C,P_i}/\mathfrak{m}_{P_i}^{m_i}$ each supported at a closed point P_i of C

$$T = \bigoplus_{i=1}^{m} \mathcal{O}_{C,P_i}/\mathfrak{m}_{P_i}^{m_i}.$$

By the length of T we mean the length of T as $H^0(C,\mathcal{O}_C) = k$ module, and hence $\ell(T) = \mathrm{length}(T) = \chi(T)$, where the Euler characteristic is taken with respect to the base field k.

Proposition 1.3.2. *Let G be a semi-stable vector bundle on C of rank r and degree 0. Then we have the following estimate*

$$\dim_k \Gamma(C,G) \le r.$$

Proof. Let $G = \mathrm{FM}_{\mathcal{G}}(T)$, where T is a torsion sheaf of length $\ell(T) = \chi(G) = r$. By Proposition 1.3.1 we have

$$\begin{aligned} H^0(C,G) &= \mathrm{Hom}(\mathcal{O}_C,G) \\ &= \mathrm{Hom}(\mathrm{FM}_{\mathcal{G}}(k_{P_0}),\mathrm{FM}_{\mathcal{G}}(T)) \\ &= \mathrm{Hom}(k_{P_0},T). \end{aligned}$$

where k_{P_0} is a torsion sheaf of length 1 (P_0 is a k-rational point of C) associated to the line bundle $\mathcal{O}_C$. From the short exact sequence

$$0 \to \mathcal{O}_C(-P_0) \to \mathcal{O}_C \to k_{P_0} \to 0$$

we obtain the inclusion

$$0 \to \mathrm{Hom}(k_{P_0}, T) \to \mathrm{Hom}(\mathcal{O}_C, T)$$

As

$$\mathrm{Hom}(\mathcal{O}_C, T) = H^0(C, T) = k^{\oplus r}$$

we have $\dim_k \mathrm{Hom}(k_{P_0}, T) \leq r$. $\square$

Chapter 2

Elliptic curves over function fields of elliptic curves

In this chapter we focus on elliptic curves over function fields of elliptic curves over finite fields. We fix an elliptic curve (C, O) defined over a finite field k. By $k(C)$ we denote the function field of C over k and (E, O) will always indicate an elliptic curve defined over the function field $k(C)$. Firstly, this chapter provides the reader basic facts about elliptic curves over function fields of elliptic curves. For any (E, O) over $k(C)$ we will define a smooth projective surface $\mathcal{E}$ over k with a morphism $f : \mathcal{E} \to C$ whose generic fiber is E. We will describe the connection between the arithmetic of $\mathcal{E}$ and that of E. Although $\mathcal{E}$ has a higher dimension than E, it is defined over the finite field k and as a result we have better control over its arithmetic. The results contained here are used in the later chapters and necessarily to prove the main result in Section 4, Chapter 3. Secondly, we will show, that the following bound holds.

Proposition. 2.3.1 *Let d be a positive integer and let $\#C(k)$ denote the number of k-rational points of C. Then*

$$\sum_{[(E,O)];h(E)=d} \frac{1}{\#\mathrm{Aut}(E,O)} \leq \#C(k)\frac{q^{10d}}{q-1},$$

the sum is taken over isomorphism classes $[(E,O)]$ of elliptic curves (E,O) of height d over the function field $k(C)$ and $\#\mathrm{Aut}(E,O)$ denotes the number of automorphisms of (E,O).

In other words we give an upper bound for the number of isomorphism classes of elliptic curves of height $d \geq 0$ defined over $k(C)$, where we always count $[(E,O)]$ with the weight $1/\#\mathrm{Aut}(E,O)$. It is clear, that the number of automorphisms is preserved in the isomorphism class. The definition of the height of an elliptic curve can be found in Definition 2.1.10.

2.1 The minimal model of a relative elliptic curve

Let k be a finite field. Let (C, O) be an elliptic curve over k and let (E, O) be an elliptic curve over $k(C)$.

Proposition 2.1.1. *[Liu, Chapter 10, Theorem 2.8] Let (E, O) be as above. Then* *(i) For any field extension $k(C) \to K'$, and for any $(P, P') \in E(K') \times E(K')$, there exists a unique point $m_{K'}(P, P') \in E(K')$ such that*

$$m_{K'}(P, P') + O \sim P + P'$$

as Cartier divisors on $E_{K'}$. The map $m_{K'}$ makes $E(K')$ into a commutative group, with the unit element O. Moreover if $K' \subseteq K''$, then $m_{K'}$ is the restriction of $m_{K''}$.

(ii) E has a structure of an abelian variety over $k(C)$ such that O is the unit element and that for any extension K' of $k(C)$, the group law on $E(K')$ is induced by the algebraic group structure on E.

(iii) Let $P \in E(k(C))$. Then there exists an automorphism of $k(C)$-schemes $T_P : E \to E$, called the translation by P, such that for any extension K' of $k(C)$ the map $E(K') \to E(K')$ induced by T_P is the translation by P. Moreover by considering P as a point $P' \in E(K')$, $T_{P'}$ is obtained from T_P by a base change.

An elliptic curve (E, O) over $k(C)$ we can naturally associate with an elliptic surface $f : \mathcal{E} \to C$ (the Kodaira-Néron model of $E/k(C)$), whose generic fibre is E.

Proposition 2.1.2. *[Ulm, Lecture 3, Proposition 1.1] Let E be an elliptic curve over $k(C)$, there exists a surface $\mathcal{E}$ defined over k and a morphism $f : \mathcal{E} \to C$ with the following properties: $\mathcal{E}$ is smooth, absolutely irreducible, and projective over k, f is surjective and relatively minimal, and the generic fiber $\mathcal{E}_\eta$ of f is isomorphic to E. The surface $\mathcal{E}$ and the morphism f are uniquely determined up to isomorphism by these requirements.*

The generic fibre of $f : \mathcal{E} \to C$ means, the fiber product

$$\begin{tikzcd}
\mathcal{E}_\eta = \eta \times_C \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "f"] \\
\eta = \operatorname{Spec}(k(C)) \arrow[r] & C
\end{tikzcd} .$$

Remark 2.1.3. Let (E, O) be an elliptic curve over $k(C)$. Since $g(C) = 1 > 0$ by [Liu, Section 9.3, Corollary 3.24] the relatively minimal surface $f : \mathcal{E} \to C$ from Proposition 2.1.2 is in fact minimal.

Let $f : \mathcal{E} \to C$ be the minimal elliptic surface associated with an elliptic curve $E/k(C)$ (as in Proposition 2.1.2) and let $\mathcal{E}(C)$ denotes the set of sections of the structure morphism f:

$$\mathcal{E}(C) := \{ \text{ morphisms } \tau : C \to \mathcal{E} \text{ defined over } k \text{ such that } f \circ \tau = \mathrm{id}_C \}.$$

Then every $\tau \in \mathcal{E}(C)$ defines an effective Cartier divisor in $\mathcal{E}/C$ (which we also denote by τ) that is proper over C and has degree 1 (see: [KM, Lemma 1.2.7]). Moreover elements of $\mathcal{E}(C)$ are in a natural one-to-one correspondence with the $k(C)$-rational points of E.

Proposition 2.1.4. *Let (E, O) be an elliptic curve over $k(C)$. Let $f : \mathcal{E} \to C$ be the associated minimal elliptic surface from Proposition 2.1.2, then there is a bijection*

$$\mathcal{E}(C) \cong E(k(C)).$$

Proof. Let $\tau : C \to \mathcal{E}$ be a section of f ($f \circ \tau = \mathrm{id}_C$), then the restriction $P := \tau|_{\mathrm{Spec}\,(k(C))}$ is a $k(C)$-rational point of E. Conversely, take a point $P \in E(k(C))$ and let $\overline{\{P\}}$ be the Zariski closure of $\{P\}$ in $\mathcal{E}$, endowed with the reduced subscheme structure. Then the restriction

$$f|_{\overline{\{P\}}} : \overline{\{P\}} \to C$$

is a proper birational morphism onto a non-singular curve C, which by Zariski's Main Theorem [Liu, Corollary 4.6] is an isomorphism. By taking the inverse we obtain a unique morphism

$$\tau : C \to \mathcal{E}$$

such that $\tau(C) = \overline{\{P\}}$ and then the equality

$$\tau \circ f = \mathrm{id}_C$$

follows automatically. □

Definition 2.1.5. *Let (E, O) be an elliptic curve over $k(C)$. Let $f : \mathcal{E} \to C$ be the associated minimal elliptic surface from Proposition 2.1.2 and let $\sigma : C \to \mathcal{E}$ be the section of f associated with the $k(C)$-rational point O of E. The pair*

$$(\mathcal{E}, \sigma) := (f : \mathcal{E} \to C, \sigma)$$

is called the minimal model for (E, O).

Furthermore the following proposition relates the automorphism group of (E, O) and the automorphism group of its minimal model $(\mathcal{E}, \sigma)$.

Proposition 2.1.6. *Let (E, O) be an elliptic curve over $k(C)$ and let $(\mathcal{E}, \sigma)$ be the minimal model for (E, O). Then there is a bijection*

$$\mathrm{Aut}_C(\mathcal{E}, \sigma) \cong \mathrm{Aut}_{k(C)}(\mathcal{E}_\eta, \eta(\sigma)) \cong \mathrm{Aut}(E, O).$$

Proof. Every automorphism $\varphi : \mathcal{E} \to \mathcal{E}$ over C of the minimal model $\mathcal{E}$ is compatible with the C-scheme structure of $\mathcal{E}$. If $\varphi \circ \sigma = \sigma$, in particular we get $\varphi(\sigma(\eta)) = \sigma(\eta)$. Conversely, let $\alpha : \mathcal{E}_\eta \to \mathcal{E}_\eta$ be an automorphism over $k(C)$, such that $\alpha(\sigma(\eta)) = \sigma(\eta)$. It then induces a birational morphism $\alpha' : \mathcal{E} \dashrightarrow \mathcal{E}$ and as $f : \mathcal{E} \to C$ is minimal α' is a morphism. Applying the same argument to α^{-1}, we see that α' is an automorphism. The equality $\varphi \circ \sigma = \sigma$ follows from the separateness of f. □

Proposition 2.1.7. *[Liu, Chapter 10, Theorem 2.14] Let (E, O) be an elliptic curve over $k(C)$. Let $(\mathcal{E}, \sigma)$ be its minimal model over C. Then the open subscheme $\mathcal{N}$ of smooth points of $\mathcal{E}$ over C is the Néron model of (E, O) over C.*

Proposition 2.1.8. *[Liu, Chapter 10, Lemma 2.12] Let (E, O) be an elliptic curve over $k(C)$, $\mathcal{E}$ its minimal model over C and $\mathcal{N}$ its Néron model over C. Then the following are true.*

(i) The canonical maps $\mathcal{N}(C) \to \mathcal{E}(C) \to E(k(C))$ are bijective.

(ii) For any section $\tau \in \mathcal{E}(C)$, the translation $T_{\tau_{k(C)}} : E \to E$ associated with $\tau_{k(C)} \in E(k(C))$ extends to an automorphism $T_\tau : \mathcal{E} \to \mathcal{E}$.

(iii) Let $m : E \times_{k(C)} E \to E$ be the algebraic group law on E. Then the automorphism $t = (m, pr_2) : E \times_{k(C)} E \to E \times_{k(C)} E$, where $pr_2 : E \times_{k(C)} E \to E$ is the second projection, extends to an automorphism $t : \mathcal{E} \times_C \mathcal{N} \to \mathcal{E} \times_C \mathcal{N}$.

(iv) Let $\mathrm{pr}_1 : \mathcal{N} \times_C \mathcal{N} \to \mathcal{N}$ be the first projection. Then t induces an automorphism $\iota : \mathcal{N} \times_C \mathcal{N} \to \mathcal{N} \times_C \mathcal{N}$ and $\mathrm{pr}_1 \circ \iota$ defines a smooth group scheme structure on $\mathcal{N} \to C$.

Let $f : \mathcal{E} \to C$ be a relatively minimal elliptic surface with a section $\sigma : C \to \mathcal{E}$. The sheaf $\mathrm{R}^1 f_* \mathcal{O}_{\mathcal{E}}$ is a line bundle on C (see Proposition 2.2.3 (iii)). Moreover Grothendieck's Duality ([Con, Theorem 5.1.2]) gives a canonical isomorphism

$$(\mathrm{R}^1 f_* \mathcal{O}_{\mathcal{E}})^{\vee} = f_* \omega_{\mathcal{E}/C},$$

where $\omega_{\mathcal{E}/C}$ is the relative dualizing sheaf for $f : \mathcal{E} \to C$.

Definition 2.1.9. *Let $\mathcal{E} \to C$ be a minimal elliptic surface with a section. Let*

$$\omega := \left(\mathrm{R}^1 f_* \mathcal{O}_{\mathcal{E}}\right)^{\vee} \in \mathrm{Pic}(C).$$

We define the height of the pair $(\mathcal{E}, \sigma)$ by

$$h(\mathcal{E}, \sigma) = \deg_C(\omega).$$

Definition 2.1.10. *Let (E, O) be an elliptic curve defined over $k(C)$. The height $h(E)$ of the elliptic curve (E, O) is the height of the minimal model $(\mathcal{E}, \sigma)$ of (E, O)*

$$h(E) := h(E, O) := h(\mathcal{E}, \sigma).$$

Lemma 2.1.11. *The height $h(E, O)$ of an elliptic curve (E, O) over $k(C)$ is non-negative.*

Proof. Let $f : \mathcal{E} \to C$ be the minimal model of (E, O). It suffices to check, that $\deg_C \omega \geq 0$ after the base change of $\mathcal{E}$ to the algebraic closure $\bar{k}$ of k, i.e. for $\mathcal{E} = \overline{\mathcal{E}} \to \overline{C} = C$. There we have $\deg_C \omega = \chi(\mathcal{O}_{\mathcal{E}})$ and $\omega_{\mathcal{E}/C} = f^* \omega$. Furthermore the formula $\chi(\mathcal{O}_{\mathcal{E}}) = \frac{1}{12}(\mathrm{K}_{\mathcal{E}}^2 + c_2(\mathcal{E}))$ (see e.g. [Har, App A, Example 4.1.2]) yields $\deg_C \omega = \frac{1}{12} c_2(\mathcal{E})$, which is non-negative by [CD, Proposition 5.1.6]. □

Remark 2.1.12. Every minimal elliptic surface $(f : \mathcal{E} \to C, \sigma)$ of height d determines a unique up to isomorphism elliptic curve (E, O) over $k(C)$ (its generic fiber) of height d. Moreover by Proposition 2.1.2 and Proposition 2.1.4 every elliptic curve (E, O) over $k(C)$ determines a minimal elliptic surface $(\mathcal{E}, \sigma)$ of height d, that is unique up to isomorphism. That gives the following one-to-one correspondence

$$\left\{ \begin{array}{c} \text{elliptic curves} \\ (E, O) \text{ over } k(C) \\ \text{of height } d \end{array} \right\}_{/\cong} \longleftrightarrow \left\{ \begin{array}{c} \text{minimal elliptic surfaces} \\ (\mathcal{E}, \sigma) \text{ over } k \\ \text{of height } d \end{array} \right\}_{/\cong}$$

2.2 The Weierstraß equation of a relative elliptic curve

In this section we recall how to obtain the Weierstraß equation of a minimal pair $(\mathcal{E}, \sigma)$.

First of all we recall the standard result about the Picard group of an elliptic curve.

Theorem 2.2.1. *[KM, Theorem 2.1.2] Let (C,O) be an elliptic curve over k. Then for any integer d there is a bijection*

$$\mathrm{Pic}^d(C)(k) \cong C(k).$$

Let $\#C(k)$ be the number of k-rational points of C. Take an integer d and fix line bundles $\omega_{d,j}$; $j \in \{1,...,\#C(k)\}$ on C such that

$$\mathrm{Pic}^d(C)(k) = \{\omega_{d,1},...,\omega_{d,\#C(k)}\}/_{\cong}. \tag{2.1}$$

Let (C,O) be an elliptic curve over k. Let (E,O) be an elliptic curve over $k(C)$ and let $(f:\mathcal{E}\to C,\sigma)$ be the minimal model of (E,O). By [Liu, Corollary 3.6, Chapter 8] for each closed point $c\in C$ the fiber $\mathcal{E}_c$ is a geometrically connected projective curve over $k(c)$ of arithmetic genus 1. Moreover the existence of the section σ ensures reduceness of the fibers. Indeed, by [Liu, Corollary 1.32, chapter 9] $\mathcal{E}_c\cap\sigma$ is reduced to a point $p\in\mathcal{E}_c(k(c))$, that belongs to a single irreducible component of the fiber $\mathcal{E}_c$ and is of multiplicity 1 in $\mathcal{E}_c$, hence $\mathcal{E}_c$ is reduced and as k_c is perfect it is geometrically reduced. Moreover as $\mathcal{E}$ is regular [Liu, Corollary 3.6, Chapter 8] the morphisms $\mathcal{E}\to C$ and $\mathcal{E}_c\to \mathrm{Spec}\,k(c)$ are local complete intersections and we have $\omega_{\mathcal{E}_c/k(c)} = \omega_{\mathcal{E}/C}|_{\mathcal{E}_c}$ for the dualizing sheaves. Furthermore, by [Liu, Chapter 7, Corollary 3.31] for each closed point $c\in C$ the degree of $\omega_{\mathcal{E}_c/k(c)}$ is equal $\deg(\omega_{\mathcal{E}_c/k(c)}) = 2(p_a(\mathcal{E}_c)-1) = 0$ (as the arithmetic genus p_a of $\mathcal{E}_c$ is 1). Moreover, we have the following lemma.

Lemma 2.2.2. *Let X be a projective, geometrically connected and geometrically reduced $1-$ dimensional Gorenstein scheme over a perfect field k and assume that its dualizing sheaf is trivial. Let D be an effective Cartier Divisor of positive degree on X. Then*

$$H^0(X,\mathcal{O}_X) = k \text{ and } H^1(X,\mathcal{O}_X(D)) = 0.$$

Proof. First we prove, that $H^0(X,\mathcal{O}_X(D)) = k$. Indeed since X is projective $A := H^0(X,\mathcal{O}_X)$ is an Artin k-algebra. Since $X\to k$ is geometrically connected and factors as $X\to A\to k$, it follows that A is a local Artin ring whose residue field is purely inseparable over k. Since k is perfect and X is reduced we get $A = k$. Since D is effective we have an injective map $s:\mathcal{O}_X\to\mathcal{O}_X(D)$. Assume, that $H^1(X,\mathcal{O}_X(D))$ is not zero, then using Serre-Duality and the fact $\omega_X = \mathcal{O}_X$ we get a non-trivial section of $H^0(X,\mathcal{O}_X(-D))$ i.e. a map $\mathcal{O}_X\to\mathcal{O}_X(-D)$ which gives a map $t:\mathcal{O}_X(D)\to\mathcal{O}_X$ and since $\mathcal{O}_X(D)$ is an invertible sheaf $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(D),\mathcal{O}_X(D)) = H^0(X,\mathcal{O}_X) = k$. The map s is injective and t is not zero, therefore st is a multiplication by a non-zero scalar. In particular t is injective, and in the same way we conclude, that ts is a multiplication by a non-zero scalar. Thus $\mathcal{O}_X = \mathcal{O}_X(D)$. In particular we get, that D is sent to zero under the degree map

$$\mathrm{Pic}(X)\to \mathrm{CH}^1(X)\xrightarrow{\deg}\mathbb{Z} = \mathrm{CH}_0(k)$$

which is a contradiction to the assumption, that D has a positive degree. □

Proposition 2.2.3. *Let $\mathcal{E}\to C$ be a minimal elliptic surface with a section σ. Then*

(i) For $n\geq 1$ the sheaf $f_*\mathcal{O}_{\mathcal{E}}(n\sigma)$ is locally free of rank n.

(ii) $R^1f_*\mathcal{O}_{\mathcal{E}}(n\sigma) = (0)$ for $n>0$, and locally free of rank 1 for $n=0$.

(iii) For $i>1$ and all n we have $R^if_*\mathcal{O}_{\mathcal{E}}(n\sigma) = (0)$.

Proof. For dimension reasons $R^i f_* \mathcal{O}_{\mathcal{E}}(n\sigma) = (0)$ for all $i \geq 2$ and all n. Assume $n \geq 1$ as $R^2 f_* \mathcal{O}_{\mathcal{E}}(n\sigma) = (0)$, then by [Mum, Corollary, page 47] for all $c \in C$ we have an isomorphism

$$R^1 f_* \mathcal{O}_{\mathcal{E}}(n\sigma) \otimes_{\mathcal{O}_C} k(c) \cong H^1(\mathcal{E}_c, \mathcal{O}_{\mathcal{E}}(n\sigma)|_{\mathcal{E}_c}).$$

Let p be the $k(c)$–rational point of the intersection $\mathcal{E}_c \cap \sigma$, then Lemma 2.2.2 leads

$$\dim_{k(c)} H^1(\mathcal{E}_c, \mathcal{O}_{\mathcal{E}}(n\sigma)|_{\mathcal{E}_c}) = \dim_{k(c)} H^1(\mathcal{E}_c, \mathcal{O}_{\mathcal{E}}(np)) = 0,$$

Thus the coherent sheaf $R^1 f_* \mathcal{O}_{\mathcal{E}}(n\sigma)$ has all fibers zero and hence it vanishes by Nakayama Lemma. This implies [EGA III, Exp. II, Corollaire (7.9.10)] that $f_* \mathcal{O}_{\mathcal{E}}(n\sigma)$ is locally free of formation compatible with arbitrary change of base. By Riemann-Roch theorem and Lemma 2.2.2 we have

$$\dim_{k(c)} H^0(\mathcal{E}_c, \mathcal{O}_{\mathcal{E}_c}(np)) = \dim_{k(c)} H^1(\mathcal{E}_c, \mathcal{O}_{\mathcal{E}_c}(np)) + \deg(\mathcal{O}_{\mathcal{E}_c}(np)) = n,$$

Therefore the rank of $f_* \mathcal{O}_{\mathcal{E}}(n\sigma)$ is n. Assume $n = 0$, then as $R^2 f_* \mathcal{O}_{\mathcal{E}} = (0)$ by [Mum, Corollary, page 47] we have

$$R^1 f_* \mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_C} k(c) \cong H^1(\mathcal{E}_c, \mathcal{O}_{\mathcal{E}}|_{\mathcal{E}_c}).$$

Furthermore Serre-Duality and the fact that $\omega_{\mathcal{E}_c/k(c)} = \mathcal{O}_{\mathcal{E}_c}$ for all $c \in C$ implies

$$H^1(\mathcal{E}_c, \mathcal{O}_{\mathcal{E}}|_{\mathcal{E}_c}) = H^0(\mathcal{E}_c, \omega_{\mathcal{E}/k(c)}|_{\mathcal{E}_c}) = H^0(\mathcal{E}_c, \mathcal{O}_{\mathcal{E}/k(c)}|_{\mathcal{E}_c}) = k(c)$$

and hence by [Mum, Corollary, page 47] the sheaf $R^1 f_* \mathcal{O}_{\mathcal{E}}$ is a locally free sheaf on C of formation compatible with arbitrary change of base and hence necessary of rank 1. This leads to an isomorphism

$$f_* \mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_C} k(c) \cong H^0(\mathcal{E}_c, \mathcal{O}_{\mathcal{E}}|_{\mathcal{E}_c}).$$

Furthermore ([Liu, Corollary 3.6, Chapter 8]) $H^0(\mathcal{E}_c, \mathcal{O}_{\mathcal{E}}|_{\mathcal{E}_c}) = k(c)$ for all $c \in C$ and hence the sheaf $f_* \mathcal{O}_{\mathcal{E}}$ is locally free of formation compatible with arbitrary change of base, thus necessary of rank 1 □

Proposition 2.2.4. *With notation as above for $n \geq 2$ the sequence*

$$0 \to f_* \mathcal{O}_{\mathcal{E}}((n-1)\sigma) \to f_* \mathcal{O}_{\mathcal{E}}(n\sigma) \to \omega^{-\otimes n} \to 0$$

is exact.

Proof. For $n \geq 2$ the sequence

$$0 \to \mathcal{O}_{\mathcal{E}}(-\sigma) \to \mathcal{O}_{\mathcal{E}} \to \mathcal{O}_\sigma \to 0$$

is an exact sequence of $\mathcal{O}_{\mathcal{E}}$ modules. By twisting with $\mathcal{O}_{\mathcal{E}}(n\sigma)$ we obtain the sequence

$$0 \to \mathcal{O}_{\mathcal{E}}((n-1)\sigma) \to \mathcal{O}_{\mathcal{E}}(n\sigma) \to \mathcal{O}_\sigma(n\sigma) \to 0. \tag{2.2}$$

Applying f_* to (2.2) with $n = 1$ yields

$$0 \to f_* \mathcal{O}_{\mathcal{E}} \xrightarrow{\alpha} f_* \mathcal{O}_{\mathcal{E}}(\sigma) \xrightarrow{\beta} f_* \mathcal{O}_\sigma(\sigma) \xrightarrow{\gamma} R^1 f_* \mathcal{O}_{\mathcal{E}} \to R^1 f_* \mathcal{O}_{\mathcal{E}}(\sigma) \to \dots \ .$$

By Proposition 2.2.3 $R^1 f_* \mathcal{O}_{\mathcal{E}}(\sigma) = (0)$ and $R^1 f_* \mathcal{O}_{\mathcal{E}}$ is an invertible sheaf on C. Hence γ is a surjective map of line bundles, which clearly needs to be an isomorphism. This forces the map β to be zero and α to be an isomorphism. Hence

$$f_* \mathcal{O}_{\mathcal{E}} \cong f_* \mathcal{O}_{\mathcal{E}}(\sigma) \text{ and } f_* \mathcal{O}_\sigma(\sigma) \cong R^1 f_* \mathcal{O}_{\mathcal{E}}.$$

Assume $n \geq 2$, then by applying f_* to the short exact sequence (2.2) we obtain the long exact sequence of higher direct images

$$0 \to f_*\mathcal{O}_{\mathcal{E}}((n-1)\sigma) \to f_*\mathcal{O}_{\mathcal{E}}(n\sigma) \to f_*\mathcal{O}_{\sigma}(n\sigma) \to R^1 f_*\mathcal{O}_{\mathcal{E}}((n-1)\sigma) \to \dots \ .$$

By Proposition 2.2.3 for $n \geq 2$ we have $R^1 f_*\mathcal{O}_{\mathcal{E}}((n-1)\sigma) = (0)$. Furthermore it is clear, that

$$f_*\mathcal{O}_{\sigma}(n\sigma) = (f_*\mathcal{O}_{\sigma}(\sigma))^{\otimes n}$$

and as $\left(R^1 f_*\mathcal{O}_{\mathcal{E}}\right)^{\otimes n} = \omega^{-\otimes n}$ the statement follows. □

Proposition 2.2.5. *For $n = 2, 3$ the exact sequence*

$$0 \to f_*\mathcal{O}_{\mathcal{E}}((n-1)\sigma) \to f_*\mathcal{O}_{\mathcal{E}}(n\sigma) \to \omega^{-n} \to 0$$

splits.

Proof. For $k = \bar{k}$ the proposition is proven in [CD, Rmk 5.5.2] and the proof works as well for non-algebraically closed fields. □

Definition 2.2.6. *Let G be a vector bundle on C. Assume moreover, that G is a direct sum*

$$G = L_0 \oplus L_1 \oplus \dots \oplus L_r$$

with $L_i \in \mathrm{Pic}(C)$. Let $P = P_G$ be the projective bundle $P_G = \mathrm{Proj}_C\left(\mathrm{Sym}^*(G)\right)$ and let

$$\pi : P \to C$$

be the structure morphism. For each i let $\eta_i \in H^0(C, G \otimes L_i^\vee)$ be the section, that corresponds to the inclusion of L_i into G under the natural isomorphism

$$G \otimes_{\mathcal{O}_C} L_i^\vee \cong \mathrm{Hom}_{\mathcal{O}_C}(L_i, G).$$

Then for each i the canonical epimorphism $\pi^* G \to \mathcal{O}_P(1)$ induces a homomorphism

$$\pi^*\left(G \otimes_{\mathcal{O}_C} L_i^\vee\right) \cong \pi^* G \otimes_{\mathcal{O}_P} \pi^* L_i^\vee \to \mathcal{O}_P(1) \otimes_{\mathcal{O}_P} \pi^* L_i^\vee.$$

If $X_i \in H^0(P, \mathcal{O}_P(1) \otimes_{\mathcal{O}_P} \pi^* L_i^\vee)$ is the image of η_i by this homomorphism, then $(X_0, \dots, X_r)$ is called the global coordinate system of P relative to $(L_0, L_1, \dots, L_r)$.

Let (E, O) be an elliptic curve over $k(C)$ and let $(f : \mathcal{E} \to C, \sigma)$ be its minimal model with $\omega = f_*\omega_{\mathcal{E}/C}$ of degree d. Let $\sigma = \sigma(C)$ be the effective Cartier divisor associated with the section σ of f. We fix an isomorphism

$$\omega \cong \omega_0 \tag{2.3}$$

where $\omega_0 \in \{\omega_{d,j}\}_{j \in \{1, \dots, \#C(k)\}}$. By Proposition 2.2.5 the exact sequences

$$0 \to \mathcal{O}_{\mathcal{E}} \to f_*\mathcal{O}_{\mathcal{E}}(2\sigma) \to \omega_0^{\otimes -2} \to 0 \tag{2.4}$$

and

$$0 \to f_*\mathcal{O}_{\mathcal{E}}(2\sigma) \to f_*\mathcal{O}_{\mathcal{E}}(3\sigma) \to \omega_0^{\otimes -3} \to 0 \tag{2.5}$$

split. After choosing splittings we obtain an isomorphism

$$f_*\mathcal{O}_{\mathcal{E}}(3\sigma) \cong \mathcal{O}_C \oplus \omega_0^{\otimes -2} \oplus \omega_0^{\otimes -3}. \tag{2.6}$$

Furthermore the natural map

$$\mathrm{Sym}^3 f_*\mathcal{O}_{\mathcal{E}}(3\sigma) \to f_*\left(\mathcal{O}_{\mathcal{E}}(3\sigma)^{\otimes 3}\right) = f_*\mathcal{O}_{\mathcal{E}}(9\sigma) \tag{2.7}$$

is surjective. As $\mathrm{rk}\left(\mathrm{Sym}^3(f_*\mathcal{O}_{\mathcal{E}}(3\sigma)\right) = 10$ the kernel of (2.7) is a line bundle on C. By filtering $f_*\mathcal{O}_{\mathcal{E}}(9\sigma)$ by the pole order along σ we deduce, that $\det\left(f_*\mathcal{O}_{\mathcal{E}}(9\sigma)\right) = \omega^{\otimes -44}$, similarly $\det\left(\mathrm{Sym}^3(f_*\mathcal{O}_{\mathcal{E}}(3\sigma)\right) = \omega^{\otimes -50}$. Thus the kernel of (2.7) is canonically isomorphic to $\omega_0^{\otimes -6}$. Now, by twisting

$$0 \to \omega_0^{\otimes -6} \to \mathrm{Sym}^3 f_*\mathcal{O}_{\mathcal{E}}(3\sigma) \to f_*\mathcal{O}_{\mathcal{E}}(9\sigma) \to 0$$

with $\omega_0^{\otimes 6}$ we obtain an injective map

$$\mathcal{O}_C \to \mathrm{Sym}^3 f_*\mathcal{O}_{\mathcal{E}}(3\sigma) \otimes \omega_0^{\otimes 6}$$

of vector bundles and therefore a global section

$$F \in H^0(C, \mathrm{Sym}^3 f_*\mathcal{O}_{\mathcal{E}}(3\sigma) \otimes \omega_0^{\otimes 6}).$$

Consider the vector bundle

$$f_*\mathcal{O}_{\mathcal{E}}(3\sigma) \cong \mathcal{O}_C \oplus \omega_0^{\otimes -2} \oplus \omega_0^{\otimes -3}$$

and the smooth projective threefold

$$P = \mathrm{Proj}_C\left(\mathrm{Sym}^*\left(f_*\mathcal{O}_{\mathcal{E}}\left(3\sigma\right)\right)\right) \xrightarrow{\pi} C.$$

Let $\mathcal{O}_P(1)$ be the invertible sheaf on P such that $\pi_*\mathcal{O}_P(1) = f_*\mathcal{O}_{\mathcal{E}}(3\sigma)$ and let (z, x, y) be the global coordinate system of P relative to $(\mathcal{O}_C, \omega^{\otimes -2}, \omega^{\otimes -3})$. Recall, that by the projection formula for any line bundle $M \in \mathrm{Pic}(C)$ we have

$$H^0(P, \mathcal{O}_P(n) \otimes \pi^* M) = H^0\left(C, \mathrm{Sym}^n\left(f_*\mathcal{O}_{\mathcal{E}}(3\sigma)\right) \otimes M\right),$$

hence

$$H^0(P, \mathcal{O}_P(3) \otimes \pi^*\omega_0^{\otimes 6}) = H^0(C, \mathrm{Sym}^3(f_*\mathcal{O}_{\mathcal{E}}(3\sigma)) \otimes \omega_0^{\otimes 6}).$$

and we can write F as

$$F = A_{-3}y^3 + A_{-2}xy^2 + A_{-1}x^2y + A_0zy^2 + A_0x^3 + A_1xyz + A_2x^2z + A_3yz^2 + A_4xz^2 + A_6z^3$$

with $A_i \in H^0(C, \omega_0^{\otimes i})$. Furthermore it is shown in [KM, (2.2.5), (2.2.6)] that we can take $A_0 = A_0' = 1$ as well as $A_1 = A_2 = A_3 = 0$, then the polynomial F has the standard familiar shape

$$F = x^3 + A_4xz^2 + A_6z^3 - y^2z,$$

with $A_i \in H^0(C, \omega_0^{\otimes i})$ and

$$4{A_4}^3 + 27{A_6}^2 \neq 0 \text{ in } \Gamma(C, {\omega_0}^{\otimes 12}). \tag{2.8}$$

Definition 2.2.7. *A Weierstraß polynomial for a minimal elliptic surface $(f : \mathcal{E} \to C, \sigma)$ is any section*

$$F \in H^0(C, f_*\mathcal{O}_C(3\sigma) \otimes \omega_0^{\otimes 6}) \tag{2.9}$$

that arises via the choice of an isomorphism $\omega \cong \omega_0$ and of splittings of (2.4) and (2.5) as above and which has the form

$$F = x^3 + A_4xz^2 + A_6z^3 - y^2z$$

with $A_i \in \Gamma(C, \omega_0^{\otimes_0 i})$.

