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Stability and Moduli of Decorated Principal Bundles on Projective Schemes

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Hiermit versichere ich, diese Dissertation selbständig verfasst sowie alle verwendeten Hilfsmittel und Hilfen angegeben zu haben. Desweiteren habe ich diese Arbeit nicht in einem früheren Promotionsverfahren eingereicht.

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Introduction

In the last few decades, theoretical physics has provided a significant input to mathematical research. In this thesis, we will investigate two objects whose appeareance in algebraic geometry was influenced by physical theories, namely instanton bundles and Higgs bundles.

Instanton Bundles. Originally instantons were defined by physicists as self-dual solutions of the Yang-Mills equations. Mathematically, an instanton is a self-dual or anti-self-dual connection on a principal bundle on a four dimensional manifold. Via the Penrose-Ward correspondence on S^4 they give rise to algebraic vector bundles on odd dimensional projective spaces. In 1978, Atiyah, Drinfeld, Hitchin and Manin gave a construction of instanton bundles involving only linear algebra [AHDM78]. A few years later, Okonek and Spindler defined mathematical instanton bundles and studied their moduli spaces [OS86].

The first definition of an instanton bundle was a rank two vector bundle on three dimensional projective space satisfying some instanton conditions. These conditions include being simple and symplectic; that is, it is isomorphic to its dual bundle via an anti-symmetric isomorphism. We call a vector bundle autodual if it is isomorphic to its dual bundle. Important examples include orthogonal and symplectic bundles.

The results in the second chapter are from a joint work with S. Marchesi and M. Jardim [JMW14]. Our goal is to explicitly describe the set of isomorphism classes of autodual instanton bundles using certain linear algebra data. The construction we use has its origin in the work of Atiyah, Drinfeld, Hitchin and Manin in the 1970s. The ADHM construction of instanton bundles via monads was developed by Donaldson [Don84] on the projective plane, and by Jardim in general [FJ08, HJM14]. Using the ADHM-datum given by an instanton bundle and the autoduality structure, we can prove the first result.

Main Theorem. *Framed autodual instanton bundles on complex projective space can be parametrised by an extended* ADHM-*datum.*

Moreover, the parametrisation can be refined in the orthogonal and symplectic cases. Using this description of an orthogonal instanton bundle, we prove the following results.

Main Theorem. *There are no rank two orthogonal instanton bundles of trivial splitting type on the projective plane.*

Main Theorem. *There are no orthogonal instanton bundles of trivial splitting type with odd second Chern class.*

However, we will provide an example of a rank $2n$ orthogonal instanton bundle of non-trivial splitting type on $\mathbb{P}^n$ with odd second Chern class. Finding examples of orthogonal bundles is still a non-trivial task, for example in [FFM09] the authors prove that there are no orthogonal instanton bundles of rank $2n$ on $\mathbb{P}^{2n+1}$. Orthogonal vector bundles on curves and their moduli spaces have also been studied recently in [CH12, CH14] and [Ser08].

Whereas the moduli space of symplectic instanton bundles has been studied thoroughly (we only mention [BMT12] for a recent reference), the moduli space of orthogonal instanton bundles is less explored. Orthogonal instanton bundles on the projective plane have been studied recently in [AB13], where the authors show smoothness and irreducibility and provide examples for particular ranks and charges.

We hope that our results may lead to new insights into the geometry of the moduli spaces of instantons.

Higgs Bundles. In 1986 Higgs bundles were defined by Hitchin in his seminal paper [Hit87] as solutions to certain self-duality equations obtained from the Yang-Mills equations via some dimensional reduction. He named them after Peter Higgs because of an analogy with the Higgs boson. However, the term "Higgs bundle" was only later introduced by Simpson. Since their first appearance, Higgs bundles have played an important role in many areas of mathematics, most recently in the Langlands program. The moduli space of Higgs bundles carries a deep geometric structure providing an example of a non-compact Hyperkähler manifold.

Our goal is to prove the existence and uniqueness of a canonical reduction for Higgs bundles on smooth projective curves using the complementary polyhedron technique. Behrend introduced this construction in the case of reductive group schemes [Beh95]. Harder and Stuhler [HS03] generalised this to the arithmetic situation of Arakelov group schemes. Here one needs to be more careful, since the automorphism group of an Arakelov bundle is non-reductive in general. The automorphism group of a Higgs bundle can be non-reductive, too. However, this does not seem to play a big role in our constructions.

The canonical reduction can then be used to obtain a stratification on the moduli stack of Higgs bundles, as done by Behrend in his Phd thesis for the moduli stack of principal bundles. Our first result is the following.

Main Theorem. *A principal Higgs bundle defines a complementary Higgs polyhedron that is equal to Behrend's complementary polyhedron if the Higgs structure is zero.*

The complementary polyhedron will depend on the choice of a maximal torus. Along the way we will also prove the following result.

Main Theorem. *Given a principal bundle with a global reduction to a maximal torus, its complementary polyhedron with respect to this torus reduces to a point.*

The complementary polyhedron enables us to finally prove the main result of the last chapter.

Main Theorem. *Any principal Higgs bundle has a canonical Higgs reduction that is unique in a natural sense.*

In the case of vector bundles, the canonical reduction is given by the Harder-Narasimhan filtration [HN75]. For Higgs vector bundles, the existence and uniqueness of a Harder-Narasimhan filtration is proven by Simpson in [Sim94].

In [DP05] the authors prove the existence of a canonical reduction for principal Higgs bundles. Although they claim in the abstract that they also show uniqueness, they only show uniqueness of the canonical Higgs reduction for "semi-harmonic" principal Higgs bundles, i.e. principal Higgs bundles with vanishing Chern classes. Since they reduce the stability of principal Higgs bundles to stability of the associated Higgs vector bundle, their proof can not be generalised to positive characteristic. Note that the construction of Behrend that we use in this work applies to positive characteristic.

Overview. We will now give a short overview of the contents of this thesis.

In the first chapter, we introduce the main objects of study, namely vector and principal bundles on projective schemes. We give the necessary definitions of linear algebraic groups and group schemes. We also explain stability conditions for these and explain how autodual vector bundles and Higgs bundles can be interpreted as principal bundles with decorations.

The second chapter is devoted to the study of autodual instanton bundles on projective space. We explain how instanton bundles of trivial splitting type can be constructed from ADHM-data. After that we investigate how the autoduality structure is reflected in the ADHM-datum and obtain an extended datum. For symplectic and orthogonal instanton bundles these extended data can be refined. Finally we take a look at the construction of examples of symplectic and orthogonal instanton bundles from an extended ADHM-datum.

In the last chapter, we investigate principal Higgs bundles on smooth projective curves. We start by introducing root systems and complementary polyhedra and explain how a connected reductive algebraic group equipped with a maximal torus defines a root system. We then explain Behrend's construction of the complementary polyhedron associated to a principal bundles and compute some examples. A section is devoted to the study of torus reductions. Then we give an original construction of a complementary polyhedron associated to a principal

Higgs bundle. Finally, we state some consequences of the complementary Higgs polyhedron, i.e. the existence and uniqueness of a canonical Higgs reduction.

Chapter 1

Decorated Principal Bundles

We start by introducing decorated principal bundles. These are principal bundles equipped with a section of an associated fibre bundle. We also explain how vector and principal bundles fit into the picture of group schemes and coherent sheaves. In this chapter, $\mathbb{K}$ is an algebraically closed field of characteristic zero and X is projective smooth of arbitrary dimension $\dim X \geq 1$.

1.1 Coherent Sheaves and Vector Bundles

We will mainly use the notation of [HL10]. Let $\mathscr{E}$ be a coherent sheaf on X and denote by $\mathscr{E}_x$ the stalk at a point $x \in X$. We will assume that all coherent sheaves satisfy

$$\dim(\mathrm{supp}(\mathscr{E})) := \dim\{x \in X \mid \mathscr{E}_x \neq 0\} = \dim X.$$

In other words, we only consider sheaves that are *pure* of dimension $\dim(X)$ in the sense of [HL10] which is equivalent to being torsion-free. The sheaf $\mathscr{E}$ is *torsion-free* if for all $x \in X$ and sections $s \in \mathscr{O}_{X,x} \setminus \{0\}$ multiplication with s is an injective morphism $\mathscr{E}_x \to \mathscr{E}_x$. Finally, $\mathscr{E}$ is *locally free* of rank r if for all $x \in X$: $\mathscr{E}_x \cong \mathscr{O}_{X,x}^{\oplus r}$.

For a coherent sheaf $\mathscr{E}$ we define its dual as $\mathscr{E}^\vee := \mathrm{Hom}(\mathscr{E}, \mathscr{O}_X)$. There is a natural morphism $\mathscr{E} \to \mathscr{E}^{\vee\vee}$. We say that $\mathscr{E}$ is *reflexive* if $\mathscr{E} \to \mathscr{E}^{\vee\vee}$ is an isomorphism.

1.1.1 Lemma. *[HL10, Chapter 1.1] Let $\mathscr{E}$ be a coherent sheaf on X. There is the following chain of implications of properties of $\mathscr{E}$:*

$$\textit{locally free} \;\Rightarrow\; \textit{reflexive} \;\Rightarrow\; \textit{torsion-free}.$$

1.1.2 Example. If $X = C$ is a smooth projective curve, any coherent sheaf decomposes as $\mathscr{E} = \mathscr{T}(\mathscr{E}) \oplus \mathscr{E}/\mathscr{T}(\mathscr{E})$ where $\mathscr{T}(\mathscr{E})$ is the torsion subsheaf. Thus, in the case $\dim X = 1$, being torsion-free is equivalent to being locally free. If $X = S$ is a smooth surface, then being reflexive is equivalent to being locally free [HL10, Example 1.1.16].

1.1.3 DEFINITION (Vector bundle). A vector bundle on X is a locally free coherent sheaf on X. A line bundle is a vector bundle of rank 1. For a vector bundle $\mathscr{E}$, we define its determinant to be the line bundle $\det(\mathscr{E}) := \bigwedge^{\operatorname{rank}\mathscr{E}} \mathscr{E}$.

Choose an ample line bundle $\mathscr{O}_X(1)$ on X. We let $\mathscr{O}_X(m) := \bigotimes^m \mathscr{O}_X(1)$ for $m \geq 0$ and $\mathscr{O}_X(-1) := \mathscr{O}_X(1)^\vee$. The *Hilbert polynomial* of a coherent sheaf $\mathscr{E}$ is defined as

$$P(\mathscr{E})(m) := \chi(\mathscr{E} \otimes \mathscr{O}_X(m)) = \sum_{i=0}^{\dim X} (-1)^i h^i(X, \mathscr{E}(m)) = \sum_{i=0}^{\dim X} \frac{a_i(\mathscr{E})}{i!} m^i.$$

This enables us to define the rank of an arbitrary (not necessarily locally free) coherent sheaf $\mathscr{E}$ as

$$\operatorname{rank}(\mathscr{E}) := \frac{a_{\dim X}(\mathscr{E})}{a_{\dim X}(\mathscr{O}_X)}.$$

The definitions of the rank agree whenever $\mathscr{E}$ is locally free. We also define the degree of $\mathscr{E}$ as

$$\deg \mathscr{E} := a_{\dim X - 1}(\mathscr{E}) - \operatorname{rank}(\mathscr{E}) a_{\dim X - 1}(\mathscr{O}_X).$$

This implies $\deg \mathscr{E} = \deg(\det(\mathscr{E}))$ for a locally free coherent sheaf $\mathscr{E}$.

1.1.4 DEFINITION (Stability). A nonzero coherent sheaf $\mathscr{E}$ will be called (semi)-stable if it is torsion-free and for all coherent subsheaves $\mathscr{F} \subset \mathscr{E}$ with $\operatorname{rank}\mathscr{F} < \operatorname{rank}\mathscr{E}$:

$$\operatorname{rank}\mathscr{E} \deg \mathscr{F} \underset{(-)}{<} \operatorname{rank}\mathscr{F} \deg \mathscr{E}.$$

We call $\mathscr{E}$ simple if $\operatorname{End}(\mathscr{E}) = \operatorname{Hom}(\mathscr{E}, \mathscr{E}) \cong \mathbb{K}$ via the isomorphism $\lambda \mapsto \lambda \cdot \mathrm{id}_{\mathscr{E}}$.

If $\mathscr{E}$ is a vector bundle, the (semi)-stability is usually formulated as follows: For all saturated subsheaves $0 \subsetneq \mathscr{F} \subsetneq \mathscr{E}$ (that is subsheaves $\mathscr{F}$ such that $\mathscr{E}/\mathscr{F}$ is pure of $\dim(\operatorname{supp}(\mathscr{E}))$ or zero), there is an inequality of the so called slopes

$$\frac{\deg \mathscr{F}}{\operatorname{rank}\mathscr{F}} =: \mu(\mathscr{F}) \underset{(-)}{<} \mu(\mathscr{E}) := \frac{\deg \mathscr{E}}{\operatorname{rank}\mathscr{E}}.$$

1.1.5 Lemma. *Let $\mathscr{F}$ and $\mathscr{G}$ be semistable vector bundles on X.*

1. *If $\mu(\mathscr{F}) > \mu(\mathscr{G})$, then* $\operatorname{Hom}(\mathscr{F}, \mathscr{G}) = 0$.

2. *If $\mu(\mathscr{F}) = \mu(\mathscr{G})$ and $f : \mathscr{F} \to \mathscr{G}$ is non-trivial, then f is injective if $\mathscr{F}$ is stable and f is surjective if $\mathscr{G}$ is stable.*

Proof. [HL10, Prop 1.2.7],[LP97, Prop 5.3.3] □

1.1.6 Corollary. *If $f : \mathscr{F} \to \mathscr{G}$ is a non-trivial homomorphism of vector bundles with $\mu(\mathscr{F}) = \mu(\mathscr{G})$ and $\mathscr{F}$ or $\mathscr{G}$ is stable, then f is an isomorphism. In particular any stable vector bundle is simple.*

1.1.7 Remark. For a point $x \in X$, we let $\mathbb{K}(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ be the residue field. If $\mathscr{F}$ is a coherent sheaf, then for any point $x \in X$, the localised stalk $\mathscr{F}_x \otimes \mathbb{K}(x)$ is a finite dimensional vector space. In case of a locally free sheaf, the dimensions of the stalks are locally constant, which motivates the name vector bundle.

For the construction of moduli spaces, one needs to restrict to the (semi-) stable objects. Not every coherent sheaf is semistable. However, for vector bundles on curves the semistable objects can be considered as the building blocks of all vector bundles due to the following result [HN75].

1.1.8 Theorem (Harder-Narasimhan filtration for vector bundles)**.** *Let X be a smooth projective curve and $\mathscr{E}$ a vector bundle on X. There is a unique filtration of $\mathscr{E}$ into subbundles*

$$0 = \mathscr{E}_0 \subsetneq \mathscr{E}_1 \subsetneq \ldots \subsetneq \mathscr{E}_s = \mathscr{E}$$

with the following properties:

1. *The quotients $\mathscr{E}_i/\mathscr{E}_{i-1}$ are semistable for $i = 1, \ldots, s$.*
2. $\mu(\mathscr{E}_1/\mathscr{E}_0) > \mu(\mathscr{E}_i/\mathscr{E}_{i-1}) > \ldots > \mu(\mathscr{E}_s/\mathscr{E}_{s-1})$.

Proof. The family of vector bundles isomorphic to $\mathscr{E}$ is bounded, meaning that there is a constant C with

$$\mu_{\max}(\mathscr{E}) := \max\left\{ \mu(\mathscr{F}) \,\middle|\, 0 \subsetneq \mathscr{F} \subset \mathscr{E} \right\} \leq \mu(\mathscr{E}) + C.$$

We can find a subbundle $\mathscr{E}_1 \subset \mathscr{E}$ such that $\mu(\mathscr{E}_1) = \mu_{\max}(\mathscr{E})$, and such that $\mathscr{E}_1$ has maximal rank with respect to this property. This gives the first step in the filtration and we proceed with $\mathscr{E}/\mathscr{E}_1$. □

More generally, a similar filtration exists for pure coherent sheaves on higher dimensional bases (see [HL10]). Here a pure sheaf $\mathscr{E}$ is (semi)stable if for all proper subsheaves $\mathscr{F} \subsetneq \mathscr{E}$, we have $p(\mathscr{F})(\leq)p(\mathscr{E})$, where $p(\mathscr{E}) := P(\mathscr{E})/a_{\dim X}(\mathscr{E})$ denotes the reduced Hilbert polynomial. Note that it also exists in the case of slope stability as defined above.

1.1.9 Example. We provide some easy examples for Harder-Narasimham filtrations. If $\mathscr{E}$ itself is semistable, then we get the trivial filtration $0 \subsetneq \mathscr{E}$.

Consider the vector bundle $\mathscr{E} := \mathcal{O}_X(1) \oplus \mathcal{O}_X(1) \oplus \mathcal{O}_X(-1)$ which is not semistable. One computes the slopes of the subbundles $\mathscr{F}_1 = \mathcal{O}_X(1) \oplus \mathcal{O}_X(1)$, $\mathscr{F}_2 = \mathcal{O}_X(1) \oplus \mathcal{O}_X(-1)$, $\mathscr{F}_3 = \mathcal{O}_X(1)$ and $\mathscr{F}_4 = \mathcal{O}_X(-1)$ to get the filtration

$$0 \subsetneq \mathcal{O}_X(1) \oplus \mathcal{O}_X(1) \subsetneq \mathcal{O}_X(1) \oplus \mathcal{O}_X(1) \oplus \mathcal{O}_X(-1) = \mathscr{E}.$$

1.2 Linear Algebraic Groups

A principal bundle is a geometric object that is equipped with a structure group. We recall the most important definitions and facts about linear algebraic groups (cf. [Bor91, Spr09]).

1.2.1 DEFINITION (Linear Algebraic Group). A (linear) algebraic group is a group object in the category of affine varieties over $\mathbb{K}$. To be more precise, an algebraic group G consists of the datum $(G, \mathrm{mult}, \mathrm{inv}, e_G)$ where $e_G \in G(\mathbb{K})$ and $\mathrm{mult}\colon G \times G \to G$, $\mathrm{inv}\colon G \to G$ are morphisms of varieties such that the usual group axioms hold.

The most important example of an algebraic group is the general linear group $\mathrm{GL}(V)$ of a vector space V. Furthermore, any closed subgroup $G \subset \mathrm{GL}(V)$ is again algebraic and the following theorem shows that all examples of linear algebraic groups are of this form.

1.2.2 Theorem. *[Bor91, Spr09] If G is a linear algebraic group, then there is a finite dimensional vector space V and an embedding* $G \hookrightarrow \mathrm{GL}(V)$.

1.2.3 Example. As already mentioned above, for a finite dimensional vector space V, the general linear group

$$\mathrm{GL}(V) := \{A\colon V \to V \mid A \text{ linear isomorphism}\}$$

and its closed subgroups are algebraic groups. We also define $\mathrm{GL}(n) := \mathrm{GL}(\mathbb{K}^n)$. In particular, the multiplicative group $\mathbb{G}_m := \mathrm{GL}(1) = (\mathbb{K}, \cdot)$ is an algebraic group. The additive group $\mathbb{G}_a := (\mathbb{K}, +)$ is algebraic and there is an embedding

$$\mathbb{G}_a \to \mathrm{GL}(2),\ a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

Products of algebraic groups are again algebraic. The algeraic group $T := (\mathbb{G}_m)^n = \mathbb{G}_m \times \ldots \times \mathbb{G}_m$ is called n-dimensional torus.

A group G is *solvable* if there is a finite composition series $\{e_G\} = G_0 \subset G_1 \subset \ldots \subset G_s = G$ such that G_{j-1} is normal in G_j and the quotient G_j/G_{j-1} is an abelian group for $j = 1, \ldots, s$.

If $B \subset G$ is a connected solvable subgroup such that the quotient G/B is projective, then B is called a *Borel subgroup*. Any Borel subgroup contains a maximal torus. A closed subgroup $P \subset G$ is called *parabolic* if it contains a Borel subgroup [Hum75, Corollary B, page 135]. In particular G itself is parabolic. If $P \subset G$ is parabolic then the quotient G/P is projective.

Recall that a subgroup $G \subset \mathrm{GL}(V)$ is *unipotent* if all its elements are unipotent, meaning that $(g-1)^r = 0$ for some $r > 0$.

1.2.4 DEFINITION (Reductive Group). The radical of an algebraic group $R(G)$ is the largest connected solvable normal subgroup of G, its unipotent part $R(G)_u$ is the unipotent radical of G. The group G is defined to be *semisimple* if $R(G) = \{e_G\}$. If $R(G)_u = \{e_G\}$, then G is *reductive*.

If G is an algebraic group, there is a decomposition $G = R(G)_u \rtimes L$. Here L is a reductive subgroup of G that is unique up to conjugation. This decomposition is called a *Levi decomposition* and L the *Levi factor* of G.

1.2.5 Example. Consider $G = \mathrm{GL}(n)$, which is not semisimple for $n \geq 1$. We have

$$R(\mathrm{GL}(n)) = \left\{ \lambda \cdot \mathrm{id} \,\middle|\, \lambda \in \mathbb{K}^* \right\}.$$

However, $\mathrm{GL}(n)$ is reductive for any $n \geq 1$. The special linear group $\mathrm{SL}(n) := \{A \in \mathrm{GL}(n) \mid \det(A) = 1\}$ is semisimple and hence reductive. For a parabolic subgroup $P \subset \mathrm{GL}(n)$, there are $1 \leq r_i \leq n$ with $\sum_{i=1}^k r_i = n$ such that P is conjugate to the subgroup of upper block matrizes

$$\left\{ \begin{pmatrix} M_1 & * & * \\ \mathbf{0} & \ddots & * \\ \mathbf{0} & \mathbf{0} & M_k \end{pmatrix} \middle| M_i \in \mathrm{GL}(r_i) \right\},$$

which stabilises a flag $0 = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_k = \mathbb{K}^n$ where $\dim(V_i) = \sum_{k=1}^i r_k$. In particular, $\dim(V_i/V_{i-1}) = r_i$ holds for $i = 1, \ldots, k$. Conversely, for any partition of n, the subgroup of block upper triangular matrizes is a parabolic subgroup. The standard Borel subgroup $B \subset \mathrm{GL}(n)$ is given by the subgroup of upper triangular matrices. A Levi decomposition is given by $B = T \ltimes U$, where $T = (\mathbb{G}_m)^n$ is the maximal torus of diagonal matrices and U is the unipotent subgroup of upper triangular matrices with ones on the diagonal.

A *character* is a morphism of algebraic groups $\chi\colon G \to \mathbb{G}_m$, whereas a *one parameter subgroup* is a morphism $\lambda\colon \mathbb{G}_m \to G$ of algebraic groups. We denote by $X^*(G)$ the group of characters of G and by $X_*(G)$ the set of one parameter subgroups. For a d-dimensional torus T, there are isomorphisms (of abelian groups!)

$$\mathbb{Z}^d \to X^*(T),\ (a_1, \ldots, a_d) \mapsto \left(\mathrm{diag}(t_1, \ldots, t_d) \mapsto \prod (t_i)^{a_i}\right),$$
$$\mathbb{Z}^d \to X_*(T),\ (a_1, \ldots, a_d) \mapsto \left(t \mapsto \mathrm{diag}(t^{a_1}, \ldots, t^{a_d})\right).$$

Given a parabolic subgroup $T \subset P \subset G$ that contains a torus T, then $X^*(P) \subset X^*(T)$ by restricting a character to T. The converse inclusion does not need to hold. The characters are identical only if $P = B$ is a Borel subgroup.

1.2.6 Lemma. *Let G be a linear algebraic group and $T \subset B$ a maximal torus that is contained in the Borel subgroup B. Any character $\chi\colon T \to \mathbb{G}_m$ can be lifted uniquely to a character of B. In other words, $X^*(T) \cong X^*(B)$.*

Proof. First of all, note that B decomposes as $B = T \ltimes U$, where U is its unipotent radical. Thus, any character $\chi\colon T \to \mathbb{G}_m$ can be lifted to B by defining it to be trivial on U. Conversely, given two characters $\chi_1, \chi_2\colon B \to \mathbb{G}_m$ whose restrictions to T are identical, then $\chi_1 \cdot \chi_2^{-1}$ defines a character on U. Since U is unipotent, there is a nonzero $x \in \mathbb{K}$ that is fixed by $(\chi_1 \cdot \chi_2^{-1})(u)$ for all $u \in U$ (see [Spr09] p.36, [Bor91] 4.8). This implies $\chi_1(u) = \chi_2(u)$ for $u \in U$ and the characters are identical on B. □

1.3 From Vector Bundles to Group Schemes

In this section we will consider a generalisation of vector bundles in another direction. First we explain how vector bundles can be seen as principal bundles and then how a principal bundle defines a group scheme.

1.3.1 DEFINITION (Etale morphism). Let Y and Z be schemes over $\mathbb{K}$. A morphism $f: Y \to Z$ is *étale* at a point $y \in Y$ if it is flat and unramified at y. This is equivalent to the following three conditions:

1. The induced map $f^\#: \mathscr{O}_{Z,f(y)} \to \mathscr{O}_{Y,y}$ is a flat morphism of rings.
2. f is locally of finite type.
3. The maximal ideal $\mathfrak{n}$ of $\mathscr{O}_{Y,y}$ is generated by $f^\#(\mathfrak{m}_{Z,f(y)})$ and $\mathscr{O}_{Z,f(y)}/\mathfrak{m}_{Z,f(y)} \to \mathscr{O}_{Y,y}/\mathfrak{n}$ is a finite separable field extension.

A morphism $f: Y \to Z$ is said to be étale if it is étale at every $y \in Y$.

If V and W are nonsingular varieties, then a morphism $\varphi\colon V \to W$ is étale at $p \in V$ if and only if the differential $d\varphi_p: T_pV \to T_{\varphi(p)}W$ is an isomorphism between tangent spaces. Similarly for V and W singular varieties, a morphism $\varphi\colon V \to W$ is étale at $p \in V$ if and only if it induces an isomorphism $C_{\varphi(p)}W \to C_pV$ of tangent cones.

By the inverse function theorem, a differentiable map between manifolds of the same dimension is a local diffeomorphism at a point if and only if the differential in this point is non-zero. Hence étale maps can be seen as the algebraic counterpart of local diffeomorphisms.

1.3.2 Example. Consider $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^1$, $x \mapsto x^n$. Then $d\varphi_x = nx^{n-1}$. Hence φ is étale at all points $x \neq 0$. Note that φ is not étale at any point if the characterisic of $\mathbb{K}$ divides n.

Let G be an linear algebraic group. Principal bundles are defined as being locally trivial in the étale topology as follows [Ser58].

1.3.3 DEFINITION (Principal bundle). A tuple $(\mathscr{P}, \pi, \mu)$ is a principal bundle with structure group G if the following holds.

1. $\mu\colon \mathscr{P} \times G \to \mathscr{P}$ is a right action.

2. $\pi\colon \mathscr{P} \to X$ is G-invariant, that is $\pi(p.g) = \pi(p)$.

3. $\mathscr{P}$ is étale locally trivial, i.e. there is an étale covering $(f_i\colon W_i \to U_i)_{i\in I}$ of X with G-equivariant isomorphisms $f_i^*(\mathscr{P}|_{U_i}) \cong W_i \times G$.

The last condition implies that there is a cartesian square (or pullback diagram)

$$\begin{array}{ccc} W_i \times G & \longrightarrow & \mathscr{P}|_{U_i} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ W_i & \xrightarrow{f_i} & U_i. \end{array}$$

We say that $(f_i\colon W_i \to U_i)_{i\in I}$ trivialises the principal bundle $\mathscr{P}$. In this situation a principal bundle is uniquely determined by cocycles $\varphi_{ij}\colon W_i \times_X W_j \to G$ with values in G. Here by cocycle we mean a collection (φ_{ij}) of morphisms satisfying $\varphi_{ik} = \varphi_{jk} \cdot \varphi_{ij}$ on $W_i \times_X W_j \times_X W_k$.

Let F be a variety equipped with a left action $G \times F \to F$ and $\mathscr{P}$ a principal G-bundle. We consider the right G-action

$$\mu\colon (\mathscr{P} \times F) \times G \to \mathscr{P} \times F,\ ((p,f),g) \mapsto (pg, g^{-1}f),$$

and define the *associated fibre bundle*

$$\mathscr{P}(F) := \mathscr{P} \times^G F = (\mathscr{P} \times F)/\mu.$$

Note that the quotient exists by [Sch08, Proposition 2.1.1.7]. This is a fibre bundle with fibre F, which is not in general a principal bundle.

An algebraic group G is called *special* if any principal G-bundle can be trivialised in the Zariski topology. Grothendieck classified all special groups in [Gro58]. Important examples include $\mathrm{GL}(n), \mathrm{SL}(n)$ and $\mathrm{Sp}(n)$.