A section F of the above form is called a (minimal) Weierstraß polynomial if and only if there exists an (minimal) elliptic surface for which F is a Weierstraß polynomial.

The pair of sections (A_4, A_6) is called the Weierstraß coefficients of the Weierstraß polynomial F. Moreover we define the discriminant of the polynomial F to be the global section

$$\Delta = 4{A_4}^3 + 27{A_6}^2$$

in $\Gamma(C, {\omega_0}^{\otimes 12})$.

Remark 2.2.8. [Kl11, Page 2, Paragraph 3]Let F be a Weierstraß polynomial

$$F = x^3 + A_4xz^2 + A_6z^3 - y^2z$$

with invertible $\Delta = 4{A_4}^3 + 27{A_6}^2$. Then the minimality condition of F is equivalent to non-existence of a function

$$f \in \Big(\bigoplus_{n \geq 0} H^0(C, \omega_0^{\otimes n}) \setminus k \Big)$$

such that f^4 divides A_4 and f^6 divides A_6.

Proposition 2.2.9. *Let $(f : \mathcal{E} \to C, \sigma)$ be a minimal elliptic surface. Then minimal Weierstraß polynomials for $(f : \mathcal{E} \to C, \sigma)$ exist. Further, given a minimal Weierstraß polynomial*

$$F = x^3 + A_4xz^2 + A_6z^3 - y^2z$$

the minimal Weierstraß polynomials constructed via different choices are of the form

$$F_u = x^3 + u^4A_4xz^2 + u^6A_6z^3 - y^2z,$$

for $u \in k^*$.

Proof. The first statement is a standard result, see [CD, Prop. 5.5.1]. For the second statement, let $(f : \mathcal{E} \to C, \sigma)$ be a minimal elliptic surface with $f_*\omega_{\mathcal{E}/C} = \omega \cong \omega_0$. Let

$$F = x^3 + A_4xz^2 + A_6z^3 - y^2z \tag{2.10}$$

be an associated minimal Weierstraß polynomial with $A_i \in \Gamma(C, \omega_0^{\otimes i})$. We claim, that any other polynomial is related by a linear transformation of the form

$$[z, x, y] \mapsto [z, u^{-2}x, u^{-3}y] \tag{2.11}$$

or

$$[z, x, y] \mapsto [z, x + B_2z, y + B_1x + B_3z] \tag{2.12}$$

with $u \in k^*$ and $B_i \in \Gamma(C, \omega^{\otimes i})$. The first transformations come from the choice of the isomorphisms (2.3) and the second from the choice of (2.6). Indeed, the hypersurface $V(F)$ is embedded in P, therefore if F' is another minimal Weierstraß polynomial associated with $(f : \mathcal{E} \to C, \sigma)$, then the schemes $V(F), V(F')$ are isomorphic over C if the polynomials F, F' are related by an automorphism of P, which for $d \geq 1$ are precisely transformations given above;

$$\mathrm{Aut}_C(P) = \mathrm{Aut}_{\mathcal{O}_C}(\pi_*\mathcal{O}_P(1))\big/ k^* = \mathrm{Aut}_{\mathcal{O}_C}(\mathcal{O}_C \oplus \omega^{-2} \oplus \omega^{-3})\big/ k^* .$$

The case $d = 0$ follows from [Sil1, Proposition] as we have $A_i, B_i \in H^0(C, \omega_0^{\otimes i}) = k$ (see also Proposition 2.3.6). Now, apply (2.5) to the polynomial F, then we obtain a polynomial

$$F_u = yz^2 - x^3 - u^4 A_4 xz^2 - u^6 A_6 z^3.$$

To finish the proof we need to show, that application of (2.12) to F, will imply $B_1 = B_2 = B_3 = 0$. Indeed in the new coordinates F has the equation

$$F' = y^2 z - x^3 + (B_1(B_1B_2 + B_3) - A_4 + 3B_2^2)xz^2 + (A_4B_2 - B_2^3 - (B_1B_2 + B_3)^2)z^3$$
$$-B_1x^2z - 2(B_1B_2 + B_3)yz^2 + 2B_1xzy + 3B_2x^2z.$$

ans the coefficients of x^2z, yz^2, xyz, x^2z have to vanish as F' needs to be a Weierstraß polynomial again, hence $B_1 = B_2 = B_1 = 0$ and the statement is proven. □

Let d be a non-negative integer. Let $(f : \mathcal{E} \to C, \sigma)$ be a minimal elliptic surface of height d. Let F be a corresponding Weierstraß polynomial, then F defines a hypersurface $\mathcal{W} = V(F)$ in

$$P = \mathrm{Proj}_C(\mathrm{Sym}^*(\mathcal{O} \oplus \omega^{\otimes -2} \oplus \omega^{\otimes -3}))$$

and we have the following proposition.

Proposition 2.2.10. *[Ulm, Proposition 1.4] Let $d \geq 0$ and let (E, O) be an elliptic curve over $k(C)$ of height d, let $(\mathcal{E}, \sigma)$ be its minimal model and let F be a Weierstraß polynomial associated with $(\mathcal{E}, \sigma)$. Then $\mathcal{W} = V(F)$ is a closed subscheme of*

$$P := \mathrm{Proj}(\mathrm{Sym}^*(\mathcal{O}_C \oplus \omega^{\otimes -2} \oplus \omega^{\otimes -3})).$$

Let $g : \mathcal{W} \to C$ be the structure morphism, then $\mathcal{W}$ is normal, absolutely irreducible, and projective over k, g is surjective, each of its fibers is isomorphic to an irreducible plane cubic, and for its generic fiber we have

$$\mathcal{W}_\eta \cong \mathcal{E}_\eta \cong E.$$

Moreover the point O of E defines a section $\sigma' : C \to \mathcal{W}$ of g.

Lemma 2.2.11. *Let (E, O) be an elliptic curve over $k(C)$. Let $(f : \mathcal{E} \to C, \sigma)$ be its minimal model. Then the number of Weierstrass polynomials corresponding to $(\mathcal{E}, \sigma)$ is*

$$\frac{q-1}{\#\mathrm{Aut}(\mathcal{E}, \sigma)}.$$

Proof. Let $(f : \mathcal{E} \to C, \sigma)$ be the minimal model of (E, O) with $\omega = f_* \omega_{\mathcal{E}/C}$. Let F be an associated minimal Weierstraß polynomial. Proposition 2.2.9 implies, that every invertible scalar $u \in k^*$ gives a minimal Weierstraß Polynomial

$$F_u = yz^2 - x^3 - u^4 A_4 xz^2 - u^6 A_6 z^3$$

and there is at most $q - 1$ of them. Observe, that some of those scalars may give the same polynomial and it happens exactly when

$$A_4 = u^4 A_4 \text{ and } A_6 = u^6 A_6.$$

This is equivalent to u being a root of unity

$$\begin{cases} u \in \mu_2(k) \text{ if } A_4 \neq 0 \text{ and } A_6 \neq 0 \\ u \in \mu_4(k) \text{ if } A_4 \neq 0 \text{ and } A_6 = 0 \\ u \in \mu_6(k) \text{ if } A_4 = 0 \text{ and } A_6 = 0, \end{cases}$$

here $\mu_n(k)$ denotes the set of nth roots of unity of k. Furthermore we have

$$\mathrm{Aut}_C(\mathcal{E},\sigma) \quad = \quad \mathrm{Aut}_{k(\eta)}(\mathcal{E}_\eta,\sigma(\eta)) \qquad (\text{ by Proposition } 2.1.6)\ .$$

and as the generic fiber of $\mathcal{E} \to C$ is an elliptic curve E over the field $k(C)$ given by the equation

$$y^2z = x^3 + A_4xz^2 + A_6z^3,$$

(here we treat A_4, A_6 as functions) and hence [Sil1, Chapter 3, Theorem 10.1]

$$\mathrm{Aut}_{k(C)}(E,O) = \begin{cases} \mu_2(k(C)) \text{ if } A_4 \neq 0 \text{ and } A_6 \neq 0 \\ \mu_4(k(C)) \text{ if } A_4 \neq 0 \text{ and } A_6 = 0 \\ \mu_6(k(C)) \text{ if } A_4 = 0 \text{ and } A_6 = 0 \end{cases}$$

The curve E is geometrically connected and geometrically integral, moreover k is perfect and therefore k is algebraically closed in $k(C)$, hence $\mu_n(k(C)) = \mu_n(k)$ and the result follows. □

2.3 Counting elliptic curves over function fields of elliptic curves

Let d be a non-negative integer. As in the previous section $\omega_{d,j}; j \in \{1, ..., \#C(k)\}$ are chosen line bundles on C such that

$$\mathrm{Pic}^d(C)(k) = \{\omega_{d,1}, ..., \omega_{d,\#C(k)}\}/_{\cong}.$$

For each $\omega_{d,j}$ we define the set of all minimal Weierstraß polynomials as follows

$$\mathcal{N}_{\omega_{d,j}} := \left\{ \begin{array}{c} \text{minimal Weierstrass polynomials} \\ F = x^3 + A_4xz^2 + A_6z^3 - y^2z \\ \text{with} \quad A_i \in \Gamma(C, \omega_{d,j}^{\otimes i}) \end{array} \right\}.$$

Each $\mathcal{N}_{\omega_{d,j}}$ is finite, as C is a projective curve and we are over a finite field k. Furthermore for $d \geq 1$ and $i \geq 1$, we have $\dim_k H^0(C, \omega_{d,j}^{\otimes i}) = id$. Denote by $\#\mathcal{N}_{\omega_{d,j}}$ the number of elements in $\mathcal{N}_{\omega_{d,j}}$, then

$$\#\mathcal{N}_{\omega_{d,j}} \leq q^{10d}. \tag{2.13}$$

For $d \geq 0$ let N_d denote the number of minimal Weierstrass polynomials; as in the formula.

$$N_d := \sum_{\omega_{d,j} \in \mathrm{Pic}^d(C)} \#\mathcal{N}_{\omega_{d,j}}. \tag{2.14}$$

and consider the weighted number of isomorphism classes of elliptic curves of height d:

$$\sum_{[(E,O)]; h(E)=d} \frac{1}{\#\mathrm{Aut}(E,O)}. \tag{2.15}$$

In the following proposition we give a trivial upper bound for the number (2.15).

Proposition 2.3.1. *Let $d \geq 1$, then*

$$\sum_{[(E,O)];h(E)=d} \frac{1}{\#\mathrm{Aut}(E,O)} \leq \#C(k)\frac{q^{10d}}{q-1}.$$

Proof. Remark 2.1.12 and Proposition 2.1.6 lead

$$\sum_{[(E,O)];h(E)=d} \frac{1}{\#\mathrm{Aut}(E,O)} = \sum_{[(\mathcal{E},\sigma)];h((\mathcal{E},\sigma))=d} \frac{1}{\#\mathrm{Aut}(\mathcal{E},\sigma)}.$$

Proposition 2.2.11 gives

$$\sum_{[(\mathcal{E},\sigma)];h((\mathcal{E},\sigma))=d} \frac{q-1}{\#\mathrm{Aut}(\mathcal{E},\sigma)} = N_d$$

and hence

$$\sum_{[(E,O)];h(E)=d} \frac{1}{\#\mathrm{Aut}(E,O)} = \frac{N_d}{q-1}.$$

Since the number of $\omega_{d,j}$ is $\#C(k)$, the inequality (2.13) gives the desired result. □

Remark 2.3.2. We have the Hasse-Weil inequality for elliptic curves

$$\#C(k) \leq 2\sqrt{q}+q+1,$$

and therefore

$$\sum_{[(E,O)];h(E)=d} \frac{1}{\#\mathrm{Aut}(E,O)} \leq (2\sqrt{q}+q+1)\frac{q^{10d}}{q-1}.$$

The case $d=0$ is a little different. Let us first recall the following lemma.

Lemma 2.3.3. *Let (E,O) be an elliptic curve over $k(C)$ of height d and let $(\mathcal{E},\sigma)$ be its minimal model with $f_*\omega_{\mathcal{E}/C} \cong \omega_{d,j}$. If $h(E)=0$, then $\omega_{d,j}$ is torsion in $\mathrm{Pic}^0(C)$ of order $1,2,3,4$ or 6.*

Proof. Recall that for a line bundle L of degree zero on an elliptic curve C we have

$$H^0(C,L) \neq 0 \Longleftrightarrow L \cong \mathcal{O}_C.$$

As in the Weierstraß equation

$$y^2z = x^3 + A_4xz^2 + A_6z^3$$

of $(\mathcal{E},\sigma)$ at least one of the coefficients $A_4 \in H^0(C,\omega_{d,j}^{\otimes 4})$ or $A_6 \in H^0(C,\omega_{d,j}^{\otimes 6})$ needs to be nonzero, hence

$$\omega_{d,j}^{\otimes 12} \cong \mathcal{O}_C.$$

This implies, that the possible orders are $1,2,3,4$ and 6. □

Definition 2.3.4. *We say that an elliptic curve (E,O) over $k(C)$ is constant if there is an elliptic curve (E_0,O) defined over k such that $E = E_0 \times_k k(C)$.*

Equivalently, (E,O) is constant if the coefficients A_i of the Weierstraß equation of $(\mathcal{E},\sigma)$ are in k.

Theorem 2.3.5. *[KM, Theorem 2.3.1] Let C be an elliptic curve defined over a finite field k of characteristic p. Let n be a non zero integer, such that $(n,p)=1$. Then over $\bar{k}$ we have an isomorphism*

$$C[n](\bar{k}) \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.$$

Proposition 2.3.6. *For $d=0$ we have*

$$q \leq \sum_{[(E,O)];h(E)=0} \frac{1}{\#\mathrm{Aut}(E,O)} \leq 5q+61.$$

Proof. This case is special, because as we have mentioned before, for a line bundle of degree zero the dimension of its global sections is either 1, if the line bundle is trivial, or it is 0 otherwise. Moreover it can happen that a line bundle is non trivial, but some tensor power of it will be. In this situation the Theorem 2.3.5 implies that, for a given ℓ, coprime to the characteristic of k, we have at most ℓ^2 of such line bundles and by Lemma 2.3.3 the possible orders of $\omega_{d,j}$ are $1,2,3,4$ and 6. The height 0 surfaces correspond to elliptic curves with constant coefficients, hence they are minimal (see Remark 2.2.8). Fix an $\omega_{d,j} \in \mathrm{Pic}^0(C)$. The case $\omega_{d,j} \cong \mathcal{O}_C$ gives q^2 polynomials (choices of A_4 and A_6), among of which q have zero discriminant, hence the number of minimal polynomials is q^2-q. Now fix $\omega_{d,j}$ in $\mathrm{Pic}^0(C)$ such that the order of $\omega_{d,j}$ is 2. Since $\omega_j^4 \cong \omega_j^6 \cong \mathcal{O}_C$, we get the same number of polynomials as in the case of order 1. Since the number of such line bundles is at most 4, the number of minimal Weierstrass polynomials will be at most $4(q^2-q)$. Let $\omega_{d,j} \in \mathrm{Pic}^0(C)$ be of order 3. Since $\omega_{d,j}^4$ is non trivial we conclude that $\Gamma(C,\omega_{d,j}^4)=0$. Hence we have q choices of A_6 and only 1 is singular. Since we have at most 9 choices for $\omega_{d,j}$, we see that the number of minimal Weierstrass polynomials is at most $9(q-1)$. Using the same argument for the cases of order 4 and 6 , we conclude that

$$N_0 \leq (q-1)(5q+61).$$

The lower bound comes from the known fact, that the number of points on the stack of elliptic curves over k is q □

Chapter 3

The set $\mathcal{A}_{C,\omega}$

Let d be a non-negative integer. Let $C := (C, O)$ be an elliptic curve over $k = \mathbb{F}_q$. In this chapter for a line bundle ω of degree d on C we define the set $\mathcal{A}_{C,\omega}$. Its elements, as we will see in the first subsection, are "Weierstraß polynomials" of more general form then in Chapter 2. We will prove that $\mathcal{A}_{C,\omega}$ is finite and as the main result of this chapter (see: Proposition 3.3.7) we will show that

$$\limsup_{d\to\infty} q^{-10d+1} \sum_{\omega\in \mathrm{Pic}^d(C)} \mathrm{A}_{C,\omega} \leq \frac{\#C(\mathbb{F}_q)A}{(q-1)^5(q+1)^2 q}$$

where $\mathrm{A}_{C,\omega}$ denotes the "weighted" number of elements in $\mathcal{A}_{C,\omega}$, $\#C(\mathbb{F}_q)$ denotes the number of $\mathbb{F}_q$-rational points of C and A is a polynomial of degree 8 in q with the leading coefficient $2\#C(\mathbb{F}_q)$.

3.1 Finiteness of the set $\mathcal{A}_{C,\omega}$

Let C be an elliptic curve over k. Fix a non-negative integer d and a line bundle ω on C of degree d. We start the construction of $\mathcal{A}_{C,\omega}$ with defining its elements.

Definition 3.1.1. *Let $\mathcal{A}$ denote the set, whose elements are pairs $(g : Y \to C, D)$ with the following properties:*

(a) the morphism $Y \xrightarrow{g} C$ is flat, proper and its generic fibre is a smooth curve,

(b) the equality $g_*\mathcal{O}_Y = \mathcal{O}_C$ holds universally,

(c) the fibres of g are Gorenstein and $\omega_{Y/C} \cong g^*\omega$,

(d) $D \subset Y$ is an effective Cartier divisor, flat over C such that $D.F = 3$, where F is a fibre of g,

(e) the sheaf $\mathcal{O}_Y(D)$ is relatively very ample for g, i.e., we obtain a canonical closed immersion over C

$$Y \hookrightarrow \mathrm{Proj}_C(\mathrm{Sym}^*(g_*\mathcal{O}_Y(D))).$$

Remarks 3.1.2.

(i) Let $(g : Y \to C, D)$ be an element of $\mathcal{A}$ then using the condition (c) for every closed point $c \in C$ we have

$$\omega_{Y/C}|_{Y_c} \cong \omega_{Y_c/k(c)} \cong \mathcal{O}_{Y_c}.$$

(ii) Let $(g : Y \to C, D)$ be an element of $\mathcal{A}$ then Y is integral. Indeed its generic fiber is a smooth projective curve, hence integral. Furthermore if we take an open affines $\operatorname{Spec}(B)$ in Y and $\operatorname{Spec}(A)$ in C such that the morphism g induces a flat map $A \to B$ then after tensoring the inclusion $A \hookrightarrow k(C) =: K$ with B over A the flatness yields an inclusion $B \hookrightarrow B \otimes_A K$. Since $B \otimes_A K$ is integral, so is B.

Definition 3.1.3. *Two elements $(Y \xrightarrow{g} C, D)$ and $(Y' \xrightarrow{g'} C, D')$ of $\mathcal{A}$ are called equivalent if and only if there exists an isomorphism $\psi : Y \to Y'$ over C such that we have the following rational equivalence relation of divisors on Y*

$$\psi^* D' \sim D + g^* D_C \tag{3.1}$$

for some $D_C \in \operatorname{Div}(C)$.

Remark 3.1.4. Observe that the divisor $g^* D_C$ is a sum of fibers with multiplicities, for that suppose $D_C = \sum_i n_i P_i$ for points $P_i \in C$ since g is flat, we have $g^* D_C = \sum_i n_i g^*(P_i) = \sum_i n_i (F_{P_i})$. Furthermore the relation (3.1) can be equivalently written in terms of isomorphism of line bundles

$$\psi^* \mathcal{O}_{Y'}(D') \cong \mathcal{O}_Y(D) \otimes g^* L$$

for $L = \mathcal{O}_C(D_C) \in \operatorname{Pic}(C)$.

Remark 3.1.5. The relation (3.1) is indeed an equivalence relation on the set of pairs $(g : Y \to C, D)$.

1. It is easy to check, that it is reflexive.
2. It is symmetric: let $(g : Y \longrightarrow C, D)$ and $(g' : Y' \longrightarrow C, D')$ be pairs as above, that are equivalent. Then there exists an isomorphism $\psi : Y \to Y'$ such that we have a rational equivalence of divisors $\psi^* D' \sim D + g^* D_C$. The inverse $\psi^{-1} : X' \to X$ is an isomorphism over C. Moreover
$$\psi^{-1*} D \sim \psi^{-1*}(\psi^* D' + g^* D_C)$$
$$= (\psi \circ \psi^{-1})^* D' + (g \circ \psi^{-1})^* D_C = D' + g'^* D_C.$$
3. To see transitivity, take $(g_1 : Y_1 \longrightarrow C, D_1)$ equivalent to $(g_2 : Y_2 \longrightarrow C, D_2)$ and $(g_3 : Y_3 \longrightarrow C, D_3)$ such that $(g_2 : Y_2 \longrightarrow C, D_2)$ is equivalent to $(g_3 : Y_3 \longrightarrow C, D_3)$. By definition we get an isomorphisms $\psi : Y_1 \to Y_2$ and $\varphi : Y_2 \to Y_3$ over C with rational equivalence relations
$$\psi^* D_2 \sim D_1 + g_1{}^* D_C \quad \text{and} \quad \varphi^* D_3 \sim D_2 + g_2{}^* G_C,$$
for some $D_C, G_C \in \operatorname{Div}(C)$. The composition
$$\alpha = \varphi \circ \psi : Y_1 \to Y_3$$
is an isomorphism over C and
$$\begin{aligned} \alpha^* D_3 &= \psi^*(\varphi^* D_3) \\ &\sim \psi^* D_2 + (g_2 \circ \psi)^* G_C \\ &\sim D_1 + g_1{}^* D_C + g_1{}^* G_C \\ &= D_1 + g_1{}^*(D_C + G_C) \end{aligned}$$
Thus $(g_1 : Y_1 \longrightarrow C, D_1)$ and $(g_3 : Y_3 \longrightarrow C, D_3)$ are equivalent.

Definition 3.1.6. *A pair* $(g : Y \to C, D)$ *is called degenerate if there exists a divisor* D' *of degree* 1 *on the generic fiber* Y_η *such that* $D_\eta \sim 3D'$ *(rational equivalence).*

Definition 3.1.7. *Let* $\mathcal{A}_{C,\omega}$ *denote the set*

$$\mathcal{A}_{C,\omega} = \left\{ \begin{array}{c} \textit{equivalence classes} \left[(Y \xrightarrow{g} C, D) \right] \textit{(Definition 3.1.3)} \\ \textit{of non-degenerate (Definition 3.1.6)} \\ \textit{pairs } (Y \xrightarrow{g} C, D) \textit{satisfying properties (a)-(e) (Definition 3.1.1)} \end{array} \right\} \tag{3.2}$$

Definition 3.1.8. *The automorphism group* $\mathrm{Aut}(Y \xrightarrow{g} C, D)$ *of a pair* $(Y \xrightarrow{g} C, D) \in \mathcal{A}$ *(Definition 3.1.1) we define as the set of automorphisms* $\psi : Y \longrightarrow Y$ *over* C *such that*

$$\psi^* D \sim D + g^* D_C \qquad (\textit{ rational equivalence })$$

for some $D_C \in \mathrm{Div}(C)$.

We will see in Section 3.2 that the set $\mathcal{A}_{C,\omega}$ is finite. We set

$$\mathrm{A}_{C,\omega} := \sum_{[(Y \xrightarrow{g} C, D)] \in \mathcal{A}_{C,\omega}} \frac{1}{\#\mathrm{Aut}(Y \xrightarrow{g} C, D)} \tag{3.3}$$

for the weighted number of elements of $\mathcal{A}_{C,\omega}$.

Remark 3.1.9. Observe, that $\mathrm{Aut}(Y \xrightarrow{g} C, D) \neq \varnothing$ as the identity automorphism id_Y satisfies the requirements in Definition 3.1.8.

Remark 3.1.10. Observe, that the sum (3.3) is well-defined. If $(g' : Y' \to C, D')$ is another representative in the equivalence class $[(Y \xrightarrow{g} C, D)]$ then each equivalence $\psi : Y \to Y'$ (in the sense of (3.1)) induces an isomorphism of automorphism groups

$$\mathrm{Aut}(g : Y \to C, D) \cong \mathrm{Aut}(g' : Y' \to C, D')$$

as follows. Let $\psi : Y \to Y'$ be an isomorphism over C with $\psi^*(D') \sim D + g^* D_C$ for some $D_C \in \mathrm{Pic}(C)$. Then every $\alpha \in \mathrm{Aut}(g : Y \to C, D)$ gives an automorphism of Y' over C by setting $\alpha' := \psi \circ \alpha \circ \psi^{-1}$,

$$\begin{tikzcd} Y \arrow[d, "\alpha"'] & Y' \arrow[l, "\psi^{-1}"'] \arrow[d, "\alpha'"] \\ Y \arrow[r, "\psi"] & Y' \end{tikzcd}$$

Moreover

$$\begin{aligned} {\alpha'}^*(D') &= (\psi \circ \alpha \circ \psi^{-1})^* D' \\ &= \left(\alpha \circ \psi^{-1}\right)^* (\psi^* D') \\ &\sim (\alpha \circ \psi^{-1})^* (D + g^* D_C) \\ &= {\psi^{-1}}^* (\alpha^* D + (g \circ \alpha)^* D_C) \\ &\sim {\psi^{-1}}^* (D + g^*(D_C + G_C)) \\ &\sim (D' - {g'}^*(D_C) + {g'}^*(D_C + G_C)) \\ &= D' + {g'}^*(G_C) \end{aligned}$$

for some $G_C \in \mathrm{Pic}(C)$ with $\alpha^* D \sim D + g^* G_C$ and hence $\alpha' \in \mathrm{Aut}(g : Y \to C, D)$. It easy to check, that this defines an isomorphism.

We have defined the set $\mathcal{A}_{C,\omega}$ to understand its elements we will need the following lemmas.

Lemma 3.1.11. *[deJ02, Lemma 8.4] Let X be a projective Gorenstein curve over a field k such that $\omega_{X/k} \cong \mathcal{O}_X$ and $H^0(X, \mathcal{O}_X) = k$. Let $D \subset X$ be an effective Cartier divisor.*

(a) If $\ell(D) \geq 2$ then $\mathcal{O}_X(D)$ is globally generated and $H^1(X, \mathcal{O}_X(D)) = (0)$.

(b) If $\ell(D) \geq 3$ then the graded k-algebra $R := \oplus_{n\geq 0}\Gamma(X, \mathcal{O}_X(D))$ is generated in degree 1.

Lemma 3.1.12. *[deJ02, Lemma 8.5] In the situation described in (b) of lemma above, the scheme $Y = \mathrm{Proj}(R)$ is a Gorenstein curve with $\omega_{Y/k} \cong \mathcal{O}_Y$ and $H^0(Y, \mathcal{O}_Y) = k$. The morphism $\varphi : X \to Y$ is dominant and induces an isomorphism of an open subscheme of X with a dense open subscheme of Y.*

Remark 3.1.13. [deJ02, Remark 8.6] With notation as in Lemma 3.1.11 and Lemma 3.1.12. By Serre-Duality we have $\dim_k H^1(Y, \mathcal{O}_Y) = 1$. The relevant terms of the Leray spectral sequence for the map φ are

$$0 \to E_2^{10} \to H^1(X, \mathcal{O}_X) \to E_2^{01} \to E_2^{20}$$

which for us is

$$0 \to H^1(Y, \varphi_*\mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to H^0(Y, R^1\varphi_*\mathcal{O}_X) \to H^2(Y, \varphi_*\mathcal{O}_X).$$

As $\mathcal{O}_Y = \varphi_*\mathcal{O}_X$ we have $\dim_k H^1(Y, \varphi_*\mathcal{O}_X) = \dim_k H^1(Y, \mathcal{O}_Y) = 1$ furthermore by Serre-Duality and the fact, that $\omega_X = \mathcal{O}_X$ we have $\dim_k H^1(X, \mathcal{O}_X) = 1$ and as Y is a curve the last term is zero. It follows, that $R^1\varphi_*\mathcal{O}_X = (0)$ and the canonical map

$$H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X)$$

is an isomorphism. The dual of this map is canonical isomorphism $\Gamma(X, \omega_X) = \Gamma(Y, \omega_Y)$.

Proposition 3.1.14. *Let $(X \xrightarrow{f} C, D)$ be a representative of a class in $\mathcal{A}_{C,\omega}$ then*

(a) The sheaf $f_*\mathcal{O}_X(D)$ is locally free of rank three on C of formation compatible with arbitrary change of base, and $R^1 f_*\mathcal{O}_X(D) = (0)$.

(b) The natural maps

$$f^* f_*\mathcal{O}_X(D) \to \mathcal{O}_X(D)$$

and

$$\bigoplus_{m\geq 0} (f_*\mathcal{O}_X(D))^{\otimes m} \to \bigoplus_{m\geq 0} f_* \left(\mathcal{O}_X(D)^{\otimes m}\right)$$

are surjective.

Proof. To prove (a) observe, that the sheaf $\mathcal{O}_X(D)$ is an invertible sheaf on X fiber-by-fiber of degree 3. First of all we show that $R^1 f_*\mathcal{O}_X(D) = (0)$. We know, that $R^2 f_*\mathcal{O}_X(D) = (0)$ hence by [Mum, Corollary, page 47] the formation of $R^1 f_*\mathcal{O}_X(D)$ is compatible with arbitrary change of base moreover, as by Lemma 3.1.12(a) the sheaf $R^1 f_*\mathcal{O}_X(D)$ is a coherent sheaf on C with all fibres zero, it vanishes by Nakayama's lemma. Now, as $R^1 f_*\mathcal{O}_X(D) = (0)$, the sheaf $f_*\mathcal{O}_X(D)$ is automatically locally free by [EGA III, Exp. II, Corollaire (7.9.10)] and by [Mum, Corollary, page 47] its formation is compatible with arbitrary change of base, so necessarily of rank 3. For (b) let u be the natural map $f^* f_* \mathcal{O}_X(D) \xrightarrow{u} \mathcal{O}_X(D)$. Take $c \in C$ then by Lemma 3.1.12(b) the map $u \otimes k_c$ is surjective, therefore by Nakayama's lemma u is surjective at c and hence everywhere. By Lemma 3.1.12(b) the sheaf of $\mathcal{O}_C$ algebras $\bigoplus_{m\geq 0} f_* \mathcal{O}_X(mD)$ is generated in degree 1 meaning that the map

$$\bigoplus_{m\geq 0} (f_* \mathcal{O}_X(D))^{\otimes m} \to \bigoplus_{m\geq 0} f_* \left(\mathcal{O}_X(D)^{\otimes m}\right)$$

is surjective. □

Let $[(g : Y \longrightarrow C, D)] \in \mathcal{A}_{C,\omega}$. By Proposition 3.1.14 the sheaf $g_* \mathcal{O}_Y(D)$ is locally free of rank 3 on C. Moreover it has a non-trivial global section since $H^0(C, g_* \mathcal{O}_Y(D)) = H^0(Y, \mathcal{O}_Y(D))$ and D is an effective Cartier divisor.

Let us denote the vector bundle $g_* \mathcal{O}_Y(D)$ by E. Let P_E denote the scheme

$$P_E = \mathrm{Proj}_C(\mathrm{Sym}^*(E)) \longrightarrow C.$$

Observe, that the equivalence class of the pair $(g : Y \to C, D)$ does not determine the vector bundle E uniquely, this is because the rational equivalence class of the divisor D is only well defined up to adding multiples of fibers. Therefore E is well defined up to twist with any line bundle from C.