1.3.4 Example. Let $\mathbb{K} = \mathbb{C}$, choose a point $x \in X$ and let $G = \pi_1(X, x)$ be the topological fundamental group. Then the universal covering $Y \to X$ is a $\pi_1(X,x)$-bundle. This also motivates why we choose principal bundles to be étale locally trivial, the Zariski topology would be too restrictive.

1.3.5 Example. Since $\mathrm{GL}(n)$ is special, for any prinicpal $\mathrm{GL}(n)$-bundle there is an open covering (U_i) of X and cocycles $\varphi_{ij}\colon U_i \cap U_j \to \mathrm{GL}(n)$ that can be used to glue a rank n vector bundle. Conversely, any rank n vector bundle $\mathscr{E}$ gives a $\mathrm{GL}(n)$-bundle via its frame bundle $\bigsqcup_{x\in X} \mathrm{Isom}(\mathbb{K}^n, \mathscr{E}_x)$. Alternatively any $\mathrm{GL}(V)$-principal bundle gives a vector bundle $\mathscr{E}$ via the standard action $\mathrm{GL}(V) \times V \to V$, $(g, v) \mapsto gv$ and letting $\mathscr{E} := \mathscr{P}(V)$ be the associated bundle with fibre V.

Let $\mathscr{P}$ be a principal G-bundle. If $H \subset G$ is a subgroup, then the projection $\pi\colon G \to G/H$ defines a principal H-bundle. For the existence of the quotient G/H see [Hum75, §12]. Let G act on the quotient via left multiplication $G \times G/H \to G/H$ and define $\mathscr{P}/H := \mathscr{P}(G/H)$. Given a section $\sigma\colon X \to \mathscr{P}/H$ consider the commutative diagram

$$\begin{array}{ccc} \sigma^*(\mathscr{P}) & \longrightarrow & \mathscr{P} \\ \big\downarrow & & \big\downarrow {\scriptstyle H\text{-bundle}} \\ X & \xrightarrow{\ \sigma\ } & \mathscr{P}/H. \end{array}$$

Then $\sigma^*(\mathscr{P}) \to X$ defines an H-bundle on X. We will say that $\sigma\colon X \to \mathscr{P}/H$ is a *reduction of structure group* of $\mathscr{P}$ to H. To give an H-bundle (where $H \subset G$ is a subgroup) is equivalent to giving a principal G-bundle and a reduction σ to H. Since all linear algebraic groups are embedded in some $\mathrm{GL}(V)$, principal bundles can be seen as vector bundles plus some extra data.

On the other hand if $f\colon G \to H$ is a morphism of groups, one can *extend the structure group*. For this we define a left G-action on H via $(g.h) := f(g)\cdot h$. Then $\mathscr{P} \times^f H := \mathscr{P}(H)$ can be shown to be a principal H-bundle.

1.3.6 DEFINITION (Group scheme). A group scheme on X is a group object in the category of schemes over X. It is given by a scheme $\mathscr{G} \to X$ together with morphisms over X

$$\mathrm{mult}\colon \mathscr{G} \times_X \mathscr{G} \to \mathscr{G},\ \mathrm{inv}\colon \mathscr{G} \to \mathscr{G},\ e\colon X \to \mathscr{G},$$

that satisfy the usual group axioms (meaning that certain diagrams commute).

Given a principal G-bundle $\pi\colon \mathscr{P} \to X$, we define its automorphism bundle $\mathrm{Aut}(\mathscr{P})$ as follows. Let G act on itself by conjugation $G \times G \to G$, $(g,h) \mapsto ghg^{-1}$ and define $\mathrm{Aut}(\mathscr{P}) = \mathscr{P}(G)$. The group structure is given by the following maps

$$\begin{aligned} \mathrm{mult}\colon & \mathrm{Aut}(\mathscr{P}) \times_X \mathrm{Aut}(\mathscr{P}) \to \mathrm{Aut}(\mathscr{P}),\ (p,g_1),(p,g_2) \mapsto (p,g_1g_2), \\ \mathrm{inv}\colon & \mathrm{Aut}(\mathscr{P}) \to \mathrm{Aut}(\mathscr{P}),\ (p,g) \mapsto (p,g^{-1}), \\ e\colon & X \to \mathrm{Aut}(\mathscr{P}),\ x \mapsto (p_x, e_G) \text{ for a fixed } p_x \in \pi^{-1}(x). \end{aligned}$$

Note that when replacing a principal bundle by its automorphism group scheme, one looses structure, as the map $G \to \mathrm{Aut}(G)$ is not bijective in general.

There is also a stability notion for principal bundles. Since the category of principal G-bundles is neither abelian nor additive, we need to find an appropriate replacement for subbundles.

1.4 Stability and Decorations

Let G be a reductive algebraic group and $\chi\colon P \to \mathbb{G}_m$ a character of the parabolic subgroup $P \subset G$. Consider the P-bundle $G \to G/P$ and define the associated line bundle

$$\mathscr{L}_\chi := (G \to G/P) \times^\chi \mathbb{G}_m.$$

We define χ to be an *antidominant character* of P if $\mathscr{L}_\chi$ is ample and χ is trivial on the center $Z(P)$. Similarly a character $\chi\colon P \to \mathbb{G}_m$ is dominant if χ^{-1} is antidominant (meaning that $\mathscr{L}_\chi^\vee$ is ample).

1.4.1 DEFINITION (Semistability of Principal Bundles). A principal G-bundle $\mathscr{P} \to X$ is *semistable* if for all parabolic subgroups $Q \subset G$, reductions $\beta\colon U \to \mathscr{P}|_U/Q$ on a big open subset $U \subset X$ (that is $\operatorname{codim}(X \setminus U) \geq 2$), and any dominant character $\chi\colon Q \to \mathbb{G}_m$, one has

$$\deg(\beta^* \mathscr{P}|_U \times^\chi \mathbb{G}_m) \leq 0.$$

Here $\beta^* \mathscr{P}|_U \times^\chi \mathbb{G}_m$ is the line bundle on U constructed from $\mathscr{P}$ using β and χ.

1.4.2 Remark. An equivalent definition of semistability is to require that for all parabolic subgroups $Q \subset G$ and reductions $\beta\colon U \to \mathscr{P}|_U/Q$ on a big open subset $U \subset X$, we have

$$\deg(\beta^* \mathscr{P}|_U \times^{\mathrm{Ad}} \mathfrak{q}) \leq 0,$$

where $\mathrm{Ad}\colon Q \to \mathrm{GL}(\mathfrak{q})$ is the adjoint action of Q on its Lie algebra and $\beta^* \mathscr{P}|_U \times^{\mathrm{Ad}} \mathfrak{q}$ the associated vector bundle [Lan09, Section 4]. We will work with the definition above since it is more suitable for our constructions. Also note that semistability is only defined for reductive G. For reductive groups the adjoint action factors through $\mathrm{SL}(\mathfrak{g})$, and thus

$$\deg(\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{g}) = \deg(\det(\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{g})) = \deg(\mathscr{O}_X) = 0.$$

Hence, $\deg(\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{g}) = 0$ for reductive G. Since $\beta^* \mathscr{P}|_U \times^{\mathrm{Ad}} \mathfrak{q}$ for a maximal parabolic subgroup corresponds to the determinant bundle of a subbundle of $\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{g}$, this shows that if $\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{g}$ is a semistable vector bundle then $\mathscr{P}$ is a semistable principal bundle. The converse does not hold in positive characteristic.

1.4.3 Example. Let $G = \mathrm{GL}(n)$ so that principal G-bundles correspond to rank n vector bundles. Consider the parabolic subgroup $Q_0 \subset \mathrm{GL}(n)$ consisting of matrizes of the form

$$M = \begin{pmatrix} M_1 & * \\ \mathbf{0} & M_2 \end{pmatrix},$$

where the first block M_1 has size $r < n$. A reduction $\beta\colon X \to \mathscr{P}/Q_0$ to Q_0 of a principal $\mathrm{GL}(n)$-bundle $\mathscr{P}$ gives a rank r subbundle $\mathscr{F} \subset \mathscr{E}$ of the corresponding vector bundle. Since Q_0 is maximal parabolic, any antidominant character of Q_0 is a positive multiple of

$$\chi \colon Q_0 \to \mathbb{G}_m,\ M \mapsto \det(M_1)^{r-n}\det(M_2)^r.$$

The associated line bundle $\mathscr{L}(\beta,\chi) = \beta^*\mathscr{P} \times^\chi \mathbb{G}_m = \det(\mathscr{F})^{\otimes(r-n)} \otimes \det(\mathscr{E}/\mathscr{F})^{\otimes r}$ then has degree

$$\deg(\mathscr{L}(\beta,\chi)) = (r-n)\deg\mathscr{F} + r\deg(\mathscr{E}/\mathscr{F}) = \operatorname{rk}\mathscr{F}\deg(\mathscr{E}) - \operatorname{rk}\mathscr{E}\deg(\mathscr{F}).$$

We see that if $\mathscr{P}$ is semistable, then $\mu(\mathscr{F}) \le \mu(\mathscr{E})$ and we recover the usual semistability of vector bundles. Conversely, if $\mu(\mathscr{F}) \le \mu(\mathscr{E})$ then $\deg(\mathscr{L}(\beta,\chi)) \ge 0$ and $\mathscr{P}$ is a semistable principal bundle (since χ is antidominant).

Note that if $Q \subset \mathrm{GL}(n)$ is a parabolic subgroup, then any antidominant character on Q is a nonnegative linear combination of antidominant characters of parabolic subgroups containing Q. Since the degree $\deg(\mathscr{L}(\beta,\chi))$ behaves linearly with respect to χ, it is enough to consider maximal parabolic subgroups.

We now turn our focus on decorations on curves. Let X be a curve, G a reductive linear algebraic group and F a projective variety that is equipped with a left G-action $G \times F \to F$ and let $\mathscr{P}(F)$ be the associated fibre bundle.

1.4.4 DEFINITION (Decoration). A pair $(\mathscr{P},\sigma)$ consisting of a principal G-bundle $\mathscr{P}$ and a section $\sigma \colon X \to \mathscr{P}(F)$ will be called *decorated principal bundle*.

1.4.5 Example. The most important examples of decorations arise from linear representations $\rho \colon G \to \mathrm{GL}(V)$. The representation induces an action $G \times V \to V$, $(g,v) \mapsto \rho(g)v$ and hence an associated vector bundle $\mathscr{P}(V)$. The section $\sigma \colon X \to \mathscr{P}(V)$ is then equivalent to a map of vector bundles $\varphi \colon \mathscr{O}_X \to \mathscr{P}(V)$.

One can also look at the induced action $G \times \mathbb{P}(V) \to \mathbb{P}(V)$, where we define $\mathbb{P}(V)$ as the projective space of hyperplanes in V in the sense of Grothendieck. This step is needed to construct projective moduli spaces. The decoration in this case is a section $\sigma \colon X \to \mathbb{P}(\mathscr{P}(V))$ of the associated projective bundle. This data is equivalent to giving a surjective map $\mathscr{P}(V) \to \mathscr{L}$ where $\mathscr{L}$ is a line bundle.

Decorated principal bundles (in particular constructions of moduli spaces) on curves have been studied by Schmitt in [Sch08]. We will focus on different aspects of these objects.

1.5 Examples

In this secton, we will introduce two examples of decorated bundles, namely autodual vector bundles and Higgs bundles. In the next two chapters of this thesis, we will take a closer look at these.

1.5.1 Autodual Vector Bundles

A vector bundle $\mathscr{E}$ is autodual if there is an isomorphism $\Phi\colon \mathscr{E} \to \mathscr{E}^\vee$ to its dual bundle. These include the important examples of symplectic bundles and orthogonal bundles.

Let $G = \mathrm{GL}(n)$ and $H\colon \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ a symmetric bilinear form. We define the orthogonal group as $\mathrm{O}(n,H) := \{A \in \mathrm{GL}(n) \mid H(Ax,Ay) = (x,y) \forall x,y \in \mathbb{K}^n\}$. If H_{std} is the standard inner product on $\mathbb{K}^n$, we let

$$\mathrm{O}(n) := \mathrm{O}(n,H_{\mathrm{std}}) = \left\{A \in \mathrm{GL}(n) \,\middle|\, A^tA = \mathbf{1}\right\}.$$

We also define $\mathrm{Sym}(n) := \left\{A \in \mathrm{GL}(n) \,\middle|\, A^t = A\right\}$ to be the symmetric matrices. The kernel of the homomorphism

$$\mathrm{GL}(n) \to \mathrm{Sym}(n),\ A \mapsto A^tA$$

is the standard orthogonal group and thus $\mathrm{GL}(n)/\mathrm{O}(n) \cong \mathrm{Sym}(n)$ since the above map is surjective by linear algebra results. Now let $\mathscr{P}$ be a principal $\mathrm{O}(n)$ bundle. We have already seen that this is the same as a $\mathrm{GL}(n)$-bundle plus a reduction $\beta\colon X \to \mathrm{GL}(n)/\mathrm{O}(n) \cong \mathrm{Sym}(n)$. This section β is therefore equivalent to an isomorphism $\Phi\colon \mathscr{E} \to \mathscr{E}^\vee$ such that $\Phi^\vee = \Phi$.

1.5.1 DEFINITION (Orthogonal Bundle). An orthogonal bundle is a vector bundle $\mathscr{E}$ together with a symmetric isomorphism $\Phi\colon \mathscr{E} \to \mathscr{E}^\vee$.

The same procedure works if $H\colon \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ is a nondegenerate antisymmetric bilinear form (and hence n is even). We let

$$\mathrm{Sp}(n,H) := \left\{A \in \mathrm{GL}(n) \,\middle|\, H(Ax,Ay) = (x,y) \forall x,y \in \mathbb{K}^n\right\}$$

be the associated symplectic group. The standard symplectic group then takes the form

$$\mathrm{Sp}(n) := \left\{A \in \mathrm{GL}(n) \,\middle|\, A^t\Omega A = \mathbf{1}\right\},\ \Omega := \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}.$$

We define $\mathrm{Asym}(n) = \left\{A \in \mathrm{GL}(n) \,\middle|\, A^t = -A\right\}$ to be the set of skew symmetric matrices and consider the map

$$\mathrm{GL}(n) \to \mathrm{Asym}(n),\ A \mapsto A^t\Omega A,$$

whose kernel is the standard symplectic group $\mathrm{Sp}(n)$. This induces an isomorphism $\Phi\colon \mathscr{E} \to \mathscr{E}^\vee$ with $\Phi^\vee = -\Phi$.

1.5.2 DEFINITION (Symplectic Bundle). A symplectic bundle is a vector bundle $\mathscr{E}$ together with an antisymmetric isomorphism $\Phi\colon \mathscr{E} \to \mathscr{E}^\vee$.

Given a vector bundle $\mathscr{E}$ and an arbitrary isomorphism $\Phi\colon \mathscr{E} \to \mathscr{E}^\vee$, one might be tempted to decompose it as $\Phi = \Phi_{\mathrm{sym}} + \Phi_{\mathrm{asym}}$ where

$$\Phi_{\mathrm{sym}} = \frac{1}{2}(\Phi + \Phi^\vee),\ \Phi_{\mathrm{asym}} = \frac{1}{2}(\Phi - \Phi^\vee)$$

and consider $(\mathscr{E}, \Phi_{\mathrm{sym}}, \Phi_{\mathrm{asym}})$ as an $\mathrm{O}(n) \times \mathrm{Sp}(n)$ bundle. The problem here is that $\mathrm{GL}(n)$ does not decompose as $\mathrm{O}(n) \times \mathrm{Sp}(n)$. For example if Φ is symmetric then $\Phi_{\mathrm{asym}} = 0$ and this does not give an element in $\mathrm{Asym}(n)$. In other words given a non-degenerate bilinear form H, one can decompose it into a symmetric and an antisymmetric part, but these do not have to be non-degenerate.

1.5.2 Higgs bundles

Let X be a smooth projective curve and G a connected reductive linear algebraic group. Consider the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$. Given a principal G-bundle $\mathscr{P}$, we obtain the adjoint vector bundle

$$\mathrm{Ad}(\mathscr{P}) := \mathscr{P} \times^{\mathrm{Ad}} \mathfrak{g}.$$

In the case $G = \mathrm{GL}(n)$ the Lie algebra $\mathfrak{gl}_n = \mathrm{Mat}(n \times n)$ consists of all $(n \times n)$-matrices and the adjoint bundle is simply the endomorphism bundle $\mathrm{End}(\mathscr{E})$ of the associated vector bundle. Hence, a decorated $\mathrm{GL}(n)$-bundle with a decoration of type Ad consists of a vector bundle $\mathscr{E}$ and an endomorphism $\mathscr{E} \to \mathscr{E}$. Consider a representation

$$\rho\colon\ \mathrm{GL}(n) \to \mathrm{GL}(\mathfrak{gl}_n \oplus \mathbb{K}).$$

The associated principal bundle $\mathscr{P}_\rho$ takes the form $\mathrm{End}(\mathscr{E}) \oplus \mathscr{O}_X^{\otimes m}$ for an integer $m \in \mathbb{Z}$. A section of $\mathscr{P}_\rho$ consists of a pair (φ, ϵ) where $\varphi \in \mathrm{End}(\mathscr{E})$ and $\epsilon\colon \mathscr{O}_X \to \mathscr{O}_X^{\otimes m}$ is a section. Using the associated projective bundle $\mathbb{P}(\mathscr{P}_\rho)$, we obtain a line bundle $\mathscr{L}$ and a surjective map $\mathrm{End}(\mathscr{E}) \oplus \mathscr{O}_X \to \mathscr{L}$ (cf. Example 1.4.5) which we can use to describe Higgs bundles (see also [Sch04, Section 3.6], [Sch08, 2.3.6.10, 2.8.2.5] and the references therein).

1.5.3 DEFINITION (Higgs bundle). Let $\mathscr{L}$ be a line bundle. A Higgs vector bundle is a pair $(\mathscr{E}, \varphi)$ consisting of a vector bundle $\mathscr{E}$ and a twisted endomorphism $\varphi\colon \mathscr{E} \to \mathscr{E} \otimes \mathscr{L}$. A principal Higgs bundle is a pair $(\mathscr{P}, \varphi)$ where $\mathscr{P}$ is a principal G-bundle and $\varphi\colon \mathscr{O}_X \to (\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{g}) \otimes \mathscr{L}$ is a section.

Chapter 2

Moduli of Autodual Instanton Bundles

In this chapter we will investigate the moduli space of autodual instanton bundles (this is joint work with M. Jardim and S. Marchesi [JMW14]). Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. We work over the n-dimensional projective space $X := \mathbb{P}^n_{\mathbb{K}}$ where $n \geq 2$.

2.1 Cohomology Bundles

Despite its age, an excellent reference for vector bundles on projective space is still the book of Okonek, Schneider and Spindler [OSS80]. We recall some concepts and notation. Another reason for the importance of instanton bundles is that they can be described effectively via monads.

2.1.1 DEFINITION (Monad). A complex of coherent sheaves

$$0 \longrightarrow \mathscr{A} \xrightarrow{\ a\ } \mathscr{B} \xrightarrow{\ b\ } \mathscr{C} \longrightarrow 0 \qquad (2.1)$$

is a *monad* if it is exact at $\mathscr{A}$ and $\mathscr{B}$, in other words a is an injective sheaf map and b is a surjective sheaf map. The coherent sheaf $\mathscr{E} := \ker b / \operatorname{im} a$ will be called *cohomology of the monad*.

Many important properties and invariants of cohomology sheaves can already be read off the defining monad, such as its rank and Chern polynomial.

2.1.2 Lemma. *If $\mathscr{E}$ is defined as the cohomology of the monad* (2.1)*, then*

$$\begin{aligned} \operatorname{rank}(\mathscr{E}) &= \operatorname{rank}(\mathscr{B}) - \operatorname{rank}(\mathscr{A}) - \operatorname{rank}(\mathscr{C}), \\ c(\mathscr{E}) &= c(\mathscr{B})\, c(\mathscr{A})^{-1}\, c(\mathscr{C})^{-1}. \end{aligned}$$

Proof. Define $\mathscr{K} := \ker b$ and $\mathscr{Q} := \operatorname{coker} a$. The monad (2.1) induces a commutative diagram (also called the *display of the monad*) with exact rows and columns

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{A} & \xrightarrow{a} & \mathscr{K} & \longrightarrow & \mathscr{E} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{A} & \xrightarrow{a} & \mathscr{B} & \longrightarrow & \mathscr{Q} & \longrightarrow & 0. \\ & & & & \downarrow{\scriptstyle b} & & \downarrow & & \\ & & & & \mathscr{C} & & \mathscr{C} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Now observe that

$$\begin{aligned} \operatorname{rank}(\mathscr{Q}) &= \operatorname{rank}(\mathscr{E}) + \operatorname{rank}(\mathscr{C}),\ c(\mathscr{Q}) = c(\mathscr{E})c(\mathscr{C}), \\ \operatorname{rank}(\mathscr{B}) &= \operatorname{rank}(\mathscr{A}) + \operatorname{rank}(\mathscr{Q}),\ c(\mathscr{B}) = c(\mathscr{A})c(\mathscr{Q}), \end{aligned}$$

which is immediate from the display. □

2.1.3 DEFINITION (Morphism of monads). A morphism of monads is a morphism of the underlying complexes. More precisely, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{A} & \xrightarrow{a} & \mathscr{B} & \xrightarrow{b} & \mathscr{C} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Phi_a} & & \downarrow{\scriptstyle \Phi_b} & & \downarrow{\scriptstyle \Phi_c} & & \\ 0 & \longrightarrow & \mathscr{A}' & \xrightarrow{a'} & \mathscr{B}' & \xrightarrow{b'} & \mathscr{C}' & \longrightarrow & 0 \end{array}$$

where the Φ_i are morphisms of coherent sheaves. We denote a morphism Φ of a monad by the tuple (Φ_a, Φ_b, Φ_c).

A morphism $\mathscr{E} \to \mathscr{E}'$ of cohomology sheaves does not need to lift to a morphism of monads. However, the morphism groups are in bijection for some special monads.

2.1.4 Lemma (Lemma 4.1.3 in [OSS80]). *Let $\mathscr{E}$ and $\mathscr{E}'$ be cohomology sheaves defined by the monads $M : 0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ and $M' : 0 \to \mathscr{A}' \to \mathscr{B}' \to \mathscr{C}' \to 0$, respectively. There is a bijection $\mathrm{Hom}(\mathscr{E},\mathscr{E}') \to \mathrm{Hom}(M,M')$ between morphisms of sheaves and morphism of monads if*

1. $\mathrm{Hom}(\mathscr{B},\mathscr{A}') = \mathrm{Hom}(\mathscr{C},\mathscr{B}') = 0.$
2. $H^1(X,\mathscr{C}^\vee \otimes \mathscr{A}') = H^1(X,\mathscr{B}^\vee \otimes \mathscr{A}') = H^1(X,\mathscr{C}^\vee \otimes \mathscr{B}') = H^2(X,\mathscr{C}^\vee \otimes \mathscr{A}') = 0.$

A cohomology sheaf $\mathscr{E}$ need not be locally free, even if the sheaves in the monad are locally free. The *degeneration locus* of (2.1) is defined as

$$\Sigma := \left\{ x \in X \;\middle|\; \text{the stalk } \mathscr{E}_x \text{ is not a free } \mathscr{O}_{X,x}-\text{module} \right\}.$$

Then $\mathscr{E}$ is locally free if and only if Σ is empty. In this case we refer to $\mathscr{E}$ as a *cohomology bundle*. We will come back to this later.

2.2 Mathematical Instanton Bundles

In their 1986 article [OS86], Okonek and Spindler defined mathematical instanton bundles as rank $2m$ locally free sheaves $\mathscr{E}$ on odd dimensional complex projective space $\mathbb{P} = \mathbb{P}^{2m+1}_{\mathbb{C}}$ that satisfy the following conditions:

1. There is a $c > 0$ such that the Chern polynomial is given by
$$\mathrm{c}(\mathscr{E}) = \frac{1}{(1-H^2)^c} = (1 + H^2 + H^4 + \cdots)^c,$$
where H is the class of a hyperplane. In particular, $\mathscr{E}$ has Chern classes $\mathrm{c}_1(\mathscr{E}) = 0$ and $\mathrm{c}_2(\mathscr{E}) = c > 0$.
2. The bundle $\mathscr{E}$ has natural cohomology in the range $-2m-1 \le k \le 0$. This means, that for all k in the range there is at most one $l \ge 0$ with $H^l(\mathbb{P},\mathscr{E}(k)) \ne 0$.
3. The bundle $\mathscr{E}$ has trivial splitting type. There is a line $l \subset \mathbb{P}$ such that $\mathscr{E}|_l \cong \mathscr{O}_l^{2m}$.

Using the Beilinson spectral sequence, Okonek and Spindler deduce the following result from these defining properties.

2.2.1 Theorem. *Any mathematical instanton bundle $\mathscr{E}$ can be represented as the cohomology of a monad of the form*

$$0 \longrightarrow H^1(\mathscr{E} \otimes \Omega^2(1)) \otimes \mathscr{O}_{\mathbb{P}}(-1) \longrightarrow H^1(\mathscr{E} \otimes \Omega^1) \otimes \mathscr{O}_{\mathbb{P}} \longrightarrow H^1(\mathscr{E}(-1) \otimes \mathscr{O}_{\mathbb{P}}(1)) \longrightarrow 0.$$

Since the monad in Theorem (2.2.1) has a special form, the following definiton seems natural.

2.2.2 DEFINITION (Linear monad). A linear monad is a monad of the form

$$0 \longrightarrow U \otimes \mathscr{O}_X(-1) \xrightarrow{\ a\ } V \otimes \mathscr{O}_X \xrightarrow{\ b\ } W \otimes \mathscr{O}_X(1) \longrightarrow 0. \tag{2.2}$$

Here U, V and W are finite dimensional $\mathbb{K}$-vector spaces so that $a \in \mathrm{Hom}(U,V) \otimes H^0(\mathscr{O}_X(1))$ and $b \in \mathrm{Hom}(V,W) \otimes H^0(\mathscr{O}_X(1))$.

In the following, we will consider more general objects than the mathematical instanton bundles of Okonek and Spindler. These have also been studied by Jardim in [Jar06], where he defines *linear sheaves* as cohomologies of linear monads.

2.2.3 DEFINITION (Instanton Sheaf). A torsion-free coherent sheaf $\mathscr{E}$ on X is called an *instanton sheaf* of charge c and rank r if it is the cohomology of a linear monad (2.2) and $\dim(U) = \dim(W) = c$, $\dim(V) = r + 2c$. Again we refer to $\mathscr{E}$ as an *instanton bundle* if it happens to be locally free.

By Lemma (2.1.2), an instanton sheaf satisfies $\mathrm{rank}(\mathscr{E}) = r$, $c_1(\mathscr{E}) = 0$ and $c_2(\mathscr{E}) = c > 0$. Since $\mathscr{E}$ does not need to be locally free, the cohomological characterisation becomes more complicated. In fact $\mathscr{E}$ satisfies (for a proof see [Jar06]):

1. For $n \geq 2$: $H^0(\mathscr{E}(-1)) = H^n(\mathscr{E}(-n)) = 0$.
2. For $n \geq 3$: $H^1(\mathscr{E}(-2)) = H^{n-1}(\mathscr{E}(1-n)) = 0$.
3. For $n \geq 4$: $H^p(\mathscr{E}(k)) = 0$ for $2 \leq p \leq n-2$ and all k.

Later, we will want to restrict our attention to instanton bundles in order to define a suitable autoduality structure. To that end, consider the degeneration locus Σ. For any $x \in X$, the maps of stalks $a_x \colon U \otimes \mathscr{O}_{X,x}(-1) \to V \otimes \mathscr{O}_{X,x}$ and $b_x \colon V \otimes \mathscr{O}_{X,x} \to W \otimes \mathscr{O}_{X,x}(1)$ are injective and surjective, respectively. The problem is that if we tensor with the residue field $\mathscr{O}_{X,x} \otimes \mathbb{K}(x) \cong \mathbb{K}$ of the point x, exactness is not preserved. Denote by $a(x) \colon U \otimes \mathscr{O}_{X,x}(-1) \otimes \mathbb{K}(x) \to V \otimes \mathscr{O}_{X,x} \otimes \mathbb{K}(x)$ and $b(x) \colon V \otimes \mathscr{O}_{X,x} \otimes \mathbb{K}(x) \to W \otimes \mathscr{O}_{X,x}(1) \otimes \mathbb{K}(x)$ the map of the fibres. One may rephrase this by saying that the sequence

$$0 \longrightarrow U \xrightarrow{\ a(x)\ } V \xrightarrow{\ b(x)\ } W \longrightarrow 0.$$

may not be exact at U for an arbitrary point $x \in X$. This discussion also shows that

$$\Sigma = \{x \in X \mid a(x) \text{ is not injective}\}.$$

Let $l \subset X$ be a line in X (which will later be fixed). Given a vector bundle $\mathscr{E}$ on X, the splitting theorem of Grothendieck [Gro57] asserts that the restriction of $\mathscr{E}$ to this line splits as a direct sum of line bundles. We say that $\mathscr{E}$ has trivial splitting type on l if $\mathscr{E}|_l \cong \mathscr{O}_X^{\operatorname{rank}\mathscr{E}}$. The instanton bundles we will construct will be of trivial splitting type on a particular line. Moreover, they will be equipped with a framing.

2.2.4 DEFINITION (Framing). An isomorphism $\Phi\colon \mathscr{E}|_l \longrightarrow \mathscr{O}_l^{\oplus r}$ is called a *framing*. We will refer to the pair $(\mathscr{E},\Phi)$ consisting of an instanton sheaf $\mathscr{E}$ and a framing Φ as a *framed instanton sheaf*. A morphism of framed instanton sheaves $(f,\overline{f})\colon (\mathscr{E}_1,\Phi_1)\to(\mathscr{E}_2,\Phi_2)$ is a morphism of sheaves $f\colon \mathscr{E}_1\to\mathscr{E}_2$ such that $\overline{f}=\Phi_2\circ f|_l\circ\Phi_1^{-1}$; in other words, $\overline{f}$ fits into the commutative diagram

$$\begin{array}{ccc} (\mathscr{E}_1)|_l & \xrightarrow{\Phi_1} & \mathscr{O}_l^{\oplus r_1} \\ \big\downarrow{\scriptstyle f|_l} & & \big\downarrow{\scriptstyle \overline{f}} \\ (\mathscr{E}_2)|_l & \xrightarrow{\Phi_2} & \mathscr{O}_l^{\oplus r_2}. \end{array}$$

An isomorphism of framed instanton sheaves is a pair (f,id), where f is an isomorphism of sheaves.

To determine the splitting type of a cohomology bundle, the following criterion is useful.

2.2.5 Lemma (4.2.3, 4.2.4 in [OSS80]). *Let $\mathscr{E}$ be a cohomology bundle on X defined by a linear monad* (2.2) *with* $\dim(U)=\dim(W)$ *and $p_1,p_2\in X$ two distinct points. If $l\subset X$ is the unique line through p_1 and p_2, then the restriction $\mathscr{E}|_l$ has trivial splitting type if and only if $b(p_1)\circ a(p_2)\in\operatorname{Hom}(U,W)$ is an isomorphism.*

In the next section, we will see how these properties of an instanton bundle can be formulated in terms of ADHM data.

2.3 ADHM Construction of Instanton Bundles

The ADHM construction provides an explicit description of the monad maps in terms of certain matrices. It has its name from the famous 1978 article *"Constructions of Instantons"* by Atiyah, Drinfeld, Hitchin and Manin. It has since been studied by Donaldson in 1984 [Don84] and more recently by Jardim in [FJ08].

Let V and W be $\mathbb{K}$-vector spaces of dimension $\dim V=c>0$ and $\dim W=r>0$. We also define $d:=n-2$, which is a non-negative integer since we chose n to be at least 2. An ADHM datum is given by a tuple of linear maps (A_k, B_k, I_k, J_k) for $k = 0, \dots, d$ where $A_k, B_k \in \operatorname{End}(V)$, $I_k \in \operatorname{Hom}(W, V)$ and $J_k \in \operatorname{Hom}(V, W)$.

$$A_k, B_k \circlearrowright V \underset{I_k}{\overset{J_k}{\rightleftarrows}} W$$

Next we choose homogeneous coordinates $[x : y : z_0 : \dots : z_d]$ on X such that the elements $z_0, \dots, z_d$ form a basis of $H^0(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1))$ and define

$$A := \sum_{i=0}^{d} A_i \otimes z_i,\ B := \sum_{i=0}^{d} B_i \otimes z_i,\ I := \sum_{i=0}^{d} I_i \otimes z_i,\ J := \sum_{i=0}^{d} J_i \otimes z_i.$$

We see these as elements of $\mathbb{B} := (\operatorname{End}(V) \oplus \operatorname{End}(V) \oplus \operatorname{Hom}(W, V) \oplus \operatorname{Hom}(V, W)) \otimes H^0(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1))$. We obtain a map $\mu \colon \mathbb{B} \to \operatorname{End}(V) \otimes H^0(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1))$ by defining $\mu(A, B, I, J) = [A, B] + IJ$.

2.3.1 Definition (ADHM datum). The subset $\mu^{-1}(0) \subset \mathbb{B}$ is called the set of all *d-dimensional* ADHM *data*. An element $(A, B, I, J) \in \mu^{-1}(0)$ satisfies the ADHM equation $[A, B] + IJ = 0$.

2.3.2 Remark. For any $p \in \mathbb{P}^d$ we can consider the evaluation map

$$\operatorname{ev}_p : H^0(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1)) \to \mathbb{K}$$

which we tensor with the identity to obtain

$$\operatorname{ev}_p : \mathbb{B} \to (\operatorname{End}(V) \oplus \operatorname{End}(V) \oplus \operatorname{Hom}(W, V) \oplus \operatorname{Hom}(V, W)).$$

So a datum $(A, B, I, J) \in \mathbb{B}$ can be seen as a family (A_p, B_p, I_p, J_p) of real ADHM data parametrised by $\mathbb{P}^d$. These real data will satisfy the real ADHM equations as developed in [Don84] and [FJ08]. On $\mathbb{P}^3$ they are parametrised by $\mathbb{P}^1 \cong \mathbb{K} \cup \{\infty\}$ and that is why they were called complex ADHM-data in [FJ08].

To construct an instanton bundle out of an element $(A, B, I, J) \in \mathbb{B}$ we define $\widetilde{W} := V \oplus V \oplus W$ and a monad

$$0 \longrightarrow V \otimes \mathcal{O}_X(-1) \xrightarrow{\ \alpha\ } \widetilde{W} \otimes \mathcal{O}_X \xrightarrow{\ \beta\ } V \otimes \mathcal{O}_X(1) \longrightarrow 0 \tag{2.3}$$

where the monad maps are given by

$$\alpha = \begin{pmatrix} A + \mathbf{1}x \\ B + \mathbf{1}y \\ J \end{pmatrix} \in \operatorname{Hom}(V, \widetilde{W}) \otimes H^0(X, \mathcal{O}_X(1)),$$

$$\beta = \begin{pmatrix} -B - \mathbf{1}y & A + \mathbf{1}x & I \end{pmatrix} \in \operatorname{Hom}(\widetilde{W}, V) \otimes H^0(X, \mathcal{O}_X(1)).$$

A straightforward calculation shows that the vanishing of $\beta\circ\alpha = 0$ is equivalent to the requirement that $(A,B,I,J)\in\mu^{-1}(0)$ or that the ADHM equation $[A,B]+IJ = 0$ holds.

As already mentioned above, we want to restrict our attention to instanton bundles. This will imply that the dual $\mathscr{E}^\vee$ of the cohomology bundle is again an instanton bundle. Let us take a closer look at the degeneration locus of the above monad $\Sigma = \{p\in X \mid \alpha(p) \text{ not injective}\}$.

Firstly, consider a point $p\in X$ that lies on the line $l_\infty = \{z_0 = \ldots = z_d = 0\}$. Then $\alpha(p) = \begin{pmatrix}\mathbf{1}x & \mathbf{1}y & 0\end{pmatrix}^T$ and $\beta(p) = \begin{pmatrix}-\mathbf{1}y & \mathbf{1}x & 0\end{pmatrix}$, and these are clearly injective and surjective, respectively, since $[x:y]\in\mathbb{P}^1$. So let $p = [x:y:z_0:\ldots:z_d]\in X$ such that at least one of the z_i is not 0. Suppose that $\alpha(p)\colon V\to V\oplus V\oplus W$ is not injective. There is a $0\neq v\in V$ with $\alpha(p)(v) = 0$ which gives

$$\begin{aligned}(A_0z_0+\ldots+A_dz_d)v &= -xv\\ (B_0z_0+\ldots+B_dz_d)v &= -yv\\ (J_0z_0+\ldots+J_dz_d)v &= 0.\end{aligned}$$

Thus v is a common eigenvector of A_k and B_k, and $v\in\ker J_k$ for all $k = 0,\ldots,d$.

We also need to check that (2.3) defines a monad; that is, the map β has to be surjective. Suppose that $\beta(p)\colon V\oplus V\oplus W\to V$ is not surjective. Then the dual map $\beta(p)^\vee$ is not injective and there is a nonzero $v\in V$ with

$$\begin{aligned}-(B_0z_0+\ldots+B_dz_d)^\vee v &= yv\\ (A_0z_0+\ldots+A_dz_d)^\vee v &= -xv\\ (I_0z_0+\ldots+I_dz_d)^\vee v &= 0.\end{aligned}$$

Again we see that v is a common eigenvector of A_k and B_k, and $v\in\ker(I_k^\vee) = (\operatorname{im} I_k)^\vee$. This justifies the following definition.

2.3.3 Definition (Regularity). An ADHM datum (A,B,I,J) is said to be *regular* if the following condition holds. There is no proper subspace $0\subsetneq S\subsetneq V$ that is A_k,B_k-invariant ($A_k(S),B_k(S)\subseteq S$) and $\operatorname{im} I_k\subseteq S$, $S\subseteq\ker(J_k)$ for all $k = 0,\ldots,d$.

The above discussion shows that the cohomology of the monad (2.3) is locally free whenever the ADHM datum (A,B,I,J) is regular.

Instanton bundles arising from ADHM data will always have trivial splitting type. They are trivial on the line $l_\infty = \{z_0 = \ldots = z_d = 0\}$ with fibre given by W. This can be seen by looking at the restriciton of (2.3) to $l_\infty\subset X$ which is

$$0\longrightarrow V\otimes\mathcal{O}_{l_\infty}(-1)\xrightarrow{\begin{pmatrix}\mathbf{1}x & \mathbf{1}y & 0\end{pmatrix}}\widetilde{W}\otimes\mathcal{O}_{l_\infty}\xrightarrow{\begin{pmatrix}-\mathbf{1}y\\ \mathbf{1}x\\ 0\end{pmatrix}}V\otimes\mathcal{O}_{l_\infty}(1)\longrightarrow 0.$$

Alternatively, l_∞ is the line through the points $p_1 = [1:0:0:\ldots:0]$ and $p_2 = [0:1:0\ldots:0]$ in X. One computes

$$\beta(p_1)\alpha(p_2) = \begin{pmatrix}0 & \mathbf{1} & 0\end{pmatrix}\begin{pmatrix}0 & \mathbf{1} & 0\end{pmatrix} = \mathbf{1}$$

and uses Lemma (2.2.5). To identify those ADHM data that produce isomorphic cohomology bundles, we let $\mathrm{GL}(V)$ act on $\mathbb{B}$ by defining

$$g.(A,B,I,J) = \left(gAg^{-1}, gBg^{-1}, gI, Jg^{-1}\right).$$

One checks that the set $\mu^{-1}(0)$ is $\mathrm{GL}(V)$-invariant. Finally, the following result is crucial for our considerations.

2.3.4 Theorem (Thm 3.8 in [HJM14]). *Let $X = \mathbb{P}^n_{\mathbb{K}}$ and $r, c > 0$ positive integers. The above construction provides a bijection between the following two sets:*

1. *Equivalence classes (A,B,I,J) of $(n-2)$-dimensional regular solutions to $[A,B] + IJ = 0$.*
2. *Isomorphism classes of framed instanton bundles $\mathscr{E}$ on X with $\mathrm{rank}(\mathscr{E}) = r$ and charge $\mathrm{c}_2(\mathscr{E}) = c$.*

Note that choosing any $H \in \mathrm{GL}(W)$ gives a framing of the cohomology bundle of the ADHM monad (2.3).

2.4 Autodual Instanton Bundles

The purpose of this section is to see how an isomorphism $\varphi\colon \mathscr{E} \to \mathscr{E}^\vee$ of an instanton bundle to its dual bundle is reflected in the corresponding monad and ADHM data. This will lead to a description of the moduli space of autodual framed instanton bundles using the resulting relations. Note that we do not consider any scheme theoretic structure on the moduli spaces, meaning that we only describe the underlying set of isomorphism classes of the objects in question.

Since we restrict ourselves to instanton bundles, the dual bundle is again an instanton bundle, which is already proven in [AO94]. However, we will need the following stronger result.

2.4.1 Lemma. *If $\mathscr{E}$ is an instanton bundle, then its dual bundle $\mathscr{E}^\vee$ is the cohomology of the monad which is dual to the monad that defines $\mathscr{E}$. In particular, $\mathscr{E}^\vee$ is again an instanton bundle.*

Proof. The bundle $\mathscr{E}$ is defined by a monad of the form (2.3) which we can split in two pairs of short exact sequences, namely

$$\begin{cases} 0 \longrightarrow \mathscr{K} \longrightarrow \widetilde{W} \otimes \mathscr{O}_X \xrightarrow{\beta} V \otimes \mathscr{O}_X(1) \longrightarrow 0, & \mathscr{K} = \ker\beta \\ 0 \longrightarrow V \otimes \mathscr{O}_X(-1) \xrightarrow{\alpha} \mathscr{K} \longrightarrow \mathscr{E} \longrightarrow 0, & \mathscr{E} = \ker\beta/\operatorname{im}\alpha \end{cases} \tag{2.4}$$

and

$$\begin{cases} 0 \longrightarrow V \otimes \mathscr{O}_X(-1) \xrightarrow{\alpha} \widetilde{W} \otimes \mathscr{O}_X \longrightarrow \mathscr{Q} \longrightarrow 0, \quad \mathscr{Q} = \operatorname{coker} \alpha \\ 0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{Q} \xrightarrow{\beta} V \otimes \mathscr{O}_X(1) \longrightarrow 0. \end{cases} \tag{2.5}$$

Since $\mathscr{E}$ is locally free, we can dualise (2.4) to obtain

$$\begin{cases} 0 \longrightarrow V^\vee \otimes \mathscr{O}_X(-1) \xrightarrow{\beta^\vee} \widetilde{W}^\vee \otimes \mathscr{O}_X \longrightarrow \mathscr{K}^\vee \longrightarrow 0, \\ 0 \longrightarrow \mathscr{E}^\vee \longrightarrow \mathscr{K}^\vee \xrightarrow{\alpha^\vee} V^\vee \otimes \mathscr{O}_X(1) \longrightarrow 0. \end{cases} \tag{2.6}$$

From the monads (2.6) and (2.5) and using the isomorphism $(\ker \beta)^\vee \cong \operatorname{coker} \beta^\vee$, we can reconstruct the following monad

$$0 \longrightarrow V^\vee \otimes \mathscr{O}_X(-1) \xrightarrow{\beta^\vee} \widetilde{W}^\vee \otimes \mathscr{O}_X \xrightarrow{\alpha^\vee} V^\vee \otimes \mathscr{O}_X(1) \longrightarrow 0, \tag{2.7}$$

which is dual to the defining monad of $\mathscr{E}$ and whose cohomology bundle is indeed $\mathscr{E}^\vee$ (see 2.6). □

2.4.2 Remark. The above argument also applies to monads $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ of locally free sheaves whose cohomology bundle $\mathscr{E}$ is locally free. However, it is false if $\mathscr{E}$ contains non-trivial torsion, because after dualising (2.4), the sequence (2.6) is not exact in the torsion case.

We shall now state the precise definitions of the objects which will be considered in the remainder of this chapter.

2.4.3 Definition (Autoduality). We call an instanton bundle $\mathscr{E}$ *autodual* if it is isomorphic to its dual, i.e. there exists an isomorphism $\varphi\colon \mathscr{E} \to \mathscr{E}^\vee$ (of framed instanton bundles). If the isomorphism φ satisfies $\varphi^\vee = -\varphi$ the instanton bundle $(\mathscr{E}, \varphi)$ is called *symplectic*. If it satisfies $\varphi^\vee = \varphi$ we call $(\mathscr{E}, \varphi)$ *orthogonal*. We denote a framed autodual instanton bundle by the triple $(\mathscr{E}, \Phi, \varphi)$, where $\Phi\colon \mathscr{E}|_l \to W \otimes \mathscr{O}_l$ denotes the framing and $\varphi\colon \mathscr{E} \to \mathscr{E}^\vee$ the isomorphism that respects the framing.

We denote by $\mathscr{M}_{\mathrm{audi}}(r,c)$ the set of isomorphism classes of framed autodual instanton bundles $(\mathscr{E}, \Phi, \varphi)$ of rank r and second Chern class $c_2(\mathscr{E}) = c$. Similarly $\mathscr{M}_{\mathrm{symp}}(r,c)$ denotes the set of isomorphism classes of framed symplectic instanton bundles and $\mathscr{M}_{\mathrm{orth}}(r,c)$ the set of isomorphism classes of framed orthogonal instanton bundles. In the following, we will often drop the term "framed", remembering that all bundles we consider come with a fixed framing.

Let $\mathscr{E}$ be an instanton bundle. By Theorem (2.3.4) there is an ADHM datum and hence a linear monad defining $\mathscr{E}$. Note that a map $\varphi\colon \mathscr{E}_1 \to \mathscr{E}_2$ between two instanton bundles can be lifted to the corresponding monads, since (compare Lemma 2.1.4)

$$H^0(X, \widetilde{W_1}^\vee \otimes V_2 \otimes \mathscr{O}_X(-1)) = H^0(X, V_1^\vee \otimes \widetilde{W}_2 \otimes \mathscr{O}_X(-1)) = 0,$$

and similarly

$$\begin{aligned}H^1(X,V_1^\vee\otimes V_2\otimes\mathscr{O}_X(-2)) &= H^1(X,\widetilde{W}_1^\vee\otimes V_2\otimes\mathscr{O}_X(-1))\\ &= H^1(X,V_1^\vee\otimes\widetilde{W}_2\otimes\mathscr{O}_X(-1))\\ &= H^1(X,V_1^\vee\otimes V_2\otimes\mathscr{O}_X(-2))\\ &= 0.\end{aligned}$$

Hence, in the situation of an autodual instanton bundle, the isomorphism $\varphi\colon\mathscr{E}\to\mathscr{E}^\vee$ lifts to an isomorphism of monads

$$\begin{array}{ccccc} V\otimes\mathscr{O}_X & \xrightarrow{\alpha} & \widetilde{W}\otimes\mathscr{O}_X & \xrightarrow{\beta} & V\otimes\mathscr{O}_X(1)\\ \downarrow G_1 & & \downarrow F & & \downarrow G_2\\ V^\vee\otimes\mathscr{O}_X(-1) & \xrightarrow{\beta^\vee} & \widetilde{W}^\vee\otimes\mathscr{O}_X & \xrightarrow{\alpha^\vee} & V^\vee\otimes\mathscr{O}_X(1). \end{array} \tag{2.8}$$

The maps G_1, G_2 and F are isomorphisms $V\to V^\vee$ and $\widetilde{W}\to\widetilde{W}^\vee$, respectively. Recall that $\widetilde{W}=V\oplus V\oplus W$ and so F takes the block form

$$F=\begin{pmatrix}F_1 & F_2 & F_3\\ F_4 & F_5 & F_6\\ F_7 & F_8 & F_9\end{pmatrix}: V\oplus V\oplus W\to V^\vee\oplus V^\vee\oplus W^\vee.$$

The commutativity of (2.8) gives us relations for the maps involved, in particular the left half gives us $F\alpha=\beta^\vee G_1$ whereas the right half gives us $G_2\beta=\alpha^\vee F$. The description of α and β given by the ADHM construction and the description of F above gives

$$F\alpha=\begin{pmatrix}F_1 & F_2 & F_3\\ F_4 & F_5 & F_6\\ F_7 & F_8 & F_9\end{pmatrix}\begin{pmatrix}A+\mathbf{1}x\\ B+\mathbf{1}y\\ J\end{pmatrix}=\begin{pmatrix}F_1A+F_1x+F_2B+F_2y+F_3J\\ F_4A+F_4x+F_5B+F_5y+F_6J\\ F_7A+F_7x+F_8B+F_8y+F_9J\end{pmatrix},$$

$$\beta^\vee G_1=\begin{pmatrix}-B^\vee-\mathbf{1}y\\ A^\vee+\mathbf{1}x\\ I^\vee\end{pmatrix}G_1=\begin{pmatrix}-B^\vee G_1-G_1y\\ A^\vee G_1+G_1x\\ I^\vee G_1\end{pmatrix},$$

from which we obtain $F_1=F_5=F_7=F_8=0$ and $F_2=-G_1$, $F_4=G_1$. The right half then reduces to

$$\alpha^\vee F = \begin{pmatrix} A^\vee + \mathbf{1}x & B^\vee + \mathbf{1}y & J^\vee \end{pmatrix} \begin{pmatrix} 0 & -G_1 & F_3 \\ G_1 & 0 & F_6 \\ 0 & 0 & F_9 \end{pmatrix}$$

$$= \begin{pmatrix} B^\vee G_1 + G_1 y & -A^\vee G_1 - G_1 x & A^\vee F_3 + F_3 x + B^\vee F_6 + F_6 y + J^\vee F_9 \end{pmatrix},$$

$$G_2\beta = G_2 \begin{pmatrix} -B - \mathbf{1}y & A + \mathbf{1}x & I \end{pmatrix}$$

$$= \begin{pmatrix} -G_2 B - G_2 y & G_2 A + G_2 x & G_2 I \end{pmatrix},$$

and we get $F_3 = F_6 = 0$ and furthermore $G_1 = -G_2$. Our calculations give the following description of autoduality in terms of monads.

2.4.4 Lemma. *Let $(\mathscr{E}, \varphi)$ be an autodual instanton bundle. Then the isomorphism $\varphi\colon \mathscr{E} \to \mathscr{E}^\vee$ is given by isomorphisms $G\colon V \to V^\vee$ and $H\colon W \to W^\vee$ that fit into the defining monad as*

$$\begin{array}{ccccc} V \otimes \mathcal{O}_X(-1) & \xrightarrow{\ \alpha\ } & \widetilde{W} \otimes \mathcal{O}_x & \xrightarrow{\ \beta\ } & V \otimes \mathcal{O}_X(1) \\ \big\downarrow{\scriptstyle -G} & & \big\downarrow{\scriptstyle F} & & \big\downarrow{\scriptstyle G} \\ V^\vee \otimes \mathcal{O}_X(-1) & \xrightarrow{\ \beta^\vee\ } & \widetilde{W}^\vee \otimes \mathcal{O}_X & \xrightarrow{\ \alpha^\vee\ } & V^\vee \otimes \mathcal{O}_X(1). \end{array}$$

where

$$F = \begin{pmatrix} 0 & G & 0 \\ -G & 0 & 0 \\ 0 & 0 & H \end{pmatrix} : V \oplus V \oplus W \to (V \oplus V \oplus W)^\vee.$$

Lemma (2.4.4) allows us to parametrise autodual instanton bundles by an extended datum (A, B, I, J, G, H). We will refine this parameterisation. The first step is to make use of the commutativity of the diagram to obtain the duality relations

$$GB = B^\vee G,\ GA = A^\vee G,\ HJ = -I^\vee G,\ GI = J^\vee H. \tag{2.9}$$

The (anti-)symmetry of $\varphi\colon \mathscr{E} \to \mathscr{E}^\vee$ is also reflected in the refined parametrisation. We take a closer look at this in the next lemma.

2.4.5 Lemma. *Let $\mathscr{E}$ be an autodual instanton bundle be given by the extended datum (A, B, I, J, G, H).*

1. *The bundle $\mathscr{E}$ is symplectic, if and only if H is antisymmetric and G is symmetric.*

2. *The bundle $\mathscr{E}$ is orthogonal, if and only if H is symmetric and G is antisymmetric.*

Proof. By dualising the map $\varphi\colon \mathscr{E} \to \mathscr{E}^\vee$, one obtains the following diagram

$$\begin{array}{ccccc}
V \otimes \mathscr{O}_X(-1) & \xrightarrow{\ \alpha\ } & \widetilde{W} \otimes \mathscr{O}_X & \xrightarrow{\ \beta\ } & V \otimes \mathscr{O}_X(1) \\
\downarrow{\scriptstyle G^\vee} & & \downarrow{\scriptstyle F^\vee} & & \downarrow{\scriptstyle -G^\vee} \\
V^\vee \otimes \mathscr{O}_X(-1) & \xrightarrow{\ \beta^\vee\ } & \widetilde{W}^\vee \otimes \mathscr{O}_X & \xrightarrow{\ \alpha^\vee\ } & V^\vee \otimes \mathscr{O}_X(1).
\end{array}$$

In other words the bundle map $\varphi^\vee : \mathscr{E} \to \mathscr{E}^\vee$ is given by the isomorphisms $-G^\vee : V \to V^\vee$ and $H^\vee : W \to W^\vee$. Now $\varphi^\vee = -\varphi$ is equivalent to $G^\vee = G$ and $H^\vee = -H$. This gives the statement in the symplectic case. The orthogonal case is completely analogous. □

We need to extend the $\mathrm{GL}(V)$-action on the ADHM datum (A,B,I,J) to these extended data. Indeed, if we consider a change of coordinates $g \in \mathrm{GL}(V)$, it induces a change of coordinates on the dual vector space $g^\vee \in \mathrm{GL}(V^\vee)$. The action is now defined as $g.G := (g^\vee)^{-1} G g^{-1} \colon V \to V^\vee$, so that the action of $\mathrm{GL}(V)$ on the extended datum is

$$g.(A,B,I,J,G,H) = (gAg^{-1}, gBg^{-1}, gI, Jg^{-1}, (g^\vee)^{-1}Gg^{-1}, H).$$

Considering everything seen so far and keeping in mind Theorem (2.3.4), we can state the following result.

2.4.6 Proposition. *Let $X = \mathbb{P}^n_{\mathbb{K}}$ and $r, c > 0$ positive integers. There is a bijection*

$$\mathscr{M}_{audi}(r,c) \cong \{(A,B,I,G,H)\}\big/_{\mathrm{GL}(V)}$$

where the datum (A,B,I,G,H) satisfies the following:

1. *$(A,B,I,-H^{-1}I^\vee G)$ is a regular ADHM datum.*

2. *$GA = A^\vee G$, $GB = B^\vee G$, $GIH^{-1} + G^\vee I(H^\vee)^{-1} = 0$.*

Proof. We have already observed that a framed autodual instanton bundle $(\mathscr{E}, \Phi, \varphi)$ gives an extended ADHM datum satisfying the duality relations (2.9). Note that we have $J = -H^{-1}I^\vee G$ by the duality relations and thus $GIH^{-1} - J^\vee = 0$, which gives the last relation. One checks that all relations are $\mathrm{GL}(V)$-invariant. Furthermore, the isomorphism $H\colon W \to W^\vee$ determines the framing of $\mathscr{E}^\vee$ by fitting into the commutative diagram

$$\begin{array}{ccc} \mathscr{E}_{|l} & \xrightarrow{\quad\Phi\quad} & W\otimes\mathscr{O}_l \\ \Big\downarrow{\scriptstyle \varphi|_l} & & \Big\downarrow{\scriptstyle H} \\ \mathscr{E}^\vee_{|l} & \xrightarrow{(\Phi^\vee)^{-1}} & W^\vee\otimes\mathscr{O}_l. \end{array}$$

Conversely, given a datum (A,B,I,G,H), we let $J := -H^{-1}I^\vee G$ and define $\mathscr{E}$ as the cohomology of the monad obtained from the ADHM-datum (A,B,I,J). Then $\mathscr{E}$ is a framed auto dual instanton bundle by construction (cf. Theorem 2.3.4), the isomorphism $\varphi\colon \mathscr{E}\to\mathscr{E}^\vee$ is determined by (G,H) and Lemma (2.4.4). □

2.5 Symplectic Instanton Bundles

In this section we will describe the set of isomorphism classes in the symplectic case. Since this is a special case of autoduality, we get by Proposition (2.4.6) that a symplectic instanton bundle gives an extended ADHM datum (A,B,I,G,H). From Lemma (2.4.5) we obtain that $H\colon W\to W^\vee$ is an antisymmetric isomorphism and $G\colon V\to V^\vee$ is a symmetric isomorphism. Moreover, considering the group action defined by

$$(g.G)^\vee = ((g^\vee)^{-1}Gg^{-1})^\vee = (g^\vee)^{-1}Gg^{-1} = g.G$$

for any $g\in \mathrm{GL}(V)$, we see that the symmetry of G is not affected by it.

Let us take a closer look at the duality relations in Proposition (2.4.6). Using the (anti)-symmetry of G respectively H, we see that the third relation

$$GIH^{-1} + G^\vee I(H^\vee)^{-1} = GIH^{-1} - GIH^{-1} = 0$$

holds for all given I. In summary, we have the following.