Proposition 3.1.15. *Let $(g : Y \to C, D)$ and let $E = g_* \mathcal{O}_Y(D)$. The scheme P_E is up to isomorphism independent of the choice of $(g : Y \to C, D)$ in its equivalence class.*

Proof. Let $(g' : Y' \to C, D')$ be another representative in the equivalence class of $(g : Y \longrightarrow C, D)$. The condition (3.1) implies, that there exists an isomorphism $\psi : Y \to Y'$ over C such that,

$$\mathcal{O}_Y(\psi^* D') \cong \mathcal{O}_Y(D) \otimes g^* L$$

for some $L \in \mathrm{Pic}(C)$. Hence we have an isomorphism

$$g_* \mathcal{O}_Y(D) \cong g_* \left(\mathcal{O}_Y(\psi^* D') \otimes g^* L^{-1}\right) \cong (g' \circ \psi)_* \left(\mathcal{O}_Y(\psi^* D')\right) \otimes L^{-1} \cong g'_* \mathcal{O}_{Y'}(D') \otimes L^{-1}$$

of vector bundles on C. Therefore $E \cong E' \otimes L$ and hence $P_E \cong P_{E'}$. □

Lemma 3.1.16. *Let d be a non-negative integer and let ω be a line bundle on C of degree d. Let E be a rank 3 vector bundle on C and let P be the projective scheme*

$$\pi : P = P_E := \mathrm{Proj}_C(\mathrm{Sym}^*(E)) \longrightarrow C$$

with $\pi_* \mathcal{O}_P(1) = E$. Assume Y is a subscheme of P so that the pair $(g : Y \longrightarrow C, D)$ is a representative of a class in $\mathcal{A}_{C,\omega}$ where $\mathcal{O}_Y(D) \cong \mathcal{O}_P(1)\,|_Y$. Then

$$g_* \mathcal{O}_Y(D) = \pi_* \mathcal{O}_P(1).$$

Proof. Let $i : Y \hookrightarrow P$ be the inclusion. The sequence

$$0 \to \mathcal{O}_P(-Y) \to \mathcal{O}_P \to i_* \mathcal{O}_Y \to 0$$

is exact. By twisting with $\mathcal{O}_P(1)$ and applying π_* we get an exact sequence

$$0 \to \pi_*(\mathcal{O}_P(-Y)(1)) \to \pi_* \mathcal{O}_P(1) \to \pi_*(i_* \mathcal{O}_Y) \otimes \mathcal{O}_P(1) \to R^1 \pi_*(\mathcal{O}_P(-Y)(1)) \to \dots .$$

We have

$$\pi_*(i_* \mathcal{O}_Y \otimes \mathcal{O}_P(1)) = \pi_*(i_*(\mathcal{O}_Y \otimes i^* \mathcal{O}_P(1))) = g_* \mathcal{O}_Y(D).$$

Therefore to prove that $\pi_* \mathcal{O}_P$ and $g_* \mathcal{O}_Y(D)$ are isomorphic it is enough to show that the sheaf $R^i \pi_*(\mathcal{O}_P(-Y)(1))$ vanishes for all $i \geq 0$. As Y has dimension 2 it is true for all $i \geq 3$. Assume, that for some $i+1$ we have

$$R^{i+1}\pi_*(\mathcal{O}_P(-Y)(1)) = (0).$$

Then by [Mum, Corollary page 47] the formation of $R^i \pi_*(\mathcal{O}_P(-Y)(1))$ is compatible with arbitrary change of base. Hence for each $c \in C$ we have

$$R^i \pi_*(\mathcal{O}_P(-Y)(1)) \otimes_{\mathcal{O}_{C,c}} k_c = H^i(P_c, \mathcal{O}_P(-Y)(1) \otimes k(c)).$$

Observe, that by Lemma 3.1.20 we have $\mathcal{O}_P(-Y)(1) \otimes k(c) \cong \mathcal{O}_{\mathbb{P}^2}(-3+1)$ and hence

$$R^i \pi_*(\mathcal{O}_P(-Y)(1)) \otimes_{\mathcal{O}_{C,c}} k_c = H^i(\mathbb{P}^2_{k(c)}, \mathcal{O}_{\mathbb{P}^2}(-2)) = 0.$$

The sheaf $R^i \pi_*(\mathcal{O}_P(-Y)(1))$ is therefore a coherent sheaf with all fibers 0 and hence it vanishes by Nakayama's lemma. This finishes the proof of the lemma. □

Definition 3.1.17. *Let E and E' be vector bundles on C. We say that E and E' are twist equivalent and write $E \sim_{\otimes} E'$ if*

$$\exists L \in \mathrm{Pic}(C) \text{ such that } \quad E' \cong E \otimes L. \tag{3.4}$$

Remark 3.1.18. By Proposition 3.1.15 and Lemma 3.1.16 the set $\mathcal{A}_{C,\omega}$ can be written in a natural way as a disjoint union

$$\bigsqcup_{[E]_{\otimes}} \{ \text{ non-degenerate pairs } (g : Y \to C, D) \text{ occurring in } P_E \} \tag{3.5}$$

where $[E]_{\otimes}$ runs through all twist-equivalence classes of rank 3 vector bundles on C and a non-degenerate pair $(g : Y \to C, D)$ "occurs in P_E" if the morphism g factors over a closed immersion $i : Y \hookrightarrow P_E$ such that $i^* \mathcal{O}_{P_E}(1) \cong \mathcal{O}_Y(D)$.

Proposition 3.1.19. *The degree of each E is bounded by $0 \leq \deg(E) \leq 2d$ and for a fixed degree there is only finitely many of equivalence classes $[E]_{\otimes}$.*

Proof. The proof is given in Sections 3.2.1, 3.2.2, 3.2.3. □

Now we will show, that for a fixed equivalence class $[E]_{\otimes}$ there is only finite number of $(g : Y \to C, D)$ occurring in P_E and the number of automorphisms $\mathrm{Aut}(g : Y \to C, D)$ of each pair $(g : Y \to C, D)$ is finite.

Lemma 3.1.20. *Let d be a non-negative integer and let ω be a line bundle on C of degree d. Let E be a rank 3 vector bundle on C. Consider the projective scheme*

$$\pi : P = P_E := \mathrm{Proj}_C(\mathrm{Sym}^*(E)) \longrightarrow C$$

and let $\mathcal{O}_P(1)$ be the invertible sheaf on P such that $\pi_* \mathcal{O}_P(1) = E$. Let $Y \subset P$ be a closed subscheme of P, let $i : Y \hookrightarrow P$ be the inclusion and $g = \pi \circ i$ the structure morphism of Y to C. Assume that $(g : Y \longrightarrow C, D)$ is a representative of a class in $\mathcal{A}_{C,\omega}$ where $\mathcal{O}_Y(D) \cong \mathcal{O}_P(1)|_Y$. Then there is a nonzero section s of

$$\Gamma\left(P, \mathcal{O}_P(3) \otimes \pi^*(\omega \otimes \det(E)^{\vee})\right)$$

such that Y is the zero scheme of s: $Y = V(s)$.

Proof. Observe, that P is a smooth projective threefold, therefore Weil and Cartier divisors agree on P. Moreover as C is smooth

$$\mathrm{Pic}(P) = \mathbb{Z} \oplus \pi^*\mathrm{Pic}(C).$$

Every Y (as above) has dimension 2 hence it is a divisor on P and we have

$$\mathcal{O}_P(Y) \cong \mathcal{O}_P(n) \otimes \pi^* M$$

for some $n \in \mathbb{N}$ and a line bundle $M \in \mathrm{Pic}(C)$. We determine n and M. We have the diagram

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[dr, "g"'] & P \arrow[d, "\pi"] \\ & C \end{tikzcd}$$

and for all $c \in C$

$$\begin{tikzcd} Y_c \arrow[r, hook, "i_c"] \arrow[dr, "g_c"'] & P_c = \mathbb{P}^2_{k(c)} \arrow[d, "\pi_c"] \\ & k(c) \end{tikzcd} \quad .$$

Observe that for each $c \in C$

$$\mathcal{O}_P(Y) \otimes_{\mathcal{O}_{C,c}} k(c) = \mathcal{O}_{P_c}(Y_c) = \mathcal{O}_{\mathbb{P}^2_{k(c)}}(n).$$

By Definition 3.1.1 (d) the intersection number of each Y_c with the hyperplane defined by the divisor D is 3 ($\mathcal{O}_Y(D) \cong i^*\mathcal{O}_P(1)$), therefore Bézout Theorem implies $n = 3$. Consider the following exact sequence

$$0 \to \Omega_{P/C} \to \pi^* E(-1) \to \mathcal{O}_P \to 0.$$

Taking determinants we get

$$\omega_{P/C} = \bigwedge^3 \pi^* E(-1) \otimes \mathcal{O}_P.$$

Consequently

$$\begin{aligned} \omega_{P/C} &= \bigwedge^3 \pi^* E(-1) \\ &= \bigwedge^3 \Big(\pi^*(E) \otimes \mathcal{O}_P(-1)\Big) \\ &= \Big(\bigwedge^3 \pi^* E\Big) \otimes \mathcal{O}_P(-3) \\ &= \pi^* \det(E) \otimes \mathcal{O}_P(-3) \end{aligned}$$

The morphism $\pi : P \to C$ is proper, flat and all of its fibers are smooth. Therefore π is a Cohen-Macaulay morphism. The morphism g is C-flat and for all $c \in C$ the fiber morphisms $Y_c \hookrightarrow \mathbb{P}^2_{k(c)}$ are (local) complete intersections, and hence (by definition) the closed immersion $i : Y \hookrightarrow P$ is transversally regular over C ([Co00, Notation and Terminology, Page 7]). By adjunction for i (see: [Co00, (2.2.1)]) we obtain

$$\begin{aligned} \omega_{Y/C} &= \big(\omega_{P/C} \otimes \mathcal{O}_P(Y)\big)\,|_Y \\ &\cong \pi^*(\det(E) \otimes M)\,|_Y \\ &= g^*(\det(E) \otimes M). \end{aligned}$$

On the other hand, by Definition 3.1.1 (c) we have $\omega_{Y/C} \cong g^*\omega$ hence

$$g^*(\det(E) \otimes M) \cong g^*\omega$$

and therefore

$$g^*(\det(E) \otimes M \otimes \omega^\vee) \cong \mathcal{O}_Y$$

Applying g_* and using the assumption (b) we get

$$g_*g^*(\det(E) \otimes M \otimes \omega^\vee) \cong g_*\mathcal{O}_Y = \mathcal{O}_C$$

Consequently by the projection formula

$$M \cong \det(E)^\vee \otimes \omega$$

and the result follows. $\square$

Let $G_E = \mathrm{Aut}(E)$ be the automorphism group of the sheaf E over C. The group $k^* \subset G_E$ is a central subgroup and we set $PG_E = G_E/k^*$. We have the natural action of the group G_E on the pair $(P, \mathcal{O}_P(1))$ and therefore also on the vector space

$$V_E := \Gamma\left(P, \mathcal{O}_P(3) \otimes \pi^*(\omega \otimes \det(E)^\vee)\right). \tag{3.6}$$

The action of the subgroup $k^* \subset G_E$ is by scalars, hence the group PG_E acts on the projective space $\mathbb{P}(V_E)$ of lines in V_E.

Lemma 3.1.21. *Let d be a non-negative integer and let ω be a line bundle on C of degree d. Let Y and Y' be two subschemes of P with divisors D and D' such that $\mathcal{O}_Y(D) \cong \mathcal{O}_P(1)|_Y$ and $\mathcal{O}_{Y'}(D') \cong \mathcal{O}_P(1)|_{Y'}$. Suppose moreover that $(g : Y \to C, D)$ and $(g' : Y' \to C, D')$ are representatives of classes in $\mathcal{A}_{C,\omega}$. Then*

$$\mathrm{Equiv}_{\mathcal{A}_{C,\omega}}\left((Y \xrightarrow{g} C, D_Y), (Y' \xrightarrow{g'} C, D_{Y'})\right) = \{v \in PG_E \mid v(Y) \subset Y'\}. \tag{3.7}$$

Proof. First we construct a map from the RHS to the LHS. Let $v \in PG_E$ be such that $v(Y) \subset Y'$. Then v induces an automorphism $\psi_v : P \to P$ of the projective bundle P such that $\psi_v^*\mathcal{O}_P(1) = \mathcal{O}_P(1)$. Then ψ_v restricts to a closed immersion $\psi_v|_Y : Y \to Y'$ between integral closed subschemes of the same dimension and hence is an isomorphism (see Remark 3.1.2(i)). Moreover $\mathcal{O}_Y(D) \cong \mathcal{O}_P(1)|_Y$ and $\mathcal{O}_{Y'}(D') \cong \mathcal{O}_P(1)|_{Y'}$ hence ψ_v gives an equivalence of $(g : Y \to C, D)$ and $(g' : Y' \to C, D_{Y'})$ This gives the first map. Now we construct a map in the other direction. Let $\psi : Y \to Y'$ be an isomorphism over C with $\mathcal{O}_Y(\psi^*D') \cong \mathcal{O}_Y(D) \otimes g^*L$ and $L \in \mathrm{Pic}(C)$. The isomorphism ψ induces a map $\mathcal{O}_{Y'} \to \psi_*\mathcal{O}_Y$ and by twisting it with $\mathcal{O}_{Y'}(D')$ we obtain a map

$$\mathcal{O}_{Y'}(D') \to \psi_*(\mathcal{O}_Y(D) \otimes g^*L) = \psi_*(\mathcal{O}_Y(D)) \otimes g'^*L. \tag{3.8}$$

The subscheme Y' is embedded in P via the relatively very ample sheaf $\mathcal{O}_{Y'}(D') \cong \mathcal{O}_P(1)|_{Y'}$. Identifying P with $\mathrm{Proj}_C(\mathrm{Sym}^*(\pi_*\mathcal{O}_P(1) \otimes L))$ we see, that Y is embedded in P via $\mathcal{O}_Y(D) \otimes g^*L$. Thus the map $\psi : Y \to Y'$ is compatible with embeddings into P and hance has to be a closed immersion. This implies, that $\mathcal{O}_{Y'} \to \psi_*\mathcal{O}_Y$ is surjective and hence so is the map (3.8). Applying g'_* to (3.8) and using Lemma 3.1.16 for the pair $(g' : Y' \to C, D')$ we obtain a surjective map

$$\pi_*\mathcal{O}_P(1) \twoheadrightarrow \pi_*(\mathcal{O}_P(1)) \otimes L$$

which induces a closed immersion

$$P = \mathrm{Proj}_C(\mathrm{Sym}^*(\pi_*\mathcal{O}_P(1) \otimes L)) \hookrightarrow P = \mathrm{Proj}_C(\mathrm{Sym}^*(\pi_*\mathcal{O}_P(1))). \tag{3.9}$$

For dimensions reasons (3.9) is an isomorphism over C. Therefore the map ψ induces an element in $\mathrm{Aut}(\pi_* \mathcal{O}_P(1))$. This gives a map from the LHS to the RHS. By construction those maps are inverse to each other and hence the lemma follows. $\square$

Lemma 3.1.22. *Let d be a non-negative integer and let ω be a line bundle on C of degree d. Let Y and Y' be two subschemes of P with divisors D and D' such that $\mathcal{O}_Y(D) \cong \mathcal{O}_P(1)|_Y$ and $\mathcal{O}_{Y'}(D') \cong \mathcal{O}_P(1)|_{Y'}$. Suppose moreover $(g : Y \to C, D)$ and $(g' : Y' \to C, D')$ are representatives of classes in $\mathcal{A}_{C,\omega}$. Let s, s' be the associated sections $s, s' \in V_E$ from Lemma 3.1.20 and let $[s]$ and $[s']$ be the associated lines in $\mathbb{P}(V_E)$. Then we have*

$$\mathrm{Equiv}_{\mathcal{A}_{C,\omega}}\left((Y \xrightarrow{g} C, D_Y), (Y' \xrightarrow{g'} C, D_{Y'})\right) = \{v \in PG_E \mid v([s]) = [s']\}. \tag{3.10}$$

Proof. The inclusion "$\subseteq$" is clear. For the other inclusion observe, that Lemma 3.1.21 implies that if ψ is an equivalence of $(g : Y \to C, D_Y)$ and $(g' : Y' \to C, D_{Y'})$ then the induced map $v : V(s) \hookrightarrow V(s')$ with $v \in PG_E$ is a closed immersion. By Remark 3.1.5 (b) the inverse morphism ψ^{-1} is an equivalence of $(g' : Y' \to C, D_{Y'})$ and $(g : Y \to C, D_Y)$ which by Lemma 3.1.21 induces a closed immersion $v^{-1} = v : V(s') \hookrightarrow V(s)$ of integral schemes of the same dimension, hence an isomorphism and the lemma follows. $\square$

Theorem 3.1.23. *Let $d \geq 0$ and let ω be an element of $\mathrm{Pic}^d(C)$. Then*

$$A_{C,\omega} \leq \sum_{[E]_\otimes} \frac{\#\mathbb{P}(V_E)}{\#PG_E} \tag{3.11}$$

where the sum is taken over twist equivalence classes (Definition 3.1.17) of rank 3 vector bundles on C.

Proof. Let $[(Y \xrightarrow{g} C, D)]$ be an element of $\mathcal{A}_{C,\omega}$. Then Y is embedded in P_E with $E = g_* \mathcal{O}_Y(D)$. Moreover by Lemma 3.1.20 Y is given by an element $[s] \in \mathbb{P}(V_E)$, $Y = V(s)$. Let $[s] \in \mathbb{P}(V_E)$ and let $\mathrm{Stab}([s])$ denote the stabilizer of $[s]$ under the action of PG_E

$$\mathrm{Stab}([s]) = \{v \in PG_E \mid v([s]) = [s]\}.$$

Then by Lemma 3.1.22 we have

$$\mathrm{Aut}(g : Y \to C, D) = \mathrm{Stab}([s])$$

and therefore

$$\sum_{\substack{(g:Y\to C,D)\subset P_E \\ (g:Y\to C,D)\in\mathcal{A}_{C,\omega}}} \frac{1}{\mathrm{Aut}(g : Y \to C, D)} \leq \sum_{[s]\in\mathbb{P}(V_E)/PG_E} \frac{1}{\mathrm{Stab}([s])} = \frac{\#\mathbb{P}(V_E)}{\#PG_E}.$$

Hence it follows from Proposition 3.1.15 and Lemma 3.1.16 (see Remark 3.1.18) that we have

$$A_{C,\omega} = \sum_{[E]_\otimes} \sum_{\substack{(g:Y\to C,D)\subset P_E \\ (g:Y\to C,D)\in\mathcal{A}_{C,\omega}}} \frac{1}{\mathrm{Aut}(g : Y \to C, D)} \leq \sum_{[E]_\otimes} \frac{\#\mathbb{P}(V_E)}{\#PG_E}.$$

$\square$

3.2 Counting elements of the set $\mathcal{A}_{C,\omega}$

In Theorem 3.1.23 we showed that for $d \geq 0$ and $\omega \in \mathrm{Pic}^d(C)$ the following bound holds

$$\mathrm{A}_{C,\omega} \leq \sum_{[E]_\otimes} \frac{\#\mathbb{P}(V_E)}{\#PG_E} \tag{3.12}$$

where $\otimes$ denotes the twist equivalence relation from Definition 3.1.17. The goal of this section is to show, that only finitely many of twist equivalence classes of vector bundles E contribute to this sum.

Let E be a rank 3 vector bundle on C. First of all depending on the length of the Harder-Narasihman filtration only the following cases occur.

(I) $0 = E_0 \subset E_1 = E$ if E is semi-stable. This case is treated in Paragraph 3.2.1

(II) $0 = E_0 \subset E_1 \subset E_2 = E$ if E is not semi-stable, but the first quotient ${}^{E_2}/_{E_1}$ is semi-stable. Discussed in Paragraph 3.2.2.

(III) $0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E$ if nor E neither the first quotient ${}^{E}/_{E_1}$ is semi-stable. The case argued in Paragraph 3.2.3.

Proposition 3.2.1. *Let E be a vector bundle on an elliptic curve. Let*

$$0 = E_0 \subsetneq E_1 \subset \ldots \subset E_n = E$$

be its Harder-Narasihman filtration. Then

$$E \cong \bigoplus_{i=1}^{n} {}^{E_i}/_{E_{i-1}}.$$

Proof. For each $i \in \{1, \ldots, n\}$ the quotient ${}^{E_i}/_{E_{i-1}}$ is semi-stable and the sequence

$$0 \to {}^{E_i}/_{E_{i-1}} \to {}^{E}/_{E_{i-1}} \to {}^{E}/_{E_i} \to 0 \tag{3.13}$$

is exact with

$$\mu\left({}^{E_i}/_{E_{i-1}}\right) > \mu\left({}^{E_{i+1}}/_{E_i}\right) \tag{3.14}$$

(see: proof of Lemma 1.3.8 in [HN74]). As C is an elliptic curve we have

$$\begin{aligned}
\mathrm{Ext}^1\left({}^{E_{i+1}}/_{E_i}, {}^{E_i}/_{E_{i-1}}\right) &\cong H^1\left(C, {}^{E_i}/_{E_{i-1}} \otimes {}^{E_{i+1}}/_{E_i}{}^{\vee}\right) \\
&\cong H^0\left(C, {}^{E_i}/_{E_{i-1}}{}^{\vee} \otimes {}^{E_{i+1}}/_{E_i}\right) \\
&= 0 && \text{(by (3.14))}
\end{aligned}$$

Therefore for each $i \in \{1, \ldots, n\}$ we have a splitting of (3.13) and as $E_0 = 0$ the Lemma follows. □

Now we will shortly recall some results on moduli spaces of geometrically indecomposable vector bundles on an elliptic curve. An indecomposable vector bundle need not be indecomposable any more after an extension of the base field. This phenomenon was studied in [ArEl92] and motivates the following definition

Definition 3.2.2. *A vector bundle E on C over k is called geometrically indecomposable if after any field extension K the vector bundle $E_K = E \otimes_{\mathcal{O}_C} \mathcal{O}_{C_K}$ is indecomposable, where $C_K = C \times_k K$.*

and recall the following properties

Lemma 3.2.3. *[HP05, Lemma 1] Let E be a vector bundle on C of rank r and degree e defined over k. Then we have the implications $(i) \implies (ii) \implies (iii) \implies (iv)$ with*

(i) E is stable over k

(ii) E is simple over k

(iii) E is indecomposable over k

(iv) E is semi-stable over k

If moreover r and e are coprime, then we also have $(iv) \implies (i)$ so that all four properties are equivalent.

In [At57] Atiyah classified vector bundles on an elliptic curve C over an algebraically closed field k later A.Tillmann [Til83] showed, that Atiyah's classification holds for geometrically indecomposable vector bundles on C over non-algebraically closed fields and S.Pumplün analyzed it for genus 1 curves over perfect fields of arbitrary index. As Tillman's work is unpublished, we refer to the paper [Pum04] of S.Pumplün.

We define the following sets of isomorphism classes of indecomposable vector bundles of rank r and degree e:

$$\Omega_C(r,e) = \Omega(r,e)$$
$$= \{ \text{ iso. classes of indecomposable vector bundles } E \text{ on } C \text{ with } \deg E = e \text{ and } \mathrm{rk} E = r\}$$

$$\overline{\Omega}_C(r,e) = \overline{\Omega}(r,e)$$
$$= \{ \text{ iso. classes of geom. indecomposable vector bundles } E \text{ on } C \text{ with } \deg E = e \text{ and } \mathrm{rk} E = r\}$$

It is clear, that

$$\overline{\Omega}(r,e) \subset \Omega(r,e).$$

Theorem 3.2.4. *[Pum04, Theorem 4.4] Let C be an elliptic curve over k. Let $r \in \mathbb{Z}_{>0}$ and $e \in \mathbb{Z}$. There is a canonical bijection between the set $\overline{\Omega}(r,e)$ and the set $C(k)$ of k–rational points on C. Via this bijection, $\overline{\Omega}(r,e)$ and $C(k)$ are identified in such a way that, the map*

$$\overline{\Omega}(r,e) \to \mathrm{Pic}^e(C)$$

$$E \mapsto \det E$$

corresponds to isogeny "multiplication by h"

$$[h] : C(k) \to C(k),$$

$$P \mapsto [h]P = P + P + \ldots + P$$

where $h = (r,e)$ is the highest common divisor of r and e.

We moreover have the following description of vector bundles of degree zero.

Theorem 3.2.5. *[Pum04, Theorem 3.12, 3.13]Let C be an elliptic curve over k*

(i) For each $r \geq 1$ there exists a vector bundle $F_r \in \overline{\Omega}_C(r,0)$ that is unique up to isomorphism, such that $H^0(C,F_r) \neq 0$. Moreover there exists an exact sequence

$$0 \to \mathcal{O}_C \to F_r \to F_{r-1} \to 0. \tag{3.15}$$

(ii) *Let $E \in \overline{\Omega}_C(r,0)$ then there is a line bundle S of degree 0 on C which is unique up to isomorphism, such that $E \cong S \otimes F_r$. In particular, E contains S as a subbundle and $\det E \cong S^r$.*

Remark 3.2.6. We have $F_1 = \mathcal{O}_C$.

Proposition 3.2.7. *[Pum04, Corollary 3.16] Let C be an elliptic curve over k*

(i) *The vector bundle F_r from Theorem 3.2.5 is self-dual.*

(ii) *For all $r,s \geq 1$ and $s < r$ we have an exact sequence*

$$0 \to F_s \to F_r \to F_{r-s} \to 0. \tag{3.16}$$

The embedding of F_s in F_r as a subbundle is uniquely determined up to isomorphisms of F_r.

(iii)
$$\dim_k H^0(C, F_r \otimes F_s) = \min(r,s).$$

Assume we have a vector bundle E over C that is indecomposable over C but splits over $\overline{C}$. It is clear that E will split over some $C^n = C \times_k k^n$ where k^n is the degree n extension of k. We will show, that in fact for $\mathrm{rk}(E) = 3$ we have $n = 3$ and if $\mathrm{rk}(E) = 2$ then $n = 2$.

Remark 3.2.8. Let C be a smooth projective curve. Let E be a rank r vector bundle on C and $A_{ij} \in \mathrm{GL}(r, \mathcal{O}_{U_j \cap U_i})$ its transition matrix in standard basis, meaning the matrix expression of the composite of the two isomorphisms $\mathcal{O}_{U_j}^{\oplus r} \simeq E|_{U_j}$ and $\mathcal{O}_{U_i}^{\oplus r} \simeq E|_{U_i}$ over $U_j \cap U_i$. The collection $\{A_{i,j}\}$ of all such matrix functions is a 1-cocycle with coefficients in the sheaf $\mathrm{GL}(r, \mathcal{O}_C)$. Hence rank r vector bundles on C are parameterized by the first cohomology set $H^1(C, \mathrm{GL}(r, \mathcal{O}_C))$.

Proposition 3.2.9. *Let C be an elliptic curve over a finite field k. Let E be a rank 3 indecomposable vector bundle on C that is not geometrically indecomposable. Then E splits into a direct sum of line bundles over $C^3 = C \times_k k^3$ that are not isomorphic to each other and do not descend to C.*

Proof. By assumption there exists a finite field extension k' of k^3 such that if we denote by $\pi : C' = C \times k' \to C$ the projection, then $\pi^* E$ splits as $L' \oplus F'$ where L' is a line bundle on C' and F' is a rank 2 vector bundle on C'. The Galois group $G(k'/k)$ acts naturally on $\pi^* E$. Let $\sigma \in G(k'/k)$ be a generator and suppose, that the composition

$$L' \hookrightarrow L' \oplus F' \cong \pi^* E \xrightarrow{\sigma} \pi^* E \cong L' \oplus F' \twoheadrightarrow L' \tag{3.17}$$

is not zero. If we denote by $[L']$ the class of L' in $H^1(C', \mathbb{G}_m)$ then σ maps $[L']$ to itself and hence $[L'] \in H^1(C', \mathbb{G}_m)^{G(k'/k)}$. As k' is a finite field and as $H^0(C', \mathbb{G}_m) = (k')^*$ we have the vanishing of

$$H^1(G(k'/k), H^0(C', \mathbb{G}_m)) = 0$$

as well as of the Brauer group of k'

$$H^2(G(k'/k), H^0(C', \mathbb{G}_m)) = 0.$$

The sheaf $\mathbb{G}_m$ is an étale sheaf on C therefore we can use the Hochschild-Serre spectral sequence

$$H^i(G(k'/k), H^j(C', \mathbb{G}_m)) \Rightarrow H^{i+j}(C, \mathbb{G}_m)$$

[Mil, Theorem 2.20, Chapter 3] to deduce, that

$$H^1(C',\mathbb{G}_m)^{G(k'/k)} = H^1(C,\mathbb{G}_m).$$

Thus

$$[L'] \in H^1(C',\mathbb{G}_m)^{G(k'/k)} = H^1(C,\mathbb{G}_m)$$

which implies that L' descends to a line bundle on C. It means, that there exists a line bundle L on C with $L' \cong \pi^* L$. This gives an isomorphism $\pi^* E \cong \pi^* L \oplus F'$ and by applying π_* on both sides we obtain an isomorphism

$$E^{\oplus[k':k]} \cong L^{\oplus[k':k]} \oplus \pi_* F'$$

which contradicts the indecomposability assumption on E. Hence we have proved, that the map (3.17) is zero. Therefore L' maps under σ to F' and since $\sigma(L')$ is a direct summand of $\pi^* E$ it has to be a direct summand of F' i.e. $\pi^* E$ is a direct sum of line bundles $L'_1 \oplus L'_2 \oplus L'_3$, $L'_i \in \mathrm{Pic}(C')$ non of which descends to C (by the above). Also σ maps L'_i isomorphically to $\sigma(L'_j)$ for some j such that $j \neq i$ but $\sigma(L_j) \ncong L_i$ and therefore permutes their classes in $H^1(C',\mathbb{G}_m)$. It follows, that these classes live in $H^1(C',\mathbb{G}_m)^{G(k'/k^3)}$ and hence L_i's descend to line bundles M_i on C^3. Furthermore if $\tau \in G(k^3/k)$ is a generator then by the same argument as above we have $M_2 \cong \tau^i M_1$ and $M_3 \cong \tau^j M_1$ with $i,j \in \{1,2\}$ and M_i cannot be isomorphic to M_j. Denote by $\rho : C' \to C^3$ the projection and by E^3 the pullback of E to C^3. Then we have an isomorphism $\rho^* E^3 \cong \rho^*(M_1 \oplus M_2 \oplus M_3)$. Thus E^3 is a twisted-form of $M := M_1 \oplus M_2 \oplus M_3$ and hence defines an element in

$$\check{\mathrm{H}}^1(C'/C^3, \underline{\mathrm{Aut}}(M)) = H^1(G(k'/k^3), \mathrm{Aut}_{C'}(\rho^* M)). \tag{3.18}$$

We have

$$\mathrm{End}_{C'}(\rho^* M) = \bigoplus_{i,j} H^0(C', \rho^* M_i \otimes \rho^* M_j^{-1}).$$

But $G(k^3/k)$ acts as a 3-cycle permutation on the set $\{[M_1],[M_2],[M_3]\}$. Furthermore since the degree on $\mathrm{Pic}(C^3)$ commutes with the Galois action, the line bundles M_i have the same degree and therefore $\rho^* M_i \otimes \rho^* M_j^{-1}$ for $i \neq j$ are degree-zero line bundles that are all non-trivial. Thus $\mathrm{End}_{C'}(\rho^* M)$ equals to $\mathrm{Diag}(3,k')$ and hence

$$\mathrm{Aut}(\rho^* M) = \mathrm{Diag}(3,k'^*).$$

Hence (3.18) vanishes by Hibert 90. This implies that any twisted-form of M on C' is isomorphic to M, in particular $E^3 = M$ on C^3. □

Using the same argument as in Proposition 3.2.9 one shows:

Proposition 3.2.10. *Let C be an elliptic curve over a finite field k. Let E be a rank 2 indecomposable vector bundle on C that is not geometrically indecomposable. Then E splits into a direct sum of line bundles over $C^2 = C \times_k k^2$ that are not isomorphic to each other and do not descend to C.*

Corollary 3.2.11. *Let E be a vector bundle of rank 3 on an elliptic curve C. Let $E = E_1 \oplus \ldots \oplus E_s$ be the indecomposable decomposition of E. We have the following cases*

(1) $s = 1$ then

- *(a) E is geometrically indecomposable;*
- *(b) or E is indecomposable over C but splits as a direct sum of line bundles over $C^3 = C \times_k k^3$ that have the same degree, but are not isomorphic and do not descend to C.*

(2) $s = 2$ *then* $E = E_1 \oplus E_2$ *with* $\mathrm{rk}E_i = i$ *with* $i \in \{1,2\}$ *and*

  (a) E_2 *is geometrically indecomposable;*

  (b) *or* E_2 *is indecomposable over* C *but splits as a direct sum of line bundles over* $C^2 = C \times_k k^2$ *that have the same degree, but are not isomorphic and do not descend to* C*;*

  (c) *or* E_2 *decomposes over* C.

(3) $s = 3$ *then* $E = E_1 \oplus E_2 \oplus E_3$ *with* $\mathrm{rk}E_i = 1$ *for* $i \in \{1,2,3\}$.

We wish to evaluate the sum

$$\sum_{[E]_\otimes} \frac{\#\mathbb{P}(V_E)}{\#PG_E}.$$

In order to do that in each equivalence class $[E]_\otimes$ we are allowed to choose a representative E'. This representative has to satisfy $H^0(C,E') \neq 0$ and for computational reasons we wish to minimize its degree.

Remark 3.2.12. Observe, that if E is a vector bundle of rank r on C then fo any $L \in \mathrm{Pic}(C)$

$$\deg(E \otimes L) = \deg E + r \cdot \deg L.$$

In this way, when considering vector bundles of rank r it is enough to consider vector bundles of degree $0, 1, ..., r-1$ then by twisting with a line bundle of appropriate degree we get all other degrees.

Proposition 3.2.13. *Let* E *be a vector bundle of rank* 3 *on an elliptic curve* C*. Assume moreover, that* E *is not as in Proposition 3.2.9. Then in the twist equivalence class* $[E]_\otimes = \{E \otimes L, L \in \mathrm{Pic}(C)\}$ *of* E *we can always find a vector bundle* E' *such that it has a minimal non-negative slope* $\mu(E') \in [0,1)$ *and satisfies* $H^0(C,E') \neq 0$*. Furthermore if* E *satisfies assumptions of Proposition 3.2.9 then* E' *can be chosen so that* $\mu(E') = 1$.

Proof. Let E be a rank 3 vector bundle on C. Assume that E is not as in Proposition 3.2.9. Let

$$0 = E_0 \subsetneq E_1 \subset ... \subset E_n = E$$

be the Harder Narasimhan filtration of E. By Lemma 3.2.1 we have

$$E \cong \bigoplus_{i=1}^{n} {}^{E_i}\!\big/_{E_{i-1}} \tag{3.19}$$

and by Theorem 1.1.9 the slopes of the semi-stable summands of (3.19) satisfy

$$\mu(E_1) > \mu({}^{E_2}\!/_{E_1}) > ... > \mu({}^{E_n}\!/_{E_{n-1}}).$$

Let p be a k-rational point on C then using the Remark 3.2.12 we can always find $m_E \in \mathbb{Z}$ such that

$$\mu(E \otimes \mathcal{O}_C(m_E p)) = \frac{m_E \mathrm{rk}E + \deg E}{\mathrm{rk}E} \in [0,1).$$

Therefore without loss of generality we may assume $\mu(E) \in [0,1)$. The proposition is clear for $n \geq 2$ as then E is not semi-stable and E_1 is a semi-stable sub-bundle with $\mu(E_1) > \mu(E) \geq 0$ implying $H^0(C,E_1) \neq 0$ and consequently $H^0(C,E) \neq 0$. If $n = 1$ and $\mu(E) > 0$ then we are also done. It remains to prove the case $n = 1$ and $\mu(E) = 0$. Meaning, that E is semi-stable of slope zero. If E is decomposable, then semi-stability implies, that all the summands have degree 0 in particular E has a line bundle L of degree 0 as a direct summand and we can take $E' = E \otimes L^\vee$. If E is geometrically indecomposable, then $E \cong F_3 \otimes L$ for some line bundle L of degree 0 and we again take $E' = E \otimes L^\vee$. If E is indecomposable but not geometrically, then by Proposition 3.2.9 $\pi^* E$ is a sum of three line bundles; here $\pi : C \times_k k^3 \to C$ is the projection. These line bundles are non-trivial and of the same degree, hence of degree 0 since $\mu(E) = \mu(\pi^* E) = 0$. Thus we can take $E' = E \otimes \mathcal{O}_C(p)$ for some $p \in C(k)$. This finishes the proof. □

3.2.1 Semistable case

Let $d \geq 0$ and $\omega \in \mathrm{Pic}^d(C)$. First of all we treat the case that E is semi-stable. By minimizing the slope of E as in Proposition 3.2.13 we have the fo llowing cases

- If E does not satisfy assumptions of Proposition 3.2.9, then $\mu(E) = \frac{1}{3}$ or $\mu(E) = \frac{2}{3}$ or $\mu(E) = \frac{0}{3} = 0$.
- If E satisfies assumptions of Proposition 3.2.9 then $\mu(E) = \frac{3}{3} = 1$.

Remark 3.2.14. For computing automorphism groups of some vector bundles on C we will use the following fact.

Let K be a Galois extension of k and consider the following diagram

$$\begin{tikzcd} \pi^* E \arrow[r] \arrow[d] & E \arrow[d] \\ C_K \arrow[r, "\pi"] \arrow[d] & C \arrow[d] \\ \mathrm{Spec}\,(K) \arrow[r] & \mathrm{Spec}\,(k) \end{tikzcd},$$

where E a vector bundle on C. Then from Galois-Descent follows, that

$$\mathrm{Aut}(E) = \mathrm{Aut}(\pi^* E)^{\mathrm{Gal}(K/k)}.$$

gcd(3,2)=gcd(3,1)=1

If the degree of E is 1 or 2 then $\gcd(3,1) = \gcd(3,2) = 1$ and hence by Lemma 3.2.3 E is stable. The degree and the rank are preserved under base extension (see Section 1.1) therefore the vector bundle E is geometrically stable and by Lemma 3.2.3 E is geometrically indecomposable. The number of isomorphism classes of geometrically indecomposable vector bundles with $(r,e) = (3,2)$ or $(r,e) = (3,1)$ is by Lemma 3.2.4 equal to $\#C(k)$. Furthermore the automorphism group of a geometrically stable vector bundle is the multiplicative group $\mathbb{G}_m/k$.

gcd(3,0)=gcd(3,3)=3

We now assume $\mu(E) = 0$.

Lemma 3.2.15. *Let E be a semi-stable vector bundle of rank 3 and degree $\deg(E) = 0$. Assume that if E is indecomposable, then it is geometrically indecomposable. Up to twist with a line bundle we have the following possible different structures on E.*

(R3a) The vector bundle E is geometrically indecomposable, $E = F_3 \in \overline{\Omega}_C(3,0)$. It is unique up to isomorphism and has the automorphism group $\#\mathrm{Aut}(F_3) = q^2(q-1)$.

(R3b) *The vector bundle E has the decomposition $E = \mathcal{O}_C \oplus G$, with $G \in \overline{\Omega}_C(2,0)$. The number of isomorphism classes of vector bundles E of the type $E = \mathcal{O}_C \oplus G$, with $G \in \overline{\Omega}_C(2,0)$ is $\#C(k)$. Furthermore*

$$\#\mathrm{Aut}(E) = \begin{cases} (q-1)^2 q^3, & \text{if } G \cong F_2, \\ (q-1)^2 q, & \text{if } G \ncong F_2. \end{cases} \tag{3.20}$$

(R3c) *The vector bundle E has the decomposition $E = \mathcal{O}_C \oplus G$, where G is a rank 2 vector bundle that is indecomposable over C but splits over $C^2 = C \times_k k^2$. The number of isomorphism classes of vector bundles E of the above type is at most*

$$\#C(k^2) - \#C(k) - 1.$$

Furthermore

$$\#\mathrm{Aut}(E) = (q-1)(q^2-1),$$

R3d) *The vector bundle E is a direct sum of line bundles, $E = \mathcal{O}_C \oplus L_1 \oplus L_2$ with $L_i \in \mathrm{Pic}^0(C)$. The number of isomorphism classes of vector bundles E of the above type is at most $\#C(k)^2$. Furthermore*

$$\#\mathrm{Aut}(E) = \begin{cases} (q^3-1)(q^3-q)(q^3-q^2), & \text{if } L_1 \cong L_2 \cong \mathcal{O}_C, \\ (q-1)^3(q+1)q, & \text{if } L_1 \cong L_2 \ncong \mathcal{O}_C, \\ (q-1)^2 q^3, & \text{if } L_1 \ncong L_2 \cong \mathcal{O}_C \\ (q-1)^3, & \text{if } \mathcal{O}_C \ncong L_1 \ncong L_2 \ncong \mathcal{O}_C. \end{cases} \tag{3.21}$$

Proof. The classification (R3a)-(R3d) follows from Corollary 3.2.11.

R3a) Assume E is geometrically indecomposable. By Theorem 3.2.5(ii) up to twist with a line bundle we have $E = F_3$. Furthermore by Theorem 3.2.5(i) the vector bundle F_3 is unique up to isomorphism.

Take $\{U_i\}_{i \in I}$ an open cover of C such that there exist isomorphisms $F_3|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus 3}$. The vector bundle F_3 fits into the exact sequence (3.16)

$$0 \to \mathcal{O}_C \xrightarrow{\gamma} F_3 \xrightarrow{\delta} F_2 \to 0. \tag{3.22}$$

Let $\{f_1^i, f_2^i\}$ be a basis of $F_2|_{U_i}$ such that $\alpha(1) = f_1^i$ and $\beta(f_2^i) = 1$ under the maps

$$0 \to \mathcal{O}_C \xrightarrow{\alpha} F_2 \xrightarrow{\beta} \mathcal{O}_C \to 0.$$

Then a transition matrix of F_2 with respect to this basis has the form

$$\begin{pmatrix} 1 & \beta_{i,j} \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathcal{O}(U_i \cap U_j)).$$

Let $\{e_1^i, e_2^i, e_3^i\}$ be a basis of $F_3|_{U_i}$ such that $1 \mapsto e_1^i$ under the map γ and $e_k^i \mapsto f_{k-1}^i$ under the map δ for $k \in \{2,3\}$. Then the transition matrix on $U_i \cap U_j$ of F_3 is given by

$$\eta_{i,j} = \begin{pmatrix} 1 & \gamma_{i,j} & \alpha_{i,j} \\ 0 & 1 & \beta_{i,j} \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3, \mathcal{O}(U_i \cap U_j)).$$

We will show, that $\eta_{i,j}$ is symmetric, i.e $\beta_{i,j} = \gamma_{i,j}$. The dual of the sequence (3.22) is the sequence

$$0 \to F_2^\vee \to F_3^\vee \to \mathcal{O}_C \to 0.$$

Furthermore as $F_2^\vee \cong F_2$ (By Proposition 3.2.7(i)) we can find a local basis for $F_3^\vee$ such that the gluing over $U_i \cap U_j$ is given by

$$\overline{\eta}_{i,j} = \begin{pmatrix} 1 & \beta_{i,j} & \delta_{i,j} \\ 0 & 1 & \tau_{i,j} \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3, \mathcal{O}(U_i \cap U_j)).$$

The existence of the isomorphism $F_3^\vee \cong F_3$ (Proposition 3.2.7(i)) implies that, there is a transformation $h_i \in \mathrm{GL}(3, \mathcal{O}_{U_i})$ such that $h_i \overline{\eta}_{i,j} h_j^{-1} = \eta_{i,j}$. Furthermore by Proposition 3.2.7 for every $s, r \geq 1$ and $s < r$ we have the diagram

$$\begin{tikzcd} 0 \arrow[r] & F_r \arrow[r] \arrow[d, "\cong"] & F_s \arrow[r] \arrow[d, "\cong"] & F_{s-r} \arrow[r] \arrow[d, "\cong"] & 0, \\ 0 \arrow[r] & F_r^\vee \arrow[r] & F_s^\vee \arrow[r] & F_{s-r}^\vee \arrow[r] & 0 \end{tikzcd}$$

where the bottom exact sequence is the dual of the sequence $0 \to F_{r-s} \to F_r \to F_s \to 0$. Now, since for each $s < r$ the embedding of F_s in F_r as a subbundle is unique up to isomorphisms of F_r (see Proposition 3.2.7(ii)) the above diagram commutes. Furthermore its commutativity implies, that we can find a local basis $\{a_1^i, a_2^i, a_3^i\}$ of $F_3|_{U_i}$ and a local basis $\{\bar{a}_1^i, \bar{a}_2^i, \bar{a}_3^i\}$ of $F_3^\vee|_{U_i}$ such that $\overline{\eta}_{i,j} = \eta_{i,j}$ hence the gluing matrix is symmetric given by

$$\eta_{i,j} = \begin{pmatrix} 1 & \beta_{i,j} & \alpha_{i,j} \\ 0 & 1 & \beta_{i,j} \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3, \mathcal{O}(U_i \cap U_j)).$$

This proves the claim. Now using the gluing matrix we will describe automorphisms of F_3. First of all the collection $\eta_{i,j}$ satisfy the cocycle condition, therefore the functions $\beta_{i,j}$ must form a cocycle $\beta = \{\beta_{i,j}\}$ in $H^1(C, \mathcal{O}_C) = k$. It can not be 0 as F_2 is a non-trivial extension. An automorphism ϕ of F_3 is locally given by a matrix $\phi(U_i) = (A_i) \in \mathrm{GL}(3, \mathcal{O}_{U_i})$ that commutes with the gluing data

$$\begin{tikzcd} \mathcal{O}_{U_i}^{\oplus 3}|_{U_i \cap U_j} \arrow[r, "A_i"] \arrow[d, "\eta_{i,j}"] & \mathcal{O}_{U_i}^{\oplus 3}|_{U_i \cap U_j} \arrow[d, "\eta_{i,j}"] \\ \mathcal{O}_{U_j}^{\oplus 3}|_{U_i \cap U_j} \arrow[r, "A_j"] & \mathcal{O}_{U_j}^{\oplus 3}|_{U_i \cap U_j} \end{tikzcd}$$

meaning

$$\begin{pmatrix} a_i & b_i & c_i \\ d_i & e_i & f_i \\ g_i & h_i & k_i \end{pmatrix} \begin{pmatrix} 1 & \beta_{i,j} & \alpha_{i,j} \\ 0 & 1 & \beta_{i,j} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \beta_{i,j} & \alpha_{i,j} \\ 0 & 1 & \beta_{i,j} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_j & b_j & c_j \\ d_j & e_j & f_j \\ g_j & h_j & k_j \end{pmatrix}.$$

Therefore we obtain the following system of equations

$$\begin{cases} g_i = g_j \\ h_i - h_j = -g_i\beta_{i,j} \\ d_i - d_j = g_j\beta_{i,j} \\ e_i - e_j = (h_j - d_i)\beta_{i,j} \\ k_i - k_j = -g_i\alpha_{i,j} - h_i\beta_{i,j} \\ a_i - a_j = d_j\beta_{i,j} + g_j\alpha_{i,j} \\ b_i - b_j = (e_j - a_i)\beta_{i,j} + h_j\beta_{i,j} \\ f_i - f_j = (k_j - e_i)\beta_{i,j} - d_i\alpha_{i,j} \\ c_i - c_j = (f_j - b_i)\beta_{i,j} + (k_j - a_i)\alpha_{i,j} \end{cases}$$

The condition $g_i = g_j$ implies that the collection $\{g_i\}$ form a global section $g \in H^0(C, \mathcal{O}_C)$. Moreover we must have $g = 0$ if not $\beta_{i,j} = \frac{h_i - h_j}{g}$ which would imply, that $\beta_{i,j}$ is a coboundary and contradict indecomposability of F_2. Consequently it follows, that $\{h_i\}$ and $\{d_i\}$ form global sections of $h, d \in H^0(C, \mathcal{O}_C)$. From the equality $e_i - e_j = (h_j - d_i)\beta_{i,j}$ follows, that $h = d$ and therefore the collection $\{e_i\}$ form a global section $e \in H^0(C, \mathcal{O}_C)$. Moreover as $g = 0$ we have $k_i - k_j = -h\beta_{i,j}$ hence $h = 0$ and the elements $\{k_i\}$ form a global section $k \in H^0(C, \mathcal{O}_C)$. From $d = g = 0$ we deduce, that $a_i = a_j$ and therefore $\{a_i\}$ form a global section $a \in H^0(C, \mathcal{O}_C)$. Now as e, a and k, e are global sections of $\mathcal{O}_C$ and $h = d = 0$ we have $b_i - b_j = (e - a)\beta_{i,j} = 0$ and $k = e$ otherwise $\frac{f_i - f_j}{(k-e)} = \beta_{i,j}$. Then $f = \{f_i\} \in H^0(C, \mathcal{O}_C)$ and also $b = \{b_i\} \in H^0(C, \mathcal{O}_C)$. As $k = e = a$ the last equation implies $f = b$ and $c = \{c_i\} \in H^0(C, \mathcal{O}_C)$. All in all we conclude, that automorphism group of F_3 is the group of 3×3 invertible matrices

$$\begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$$

with $a, b, c \in k$ that is a semidirect product of the group of strictly upper triangular matrices

$$\begin{pmatrix} 1 & b & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

and the group of invertible diagonal matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}.$$

Consequently using the natural isomorphisms $\mathrm{Aut}(F_3) = (\mathbb{G}_a(k) \times \mathbb{G}_a(k)) \rtimes \mathbb{G}_m(k)$.

R3b) Assume E is a direct sum $E = \mathcal{O}_C \oplus G$ with $G \in \overline{\Omega}_C(2, 0)$. By Theorem 3.2.5(ii) there exists a unique line bundle $L \in \mathrm{Pic}^0(C)$ such that $G = L \otimes F_2$. Furthermore it is clear, that if $E' = \mathcal{O}_C \oplus G'$ is another vector bundle of this type, then $E \cong E'$ if and only if $G \cong G'$ hence the isomorphism class of E depends only on G which is determined by L. Therefore the number of isomorphism classes of vector bundles E of the type $E = \mathcal{O}_C \oplus G$, with $G \in \overline{\Omega}_C(2, 0)$ is $\#C(k)$.

First we will describe automorphisms of F_2 and later compute automorphisms of $\mathcal{O}_C \oplus L \otimes F_2$. Take $\{U_i\}_{i \in I}$ an open cover of C such that there exist isomorphisms $F_2|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus 2}$. Let $\{f_1^i, f_2^i\}$ be a local basis of $F_2|_{U_i}$ such that $\alpha(1) = f_1^i$

and $\beta(f_2^i) = 1$ under the maps (see:(3.16))

$$0 \to \mathcal{O}_C \xrightarrow{\alpha} F_2 \xrightarrow{\beta} \mathcal{O}_C \to 0. \tag{3.23}$$

Then a transition matrix of F_2 with respect to this basis has the form

$$\delta_{i,j} = \begin{pmatrix} 1 & \beta_{i,j} \\ 0 & 1 \end{pmatrix}.$$

The cocycle condition for $\{\delta_{i,j}\}$ implies, that the elements $\{\beta_{ij}\}$ form a cocycle, $\beta = \{\beta_{i,j}\} \in \mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) \cong H^1(C, \mathcal{O}_C) = k$. It is clear, that $\beta \neq 0$ as F_2 is a non-trivial extension. An automorphism φ of F_2 is locally given by a matrix $\varphi(U_i) = (A_i)_{1 \le i \le 2} \in \mathrm{GL}(2, \mathcal{O}_{U_i})$ that commutes with the gluing data

$$\begin{tikzcd}
\mathcal{O}_{U_i}^{\oplus 2}|_{U_i \cap U_j} \arrow[r, "A_i"] \arrow[d, "\delta_{i,j}"] & \mathcal{O}_{U_i}^{\oplus 2}|_{U_i \cap U_j} \arrow[d, "\eta_{i,j}"] \\
\mathcal{O}_{U_j}^{\oplus 2}|_{U_i \cap U_j} \arrow[r, "A_j"] & \mathcal{O}_{U_j}^{\oplus 2}|_{U_i \cap U_j}
\end{tikzcd}$$

meaning

$$\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \begin{pmatrix} 1 & \beta_{i,j} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \beta_{i,j} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}.$$

From the diagram above we obtain the following system of equalities

$$\begin{cases} c_i & = c_j \\ a_i - a_j & = c_j \beta_{i,j} \\ d_i - d_j & = c_i \beta_{i,j} \\ b_i - b_j & = (d_j + a_i)\beta_{i,j} \end{cases}$$

The condition $c_i = c_j$ implies that the c_i's glue to a global section $c \in H^0(C, \mathcal{O}_C) = k$. Moreover we must have $c = 0$ otherwise the equality

$$\beta_{i,j} = \frac{a_i}{c} - \frac{a_j}{c}$$

is satisfied, which implies that $\beta_{i,j}$ is a coboundary and the exact sequence (3.23) splits. Hence $c = 0$ which consequently imply, that $a_i = a_j$ and $d_i = d_j$ and therefore a_i's and d_i's glue to give global sections $a, d \in H^0(C, \mathcal{O}_C) = k$. By applying the same argument as for c to the equation $b_i - b_j = (d - a)\beta_{i,j}$ we obtain $a = d$ and hence a global section $b \in H^0(C, \mathcal{O}_C) = k$. Therefore the automorphism group of F_2 is the group of 2×2 matrices

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

with $a, b \in k$ and $a \neq 0$. This group is a direct product of the group of strictly upper triangular matrices

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

and the group of invertible diagonal matrices

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Therefore

$$\mathrm{Aut}(F_2) = \mathbb{G}_a(k) \rtimes \mathbb{G}_m(k). \tag{3.24}$$

Let us now describe automorphisms of $E = \mathcal{O}_C \oplus F_2$. Again we take a covering $\{U_i\}_i$ of C such that $E|_{U_i} = (\mathcal{O}_C \oplus F_2)|_{U_i} \cong \mathcal{O}_{U_i} \oplus \mathcal{O}_{U_i} \oplus \mathcal{O}_{U_i}$. Take the basis $\{f_1^i, f_2^i\}$ of F_2 and complete it to the basis $\{1, f_1^i, f_2^i\}$ of $\mathcal{O}_C \oplus F_2$. As we have the direct decomposition of E a gluing on $U_i \cap U_j$ with respect to the basis $\{1, f_1^i, f_2^i\}$ and $\{1, f_1^j, f_2^j\}$ is given by

$$\zeta_{i,j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \beta_{i,j} \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$\begin{pmatrix} 1 & \beta_{i,j} \\ 0 & 1 \end{pmatrix}$$

is a transition matrix of F_2. Automorphisms of E are locally given by 3×3 matrices $(B_i) \in \mathrm{GL}(3, \mathcal{O}_{U_i})$ that commute with the gluing data

$$\begin{tikzcd}
\mathcal{O}_{U_i}^{\oplus 3}|_{U_i \cap U_j} \arrow[r, "B_i"] \arrow[d, "\zeta_{i,j}"] & \mathcal{O}_{U_i}^{\oplus 3}|_{U_i \cap U_j} \arrow[d, "\zeta_{i,j}"] \\
\mathcal{O}_{U_j}^{\oplus 3}|_{U_i \cap U_j} \arrow[r, "B_j"] & \mathcal{O}_{U_j}^{\oplus 3}|_{U_i \cap U_j}
\end{tikzcd}$$

meaning

$$\begin{pmatrix} a_i & b_i & c_i \\ d_i & e_i & f_i \\ g_i & h_i & k_i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \beta_{i,j} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \beta_{i,j} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_j & b_j & c_j \\ d_j & e_j & f_j \\ g_j & h_j & k_j \end{pmatrix}.$$

Therefore

$$\begin{cases} a_i &= a_j \\ b_i &= b_j \\ g_i &= g_j \\ h_i &= h_j \\ c_j - c_i &= b_i \beta_{i,j} \\ k_j - k_i &= h_i \beta_{i,j} \\ e_j - e_i &= h_j \beta_{i,j} \\ d_j - d_i &= g_j \beta_{i,j} \\ f_j - f_i &= (e_i - k_j)\beta_{i,j} \end{cases}$$

The first four equalities imply, that the collections $\{a_i\}, \{b_i\}, \{g_i\}, \{h_i\}$ glue to global sections $a, b, g, h \in H^0(C, \mathcal{O}_C)$. The four next equalities imply, that $b = h = g = 0$ (otherwise $\beta_{i,j}$ would be a coboundary) which consequently leads to $\{c_i\}, \{k_i\}, \{e_i\}, \{g_i\}$ forming global sections $c, k, e, d \in H^0(C, \mathcal{O}_C)$. The last equality proves, that $e = k$ and that the collection $\{f_i\}$ gives a global section f of $\mathcal{O}_C$. We therefore conclude, that every automorphism of E is given by an invertible matrix

$$\begin{pmatrix} a & 0 & c \\ d & e & f \\ 0 & 0 & e \end{pmatrix}$$

with $a, c, d, e, f \in k$. Therefore

$$\mathrm{Aut}(E) = \left\{ \begin{pmatrix} a & 0 & c \\ d & e & f \\ 0 & 0 & e \end{pmatrix} \text{ with } a, e \in k^* \text{ and } c, d, f \in k \right\}.$$

Assume now that $E = \mathcal{O}_C \oplus L \otimes F_2$ with $L \ncong \mathcal{O}_C$. Twisting the exact sequence (3.16) with L we obtain $H^0(C, L \otimes F_2) = 0$ and therefore there is no cross homomorphisms $\mathrm{Hom}(L, F_2 \otimes L) = \mathrm{Hom}(L, F_2 \otimes L) = 0$ (here we use Proposition 3.2.7(i)) which gives the direct decomposition

$$\mathrm{Aut}(\mathcal{O} \oplus L \otimes F_2) = \mathbb{G}_m(k) \times \mathrm{Aut}(F_2).$$

R3c) Assume E is a direct sum $E = \mathcal{O}_C \oplus G$, where G has rank 2 is indecomposable over C but not geometrically indecomposable. By Proposition 3.2.10 $\pi^* E = L_1 \oplus L_2$ with $\deg(L_i) = 0$, $L_1 \ncong L_2$ and $L_i \ncong \mathcal{O}_{C^2}$, where $\pi : C^2 := C^2 \times_k k^2 \to C$ is the projection. Lemma 3.2.16 implies that the isomorphism class of G determines the isomorphism class of E. Furthermore if we fix L_1 then the line bundle L_2 is determined by L_1 namely $L_2 = \sigma(L_1)$, where $\sigma \in G(k^2/k)$ is a generator. Now as $\mathrm{Pic}^0(C^2) \cong C^2$ the number of isomorphism classes of vector bundles G as above is at most

$$\#C(k^2) - \#C(k) - 1.$$

To describe the automorphism group of each E we use the Remark 3.2.14. Let $\pi : C^2 := C \times_k k^2 \to C$ be the projection, then by Proposition 3.2.9 $\pi^* G = M_1 \oplus M_2$, where $M_i \in \mathrm{Pic}^0(C^2)$ with $M_1 \ncong M_2$ and $M_i \ncong \mathcal{O}_{C^2}$ for $i = 1, 2$. Using properties of M_i's for $i \in \{1, 2\}$ we have

$$\mathrm{Hom}(\mathcal{O}_C, M_i) = \mathrm{Hom}(M_i, \mathcal{O}_C) = \mathrm{Hom}(M_i, M_j) = 0.$$

Therefore the $\mathcal{O}_{C^2}$-linear automorphisms of $\pi^* E$ are given by invertible matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix}$$

with $a, d, e \in k^2$. Furthermore the Galois group acts on $\pi^* E$ by fixing $\mathcal{O}_{C^2}$ and permuting M_1 and M_2. Its corresponding matrix representation is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and the $\mathrm{Gal}(k^2/k)$-invariant automorphisms of $\pi^* E$ are those, that satisfy the relation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \bar{a} & 0 & 0 \\ 0 & \bar{d} & 0 \\ 0 & 0 & \bar{e} \end{pmatrix},$$

where $\bar{a}$ denotes the conjugation of a. We have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Consequently automorphisms of E are invertible matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & \overline{d} \end{pmatrix}$$

with $a, d \in k^2$ and hence

$$\mathrm{Aut}(E) = \mathbb{G}_m(k) \times \mathrm{Res}_{k^2/k}\mathbb{G}_m(k),$$

where $\mathrm{Res}_{k^2/k}\mathbb{G}_m$ denotes the Weil restriction of $\mathbb{G}_m$.

R3e) Assume E is a direct sum of three line bundles of degree 0 defined over C. Therefore up to twist we have $E = \mathcal{O}_C \oplus L_1 \oplus L_2$ with $L_i \in \mathrm{Pic}^0(C)$. The number of isomorphism classes of vector bundles E of $\mu(E) = 0$ and $E = \mathcal{O}_C \oplus L_1 \oplus L_2$ with $L_i \in \mathrm{Pic}^0(C)$ is at most $\#C(k)^2$.

Assume now that $E = \mathcal{O} \oplus L_1 \oplus L_2$ and let us write $L_0 := \mathcal{O}_C$, then every $\mathcal{O}_C$-linear endomorphism of E is given by a matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{33} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

with $a_{i,j} \in \mathrm{Hom}_{\mathcal{O}}(L_i, L_j)$ for $0 \le i \le 2$ and $0 \le j \le 2$ and for $i = j$ we have $\mathrm{Hom}_{\mathcal{O}}(L_i, L_j) = k$. First of all only the following cases occur

$$\begin{cases} L_1 \cong L_2 \cong \mathcal{O}_C \\ L_1 \cong L_2 \ncong \mathcal{O}_C \\ L_1 \ncong L_2 \cong \mathcal{O}_C \\ \mathcal{O}_C \ncong L_1 \ncong L_2 \ncong \mathcal{O}_C \end{cases}$$

It is clear, that the case $E = \mathcal{O}_C^{\oplus 3}$ brings

$$\mathrm{Aut}(E) = \mathrm{GL}_3(k).$$

The second condition $L_1 \cong L_2$ but $L_i \ncong \mathcal{O}_C$ implies $\mathrm{Hom}(\mathcal{O}_C, L_i) = \mathrm{Hom}(L_i, \mathcal{O}_C) = 0$ and $\mathrm{Hom}(L_1, L_2) = \mathrm{Hom}(L_2, L_1) = k$ and therefore

$$\mathrm{Aut}(\mathcal{O}_C \oplus L_1 \oplus L_1) = \mathbb{G}_m(k) \times \mathrm{GL}_2(k).$$

Suppose $E = \mathcal{O}_C \oplus L_1 \oplus L_2$ with $L_1 \ncong L_2 \cong \mathcal{O}_C$. That gives $\mathrm{Hom}(\mathcal{O}_C, L_1) = \mathrm{Hom}(L_1, \mathcal{O}_C) = 0$ and $\mathrm{Hom}(\mathcal{O}_C, L_2) = \mathrm{Hom}(L_2, \mathcal{O}_C) = k$. Moreover as $L_1 \ncong L_2$ we have $\mathrm{Hom}(L_1, L_2) = \mathrm{Hom}(L_2, L_1) = 0$ and therefore

$$\#\mathrm{Aut}(E) = (q-1)^3 q^2.$$

In the last case $\mathcal{O}_C \ncong L_1 \ncong L_2 \ncong \mathcal{O}_C$ we have $\mathrm{Hom}(L_i, L_j) = 0$ for $i \neq j$ and $\mathrm{Hom}(L_i, L_i) = k$. Therefore in this case, the automorphism group is

$$\mathrm{Aut}(E) = \mathrm{Diag}(3, k).$$

□

Lemma 3.2.16. *Let E, E' be rank 3 vector bundles on C such $E = \mathcal{O}_C \oplus G$, $E' = \mathcal{O}_C \oplus G'$, where G, G' are indecomposable over C but not geometrically indecomposable. Then*

$$E \cong E' \iff \pi^* E \cong \pi^* E',$$

where $\pi : C^2 := C \times_k k^2 \to C$ is the projection.

Proof. The implication "$\Rightarrow$" is clear. To prove the implication "$\Leftarrow$" we must show that $H^1\left(\mathrm{Gal}(k^2/k), \underline{\mathrm{Aut}}_{C^2}(\pi^*E)\right) = 1$. By Proposition 3.2.9 $\pi^*G = M_1 \oplus M_2$, where $M_i \in \mathrm{Pic}^d(C^2)$ for some $d \in \mathbb{Z}$ with $M_1 \ncong M_2$ and $M_i \ncong \mathcal{O}_{C^2}$ for $i = 1, 2$. Assume that d is positive (the case $d < 0$ is symmetric and $d = 0$ will follow from this case). Then the $\mathcal{O}_{C^2}$-linear automorphisms of π^*E are given by invertible matrices

$$\begin{pmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix}$$

with $a, d, e \in k^2$ and $b, c \in (k^2)^d$. Therefore $\mathrm{Aut}(\pi^*E) = (\mathbb{G}_a^d \times \mathbb{G}_a^d) \rtimes (\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m)$ and we have the exact sequence of groups

$$1 \to \mathbb{G}_a^d \times \mathbb{G}_a^d \to \mathrm{Aut}(\pi^*E) \to \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m \to 1. \tag{3.25}$$

which induces the long exact cohomology sequence

$$\begin{aligned} 1 &\to H^0(\mathrm{Gal}(k^2/k), \mathbb{G}_a^d \times \mathbb{G}_a^d) \to H^0(\mathrm{Gal}(k^2/k), \mathrm{Aut}(\pi^*E)) \to H^0(\mathrm{Gal}(k^2/k), \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m) \\ &\to H^1(\mathrm{Gal}(k^2/k), \mathbb{G}_a^d \times \mathbb{G}_a^d) \to H^1(\mathrm{Gal}(k^2/k), \mathrm{Aut}(\pi^*E)) \to H^1(\mathrm{Gal}(k^2/k), \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m). \end{aligned}$$

By Hilbert's 90 we have $H^1(\mathrm{Gal}(k^2/k), \mathbb{G}_a^d \times \mathbb{G}_a^d) = H^1(\mathrm{Gal}(k^2/k), \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m) = 1$. Consequently from the above long exact sequence follows

$$H^1(\mathrm{Gal}(k^2/k), \mathrm{Aut}(\pi^*E)) = 1$$

which proves the statement. For $d < 0$ the automorphism group does not change and for $d = 0$ we have $\mathrm{Aut}(\pi^*E) = \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m$. This finishes the proof. □

Corollary 3.2.17. *Let G, G' be rank 2 vector bundles on C that are indecomposable over C but not geometrically indecomposable. Then*

$$G \cong G' \iff \pi^*G \cong \pi^*G',$$

where $\pi : C^2 := C \times_k k^2 \to C$ is the projection.

Proof. Following the proof of Proposition 3.2.16 we observe, that $\mathrm{Aut}(\pi^*G) = \mathbb{G}_m \times \mathbb{G}_m$, which by Hilbert 90 leads to $H^1(\mathrm{Gal}(k^2/k), \mathrm{Aut}(\pi^*G)) = 1$. □

Now we will treat the case of vector bundles that are indecomposable over C but not geometrically.

Lemma 3.2.18. *Let $d \geq 0$ and $\omega \in \mathrm{Pic}^d(C)$. Let E be a semi-stable vector bundle on C of slope $\mu(E) = \frac{3}{3} = 1$ that is indecomposable on C but not geometrically indecomposable. Then the number of isomorphism classes of vector bundles of this type is at most*

$$\#C(k^3) - \#C(k) - 1$$

and $\mathrm{Aut}(E) = \mathrm{Res}_{k^3/k}\mathbb{G}_m(k)$.