2.5.1 Proposition. *Let $X=\mathbb{P}^n_{\mathbb{K}}$ and $r,c>0$ positive integers. There is a bijection*

$$\mathscr{M}_{symp}(r,c)\cong\{(A,B,I,G,H)\}/_{\mathrm{GL}(V)}$$

where (A,B,I,G,H) satisfies the following:

1. $(A,B,I,-H^{-1}I^\vee G)$ *is a regular* ADHM *datum.*
2. $GA=A^\vee G$, $GB=B^\vee G$.
3. $G^\vee=G$, $H^\vee=-H$.

2.5.2 Example. We will provide some examples of ADHM data defining symplectic instanton bundles on $\mathbb{P}^2$. For simplicity, assume that the rank is $r = 2$.

First consider the case $c = 1$. Let $A, B \in \mathbb{K}$ arbitrary and consider $I = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $G = \begin{pmatrix} 1 \end{pmatrix}$, $H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then we get

$$J = -H^{-1} I^\vee G = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

and the ADHM equation reduces to $IJ = 0$. Furthermore, $\mathrm{im}(I) = \mathbb{K}, \ker(J) = 0$ and the regularity of (A, B, I, J) is immediate.

Now consider the case $c = 2$. Again, we let $G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We also choose

$$I = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \; A = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & \det(I) \end{pmatrix}, \; B = \begin{pmatrix} \det(I) & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

This forces

$$J = -H^{-1} I^\vee G = \begin{pmatrix} x_2 & x_4 \\ -x_1 & -x_3 \end{pmatrix},$$

and we compute the ADHM equation as

$$[A, B] + IJ = \begin{pmatrix} 0 & -\det(I) \\ \det(I) & 0 \end{pmatrix} + \begin{pmatrix} 0 & \det(I) \\ -\det(I) & 0 \end{pmatrix} = 0.$$

To check the regularity of (A, B, I, J), note that $\det(I) = x_1 x_4 - x_2 x_3 = \det(J)$ and choosing any $I \in \mathrm{GL}(2)$ guarantees regularity.

Let $(\mathscr{E}, \varphi)$ be a symplectic instanton bundle. We will take a closer look at the framing $\Phi \colon \mathscr{E}|_l \to W \otimes \mathscr{O}|_l$. Recall that there is a commutative diagram

$$\begin{array}{ccc} \mathscr{E}_{|l} & \xrightarrow{\Phi} & W \otimes \mathscr{O}_l \\ \varphi \downarrow & & \downarrow H \\ \mathscr{E}^\vee_{|l} & \xrightarrow{(\Phi^\vee)^{-1}} & W^\vee \otimes \mathscr{O}_l, \end{array}$$

in other words, the isomorphism $H \colon W \to W^\vee$ satisfies $H = \varphi\big|_l$. Let $h \in \mathrm{GL}(W)$ and define the action

$$h.(I, H) := (Ih^{-1}, (h^\vee)^{-1} H h^{-1}).$$

This induces an action of $\mathrm{GL}(V) \times \mathrm{GL}(W)$ on the extended ADHM datum corresponding to $\mathscr{E}$. Now the pair (W, H) defines a symplectic vector space and hence we can assume via the above action that

$$H \cong \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} =: \Omega.$$

Defining $\mathscr{F}_{\text{symp}}(r,c)$ to be the set of isomorphism classes of symplectic instanton bundles $(\mathscr{E},\varphi)$ of trivial splitting type (without any fixed framing), the discussion shows the following.

2.5.3 Proposition. *Let $X = \mathbb{P}^n_{\mathbb{K}}$ and $r,c > 0$ positive integers. There is a bijection*

$$\mathscr{F}_{symp}(r,c) \cong \{(A,B,I,G)\}/_{\mathrm{GL}(V)\times\mathrm{GL}(W)}$$

where (A,B,I,G) satisfies the following:

1. $(A,B,I,-\Omega^{-1}I^\vee G)$ *is a regular* ADHM *datum.*
2. $GA = A^\vee G$, $GB = B^\vee G$.
3. $G^\vee = G$.

Of course, the investigation of $\mathscr{M}_{\text{symp}}(r,c)$ and $\mathscr{F}_{\text{symp}}(r,c)$ is far from being complete. Recently Bruzzo, Markushevich and Tikhomirov showed in [BMT12] that the moduli space of symplectic instanton bundles of rank $2r \geq 4$ and second Chern class $c \geq r$ on $\mathbb{P}^3$ where $c \equiv r \mod 2$ contains an irreducible component of dimension $4c(r+1) - r(2r+1)$.

2.6 Orthogonal Instanton Bundles

In this section we will describe the set of isomorphism classes in the orthogonal case, concluding with some examples. Orthogonal instanton bundles occupied a special position because it was very hard to find any examples. For instance, in [FFM09] the authors proved that in the classical situtation of rank $2n$ on $\mathbb{P}^{2n+1}$, orthogonal instanton bundles do not exist. We will extend this result in various directions.

Similarly to the symplectic case, an orthogonal instanton bundle gives an extended ADHM datum (A,B,I,G,H) with $H\colon W \to W^\vee$ symmetric and $G\colon V \to V^\vee$ antisymmetric. Again, the antisymmetry of G is not affected by the action of $\mathrm{GL}(V)$. The duality relation is again fulfilled for all I, since

$$GIH^{-1} + G^\vee I(H^\vee)^{-1} = GIH^{-1} - GIH^{-1} = 0.$$

2.6.1 Proposition. *Let $X = \mathbb{P}^n_{\mathbb{K}}$ and $r,c > 0$ positive integers. There is a bijection*

$$\mathscr{M}_{orth}(r,c) \cong \{(A,B,I,G,H)\}/_{\mathrm{GL}(V)}$$

where (A,B,I,G,H) satisfies the following:

1. $(A,B,I,-H^{-1}I^\vee G)$ *is a regular* ADHM *datum.*
2. $GA = A^\vee G$, $GB = B^\vee G$.

3. $G^\vee = -G$, $H^\vee = H$.

2.6.2 Corollary. *If $\mathscr{E}$ is an orthogonal instanton bundle of trivial splitting type, then $c = c_2(\mathscr{E})$ is even. In particular $\mathscr{M}_{orth}(r,c)$ is empty for c odd.*

Proof. The ADHM datum of $\mathscr{E}$ gives an antisymmetric isomorphism $G\colon V \to V^\vee$, forcing $\dim V = c$ to be even. □

As in the symplectic case, if we forget the chosen framing of an orthogonal instanton bundle, we can get $H\colon W \to W^\vee$ into a standard form. We let GL(W) act on the pair (I,H) via

$$h.(I,H) := (Ih^{-1}, (h^\vee)^{-1} H h^{-1}).$$

The pair (W,H) can be interpreted as a vector space together with a nondegenerate symmetric bilinear form. Via the above defined action, we can assume that $H \cong \mathbf{1}_W$, since the signature of H must be $(\dim(W),0)$. Again let $\mathscr{F}_{\text{orth}}(r,c)$ be the set of isomorphism classes of orthogonal instanton bundles of trivial splitting type (wihout fixed framing) parametrising pairs $(\mathscr{E},\varphi)$.

2.6.3 Proposition. *Let $X = \mathbb{P}^n_{\mathbb{K}}$ and $r,c > 0$ positive integers. There is a bijection*

$$\mathscr{F}_{orth}(r,c) \cong \{(A,B,I,G)\}\big/_{\mathrm{GL}(V)}$$

where (A,B,I,G) satisfies the following:

1. $(A,B,I,-I^\vee G)$ *is a regular* ADHM *datum.*
2. $GA = A^\vee G$, $GB = B^\vee G$.
3. $G^\vee = -G$.

2.6.4 Example. We provide an example of an orthogonal instanton bundle of non trivial splitting type. Let $\mathscr{E}$ on $\mathbb{P}^n$ be defined as the cohomology of the monad

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha^t} \mathscr{O}_{\mathbb{P}^n}^{2n+2} \xrightarrow{\alpha} \mathscr{O}_{\mathbb{P}^n}(1) \longrightarrow 0,$$

where $[x_0 : \ldots : x_n]$ are homogeneous coordinates on $\mathbb{P}^n$ and

$$\alpha = \begin{pmatrix} x_0 & x_0 i & \ldots & x_n & x_n i \end{pmatrix}.$$

Clearly $\alpha\alpha^\vee = 0$, and $\alpha^\vee(p)$ is injective at every point $p \in \mathbb{P}^n$ so that $\mathscr{E}$ is an orthogonal instanton bundle of $\mathrm{rk}(\mathscr{E}) = 2n$ and charge $c = 1$. However, restricted to any line $l = \langle xy \rangle$ through $x, y \in \mathbb{P}^n$, the bundle $\mathscr{E}$ is not of trivial splitting type since (cf. Lemma 2.2.5)

$$\det(\alpha_l(x)\alpha_l(y)^\vee) = \sum_{i=0}^n x_i y_i + i^2 x_i y_i = 0.$$

2.7 The Rank Two Case

Let $\mathscr{E}$ be an autodual rank two instanton bundle on X. Since the first Chern class of $\mathscr{E}$ is zero, the determinant line bundle $\det(\mathscr{E}) = \Lambda^2\mathscr{E} \cong \mathscr{O}_X$ is trivial. Hence there is a natural isomorphism $\varphi\colon \mathscr{E} \to \mathscr{E}^\vee$. The map φ can be seen as an element of

$$\operatorname{Hom}(\mathscr{E}^\vee, \mathscr{E}) = H^0(\mathscr{E} \otimes \mathscr{E}) = H^0(S^2\mathscr{E}) \oplus H^0(\Lambda^2\mathscr{E}). \tag{2.10}$$

Note that $\varphi \in H^0(S^2\mathscr{E})$ if and only if $\varphi^\vee = \varphi$ is symmetric and $\varphi \in H^0(\Lambda^2\mathscr{E})$ if and only if $\varphi^\vee = -\varphi$ is antisymmetric. As remarked above

$$H^0(\Lambda^2\mathscr{E}) = H^0(\mathscr{O}_X) = \mathbb{K},$$

which says that any rank two instanton bundle admits a symplectic structure that is unique up to scaling. Similarly, the orthogonal structures of $\mathscr{E}$ are parametrised by $H^0(S^2\mathscr{E})$. Note that the decomposition (2.10) reflects the fact that any autoduality structue φ can be decomposed $\varphi = \varphi_{\text{sym}} + \varphi_{\text{asym}}$ into a symmetric and an antisymmetric part (that might be zero).

Suppose now that $\mathscr{E}$ is also simple. Then necessarily $H^0(S^2\mathscr{E}) = 0$ from (2.10) and there is no orthogonal structure on $\mathscr{E}$. Hence, if $\mathscr{E}$ is a simple rank two instanton bundle, then it does not admit an orthogonal structure.

To simplify computations, we will now focus on rank two orthogonal instanton bundles on $X = \mathbb{P}^2$. Recall that the second Chern class c has to be even by Corollary (2.6.2).

2.7.1 Case $r = 2$, $c = 2$ and $n = 2$

The easiest case is that of rank two orthogonal instanton bundles with second Chern class two on the projective plane $X = \mathbb{P}^2$.

2.7.1 Theorem. *Orthogonal instanton bundles of rank two and charge two on $\mathbb{P}^2$ do not exist. In other words $\mathscr{M}^{\mathbb{P}^2}_{orth}(2,2)$ is empty.*

Proof. All maps in an extended ADHM datum $(A,B,I,J,G,H) \in \mathscr{M}^{\mathbb{P}^2}_{\text{orth}}(2,2)$ are given by (2×2)-matrices with entries in $\mathbb{K}$. In order to obtain an orthogonal bundle, we need that G is an antisymmetric isomorphism. Equivalently, the pair (V,G) defines a symplectic vector space and hence we can choose a basis such that

$$G = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Setting $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, we compute the duality relation $GA = A^\vee G$ and get

$$\begin{pmatrix} a_3 & a_4 \\ -a_1 & -a_2 \end{pmatrix} = GA = A^\vee G = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -a_3 & a_1 \\ -a_4 & a_2 \end{pmatrix},$$

hence $A = a \cdot \mathbf{1}$ and similarly $B = b \cdot \mathbf{1}$ are multiples of the identity matrix. Then A and B commute, so the ADHM equation reduces to $IJ = 0$. The maps I and J have the same rank since $GI = J^\vee H$, and combining this with the ADHM equation we get that $\mathrm{rk}(I), \mathrm{rk}(J) \leq 1$. Choosing homogeneous coordinates $[z : x : y]$ on $\mathbb{P}^2$, the monad maps take the form

$$\alpha = \begin{pmatrix} az+x & 0 \\ 0 & az+x \\ bz+y & 0 \\ 0 & bz+y \\ i_1 z & i_2 z \\ i_3 z & i_4 z \end{pmatrix}, \quad \beta = \begin{pmatrix} -bz-y & 0 & az+x & 0 & j_1 z & j_2 z \\ 0 & -bz-y & 0 & az+x & j_3 z & j_4 z \end{pmatrix},$$

where $I = \begin{pmatrix} i_1 & i_2 \\ i_3 & i_4 \end{pmatrix}$, $J = \begin{pmatrix} j_1 & j_2 \\ j_3 & j_4 \end{pmatrix}$. We let $p = [-1 : a : b] \in \mathbb{P}^2$ and the fibre maps are

$$\alpha(p) = \begin{pmatrix} 0 \\ 0 \\ -I \end{pmatrix}, \quad \beta(p) = \begin{pmatrix} 0 & 0 & -J \end{pmatrix}.$$

Since $\mathrm{rk}(I), \mathrm{rk}(J) \leq 1$, we see that $\alpha(p)$ can never be injective and $\beta(p)$ can never be surjective, which shows that there are no regular solutions. □

2.7.2 Case $r = 2$, $c = 4$ and $n = 2$

Using Proposition (2.6.3) we will investigate the existence of orthogonal instanton bundles of rank two and charge four on $\mathbb{P}^2$. We know by now that this means considering the commutative diagram

$$\begin{array}{ccccc} V \otimes \mathcal{O}_X(-1) & \xrightarrow{\alpha} & \widetilde{W} \otimes \mathcal{O}_X & \xrightarrow{\beta} & V \otimes \mathcal{O}_X(1) \\ \downarrow{\scriptstyle -G} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle G} \\ V^\vee \otimes \mathcal{O}_X(-1) & \xrightarrow{\beta^\vee} & \widetilde{W}^\vee \otimes \mathcal{O}_X & \xrightarrow{\alpha^\vee} & V^\vee \otimes \mathcal{O}_X(1). \end{array}$$

with the following relations

$$GB = B^\vee G, \quad GA = A^\vee G, \quad HJ = -I^\vee G, \quad GI = J^\vee H,$$

such that H is symmetric and G antisymmetric. Fix a basis of the symplectic vector space V such that

$$G = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Using the first two relations in (2.9), a computation shows that A and B have the following form

$$A = \begin{pmatrix} a_1 & a_2 & 0 & a_4 \\ a_5 & a_6 & -a_4 & 0 \\ 0 & a_{10} & a_1 & a_5 \\ -a_{10} & 0 & a_2 & a_6 \end{pmatrix}, \; B = \begin{pmatrix} b_1 & b_2 & 0 & b_4 \\ b_5 & b_6 & -b_4 & 0 \\ 0 & b_{10} & b_1 & b_5 \\ -b_{10} & 0 & b_2 & b_6 \end{pmatrix}.$$

To simplify computations we also fix H to be the identity matrix and choose

$$J = \begin{pmatrix} j_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & j_8 \end{pmatrix}.$$

Note that we can recover I from the remaining relations.

After making these choices we can compute the ADHM equation and this produces a (4×4)-matrix whose entries must equal zero.

$$AB - BA + IJ = \begin{pmatrix} -a_5b_2 + a_{10}b_4 + a_2b_5 - a_4b_{10} & -a_2b_1 + a_1b_2 - a_6b_2 + a_2b_6 & 2a_4b_2 - 2a_2b_4 & -a_4b_1 + a_1b_4 - a_6b_4 + a_4b_6 \\ a_5b_1 - a_1b_5 + a_6b_5 - a_5b_6 & a_5b_2 + a_{10}b_4 - a_2b_5 - a_4b_{10} & -a_4b_1 + a_1b_4 - a_6b_4 + a_4b_6 & 2a_5b_4 - 2a_4b_5 - j_8^2 \\ 2a_{10}b_5 - 2a_5b_{10} + j_1^2 & -a_{10}b_1 + a_{10}b_6 + a_1b_{10} - a_6b_{10} & a_5b_2 - a_{10}b_4 - a_2b_5 + a_4b_{10} & -a_5b_1 + a_1b_5 - a_6b_5 + a_5b_6 \\ -a_{10}b_1 + a_{10}b_6 + a_1b_{10} - a_6b_{10} & -2a_{10}b_2 + 2a_2b_{10} & a_2b_1 - a_1b_2 + a_6b_2 - a_2b_6 & -a_5b_2 - a_{10}b_4 + a_2b_5 + a_4b_{10} \end{pmatrix}.$$

We are looking for solutions in the variables a_i and b_i such that j_1 and j_8 are not zero. Define the ideal $R \subset \mathbb{K}[a_i, b_i]$ generated by all entries of the matrix without a summand in either j_1 or j_8. After that we consider the ideal S defined as

$$S = \langle R, a_{10}b_5 - a_5b_{10}, a_5b_4 - a_4b_5 \rangle.$$

A computation using Macaulay2 [GS] shows that R and S are not the same ideal, which means that we can find solutions of the ADHM equation with j_1 and j_8 different from zero.

Furthermore, we need to look for regular solutions of the ADHM equation. To this end note that

$$I = G^{-1}J^{\vee}H = \begin{pmatrix} 0 & 0 \\ 0 & -j_8 \\ j_1 & 0 \\ 0 & 0 \end{pmatrix},$$

and hence $\operatorname{im} I = \left\langle \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^T \right\rangle = \ker J$. For the regularity of (A, B, I, J) we need that for all subspaces $S \subset V \cong \mathbb{K}^4$ satisfying $\operatorname{im} I \subset S \subset \ker J$, that S is not A,B-invariant. Since

$$A\cdot\begin{pmatrix}0\\x\\y\\0\end{pmatrix}=\begin{pmatrix}a_2x\\a_6x-a_4y\\a_{10}x+a_1y\\a_2y\end{pmatrix},$$

it is sufficient to ask that both a_2 and b_2 are not zero. However, the ideal R decomposes as $R=S\cap U$ where

$$U=\langle a_2,b_2,-a_1+a_6,-b_1+b_6,-a_{10}b_4+a_4b_{10}\rangle.$$

Since a_2 and b_2 are generators of this ideal, there cannot be regular solutions of the ADHM equation with these choices of (A,B,I,J,G,H). It remains open whether there are regular solutions for different choices.

2.7.3 Case $r=2$, $c\geq 6$ and $n=2$

The computations become very large, when using a similar algorithm for higher charges. Indeed, Macaulay2 could not finish the computations in the charge six case. Again the problem is not to find a solution but a *regular* solution of the ADHM and duality equations.

In [AB13] there are also examples of orthogonal instanton bundles on $\mathbb{P}^2$ of rank $r\geq 2$ and second Chern class $c=r$. The authors also show that the moduli space of stable orthogonal instanton bundles of trivial splitting type of rank r and even second Chern class c on the projective plane is smooth and irreducible, and compute its dimension for certain values of r and n.

2.A Macaulay2 Code

Below is the Macaulay2 Code that was used in the case of orthogonal instanton bundles of rank two and charge four.

```
// we work in the polynomial ring
R=QQ[a_0..a_10,b_0..b_10,j_1..j_8,i_1..i_8]

// the matrices A, B, G, and J  are choosen as above
A=matrix{
    {a_1,a_2,0,a_4},
    {a_5,a_6,-a_4,0},
    {0,a_10,a_1,a_5},
    {-a_10,0,a_2,a_1}
}
```

```
B=matrix{
    {b_1,b_2,0,b_4},
    {b_5,b_6,-b_4,0},
    {0,b_10,b_1,b_5},
    {-b_10,0,b_2,b_1}
}

G=matrix{
    {0,0,1,0},
    {0,0,0,1},
    {-1,0,0,0},
    {0,-1,0,0}
}
iG=inverse G

J=matrix{
    {j_1,0,0,0},
    {0,0,0,j_8}
}
TJ=transpose J

// the matrix I is determined by the choices and the duality equations
I=iG*TJ

// Q is the ideal of all solutions of the ADHM equation
M=A*B-B*A + I*J
Q=ideal{M}

// compute the generators of Q
mingens Q

// Y is the ideal R from above
Y=ideal(
    a_4*b_2-a_2*b_4,
    a_10*b_2-a_2*b_10,
    a_5*b_2-a_2*b_5,
    -a_4*b_10+a_10*b_4,
    a_10*b_1-a_10*b_6-a_1*b_10+a_6*b_10,
    a_5*b_1-a_1*b_5+a_6*b_5-a_5*b_6,
    a_4*b_1-a_1*b_4+a_6*b_4-a_4*b_6,
    a_2*b_1-a_1*b_2+a_6*b_2-a_2*b_6
)
```

```
// compute the generators of Y
mingens Y

// X is the ideal S from above
X=ideal(Y, a_10*b_5-a_5*b_10, a_5*b_4-a_4*b_5 )

// compute the generators of X
mingens X
X==Y

// decompose the ideal Y
decompose Y
```

Chapter 3

Complementary Polyhedron of Higgs Bundles

In this chapter we will investigate canonical reductions of principal Higgs bundles. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. In this chapter X will be a smooth projective curve of genus g over $\mathbb{K}$ and G a connected reductive linear algebraic group.

3.1 Complementary Polyhedra

We recall the definition of complementary polyhedra as introduced in [Beh95]. Further details on root systems and root data can be found in [Hum72].

Let $(V, \langle\cdot,\cdot\rangle)$ be a Euclidean vector space of dimension n and $\Phi \subset V$ a reduced root system in V. More precisely, Φ satisfies the following:

(R1) Φ is finite, generates V and does not contain $0 \in V$.

(R2) If $\alpha \in \Phi$, then $\lambda\alpha \in \Phi$ if and only if $\lambda = \pm 1$.

(R3) If s_α is the reflection at the line spanned by $\alpha \in \Phi$, then $s_\alpha(\Phi) = \Phi$.

(R4) For all $\alpha, \beta \in \Phi$, we have $(\alpha, \beta) := \frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z}$.

The elements of a root system are called *roots*. Note that $s_\alpha(\beta) = \beta - (\beta,\alpha)\alpha$ and hence $s_\alpha(\beta) = \beta + n\alpha$ for an integer $n \in \mathbb{Z}$. The subgroup $\mathscr{W} \subset \mathrm{GL}(V)$ generated by all reflections s_α for $\alpha \in \Phi$ is called *Weyl group*.

A subset $\Delta \subset \Phi$ is called *basis* of Φ if it satisfies:

(B1) Δ generates V.

(B2) Every root is an integral linear combination of elements of Δ and the coefficients are either all non-negative or non-positive.

Elements of a basis are called *simple roots*. The choice of a basis gives a partition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots. If $\alpha \in V$ we let $L_\alpha := \{v \in V \mid \langle v, \alpha \rangle = 0\}$ be the hyperplane orthogonal to α. The collection $(L_\alpha)_{\alpha \in \Phi}$ gives a partition of V into *facets*. Two vectors $v, w \in V$ are in the same facet if and only if for all $\alpha \in \Phi$, we have $v, w \in L_\alpha$ or $\langle v, \alpha \rangle \langle w, \alpha \rangle > 0$. The unique facet of dimension 0 is the origin $\{0\}$ and the facets of dimension n are called *Weyl chambers*.

A subset $R \subset \Phi$ is *parabolic* if it satisfies:

(P1) For all $\alpha \in \Phi$, we have $\alpha \in R$ or $-\alpha \in R$.

(P2) If $\alpha, \beta \in R$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in R$.

Note that after choosing a basis Δ, the positive roots Φ^+ with respect to Δ form a (minimal) parabolic subset.

3.1.1 Lemma (Corollary 1.8 in [Beh95])**.** *The following correspondence*

$$R : \{\textit{Facets of } \Phi\} \longrightarrow \{\textit{Parabolic subsets of } \Phi\},$$
$$P \mapsto R(P),$$

where $R(P) := \{\alpha \in \Phi \mid \langle \alpha, \lambda \rangle \geq 0 \ \forall \lambda \in P\}$ *is bijective. Furthermore, it is inclusion reversing; that is, if* $P_1 \subset P_2$ *are facets then* $R(P_1) \supset R(P_2)$*.*

If P is a facet of Φ, we define the subspace $V_P := (\operatorname{span} P)^\perp \subset V$. Then $\Phi_P := \Phi \cap V_P$ is a root system in V_P, we refer to it as the *reduction of the root system* Φ to the facet P. With this definition the parabolic subset corresponding to P decomposes as $R(P) = U(P) \cup \Phi_P$ where $U(P) = \{\alpha \in \Phi \mid \exists \lambda \in P : \langle \alpha, \lambda \rangle > 0\}$.

We let $\Lambda := \{\lambda \in V \mid (\lambda, \alpha) \in \mathbb{Z} \ \forall \alpha \in \Phi\}$ be the set of *weights*. If Δ is a basis, then $\lambda \in \Lambda$ will be called *dominant* (with respect to Δ) if $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Delta$. Suppose $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ and define elements $\lambda_1, \ldots, \lambda_l \in \Lambda$ by requiring $2\left\langle \lambda_i, \alpha_j \right\rangle = \delta_{ij} \left\langle \alpha_j, \alpha_j \right\rangle$. Recall that $(V, \langle \cdot, \cdot \rangle)$ is a Euclidean vector space and hence there is an isomorphism

$$G : V \to V^\vee, \ v \mapsto \left(w \mapsto \frac{2\langle v, w \rangle}{\langle v, v \rangle} \right).$$

The (λ_i) can then be interpreted as the dual basis of $(G(\alpha_i))_i$ and are called *fundamental dominant weights* (with respect to Δ). We let $\Lambda_{\mathrm{fd}} \subset \Lambda$ be the subset of all fundamental dominant weights. The elements $\alpha^\vee := G(\alpha) \in V^\vee$ are also called *dual roots* and do indeed form a root system on $V^\vee$. If $S := \big((\alpha_i, \alpha_j)\big)_{ij}$ denotes the Cartan matrix, the fundamental dominant weights can be computed using

$$S \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_l \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_l \end{pmatrix}.$$

For any facet $P \subset V$, we define $\operatorname{vert}(P) := \left\{ \lambda \in \Lambda_{\text{fd}} \,\middle|\, \lambda \in \overline{P} \right\}$ and call the elements *vertices* of the facet P. With this definition, $P = \{\sum a_i \lambda_i \mid a_i > 0, \lambda_i \in \operatorname{vert}(P)\}$. Note that if $\mathfrak{c}$ is a Weyl chamber corresponding to the basis $\{\alpha_1, \ldots, \alpha_l\}$, then $\operatorname{vert}\mathfrak{c} = \{\lambda_1, \ldots, \lambda_l\}$ is the set of fundamental dominant weights with respect to this basis.

3.1.2 DEFINITION (Conjugate chambers). If $\mathfrak{c}$ and $\mathfrak{d}$ are Weyl chambers that have precisely $n-1$ vertices in common and there is a unique $\alpha \in \Phi$ that is positive with respect to $\mathfrak{c}$ and negative with respect to $\mathfrak{d}$, we will call them α-conjugate.

3.1.3 Example. We consider $V = \mathbb{R}^2$ and let Φ be the reduced root system of type $C2$; that is, after defining $\alpha = (0,2)$ and $\beta = (1,-1)$, we have

$$\Phi = \left\{ \pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\alpha+2\beta) \right\}.$$

The set $\Delta = \left\{\alpha, \beta\right\}$ is a basis of Φ and hence determines a chamber B whose vertices are $\Lambda = \left\{\alpha+\beta, \alpha+2\beta\right\}$. There are unique chambers B_α with vertices $\Lambda_\alpha = \left\{\alpha+\beta, \beta\right\}$ and B_β with vertices $\Lambda_\beta = \left\{\alpha+\beta, \alpha\right\}$ (see Figure (3.1)). The corresponding bases are $\Delta_\alpha = \left\{-\alpha, \alpha+\beta\right\}$ and $\Delta_\beta = \left\{-\beta, \alpha+2\beta\right\}$. Clearly B and B_α are α-conjugate and B and B_β are β-conjugate.