Proof. Let $\pi : C^3 = C \times_k k^3 \to C$ be the projection. By Proposition 3.2.9 we have $\pi^*E = M_1 \oplus M_2 \oplus M_3$, with $M_i \in \mathrm{Pic}^0(C)$ and $M_i \ncong M_j$ for $i \neq j$ and such that $M_i's$ do not descend to C. It is clear, that the isomorphism class of E is determined by the $M_i's$ up to isomorphism. If we fix M_1, then M_2 and M_3 are determined by M_1, namely $M_2 = \sigma(M_1)$ and $M_3 = \sigma^2(M_1)$, where $\sigma \in G(k^3/k)$ is a generator. Now as $\mathrm{Pic}^0(C^2) \cong C^2$ the number of isomorphism classes of vector bundles E as above is at most $\#C(k^3) - \#C(k) - 1$. To describe automorphisms of E we use the Remark

3.2.14. First of all we have $\mathrm{Hom}(M_i, M_j) = k^3$ if $i = j$ and $\mathrm{Hom}(M_i, M_j) = 0$ if $i \neq j$. Hence automorphisms of $\pi^* E$ that are $\mathcal{O}_{C^3}$-linear are given by 3×3 invertible matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

with $a, b, c \in k^3$. The Galois group $G(k^3/k)$ acts by a 3-cycle permutation. Its corresponding matrix representation is given by

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

however

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and therefore it is enough to consider only one of them. Automorphisms of E are those invertible matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

with coefficients

$$a, b, c \in k^3,$$

that satisfy

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \bar{a} & 0 & 0 \\ 0 & \bar{b} & 0 \\ 0 & 0 & \bar{c} \end{pmatrix},$$

where $\bar{a}$ denotes the conjugate of a. Therefore

$$\begin{pmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} \bar{a} & 0 & 0 \\ 0 & \bar{b} & 0 \\ 0 & 0 & \bar{c} \end{pmatrix}$$

and hence $b = c = \bar{a}$, which implies that $\mathrm{Aut}(E) = \mathrm{Res}_{k^3/k} \mathbb{G}_m(k)$. □

Proposition 3.2.19. *Let $d \geq 0$ and $\omega \in \mathrm{Pic}^d(C)$ and let E be a rank 3 semi-stable vector bundle of degree 0, then the dimension of the vector space*

$$V_E = H^0(C, \mathrm{Sym}^3(E) \otimes \det(E)^\vee \otimes \omega)$$

is bounded by

$$\begin{cases} \dim_k V_E \leq 10, & \text{if } d = 0, \\ \dim_k V_E = 10d, & \text{if } d \geq 1. \end{cases} \tag{3.26}$$

Proof. Let $d \geq 0$ and let $\omega \in \mathrm{Pic}^d(C)$ we will bound the dimension of the vector space $V_E := H^0(C, \mathrm{Sym}^3(E) \otimes M_E)$ for a semi-stable vector bundle E of degree 0. There are two cases to consider. Assume $d = 0$, then by Theorem 1.2.4 the vector bundle $\mathrm{Sym}^3(E) \otimes M_E$ is semi-stable. We have

$$\mu(\mathrm{Sym}^3(E)) = 3\mu(E) = \frac{3 \deg(E)}{3} = \deg(E),$$

and therefore

$$\mu\big(\mathrm{Sym}^3(E)\otimes M_E\big) = \mu\big(\mathrm{Sym}^3(E)\big) + \mu(M_E) = \deg(E) - \deg(E) + d = 0.$$

Hence by Proposition 1.3.2

$$h^0(C, \mathrm{Sym}^3(E)\otimes M_E) \le 10.$$

In the case $d \ge 1$ the slope $\mu(\mathrm{Sym}^3(E)\otimes M_E) = 10d \ge 10$ and as $\mathrm{Sym}^3(E)\otimes M_E$ is semi-stable with positive degree hence

$$h^0(C, \mathrm{Sym}^3(E)\otimes M_E) = 10d.$$

□

Remark 3.2.20. All in all for the cases (R3a)-(R3d) we have

$$\begin{cases} \#\mathbb{P}(V_E)(k) \le \frac{q^{10}-1}{q-1} & \text{if } d = 0 \\ \#\mathbb{P}(V_E)(k) = \frac{q^{10d}-1}{q-1} & \text{if } d \ge 1 \end{cases} \tag{3.27}$$

3.2.2 Unstable case

Let $d \ge 0$ let $\omega \in \mathrm{Pic}^d(C)$. Let E be an unstable vector bundle of rank 3 on C with the Harder-Narasimhan filtration of the type

$$0 \subset E_{\max} \subset E,$$

where $E_{\max}$ is the maximal destabilizing subbundle of E (we also used the notation E_1 in Section 3.2) and let Q denote the quotient ${}^{E}/_{E_{\max}}$. By Proposition 3.2.1 we have $E \cong E_{\max} \oplus Q$. For convenience we assume here that $\mathrm{rk}(E_{\max}) = 2$ however the below construction work as well for the case $\mathrm{rk}(E_{\max}) = 1$. We would like to point out here that over an elliptic curve the tensor product of semi-stable vector bundles is semi-stable (see: Theorem 1.2.3) as well as are the symmetric powers of a semi-stable vector bundle (see: Theorem 1.2.4). Consider the smooth projective threefold

$$\pi : P := \mathrm{Proj}_C(\mathrm{Sym}^*(E)) \longrightarrow C.$$

and let $\mathcal{O}_P(1)$ be the invertible sheaf on P such that $\pi_*\mathcal{O}_P(1) = E$. We have surjective maps $E \longrightarrow E_{\max}$ and $E \longrightarrow Q$ which give us the following commutative diagram

$$\begin{tikzcd} P_{E_{\max}} \arrow[r, hook] \arrow[dr, "\pi_{E_{\max}}"'] & P \arrow[d, "\pi"] & P_Q \arrow[l, hook'] \arrow[dl, "\pi_Q"] \\ & C & \end{tikzcd}$$

The scheme $P_{E_{\max}} := \mathrm{Proj}_C(\mathrm{Sym}^*(E_{\max}))$ is a ruled surface and the scheme $P_Q := \mathrm{Proj}_C(\mathrm{Sym}^*(Q))$ is isomorphic to the base curve C. Denote by M_E the line bundle $\det(E)^\vee \otimes \omega$. We are interested in bounding the dimension of

$$H^0(P, \mathcal{O}_P(3)\otimes \pi^*(M_E)).$$

Every such a global section defines a family of degree three curves $Y \to C$ inside P. We are not interested in all sections but in those whose zero sets lie in $\mathcal{A}_{C,\omega}$. To bound this dimension we proceed as follows. First note, that we have the projection formula

$$H^0\left(P, \mathcal{O}_P(3) \otimes \pi^*(M_E)\right) = H^0\left(C, \mathrm{Sym}^3(E) \otimes M_E\right).$$

The vector bundle $\mathrm{Sym}^3(E) \otimes M_E$ is a rank 10 vector bundle of slope

$$\begin{aligned} \mu(\mathrm{Sym}^3(E) \otimes M_E) &= \mu(\mathrm{Sym}^3(E)) + \mu(M_E) \\ &= 3\mu(E) + \mu(M_E) \\ &= d \end{aligned} \tag{3.28}$$

$$\mathrm{Sym}^3(E) \otimes M_E =$$
$$\left[\mathrm{Sym}^3(E_{\max}) \oplus \left(\mathrm{Sym}^2(E_{\max}) \otimes Q\right) \oplus \left(E_{\max} \otimes \mathrm{Sym}^2(Q)\right) \oplus \mathrm{Sym}^3(Q)\right] \otimes M_E$$

where by Theorem 1.2.4 all the summands are semi-stable. The decomposition above gives us the decomposition of the cohomology groups

$$H^i\left(C, \mathrm{Sym}^3(E) \otimes M_E\right) = \tag{3.29}$$

$$\begin{aligned} &H^i\left(C, \mathrm{Sym}^3(E_{\max}) \otimes M_E\right) \oplus H^i\left(C, \mathrm{Sym}^2(E_{\max}) \otimes Q \otimes M_E\right) \\ &\oplus H^i\left(C, E_{\max} \otimes \mathrm{Sym}^2(Q) \otimes M_E\right) \oplus H^i\left(C, \mathrm{Sym}^3(Q) \otimes M_E\right). \end{aligned}$$

If we bound the h^1- dimensions of the summands, we will get a bound of the dimension of

$$H^1\left(C, \mathrm{Sym}^3(E) \otimes M_E\right) = H^1\left(P, \mathcal{O}_P(3) \otimes \pi^*(M_E)\right)$$

and consequently using Riemann-Roch a bound of the dimension of

$$H^0\left(C, \mathrm{Sym}^3(E) \otimes M_E\right) = H^1\left(P, \mathcal{O}_P(3) \otimes \pi^*(M_E)\right).$$

Observe, that

$$\begin{aligned} \mu\left(\mathrm{Sym}^2(E_{\max}) \otimes Q \otimes M_E\right) &= \mu\left(\mathrm{Sym}^2(E_{\max})\right) + \mu(Q) + \mu(M_E) \\ &= 2\mu(E_{\max}) + \deg(Q) - \deg(E) + d \\ &= d \geq 0. \end{aligned} \tag{3.30}$$

Proposition 3.2.21. *Let $d \geq 0$ and let $\omega \in \mathrm{Pic}^d(C)$. Assume furthermore that E is an unstable vector bundle on C with the maximal destabilizing subbundle $E_{\max}$ of rank 2. Let Q denote the quotient $Q := {}^{E}/_{E_{\max}}$. Let $\pi : P = \mathrm{Proj}_C(\mathrm{Sym}^* E) \to C$ and*

$$s \in H^0(P, \mathcal{O}_P(3) \otimes \pi^*(M_E)),$$

where $M_E = \det(E)^\vee \otimes \omega$. Moreover let $Y = V(s)$ be a closed subscheme of P and let $i : Y \hookrightarrow P$ be the inclusion. Assume that $(g : Y \to C, D)$ with $\mathcal{O}_Y(D) \cong \mathcal{O}_P(1)|_Y$ and $g = \pi \circ i$ is a representative of a class in $\mathcal{A}_{C,\omega}$. Then

(a) $\mu(E_{\max}) - \mu(Q) > 0$,

(b) $\mu\left(\mathrm{Sym}^3(E_{\max}) \otimes M_E\right) \geq 0$,

(c) If $\mu(\mathrm{Sym}^3(Q) \otimes M_E) < 0$, then C is a section of the surface Y

(d) *If* $\mu(\mathrm{Sym}^3(Q)\otimes M_E)<0$, *then* $\mu(\mathrm{Sym}^2(Q)\otimes E_{\max}\otimes M_E)\geq 0$;

(e) $\mu(\mathrm{Sym}^2(Q)\otimes E_{\max}\otimes M_E)\geq 0$.

Proof. (a) This is a property of the Harder-Narasimhan filtration.

(b) Assume $\mu\left(\mathrm{Sym}^3(E_{\max})\otimes M_E\right)<0$. We have the restriction map

$$[\mathcal{O}_P(3)\otimes\pi^*(M_E)]\,|_{P_{E_{\max}}}=\mathcal{O}_{P_{E_{\max}}}(3)\otimes\pi^*_{E_{\max}}(M_E)$$

and by assumption

$$H^0\left(P_{E_{\max}},\mathcal{O}_{P_{E_{\max}}}(3)\otimes\pi^*_{E_{\max}}(M_E)\right)=H^0\left(C,\mathrm{Sym}^3(E_{\max})\otimes M_E\right)=0.$$

Take

$$s\in H^0\left(P,\mathcal{O}_P(3)\otimes\pi^*(M_E)\right).$$

Let Y be the zero scheme of s. By the above, the restriction of s to $P_{E_{\max}}$ is zero, hence Y contains the ruled surface $P_{E_{\max}}$. This means that the generic fiber of $Y\to C$ contains a $\mathbb{P}^1$ which contradicts the assumption that the generic fiber is a smooth connected projective curve with trivial canonical sheaf.

(c) The surjection $E\to Q$ defines the inclusion of C in P and the ideal sheaf of C inside P is the ideal $\mathcal{I}_{C/P}=\widetilde{I}$, where I is generated by the kernel $E_{\max}\subseteq\mathrm{Sym}^1(E)$. Let

$$s\in H^0\left(C,\mathrm{Sym}^3(E)\otimes M_E\right).$$

Now as we assume $\mu(\mathrm{Sym}^3(Q)\otimes M_E)<0$ semistability of $\mathrm{Sym}^3(Q)\otimes M_E$ give us vanishing of

$$H^0(\mathrm{Sym}^3(Q)\otimes M_E)=0$$

and hence

$$H^0\left(C,\mathrm{Sym}^3(E)\otimes M_E\right)=H^0\left(C,\mathrm{Sym}^3(E_{\max})\otimes M_E\right)\oplus$$
$$H^0\left(C,\mathrm{Sym}^2(E_{\max})\otimes Q\otimes M_E\right)\oplus H^0\left(C,E_{\max}\otimes\mathrm{Sym}^2(Q)\otimes M_E\right).$$

The section s generates a homogeneous ideal J in $\mathrm{Sym}^*(E)\otimes M_E$. Let Y be the closed subscheme defined by the sheaf of ideals $\mathcal{J}_{Y/P}=\widetilde{J}$. By the above the surjective map

$$\mathrm{Sym}^3(E)\otimes M_E\to\mathrm{Sym}^3(Q)\otimes M_E$$

sends s to zero, hence we have an inclusion of ideals $J\subseteq I$. Since the statement is local, we conclude that C is a section of the surface Y.

(d) By assumption $Y\to C$ is generically smooth and therefore $C\hookrightarrow Y$ is generically a regular embedding. Let $\mathcal{J}_{Y/P}=\widetilde{J}$ be the ideal sheaf of Y in P, $J\subset\mathrm{Sym}^*(E)\otimes M_E$ it is a homogeneous ideal generated by the section s. Let $\mathcal{I}_{C/P}=\widetilde{I}$ be the ideal sheaf of C inside P i.e. I is generated by the kernel $E_{\max}\subseteq\mathrm{Sym}^1(E)$. Assume $\mu(\mathrm{Sym}^3(Q)\otimes M_E)<0$ and $\mu(\mathrm{Sym}^2(Q)\otimes E_{\max}\otimes M_E)<0$, then we have the decomposition

$$H^0\left(C,\mathrm{Sym}^3(E)\otimes M_E\right)=H^0\left(C,\mathrm{Sym}^3(E_{\max})\otimes M_E\right)\oplus H^0\left(C,\mathrm{Sym}^2(E_{\max})\otimes Q\otimes M_E\right).$$

Furthermore the above decomposition implies that the inclusion $J\subseteq I^2$ holds. We claim that then Y is singular along the section C. According to [EGA IV, Exp. IV, Proposition 16.2.7] locally on C the conormal bundle $\mathcal{N}_{C/Y}$ of C inside Y is given by

$$I\big/\left(I^2+J\right).$$

If Y were smooth along C [SGA I, Corollaire 4.11] then at every point $x \in C$ the dimension of $\mathcal{N}_{C/Y_x}$ would be equal the codimension of C in Y which is 1. Using $J \subseteq I^2$ and the fact that C is smooth in P we get

$$\dim \mathcal{N}_{C/Y_x} = \dim \mathcal{N}_{C/P_x} = 2.$$

This implies that Y is singular along C which is a contradiction

(e) Observe that we have

$$\begin{aligned}
\mu(\mathrm{Sym}^3(Q) \otimes M_E) &= \mu(\mathrm{Sym}^3(Q)) + \mu(M_E) \\
&= 3\mu(Q) - \deg(E) + d \\
&= 3\deg(Q) - \deg(Q) - \deg(E_{\max}) + d \\
&= 2\deg(Q) - \deg(E_{\max}) + d.
\end{aligned}$$

and

$$\begin{aligned}
\mu(\mathrm{Sym}^2(Q) \otimes E_{\max} \otimes M_E) &= \mu(\mathrm{Sym}^2(Q)) + \mu(E_{\max}) + \mu(M_E) \\
&= 2\mu(Q) + \mu(E_{\max}) + \mu(M_E) \\
&= 2\deg(Q) + \mu(E_{\max}) - \deg(E) + d \\
&= \frac{1}{2}\left(4\deg(Q) + \deg(E_{\max}) - 2\deg(E) + 2d\right) \\
&= \frac{1}{2}\left(2\deg(Q) - \deg(E_{\max}) + 2d\right).
\end{aligned}$$

Therefore if $\mu(\mathrm{Sym}^3(Q) \otimes M_E) \geq 0$ so is $\mu(\mathrm{Sym}^2(Q) \otimes E_{\max} \otimes M_E) \geq 0$ which together with (d) gives that the slope

$$\mu(\mathrm{Sym}^2(Q) \otimes E_{\max} \otimes M_E) \geq 0.$$

is always non-negative.

$\square$

By the above we can conclude, that if E is a vector bundle on C such that E is not semi-stable and the maximal destabilizing subbundle $E_{\max}$ of E is of rank 2 then E gives a contribution to the sum from Theorem 3.1.23 if the slope inequalities from Proposition 3.2.21 are satisfied. We will analyze those inequalities in the next subsection.

We also have a similar statement, when $\mathrm{rk}(E_{\max}) = 1$.

Proposition 3.2.22. *Let $d \geq 0$ and $\omega \in \mathrm{Pic}^d(C)$. Assume, E is an unstable vector bundle on C with the maximal destabilizing subbundle $E_{\max}$ of rank 1. Let Q denote the quotient $Q := E\big/E_{\max}$. Let $\pi : P = \mathrm{Proj}_C(\mathrm{Sym}^* E) \to C$ and*

$$s \in H^0(P, \mathcal{O}_P(3) \otimes \pi^*(M_E)).$$

Moreover let $Y = V(s)$ be a closed subscheme of P and let $i : Y \hookrightarrow P$ be the inclusion. Assume that $(g : Y \to C, D)$ with $\mathcal{O}_Y(D) \cong \mathcal{O}_P(1)|_Y$ and $g = \pi \circ i$ is a representative of a class in $\mathcal{A}_{C,\omega}$. Then

(a) $\mu(E_{\max}) - \mu(Q) > 0$

(b) $\mu\left(\mathrm{Sym}^3(Q) \otimes M_E\right) \geq 0$;

(c) *If* $\mu(\mathrm{Sym}^3(E_{\max}) \otimes M_E) < 0$*, then* C *is a section of the surface* Y

(d) *If* $\mu(\mathrm{Sym}^3(E_{\max}) \otimes M_E) < 0$ *then* $\mu(\mathrm{Sym}^2(E_{\max}) \otimes Q \otimes M_E) \geq 0$.

(e) $\mu(\mathrm{Sym}^2(E_{\max}) \otimes Q \otimes M_E) \geq 0$.

Proof. Follow the proof of Proposition 3.2.21 with $E_{\max}$ replaced with Q and Q replaced with $E_{\max}$. □

Remark 3.2.23. Notice, that we can as well apply the whole construction to a twist of E then we get a bound for slopes of twisted summands.

Unstable case with the maximal destabilizing subbundle of rank 2

Let E be an unstable vector bundle on C with $\mathrm{rk}(E_{\max}) = 2$ and the quotient $Q := E/E_{\max} \in \mathrm{Pic}(C)$. Then by Proposition 3.2.1 we have $E \cong E_{\max} \oplus Q$ and up to twist with a line bundle

$$E = \mathcal{O}_C \oplus F,$$

where F is a semi-stable rank 2 vector bundle of degree $\deg(F) > \deg(\mathcal{O}_C) = 0$.

Lemma 3.2.24. *Consider a vector bundle* $E = \mathcal{O}_C \oplus F$*, such that* F *is semi-stable with* $\mathrm{rk}(F) = 2$ *and* $\mu(F) > \mu(\mathcal{O}_C) = 0$*. Then we have the following cases.*

(L2a) $E = \mathcal{O}_C \oplus F$*, with* $\gcd(2, \deg(F)) = 1$ *and hence* F *is geometrically stable,*

(L2b) *or* $E = \mathcal{O}_C \oplus F$*, with* $\gcd(2, \deg(F)) = 2$ *and* F *is geometrically indecomposable*

(L2c) *or* $E = \mathcal{O}_C \oplus F$*, with* $\gcd(2, \deg(F)) = 2$ *where* F *is indecomposable over* C *but splits over* $C^2 := C \times_k k^2$.

(L2d) $E = \mathcal{O}_C \oplus F$*, with* $\gcd(2, \deg(F)) = 2$ *where* $F = L_1 \oplus L_2$ *with* $L_1, L_2 \in \mathrm{Pic}^l(C)$ *and* $l > 0$.

Furthermore let $d \geq 0$ *and* $\omega \in \mathrm{Pic}^d(C)$ *and let* $\pi : P = \mathrm{Proj}_C(\mathrm{Sym}^* E) \to C$ *and*

$$s \in H^0(P, \mathcal{O}_P(3) \otimes \pi^*(M_E)),$$

where $M_E = \det(E)^\vee \otimes \omega$*. Moreover let* $Y = V(s)$ *be a closed subscheme of* P *and* $i : Y \hookrightarrow P$ *be the inclusion. Assume that* $(g : Y \to C, D)$ *with* $\mathcal{O}_Y(D) \cong \mathcal{O}_P(1)|_Y$ *and* $g = \pi \circ i$ *is a representative of a class in* $\mathcal{A}_{C,\omega}$*. Then*

(a) *The degree of* E *is bounded by* $0 < \deg E \leq 2d$.

(b) *Let* $V_E = H^0(P, \mathcal{O}_P(3) \otimes \pi^*(M_E))$ *then*

$$\dim_k V_E = \begin{cases} 11d + 2 & \text{if } \deg E = 2d \\ 10d & \text{if } 0 < \deg E < d \\ \leq 10d + 1 & \text{if } \deg E = d \\ 9d + \deg E & \text{if } d < \deg E < 2d \end{cases}$$

(c) *The number of automorphisms of E in the respective cases is as follows*

$$\#\mathrm{Aut}(E) = \begin{cases} (q-1)^2 q^{\deg E}, & \textit{in the case (L2a);} \\ \geq (q-1)^2 q^{\deg E}, & \textit{in the case (L2b);} \\ (q-1)^2(q+1)q^{\deg E}, & \textit{in the case (L2c);} \\ (q-1)^3(q+1)q^{\deg E+1}, & \textit{in the case (L2d) with } L_1 \cong L_2; \\ (q-1)^3 q^{\deg E}, & \textit{in the case (L2d) with } L_1 \ncong L_2. \end{cases}$$

(d) *For a fixed degree of E,* $\deg(E) \in (0, 2d]$ *the number of isomorphism classes of vector bundles* $E = \mathcal{O}_C \oplus F$ *in the respective cases is*

$$\begin{cases} \#C(k), & \textit{in the case (L2a);} \\ \leq \#C(k), & \textit{in the case (L2b);} \\ \leq \#C(k^2) - \#C(k) - 1, & \textit{in the case (L2c);} \\ \leq \#C(k)^2, & \textit{in the case (L2d);} \end{cases}$$

Proof. We know, that F is semi-stable, therefore if $\gcd(\deg(F), 2) = 1$ then F is geometrically stable. If $\gcd(\deg(F), 2) = 2$ then there are three possible cases F might be geometrically indecomposable, or might be indecomposable over C but not geometrically indecomposable or E is a sum of two line bundles defined over C.

For (a) it is clear, that $\deg(E) > 0$. Furthermore observe that by Proposition 3.2.21(d) we have

$$\mu(\mathrm{Sym}^2(\mathcal{O}_C) \otimes F \otimes M_E) = \frac{1}{2}\Big(2\deg \mathcal{O}_C - \deg(F) + 2d\Big) = \frac{1}{2}\Big(-\deg(F) + 2d\Big) \geq 0$$

hence $\deg F = \deg E$ which leads to

$$0 < \deg E \leq 2d.$$

To prove (b) recall that we have the splitting (3.29). The slope $\mu(\mathrm{Sym}^3(\mathcal{O}_C) \otimes M_E) = d - \deg(E)$ can be negative. The slopes of the other summands

$$\begin{cases} \mu\left(\mathrm{Sym}^3(F) \otimes M_E\right) = 3\mu(F) + \mu(M_E) = \frac{\deg E + 2d}{2} > 0 \\ \mu(\mathrm{Sym}^2(\mathcal{O}_C) \otimes F \otimes M_E) = \mu(F) + \mu(M_E) = \frac{-\deg E + 2d}{2} \geq 0 \\ \mu\left(\mathrm{Sym}^2(F) \otimes \mathcal{O}_C \otimes M_E\right) = \mu(\mathrm{Sym}^2(F)) + \mu(M_E) = d > 0 \end{cases}$$

Therefore by Riemann-Roch, Proposition 1.1.10 and Proposition 1.3.2 the dimension of $V_E := H^0(\mathrm{Sym}^3(E) \otimes M_E)$ is a follows

$$\dim_k V_E = \begin{cases} 11d + 2 & \text{if } \deg E = 2d \\ 10d & \text{if } 0 < \deg E < d \\ \leq 10d + 1 & \text{if } \deg E = d \\ 10d + \deg E - d & \text{if } d < \deg E < 2d \end{cases}$$

(c)To count automorphisms of E we use its direct decomposition. The fact, that $\mu(F) > 0$ and semi-stability of F imply $\mathrm{Hom}(F, \mathcal{O}_C) = 0$. Furthermore $\mathrm{Hom}(\mathcal{O}_C, F) = H^0(C, F)$ and as F is semi-stable with positive slope $\dim_k H^0(C, F) = \deg(F)$. Hence $\mathrm{End}(E) = k \oplus \mathrm{Hom}(\mathcal{O}_C, F) \oplus \mathrm{End}(F)$ and therefore

$$\#\mathrm{Aut}(E) = (q-1)q^{\deg(F)}\#\mathrm{Aut}(F).$$

Furthermore if F is geometrically stable, then $\mathrm{Aut}(F) = \mathbb{G}_m(k)$. If F is geometrically indecomposable, then $\mathbb{G}_m(k) \subset \mathrm{Aut}(F)$ and hence $\#\mathrm{Aut}(E) \geq q - 1$. Assume that F is indecomposable over C but not geometrically indecomposable, then by Proposition 3.2.10 $\pi^* F = M_1 \oplus M_2$ where $\pi : C^2 := C \times_k k^2 \to C$ is the projection and $M_i \in \mathrm{Pic}^d(C^2)$ are such that $M_1 \ncong M_2$, $M_i \ncong \mathcal{O}_C$ and $d > 0$. Furthermore as $M_1 \oplus M_2$ is semi-stable, so is by Proposition 1.1.4 the vector bundle F. Automorphisms of $\pi^* F$ are therefore invertible 2×2 matrices

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

with $a, b \in k^2$. The Galois group $G(k^2/k)$ acts on F by permuting M_1 and M_2. It naturally acts on $\mathrm{End}(\pi^* F)$ by conjugation and $\mathrm{Aut}(G) = \mathrm{Aut}(\pi^* F)^{G(k^2/k)}$. Therefore automorphisms of F must satisfy

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \bar{a} & 0 \\ 0 & \bar{b} \end{pmatrix},$$

where $\bar{a}$ denotes the conjugation of the element a of k^2. Consequently

$$\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} \bar{a} & 0 \\ 0 & \bar{b} \end{pmatrix}.$$

Therefore $b = \bar{a}$ and hence $\mathrm{Aut}(F) = \mathrm{Res}_{k^2/k}\mathbb{G}_m(k)$, where $\mathrm{Res}_{k^2/k}\mathbb{G}_m$ is the Weil restriction of $\mathbb{G}_m$. For the last case suppose, that $F = L_1 \oplus L_2$ then semi-stability implies, that $\deg(L_1) = \deg(L_2)$. Therefore we have two cases $L_1 \cong L_2$ or $L_1 \ncong L_2$. Then automorphism group is as follows

$$\mathrm{Aut}(F) = \begin{cases} \mathrm{GL}(2,k), & \text{if } L_1 \cong L_2; \\ \mathbb{G}_m(k) \times \mathbb{G}_m(k), & \text{if } L_1 \ncong L_2. \end{cases}$$

To prove the last statement fix the degree of E, $\deg(E) \in (0, 2d]$. The vector bundle F is semi-stable, therefore if $\gcd(2, \deg(F)) = 1$ then F is geometrically stable. The number geometrically stable vector bundles of fixed degree is $\#C(k)$. By the uniqueness of the Harder-Narasimhan filtration the vector bundle E is up to isomorphism uniquely determined by F. If $\gcd(2, \deg(F)) = 2$ then we have three cases: F is geometrically indecomposable or F splits into a sum of line bundles after degree two extension of k or F is a direct sum of two line bundles defined over k. If F is geometrically indecomposable, then the number of isomorphism classes of geometrically indecomposable vector bundles of rank 2 and degree $\deg(F)$ is by Theorem 3.2.4 equal to $\#C(k)$. Here again by the uniqueness of the Harder-Narasimhan filtration E is up to isomorphism uniquely determined by F. Assume now, that F is indecomposable over C but not geometrically indecomposable, then by Proposition 3.2.10 $\pi^* F = M_1 \oplus M_2$ where $\pi : C^2 := C \times_k k^2 \to C$ is the projection and $M_i \in \mathrm{Pic}^d(C^2)$ are such that $M_1 \ncong M_2$, $M_i \ncong \mathcal{O}_C$ and $d > 0$. Furthermore by Lemma 3.2.16 the vector bundle E is up to isomorphism uniquely determined by the pair (M_1, M_2). The number of possible pairs (M_1, M_2) is at most $\#C(k^2) - \#C(k) - 1$ (see: proof of Lemma 3.2.15(R3c)). For the last case assume, that F is a direct sum of two line bundles $L_1 \oplus L_2$, $L_1, L_2 \in \mathrm{Pic}(C)$. By the uniqueness of the Harder-Narasimhan filtration the vector bundle $E = \mathcal{O}_C \oplus F$ is up to isomorphism uniquely determined by the pair (L_1, L_2) and the number of possible choices of (L_1, L_2) is at most $\#C(k)^2$. □

Unstable case with the maximal destabilizing subbundle of rank 1

Assume E is an unstable vector bundle on C with the maximal destabilizing subbundle $E_{\max}$ of rank $\mathrm{rk}(E_{\max}) = 1$ and let $Q := E/E_{\max}$ be the quotient. As $\mathrm{rk}(Q) =$

2 depending on whether the degree of Q is even or odd, we can twist Q with a line bundle such that we make its degree 0 or 1.

deg(Q)=1

Assume we can normalize E such that $E = L \oplus G$, where L is a line bundle, G is a rank 2 vector bundle and $\mu(L) > \mu(G) = \frac{1}{2}$.

Lemma 3.2.25. *Let $d \geq 0$ and $\omega \in \mathrm{Pic}^d(C)$. Consider a vector bundle $E = L \oplus G$, where L is a line bundle, G is a rank 2 vector bundle and $\mu(L) > \mu(G) = \frac{1}{2}$. Let $\pi : P = \mathrm{Proj}_C(\mathrm{Sym}^* E) \to C$ and*

$$s \in H^0(P, \mathcal{O}_P(3) \otimes \pi^*(M_E)).$$

Moreover let $Y = V(s)$ be a closed subscheme of P and $i : Y \hookrightarrow P$ be the inclusion. Assume that $(g : Y \to C, D)$ with $\mathcal{O}_Y(D) \cong \mathcal{O}_P(1)|_Y$ and $g = \pi \circ i$ is a representative of a class in $\mathcal{A}_{C,\omega}$. Then

(a) The degree of E is bounded by

$$1 < \deg E \leq d + 1$$

(b) For $V_E = H^0(P, \mathcal{O}_P(3) \otimes \pi^*(M_E))$ we have $\dim_k V_E = 10d$.

(c) Moreover

$$\#\mathrm{Aut}(E) = (q-1)^2 q^{2\deg L - 1}.$$

(d) Fix the degree of E, $\deg(E) \in (1, d+1]$ then the number of isomorphism classes of vector bundles E as above is at most $\#C(k)^2$.

Proof. For (a) by Proposition 3.2.22(b) we have

$$\mu\big(\mathrm{Sym}^3(G) \otimes M_E\big) = \frac{1}{2}\big(3\deg G - 2\deg E + 2d\big) \geq 0.$$

Therefore $2\deg L \leq 2d + \deg G = 2d + 1$ and hence

$$1 < \deg(L) \leq d + 1 \Longleftrightarrow 2 < \deg E \leq d + 2. \tag{3.31}$$

To show (b) we use (3.31) it gives

$$\begin{cases} \mu\left(\mathrm{Sym}^3(L) \otimes M_E\right) = 2\deg L + d > 0 \\ \mu(\mathrm{Sym}^2(L) \otimes G \otimes M_E) = 2\deg L - \frac{1}{2} + 2d > 0 \\ \mu\left(\mathrm{Sym}^2(G) \otimes L \otimes M_E\right) = d > 0 \end{cases} \tag{3.32}$$

and therefore

$$\dim_k V_E = 10d.$$

To prove (c) observe, that the vector bundles L and G are geometrically stable with $\mu(L) > \mu(G) = 1$ hence $\mathrm{Hom}(G,G) = k$ and $\mathrm{Hom}(L,G) = 0$ and therefore $\mathrm{Hom}(E,E) = k \oplus k \oplus \mathrm{Hom}(G,L)$ which implies

$$\#\mathrm{Aut}(E) = (q-1)^2 q^{2\deg L - 1}.$$

To show (d) observe that from the uniqueness of the Harder-Narasimhan filtration follows, that up to isomorphism E is uniquely determined by L and G. As the degree of G is 1 hence $\gcd(2,1) = 1$ which means that G is geometrically stable, moreover any line bundle on C is geometrically stable. Therefore for a fixed degree of E (which is $\deg E = \deg L + 1$) the number of isomorphism classes of such vector bundles is at most $\#C(k)^2$. ☐

deg(Q) = 0

We normalize E such that $E = L \oplus G$, where L is a line bundle, G is a rank 2 vector bundle of slope $\mu(G) = 0$ and $\mu(L) > \mu(G) = 0$.

Lemma 3.2.26. *Let $d \geq 0$ and $\omega \in \mathrm{Pic}^d(C)$. Consider a vector bundle $E = L \oplus G$, such that L is a line bundle, G is a rank 2 vector bundle and $\mu(L) > \mu(G) = 0$. Then up to a twist with a degree 0 line bundle we have the following cases.*

(R2a) *$E = L \oplus G$ where $G \cong F_2$ i.e. G is geometrically indecomposable;*

(R2b) *or $E = L \oplus G$ with G is an indecomposable vector bundle of rank 2 over C but splits into a direct sum of line bundles over $C^2 = C \times_k k^2$.*

(R2c) *or $E = L \oplus G$ where G is a sum of line bundles over C and hence $G = \mathcal{O}_C \oplus M$ with $M \in \mathrm{Pic}^0(C)$.*

Furthermore, let $\pi : P = \mathrm{Proj}_C(\mathrm{Sym}^* E) \to C$ and

$$s \in H^0(P, \mathcal{O}_P(3) \otimes \pi^*(M_E)).$$

Moreover let $Y = V(s)$ be a closed subscheme of P and $i : Y \hookrightarrow P$ be the inclusion. Assume that $(g : Y \to C, D)$ with $\mathcal{O}_Y(D) \cong \mathcal{O}_P(1)|_Y$ and $g = \pi \circ i$ is a representative of a class in $\mathcal{A}_{C,\omega}$. Then

(a) *The degree of E is bounded by*

$$1 \leq \deg(E) \leq d$$

(b) *For the vector space $V_E = H^0(P, \mathcal{O}_P(3) \otimes \pi^*(M_E))$ we have $\dim_k V_G = 10d$.*

(c) *The number of automorphisms of E in the respective cases is as follows*

$$\#\mathrm{Aut}(E) = \begin{cases} (q-1)q^{\deg L+1} & \text{in the case (R2a)} \\ (q-1)(q^2-1)q^{\deg L} & \text{in the case (R2b)} \\ (q-1)q(q^2-1)q^{\deg L} & \text{in the case (R2c) with } M \cong \mathcal{O}_C \\ (q-1)^2 q^{\deg L} & \text{in the case (R2c) with } M \ncong \mathcal{O}_C \end{cases}$$

(d) *Fix the degree of E, $\deg(E) \in (1, d]$ then the number of isomorphism classes of E is as follows*

$$\#E = \begin{cases} \#C(k) & \text{in the case (R2a)} \\ \leq \#C(k)(\#C(k^2) - \#C(k) - 1) & \text{in the case (R2b)} \\ \leq \#C(k)^2 & \text{in the case (R2c)} \end{cases}$$

Proof. For (a) observe, that as rk$G = 2$ the possible structures on G are (R2a-R2c). By Proposition 3.2.22(b) we have

$$\begin{aligned} \mu\big(\mathrm{Sym}^3(G) \otimes M_E\big) &= \mu(\mathrm{Sym}^3(G)) + \mu(M_E) \\ &= 3\mu(G) + \deg(M_E) \\ &= 3\frac{1}{2} - \deg L - 1 + d \\ &= d + \frac{1}{2} - \deg L \geq 0. \end{aligned}$$

We therefore get $\deg L \le d + \frac{1}{2}$ which as $\deg L \in \mathbb{Z}$ leads to

$$1 \le \deg(L) \le d.$$

Then we also have

$$\begin{cases} \mu\left(\mathrm{Sym}^3(L) \otimes M_E\right) = \mu(\mathrm{Sym}^3(L)) + \mu(E) = 2\deg L + d - 1 > 0 \\ \mu(\mathrm{Sym}^2(L) \otimes G \otimes M_E) = \mu(\mathrm{Sym}^2(L) \otimes M_E) + \mu(G) = 2\deg L + 2d - \dfrac{3}{2} > 0 \\ \mu\left(\mathrm{Sym}^2(G) \otimes L \otimes M_E\right) = \mu(\mathrm{Sym}^2(G)) + \mu(L \otimes M_E) = 2\mu(G) - 1 + d = d > 0 \end{cases}$$

and hence

$$\dim_k V_E = 10d$$

To prove (c) we first describe automorphisms of G the respective cases are

(R2a) The case treated in Section 3.2.1, Case (R3c), (3.24). We have

$$\mathrm{Aut}(F_2) = \mathbb{G}_m(k) \times \mathbb{G}_a(k).$$

(R2b) Here we have $\mathrm{Aut}(G) = \mathrm{Res}_{k^2/k}\mathbb{G}_m$, where $\mathrm{Res}_{k^2/k}\mathbb{G}_m$ is the Weil restriction of $\mathbb{G}_m$. The proof the reader may find in the proof of Lemma 3.2.24.

(R2c) In the last case we have

$$\mathrm{Aut}(\mathcal{O}_C \oplus M) = \begin{cases} \mathrm{GL}_2(k) & \text{if } M \cong \mathcal{O}_C \\ \mathbb{G}_m(k) \times \mathbb{G}_m(k) & \text{if } M \ncong \mathcal{O}_C \end{cases}$$

To describe automorphisms of E observe that as L and G are semi-stable with $\mu(L) > \mu(G)$ hence there are no non-trivial morphisms $L \to G$. Therefore

$$\mathrm{End}(E) = k \oplus \mathrm{Hom}(G, L) \oplus \mathrm{Hom}(G, G).$$

Furthermore as the vector bundle $G^\vee \otimes L$ is semi-stable with slope $\mu(L) - \mu(G) > 0$ we have $\dim_k H^0(C, G^\vee \otimes L) = \deg(G^\vee \otimes L) = \deg L$. Then we have the following list

$$\#\mathrm{Aut}(E) = \begin{cases} (q-1)^2 q^{\deg L + 1} & \text{in the case (R2a)} \\ (q-1)(q^2-1)q^{\deg L} & \text{in the case (R2b)} \\ (q-1)^3(q+1)q^{\deg L+1} & \text{in the case (R2c) with } M \cong \mathcal{O}_C \\ (q-1)^3 q^{\deg L} & \text{in the case (R2c) with } M \ncong \mathcal{O}_C \end{cases}$$

For (d) fix the degree of E, $\deg E \in (1, d]$ then the respective numbers of isomorphism classes of E's are

R2a) The vector bundle F_2 is unique up to isomorphism. Furthermore by the uniqueness of the Harder-Narasimhan filtration E is up to isomorphism uniquely determined by L. As the number of isomorphism classes of line bundles of given degree is $\#C(k)$ so is the number of isomorphism classes of E's.

R2b) The number of isomorphism classes of vector bundles G of the type (R2b) is at most (see: proof of Lemma 3.2.15(R3c)) $(\#C(k^2) - \#C(k) - 1)$. By the uniqueness of the Harder-Narasimhan filtration up to isomorphism E is determined uniquely by L and G. Therefore for a fixed degree the number of isomorphism classes of $E = L \oplus G$ is at most

$$\#C(k)(\#C(k^2) - \#C(k) - 1).$$

(R2c) By the uniqueness of the Harder-Narasimhan filtration the isomorphism class of E depends only on L and M. Furthermore by the uniqueness of the JOrdan Hölder filtration of a semi-stable vector bundle the vector bundle $\mathcal{O}_C \oplus M$ is uniquely determined by M. Therefore the number of isomorphism classes of vector bundles $E = L \oplus M \oplus \mathcal{O}_C$ is at most $\#C(k)^2$.

□

3.2.3 Sum of three line bundles

Let $d \geq 0$, let $\omega \in \mathrm{Pic}^d(C)$. Assume, that neither E nor the first quotient Q in the Harder-Narasimhan filtration of E are semi-stable, then E is a sum of three line bundles and we can always normalize E such that

$$E = \mathcal{O}_C \oplus L_1 \oplus L_2$$

with

$$0 < \deg L_1 < \deg L_2.$$

We follow here the argument of J.A. de Jong, the author uses in Section 7 of [deJ02].

Lemma 3.2.27. *Consider a vector bundle*

$$E = \mathcal{O}_C \oplus L_1 \oplus L_2$$

with $\deg(L_i) = l_i$ *such that* $0 < l_1 < l_2$. *Let* $\pi : P = \mathrm{Proj}_C(\mathrm{Sym}^* E) \to C$ *and*

$$s \in H^0(P, \mathcal{O}_P(3) \otimes \pi^*(M_E)).$$

Moreover let $Y = V(s)$ *be a closed subscheme of* P *and* $i : Y \hookrightarrow P$ *be the inclusion. Assume that* $(g : Y \to C, D)$ *with* $\mathcal{O}_Y(D) \cong \mathcal{O}_P(1)|_Y$ *and* $g = \pi \circ i$ *is a representative of a class in* $\mathcal{A}_{C,\omega}$. *Then*

(a) The degrees of the summands of E are bounded by

$$\begin{cases} 0 < l_1 < l_2 \\ l_1 \leq d \\ l_2 \leq d + l_1 \end{cases}$$

(b) For $V_E = H^0(P, \mathcal{O}_P(3) \otimes \pi^*(M_E))$ *we have*

$$\dim_k V_E = \begin{cases} 10d & \text{if } l_1 + l_2 < d \\ \leq 10d + 1 & \text{if } l_1 + l_2 = d \\ 10d + (l_1 + l_2 - d) & \text{if } l_1 + l_2 > d \text{ and } l_2 < d \\ \leq 9d + (l_1 + l_2 + 1) & \text{if } l_1 + l_2 > d \text{ and } l_2 = d \\ 10d + (l_1 + l_2 - d) + (l_2 - d) & \text{if } l_1 + l_2 > d \text{ and } l_2 > d \end{cases}$$

(c) Moreover

$$\#\mathrm{Aut}(E) = (q-1)^3 q^{2l_2}.$$

(d) Fix the degree of E then for the fixed degree the number of isomorphism classes of vector bundles E as above is at most $\#C(k)^2$.

Proof. We have

$$\mathrm{Sym}^3(E)\otimes\big((L_1\otimes L_2)^\vee\otimes\omega\big)$$
$$=\big[\mathcal{O}_C\oplus L_1\oplus\big(L_1^{\otimes 2}\big)\oplus(L_1\otimes L_2)\oplus\big(L_2^{\otimes 2}\big)\oplus\big(L_1^{\otimes 3}\big)\oplus\big(L_1^{\otimes 2}\otimes L_2\big)\oplus L_2\oplus\big(L_1\otimes L_2^{\otimes 2}\big)\oplus\big(L_2^{\otimes 3}\big)\big]\otimes\big((L_1\otimes L_2)^\vee\otimes\omega\big).$$

then we can write

$$s=Ay^3+By^2x+Cy^2z+Dyx^2+Exyz+Fyz^2+Gx^3+Hx^2z+Ixz^2+Jz^3,$$

with

$$A\in H^0(C,\omega\otimes(L_1\otimes L_2)^\vee), B\in H^0(C,\omega\otimes L_2^\vee), C\in H^0(C,\omega\otimes L_1^\vee),$$
$$D\in H^0(C,\omega\otimes L_1\otimes L_2^\vee), E\in H^0(C,\omega), F\in H^0(C,\omega\otimes L_1^\vee\otimes L_2), G\in H^0(C,\omega\otimes L_1^{\otimes 2}\otimes L_2^\vee),$$
$$H\in H^0(C,\omega\otimes L_1), I\in H^0(C,\omega\otimes L_2), J\in H^0(C,\omega\otimes L_1^\vee\otimes L_2^{\otimes 2}).$$

Observe that

(a) If $l_1+l_2>d$ then $A=0$ hence the point "$(0:1:0)$" defines a section σ of Y over C.

(b) If $l_1>d$ then A, B and C are all zero (recall that $l_2>l_1>0$). Thus Y is singular along σ which is a contradiction, since Y is generically smooth over C and hence σ is generically a regular embedding.

(c) If $d+l_1-l_2<0$ then $A=0$, $B=0$ and $D=0$. This means that σ is a flex point of the generic fiber of $Y\to C$. (The ex line is $z=0$.) This contradicts the non-degeneracy assumption for elements of $\mathcal{A}_{C,\omega}$.

By the above we see that the pairs (l_1,l_2) that will occur are subject to the following system of inequalities:

$$\begin{cases}0<l_1<l_2\\ l_1\le d\\ l_2\le d+l_1\end{cases}\tag{3.33}$$

For (b) we have $0<l_1<l_2$ and hence $l_2-l_1>0$ moreover by the inequalities (3.33) the inequality $d+l_1-l_2\ge 0$ hold, therefore

$$\begin{cases}
\deg(\omega\otimes(L_1\otimes L_2)^\vee) & =d-l_1-l_2 & \\
\deg(\omega\otimes L_2^\vee) & =d-l_2 & \\
\deg(\omega\otimes L_1^\vee) & =d-l_1 & \ge 0\\
\deg(\omega\otimes L_1\otimes L_2^\vee) & =d+l_1-l_2 & \ge 0\\
\deg(\omega) & =d & >0\\
\deg(\omega\otimes L_1^\vee\otimes L_2) & =d-l_1+l_2 & >0\\
\deg(\omega\otimes L_1^{\otimes 2}\otimes L_2^\vee) & =d+2l_1-l_2 & >0\\
\deg(\omega\otimes L_1) & =d+l_1 & >0\\
\deg(\omega\otimes L_2) & =d+l_2 & >0\\
\deg(\omega\otimes L_1^\vee\otimes L_2^{\otimes 2}) & =d-l_1+2l_2 & >0.
\end{cases}$$

If $l_1 + l_2 < d$ then all of the slopes above are positive and hence $\dim_k V_E = 10d$. If $d = l_1 + l_2$ then $d - l_1 - l_2 = 0$ and the rest of the slopes are positive, hence $\dim_k V_E \leq 10d + 1$. If $l_1 + l_2 > d$ and $l_2 < d$ then $d - l_1 - l_2 < 0$ and the rest of the slopes are positive therefore $\dim_k V_E = 10d + (l_1 + l_2 - d)$. If $l_1 + l_2 > d$ and $l_2 = d$ then $d - l_1 - l_2 < 0$ and $l_2 - d = 0$ hence $\dim_k V_E \leq 9d + (l_1 + l_2 + 1)$. Let now $l_2 > d$ then

$$\begin{cases} d - l_1 - l_2 & < 0 \\ d - l_2 & < 0 \end{cases}$$

and hence $\dim_k V_E = 10d + (l_1 + l_2 - d) + (l_2 - d)$. For (c) observe, that as $0 < l_1 < l_2$ we have $\mathrm{Hom}(L_1, \mathcal{O}_C) = \mathrm{Hom}(L_2, \mathcal{O}_C) = \mathrm{Hom}(L_2, \mathcal{O}_C) = 0$. Then every endomorphism of E is given by a matrix

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}.$$

with $a, s, f \in \mathrm{Hom}(\mathcal{O}_C, \mathcal{O}_C)$,$b \in \mathrm{Hom}(\mathcal{O}_C, L_1)$, $c \in \mathrm{Hom}(\mathcal{O}_C, L_1)$ and $e \in \mathrm{Hom}(L_1, L_2)$. Therefore

$$\mathrm{End}(E) = k \oplus k \oplus k \oplus \mathrm{Hom}(\mathcal{O}_C, L_1) \oplus \mathrm{Hom}(\mathcal{O}_C, L_1) \oplus \mathrm{Hom}(L_1, L_2).$$

By dimension counting

$$\#\mathrm{Aut}(E) = (q-1)^3 q^{l_2} q^{l_1} q^{l_2 - l_1} = (q-1)^3 q^{2l_2}$$

$\square$

3.3 The asymptotic limit

Let (C, O) be an elliptic curve over a finite field $\mathbb{F}_q$ of characteristic $p \geq 5$. Recall that for a fixed line bundle ω on C of degree $d \geq 0$ by $\mathrm{A}_{C,\omega}$ we denote the weighted number of elements in the set $\mathcal{A}_{C,\omega}$ (3.3).

Definition 3.3.1. *Let $d \geq 0$, let $\omega \in \mathrm{Pic}^d(C)$. Let E be a rank 3 vector bundle on C. We define the dimension of E with respect to ω, $\dim_\omega(E)$ to be the highest exponent of q in the expression $\frac{\#\mathbb{P}(V_E)}{\#PG_E}$ (when written out as a Laurent series in q) with*

$$V_E = H^0(C, \mathrm{Sym}^3(E) \otimes \det(E)^\vee \otimes \omega)$$

and

$$PG_E = \mathrm{Aut}(E) \big/ \mathbb{F}_q^* .$$

Let $d \geq 1$, then analyzing case by case one computes that the maximal dimension, in the sense of the Definition 3.3.1, is attained by geometrically stable vector bundles E from Section 3.2.1. It is equal to $\dim_\omega(E) = 10d - 1$. For those vector bundles we obtain $\dim_k V_E = 10d$ and they have the minimal number of automorphisms, namely $q - 1$.

The goal of this section is to bound the asymptotic limit

$$\limsup_{d \to \infty} q^{-10d+1} \mathrm{A}_{C,d} \tag{3.34}$$

where $\mathrm{A}_{C,d} = \sum_{\omega \in \mathrm{Pic}^d(C)} \mathrm{A}_{C,\omega}$. We will do it case by case counting the possible contributions from Section 3.2.1, Section 3.2.2 and Section 3.2.3.

We start with semi-stable vector bundles, Section 3.2.1, it gives the following contribution.

Lemma 3.3.2. *Let $d \geq 1$. Then*

$$\limsup_{d\to\infty} q^{-10d+1} \sum_{\substack{E \text{ semi-stable} \\ 0\leq \deg E\leq 3}} \frac{\#\mathbb{P}(V_E)}{\#PG_E}$$

$$= q\left(\frac{2\#C(\mathbb{F}_q)}{q-1}+\frac{1}{(q-1)q^2}+\frac{\#C(\mathbb{F}_q)}{(q-1)^2}+\frac{\#C(\mathbb{F}_q^2)}{(q-1)^3}+\frac{\#C(\mathbb{F}_{q^2})^2}{(q-1)^3}+\frac{\#C(\mathbb{F}_q)^2}{(q-1)^2}+\frac{\#C(\mathbb{F}_{q^3})^3}{(q-1)^3}\right).$$

Proof. Let

$$S := \sum_{\substack{E \text{ semi-stable} \\ 1\leq \deg E\leq 2}} \frac{\#\mathbb{P}(V_E)}{\#PG_E} + \sum_{\substack{E \text{ semi-stable} \\ \deg E=0}} \frac{\#\mathbb{P}(V_E)}{\#PG_E} + \sum_{\substack{E \text{ semi-stable} \\ \deg E=3}} \frac{\#\mathbb{P}(V_E)}{\#PG_E}$$

As $d \geq 1$ by Proposition 3.2.19 for each E we have $\#P(V_E)(k) = \frac{q^{10d-1}}{q-1}$. Then

$$S = (q^{-10d+1}-1)\left(\frac{2\#C(\mathbb{F}_q)}{q-1}+\frac{1}{(q-1)q^2}+\frac{\#C(\mathbb{F}_q)-1}{(q-1)^2}+\frac{1}{(q-1)^2q^3}\right.$$

$$+\frac{(\#C(\mathbb{F}_{q^2})-\#C(\mathbb{F}_q)-1)(\#C(\mathbb{F}_{q^2})-\#C(\mathbb{F}_q)-2)}{(q-1)(q^2-1)}+\frac{q}{(q^3-q^2)(q^3-q)(q^3-1)}$$

$$+\frac{(\#C(\mathbb{F}_q)-1)}{(q-1)^3(q+1)}+\frac{(\#C(\mathbb{F}_q)-1)}{(q-1)^2q^2}+\frac{(\#C(\mathbb{F}_q)^2-2\#C(\mathbb{F}_q)+1)}{(q-1)^3(q+1)}$$

$$\left.+\frac{(\#C(\mathbb{F}_{q^3})-\#C(\mathbb{F}_q)-1)(\#C(\mathbb{F}_{q^3})-\#C(\mathbb{F}_q)-2)}{(q^3-1)^3(q+1)}\right).$$

By simple algebraic computations follows

$$S \leq (q^{10d}-1)\left(\frac{2\#C(\mathbb{F}_q)}{q-1}+\frac{1}{(q-1)q^2}+\frac{\#C(\mathbb{F}_q)}{(q-1)^2}+\frac{\#C(\mathbb{F}_q^2)}{(q-1)^3}+\frac{\#C(\mathbb{F}_{q^2})^2}{(q-1)^3}+\frac{\#C(\mathbb{F}_q)^2}{(q-1)^2}+\frac{\#C(\mathbb{F}_{q^3})^3}{(q-1)^3}\right).$$

Therefore

$$q^{-10d+1}S \leq q^{-10d+1}(q^{10d}-1)\left(\frac{2\#C(\mathbb{F}_q)}{q-1}+\frac{1}{(q-1)q^2}+\frac{\#C(\mathbb{F}_q)}{(q-1)^2}+\frac{\#C(\mathbb{F}_q^2)}{(q-1)^3}+\frac{\#C(\mathbb{F}_{q^2})^2}{(q-1)^3}+\frac{\#C(\mathbb{F}_q)^2}{(q-1)^2}+\frac{\#C(\mathbb{F}_{q^3})^3}{(q-1)^3}\right)$$

$$= q\left(\frac{2\#C(\mathbb{F}_q)}{q-1}+\frac{1}{(q-1)q^2}+\frac{\#C(\mathbb{F}_q)}{(q-1)^2}+\frac{\#C(\mathbb{F}_q^2)}{(q-1)^3}+\frac{\#C(\mathbb{F}_{q^2})^2}{(q-1)^3}+\frac{\#C(\mathbb{F}_q)^2}{(q-1)^2}+\frac{\#C(\mathbb{F}_{q^3})^3}{(q-1)^3}\right)$$

$$+q^{-10d+1}\left(\frac{2\#C(\mathbb{F}_q)}{q-1}+\frac{1}{(q-1)q^2}+\frac{\#C(\mathbb{F}_q)}{(q-1)^2}+\frac{\#C(\mathbb{F}_q^2)}{(q-1)^3}+\frac{\#C(\mathbb{F}_{q^2})^2}{(q-1)^3}+\frac{\#C(\mathbb{F}_q)^2}{(q-1)^2}+\frac{\#C(\mathbb{F}_{q^3})^3}{(q-1)^3}\right)$$

As $\lim_{d\to\infty} q^{-10d+1} = 0$ the result follows. □

Now we move to the Section 3.2.2. There are two possible cases, first, that the maximal destabilizing bundle is of rank 2, second, the maximal destabilizing bundle is of rank 1.

Lemma 3.3.3. *Let $d \geq 1$ and let E be a vector bundle on C as in Lemma 3.2.24. Then*

$$\limsup_{d\to\infty} q^{-10d+1} \sum_{E\in(3.2.24)} \frac{\#\mathbb{P}(V_E)}{\#PG_E} \leq \frac{q^2}{q-1}\left(\frac{\#C(\mathbb{F}_q)+\#C(\mathbb{F}_{q^2})^2+\#C(\mathbb{F}_q)^2}{(q-1)^3}\right).$$

Proof. By Lemma 3.2.24 we have $0 \leq \deg E \leq 2d$. Let M be the following number

$$M := \frac{\#C(\mathbb{F}_q)}{(q-1)^2} + \frac{(\#C(\mathbb{F}_{q^2})-\#C(\mathbb{F}_q)-1)(\#C(\mathbb{F}_{q^2})-\#C(\mathbb{F}_q)-2)}{(q-1)^2(q+1)} + \frac{\#C(\mathbb{F}_q)}{(q-1)^3(q+1)q} + \frac{\#C(\mathbb{F}_q)(\#C(\mathbb{F}_q)-1)}{(q-1)^3}.$$

Then

$$\sum_{E\in(3.2.24)} \frac{\#\mathbb{P}(V_E)}{\#PG_E} \leq M\left(\sum_{0<\deg(E)<d} \frac{q^{10d}-1}{q^{\deg(E)}} + \sum_{d\leq\deg(E)\leq 2d} \frac{q^{10d+\deg(E)-d+2}-1}{q^{\deg(E)}}\right).$$

By multiplying with q^{-10d+1} we obtain

$$q^{-10d+1} \sum_{E\in(3.2.24)} \frac{\#\mathbb{P}(V_E)}{\#PG_E} \leq q^{-10d+1} M\left(\sum_{0<\deg(E)<d} \frac{q^{10d}-1}{q^{\deg(E)}} + \sum_{d\leq\deg(E)\leq 2d} \frac{q^{10d+\deg(E)-d+2}-1}{q^{\deg(E)}}\right)$$

$$= qM\left[\sum_{0<\deg(E)<d}\left(q^{-\deg(E)} - q^{-\deg(E)-10d}\right) + \sum_{d\leq\deg(E)\leq 2d}\left(q^{2-d} - q^{-\deg(E)-10d}\right)\right].$$

For $0 < \deg(E) < d$ we have

$$q^{-\deg(E)-10d} \leq q^{-10d}$$

moreover for $d \leq \deg(E) \leq 2d$ we have

$$q^{2-d} - q^{-\deg(E)-10d} \leq q^{2-d} - q^{-11d}.$$

And hence for large d those two terms vanish. Furthermore

$$\sum_{0\leq\deg(E)\leq d} q^{-\deg(E)} = \frac{(q-q^{-d})}{(q-1)} = \frac{q}{(q-1)}\left(1-q^{-d-1}\right)$$

and by simple algebraic computations one shows

$$M \leq \frac{2\#C(\mathbb{F}_q)+\#C(\mathbb{F}_{q^2})^2+\#C(\mathbb{F}_q)^2}{(q-1)^3}.$$

Hence the limit is

$$\lim_{d\to\infty} q^{-10d+1} \sum_{E\in(3.2.24)} \frac{\#\mathbb{P}(V_E)}{\#PG_E} \leq \frac{q^2}{q-1} \cdot \frac{2\#C(\mathbb{F}_q)+\#C(\mathbb{F}_{q^2})^2+\#C(\mathbb{F}_q)^2}{(q-1)^2}.$$

□

Now we move to case of unstable vector bundles with the maximal destabilizing subbundle of rank 1.

Lemma 3.3.4.

$$\limsup_{d\to\infty} q^{-10d+1} \sum_{E\in(3.2.25)} \frac{\#\mathbb{P}(V_E)}{\#PG_E} \leq \#C(\mathbb{F}_q)^2 \frac{q^2}{(q-1)^3}$$

Proof. It is clear, that the sum of isomorphism classes of vector bundles from Lemma 3.2.25 is at most

$$U_2 := \#C(\mathbb{F}_q)^2 \sum_{1\le \deg(L)\le d} \frac{q^{10d}-1}{(q-1)^2 q^{2\deg(L)-1}}.$$

Then

$$q^{-10d+1}U_2 = \#C(\mathbb{F}_q)^2 \sum_{0\le \deg(L)\le d} \left(\frac{1}{(q-1)^2 q^{2\deg(L)}} - \frac{q^{-10d}+1}{(q-1)^2 q^{\deg(L)}} \right).$$

If $0 \le \deg(L) \le d$, then

$$\frac{q^{-10d+1}}{(q-1)^2 q^{\deg(L)}} \le \frac{q^{-10d+1}}{(q-1)^2}.$$

Moreover

$$\sum_{0\le \deg(L)\le d-1} \frac{1}{(q-1)^2 q^{2\deg(L)}} = \frac{1}{(q-1)^2} \frac{1-q^{-(2(d-1)+2)}}{1-q^2} = \frac{q^2}{(q-1)^2(q^2-1)} \left(1-q^{-2d}\right)$$

And therefore by passing to the limit, we have

$$\lim_{d\to\infty} q^{-10d+1} U_2 \le \#C(\mathbb{F}_q)^2 \frac{q^2}{(q-1)^3(q+1)} \le \frac{q^2}{(q-1)^2}.$$

□

Lemma 3.3.5.

$$\limsup_{d\to\infty} q^{-10d+1} \sum_{E\in (3.2.26)} \frac{\#\mathbb{P}(V_E)}{\#PG_E} \le \frac{q}{(q-1)} \frac{\#C(\mathbb{F}_q)^2 + \#C(\mathbb{F}_q) + \#C(\mathbb{F}_q)\#C(\mathbb{F}_{q^2})}{(q-1)^2}.$$

Proof. Let

$$N := \frac{\#C(\mathbb{F}_q)}{(q-1)q} + \frac{\#C(\mathbb{F}_q)(\#C(\mathbb{F}_q^2) - \#C(\mathbb{F}_q) - 1)(\#C(\mathbb{F}_q^2) - \#C(\mathbb{F}_q) - 2)}{(q-1)^2(q+1)}$$
$$+ \frac{\#C(\mathbb{F}_q)}{(q-1)^2(q+1)q} + \frac{\#C(\mathbb{F}_q)(\#C(\mathbb{F}_q)-1)}{(q-1)^2}.$$

Then

$$\sum_{E\in (3.2.26)} \frac{\#\mathbb{P}(V_E)}{\#PG_E} \le N \sum_{\deg(L)=1}^{d} \frac{q^{10d}-1}{(q-1)^2 q^{2\deg(L)+1}}$$

By the same methods we use above

$$q^{-10d+1} \sum_{E\in (3.2.26)} \frac{\#\mathbb{P}(V_E)}{\#PG_E} \le N q^{-10d+1} \sum_{\deg(L)=1}^{d} (q^{10d}-1) q^{-\deg(L)}$$

$$= Nq \sum_{\deg(L)=1}^{d} (1-q^{-10d}) q^{-\deg(L)}.$$

Moreover for $1 \le \deg(L) \le d$

$$q^{-10d-\deg(L)} \le q^{-10d}.$$

Therefore by passing to the limit, we have

$$\lim_{d\to\infty} q^{-10d+1}\sum_{E\in\,(3.2.26)}\frac{\#\mathbb{P}(V_E)}{\#PG_E} \le qN\limsup_{d\to\infty}\sum_{\deg(L)=1}^{d} q^{-\deg(L)}$$
$$= qN\limsup_{d\to\infty}\left(\frac{1}{q-1}+\frac{q^{-d}}{q-1}\right)$$
$$= \frac{qN}{q-1}.$$

Furthermore by simple algebraic computations one shows

$$N \le \frac{\#C(\mathbb{F}_q)^2+\#C(\mathbb{F}_q)+\#C(\mathbb{F}_q)\#C(\mathbb{F}_{q^2})}{(q-1)^2}$$

and hence the result follows. □

The last case is the case of a direct sum of line bundles.

Lemma 3.3.6.

$$\limsup_{d\to\infty} q^{-10d+1}\sum_{E\in\,(3.2.27)}\frac{\#\mathbb{P}(V_E)}{\#PG_E} \le \frac{\#C(\mathbb{F}_q)^2 q}{(q-1)^3(q^2-1)^2}$$

Proof. Let E be a sum of line bundles, $E=\mathcal{O}_C\oplus L_1\oplus L_2$ and the inequalities (3.33) are satisfied.

First of all suppose that $l_1+l_2>d$ and $l_2\le d$, then for fixed l_1,l_2 the contribution of such a term is at most

$$\#C(\mathbb{F}_q)^2 q^{-10d+1}\frac{q^{10d+1+l_1+l_2-d}-1}{(q-1)^3q^{2l_2}} \le \#C(\mathbb{F}_q)^2\frac{q^{1-d}}{(q-1)^3}$$

as $l_1<l_2$ also $9d+(l_1+l_2-d+1)\le 10d+(l_1+l_2-d)$. The number of pairs (l_1,l_2) like this is at most d^2, therefore the contribution is vanishingly small and can be disregarded from the limit. Similarly for the case $0<l_1<l_2$ and $l_2>d$ we have

$$\#C(\mathbb{F}_q)q^{-10d+1}\frac{q^{10d+(l_1+l_2-d)+(l_2-d)}-1}{(q-1)q^{2l_2}} \le \#C(\mathbb{F}_q)^2\frac{q^{1-d}}{(q-1)^3},$$

as $l_1\le d$. Now only the case $l_1+l_2\le d$ is left. Here for every fixed $l_2\ge 2$ the pairs (l_1,l_2) with $0\le l_1\le l_2$ contribute to the limit the quantity

$$\lim_{d\to\infty} q^{-10d+1}\sum_{l_2=2}^{d}\#C(\mathbb{F}_q)^2\frac{(l_2-1)(q^{10d}-1)}{(q-1)^3q^{2l_2}} = \lim_{d\to\infty}\frac{\#C(\mathbb{F}_q)^2q}{(q-1)^3}\left[\sum_{l_2=2}^{d}\frac{l_2-1}{q^{2l_2}}\right],$$

(as $l_2\ge 2$ we have $9d+1\le 10d$). Furthermore

$$\sum_{l_2=2}^{d}\frac{l_2-1}{q^{2l_2}} = \frac{1}{(q^2-1)^2}-\frac{q^{-2d}(1-d+dq^2)}{(q^2-1)^2}.$$

and hence the contribution of this term is

$$\frac{\#C(\mathbb{F}_q)^2}{(q-1)^3}\frac{q}{(q^2-1)^2}.$$

□

Proposition 3.3.7. *Let (C,O) be an elliptic curve over a finite field $k=\mathbb{F}_q$ with q elements and of characteristic $p>7$. Let $\#C(\mathbb{F}_{q^n})$ denote the number of $\mathbb{F}_{q^n}$-rational points of C. Then*

$$\limsup_{d\to\infty} q^{-10d+1}\mathrm{A}_{C,d} \leq \frac{\#C(\mathbb{F}_q)A}{(q-1)^5(q+1)^2q},$$

where A is given by

$$A = 2\#C(\mathbb{F}_q)q^8 + \left(2\#C(\mathbb{F}_q)^2 + \#C(\mathbb{F}_{q^2})^2\right)q^7$$

$$+\left(\#C(\mathbb{F}_q)\#C(\mathbb{F}_{q^2}) + \#C(\mathbb{F}_{q^3})^3 + 1 - 2\#C(\mathbb{F}_q) + \#C(\mathbb{F}_{q^2})^2\right)q^6 + \left(-2-2\#C(\mathbb{F}_{q^2})^2 - 4\#C(\mathbb{F}_q)^2\right)q^5$$

$$+\left(-2\#C(\mathbb{F}_{q^3})^3 - 1 - 2\#C(\mathbb{F}_q) - 2\#C(\mathbb{F}_{q^2})^2 - 2\#C(\mathbb{F}_q)\#C(\mathbb{F}_{q^2})\right)q^4$$

$$+\left(\#C(\mathbb{F}_{q^2})^2 + 2\#C(\mathbb{F}_q)^2 + 4\right)q^3$$

$$+\left(\#C(\mathbb{F}_{q^3})^3 + \#C(\mathbb{F}_{q^2})^2 + \#C(\mathbb{F}_q)^2 - 1 + 2\#C(\mathbb{F}_q) + \#C(\mathbb{F}_q)\#C(\mathbb{F}_{q^2})\right)q^2 - 2q + 1$$

Proof. As we are interested in the asymptotic limit we may assume $d \geq 1$. The number of isomorphism classes of line bundles of given degree on C is equal to $\#C(\mathbb{F}_q)$, hence for a given d we have

$$\mathrm{A}_{C,d} = \sum_{\omega\in\mathrm{Pic}^d(C)} \mathrm{A}_{C,\omega} \leq \#C(\mathbb{F}_q)\mathrm{A}_{C,\omega}.$$

Then the above bound is a result of summing up all possible cases from Lemma 3.3.2, 3.3.3, 3.3.5 and Lemma 3.3.6. ☐

Remark 3.3.8. The assumption that $p>7$ is made for computational reasons.