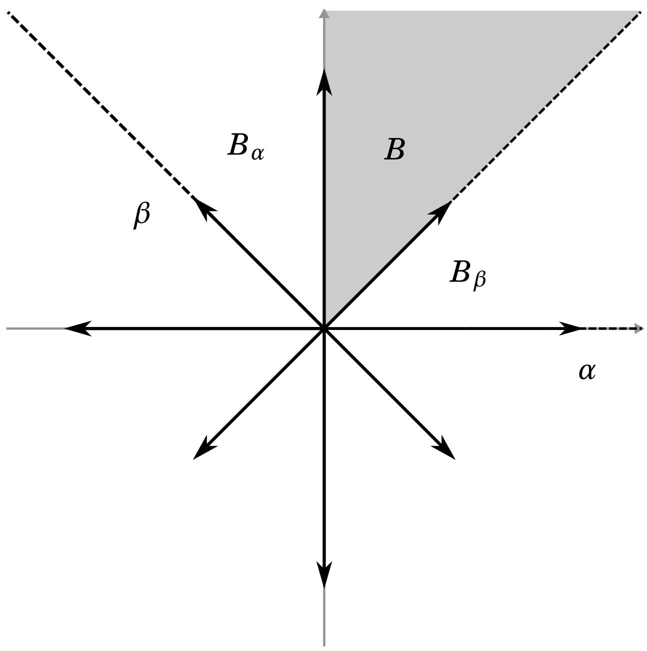

Figure 3.1: Conjugate chambers in C_2

We are now ready to define the object of interest, which will occupy the rest of the chapter.

3.1.4 DEFINITION (Complementary Polyhedron). We let $\mathfrak{C}$ be the set of Weyl chambers of the root system Φ. A family $d = d(\mathfrak{c})_{\mathfrak{c}\in\mathfrak{C}}$ of points in $V^\vee$ is called a *complementary polyhedron* on Φ if the following holds:

(C1) If λ is a vertex of the Weyl chambers $\mathfrak{c}$ and $\mathfrak{d}$, then

$$d(\mathfrak{c})(\lambda) = d(\mathfrak{d})(\lambda).$$

(C2) If the Weyl chambers $\mathfrak{c}$ and $\mathfrak{d}$ are α-conjugate and $\alpha \in \Phi$ is positive with respect to $\mathfrak{c}$ and negative with respect to $\mathfrak{d}$, then

$$d(\mathfrak{c})(\alpha) \leq d(\mathfrak{d})(\alpha).$$

We define $F(P) := \operatorname{conv}\{d(\mathfrak{c}) \mid P \subset \mathfrak{c}\}$ for any facet P and also $F := F(\{0\}) = \operatorname{conv}\{d(\mathfrak{c}) \mid \mathfrak{c} \in \mathfrak{C}\}$, which is the polyhedron that justifies the terminology. Given a complementary polyhedron d and a facet P, the inclusion $\iota\colon V_P \hookrightarrow V$ induces a complementary polyhedron d_P on Φ_P by letting $d_P(\mathfrak{c}) = (d \circ \iota)(\mathfrak{c})$. We call this *reduction of the complementary polyhedron* to the facet P.

3.1.5 DEFINITION (Stability). Let d be a complementary polyhedron on the root system Φ. For a facet P of Φ, we define its *degree* by

$$\deg(P) := \sum_{\alpha\in R(P)} d(\mathfrak{c})(\alpha),$$

where $\mathfrak{c} \in \mathfrak{C}$ is any chamber with $P \subset \mathfrak{c}$. By the first property $(C1)$ and because $\sum_{\alpha\in R(P)} \alpha \in P$, this is well-defined. The root system (Φ, d) with complementary polyhedron is *semistable* if for every facet P of Φ, we have $\deg(P) \leq 0$. Note that the root system (Φ, d) is semistable if and only if $0 \in F$.

3.1.6 DEFINITION (Special facet). Let d be a complementary polyhedron on the root system Φ and $F \subset V^\vee$ the corresponding polyhedron. Let $y(d)$ be the unique point of $V^\vee$ in F closest to the origin. A facet P of Φ is called *special* (with respect to d) if $y(d) \in F(P)$.

3.1.7 Proposition (Proposition 3.14 in [Beh95])**.** *A root system with complementary polyhedron (Φ, d) has a unique special facet.*

Thus, our strategy to prove existence and uniqueness of canonical reductions for principal G-Higgs bundles is to define a complementary polyhedron on a suitable root system and show the equivalence of the two stability concepts.

3.1.8 DEFINITION (Numerical invariant). Let P be a facet of Φ and $\lambda \in \operatorname{vert}(P)$ a vertex of P. Define $\Psi(P,\lambda) = \left\{\alpha \in \Phi \,\middle|\, (\alpha,\lambda) = 1, (\alpha,\mu) = 0 \forall \mu \in \operatorname{vert}(P) \setminus \lambda\right\}$. If Φ is equipped with a complementary polyhedron d and $P \subset \mathfrak{c}$ is any chamber, we define the numerical invariant of P with respect to λ and d as

$$n(P,\lambda) = \sum_{\alpha\in\Psi(P,\lambda)} d(\mathfrak{c})(\alpha).$$

Let P be a facet of Φ with vertices $\mathrm{vert}(P) = \{\lambda_1, \ldots, \lambda_r\}$. Consider the dual root system $\Phi^\vee$ in $V^\vee$, for each vertex λ_i there is a corresponding dual vertex $\lambda_i^\vee$. Note that in general $G(\lambda_i) \neq \lambda_i^\vee$. We define the *dual facet* $P^\vee$ of P by letting $\mathrm{vert}(P^\vee) := \{\lambda_i^\vee, \ldots, \lambda_r^\vee\}$.

The following characterisation of the special facet will be more suitable for our considerations.

3.1.9 Lemma (Section 3 in [Beh95]). *A facet P of Φ is special with respect to a complementary polyhedron d if and only if the following holds:*

1. *For all vertices $\lambda \in \mathrm{vert}(P)$ the numerical invariants $n(P,\lambda) > 0$ are positive.*
2. *The reduction (Φ_P, d_p) of d to P is semistable.*

These two properties are equivalent to $P^\vee \cap F(P) \neq \emptyset$.

To become more acquainted with the terminology, we will provide an easy example of a root system with a complementary polyhedron.

3.1.10 Example. Let $V = \mathbb{R}$ with the standard inner product given by multiplication and define $\Phi := \{1, -1\}$. This is a root system in V (of type A_1). There are exactly three facets, namely the origin $\{0\}$ and $P^\pm := \mathbb{R}_{>0} \cdot (\pm 1)$. The Cartan matrix is $S = (2)$ and we can compute the fundamental dominant weights as $\Lambda_{\mathrm{fd}} = \{\pm\frac{1}{2}\}$.

To give a complementary polyhedron on Φ is to give two points $d^\pm \in V^\vee$. Suppose that $d^\pm = x^\pm$. The first condition $(C1)$ in this case is void and the second condition $(C2)$ means we have to compute the relation $d^+(1) \leq d^-(1)$, giving $2x^+ \leq 2x^-$. Hence the polyhedron is the interval $F = [x^+, x^-]$. We see that (Φ, d) is unstable if $x^+ > 0$ or $x^- < 0$. The facet P^+ is special if $x^+ > 0$, and in the case $x^- < 0$ the special facet is P^-. Semistability is equivalent to $x^+ \leq 0 \leq x^-$.

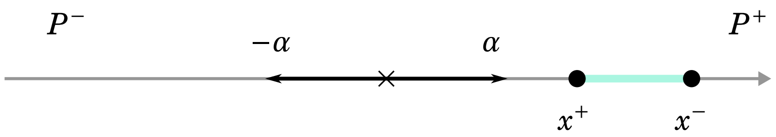

Figure 3.2: Complementary polyhedron in A_1

3.2 Root Data and Parabolic Subgroups

We let G be a connected reductive linear algebraic group with Lie algebra $\mathfrak{g} = T_e(G)$. We also fix a maximal torus $T \subset G$ with Lie algebra $\mathfrak{t}$ and let $X^*(T) = \mathrm{Hom}(T, \mathbb{K}^*)$ be the character group. Consider the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$. The restriction to the torus T gives a decomposition into root spaces

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{Ad}(t)X = \alpha(t)X \,\forall t \in T\}$. Then the *roots* (of G with respect to T) are the elements of $\Phi(G,T) := \{\alpha \in X^*(T) \mid \mathfrak{g}_\alpha \neq 0\}$. There are only finitely many roots because the dimension of $\mathfrak{g}$ is finite. Recall for the following that the *normaliser* of a subset $S \subset G$ is defined as $N_G(S) := \{g \in G \mid gSg^{-1} = S\}$.

3.2.1 Proposition. *Let $T \subset G$ be a connected reductive linear algebraic group with fixed maximal torus. Define $V := (\operatorname{span}\Phi(G,T)) \otimes_{\mathbb{Z}} \mathbb{R}$ equipped with the standard inner product. Then $\Phi(G,T)$ is a reduced root system in V. Its rank* $\dim V$ *is equal to the semisimple rank of G and the Weyl group is isomorphic to $N_G(T)/T$.*

We will explain two constructions of parabolic subgroups using the root data of G. A good exposition of these can be found for example in [Spr09].

Since all parabolic subgroups contain a Borel subgroup, it seems natural to start with one of those. Let $T \subset B \subset G$ be a Borel subgroup containing the fixed torus and let $\Delta \subset \Phi(G,T)$ be the corresponding basis. For a root $\alpha \in \Phi$, we denote by S_α the element in $N_G(T)/T$ given by the reflection s_α. Choose a subset $I \subset \Delta$ and define W_I as the subgroup generated by all S_α for $\alpha \in I$. Then $P_I := BW_IB$ is a parabolic subgroup of G, and in particular $P_\emptyset = B$ and $P_\Delta = G$. Furthermore, the roots of P_I with respect to T are $\Phi^+ \cup (\Phi^- \cap \Phi_I)$, where Φ_I is the set of roots that are integral linear combinations of elements of I. All parabolic subgroups of G containing B are of this form.

The second construction of parabolic subgroups involves a one parameter subgroup $\lambda\colon \mathbb{G}_m \to G$. Define the subgroup

$$P(\lambda) := \left\{ g \in G \;\middle|\; \lim_{z\to 0} \lambda(z) g \lambda(z)^{-1} \in G \right\}.$$

Suppose that $\operatorname{im}(\lambda) \subset T$ with λ non-trivial. Choose a set Φ^+ of positive roots such that all roots α with $\langle \alpha, \lambda \rangle > 0$ are contained in Φ^+. Let $\Delta \subset \Phi^+$ be a basis and consider $I := \{\alpha \in \Delta \mid \langle \alpha, \lambda \rangle = 0\}$. Then $P(\lambda) = P_I$ and again all parabolic subgroups are of this form.

The following correspondence is the first step towards the identification of the two stability concepts.

3.2.2 Lemma. *Let $T \subset G$ be a maximal torus and $\Phi(G,T)$ the corresponding root system. There is a unique bijective (inclusion reversing) correspondence between parabolic subgroups of G containing T and facets of the root system $\Phi(G,T)$. If $B \subset P \subset G$ is a parabolic subgroup, its Lie algebra $\mathfrak{p} \subset \mathfrak{g}$ decomposes as*

$$\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R(P)} \mathfrak{g}_\alpha,$$

where $R(P)$ is a parabolic subset of $\Phi(G,T)$. If Δ is the basis corresponding to B and $P = P_I$ for a subset $I \subset \Delta$, then $R(P_I) := \Phi^+ \cup (\Phi^- \cap \Phi_I)$.

3.2.3 Remark. Lemma (3.2.2) can be generalised for reductive group schemes $\mathscr{G} \to S$ with connected fibres and connected base scheme S. Given a split maximal torus $\mathscr{T}$ of $\mathscr{G}$ one can define a root system $\Phi(\mathscr{G},\mathscr{T})$ to obtain a bijective correspondence between parabolic subgroups of $\mathscr{G}$ containing $\mathscr{T}$ and parabolic subsets of $\Phi(\mathscr{G},\mathscr{T})$ (cf. [sga70b],[Beh95, Lemma 5.2] and [Con14, Proposition 5.2.3]). Here a parabolic subgroup is a subscheme $\mathscr{Q} \subset \mathscr{G}$ such that $\mathscr{Q}_s$ is parabolic in $\mathscr{G}_s$ for all geometric points $s\colon \operatorname{Spec}(\mathbb{L}) \to S$. We will use this correspondence in the construction of the complementary polyhedron of a (Higgs) principal bundle.

The case of the general linear group is especially important and hence we will describe its root system in the next example.

3.2.4 Example (Root system of type A_{n-1}). Let $n > 1$ and $G = \mathrm{GL}(n)$ be the general linear group with Lie algebra $\mathfrak{gl}_n = \mathrm{Mat}(n \times n)$ and $T \subset G$ the maximal torus of diagonal matrices. Define $E_{ij} \in \mathfrak{gl}_n$ as the matrix having a 1 in the (i,j)-th entry and zeros elsewhere. For $1 \le i \ne j \le n$ we define the character

$$\alpha_{ij}\colon T \to \mathbb{G}_m,\ \alpha_{ij}(\operatorname{diag}(t_1,\dots,t_n)) = t_i t_j^{-1}.$$

The roots of G with respect to T are $\Phi(G,T) = \left\{ \alpha_{ij} \,\middle|\, 1 \le i \ne j \le n \right\}$ and the root spaces $\mathfrak{gl}_{\alpha_{ij}}$ are one dimensional and spanned by E_{ij}. The Weyl group W is isomorphic to the symmetric group $\mathscr{S}_n$ and therefore has $n!$ elements. Let $e_i \in \mathbb{K}^n$ denote the i-th standard basis vector. Given a permutation $\sigma \in \mathscr{S}_n$, the corresponding element in W can be identified with the permutation matrix

$$P_\sigma = \begin{pmatrix} e_{\sigma(1)} & e_{\sigma(2)} & \cdots & e_{\sigma(n)} \end{pmatrix}.$$

The subset $\Delta := \{ \alpha_{ii+1} \mid 1 \le i \le n-1 \}$ is a basis of $\Phi(G,T)$ and corresponds to the Borel subgroup B of upper triangular matrices. Consider $I := \{\alpha_{12}\} \subset \Delta$, the corresponding parabolic subgroup P_I has one additional root $-\alpha_{12} = \alpha_{21}$. This means that P_I takes the form

$$P_I = \begin{pmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * & * \\ 0 & \cdots & 0 & 0 & * \end{pmatrix}.$$

There are exactly $n!$ Borel subgroups containing T, which can be computed via $B_\sigma = P_\sigma B P_\sigma^{-1}$. Note that $B_{\mathrm{id}} = B$ is the standard Borel subgroup of upper triangular matrices. To give a complementary polyhedron on $\Phi(G,T)$ is to give $n!$ points $d(B_\sigma)$ in $V^\vee = \left\{ (x_1,\dots,x_n)^t \in \mathbb{R}^n \,\middle|\, x_1 + \ldots + x_n = 0 \right\}$ satisfying the properties $(C1)$ and $(C2)$.

3.3 Canonical Reduction for Principal Bundles

Let $\mathscr{P}$ be a principal G-bundle on X. Since a principal $\mathrm{GL}(n)$-bundle can be identified with a vector bundle, we want to generalise the existence of a Harder-Narasimhan filtration. The stability of principal bundles depends on reductions to parabolic subgroups, hence we are looking for a reduction that is canonical in some sense.

The following result is the analogous statement to the existence of a maximal destabilising subbundle of a vector bundle. It grants the existence of a maximal destabilising reduction.

3.3.1 Lemma (Lemma 4.3 in [Beh95]). *Let $\mathscr{P}$ be a principal G-bundle. There is a positive constant $D > 0$ such that*

$$\deg(\beta^*\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{q}) \leq D$$

for all parabolic subgroups $Q \subset G$ and reductions $\beta\colon X \to \mathscr{P}/Q$.

Proof. For any reduction $\beta\colon X \to \mathscr{P}/Q$ to a parabolic $Q \subset G$, the associated (vector) bundle $\beta^*\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{q}$ is a subbundle of $\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{g}$, and hence we can compute the degree using Riemann-Roch

$$\begin{aligned}\deg(\beta^*\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{q}) &= h^0(X, \beta^*\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{q}) - h^1(X, \beta^*\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{q}) + \mathrm{rk}(\beta^*\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{q})(g-1)\\ &\leq h^0(X, \beta^*\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{q}) + \mathrm{rk}(\beta^*\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{q})(g-1)\\ &\leq h^0(X, \mathscr{P} \times^{\mathrm{Ad}} \mathfrak{g}) + (\mathrm{rk}(\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{g}) - 1)(g-1).\end{aligned}$$

Since the right hand side does not depend on Q or β, the statement follows. □

3.3.2 DEFINITION (Canonical reduction). Let $\mathscr{P}$ be a principal G-bundle. The non-negative integer

$$\mathrm{ideg}(\mathscr{P}) := \max\left\{ \deg(\beta^*\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{q}) \,\middle|\, Q \subset G \text{ parabolic and } \beta\colon X \to \mathscr{P}/Q \text{ reduction} \right\}$$

is called the *degree of instability* of $\mathscr{P}$. A pair (β, Q) consisting of a parabolic subgroup $Q \subset G$ and a reduction $\beta\colon X \to \mathscr{P}/Q$ is called *canonical reduction* if $\deg(\beta^*\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{q}) = \mathrm{ideg}(\mathscr{P})$.

The following lemma is the analogous statement of Lemma (3.1.9) and illustrates the identification of a canonical reduction and with a special facet.

3.3.3 Lemma. *A canonical reduction (β, Q) of the principal G-bundle $\mathscr{P}$ satisfies the following properties.*

1. *For any dominant character $\chi\colon Q \to \mathbb{G}_m$, let $\mathscr{L}(\beta,\chi) := \beta^*\mathscr{P} \times^{\chi} \mathbb{K}$ be the associated line bundle. Then* $\deg(\mathscr{L}(\beta,\chi)) > 0$.

2. *The extension of the Q-bundle $\beta^*\mathscr{P}$ to the Levi quotient $L = Q/R_u(Q)$ is a semistable L-bundle.*

Proof. See [Beh95, Proposition 7.2] and [Hei08, Lemma 4]. □

3.3.4 Example. We consider the situation of Example (1.4.3) and compare the properties of the Harder-Narasimhan filtration and canonical reduction.

1. As Levi quotient L of the parabolic subgroup Q, we can choose the subgroup of block diagonal matrices with the same block sizes

$$\left\{ \left. \begin{pmatrix} M_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_k \end{pmatrix} \right| M_i \in \mathrm{GL}(r_i), \sum_{i=1}^{k} r_i = n \right\}.$$

Let (β, Q) be the canonical reduction. This gives a filtration of the associated vector bundle

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \ldots \subset \mathscr{E}_k = \mathscr{E},$$

such that $\mathrm{rk}(\mathscr{E}_i/\mathscr{E}_{i-1}) = r_i$ for $i = 1, \ldots, k$. The extension to L is a $\mathrm{GL}(r_1) \times \ldots \times \mathrm{GL}(r_k)$-bundle, which in our situation corresponds to the direct sum of the quotient vector bundles

$$\mathscr{E}_1/\mathscr{E}_0 \oplus \ldots \oplus \mathscr{E}_k/\mathscr{E}_{k-1}.$$

This is a semistable principal L-bundle, which gives that $\mathscr{E}_i/\mathscr{E}_{i-1}$ is a semistable vector bundle for all $i = 1, \ldots, k$.

2. To simplify computations, we will assume that (β, Q_0) is the canonical reduction to the maximal parabolic subgroup Q_0. This means the associated filtration has only one step $0 \subset \mathscr{F} \subset \mathscr{E}$ and it is enough to consider the antidominant character

$$\beta \colon Q_0 \to \mathbb{G}_m,\ M \mapsto \det(M_1)^{r-n} \det(M_2)^r.$$

The property $\deg \mathscr{L}(\beta, \chi) < 0$ is equivalent to $\mathrm{rk}(\mathscr{F})\deg(\mathscr{E}) < \mathrm{rk}(\mathscr{E})\deg(\mathscr{F})$ and thus

$$\begin{aligned}\mu(\mathscr{E}/\mathscr{F}) - \mu(\mathscr{F}) &= \frac{\deg(\mathscr{E}) - \deg(\mathscr{F})}{\mathrm{rk}(\mathscr{E}) - \mathrm{rk}(\mathscr{F})} - \frac{\deg(\mathscr{F})}{\mathrm{rk}(\mathscr{F})} \\ &= \frac{\mathrm{rk}(\mathscr{F})\deg(\mathscr{E}) - \deg(\mathscr{F})\mathrm{rk}(\mathscr{E})}{\mathrm{rk}(\mathscr{F})(\mathrm{rk}(\mathscr{E}) - \mathrm{rk}(\mathscr{F}))} \\ &< 0,\end{aligned}$$

which is the second defining property of the Harder-Narasimhan filtration for vector bundles.

3.3.5 Theorem (Proposition 8.2 in [Beh95]). *Any principal G-bundle $\mathscr{P}$ has a canonical reduction (β, Q) to a parabolic subgroup $Q \subset G$ such that*

1. *For all dominant characters $\chi\colon Q \to \mathbb{G}_m$, the line bundles $\mathscr{L}(\beta,\chi) = \beta^*\mathscr{P} \times^\chi \mathbb{K}$ have positive degree.*

2. *The extension of the Q-bundle $\beta^*\mathscr{P}$ to the Levi quotient $L = Q/R_u(Q)$ is a semistable L-bundle.*

It is unique in the sense that given two pairs (β_1, Q_1) and (β_2, Q_2) consisting of parabolic subgroups $Q_i \subset G$ and sections $\beta_i\colon X \to \mathscr{P}/Q_i$, we have $\beta_1^*(\mathscr{P}) \cong \beta_2^*(\mathscr{P})$.

For the proof one constructs a suitable complementary polyhedron, which we describe in the next section.

3.3.1 Construction of the Complementary Polyhedron

Let G be a connected reductive linear algebraic group, $\mathscr{P}$ a principal G-bundle and $\eta \in X$ the generic point. Since for a parabolic subgroup, the quotient G/Q is projective and reductions correspond to sections of $\mathscr{P}/Q$, any reduction of $\mathscr{P}$ to a parabolic subgroup $Q \subset G$ in η can be extended uniquely to X [sga70b, Exp. XXVI]. Hence we can consider the restriction $\mathscr{P}|_\eta \cong G \otimes \mathbb{K}(X) =: G(\eta)$ where $\mathbb{K}(X)$ denotes the function field of X.

Recall that there is an étale cover $(W_i \to U_i)_i$ of X such that the principal G-bundle on X is determined by a cocyle $(\varphi_{ij}\colon W_i \times_X W_j \to G)$ and hence by an element of $H^1(X,G)$. To give a principal G-bundle over η reduces to giving an element of $H^1(\mathbb{K}(X),G)$. Steinberg showed in [Ste65, §10] that $H^1(\mathbb{K}(X),G) = 0$ (see also [DS95]). Hence over η any principal bundle becomes trivial.

Given a split maximal torus $T \subset G(\eta)$ (which exists by [sga70a, Exp. XIV, Théorème 1.1]; see also [Con14, Theorem A.1.1]), we let $\Phi(G(\eta),T)$ be the root system of $G(\eta)$ with respect to T. If $T \subset B \subset G(\eta)$ is a generic Borel subgroup, we can interpret it as a generic principal B-bundle $\mathscr{P}_B$. Note that B has a Levi decompositon $B = T \ltimes U$ and hence given a character $\chi\colon T \to \mathbb{G}_m$, we can extend it to B by letting it be trivial on the unipotent part U. We then define

$$\mathscr{L}_B(\chi) := \overline{\mathscr{P}_B} \times^\chi \mathbb{K}$$

to be the line bundle obtained from the extension $\overline{\mathscr{P}_B}$ of the generic B-bundle to X and the character χ.

3.3.6 Proposition (Proposition 6.6 in [Beh95])**.** *Let $\mathscr{P}$ be a principal G-bundle and $T \subset G(\eta)$ a maximal torus. Define the map*

$$\begin{aligned} d\colon \{T \subset B \subset G(\eta) \text{ Borel subgroup}\} &\longrightarrow X^*(T)^\vee \otimes \mathbb{R} \\ B &\longmapsto (\chi \mapsto \deg(\mathscr{L}_B(\chi)). \end{aligned}$$

Then this defines a complementary polyhedron on $\Phi(G(\eta),T)$.

Proof. We show the two properties of a complementary polyhedron. In this case they read as follows:

(C1) If B and B' are two Borel subgroups contained in the parabolic $P \subset G(\eta)$ and $\chi\colon P \to \mathbb{G}_m$ is a character, then

$$d(B)(\chi) = d(B')(\chi).$$

(C2) Let B and B' be Borel subgroups such that the simple roots of B are precisely $I_B = \{\alpha, \alpha_1, \ldots, \alpha_{r-1}\}$ and $\{-\alpha\} = -I_B \cap \Phi_{B'}$. Then

$$d(B)(\alpha) + d(B')(-\alpha) \leq 0 \;\Leftrightarrow\; d(B)(\alpha) \leq d(B')(\alpha).$$

1. The parabolic $P \subset G(\eta)$ gives the line bundle $\mathscr{L}_P(\chi) := (\mathscr{P}_B \times^B P) \times^\chi \mathbb{K}$ and since B and B' are conjugate inside P, this agrees with $(\mathscr{P}_{B'} \times^{B'} P) \times^\chi \mathbb{K}$. Since χ is a character of P, we get

$$d(B)(\chi) = \deg(\mathscr{L}_B(\chi)) = \deg(\mathscr{L}_P(\chi)) = \deg(\mathscr{L}_{B'}(\chi)) = d(B')(\chi).$$

2. The claim is equivalent to $\deg \mathscr{L}_B(\alpha) \leq \deg \mathscr{L}_{B'}(\alpha)$. By induction we can assume that $G(\eta)$ has semisimple rank one as follows (see also [HS10]). Let $P := BB'$ be the parabolic subgroup generated by B and B'. Then P is of semisimple rank one with two Borel subgroups B and B' that are also maximal parabolic inside P. Let L be a Levi factor of P and consider the L-bundle $\mathscr{P}_L = \mathscr{P}_P/L$.

Now B corresponds to a subbundle $\mathscr{L}_1 \subset \mathscr{P}_L$ and B' corresponds to a subbundle $\mathscr{L}_2 \subset \mathscr{P}_L$. Furthermore, they are generically opposite. Then $\mathscr{L}_B(\alpha) = \mathscr{L}_1 \otimes (\mathscr{P}_L/\mathscr{L}_2)^\vee$ and $\mathscr{L}_{B'}(\alpha) = (\mathscr{P}_L/\mathscr{L}_1) \otimes \mathscr{L}_2$. Thus $\deg \mathscr{L}_B(\alpha) \leq \deg \mathscr{L}_{B'}(\alpha)$ follows from Lemma (3.3.7). □

3.3.7 Lemma. *Let $\mathscr{E}$ be a rank two vector bundle on X and $\mathscr{L}_1, \mathscr{L}_2 \subset \mathscr{E}$ two line subbundles that are generically opposite, meaning that over the generic point $\mathscr{E}_\eta \cong \mathscr{L}_{1,\eta} \oplus \mathscr{L}_{2,\eta}$. Then $\deg(\mathscr{L}_1) + \deg(\mathscr{L}_2) \leq \deg(\mathscr{E})$ with equality if and only if $\mathscr{E} \cong \mathscr{L}_1 \oplus \mathscr{L}_2$.*

Proof. Since $\mathscr{L}_1$ is a subbundle of $\mathscr{E}$, we can consider the short exact sequence

$$0 \longrightarrow \mathscr{L}_1 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}/\mathscr{L}_1 \longrightarrow 0.$$

Then the composition map $\mathscr{L}_2 \to \mathscr{E}/\mathscr{L}_1$ of line bundles is non-zero and hence $\deg(\mathscr{L}_2) \leq \deg(\mathscr{E}) - \deg(\mathscr{L}_1)$. If equality holds, then necessarily $\mathscr{L}_2 \cong \mathscr{E}/\mathscr{L}_1$ since they are both line bundles and the sequence splits. □

The unique special facet of $(\Phi(G(\eta), T), d)$ gives a parabolic subgroup $T \subset P \subset G(\eta)$. As already explained above, we can uniquely extend this generic principal bundle to a parabolic principal bundle $\mathscr{P}_Q$ on X, and there is a section $\beta\colon X \to \mathscr{P}/Q$ with $\beta^*(\mathscr{P}) \cong \mathscr{P}_Q$. By construction the pair (β, Q) satisfies the conditions of Theorem (3.3.5).

3.3.8 Example. We will explicitly describe Behrend's construction for the case $G = \mathrm{GL}(3)$ where we are dealing with rank three vector bundles.