Chapter 4

Three kinds of sets

In this section we do not claim any originality. The idea of the presented construction is taken from [deJ02, Section 5], however in [deJ02, Section 5] the author considers only elliptic surfaces over $\mathbb{P}^1_k$ and in our case the base curve is an elliptic curve. It is not a trivial matter, that the construction works for the elliptic case as well. The main difference, as we will see later, is that over $\mathbb{P}^1_k$ up to an isomorphism there is only one line bundle of a given degree, whereas over an elliptic we have as many line bundles as its k-rational points.

Let (C,O) be an elliptic curve over $\mathbb{F}_q$. The aim of this section is to obtain the following inequality (see Theorem 4.5.10).

$$\limsup_{d\to\infty} q^{-10d+1}\left(\sum_{[(E,O)],h(E)=d} \frac{3^{r(E)}-1}{\#\mathrm{Aut}(E,O)}\right) \le \frac{\#C(\mathbb{F}_q)A}{(q-1)^5(q+1)^2q},$$

where the sum is taken over isomorphism classes of elliptic curves (E,O) over $\mathbb{F}_q(C)$ of height d, $r(E)$ is the Mordell-Weil rank of an elliptic curve (E,O) over $\mathbb{F}_q(C)$ and A is a polynomial of degree 8 in q (see also Section 3.3). To prove this fact we introduce three finite sets, whose elements are families of elliptic curves. We show inclusions between them and bound the number of elements of the largest set, which together with results from the previous section will implicate the bound.

4.1 The set $\mathcal{A}_{C,\omega}$

For convenience we recall the construction of the set $\mathcal{A}_{C,\omega}$ however all necessary facts about this set are proven in Section 3.1 and we do not repeat them here.

Definition 4.1.1. *Let $d \ge 0$ and $\omega \in \mathrm{Pic}^d(C)$. Let $\mathcal{A}$ denote the set whose elements are pairs $(g : Y \to C, D)$ with the following properties:*

(a) the morphism $Y \xrightarrow{g} C$ is flat, proper and its generic fibre is a smooth curve,

(b) the equality $g_*\mathcal{O}_Y = \mathcal{O}_C$ holds universally,

(c) the fibres of g are Gorenstein and $\omega_{Y/C} \cong g^*\omega$,

(d) $D \subset Y$ is an effective Cartier divisor, flat over C such that $D.F = 3$, where F is a fibre of g,

(e) the sheaf $\mathcal{O}_Y(D)$ is relatively very ample for g, i.e. we obtain a canonical closed immersion

$$Y \hookrightarrow \mathrm{Proj}_C(\mathrm{Sym}^*(g_*\mathcal{O}_Y(D))).$$

Definition 4.1.2. *Two elements $(Y \xrightarrow{g} C, D)$ and $(Y' \xrightarrow{g'} C, D')$ of $\mathcal{A}$ are called equivalent if and only if there exists an isomorphism $\psi : Y \to Y'$ over C such that we have the following rational equivalence relation of divisors on Y*

$$\psi^* D' \sim D + g^* D_C$$

for some $D_C \in \mathrm{Div}(C)$.

Definition 4.1.3. *An element $(g : Y \to C, D)$ of the set $\mathcal{A}$ is called degenerate if there exists a divisor D' of degree 1 on the generic fiber Y_η such that $D_\eta \sim 3D'$ (rational equivalence).*

Definition 4.1.4. *Let $d \geq 0$ and for $\omega \in \mathrm{Pic}^d(C)$ let $\mathcal{A}_{C,\omega}$ be the set*

$$\mathcal{A}_{C,\omega} = \left\{ \begin{array}{c} \text{equivalence classes} \left[(Y \xrightarrow{g} C, D) \right] \text{ (Definition 4.1.2)} \\ \\ \text{of non-degenerate (Definition 4.1.3)} \\ \\ \text{pairs } (Y \xrightarrow{g} C, D) \text{ satisfying properties (a)-(e) (Definition 4.1.1)} \end{array} \right\}$$

Definition 4.1.5. *The automorphism group $Aut(Y \xrightarrow{g} C, D)$ of a pair $(Y \xrightarrow{g} C, D) \in \mathcal{A}$ (Definition 4.1.1) we define as the set of automorphisms $\psi : Y \longrightarrow Y$ over C such that*

$$\psi^* D \sim D + g^* D_C \qquad (\text{ rational equivalence })$$

for some $D_C \in \mathrm{Div}(C)$.

In Chapter 3.2 we proved, that the set $\mathcal{A}_{C,\omega}$ is finite. We set

$$\mathrm{A}_{C,\omega} := \sum_{[(Y \xrightarrow{g} C, D)] \in \mathcal{A}_{C,\omega}} \frac{1}{\#\mathrm{Aut}(Y \xrightarrow{g} C, D)} \tag{4.1}$$

for the weighted number of elements of $\mathcal{A}_{C,\omega}$.

4.2 The set $\mathcal{B}_{C,\omega}$

Definition 4.2.1. *Let $d \geq 0$ and $\omega \in \mathrm{Pic}^d(C)$. Let $\mathcal{B}$ denote the set whose elements are pairs $(f : X \to C, D)$ with the following properties:*

(a) *$f : X \to C$ is a relatively minimal elliptic surface,*

(b) *the equality $f_* \mathcal{O}_Y = \mathcal{O}_C$ holds universally,*

(c) *$\omega_{X/C} \cong f^* \omega$,*

(d) *$D \subset X$ is an effective Cartier divisor, flat over C such that $D.F = 3$, where F is a fibre of f.*

The concept of equivalence, automorphisms and degeneracy is the same as for the set $\mathcal{A}_{C,\omega}$, Section 4.1.

Definition 4.2.2. *Let $d \geq 0$ and for $\omega \in \mathrm{Pic}^d(C)$ let $\mathcal{B}_{C,\omega}$ be the set*

$$\mathcal{B}_{C,\omega} = \left\{ \begin{array}{c} \textit{equivalence classes} \left[(X \xrightarrow{f} C, D) \right] \textit{(Definition 4.1.2)} \\ \\ \textit{of non-degenerate (Definition 4.1.3)} \\ \\ \textit{pairs } (X \xrightarrow{f} C, D) \textit{ satisfying properties (a)-(d) (Definition 4.2.1)} \end{array} \right\}$$

In Section 4.4 we will show, that $\mathcal{B}_{C,\omega}$ is a "weighted" subset of $\mathcal{A}_{C,\omega}$ and therefore finite. We set

$$\mathrm{B}_{C,\omega} := \sum_{[(X \xrightarrow{f} C,D)] \in \mathcal{B}_{C,\omega}} \frac{1}{\#\mathrm{Aut}(X \xrightarrow{f} C, D)} \tag{4.2}$$

for weighted number of elements of $\mathcal{B}_{C,\omega}$.

4.3 The set $\mathcal{C}_{C,\omega}$

Definition 4.3.1. *Let $d \geq 0$ and $\omega \in \mathrm{Pic}(C)$. Let $\mathcal{C}$ denote the set whose elements are triples $(f : X \to C, \sigma, \tau)$ with the following properties:*

(a) $f : X \to C$ is a relatively minimal elliptic surface,

(b) $f_* \omega_{X/C} \cong \omega$,

(c) (σ, τ) is a pair of sections of f.

For any triple $(f : X \to C, \sigma, \tau)$ satisfying properties (a), (b) and (c) we denote by $(E, O) = (X_\eta, \sigma(\eta))$ the associated elliptic curve over $k(C)$ (see: Remark 2.1.12).

Definition 4.3.2. *Two elements $(X \xrightarrow{f} C, \sigma, \tau)$ and $(X' \xrightarrow{f'} C, \sigma', \tau')$ of $\mathcal{C}$ are called equivalent if and only if there exists an isomorphism $\varphi : X \longrightarrow X'$ over C and a section $\rho : C \to X'$ such that*

(a) $\varphi \circ \sigma = \sigma'$,

(b) on the generic fibre $(E', O') = (X'_\eta, \sigma'(\eta))$ we have $3\rho = \tau' - \varphi \circ \tau$ (addition in the group law of (E', O')),

(c) for every closed point $c \in C$ the sections ρ and σ' meet the same irreducible component of X'_c.

Remark 4.3.3. The relation above is an equivalence relation on the set of all triples $(f : X \to C, \sigma, \tau)$.

1. It is easy to check, that it is reflexive.

2. It is symmetric: assume we are given $(X \xrightarrow{f} C, \sigma, \tau) \in \mathcal{C}$ and $(X' \xrightarrow{f'} C, \sigma', \tau') \in \mathcal{C}$, that are equivalent. The inverse $\varphi^{-1} : X' \to X$ of φ (φ as in Definition 4.3.2) is an isomorphism over C, moreover by (a) we have $\varphi^{-1} \circ \sigma' = \varphi^{-1} \circ \varphi \circ \sigma = \sigma$. Using (b) on the generic fibre $(X_\eta, \sigma(\eta))$ we obtain an equality

$$\tau - \varphi^{-1} \circ \tau' = \tau - \varphi^{-1}(\varphi \circ \tau + 3\rho) = \tau - \tau - 3\varphi^{-1} \circ \rho = -3\varphi^{-1} \circ \rho.$$

To complete the proof we need to show that for every $c \in C$ the sections σ and $-\varphi^{-1} \circ \rho$ meet the same irreducible component of X_c. By Proposition 2.1.7 the Néron model $\mathcal{N}$ of $X \to C$ is the open subscheme of $X \to C$ made up of the points that are smooth over C. It is unique up to isomorphism and by Proposition 2.1.8(iv) the algebraic group structure of X_η extends in a unique way to the structure of a smooth group scheme on $\mathcal{N} \to C$. As X is a minimal elliptic surface with a section there are no multiple fibers (see: Section 2.2, discussion before Lemma 2.2.2) therefore $\mathcal{N}_c$ is obtained from X_c by removing singular points. We respectively consider the Néron model $\mathcal{N}' \to C$ of $X' \to C$. The map φ^{-1} is an isomorphism and maps σ' to σ therefore it maps the identity component of each $\mathcal{N}'_c$ to the identity component of $\mathcal{N}_c$. By assumption the sections ρ and σ' meet the same irreducible component of each X'_c hence ρ meets the identity component of each $\mathcal{N}'_c$ and therefore $-\varphi^{-1} \circ \rho$ meets the identity component of each $\mathcal{N}_c$. This implies, that for every closed point $c \in C$ the sections σ and $-\varphi^{-1} \circ \rho$ meet the same irreducible component of X_c.

3. To see transitivity, take $(f : X \to C, \sigma, \tau)$ equivalent to $(f' : X' \to C, \sigma', \tau')$ and $(f'' : X'' \to C, \sigma'', \tau'')$ such that $(f' : X' \to C, \sigma', \tau')$ is equivalent to $(f'' : X'' \to C, \sigma'', \tau'')$. By definition we have isomorphisms $\psi : X \to X'$ and $\varphi : X' \to X''$ over C with relations

$$\psi \circ \sigma = \sigma' \qquad \text{and} \qquad \varphi \circ \sigma' = \sigma''.$$

Therefore $\varphi \circ \psi : X \to X''$ is an isomorphism over C satisfying

$$\varphi \circ \psi \circ \sigma = \varphi \circ \sigma' = \sigma''$$

which gives the condition (a) of an equivalence between $(f : X \to C, \sigma, \tau)$ and $(f'' : X'' \to C, \sigma'', \tau'')$. Furthermore by definition there exist sections $\rho : C \to X'$ and $\varrho : C \to X''$ such that on $(E', O') = (X'_\eta, \sigma'(\eta))$ we have

$$3\rho = \tau' - \psi \circ \tau$$

and on $(E'', O'') = (X''_\eta, \sigma''(\eta))$ we have

$$3\varrho = \tau'' - \varphi \circ \tau'.$$

Hence

$$\begin{aligned} \tau'' - \varphi \circ \psi \circ \tau &= \tau'' - \varphi \circ (\tau' - 3\rho) \\ &= \tau'' - (\tau'' - 3\varrho - 3\varphi \circ \rho) \\ &= 3(\varrho + \varphi \circ \rho) \end{aligned}$$

and therefore the section $\varrho + \varphi \circ \rho : C \to X''$ satisfies the requirement (b) of an equivalence between $(f : X \to C, \sigma, \tau)$ and $(f'' : X'' \to C, \sigma'', \tau'')$. To finish the proof of transitivity we need to show that the requirement (c) holds, i.e. for all closed points $c \in C$ the sections $\varrho + \varphi \circ \rho$ and σ'' meet the same irreducible components of each fiber X''_c. Again we will use the Néron model

$\mathcal{N} \to C$ (resp. $\mathcal{N}' \to C$ and $\mathcal{N}'' \to C$) of $X \to C$ (resp. $X' \to C$ and $X'' \to C$). By assumption $\varphi : X' \to X''$ is an isomorphism over C such that $\varphi \circ \sigma' = \sigma''$ which means that φ maps the identity component of each $\mathcal{N}'_c$ to the identity component of $\mathcal{N}''_c$. Furthermore by assumption the sections ρ and σ' meet the same irreducible component of each $\mathcal{N}'_c$ which means that ρ meets the identity component of each $\mathcal{N}'_c$. It therefore follows, that $\varphi \circ \rho$ meets the identity component of each $\mathcal{N}''_c$. Furthermore by assumption ϱ meets the identity component of each $\mathcal{N}''_c$ and hence translation by ϱ maps each irreducible component of $\mathcal{N}''_c$ to itself and hence $\varphi \circ \rho + \varrho$ meets the identity component of each $\mathcal{N}''_c$. Consequently the property (c) of an equivalence between $(f : X \to C, \sigma, \tau)$ and $(f'' : X'' \to C, \sigma'', \tau'')$ is proven and hence the transitivity of the relation from Definition 4.3.1 follows.

Definition 4.3.4. *We say that an element $(X \to C, \sigma, \tau)$ of $\mathcal{C}$ is degenerate if τ is divisible by 3 in the Mordell-Weil group of $(E, O) = (X_\eta, \sigma(\eta))$.*

Definition 4.3.5. *Let $d \geq 0$ and for $\omega \in \mathrm{Pic}^d(C)$ let $\mathcal{C}_{C,\omega}$ be the set*

$$\mathcal{C}_{C,\omega} = \left\{ \begin{array}{c} \textit{equivalence classes } [(f : X \to C, \sigma, \tau)] \textit{ (Definition 4.3.2)} \\ \\ \textit{of non-degenerate (Definition 4.3.4)} \\ \\ \textit{triples } (f : X \to C, \sigma, \tau) \textit{ satisfying properties (a)-(b) (Definition 4.3.1)} \end{array} \right\}$$

Definition 4.3.6. *The automorphism group $\mathrm{Aut}(f : X \to C, \sigma, \tau)$ of an element $(f : X \to C, \sigma, \tau)$ of $\mathcal{C}$ is the set of pairs (φ, ρ) with $\varphi : X \to X$ and $\rho : C \to X$ a section satisfying the requirements (a) (b) and (c) of Definition 4.3.2.*

Remark 4.3.7. Observe that the group operation on $\mathrm{Aut}(f : X \to C, \sigma, \tau)$ is defined by

$$(\varphi_1, \rho_1) \star (\varphi_2, \rho_2) := (\varphi_2 \circ \varphi_1, \rho_2 + \varphi_2 \circ \rho_1)$$

and the neutral element of the group $\mathrm{Aut}(f : X \to C, \sigma, \tau)$ is the pair (id_X, σ).

In Section 4.4 we will show that $\mathcal{C}_{C,\omega}$ is a "weighted" subset of $\mathcal{B}_{C,\omega}$ and therfore finite. We denote by $\mathrm{C}_{C,\omega}$ the weighted number of equivalence classes in $\mathcal{C}_{C,\omega}$

$$\mathrm{C}_{C,\omega} := \sum_{[(f:X\to C,\sigma,\tau)]\in\mathcal{C}_{C,\omega}} \frac{1}{\#\mathrm{Aut}(f : X \to C, \sigma, \tau)}. \tag{4.3}$$

4.4 Maps between the three sets

In this section we construct maps between the three sets defined in Sections 4.1, 4.2 and 4.3, namely

$$\mathcal{C}_{C,\omega} \to \mathcal{B}_{C,\omega} \tag{4.4}$$

and

$$\mathcal{B}_{C,\omega} \to \mathcal{A}_{C,\omega}. \tag{4.5}$$

Moreover we will prove, that $\mathcal{C}_{C,\omega} \subset \mathcal{B}_{C,\omega} \subset \mathcal{A}_{C,\omega}$.

We start with the first map (4.5). We recall the following facts from [deJ02, Appendix A].

Lemma 4.4.1. *[deJ02, Lemma 8.4] Let X be a projective Gorenstein curve over a field k such that $\omega_{X/k} \cong \mathcal{O}_X$ and $H^0(X, \mathcal{O}_X) = k$. Let $D \subset X$ be an effective Cartier divisor.*

(a) *If $\ell(D) \geq 2$ then $\mathcal{O}_X(D)$ is globally generated and $H^1(X, \mathcal{O}_X(D)) = (0)$.*

(b) *If $\ell(D) \geq 3$ then the graded k-algebra $R := \oplus_{n\geq 0}\Gamma(X, \mathcal{O}_X(D))$ is generated in degree 1.*

Lemma 4.4.2. *[deJ02, Lemma 8.5] In the situation described in (b) of lemma above, the scheme $Y = \mathrm{Proj}(R)$ is a Gorenstein curve with $\omega_{Y/k} \cong \mathcal{O}_Y$ and $H^0(Y, \mathcal{O}_Y) = k$. The morphism $\varphi : X \to Y$ is dominant and induces an isomorphism of an open subscheme of X with a dense open subscheme of Y.*

Remark 4.4.3. [deJ02, Remark 8.6] With notation as in Lemma 4.4.1 and Lemma 4.4.2. By Serre-Duality we have $\dim_k H^1(Y, \mathcal{O}_Y) = 1$. The relevant terms of the Leray spectral sequence for the map φ are

$$0 \to E_2^{10} \to H^1(X, \mathcal{O}_X) \to E_2^{01} \to E_2^{20}$$

which for us is

$$0 \to H^1(Y, \varphi_*\mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to H^0(Y, R^1\varphi_*\mathcal{O}_X) \to H^2(Y, \varphi_*\mathcal{O}_X).$$

As $\mathcal{O}_Y = \varphi_*\mathcal{O}_X$ we have $\dim_k H^1(Y, \varphi_*\mathcal{O}_X) = \dim_k H^1(Y, \mathcal{O}_Y) = 1$, furthermore by Serre-Duality and the fact, that $\omega_X = \mathcal{O}_X$ we have $\dim_k H^1(X, \mathcal{O}_X) = 1$ and as Y is a curve the last term is zero. It follows, that $R^1\varphi_*\mathcal{O}_X = (0)$ and the canonical map

$$H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X)$$

is an isomorphism. The dual of this map is the canonical isomorphism $\Gamma(X, \omega_X) = \Gamma(Y, \omega_Y)$.

Proposition 4.4.4. *Let $(X \xrightarrow{f} C, D)$ be a representative of an equivalence class in $\mathcal{B}_{C,\omega}$ then*

(a) *The sheaf $f_*\mathcal{O}_X(D)$ is locally free of rank three on C of formation compatible with arbitrary change of base, and $R^1 f_*\mathcal{O}_X(D) = (0)$.*

(b) *The natural maps*
$$f^* f_*\mathcal{O}_X(D) \to \mathcal{O}_X(D)$$
and
$$\bigoplus_{m\geq 0} (f_*\mathcal{O}_X(D))^{\otimes m} \to \bigoplus_{m\geq 0} f_*\left(\mathcal{O}_X(D)^{\otimes m}\right)$$
are surjective.

Proof. The reader can find the proof in Chapter 3, Proposition 3.1.14. Notice, that the requirements we put on elements of the set $\mathcal{A}_{C,\omega}$ are weaker, then the requirements on elements of $\mathcal{B}_{C,\omega}$. □

Proposition 4.4.5. *Let $(X \xrightarrow{f} C, D)$ be a representative of an equivalence class in $\mathcal{B}_{C,\omega}$. Let*

$$Y := \mathrm{Proj}_C\left(\bigoplus_{m\geq 0} f_*\mathcal{O}_X(mD)\right) \xrightarrow{g} C$$

and let $D_Y \subset Y$ be the divisor corresponding to the canonical section

$$1_D \in \Gamma(X, \mathcal{O}_X(D)) = \Gamma(Y, \mathcal{O}_Y(1)).$$

Then

$$(Y \xrightarrow{g} C, D_Y)$$

is a representative of an equivalence class in $\mathcal{A}_{C,\omega}$.

Proof. By Proposition 4.4.4 there is a morphism $\pi : X \to Y$. Lemma 4.4.2 implies that the morphism π is birational, fibres of $Y \xrightarrow{g} C$ are Gorenstein, $g_*(\mathcal{O}_Y) = \mathcal{O}_C$ holds universally and $g^* g_* \omega_{Y/C} = \omega_{Y/C}$. By the construction of Y the morphism $g : Y \to C$ is flat and proper, its generic fibre is a smooth curve. Furthermore by Lemma 4.4.1(b) the divisor $D|_{X_\eta}$ is very ample and hence $X_\eta \cong Y_\eta$. By Remark 4.4.3 there is a canonical isomorphism $g_* \omega_{Y/C} = f_* \omega_{X/C}$ and therefore

$$\omega_{Y/C} = g^* g_* \omega_{Y/C} = g^* f_* \omega_{X/C} = g^* \omega.$$

The morphism π is a resolution of singularities of Y in fact it is a minimal resolution, since $f : X \to C$ is a minimal surface. As the generic fibers $X_\eta \cong Y_\eta$ are isomorphic, we conclude, that $(f : X \to C, D)$ is non-degenerate if and only if $(g : Y \to C, D_Y)$ is non-degenerate. □

Proposition 4.4.6. *Let $(f : X \to C, D)$, $(f' : X' \to C, D')$ be representatives of equivalence classes in $\mathcal{B}_{C,\omega}$ let $(g : Y \to C, D_Y)$, $(g' : Y' \to C, D_{Y'})$ be the associated pairs from Proposition 4.4.5. Then*

$$\operatorname{Equiv}_{\mathcal{B}_{C,\omega}}\left((X \xrightarrow{f} C, D), (X' \xrightarrow{f'} C, D')\right) \tag{4.6}$$

$$= \operatorname{Equiv}_{\mathcal{A}_{C,\omega}}\left((Y \xrightarrow{g} C, D_Y), (Y' \xrightarrow{g'} C, D_{Y'})\right).$$

Therefore $\mathcal{B}_{C,\omega}$ is a "weighted" subset of $\mathcal{A}_{C,\omega}$.

Remark 4.4.7. Here the equality (4.6) means that whenever $(f : X \to C, D)$ and $(f' : X' \to C, D')$ are equivalent, the corresponding pairs $(g : Y \to C, D_Y)$ and $(g' : Y' \to C, D_{Y'})$ are equivalent and vice-versa.

Proof. Suppose $(f : X \to C, D)$ and $(f' : X' \to C, D')$ are equivalent. By definition there exists an isomorphism $\psi : X \to X'$ over C such that $\mathcal{O}_X(\psi^*(D')) \otimes f^* L \cong \mathcal{O}_X(D)$ for some $L = \mathcal{O}_C(D_C) \in \operatorname{Pic}(C)$. Therefore

$$\begin{aligned}
Y &= \operatorname{Proj}_C\left(\oplus_{m\geq 0} f_* \mathcal{O}_X(D)^{\otimes m}\right) \\
&= \operatorname{Proj}_C\left(\oplus_{m\geq 0} f_* \mathcal{O}_X(\psi^* D')^{\otimes m} \otimes L^{\otimes m}\right) \qquad \text{(by projection formula)} \\
&= \operatorname{Proj}_C\left(\oplus_{m\geq 0} (f' \circ \psi)_* \mathcal{O}_X(\psi^* D')^{\otimes m} \otimes L^{\otimes m}\right) \\
&= \operatorname{Proj}_C\left(\oplus_{m\geq 0} f'_* \mathcal{O}_{X'}(D')^{\otimes m} \otimes L^{\otimes m}\right) \\
&\cong Y'
\end{aligned}$$

Moreover if α is an isomorphism from Y to Y' then $\alpha^* \mathcal{O}_{Y'}(1) \cong \mathcal{O}_Y(1) \otimes g^* L$ and therefore $(g : Y \to C, D_Y)$ and $(g' : Y' \to C, D_{Y'})$ are equivalent.

On the other hand assume, that $(g : Y \to C, D_Y)$ and $(g' : Y' \to C, D_{Y'})$ are equivalent as elements of $\mathcal{A}$ then as X and X' are the unique minimal models of Y and Y' hence they must be isomorphic over C. By the definition of D_Y and D'_Y the rational equivalence relation of D and D' holds. That proves the claim. □

By Proposition 4.4.5 and Proposition 4.4.6 we have

$$\mathrm{B}_{C,\omega} \leq \mathrm{A}_{C,\omega}. \tag{4.7}$$

Now we construct the map (4.18) $\mathcal{C}_{C,\omega} \to \mathcal{B}_{C,\omega}$.

Proposition 4.4.8. *Let $(f : X \to C, \sigma, \tau)$ be a triple, that is a representative of an equivalence class in $\mathcal{C}_{C,\omega}$. Set*

$$D := 2[\sigma] + [\tau],$$

where $[\sigma]$ denotes the effective Cartier divisor $\sigma(C)$ on X. Then

$$(f : X \to C, D)$$

is a representative of an equivalence class in $\mathcal{B}_{C,\omega}$.

Proof. The surface $f : X \to C$ is a relatively minimal elliptic surface with a section, hence by [Liu, Section 9.3, Proposition 3.27] we have $\omega_{X/C} \cong f^* f_* \omega_{X/C} \cong f^* \omega$. For each closed point $c \in C$ the fiber X_c is a geometrically connected projective curve over $k(c)$ and therefore $f_* \mathcal{O}_X = \mathcal{O}_C$ holds universally (see Section 2.2). Furthermore $D = 2[\sigma] + [\tau]$ is fibre-by-fibre of degree 3 so the pair $(f : X \to C, D)$ satisfies conditions $(a) - (d)$ of the set $\mathcal{B}_{C,\omega}$. $\square$

Proposition 4.4.9. *Suppose we have given two triples $(X \xrightarrow{f} C, \sigma, \tau)$, $(X' \xrightarrow{f'} C, \sigma', \tau')$ that are representatives of equivalence classes in $\mathcal{C}_{C,\omega}$. Let $(f : X \to C, D)$ and $(f' : X' \to C, D')$ be the corresponding pairs from Proposition 4.4.8. Then*

$$\begin{aligned} &\mathrm{Equiv}_{\mathcal{C}_{C,\omega}}\left((X \xrightarrow{f} C, \sigma, \tau), (X' \xrightarrow{f'} C, \sigma', \tau')\right) \\ &= \mathrm{Equiv}_{\mathcal{B}_{C,\omega}}\left((X \xrightarrow{f} C, D), (X' \xrightarrow{f'} C, D')\right). \end{aligned} \tag{4.8}$$

Therefore $\mathcal{C}_{C,\omega}$ is a "weighted" subset of $\mathcal{B}_{C,\omega}$.

Proof. "$\subseteq$" Let $(f : X \to C, D)$ and $(f' : X' \to C, D')$ be equivalent. There exists an isomorphism $\psi : X \to X'$ over C and a divisor $D_C \in \mathrm{Div}(C)$ such that $\psi^*(D') \sim D + f^* D_C$. We set $\rho := \psi \circ \sigma$ and $\varphi := T_{-\rho} \circ \psi$, where $T_{-\rho}$ denotes translation by $-\rho$ on the generic fiber $(E', O) = (X'_\eta, \sigma'(\eta))$ that by the uniqueness of the relatively minimal models extends to an automorphism of X' [Liu, Proposition 3.13]. By definition we have

$$\psi^*(2[\sigma'] + [\tau']) \sim 2[\sigma] + [\tau] + f^* D_C$$

and this is equivalent to

$$\psi_*\left(\psi^*(2[\sigma'] + [\tau'])\right) \sim \psi_*(2[\sigma] + [\tau] + f^* D_C)$$

that as $\psi_* \circ \psi^* = \mathrm{id}$ and $f = f' \circ \psi$ is equivalent to

$$2[\sigma'] + [\tau'] \sim 2[\psi \circ \sigma] + [\psi \circ \tau] + f'^* D_C.$$

The way we have defined ρ and φ leads to

$$2[\rho] + [\varphi \circ \tau +_{E'} \rho] \sim 2[\sigma'] + [\tau'] - f'^* D_C \tag{4.9}$$

and by restricting it to the generic fibre $(E', O') = (X'_\eta, \sigma'(\eta))$ we obtain the equality

$$3\rho = -(\varphi \circ \tau) + \tau'$$

which is exactly the condition (b) for equivalence in $\mathcal{C}_{C,\omega}$.

Now we claim, that ρ and σ' meet the same irreducible component of each fiber X'_c of f'. In a way of contradiction assume that Γ is an irreducible component of a fiber X'_c of f' such that ρ meets Γ and σ' does not meet Γ. Then we have

$$(2[\rho] + [\varphi \circ \tau +_{E'} \rho]).\Gamma \geq 2 \text{ and } (2[\sigma'] + [\tau']).\Gamma \leq 1,$$

however this is a contradiction since the rational equivalence of divisors (4.9) holds. Therefore ρ and σ' meet the same irreducible component of each fiber.

We have proven, that (φ, ρ) satisfies properties $(a), (b)$ and (c) of Definition 4.3.2 and hence is an equivalence between $(f : X \to C, \sigma, \tau)$ and $(f' : X' \to C, \sigma', \tau')$.

$'' \supseteq''$ Suppose we are given an equivalence (φ, ρ) (Definition 4.3.2) between $(f : X \to C, \sigma, \tau)$ and $(f' : X' \to C, \sigma', \tau')$. By definition there exists an isomorphism $\psi : X \to X'$ over C and a section $\rho : C \to X'$ such that

(a) $\varphi \circ \sigma = \sigma'$

(b) on the generic fibre $(E', O') = (X'_\eta, \sigma'(\eta))$ we have $3\rho = \tau' - \varphi \circ \tau$ (addition in the group law of E')

(c) for every closed point $c \in C$ the sections ρ and σ' meet the same irreducible component of X'_c.

We set $\psi = T_\rho \circ \varphi$ and claim that this an equivalence between $(f : X \to C, D)$ and $(f' : X' \to C, D')$. We need to show that there exists a divisor $D_C \in \mathrm{Div}(C)$ such that

$$\psi^* D' \sim D + f^* D_C. \tag{4.10}$$

As $D = 2[\sigma] + [\tau]$ and $D' = 2[\sigma'] + [\tau]$ this is equivalent to

$$\psi^*(2[\sigma'] + [\tau']) \sim 2[\sigma] + [\tau] + f^* D_C$$

that by applying ψ_* is equivalent to

$$2[\sigma'] + [\tau'] \sim 2[\psi \circ \sigma] + [\psi \circ \tau] + f'^* D_C.$$

By condition (b) for an equivalence in $\mathcal{C}_{C,\omega}$ on the generic fiber $(E', O') = (X'_\eta, \sigma'(\eta))$ we have

$$\tau' = 2\rho + \varphi \circ \tau + \rho.$$

therefore there is a divisor F' supported on fibers of the morphism $f' : X' \to C$ such that

$$2[\sigma'] + [\tau'] \sim 2[\sigma' +_{E'} \rho] + [\varphi \circ \tau +_{E'} \rho] + F'. \tag{4.11}$$

As $\psi = T_\rho \circ \varphi$ the divisor F' is our candidate for $f'^* D_C$.

First of all we claim that for any irreducible component Γ of a fiber X'_c the intersection number of Γ with either side of (4.11) is the same. By condition (c) of equivalence in $\mathcal{C}_{C,\omega}$ the section ρ lies in the connected component of identity σ. Therefore translation by ρ maps the irreducible components of each reducible fibre of $f' : X' \to C$ to itself. Hence

$$\sigma' +_{E'} \rho \text{ and } \sigma'$$

meet the same irreducible component and

$$\tau' = \varphi \circ \tau +_{E'} 2\rho \text{ and } \varphi \circ \tau +_{E'} \rho$$

meet the same irreducible component, and so the claim ifollows.