Given a rank three vector bundle $\mathscr{E}$, we restrict it to the generic point η and obtain a three dimensional vector space E_η. Let $\{b_1, b_2, b_3\}$ be a basis of E_η. Note that a T-bundle corresponds to a rank three vector bundle that splits into a direct sum of line bundles. Hence choosing a T-reduction of E_η is equivalent to choosing a direct sum composition

$$E_\eta = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus \langle b_3 \rangle .$$

Any B-bundle corresponds to a rank three vector bundle together with a complete flag $0 \subsetneq \mathscr{E}_1 \subsetneq \mathscr{E}_2 \subsetneq \mathscr{E}$. So given a T- bundle, one obtains a B-bundle by choosing an order of the line bundles or of the basis vectors of E_η. There are exactly $|\mathscr{S}_3| = 3! = 6$ possibilites. For instance, the flag $0 \subsetneq \langle b_2 \rangle \subsetneq \langle b_1, b_2 \rangle \subsetneq \langle b_1, b_2, b_3 \rangle = E_\eta$ corresponds to the Borel subgroup

$$B_{12} = \begin{pmatrix} * & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix},$$

which corresponds to the permutation exchanging 1 for 2. Note that we can extend a B-reduction at the generic point to the whole curve, but not a T-reduction (since $\mathscr{E}$ might not split as a vector bundle). Finally, we end up with a complete flag

$$0 = \mathscr{E}_0^\sigma \subset \mathscr{E}_1^\sigma \subset \mathscr{E}_2^\sigma \subset \mathscr{E}_3^\sigma = \mathscr{E}$$

for each permutation $\sigma \in \mathscr{S}_3$. The complementary polyhedron is defined as a map

$$d \colon \mathscr{S}_3 \to X^*(T)^\vee,\ \sigma \mapsto d(B_\sigma),$$

where $X^*(T) \cong \mathbb{Z}^3$ is the character group of T. The complementary polyhedron from Proposition (3.3.6) now takes the form

$$\begin{aligned} d(B_\sigma) \colon X^*(T) &\to \mathbb{Z}, \\ (\alpha_1, \alpha_2, \alpha_3) &\mapsto \sum_{i=1}^{3} \alpha_i \deg(\mathscr{E}_{(i)}^\sigma / \mathscr{E}_{\sigma(i)-1}^\sigma). \end{aligned}$$

Let us show that this indeed defines a complementary polyhedron. The two properties read as follows:

(C1) If $T \subset B, B' \subset P$ are contained in the parabolic subgroup $P \subset \mathrm{GL}(3)$ and $\alpha \in X^*(P)$ is a character, then $d(B)(\alpha) = d(B')(\alpha)$.

(C2) Let B be a Borel subgroup with corresponding basis $\Delta_B = \{\alpha, \beta\}$. If B', B'' are the Borel subgroups with $\Delta_{B'} = \{\alpha + \beta, -\alpha\}$, $\Delta_{B''} = \{\alpha + \beta, -\beta\}$ then

$$\begin{aligned} d(B)(\alpha) &\le d(B')(\alpha) \\ d(B)(\beta) &\le d(B'')(\beta). \end{aligned}$$

Proof. ($C1$). We consider the maximal parabolic subgroup

$$P = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}.$$

The characters of P are given by tuples $(\alpha_1, \alpha_1, \alpha_2) \in \mathbb{Z}^3$ and there are two Borel subgroups contained in P, namely

$$B_{\mathrm{id}} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}, \; B_{12} = \begin{pmatrix} * & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}.$$

If B_{id} corresponds to the flag $0 \subset \mathscr{E}_1 \subset \mathscr{E}_2 \subset \mathscr{E}$, then the flag of B_{12} takes the form $0 \subset \mathscr{F}_1 \subset \mathscr{E}_2 \subset \mathscr{E}$. Hence we can compute

$$\begin{aligned} d(B_{\mathrm{id}})(\alpha_1, \alpha_1, \alpha_2) &= \alpha_1 \deg(\mathscr{E}_1) + \alpha_1 \deg(\mathscr{E}_2/\mathscr{E}_1) + \alpha_2 \deg(\mathscr{E}/\mathscr{E}_2) = \\ &= (\alpha_1 - \alpha_2)\deg(\mathscr{E}_2) + \alpha_2 \deg(\mathscr{E}), \\ d(B_{12})(\alpha_1, \alpha_1, \alpha_2) &= \alpha_1 \deg(\mathscr{E}_2/\mathscr{F}_1) + \alpha_1 \deg(\mathscr{F}_1) + \alpha_2 \deg(\mathscr{E}/\mathscr{E}_2) = \\ &= (\alpha_1 - \alpha_2)\deg(\mathscr{E}_2) + \alpha_2 \deg(\mathscr{E}). \end{aligned}$$

The other cases can be obtained similarly.

($C2$). If $B = B_{\mathrm{id}}$ is the Borel subgroup of upper triangular matrices, then $\Delta_B = \{\alpha, \beta\}$ where $\alpha = (1,-1,0)$ and $\beta = (0,1,-1)$. With $\alpha + \beta = (1,0,-1)$, we obtain

$$B' = \begin{pmatrix} * & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \; B'' = \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix}.$$

So if $0 \subset \mathscr{E}_1 \subset \mathscr{E}_2 \subset \mathscr{E}$ is the filtration given by B, this means that the filtrations of B' and B'' are

$$\begin{aligned} B' &: \; 0 \subset \mathscr{F}_1 \subset \mathscr{E}_2 \subset \mathscr{E}, \\ B'' &: \; 0 \subset \mathscr{E}_1 \subset \mathscr{F}_2 \subset \mathscr{E}, \end{aligned}$$

and furthermore $\mathscr{F}_1, \mathscr{E}_1 \subset \mathscr{E}_2$ and $\mathscr{F}_2/\mathscr{E}_1, \mathscr{E}_2/\mathscr{E}_1 \subset \mathscr{E}/\mathscr{E}_1$ are generically opposite subbundles. By Lemma (3.3.7) we obtain the inequalities

$$\begin{aligned} &\deg(\mathscr{F}_1) + \deg(\mathscr{E}_1) \leq \deg(\mathscr{E}_2), \\ &\deg(\mathscr{F}_2) + \deg(\mathscr{E}_2) \leq \deg(\mathscr{E}) + \deg(\mathscr{E}_1). \end{aligned}$$

We can now compute

$$\begin{aligned}
d(B)(\alpha) &= \deg(\mathscr{E}_1) - \deg(\mathscr{E}_2/\mathscr{E}_1) \\
&= 2\deg(\mathscr{E}_1) - \deg(\mathscr{E}_2) \\
&\leq \deg(\mathscr{E}_2) - 2\deg(\mathscr{F}_1) \\
&= d(B')(\alpha), \\
d(B)(\beta) &= \deg(\mathscr{E}_2/\mathscr{E}_1) - \deg(\mathscr{E}/\mathscr{E}_2) \\
&= 2\deg(\mathscr{E}_2) - \deg(\mathscr{E}_1) - \deg(\mathscr{E}) \\
&\leq \deg(\mathscr{E}) + \deg(\mathscr{E}_1) - 2\deg(\mathscr{F}_2) \\
&= d(B'')(\beta).
\end{aligned}$$

Again the other cases can be obtained by similar calculations. □

3.3.2 Alternative description

We give an alternative description of the complementary polyhedron in the case $G = \mathrm{GL}(n)$ using the automorphism group scheme. Let $\mathscr{E}$ be a rank n vector bundle and consider its automorphism group scheme $\mathrm{Aut}(\mathscr{E})$, which has the following Lie algebra (cf. [Con14, Chapter 4-5])

$$\mathrm{Lie}(\mathrm{Aut}(\mathscr{E})) = \mathrm{Hom}(\mathscr{E},\mathscr{E}) = \mathscr{E}^\vee \otimes \mathscr{E}.$$

Note that this is a vector bundle on X. Furthermore, after choosing a generic splitting $\mathscr{E}_\eta \cong (\mathscr{L}_1 \oplus \ldots \oplus \mathscr{L}_n)_\eta$ (which corresponds to the choice of a maximal torus of $\mathrm{GL}(n)(\eta)$) it decomposes into subbundles

$$\mathrm{Lie}(\mathrm{Aut}(\mathscr{E}))_\eta = \left(\mathscr{O}_X^n \oplus \bigoplus_{\alpha\in\Phi(\mathrm{GL}(n)(\eta),T)} \mathscr{L}_\alpha\right)_\eta, \tag{3.1}$$

where the root bundles $\mathscr{L}_\alpha$ can be seen as line bundles on X. Let $T \subset B \subset \mathrm{GL}(n)(\eta)$ be a Borel subgroup corresponding to the basis $\Delta = \{\alpha_1,\ldots,\alpha_r\} \subset \Phi(\mathrm{GL}(n)(\eta),T)$ with vertices $\Lambda = \{\lambda_1,\ldots,\lambda_r\} \subset V$ and let $\{\lambda_1^\vee,\ldots,\lambda_r^\vee\} \subset V^\vee$ be the dual vertices. For any root $\alpha \in \Delta$, we define

$$v(\alpha) := \sum_{i=0}^{r} \lambda_i^\vee(\alpha)\lambda_i \in \Lambda$$

and the *numerical invariant* of B with respect to $v(\alpha)$ as

$$n(B,v(\alpha)) := \deg(\mathscr{L}_\alpha).$$

Note that $v\colon \Delta \to \Lambda$ is a bijection. The complementary polyhedron of $\mathscr{E}$ with respect to T is then defined by letting

$$d(B) := \sum_{i=0}^{l} n(B,\lambda_i)\lambda_i^\vee \in V^\vee.$$

If G is an arbitrary connected reductive linear algebraic group and $\mathscr{P}$ is a G-bundle, the automorphism group scheme $\mathrm{Aut}(\mathscr{P})$ has a similar decomposition of the form (3.1) and the complementary polyhedron can be described in the same manner using numerical invariants. In particular, a complementary polyhedron is determined by the numerical invariants of the Borel subgroups of $G(\eta)$ [Beh95, Note 3.8].

3.3.9 Example. We let $\mathscr{E} = \mathscr{O}_X(m) \oplus \mathscr{O}_X(-m)$ for a non-negative integer $m \geq 0$. We can interpret this as an SL(2)-bundle which already has a global T-reduction. The root system $\Phi = \Phi(\mathrm{SL}(2), T)$ consists of two roots which we denote by $\alpha_1 = (1, -1)$ and $\alpha_2 = (-1, 1)$. The set of weights is then given by $\Lambda = \{\lambda_1, \lambda_2\}$ where $2\lambda_i = \alpha_i$. Since all roots have inner product two, we have that the dual root system is given by $\Phi^\vee \cong \Phi$ and the dual weights are $\Lambda^\vee \cong \Lambda$.

Consider the associated group scheme $\mathrm{Aut}(\mathscr{E})$. Its Lie algebra is given by

$$\mathrm{Lie}(\mathrm{Aut}(\mathscr{E})) = \mathrm{Hom}(\mathscr{E}, \mathscr{E}) = \mathscr{E} \otimes \mathscr{E}^\vee = \mathscr{O}_X^2 \oplus \mathscr{O}_X(2m) \oplus \mathscr{O}_X(-2m)$$

and decomposes as $\mathrm{Lie}(\mathrm{Aut}(\mathscr{E})) = \mathscr{O}_X^2 \oplus \mathscr{L}_{\alpha_1} \oplus \mathscr{L}_{\alpha_2}$ where the root bundles are $\mathscr{L}_{\pm\alpha_1} = \mathscr{O}_X(\pm 2m)$.

There are two Borel subgroups B^+ and B^- of upper triangular and lower triangular matrices, respectively, with corresponding bases $\Delta(B^+) = \{\alpha_1\}$ and $\Delta(B^-) = \{\alpha_2\}$. For the complementary polyhedron we need to define the numerical invariants $n(B, \lambda)$.

1. Case B^+: Here λ_1 is the only weight. Furthermore $v(\alpha_1) = \lambda_1^\vee(\alpha_1)\lambda_1 = \lambda_1$. Hence $n(B^+, \lambda_1) = \deg(\mathscr{L}_{\alpha_1}) = 2m$.

2. Case B^-: The same calculations give $n(B^-, \lambda_2) = \deg(\mathscr{L}_{\alpha_2}) = -2m$.

We can now define $d(B^\pm)$ as

$$\begin{aligned} d(B^+) &= n(B^+, \lambda_1)\lambda_1^\vee = m\alpha_1^t = (m, -m)^t, \\ d(B^-) &= n(B^-, \lambda_2)\lambda_2^\vee = -m\alpha_2^t = (m, -m)^t. \end{aligned}$$

The complementary polyhedron reduces to one point $F = \left\{(m, -m)^t\right\}$. We see that $\mathscr{E}$ is semistable if and only if $m = 0$, and B^+ is canonical if $m > 0$.

Note that B^+ corresponds to the filtration $0 \subsetneq \mathscr{O}_X(m) \subsetneq \mathscr{E}$, which is indeed the Harder-Narasimhan filtration. This can be seen from the fact that sections of $\mathscr{L}_{\alpha_1} = \mathscr{O}_X(2m)$ correspond to elements of $\mathrm{Hom}(\mathscr{O}_X(-m), \mathscr{O}_X(m))$ and these leave the filtration invariant.

3.3.10 Example. Let $\mathscr{E} = \mathscr{O}_X(m) \oplus \mathscr{O}_X(n) \oplus \mathscr{O}_X(-(m+n))$ for $m \leq n$, which we interpret as an SL(3)-bundle with a global T-reduction. The root system of SL(3) is given by $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3\}$ where

$$\alpha_1 = (1, -1, 0), \ \alpha_2 = (0, 1, -1), \ \alpha_3 = (1, 0, -1).$$

The corresponding weights are $\Lambda = \{\pm\lambda_1, \pm\lambda_2, \pm\lambda_3\}$ with

$$3\lambda_1 = (2,-1,-1),\ 3\lambda_2 = (1,1,-2),\ 3\lambda_3 = (1,-2,1).$$

Once again $\Phi^\vee = \Phi$ and $\Lambda^\vee = \Lambda$. The Lie algebra of $\mathrm{Aut}(\mathscr{E})$ decomposes as

$$\mathrm{Lie}(\mathrm{Aut}(\mathscr{E})) = \mathscr{O}_X^3 \bigoplus_{i=1}^{3} (\mathscr{L}_{\alpha_i} \oplus \mathscr{L}_{-\alpha_i}).$$

We see furthermore that

$$\begin{aligned}\mathscr{L}_{\pm\alpha_1} &= \mathscr{O}_X(\pm(m-n)),\\ \mathscr{L}_{\pm\alpha_2} &= \mathscr{O}_X(\pm(m+2n)),\\ \mathscr{L}_{\pm\alpha_3} &= \mathscr{O}_X(\pm(2m+n)).\end{aligned}$$

Suppose now that B is a Borel subgroup given by the basis $\Delta(B) = \{\alpha, \beta\}$. The weights of B are then λ_α, λ_β where $3\lambda_\alpha = 2\alpha + \beta$ and $3\lambda_\beta = \alpha + 2\beta$. We compute

$$\begin{aligned}v(\alpha) &= \lambda_\alpha^\vee(\alpha)\lambda_\alpha + \lambda_\beta^\vee(\alpha)\lambda_\beta = \lambda_\alpha,\\ v(\beta) &= \lambda_\alpha^\vee(\beta)\lambda_\alpha + \lambda_\beta^\vee(\beta)\lambda_\beta = \lambda_\beta,\end{aligned}$$

which gives $n(B,\lambda_\alpha) = \deg(\mathscr{L}_\alpha)$, $n(B,\lambda_\beta) = \deg(\mathscr{L}_\beta)$ and finally

$$d(B) = \deg(\mathscr{L}_\alpha)\lambda_\alpha^\vee + \deg(\mathscr{L}_\beta)\lambda_\beta^\vee.$$

There are exactly $|\mathscr{S}_3| = 3! = 6$ Borel subgroups containing the given torus (cf. 3.A). Denote these by B_σ for $\sigma \in \mathscr{S}_3$. One computes

$$\begin{aligned}d(B_{\mathrm{id}}) = (m,n,-(m+n)),\ d(B_{(123)}) = (m,n,-(m+n)),\ d(B_{(132)}) = (m,n,-(m+n)),\\ d(B_{(12)}) = (m,n,-(m+n)),\ d(B_{(13)}) = (m,n,-(m+n)),\ d(B_{(23)}) = (m,n,-(m+n)).\end{aligned}$$

We see that the complementary polyhedron is given by $F = \{(m,n,-(m+n))\} = \{m\alpha_3 + n\alpha_2\}$. There are several cases to distinguish (suppose that $m \leq n$).

1. $m = n = 0$. Then $F = 0$ and $\mathscr{E}$ is semistable.

2. $m = n \neq 0$. Then $F = \{m(1,1,-2)\}$ which gives the filtration $0 \subsetneq \mathscr{O}_X(m) \oplus \mathscr{O}_X(m) \subsetneq \mathscr{E}$.

3. $0 \leq m < n$. Then F lies in the chamber with corners λ_2 and $-\lambda_3$, which corresponds to the filtration $0 \subsetneq \mathscr{O}_X(n) \subsetneq \mathscr{O}_X(n) \oplus \mathscr{O}_X(m) \subsetneq \mathscr{E}$.

4. $m < 0 < n$. Then F lies in the chamber with corners $-\lambda_1$ and $-\lambda_3$, which corresponds to the filtration $0 \subsetneq \mathscr{O}_X(n) \subsetneq \mathscr{O}_X(n) \oplus \mathscr{O}_X(-(m+n)) \subsetneq \mathscr{E}$.

5. $m < n \leq 0$. Then F lies in the chamber with corners $-\lambda_1$ and $-\lambda_2$, which corresponds to the filtration $0 \subsetneq \mathscr{O}_X(-(m+n)) \subsetneq \mathscr{O}_X(-(m+n)) \oplus \mathscr{O}_X(n) \subsetneq \mathscr{E}$.

3.3.11 Remark. In the previous examples, we restricted ourselves to degree zero vector bundles, or in other words to SL(n) rather than GL(n). The computations for the GL(n) case are basically the same, but can also be obtained by a projection method. We demonstrate this for a vector bundle that splits as a direct sum of line bundles. It also works for a non-split bundle (see Example 3.3.12).

Let $\mathscr{E}$ be a GL(n)-bundle with a global T-reduction; that is, a vector bundle that splits as $\mathscr{E} = \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_n$. The complementary polyhedron can be obtained by letting

$$F = (\deg(\mathscr{L}_1), \ldots, \deg(\mathscr{L}_n)) \in \mathbb{R}^n$$

and projecting this point to the hyperplane $H_0 := \{x_1 + \ldots + x_n = 0\}$ (see Figure 3.3). For example, in the case $n = 2$, the point $(\deg(\mathscr{L}_1), \deg(\mathscr{L}_2)) \in \{x_1 + x_2 = \deg(\mathscr{E})\}$ maps to

$$F_0 = \left(\frac{\deg(\mathscr{L}_1) - \deg(\mathscr{L}_2)}{2}, \frac{\deg(\mathscr{L}_2) - \deg(\mathscr{L}_1)}{2} \right) \in H_0.$$

This also reflects the fact that the vector bundle $\mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_n$ is semistable if and only if $\deg(\mathscr{L}_1) = \cdots = \deg(\mathscr{L}_n)$. If $\mathscr{E}$ is an SL(n)-bundle, then $\deg(\mathscr{E}) = 0$ and clearly $F = F_0$.

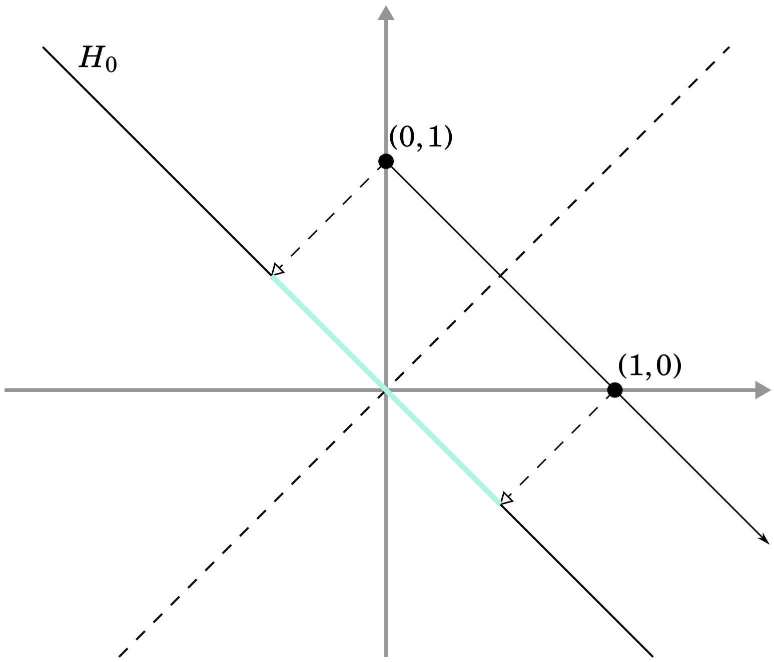

Figure 3.3: Projection of complementary polyhedron

3.3.12 Example. (A non-split example [HL10, page 13]) Let $\mathscr{E}$ be a vector bundle given by a non-trivial extension $0 \to \mathscr{L}_0 \to \mathscr{E} \to \mathscr{L}_1 \to 0$, where the $\mathscr{L}_i$ are line bundles with $\deg \mathscr{L}_0 = 0$ and $\deg \mathscr{L}_1 = 1$. Such an extension exists whenever the genus $g \geq 2$. In this case $\mathscr{E}$ is semistable with slope $\mu(\mathscr{E}) = \frac{1}{2}$.

We can choose a generic splitting $\mathscr{E}_\eta \cong (\mathscr{L}_0 \oplus \mathscr{L})_\eta$ such that the Borel subgroup B^+ of upper triangular matrices corresponds to $0 \subset \mathscr{L}_0 \subset \mathscr{E}$ and the opposite Borel B^- corresponds to the subbundle $0 \subset \mathscr{L} \subset \mathscr{E}$ of degree $\deg \mathscr{L} = l < 1$. Then $d(B^+) = (\deg(\mathscr{L}_0), \deg(\mathscr{E}/\mathscr{L}_0)) = (0,1)$ and $d(B^-) = (\deg(\mathscr{E}/\mathscr{L}), \deg(\mathscr{L})) = (1-l, l)$. The complementary polyhedron is obtained by defining (see also Figure 3.3)

$$F = \{(0,1) + t(1-l, l-1) \mid t \in [0,1]\}.$$

The projection to the hyperplane $\{(x,y) \mid x+y=0\} \cong \mathbb{R}$ is given by $(x,y) \mapsto \frac{1}{2}(x-y, y-x)$ and maps F to the complementary polyhedron $[-\frac{1}{2}, \frac{1}{2}(1-l)]$, which obviously contains zero.

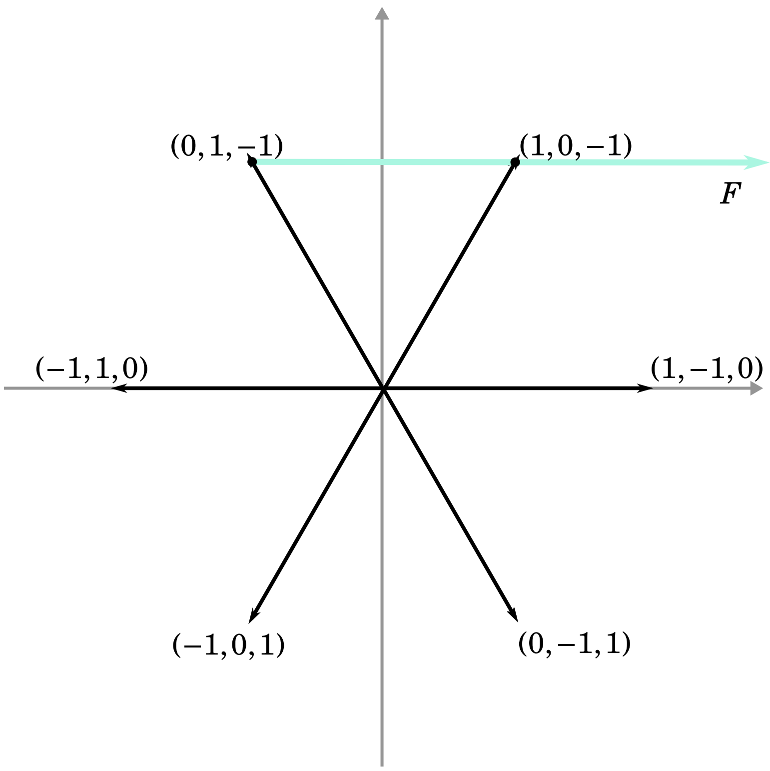

Figure 3.4: Complementary polyhedron of Example 3.3.13

3.3.13 Example. Let $\mathscr{E} = \mathscr{F} \oplus \mathscr{O}_X(-1)$ where $\mathscr{F}$ is the rank two bundle of Example (3.3.12). We can choose a generic splitting such that the six Borel subgroups B_σ

correspond to the following filtrations (cf. 3.A):

$$\begin{aligned}
B_{\mathrm{id}} &: 0 \subset \mathscr{L}_0 \subset \mathscr{F} \subset \mathscr{E} \\
B_{(12)} &: 0 \subset \mathscr{L} \subset \mathscr{F} \subset \mathscr{E} \\
B_{(23)} &: 0 \subset \mathscr{L}_0 \subset \mathscr{L}_0 \oplus \mathscr{O}_X(-1) \subset \mathscr{E} \\
B_{(13)} &: 0 \subset \mathscr{O}_X(-1) \subset \mathscr{L} \oplus \mathscr{O}_X(-1) \subset \mathscr{E} \\
B_{(123)} &: 0 \subset \mathscr{O}_X(-1) \subset \mathscr{L}_0 \oplus \mathscr{O}_X(-1) \subset \mathscr{E} \\
B_{(132)} &: 0 \subset \mathscr{L} \subset \mathscr{L} \oplus \mathscr{O}_X(-1) \subset \mathscr{E}.
\end{aligned}$$

We can then compute the complementary polyhedron F using Example (3.3.8) to obtain the following points:

$$\begin{aligned}
d(B_{\mathrm{id}}) &= (0, 1, -1) \\
d(B_{(12)}) &= (1-l, l, -1) \\
d(B_{(23)}) &= (0, 1, -1) \\
d(B_{(13)}) &= (1-l, l, -1) \\
d(B_{(123)}) &= (0, 1, -1) \\
d(B_{(132)}) &= (1-l, l, -1).
\end{aligned}$$

Note that projecting the polyhedron to the hyperplane $\{(x, -x, 0)\} \subset \mathbb{R}^3$ (which means projecting $\mathscr{E}$ to its direct summand $\mathscr{F}$) gives the complementary polyhedron of $\mathscr{F}$. We also see that $\mathscr{E}$ is not stable and the special facet is the positive span of $(1, 1, -2)$, which gives the Harder-Narasimhan filtration $0 \subset \mathscr{F} \subset \mathscr{E}$.

3.4 Torus reductions

Let $T \subset G$ be a maximal torus, that is $T \cong (\mathbb{G}_m)^r$. Since $\mathbb{G}_m = \mathrm{GL}(1)$ a principal T-bundle can be interpreted as a direct sum of line bundles. We will say that a principal G-bundle $\mathscr{P}$ admits a T-reduction if there is a T-bundle $\mathscr{P}_T$ and a reduction $\beta \colon X \to \mathscr{P}/T$ with $\beta^*(\mathscr{P}) \cong \mathscr{P}_T$. For vector bundles the problem of admitting a T-reduction is equivalent to splitting as a direct sum of line bundles.

As we have seen before in some examples, the complementary polyhedron associated to a principal G-bundle $\mathscr{P}$ reduces to a point if $\mathscr{P}$ admits a global T-reduction. In this section we will investigate the converse statement.

3.4.1 Proposition. *Let $\mathscr{P}$ be a principal G-bundle and $T \subset G$ a maximal torus. If $\mathscr{P}$ admits a T-reduction, then its complementary polyhedron d on $\Phi(G, T)$ reduces to a point.*

Proof. Recall that if $T \subset B \subset G$ is a Borel subgroup, then $d(B) \colon X^*(T) \to \mathbb{R}$ is defined by $d(B)(\alpha) = \deg(\mathscr{L}_B(\alpha))$. The line bundle $\mathscr{L}_B(\alpha)$ is defined as $\mathscr{L}_B(\alpha) = \mathscr{P}_B \times^{\alpha} \mathbb{K}$ where $\mathscr{P}_B$ is the B-bundle constructed from $\mathscr{P}$. If $\mathscr{P}$ has a T-reduction, then $\mathscr{P}_B = \mathscr{P}_T \times^T B$ for a T-bundle $\mathscr{P}_T$ and hence $\mathscr{L}_B(\alpha) = (\mathscr{P}_T \times^T B) \times^\alpha \mathbb{K} = \mathscr{P}_T \times^\alpha \mathbb{K}$, which implies $d(B) = d(B')$ for all Borel subgroups $T \subset B, B' \subset G$. □

3.4.2 Corollary. *Let $\mathscr{P}$ be a principal G-bundle on $\mathbb{P}^1$. Then for any choice of a generic maximal torus, the complementary polyhedron of $\mathscr{P}$ reduces to a point.*

Proof. Since any G-bundle on $\mathbb{P}^1$ has a global reduction to a maximal torus (see [Gro57] and more recently also [Ram83, MS02]), this follows from Proposition (3.4.1). □

In case $G = \mathrm{SL}(2)$ the converse statement can be easily proven by hand.