This implies, that $F'^2 = 0$ and by [Sil2, Chapter III, Proposition 8.2(b)] there exist $a, m \in \mathbb{Z}$ such that $aF' = mf'^* G_C$ for some $G_C \in \mathrm{Div}(C)$. Furthermore as $f' : X' \to C$ does not have multiple fibers (it is a minimal elliptic surface with a section see: Chapter 2 Section 2.2) the equality $a = 1$ must hold. Hence there exists $D_C \in \mathrm{Div}(C)$ such that the rational equivalence (4.10) holds. Consequently $(f : X \to C, D)$ and $(f' : X' \to C, D')$ are equivalent. $\square$

Using Proposition 4.4.8 and Proposition 4.4.9 we conclude that

$$C_{C,\omega} \leq B_{C,\omega}. \tag{4.12}$$

4.5 $\mathcal{C}_{C,\omega}$ and Mordell Weil groups

Let (C, O) be an elliptic curve over $k = \mathbb{F}_q$ and let $k(C)$ be the function field of C over k. For an elliptic curve (E, O) over $k(C)$ of height $d \geq 0$ we write $E(k(C))$ for the group of $k(C)$-rational points of E and we call $E(k(C))$ the Mordell-Weil group of (E, O) over $k(C)$.

Néron proved [Ner] that the group $E(k(C))$ is a finitely generated abelian group. Let (Torsion) denotes the subgroup of $E(k(C))$ of elements that have finite order. Then any automorphism $\varphi \in \operatorname{Aut}(E, O)$ induces an automorphism φ of $E(k(C))$ that fixes the subgroup (Torsion) and hence induces an automorphism

$$\overline{\varphi} : {}^{E(k(C))}\!/_{(\text{Torsion})} \to {}^{E(k(C))}\!/_{(\text{Torsion})}.$$

Furthermore the quotient

$${}^{E(k(C))}\!/_{(\text{Torsion})}$$

is a free $\mathbb{Z}$ module of finite rank and therefore $\overline{\varphi}$ induces an automorphism $\overline{\varphi}_3$ of the free ${}^{\mathbb{Z}}\!/_{3\mathbb{Z}}$ module

$$\overline{\varphi}_3 : {}^{E(k(C))}\!/_{(\text{Torsion})} \otimes_{\mathbb{Z}} {}^{\mathbb{Z}}\!/_{3\mathbb{Z}} \to {}^{E(k(C))}\!/_{(\text{Torsion})} \otimes_{\mathbb{Z}} {}^{\mathbb{Z}}\!/_{3\mathbb{Z}}.$$

In this way we obtain an action

$$\operatorname{Aut}(E, O) \times V_E \to V_E$$

by mapping

$$(\varphi, P) \longmapsto \overline{\varphi}_3(P). \tag{4.13}$$

where for an elliptic curve (E, O) over $k(C)$ by V_E we denote the $\mathbb{F}_3$ vector space

$$V_E = {}^{E(k(C))}\!/_{(\text{Torsion})} \otimes_{\mathbb{Z}} {}^{\mathbb{Z}}\!/_{3\mathbb{Z}}. \tag{4.14}$$

Moreover we denote by $r(E)$ the rank of the $\mathbb{Z}$-module ${}^{E(k(C))}\!/_{(\text{Torsion})}$ and call it the rank of the Mordell-Weil group of (E, O) over $k(C)$. Observe, that

$$\dim_{\mathbb{F}_3} V_E = r(E).$$

Let us fix an integer $d \geq 0$ and a line bundle $\omega \in \operatorname{Pic}^d(C)$. Let (E, O) be an elliptic curve over $k(C)$ of height d such that for its minimal model $(f : \mathcal{E} \to C, \sigma)$ we have $f_* \omega_{\mathcal{E}/C} = \omega$. Furthermore let $r(E)$ denotes the rank of the Mordell-Weil group of (E, O) over $k(C)$. For $v \in V_E \smallsetminus \{0\}$ let $P_v \in E(k(C))$ be any lift of v and denote by τ_v the section of $f : \mathcal{E} \to C$ corresponding to the $k(C)$-rational point P_v then the triple

$$\mathcal{E}_v := (f : \mathcal{E} \to C, \sigma, \tau_v) \tag{4.15}$$

satisfies conditions (a),(b) and (c) of Definition 4.3.1 and as the point

$$\tau_v|_{(\mathcal{E}_\eta, \sigma(\eta)) = (E,O)} = P_v$$

is not divisible by 3 in $E(k(C))$ hence $\mathcal{E}_v$ is non-degenerate in the sense of Definition 4.3.4 and we denote by $[\mathcal{E}_v] = [(f : \mathcal{E} \to C, \sigma, \tau_v)]$ the equivalence class of $\mathcal{E}_v$ in $\mathcal{C}_{C,\omega}$.

Remark 4.5.1. The section τ_v obviously depends on the choice of P_v however as we will see later we are only interested in counting and results we prove are independent of which lift we choose.

Proposition 4.5.2. *For $d \geq 0$ let $\omega \in \operatorname{Pic}^d(C)$ and let (E, O) be an elliptic curve over $k(C)$ with the minimal model $(f : \mathcal{E} \to C, \sigma)$ such that $f_* \omega_{\mathcal{E}/C} = \omega$. Furthermore consider the action of* $\operatorname{Aut}(E, O)$ *on V_E. We keep the notation as in (4.14) and (4.15).*

(i) The number of equivalence classes of $\mathcal{E}_v$ *in* $\mathcal{C}_{C,\omega}$ *is at least the number of orbits of* $\mathrm{Aut}(E,O)$ *on* $V_E \smallsetminus \{0\}$.

(ii) For $d \geq 1$ *the number* $\#\mathrm{Aut}(\mathcal{E}_v)$ *of automorphisms of* $\mathcal{E}_v$ *is at most the number of elements in the stabilizer* $\mathrm{Stab}(v)$ *of* v *under* $\mathrm{Aut}(E,O)$.

(iii) Assume that $d \geq 1$ *then*

$$\sum_{\substack{[\mathcal{E}_v]\in\mathcal{C}_{C,\omega} \\ v\in V_E\smallsetminus\{0\}}} \frac{1}{\#\mathrm{Aut}(\mathcal{E}_v)} \geq \frac{3^{r(E)}-1}{\#\mathrm{Aut}(E,O)}. \tag{4.16}$$

Proof. (i)For $v,w \in V_E$ such that $w \neq 0$ and $v \neq 0$ choose lifts P_v and P_w in $E(k(C))$. Let $\mathcal{E}_v = (f : \mathcal{E} \to C, \sigma, \tau_v)$ (resp. $\mathcal{E}_w = (f : \mathcal{E} \to C, \sigma, \tau_w)$) be the associated triple and $[(\mathcal{E}_v)]$ (resp $[(\mathcal{E}_w)]$) its equivalence class in $\mathcal{C}_{C,\omega}$. Assume furthermore, that $[(\mathcal{E}_v)] = [(\mathcal{E}_w)]$ then by definition (see: Definition (4.3.2)) there exists an automorphism $\varphi \in \mathrm{Aut}(\mathcal{E},\sigma)$ and a section $\rho : C \to \mathcal{E}$ such that on the generic fiber $(\mathcal{E}_\eta, \sigma(\eta)) = (E,O)$ we have

$$3\rho = \tau_v - \varphi \circ \tau_w.$$

Then the identifications $\mathrm{Aut}(\mathcal{E},\sigma) = \mathrm{Aut}(E,O)$, $\tau_v\big|_{(E,O)} = P_v$ and $\tau_w\big|_{(E,O)} = P_w$ lead to the equality

$$\varphi(P_v) = P_w + 3\rho.$$

It is clear then that under the given asction the pair (φ, v) is mapped to w. Hence $w \in \mathrm{Orb}(v)$ which means that $\mathrm{Orb}(v) = \mathrm{Orb}(w)$ and the first claim is proven.

Now we will prove the statement (ii). Let $v \in V_E$ be such that $v \neq 0$, let $P_v \in E(k(C))$ be a lift of v and let $\mathcal{E}_v$ be the corresponding triple. Then

$$\mathrm{Aut}(\mathcal{E}_v) = \Big\{(\varphi,\rho) \mid \varphi \in \mathrm{Aut}(\mathcal{E},\sigma), \rho \in \mathcal{E}(C) \text{ with } \varphi\circ\tau_v = \tau_v + 3\rho \text{ on } (\mathcal{E}_\eta, \sigma(\eta)) \text{ and } (*)\Big\},$$

where

$$(*) \forall_{c\in|C|} \quad \sigma \text{ and } \rho \text{ meet the same irreducible component of each } \mathcal{E}_c$$

Identify $\mathrm{Aut}(\mathcal{E},\sigma) = \mathrm{Aut}(E,O)$ and $\tau_v\big|_{(E,O)} = P_v$, then by forgetting $(*)$ we obtain a map

$$\mathrm{Aut}(\mathcal{E}_{\overline{a}}) \to \{\varphi \in \mathrm{Aut}(E,O) \mid \varphi(P_v) = P_v + 3\rho \text{ for some } \rho \in E(k(C))\} \tag{4.17}$$

$$(\varphi,\rho) \longmapsto \varphi.$$

Both sets in (4.17) form a group and the map (4.17) is a group homomorphism. The kernel of (4.17) is

$$\{(\varphi,\rho) \mid \varphi = \mathrm{id}_{\mathcal{E}},\ \ 3\rho = \sigma \ \text{ on } (\mathcal{E}_\eta,\sigma(\eta)) \text{ and } \rho \ \text{ lies in the connected component of the identity } \sigma\}.$$

From Lemma 4.5.3 below follows that the kernel is trivial and therefore the map (4.17) is injective. Furthermore it clear that

$$\{\varphi \in \mathrm{Aut}(E,O) \mid \varphi(P_v) = P_v + 3\rho \text{ for some } \rho \in E(k(C))\} \subseteq \mathrm{Stab}(v),$$

where $\mathrm{Stab}(v)$ denotes the stabilizer of v under the action of $\mathrm{Aut}(E,O)$. This proves the claim (ii).

For the last case (iii) observe, that

$$\sum_{v^*} \frac{1}{\#\mathrm{Stab}(v^*)} = \frac{3^{r(E)}-1}{\#\mathrm{Aut}(E,O)}.$$

where the sum is taken over representatives v^* of orbits in the quotient

$$\left(V_E \setminus \{0\}\right) \Big/ \mathrm{Aut}(E,O).$$

Furthermore by (i) and (ii) follows, that

$$\sum_{v^*} \frac{1}{\#\mathrm{Stab}(v^*)} \leq \sum_{\substack{[\mathcal{E}_v] \in \mathcal{C}_{C,\omega} \\ v \in V_E \setminus \{0\}}} \frac{1}{\#\mathrm{Aut}(\mathcal{E}_v)},$$

That proves the remaining statement. □

Lemma 4.5.3. *[deJ02, Lemma 5.15] Let (E,O) be an elliptic curve over C. Let P be a rational torsion point of E which extends to a section of the connected component of the Néron model $\mathcal{N}$ of E. Then $P = O$ or the height of E is $d = 0$.*

Proof. The proof in [deJ02, Lemma 5.15] is stated only for elliptic curves over $k(\mathbb{P}^1)$ however it works word by word also for elliptic curves over $k(C)$. □

Remark 4.5.4. We will show, that for non-isomorphic elliptic curves equivalence classes of associated triples in $\mathcal{C}_{C,\omega}$ are disjoint.

We keep the assumptions on (E,O) and let (E',O') be another elliptic curve over $k(C)$ with the minimal model $(f' : \mathcal{E}' \to C, \sigma')$ and $f_*\omega_{\mathcal{E}'/C} = \omega$ such that $(E',O') \ncong (E,O)$. Let $v \in V_E$ and $v' \in V_{E'}$ be such that $v \neq 0$ and $v' \neq 0$ and let $P_v \in E(k(C))$ be a lift of v and $P'_{v'} \in E'(k(C))$ a lift of v'. Then $\mathcal{E}'_{v'} := (f' : \mathcal{E}' \to C, \sigma', \tau'_{v'})$ and $\mathcal{E}_v := (f : \mathcal{E} \to C, \sigma, \tau_v)$ can not be equivalent because $f : \mathcal{E} \to C$ and $f' : \mathcal{E}' \to C$ are not isomorphic. Therefore they have disjoint equivalence classes in $\mathcal{C}_{C,\omega}$.

Remark 4.5.5. Let (E,O) be an elliptic curve over $k(C)$ and consider the action from (4.13). We will show, that the given action induces an action for each elliptic curve in the isomorphism class of (E,O) that these actions are compatible and furthermore that the weighted numbers of associated triples with respect to these actions are equal.

We keep the assumptions on (E,O) and let (E',O') be another elliptic curve over $k(C)$ with the minimal model $(f' : \mathcal{E}' \to C, \sigma')$ such that $f'_*\omega_{\mathcal{E}'/C} = \omega$. We assume furthermore, that they are isomorphic and let $\alpha : E \to E'$ be an isomorphism over $k(C)$ such that $\alpha(O) = O'$ then α induces an isomorphism

$$\overline{\alpha}_3 : V_E \to V_{E'}$$

(we keep the notation from (4.14)) and an isomorphism of automorphism groups

$$\begin{aligned} \mathrm{Aut}(E,O) &\longrightarrow \mathrm{Aut}(E',O') \\ \varphi &\longmapsto \varphi' := \alpha \circ \varphi \circ \alpha^{-1}. \end{aligned}$$

For $\varphi' = \alpha \circ \varphi \circ \alpha^{-1}$ and $v' = \overline{\alpha}_3(v)$ with $v \in V_E$ we have an induced action given by

$$\begin{aligned} \mathrm{Aut}(E,O) \times V_{E'} &\longrightarrow V_{E'} \\ (\varphi', v') &\longmapsto \overline{\varphi'}_3(v'). \end{aligned}$$

It is not hard to show, that these actions are compatible i.e. $\overline{\varphi'}_3(v') = \overline{\alpha}_3(\overline{\varphi}_3(v))$ (notation as in (4.13)).

Now we show, that the number

$$\sum_{\substack{[\mathcal{E}_v]\in\mathcal{C}_{C,\omega}\\ v\in V_E\smallsetminus\{0\}}} \frac{1}{\#\mathrm{Aut}(\mathcal{E}_v)}$$

is well defined in the isomorphism class of $(f:\mathcal{E}\to C,\sigma)$. For that we assume (E,O) to be an elliptic curve over $k(C)$ with the minimal model $(f:\mathcal{E}\to C,\sigma)$ such that $f_*\omega_{\mathcal{E}/C}=\omega$. Recall that

$$[(f:\mathcal{E}\to C,\sigma)] = \{(f':\mathcal{E}'\to C,\sigma') \mid \exists\alpha:\mathcal{E}\xrightarrow{\simeq}\mathcal{E}' \text{ over } C \text{ with } \alpha\circ\sigma=\sigma'\}.$$

Let us choose P_v to be lifts of $v\in V_E\smallsetminus\{0\}$ to $E(k(C))$ and let τ_v be sections of $f:\mathcal{E}\to C$ such that $\tau_v|_{(E,O)}=P_v$. Consider the map

$$[(f:\mathcal{E}\to C,\sigma)]\longrightarrow\mathbb{Z}$$

given by

$$(f':\mathcal{E}'\to C,\sigma')\longmapsto \sum_{\substack{[(f':\mathcal{E}'\to C,\sigma',\alpha\circ\tau_v)]\in\mathcal{C}_{C,\omega}\\ v\in V_E\smallsetminus\{0\}}} \frac{1}{\#\mathrm{Aut}(f':\mathcal{E}'\to C,\sigma',\alpha\circ\tau_v)}. \tag{4.18}$$

where α is any isomorphism between $(f:\mathcal{E}\to C,\sigma)$ and $(f:\mathcal{E}'\to C,\sigma')$.

First of all this map is well defined, indeed if $\beta:\mathcal{E}\to\mathcal{E}'$ is another isomorphism over C such that $\beta\circ\sigma=\sigma'$ then for every $v\in V_E\smallsetminus\{0\}$ the pair $(\alpha^{-1}\circ\beta,\sigma')$ is an equivalence between $(f':\mathcal{E}'\to C,\sigma',\alpha\circ\tau_v)$ and $(f':\mathcal{E}'\to C,\sigma',\beta\circ\tau_v)$ in the sense of Definition 4.3.2 and therefore

$$\sum_{\substack{[(f':\mathcal{E}'\to C,\sigma',\alpha\circ\tau_v)]\in\mathcal{C}_{C,\omega}\\ v\in V_E\smallsetminus\{0\}}} \frac{1}{\#\mathrm{Aut}(f':\mathcal{E}'\to C,\sigma',\alpha\circ\tau_v)} = \sum_{\substack{[(f':\mathcal{E}'\to C,\sigma',\beta\circ\tau_v)]\in\mathcal{C}_{C,\omega}\\ v\in V_E\smallsetminus\{0\}}} \frac{1}{\#\mathrm{Aut}(f':\mathcal{E}'\to C,\sigma',\beta\circ\tau_v)}.$$

Furthermore this map is constant. Indeed, for $v\in V_E\smallsetminus\{0\}$ the pair (α,σ) is an equivalence between the triples $(f:\mathcal{E}\to C,\sigma,\tau_v)$ and $(f':\mathcal{E}'\to C,\sigma',\alpha\circ\tau_v)$ in the sense of Definition 4.3.2 hence

$$\sum_{\substack{[(f':\mathcal{E}'\to C,\sigma',\alpha\circ\tau_v)]\in\mathcal{C}_{C,\omega}\\ v\in V_E\smallsetminus\{0\}}} \frac{1}{\#\mathrm{Aut}(f':\mathcal{E}'\to C,\sigma',\alpha\circ\tau_v)} = \sum_{\substack{[(f:\mathcal{E}\to C,\sigma,\tau_v)]\in\mathcal{C}_{C,\omega}\\ v\in V_E\smallsetminus\{0\}}} \frac{1}{\#\mathrm{Aut}(f:\mathcal{E}\to C,\sigma,\tau_v)}$$

and the claim is proven.

Theorem 4.5.6. *Let (C,O) be an elliptic curve over a finite field $\mathbb{F}_q$. Let $d\geq 1$ and let $\mathrm{C}_{C,\omega}$ be the weighted number of elements of the set $\mathcal{C}_{C,\omega}$ (see: Section 4.3). Then*

$$\sum_{[(E,O)],h(E)=d} \frac{3^{r(E)}-1}{\#\mathrm{Aut}(E,O)} \leq \sum_{\omega\in\mathrm{Pic}^d(C)} \mathrm{C}_{C,\omega},$$

where the sum is taken over isomorphism classes of elliptic curves over $\mathbb{F}_q(C)$ of height d and $r(E)$ denotes the Mordell-Weil rank of (E,O) over $\mathbb{F}_q(C)$.

Proof. By Chapter 2, Remark 2.1.12 and Proposition 2.1.6 we have

$$\sum_{[(E,O)];h(E)=d} \frac{3^{r(E)}-1}{\#\mathrm{Aut}(E,O)} = \sum_{\substack{[(f:\mathcal{E}\to C,\sigma)]\\ h(\mathcal{E})=d}} \frac{3^{r(\mathcal{E}_\eta)}-1}{\#\mathrm{Aut}(\mathcal{E}_\eta,\sigma(\eta))},$$

where the sum on the LHS is taken over isomorphism classes of elliptic curves (E,O) over $\mathbb{F}_q(C)$ of height d and the sum on the RHS is taken over isomorphism classes of minimal elliptic surfaces $(f:\mathcal{E}\to C,\sigma)$ over k of height d. Furthermore as a consequence of Lemma 4.5.7 below the RHS can be rewritten as

$$\sum_{\substack{[(f:\mathcal{E}\to C,\sigma)]\\ h(\mathcal{E})=d}} \frac{3^{r(\mathcal{E}_\eta)}-1}{\#\mathrm{Aut}(\mathcal{E}_\eta,\sigma(\eta))} = \sum_{\omega\in\mathrm{Pic}^d(C)} \sum_{\substack{[(f:\mathcal{E}\to C,\sigma)]\\ f_*\omega_{\mathcal{E}/C}\cong\omega}} \frac{3^{r(\mathcal{E}_\eta)}-1}{\#\mathrm{Aut}(\mathcal{E}_\eta,\sigma(\eta))}.$$

where we first sum over isomorphism classes of line bundles on C of degree d and the index $\substack{[(f:\mathcal{E}\to C,\sigma)]\\ f_*\omega_{\mathcal{E}/C}\cong\omega}$ will always mean that we sum over isomorphism classes of minimal elliptic surfaces $(f:\mathcal{E}\to C,\sigma)$ over k with $f_*\omega_{\mathcal{E}/C}\cong\omega$.

Fix an $\omega\in\mathrm{Pic}^d(C)$ and let $f:\mathcal{E}\to C$ be a minimal elliptic surface with a section σ such that $f_*\omega_{\mathcal{E}/C}=\omega$ and for each $v\in V_{\mathcal{E}_\eta}\setminus\{0\}$ choose a lift $P_v\in\mathcal{E}_\eta(k(C))$ (notation as in (4.14)). Proposition 4.5.2 imply, that

$$\sum_{\substack{[(f:\mathcal{E}\to C,\sigma,\tau_v)]\in\mathcal{C}_{C,\omega}\\ v\in V_{\mathcal{E}_\eta}\setminus\{0\}}} \frac{1}{\#\mathrm{Aut}(f:\mathcal{E}\to C,\sigma,\tau_v)} \geq \frac{3^{r(\mathcal{E}_\eta)}-1}{\#\mathrm{Aut}(\mathcal{E}_\eta,\sigma(\eta))},$$

furthermore in Remark 4.5.5 we proved, that the number

$$\sum_{\substack{[(f:\mathcal{E}\to C,\sigma,\tau_v)]\in\mathcal{C}_{C,\omega}\\ v\in V_{\mathcal{E}_\eta}\setminus\{0\}}} \frac{1}{\#\mathrm{Aut}(f:\mathcal{E}\to C,\sigma,\tau_v)}$$

is well defined in the isomorphism class of $(f:\mathcal{E}\to C,\sigma)$ and therefore we can take the sum

$$\sum_{\substack{[(f:\mathcal{E}\to C,\sigma)]\\ f_*\omega_{\mathcal{E}/C}\cong\omega}} \sum_{\substack{[(f:\mathcal{E}\to C,\sigma,\tau_v)]\in\mathcal{C}_{C,\omega}\\ v\in V_{\mathcal{E}_\eta}\setminus\{0\}}} \frac{1}{\#\mathrm{Aut}(f:\mathcal{E}\to C,\sigma,\tau_v)} \geq \sum_{\substack{[(f:\mathcal{E}\to C,\sigma)]\\ f_*\omega_{\mathcal{E}/C}\cong\omega}} \frac{3^{r(\mathcal{E}_\eta)}-1}{\#\mathrm{Aut}(\mathcal{E}_\eta,\sigma(\eta))}.$$

On the other hand recall (see: Definition 4.3) that

$$\mathrm{C}_{C,\omega} := \sum_{[(f:X\to C,\sigma,\tau)]\in\mathcal{C}_{C,\omega}} \frac{1}{\#\mathrm{Aut}(f:X\to C,\sigma,\tau)}$$

then Remark 4.5.4 gives

$$\mathrm{C}_{C,\omega} \geq \sum_{\substack{[(f:\mathcal{E}\to C,\sigma)]\\ f_*\omega_{\mathcal{E}/C}\cong\omega}} \sum_{\substack{[(f:\mathcal{E}\to C,\sigma,\tau_v)]\in\mathcal{C}_{C,\omega}\\ v\in V_{\mathcal{E}_\eta}\setminus\{0\}}} \frac{1}{\#\mathrm{Aut}(f:\mathcal{E}\to C,\sigma,\tau_v)}.$$

And as a consequence of the above inequalities

$$\sum_{\substack{[(f:\mathcal{E}\to C,\sigma)]\\ f_*\omega_{\mathcal{E}/C}\cong\omega}} \frac{3^{r(\mathcal{E}_\eta)}-1}{\#\mathrm{Aut}(\mathcal{E}_\eta,\sigma(\eta))} \leq \mathrm{C}_{C,\omega}.$$

To finish the proof we sum over $\omega\in\mathrm{Pic}^d(C)$ and obtain

$$\sum_{\omega\in\mathrm{Pic}^d(C)} \sum_{\substack{[(f:\mathcal{E}\to C,\sigma)]\\ f_*\omega_{\mathcal{E}/C}\cong\omega}} \frac{3^{r(\mathcal{E}_\eta)}-1}{\#\mathrm{Aut}(\mathcal{E}_\eta,\sigma(\eta))} \leq \sum_{\omega\in\mathrm{Pic}^d(C)} \mathrm{C}_{C,\omega}.$$

Hence

$$\sum_{[(E,O)];h(E)=d} \frac{3^{r(E)}-1}{\#\mathrm{Aut}(E,O)} = \sum_{\omega\in\mathrm{Pic}^d(C)} \sum_{\substack{[(f:\mathcal{E}\to C,\sigma)]\\ f_*\omega_{\mathcal{E}/C}\cong\omega}} \frac{3^{r(\mathcal{E}_\eta)}-1}{\#\mathrm{Aut}(\mathcal{E}_\eta,\sigma(\eta))} \leq \sum_{\omega\in\mathrm{Pic}^d(C)} \mathrm{C}_{C,\omega}.$$

That finishes the proof of the theorem. □

Lemma 4.5.7. *If $\omega_1,\omega_2 \in \mathrm{Pic}^d(C)$ are such that $\omega_1 \ncong \omega_2$, then $\mathcal{C}_{C,\omega_1} \cap \mathcal{C}_{C,\omega_2} = \varnothing$.*

Proof. To prove it we assume in a way of contradiction, that $\mathcal{C}_{C,\omega_1} \cap \mathcal{C}_{C,\omega_2} \neq \varnothing$, then there exists a pair $(f:\mathcal{E}\to C,\sigma)$ such that $\omega_1 \cong f_*\omega_{\mathcal{E}/C} \cong \omega_2$, which contradicts the assumption on ω_i's. □

Lemma 4.5.8. *Assume ω_1,ω_2 are line bundles on C of degree d. Then*

$$\omega_1 \ncong \omega_2 \Longrightarrow \mathcal{B}_{C,\omega_1} \cap \mathcal{B}_{C,\omega_2} = \varnothing$$

and

$$\omega_1 \ncong \omega_2 \Longrightarrow \mathcal{A}_{C,\omega_1} \cap \mathcal{A}_{C,\omega_2} = \varnothing.$$

Proof. Assume $\mathcal{A}_{C,\omega_1} \cap \mathcal{A}_{C,\omega_2} \neq \varnothing$. Then there exists a pair $(g:Y\to C,D)$ (see Definition 4.1.1), such that

$$g^*\omega_1 \cong g^*\omega_2.$$

However, then

$$\begin{aligned} g_*g^*\omega_1 &\cong g_*g^*\omega_2 \\ \Rightarrow g_*(g^*\omega_1\otimes\mathcal{O}_Y) &\cong g_*(g^*\omega_2\otimes\mathcal{O}_Y) \\ \Rightarrow \omega_1\otimes g_*\mathcal{O}_Y &\cong \omega_2\otimes g_*\mathcal{O}_Y \quad (\text{ by projection formula}) \\ \Rightarrow \omega_1 &\cong \omega_2 \quad (\text{ as } g_*\mathcal{O}_Y\cong\mathcal{O}_C) \end{aligned}$$

that contradicts the assumption on ω_i's and proves the claim. The same argument works for elements of $\mathcal{B}_{C,\omega}$. □

Using Lemma 4.5.8 for $d\geq 0$ we define

$$\mathcal{A}_{C,d} := \bigsqcup_{\omega\in\mathrm{Pic}^d(C)} \mathcal{A}_{C,\omega} \tag{4.19}$$

and denote by $\mathrm{A}_{C,d}$ its number of elements. Then

$$\mathrm{A}_{C,d} = \sum_{\omega\in\mathrm{Pic}^d(C)} \mathrm{A}_{C,\omega},$$

where $\mathrm{A}_{C,\omega}$ is the weighted number of elements in $\mathcal{A}_{C,\omega}$ (see Section 4.1)

Theorem 4.5.9. *Let (C,O) be an elliptic curve over a finite field $\mathbb{F}_q$. Let $d\geq 1$ and let $\mathrm{A}_{C,d}$ be the weighted number of elements in $\mathcal{A}_{C,d}$ (as in (4.19)). Then*

$$\sum_{[(E,O)],h(E)=d} \frac{3^{r(E)}-1}{\#\mathrm{Aut}(E,O)} \leq \mathrm{A}_{C,d},$$

where the sum is taken over isomorphism classes of elliptic curves over $\mathbb{F}_q(C)$ of height d and $r(E)$ denotes the Mordell-Weil rank of (E,O) over $\mathbb{F}_q(C)$.

Proof. The bound is a consequence of Theorem 4.5.6 and the inequalities

$$C_{C,\omega} \leq B_{C,\omega} \leq A_{C,\omega},$$

obtained in Section 4.4; (4.7) and (4.12). $\square$

Theorem 4.5.10. *Let (C, O) be an elliptic curve over a finite field $\mathbb{F}_q$ of characteristic $p > 7$. Let $\#C(\mathbb{F}_{q^n})$ denote the number of $\mathbb{F}_{q^n}$-rational points of C. Then*

$$\limsup_{d\to\infty} q^{-10d+1}\left(\sum_{[(E,O)];h(E)=d} \frac{3^{r(E)}-1}{\#\mathrm{Aut}(E,O)}\right) \leq \frac{\#C(\mathbb{F}_q)A}{(q-1)^5(q+1)^2q},$$

where the sum is taken over isomorphism classes of elliptic curves (E, O) over $\mathbb{F}_q(C)$ of height d and $r(E)$ denotes the Mordell-Weil rank of (E, O) over $\mathbb{F}_q(C)$, furthermore A is given by

$$A = 2\#C(\mathbb{F}_q)q^8 + \left(2\#C(\mathbb{F}_q)^2 + \#C(\mathbb{F}_{q^2})^2\right)q^7$$
$$+\left(\#C(\mathbb{F}_q)\#C(\mathbb{F}_{q^2}) + \#C(\mathbb{F}_{q^3})^3 + 1 - 2\#C(\mathbb{F}_q) + \#C(\mathbb{F}_{q^2})^2\right)q^6 + \left(-2 - 2\#C(\mathbb{F}_{q^2})^2 - 4\#C(\mathbb{F}_q)^2\right)q^5$$
$$+\left(-2\#C(\mathbb{F}_{q^3})^3 - 1 - 2\#C(\mathbb{F}_q) - 2\#C(\mathbb{F}_{q^2})^2 - 2\#C(\mathbb{F}_q)\#C(\mathbb{F}_{q^2})\right)q^4$$
$$+\left(\#C(\mathbb{F}_{q^2})^2 + 2\#C(\mathbb{F}_q)^2 + 4\right)q^3$$
$$+\left(\#C(\mathbb{F}_{q^3})^3 + \#C(\mathbb{F}_{q^2})^2 + \#C(\mathbb{F}_q)^2 - 1 + 2\#C(\mathbb{F}_q) + \#C(\mathbb{F}_q)\#C(\mathbb{F}_{q^2})\right)q^2 - 2q + 1$$

Proof. As we are interested only in the asymptotic limit we may assume that $d \geq 1$. Then the proof of the above bound follows from the inequality

$$\limsup_{d\to\infty} q^{-10d+1}\left(\sum_{[(E,O)];h(E)=d} \frac{3^{r(E)}-1}{\#\mathrm{Aut}(E,O)}\right) \leq \limsup_{d\to\infty} q^{-10d+1}\mathrm{A}_{C,d} \leq \frac{\#C(\mathbb{F}_q)A}{(q-1)^5(q+1)^2q},$$

which is a consequence of Theorem 4.5.9 and Proposition 3.3.7 from Section 3.3. $\square$

Remark 4.5.11. The assumption that $p > 7$ is made for computational reasons for the bound from Proposition 3.3.7 to hold.

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Zusammenfassung

Die rationalen Punkte auf elliptischen Kurven gehören zu den wichtigsten Objekten der arithmetischen Theorie. Ein Messwert für die Größe der Gruppe der rationalen Punkte einer elliptischen Kurve ist ihr Rang. Dieser ist definiert, als die minimale Anzahl an rationalen Punkten auf der gegebenen elliptischen Kurve, die benötigt werden, um all solche Punkte zu erzeugen.

Es gibt bereits viele Heuristiken und Vermutungen bezüglich des Ranges, doch es ist keine allgemeine Methode bekannt, ihn zu bestimmen. Um das Problem zu vereinfachen, stellt man sich folgende Frage: Wie verhält sich der Rang im Durchschnitt? Ordnet man die über den rationalen Zahlen definierten elliptischen Kurven nach ihrer Höhe, so besagt ein erst kürzlich erzieltes Resultat von Bhargava und Shankar [BS10], dass der Rang nach oben durch 1,5 beschränkt ist.

Selbstverständlich würde man auch gerne in anderen Fällen eine ähnliche Aussage erlangen, zum Beispiel im Falle eines Funktionenkörpers. Man betrachte eine glatte, geometrisch zusammenhängende, projektive Kurve über einem endlichen Körper und ordne die elliptischen Kurven über dem Funktionenkörper dieser gegebenen Kurve nach ihrer Höhe. In [deJ02] gibt de Jong eine obere Schranke für den durchschnittlichen Rang elliptischer Kurven über dem Funktionenkörper der projektiven Geraden an. Für die Berechnung dieser Schranke, zählt er integre Modelle geometrischer Objekte, denn diese repräsentieren Elemente gewisser Gruppen.

In erster Linie beschäftigt sich diese Dissertation mit elliptischen Kurven über Funktionenkörpern elliptischer Kurven und liefert Fortschritte über Schranken für den durchschnittlichen Rang solcher Kurven. Dabei wird hauptsächlich de Jongs Methode verwendet.

Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, die vorliegende Arbeit selbstständig verfasst zu haben. Alle verwendeten Hilfsmittel sind aufgeführt. Des Weiteren versichere ich, dass diese Arbeit nicht in dieser oder ähnlicher Form an einer weiteren Universität im Rahmen eines Prüfungsverfahrens eingereicht wurde.

Anna Fluder