3.4.3 Lemma. *Let $\mathscr{P}$ be an* SL(2)*-bundle and T a generic torus such that the complementary polyhedron of $\mathscr{P}$ with respect to T reduces to a point. Then $\mathscr{P}$ admits a global reduction to T.*

Proof. The root system $\Phi(\mathrm{SL}(2)(\eta), T)$ is of type A_1, so we are in the situation of Example (3.1.10).

Let $\mathscr{E}$ be the rank two vector bundle associated to $\mathscr{P}$. The generic torus gives a generic splitting $\mathscr{E}_\eta = (\mathscr{L}^+ \oplus \mathscr{L}^-)_\eta$ for two line subbundles $\mathscr{L}^\pm$ of $\mathscr{E}$ and the Borel subgroups $B^\pm$ of upper (and lower) triangular matrices then correspond to the filtrations $0 \subset \mathscr{L}^\pm \subset \mathscr{E}$. The complementary polyhedron is given by $d^+ = (\deg(\mathscr{L}^+), -\deg(\mathscr{L}^+))$ and $d^- = (-\deg(\mathscr{L}^-), \deg(\mathscr{L}^-))$. Since we assume that it reduces to a point, this gives

$$\deg(\mathscr{L}^+) + \deg(\mathscr{L}^-) = 0 = \deg(\mathscr{E}),$$

and we can conclude with Lemma (3.3.7). □

3.4.4 Lemma. *Let d be a complementary polyhedron in the root system $\Phi \subset V$ such that equality holds in the second property ($C2$). Then $F = \mathrm{conv}_{T\subset B\ Borel}\{d(B)\}$ reduces to a point.*

Proof. Let B be a Borel subgroup and B_α the unique Borel subgroup that is α-conjugate to B. If the vertices of B are $\lambda_1, \ldots, \lambda_n$, then the vertices of B_α are $\sigma(\lambda_1), \lambda_2, \ldots, \lambda_n$ where σ is the reflection fixing the hyperplane orthogonal to α.

The point $d(B_\alpha)$ is uniquely determined by its values on a basis of V. Since α is a root of B and $-\alpha$ is a root of B_α, α is not contained in $\langle \lambda_2, \ldots, \lambda_n \rangle$. Since the λ_i are linearly independent, the set $\{\pm\alpha, \lambda_2, \ldots, \lambda_n\}$ is a basis of V. We already know that $d(B_\alpha)(\lambda_i) = d(B)(\lambda_i)$ for $i = 2, \ldots, n$ from the property ($C1$). But the equality in ($C2$) means that $d(B_\alpha)(\alpha) = d(B)(\alpha)$ and hence $d(B) = d(B_\alpha)$. Since any Borel subgroup has exactly n conjugate subgroups, this implies that $d(B) = d(B')$ for all borel subgroups B, B' and $F = \{d(B)\}$. □

3.4.5 Conjecture. *Let $\mathscr{P}$ be a principal G-bundle and $T \subset G(\eta)$ a maximal torus. Denote by $F(T)$ the complementary polyhedron given by $\mathscr{P}$ in the root system $\Phi(G(\eta), T)$. Then $F(T)$ is a point if and only if T can be extended to X; that is, $\mathscr{P}$ admits a global torus reduction.*

In the case $G = \mathrm{GL}(n)$, Conjecture (3.4.5) could be proven by an induction argument using Lemma (3.4.3) and (3.4.4). It would also be interesting to study the analogous statement for principal Higgs bundles; that is, the Higgs polyhedron reduces to a point if and only if the Higgs bundle admits a Higgs reduction to a torus.

3.5 Higgs Bundles

Let G be a connected reductive linear algebraic group and $\mathscr{L}$ a line bundle on X of arbitrary degree. In this section we will construct a complementary polyhedron for the following objects.

3.5.1 DEFINITION (Higgs vector bundle). A pair $(\mathscr{E}, \varphi\colon \mathscr{E} \to \mathscr{E} \otimes \mathscr{L})$ consisting of a vector bundle $\mathscr{E}$ and a twisted endomorphism $\varphi\colon \mathscr{E} \to \mathscr{E} \otimes \mathscr{L}$ is a Higgs bundle. A Higgs subbundle is given by a φ-invariant subbundle $\mathscr{F} \subset \mathscr{E}$, meaning that $\varphi(\mathscr{F}) \subset \mathscr{F} \otimes \mathscr{L}$. The Higgs bunde $(\mathscr{E}, \varphi)$ is semistable if for all Higgs subbundles $\mathscr{F} \subset \mathscr{E}$

$$\mu(\mathscr{F}) \leq \mu(\mathscr{E}).$$

A filtration $0 = \mathscr{E}_0 \subsetneq \mathscr{E}_1 \subsetneq \ldots \subsetneq \mathscr{E}_s = \mathscr{E}$ of the Higgs bundle $(\mathscr{E}, \varphi)$ into Higgs subbundles is called Harder-Narasimhan (HNF) if the following holds:

1. The quotients $\mathscr{E}_i/\mathscr{E}_{i-1}$ are semistable Higgs bundles for $i = 1, \ldots, s$.

2. $\mu(\mathscr{E}_1/\mathscr{E}_0) > \mu(\mathscr{E}_i/\mathscr{E}_{i-1}) > \ldots > \mu(\mathscr{E}_s/\mathscr{E}_{s-1})$.

An automorphism of a Higgs bundle $(\mathscr{E}, \varphi)$ is an automorphism $f : \mathscr{E} \to \mathscr{E}$ of the underlying vector bundle that is compatible with the Higgs structure, meaning that there is a commutative diagram

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\ \varphi\ } & \mathscr{E} \otimes \mathscr{L} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes \mathrm{id}_{\mathscr{L}}} \\ \mathscr{E} & \xrightarrow{\ \varphi\ } & \mathscr{E} \otimes \mathscr{L}. \end{array}$$

We denote the automorphism group of a Higgs bundle $(\mathscr{E}, \varphi)$ by $\mathrm{Aut}(\mathscr{E}, \varphi)$.

The proof of the existence and uniqueness of the Harder-Narasimhan filtration in the Higgs vector bundle case is the same as in the vector bundle case (see [Sim94]).

3.5.2 DEFINITION (Principal Higgs bundle). A principal G-Higgs bundle $(\mathscr{P},\varphi)$ consists of a principal G-bundle $\mathscr{P}$ and a section $\varphi\colon \mathscr{O}_X \to (\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{g}) \otimes \mathscr{L}$. A pair (β,Q) consisting of a parabolic subgroup $Q \subset G$ and a section $\beta\colon X \to \mathscr{P}/Q$ is a Higgs reduction if φ factors through $\beta^*\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{q} \to \mathscr{P} \times^{\mathrm{Ad}} \mathfrak{g}$, i.e. there is a morphism φ_Q such that the following diagram commutes

$$\begin{array}{ccc} (\beta^*\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{q}) \otimes \mathscr{L} & \longrightarrow & (\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{g}) \otimes \mathscr{L}. \\ & {\scriptstyle \varphi_Q} \nwarrow \quad \nearrow {\scriptstyle \varphi} & \\ & \mathscr{O}_X & \end{array}$$

In other words, $(\beta^*\mathscr{P},\varphi_Q)$ is a principal Q-Higgs bundle. A principal G-Higgs bundle is semistable if for all Higgs reductions (β,Q) the following holds

$$\deg(\beta^*\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{q}) \le 0.$$

The nonnegative integer

$$\mathrm{ideg}^H(\mathscr{P}) := \max\left\{ \deg(\beta^*\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{q}) \,\middle|\, (\beta,Q) \text{ Higgs reduction} \right\}$$

is called the *Higgs degree of instability* of $\mathscr{P}$. A canonical Higgs reduction is a Higgs reduction (β,Q) such that $\deg(\beta^*\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{q}) = \mathrm{ideg}^H(\mathscr{P})$.

Note that a canonical Higgs reduction satisfies the following properties:

1. For any dominant character $\chi\colon Q \to \mathbb{G}_m$, let $\mathscr{L}(\beta,\chi) := \beta^*\mathscr{P} \times^{\chi} \mathbb{K}$ be the associated line bundle. Then $\deg(\mathscr{L}(\beta,\chi)) > 0$.

2. The extension of the Q-bundle $\beta^*\mathscr{P}$ to the Levi quotient $L = Q/R_u(Q)$ is a semistable L-Higgs bundle. Here the Higgs structure is induced by φ_Q, which is possible since the Lie algebra decomposes as $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{r}$ where $\mathfrak{l} = \mathrm{Lie}(Q)$ and $\mathfrak{r} = \mathrm{Lie}(R_u(Q))$.

3.5.3 Example (Non-reductive automorphism group). Behrend has constructed complementary polyhedra only for group schemes that are reductive. However, the automorphism group of a Higgs bundle does not have to be reductive, as the following example shows.

Let V be a two dimensional $\mathbb{K}$-vector space and $\mathscr{E} = V \otimes \mathscr{O}_X$, $\mathscr{L} = \mathscr{O}_X$. Hence, a Higgs structure on $\mathscr{E}$ is the same as an endomorphism $\varphi\colon V \to V$, and the automorphism group of $(\mathscr{E},\varphi)$ is

$$\mathrm{Aut}(\mathscr{E},\varphi) = \left\{ g \in \mathrm{GL}(2) \,\middle|\, g\varphi = \varphi g \right\}.$$

1. Suppose that φ has Jordan normal form $\varphi = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. For an arbitrary

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2)$$

we compute

$$g\varphi = \begin{pmatrix} \lambda a & a+\lambda b \\ \lambda c & c+\lambda d \end{pmatrix} = \begin{pmatrix} c+\lambda a & d+\lambda b \\ \lambda c & \lambda d \end{pmatrix} = \varphi g,$$

which forces $c = 0$, $d = a$ and hence

$$g = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Finally this implies $\mathrm{Aut}(\mathscr{E},\varphi) = \mathbb{G}_m \ltimes \mathbb{G}_a$, which is a non-reductive algebraic group.

2. If φ is diagonalisable as

$$\varphi = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},$$

one computes $\mathrm{Aut}(\mathscr{E},\varphi) = \mathrm{GL}(2)$ if $\lambda = \mu$ and $\mathrm{Aut}(\mathscr{E},\varphi) = \mathbb{G}_m \times \mathbb{G}_m$ if φ has two eigenvalues $\lambda \neq \mu$. Both automorphism groups are reductive.

Our construction of the Higgs polyhedron will not use the automorphism group of the Higgs bundle. The automorphism group will not be of importance in the following.

3.5.1 Construction of the Higgs Polyhedron

Let $(\mathscr{P},\varphi)$ be a principal G-Higgs bundle which will be fixed for the remainder of this section. We fix a maximal torus (or a generic splitting) $T \subset G(\eta)$ and denote by d the complementary polyhedron of $\mathscr{P}$ with respect to T without a Higgs structure as constructed by Behrend. Broadly speaking, we will distort the polyhedron where the distortion is determined by the Higgs structure φ. Recall that the Higgs polyhedron will be a function

$$d^H\colon \{B \mid T \subset B \subset G(\eta) \text{ Borel}\} \to X^*(T)\vee \otimes \mathbb{R} =: V^\vee.$$

The automorphisms $\mathrm{Aut}(\mathscr{P})$ form a reductive group scheme over X whose Lie algebra $\mathrm{Lie}(\mathrm{Aut}(\mathscr{P})) = \mathscr{P} \times^{\mathrm{Ad}} \mathfrak{g}$ has a generic decomposition into root bundles

$$(\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{g})_\eta \;=\; \left(\mathscr{O}_X^n \oplus \bigoplus_{\alpha\in\Phi(G(\eta),T)} \mathscr{L}_\alpha\right)_\eta,$$

where the $\mathscr{L}_\alpha$ are one dimensional and hence can be seen as line bundles on X. The Higgs structure φ is a section of the twisted adjoint bundle and we can consider the projections

$$\varphi_\alpha : \mathscr{O}_{X,\eta} \to (\mathscr{L}_\alpha \otimes \mathscr{L})_\eta \subset (\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{g})_\eta \otimes \mathscr{L}$$

for any root $\alpha \in \Phi(G(\eta), T)$. We define the numbers

$$\epsilon(\alpha) := \begin{cases} 0 & \text{if for all decompositions } \alpha = \alpha_1 + \ldots + \alpha_s \text{ there is an } i \text{ with } \varphi_{\alpha_i} = 0, \\ 1 & \text{if there is a decomposition } \alpha = \alpha_1 + \ldots + \alpha_s \text{ with } \varphi_{\alpha_i} \neq 0 \text{ for all } i. \end{cases}$$

3.5.4 DEFINITION (Higgs polyhedron). Let $T \subset B \subset G(\eta)$ be a Borel subgroup and $R(B) \subset \Phi(G(\eta), T)$ the corresponding parabolic subset. Recall that $\alpha^\vee \in V^\vee$ is the dual root of α and define the *Higgs polyhedron* as

$$d^H(B) := d(B) + |\deg(\mathscr{L})| \sum_{\alpha \notin R(B)} \epsilon(\alpha)\alpha^\vee.$$

3.5.5 Remark. In the definition of the Higgs polyhedron we can replace the constant $|\deg(\mathscr{L})|$ by any positive $C > 0$ and still get a complementary polyhedron. However, we will need that $C \geq |\deg(\mathscr{L})|$ in the proof of Lemma (3.5.15).

Before showing that the Higgs polyhedron indeed defines a complementary polyhedron on the root system $\Phi(G(\eta), T)$, we give some examples to illustrate the computation of the Higgs polyhedron.

3.5.6 Example. Let $\mathscr{E}$ be an unstable rank two vector bundle of degree zero and $\Phi := \{\pm\alpha\}$ the root system of SL(2) (cf. Example 3.1.10). Denote by $\mathscr{F}_{\max}$ the maximal destabilising subbundle, i.e. the maximal subbundle with $x := \deg(\mathscr{F}_{\max}) > \deg(\mathscr{E}) = 0$. We can choose a generic splitting $\mathscr{E}_\eta = (\mathscr{F}_{\max} \oplus \mathscr{F})_\eta$ so that the two Borel subgroups $B^\pm$ correspond to the following filtrations

$$B^+ : \ 0 \subset \mathscr{F}_{\max} \subset \mathscr{E}, \ \ B^- : \ 0 \subset \mathscr{F} \subset \mathscr{E},$$

and $\deg(\mathscr{F}) \leq -\deg(\mathscr{F}_{\max}) = -x < 0$ since $\mathscr{F}$ is generically opposite to $\mathscr{F}_{\max}$. We also suppose that $\mathscr{E}$ is equipped with a Higgs structure $\varphi : \mathscr{E} \to \mathscr{E} \otimes \mathscr{L}$ and consider the following cases:

1. $\mathscr{F}_{\max}$ is φ-invariant.

2. $\mathscr{F}_{\max}$ is not φ-invariant and $\mathscr{F}$ is φ-invariant.

3. $\mathscr{F}_{\max}$ is not φ-invariant and $\mathscr{F}$ is not φ-invariant.

1. If $\mathscr{F}_{\max}$ is φ-invariant, we get $d^H(B^+) = (\deg(\mathscr{F}_{\max}), -\deg(\mathscr{F}_{\max}))$. For the second Borel subgroup, $d^H(B^-) = (-\deg(\mathscr{F}), \deg(\mathscr{F}))$ if $\mathscr{F}$ is invariant, and $d^H(B^-) = (-\deg(\mathscr{F}) + |\deg(\mathscr{L})|, \deg(\mathscr{F}) - |\deg(\mathscr{L})|)$ if it is not invariant. But since

$0 < \deg(\mathscr{F}_{\max}) \leq -\deg(\mathscr{F}) \leq -\deg(\mathscr{F}) + |\deg(\mathscr{L})|$, in both of these cases the Higgs bundle $\mathscr{E}$ is unstable with Harder-Narasimhan filtration $0 \subset \mathscr{F}_{\max} \subset \mathscr{E}$.

2. Consider the map $\mathscr{F}_{\max} \overset{\varphi}{\to} \mathscr{E} \otimes \mathscr{L} \to \mathscr{F} \otimes \mathscr{L}$. It is not generically zero because $\mathscr{F}_{\max}$ is not φ-invariant. Therefore, it cannot be zero. A non-zero map of line bundles implies an inequality of the slopes, hence

$$0 < x = \deg(\mathscr{F}_{\max}) \leq \deg(\mathscr{F}) + \deg(\mathscr{L}).$$

In particular, $\deg(\mathscr{L}) \geq -\deg(\mathscr{F}) > 0$. As $\mathscr{F}$ is φ-invariant, we can now compute the Higgs polyhedron as

$$d^H(B^+) = (\deg(\mathscr{F}_{\max}), -\deg(\mathscr{F}_{\max})) + \deg(\mathscr{L})(-1,1) = (x - \deg(\mathscr{L}), \deg(\mathscr{L}) - x),$$
$$d^H(B^-) = (-\deg(\mathscr{F}), \deg(\mathscr{F})).$$

From the inequalities $-\deg(\mathscr{F}) > 0$ and $x - \deg(\mathscr{L}) \leq \deg(\mathscr{F}) < 0$, we see that the convex hull of these two points contains zero. Note that in this case $\mathscr{E}$ is indeed a semistable Higgs bundle.

3. In this case we again have $x = \deg(\mathscr{F}_{\max}) \leq \deg(\mathscr{F}) + \deg(\mathscr{L})$ and $\deg(\mathscr{L}) > 0$. Furthermore, $\mathscr{F}$ is not φ-invariant and thus

$$\begin{aligned} d^H(B^+) &= (\deg(\mathscr{F}_{\max}), -\deg(\mathscr{F}_{\max})) + \deg(\mathscr{L})(-1,1) \\ &= (x - \deg(\mathscr{L}), \deg(\mathscr{L}) - x), \\ d^H(B^-) &= (-\deg(\mathscr{F}), \deg(\mathscr{F})) + \deg(\mathscr{L})(1,-1) \\ &= (-\deg(\mathscr{F}) + \deg(\mathscr{L}), \deg(\mathscr{F}) - \deg(\mathscr{L})). \end{aligned}$$

Again $\mathscr{E}$ is a semistable Higgs bundle, since $-\deg(\mathscr{F}) + \deg(\mathscr{L}) > 0$.

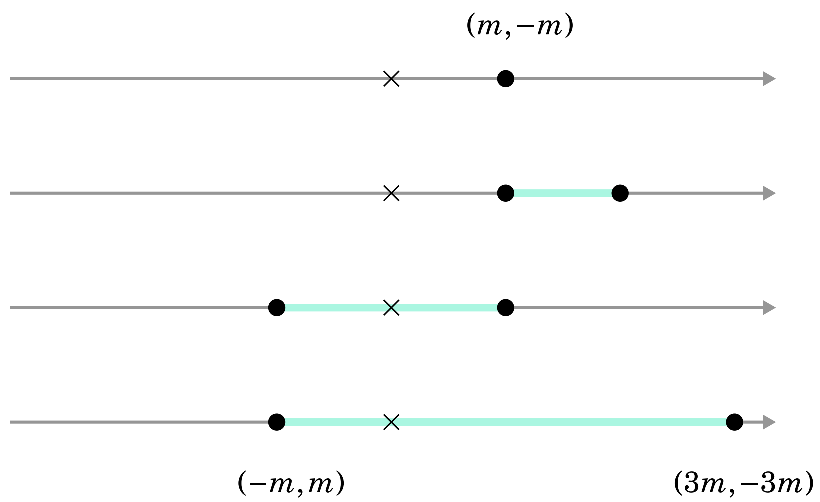

Figure 3.5: The four possible Higgs polyhedra in Example 3.5.7

3.5.7 Example. Let $\mathscr{E} = \mathscr{O}_X(m) \oplus \mathscr{O}_X(-m)$ which we interpret as an SL(2)-bundle with a global T-reduction. Since $\mathscr{E}$ is semistable if $m = 0$, we assume that $m > 0$ be positive. Then $\mathscr{O}_X(m)$ is the maximal destabilising subbundle. If $\varphi\colon \mathscr{E} \to \mathscr{E} \otimes \mathscr{L}$ is a Higgs structure, we get the following possible Higgs polyhedra.

1. If $\mathscr{O}_X(m)$ is invariant, then $d^H(B^+) = (m, -m)$, and if it is not invariant:
$$d^H(B^+) = (m - \deg(\mathscr{L}), \deg(\mathscr{L}) - m) \text{ and } \deg(\mathscr{L}) \geq 2m.$$
2. If $\mathscr{O}_X(-m)$ is invariant, then $d^H(B^-) = (m, -m)$, and if it is not invariant:
$$d^H(B^-) = (m + |\deg(\mathscr{L})|, -(m + |\deg(\mathscr{L})|)) \text{ and } \deg(\mathscr{L}) \geq -2m.$$

3.5.8 Example. Let $\mathscr{E} = \mathscr{O}_X(m) \oplus \mathscr{O}_X \oplus \mathscr{O}_X(-m)$ for a positive integer $m > 0$ and define a Higgs structure $\varphi\colon \mathscr{E} \to \mathscr{E} \otimes \mathscr{O}_X(m)$ by
$$\varphi = \begin{pmatrix} 0 & 0 & 0 \\ \psi_1 & 0 & 0 \\ 0 & \psi_2 & 0 \end{pmatrix}$$
where $\psi_1\colon \mathscr{O}_X(m) \to \mathscr{O}_X \otimes \mathscr{O}_X(m)$ and $\psi_2\colon \mathscr{O}_X \to \mathscr{O}_X(-m) \otimes \mathscr{O}_X(m)$ are nonzero maps. Since $\mathscr{E}$ has a global torus reduction, we can assume that the Borel subgroups correspond to the flags (cf. 3.A)
$$\begin{aligned} B_{\mathrm{id}} &: 0 \subset \mathscr{O}_X(m) \subset \mathscr{O}_X(m) \oplus \mathscr{O}_X \subset \mathscr{E}, \\ B_{(12)} &: 0 \subset \mathscr{O}_X \subset \mathscr{O}_X(m) \oplus \mathscr{O}_X \subset \mathscr{E}, \\ B_{(23)} &: 0 \subset \mathscr{O}_X(m) \subset \mathscr{O}_X(m) \oplus \mathscr{O}_X(-m) \subset \mathscr{E}, \\ B_{(13)} &: 0 \subset \mathscr{O}_X(-m) \subset \mathscr{O}_X \oplus \mathscr{O}_X(-m) \subset \mathscr{E}, \\ B_{(123)} &: 0 \subset \mathscr{O}_X(-m) \subset \mathscr{O}_X(m) \oplus \mathscr{O}_X(-m) \subset \mathscr{E}, \\ B_{(132)} &: 0 \subset \mathscr{O}_X \subset \mathscr{O}_X \oplus \mathscr{O}_X(-m) \subset \mathscr{E}. \end{aligned}$$

Notice that $\psi_1 = \varphi_{(-1,1,0)}$, $\psi_2 = \varphi_{(0,-1,1)}$ which implies $\epsilon(1,-1,0) = \epsilon(1,0,-1) = \epsilon(0,1,-1) = 0$ and $\epsilon(-1,1,0) = \epsilon(0,-1,1) = \epsilon(-1,0,1) = 1$. This gives the following corners of the Higgs polyhedron:
$$\begin{aligned} d^H(B_{\mathrm{id}}) &= (m,0,-m) + m(-2,0,2) &= (-m,0,m) \\ d^H(B_{(12)}) &= (m,0,-m) + m(-1,-1,2) &= (0,-m,m) \\ d^H(B_{(23)}) &= (m,0,-m) + m(-2,1,1) &= (-m,m,0) \\ d^H(B_{(13)}) &= (m,0,-m) + 0 &= (m,0,-m) \\ d^H(B_{(123)}) &= (m,0,-m) + m(-1,1,0) &= (0,m,-m) \\ d^H(B_{(132)}) &= (m,0,-m) + m(0,-1,1) &= (m,-m,0) \end{aligned}$$

The convex hull of these points contains zero (see Figure 3.6) and $(\mathscr{E},\varphi)$ is a semistable Higgs bundle (it is not a semistable vector bundle with the Harder-Narasimhan filtration $0 \subset \mathscr{O}_X(m) \subset \mathscr{O}_X(m) \oplus \mathscr{O}_X \subset \mathscr{E}$ that corresponds to the Borel subgroup B_{id}).

One could also see this by looking at the proper Higgs subbundles, which are $\mathscr{O}_X(-m)$ and $\mathscr{O}_X(-m) \oplus \mathscr{O}_X$. Then the semistability of $\mathscr{E}$ follows from $\deg(\mathscr{O}_X(-m)) = \deg(\mathscr{O}_X(-m) \oplus \mathscr{O}_X) = -m < \deg(\mathscr{E}) = 0$.

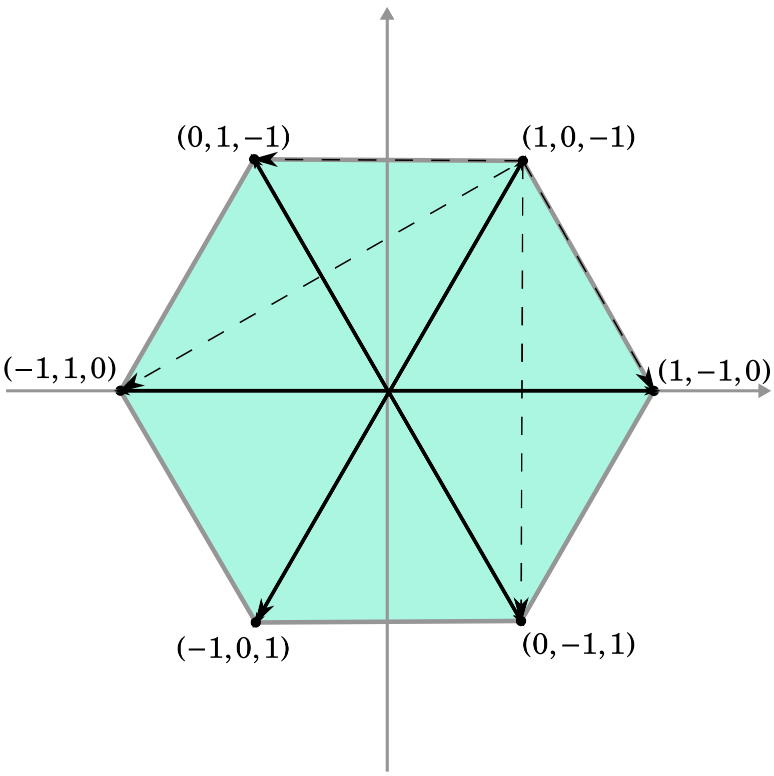

Figure 3.6: Complementary Higgs polyhedron of Example 3.5.8 with $m = 1$

3.5.9 Lemma. *Let $(\mathscr{P},\varphi)$ be a principal Higgs bundle and d^H the Higgs polyhedron as constructed above. Then d^H defines a complementary polyhedron in $\Phi(G(\eta),T)$.*

Proof. We will show the two defining properties of a complementary polyhedron.

To show $(C1)$, consider two Weyl chambers that share a common vertex. In other words, we are given two Borel subgroups $B,B' \subset P \subset G(\eta)$ that are contained in the parabolic subgroup P and a vertex $\lambda \in \mathrm{vert}(P) \subset \mathrm{vert}(B),\mathrm{vert}(B')$. Then

$$d^H(B)(\lambda) = d(B)(\lambda) + |\deg(\mathscr{L})| \sum_{\alpha \notin R(B)} \epsilon(\alpha)\alpha^\vee(\lambda).$$

Since $R(B)$ is a minimal parabolic subset, we know that $\alpha \notin R(B)$ is equivalent to $-\alpha \in R(B)$. We already know that d is a complementary polyhedron and thus $d(B)(\lambda) = d(B')(\lambda)$. Combining these two facts leaves us to show that

$$\sum_{\alpha \in R(B)} \epsilon(\alpha)\alpha^\vee(\lambda) = \sum_{\alpha \in R(B')} \epsilon(\alpha)\alpha^\vee(\lambda).$$

Recall that $R(B) = \{\alpha \in \Phi(G(\eta),T) \mid \forall \lambda \in \mathrm{vert}(B) : \langle \alpha, \lambda \rangle \geq 0\}$ and consider a root $\alpha \in R(B) \setminus R(B')$. Then $-\alpha \in R(B')$ and thus $\langle \alpha, \lambda \rangle \leq 0$. Since $\lambda \in \mathrm{vert}(B) \cap \mathrm{vert}(B')$ we must have $\alpha^\vee(\lambda) = 0$, which shows $d^H(B)(\lambda) = d^H(B')(\lambda)$.

To show $(C2)$, consider two α-conjugate chambers B and B'. This means that $R(B) = (R(B') \setminus \{-\alpha\}) \cup \{\alpha\}$ and α is the unique positive root of B that is negative with respect to B'. Using this we can compute

$$\begin{aligned}\left(\sum_{\beta \notin R(B)} \epsilon(\beta)\beta^\vee(\alpha) - \sum_{\beta \notin R(B')} \epsilon(\beta)\beta^\vee(\alpha)\right) &= \big(\epsilon(-\alpha)(-\alpha)^\vee(\alpha) - \epsilon(\alpha)\alpha^\vee(\alpha)\big) \\ &= -2\,(\epsilon(-\alpha) + \epsilon(\alpha)).\end{aligned}$$

Again since d is a complementary polyhedron, $d(B)(\alpha) - d(B')(\alpha) \leq 0$ and we finally obtain

$$\begin{aligned} d^H(B)(\alpha) - d^H(B')(\alpha) &= d(B)(\alpha) - d(B')(\alpha) - 2\,|\deg(\mathscr{L})|\,(\epsilon(-\alpha) + \epsilon(\alpha)) \\ &\leq -2\,|\deg(\mathscr{L})|\,(\epsilon(-\alpha) + \epsilon(\alpha)) \\ &\leq 0, \end{aligned}$$

which finishes the proof. □

Proposition (3.1.7) implies that there is a unique facet $P \subset V = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ that is special with respect to the Higgs polyhedron d^H. By abuse of notation we also denote by $T \subset P \subset G(\eta)$ the parabolic subgroup that corresponds to the facet P. Using the projectivity of $G(\eta)/P$, we can uniquely extend the parabolic $P \subset G(\eta)$ to obtain a global reduction $\beta \colon X \to \mathscr{P}/Q$ for a parabolic subgroup $Q \subset G$.

The next step is to show that this is indeed a Higgs reduction of the Higgs bundle $(\mathscr{P}, \varphi)$; that is, φ factors through $(\beta^* \mathscr{P} \times^{\mathrm{Ad}} \mathfrak{q}) \otimes \mathscr{L}$. We say that the parabolic subgroup $P \subset G(\eta)$ *admits a Higgs reduction* if the extended reduction (β, Q) defines a Higgs reduction of the Higgs bundle $(\mathscr{P}, \varphi)$.

First of all we are now able to define the degree and the numerical invariant of a parabolic subgroup with respect to the Higgs polyhedron (cf. Section 3.1).

3.5.10 DEFINITION (Higgs degree). Let $T \subset P \subset G(\eta)$ be a parabolic subgroup with corresponding parabolic subset $R(P) \subset \Phi(G(\eta), T)$. We define the Higgs degree to be

$$\deg^H(P) = \sum_{\alpha \in R(P)} d^H(B)(\alpha)$$

where $B \subset P$ is a Borel subgroup contained in P and d^H is the Higgs polyhedron.

3.5.11 Definition (Numerical Higgs invariant). Let $T \subset P \subset G(\eta)$ be a parabolic subgroup and $\lambda \in \mathrm{vert}(P)$ a vertex. Let $B \subset P$ be a Borel subgroup and define the numerical Higgs invariant as

$$n^H(P,\lambda) := \sum_{\alpha \in \Psi(P,\lambda)} d^H(B)(\alpha),$$

where again $\Psi(P,\lambda) := \{\alpha \in \Phi(G(\eta),T) \mid \langle \lambda, \alpha\rangle = 1, \langle \mu, \alpha\rangle = 0 \forall \mu \in \mathrm{vert}(P) \setminus \lambda\}$.

3.5.12 Definition (Higgs facet and vertex). Let P be a facet of the root system $\Phi(G(\eta),T)$. We say that P is a Higgs facet if the corresponding parabolic subgroup $P \subset G(\eta)$ admits a Higgs reduction. If λ is a vertex of $\Phi(G(\eta),T)$, we say that λ is a Higgs vertex if the one dimensional facet spanned by it is a Higgs facet.

We also need to check that the degree and numerical invariants of a parabolic subset do not get changed if it admits a Higgs reduction, which is the content of the next lemma.

3.5.13 Lemma. *Let $P \subset G(\eta)$ be a parabolic subgroup. If P admits a Higgs reduction, then* $\deg^H(P) = \deg(P)$ *and* $n^H(P,\lambda) = n(P,\lambda)$*, where* deg *and n denotes the degree and numerical invariant, respectively, with respect to Behrend's complementary polyhedron.*

Proof. Let $P \subset G(\eta)$ be a parabolic subgroup with corresponding parabolic subset $R(P) \subset \Phi(G(\eta),T)$. Recall that it decomposes as $R(P) = U(P) \cup \Phi_P$ where

$$U(P) = \{\alpha \in \Phi(G(\eta),T) \mid \exists \lambda \in \mathrm{vert}(P) : \langle \alpha, \lambda\rangle > 0\}$$

and $\Phi_P = (\mathrm{span} P)^\perp \cap \Phi(G(\eta),T) = \{\alpha \in \Phi(G(\eta),T) \mid \langle \alpha, \lambda\rangle = 0 \forall \lambda \in \mathrm{vert}(P)\}$.

Assume that P admits a Higgs reduction. Then there is a $\beta \colon X \to \mathscr{P}/Q$ such that φ factors through $\varphi_Q : \mathscr{O}_X \to (\beta^*\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{q}) \otimes \mathscr{L}$ and $\mathfrak{q}_\eta = \mathfrak{p}$. Using the fact that

$$\mathfrak{p} = \mathscr{O}_X^n \oplus \bigoplus_{\alpha \in R(P)} \mathscr{L}_\alpha,$$

this implies $\varphi_\alpha = 0$ for all $\alpha \notin R(P)$. If $B \subset P$ is any Borel subgroup, then

$$\deg^H(P) = \deg(P) + |\deg(\mathscr{L})| \sum_{\alpha \in R(P)\setminus R(B)} \epsilon(\alpha)\alpha^\vee \left(\sum_{\beta \in R(P)} \beta\right) = \deg(P),$$

because $\sum_{\beta \in R(P)} \beta \in P$ [Beh95, Proposition 1.9] and all roots that are in $R(P)\setminus R(B)$ are actually in Φ_P and hence in $(\mathrm{span} P)^\perp$.

Let λ be a vertex of P and $\eta = \sum_{\alpha \in \Psi(P,\lambda)} \alpha$. Note that $\eta \in P$ from [Beh95, Lemma 3.6] and as above $\alpha^\vee(\eta) = 0$ for all $\alpha \in R(P) \setminus R(B)$. We immediately obtain

$$n^H(P,\lambda) = n(P,\lambda) + |\deg(\mathscr{L})| \sum_{\alpha \in R(P)\setminus R(B)} \epsilon(\alpha)\alpha^\vee(\eta) = n(P,\lambda),$$

which finishes the proof. □

3.5.14 Remark. If $P \subset G(\eta)$ is a parabolic subgroup that does not admit a Higgs-reduction then there is an $\alpha \not\in R(P)$ with $\varphi_\alpha \neq 0$. Then $-\alpha \in R(P)$ because $R(P)$ is a parabolic subset. Recall also that $\sum_{\beta \in R(P)} \beta \in P$ lies in the facet corresponding to P and that $R(P) = \{\alpha \in \Phi(G(\eta), T) \mid \langle \alpha, \lambda \rangle \geq 0 \forall \lambda \in P\}$. This implies

$$\deg^H(P) = \deg(P) + |\deg(\mathscr{L})| \sum_{\alpha \not\in R(P)} \epsilon(\alpha)\alpha^\vee \left(\sum_{\beta \in R(P)} \beta \right) \leq \deg(P) - |\deg(\mathscr{L})|,$$

$$n^H(P,\lambda) = n(P,\lambda) + |\deg(\mathscr{L})| \sum_{\alpha \not\in R(P)} \epsilon(\alpha)\alpha^\vee \left(\sum_{\beta \in \Psi(P,\lambda)} \beta \right),$$

and it can happen that $n^H(P,\lambda) > n(P,\lambda)$. However, if $\deg(\mathscr{L}) \neq 0$, then $\deg^H(P) < \deg(P)$ and $n^H(P,\lambda) \neq n(P,\lambda)$. Note also that we may assume without any loss of generality that $\deg(\mathscr{L}) \neq 0$. This follows from the fact that, if $\deg(\mathscr{L}) = 0$, a principal Higgs bundle $(\mathscr{P}, \varphi)$ is semistable if and only if $\mathscr{P}$ is a semistable principal bundle (cf. [FGPN13a, Proposition 4.1], [FGPN13b, Proposition 3.1] and [Sch08, Example 2.5.6.7]).

3.5.15 Lemma. *Let $B \subset G(\eta)$ be a Borel subgroup that does not admit a Higgs reduction. Then there is a vertex $\lambda \in \mathrm{vert}(B)$ with $n^H(B,\lambda) \leq 0$. In particular, any Borel subgroup that does not admit a Higgs reduction cannot be special.*

Proof. Let $\alpha \not\in R(B)$ with $\varphi_\alpha \neq 0$. Since $\varphi_\alpha : \mathscr{O}_X \to \mathscr{L}_\alpha \otimes \mathscr{L}$ is a nonzero map of line bundles, this implies

$$\deg(\mathscr{L}) \geq -\deg(\mathscr{L}_\alpha) = \deg(\mathscr{L}_{-\alpha}).$$

Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be the basis corresponding to B. Since $-\alpha \in R(B)$, there are $c_i \geq 0$ with $-\alpha = \sum c_i \alpha_i$. Then

$$\begin{aligned} \sum_{i=1}^n c_i \cdot n^H(B, v(\alpha_i)) &= \deg(\mathscr{L}_{-\alpha}) + |\deg(\mathscr{L})| \sum_{\beta \not\in R(B)} \epsilon(\beta)\beta^\vee(-\alpha) \\ &\leq \deg(\mathscr{L}_{-\alpha}) - |\deg(\mathscr{L})| \\ &\leq 0, \end{aligned}$$

because $n(B, v(\beta)) = \deg(\mathscr{L}_\beta)$ by the definition of Behrend's complementary polyhedron and $\sum_{\beta \not\in R(B)} \epsilon(\beta)\beta^\vee(-\alpha) \leq -1$ (cf. Remark 3.5.14). Hence there is a $\lambda \in \mathrm{vert}(B)$ with $n^H(B,\lambda) \leq 0$. □

We can now return to the special facet of the Higgs polyhedron.

3.5.16 Lemma. *Let d^H be the Higgs polyhedron as constructed above and $P \subset G(\eta)$ the canonical parabolic subgroup corresponding to the special facet. Then P admits a Higgs reduction.*

Proof. Since $G(\eta)$ always admits a Higgs reduction and a Borel subgroup is special if and only if $n(B,\lambda) > 0$ for all $\lambda \in \mathrm{vert}(B)$, we can assume that the special facet P corresponds to a proper parabolic subgroup $B \subsetneq P \subsetneq G(\eta)$ by Lemma (3.5.15).

Let Q be a parabolic subgroup corresponding to a non-Higgs facet. Then there is an $\alpha \notin R(Q)$ with $\varphi_\alpha \neq 0$. If $B \subset Q$ is any Borel subgroup, then $R(B) \subset R(Q)$. Thus Q only contains Borel subgroups that correspond to non-Higgs chambers. By Lemma (3.5.15), there is a $\lambda \in \mathrm{vert}(B)$ with $n^H(B,\lambda) \leq 0$, and since

$$d^H(B) = \sum_{\lambda \in \mathrm{vert}(B)} n^H(B,\lambda)\lambda^\vee$$

we see that $d^H(B) \notin B^\vee$ for any Borel subgroup $B \subset Q$. Note that $Q^\vee = \bigcap_{B \subset Q} B^\vee$, hence $F(Q) \cap Q^\vee = \emptyset$ for any non-Higgs facet.

Because a facet is special if and only if $F(P) \cap P^\vee \neq \emptyset$, the special facet P must admit a Higgs reduction. □

We are finally able to prove the main result of this section.

3.5.17 Proposition. *Let G be a connected reductive linear algebraic group and $(\mathscr{P},\varphi)$ a principal G-Higgs bundle. Then $\mathscr{P}$ has a canonical Higgs reduction. It is unique in the sense that if (β_1,Q_1) and (β_2,Q_2) are two canonical Higgs reductions, then there is a $g \in G$ with $Q_1 = gQ_2g^{-1}$ and $\beta_1 = g\beta_2 g^{-1}$.*

Proof. Since we already know that a canonical Higgs reduction exists (the degree of instability is finite by Lemma 3.3.1), only the uniqueness part remains to be proven.

Let (β_1,Q_1) and (β_2,Q_2) be two canonical Higgs reductions. By [sga70b, Exposé XXVI, Lemme 4.1.1], there is a torus $T \subset (Q_1 \cap Q_2)_\eta$ and we can construct the Higgs polyhedron d^H with respect to T on $\Phi(G(\eta),T)$. Lemma (3.5.13) and (3.5.16) imply that $(Q_1)_\eta$ and $(Q_2)_\eta$ give a canonical facet of d^H and hence $(Q_1)_\eta = (Q_2)_\eta$ from Proposition (3.1.7). Since reductions are determined generically up to conjugation, this implies that there is a $g \in G$ with $Q_1 = gQ_2g^{-1}$ and $\beta_1 = g\beta_2 g^{-1}$. □

Note that if we fix a maximal torus $T \subset G$, then there is a unique canonical Higgs reduction (β,Q) with $T \subset Q$.

It might be worth noting that although we restricted ourselves to characteristic zero, the construction of Behrend, and hence also the Higgs polyhedron, can be applied to positive characteristic.

3.6 Decorated Bundles

After defining a complementary Higgs polyhedron, it would be desirable to also define a complementary polyhedron for other kinds of decorated bundles. The semistability condition of more general decorated bundles will often depend on a parameter. In closing, we will give a short overview of some ideas for future work.

Weighted filtrations. Let V be a finite dimensional $\mathbb{K}$-vector space and consider a representation $\rho\colon G \to \mathrm{GL}(V)$. Since some parameter dependent stability conditions will depend on weighted filtrations, we will clarify the connection between weighted filtrations of V and parabolic subgroups of G.

Given a one parameter subgroup $\lambda\colon \mathbb{G}_m \to G$, we let $Q(\lambda) := P(-\lambda)$ be the associated parabolic subgroup of G. The one parameter subgroup defines a weighted flag of V as follows. The composition $\rho \circ \lambda\colon \mathbb{G}_m \to \mathrm{GL}(V)$ gives a decomposition of V into eigenspaces

$$V = V_{\gamma_1} \oplus \cdots \oplus V_{\gamma_s},$$

where $V_{\gamma_i} = \left\{ v \in V \;\middle|\; (\rho \circ \lambda)(z)v = z^{\gamma_i} v \;\; \forall z \in \mathbb{G}_m \right\}$ for some weight $\gamma_i \in \mathbb{Z}$. We suppose that $\gamma_1 < \ldots < \gamma_s$ and let $V_i := V_{\gamma_1} \oplus \cdots \oplus V_{\gamma_i}$. This gives the filtration

$$0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_s = V,$$

and to each subspace V_i we associate a rational weight defined as

$$\alpha_i := \frac{\gamma_{i+1} - \gamma_i}{\dim V}, \; i = 1, \ldots, s-1.$$

δ-Semistability. We will restrict to the case $G = \mathrm{GL}(n)$ and $\rho\colon \mathrm{GL}(n) \to \mathrm{GL}(V)$. We let $(\mathscr{E}, \sigma)$ be a decorated vector bundle where $\sigma\colon X \to \mathbb{P}(\mathscr{E}(V))$; that is, we let ρ act on $\mathbb{P}(V)$ and consider the associated projective bundle. Fix a positive rational $\delta \in \mathbb{Q}_{\geq 0}$ – this will be the parameter in the stability condition. A weighted filtration $(\mathscr{E}^\bullet, \alpha_\bullet)$ of $\mathscr{E}$ consists of a filtration

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \ldots \subset \mathscr{E}_s = \mathscr{E}$$

into subbundles and positive rational weights $\alpha_i \in \mathbb{Q}_{>0}$ for $i = 1, \ldots, s$. Given a weighted filtration, we also define

$$M(\mathscr{E}^\bullet, \alpha_\bullet) := \sum_{i=1}^{s} \alpha_i (\deg(\mathscr{E}) \mathrm{rk}(\mathscr{E}_i) - \deg(\mathscr{E}_i) \mathrm{rk}(\mathscr{E})).$$

We call $(\mathscr{E}, \sigma)$ δ-semistable if for every weighted filtration $(\mathscr{E}^\bullet, \alpha_\bullet)$ of $\mathscr{E}$

$$\delta\text{-}\deg(\mathscr{E}^\bullet, \alpha_\bullet) := M(\mathscr{E}^\bullet, \alpha_\bullet) + \delta \cdot \mu(\mathscr{E}^\bullet, \alpha_\bullet, \sigma) \geq 0,$$

where $\mu(\mathscr{E}^\bullet, \alpha_\bullet, \sigma)$ is a certain function that also depends on the representation ρ (cf. [Sch04],[Sch08]).

3.6.1 Remark. Note that unlike $M(\mathscr{E}^\bullet, \alpha_\bullet)$, the function μ is not linear in α.

Hitchin Pairs. Let us return to Higgs bundles. Consider the representation $\rho\colon \mathrm{GL}(n) \to \mathrm{GL}(\mathfrak{gl}_n \oplus \mathbb{K})$. As we have seen before (1.5.2), the decorated principal bundles (with affine fibres) obtained from ρ correspond to Higgs vector bundles. The decorated principal bundles obtained from ρ with projective fibres are the so called *Hitchin pairs*. Hence for given $\delta \in \mathbb{Q}_{\geq 0}$, we can define a δ-semistability condition for these objects.

3.6.2 DEFINITION. A *Hitchin pair* is a tuple $(\mathscr{P}, \varphi, \epsilon)$ where $(\mathscr{P}, \varphi)$ is a principal Higgs bundle and $\epsilon \in \mathbb{C}$ a complex number. It is called *semistable* if

1. $(\mathscr{P}, \varphi)$ is a semistable Higgs bundle.

2. If $\epsilon = 0$, then φ is not nilpotent.

Recall that φ is nilpotent if it factors through $\beta^*\mathscr{P} \times^{\mathrm{Ad}} \mathfrak{r}$, where $\mathfrak{r}$ denotes the unipotent radical of $\mathfrak{gl}_n$ (nilpotent matrices in $\mathfrak{gl}_n$). Obviously any (semistable) Higgs bundle gives a (semistable) Hitchin pair via $(\mathscr{P}, \varphi) \mapsto (\mathscr{P}, \varphi, 1)$.

3.6.3 Theorem (Lemma 3.13 in [Sch04], Lemma 2.7.4.2 in [Sch08])**.** *There exists a $\Delta > 0$ such that for all $\delta \geq \Delta$ the following are equivalent:*

1. *The principal Higgs bundle $(\mathscr{P}, \varphi)$ is Higgs semistable.*

2. *The Hitchin pair $(\mathscr{P}, \varphi, 1)$ is δ-semistable.*

To show the existence and uniqueness of a canonical filtration, one strategy would be to show that the δ-stability defines a complementary polyhedron. The problem here is that μ is not linear in α but only piecewise linear and by definition the complementary polyhedron is a linear form. For future work, [RR84] might be helpful.

Affine Bumps. Another strategy could be to define a complementary polyhedron for *affine bumps* which should work in the same manner as the Higgs polyhedron. After that one might try to reduce decorated bundles with projective fibre to this case (cf. [Sch08, Remark 2.1.2.5]). Let G be a reductive linear algebraic group.

3.6.4 DEFINITION (Affine bump). Let $\rho\colon G \to \mathrm{GL}(V)$ be an irreducible representation and $\mathscr{L}$ a line bundle on X. An affine ρ-bump is a pair $(\mathscr{P}, \varphi)$ consisting of a principal G-bundle $\mathscr{P}$ and a morphism $\varphi\colon \mathscr{P}_\rho \to \mathscr{L}$. Here $\mathscr{P}_\rho$ is the vector bundle associated to $\mathscr{P}$ using the representation ρ.

The stability condition for affine bumps will depend on a rational character $\chi \in X^*(G) \otimes \mathbb{Q}$. The affine bump $(\mathscr{P}, \varphi)$ is called χ-semistable if for all one parameter subgroups $\lambda\colon \mathbb{G}_m \to G$ and reductions $\beta\colon X \to \mathscr{P}/Q(\lambda)$ with $\mu(\beta, \varphi) \leq 0$ we have

$$M(\mathscr{E}^\bullet(\beta), \alpha_\bullet(\beta)) + \langle \lambda, \chi \rangle \geq 0.$$

Here $(\mathscr{E}^\bullet(\beta), \alpha_\bullet(\beta))$ is the weighted filtration given by the reduction β and the function $\mu(\beta, \varphi)$ again depends on the representation ρ (see [Sch08, p. 289]). Note that here $\langle \lambda, \chi \rangle$ is linear in both λ and χ.

3.A Root Datum and Parabolic Subgroups of SL(3)

For reference we give the root datum of $\mathrm{SL}(3) = \{A \in \mathrm{GL}(3) \mid \det(A) = 1\}$. Let $T \subset \mathrm{SL}(3)$ be the two dimensional torus of diagonal matrices. Its character group is $X^*(T) = \left\{(a_1, a_2, a_3) \in \mathbb{Z}^3 \,\middle|\, a_1 + a_2 + a_3 = 0\right\} \cong \mathbb{Z}^2$. The corresponding root system in $V = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^2$ takes the form

$$\Phi = \{\pm(1,-1,0), \pm(0,1,-1), \pm(1,0,-1)\}.$$

Choose the scalar product on V given by the standard inner product on $\mathbb{R}^3$, that is $\langle (x_1,x_2,x_3),(y_1,y_2,y_3)\rangle = \sum x_i y_i$. A basis of Φ is given by two roots α, β such that $\langle \alpha, \beta \rangle = -1$. The Cartan matrix is

$$S = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The vertices can be computed as $\Lambda = \{\lambda_1, \dots, \lambda_6\}$ where

$$\begin{aligned} 3\lambda_1 &= (1,1,-2), \\ 3\lambda_2 &= (2,-1,-1), \\ 3\lambda_3 &= (1,-2,1), \\ 3\lambda_4 &= (-1,-1,2), \\ 3\lambda_5 &= (-2,1,1), \\ 3\lambda_6 &= (-1,2,-1). \end{aligned}$$

We now examine the parabolic subgroups $T \subset P \subset \mathrm{SL}(3)$.

The Weyl group is isomorphic to $\mathscr{S}_3$, which has six elements. Hence there are six Borel subgroups containing the torus, each of which correspond to a permutation $\sigma \in \mathscr{S}_3$. We list the Borel subgroups with the corresponding bases and vertices below.

$$B_{\mathrm{id}} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \qquad \Delta_{\mathrm{id}} = \{(1,-1,0),(0,1,-1)\}\ \Lambda_{\mathrm{id}} = \{\lambda_2, \lambda_1\}$$

$$B_{(12)} = \begin{pmatrix} * & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \qquad \Delta_{(12)} = \{(-1,1,0),(1,0,-1)\}\ \Lambda_{(12)} = \{\lambda_6, \lambda_1\}$$

$$B_{(23)} = \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix} \qquad \Delta_{(23)} = \{(0,-1,1),(1,0,-1)\}\ \Lambda_{(23)} = \{\lambda_3, \lambda_2\}$$

$$B_{(13)} = \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \qquad \Delta_{(13)} = \{(-1,1,0),(0,-1,1)\}\ \Lambda_{(13)} = \{\lambda_5, \lambda_4\}$$

$$B_{(123)} = \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ * & * & * \end{pmatrix} \qquad \Delta_{(123)} = \{(1,-1,0),(-1,0,1)\}\ \Lambda_{(123)} = \{\lambda_3, \lambda_4\}$$

$$B_{(132)} = \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & 0 & * \end{pmatrix} \qquad \Delta_{(132)} = \{(0,1,-1),(-1,0,1)\}\ \Lambda_{(132)} = \{\lambda_6, \lambda_5\}$$

Similarly, there are six maximal parabolic subgroups corresponding to a proper subspace of $\mathbb{K}^3$. Maximal parabolic subgroups have exactly one vertex.

$$P_1 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \qquad R(P_1) = \{\pm(1,-1,0),(0,1,-1),(1,0,-1)\} \;\; \Lambda_1 = \{\lambda_1\}$$

$$P_2 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \qquad R(P_2) = \{\pm(0,1,-1),(1,-1,0),(1,0,-1)\} \;\; \Lambda_2 = \{\lambda_2\}$$

$$P_3 = \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ * & * & * \end{pmatrix} \qquad R(P_3) = \{\pm(1,0,-1),(1,-1,0),(0,-1,1)\} \;\; \Lambda_3 = \{\lambda_3\}$$

$$P_4 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \qquad R(P_4) = \{\pm(1,-1,0),(0,-1,1),(-1,0,1)\} \;\; \Lambda_4 = \{\lambda_4\}$$

$$P_5 = \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \qquad R(P_5) = \{\pm(0,1,-1),(-1,1,0),(-1,0,1)\} \;\; \Lambda_5 = \{\lambda_5\}$$

$$P_6 = \begin{pmatrix} * & 0 & * \\ * & * & * \\ * & 0 & * \end{pmatrix} \qquad R(P_6) = \{\pm(1,0,-1),(-1,1,0),(0,1,-1)\} \;\; \Lambda_6 = \{\lambda_6\}$$

Since we are also interested in the filtrations that get stabilised by the Borel and parabolic subgroups, we will list them below. For this we decompose $\mathbb{K}^3 = V_1 \oplus V_2 \oplus V_3$ according to the chosen torus. The filtrations are as follows.

$$\begin{aligned} B_{\mathrm{id}}&: 0 \subset V_1 \subset V_1 \oplus V_2 \subset V & P_1&: 0 \subset V_1 \oplus V_2 \subset V \\ B_{(12)}&: 0 \subset V_2 \subset V_2 \oplus V_1 \subset V & P_2&: 0 \subset V_1 \subset V \\ B_{(23)}&: 0 \subset V_1 \subset V_1 \oplus V_3 \subset V & P_3&: 0 \subset V_1 \oplus V_3 \subset V \\ B_{(13)}&: 0 \subset V_3 \subset V_3 \oplus V_2 \subset V & P_4&: 0 \subset V_3 \subset V \\ B_{(123)}&: 0 \subset V_3 \subset V_3 \oplus V_1 \subset V & P_5&: 0 \subset V_2 \oplus V_3 \subset V \\ B_{(132)}&: 0 \subset V_2 \subset V_2 \oplus V_3 \subset V & P_6&: 0 \subset V_2 \subset V \end{aligned}$$

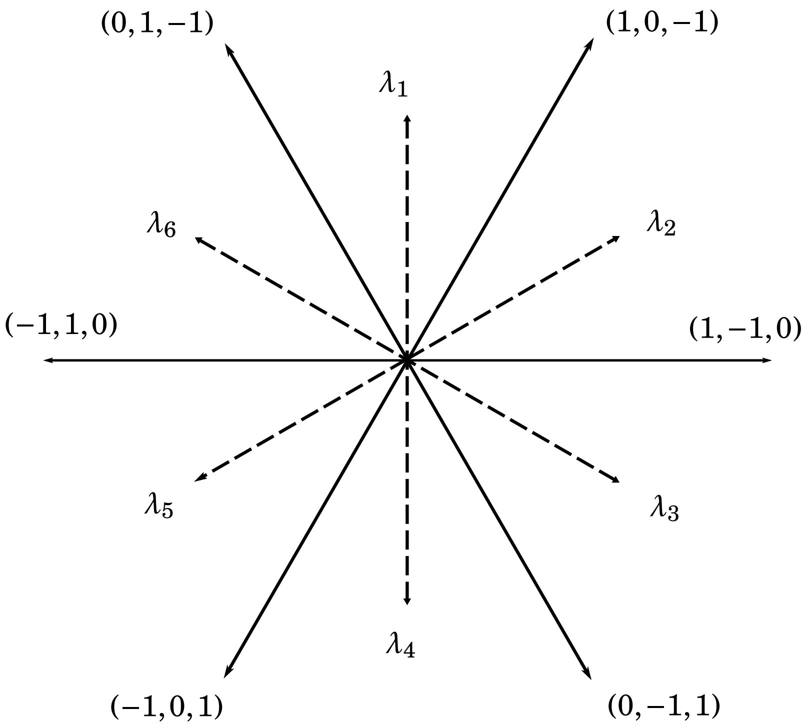

Figure 3.7: Root system of SL(3)

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Summary

This thesis investigates certain decorated principal bundles on smooth projective schemes whose appearance in algebraic geometry was influenced by theoretical physics. We will consider the moduli space of autodual instanton bundles on projective space and canonical reductions of principal Higgs bundles on smooth projective curves.

In the first chapter, we introduce the main objects of study, namely vector and principal bundles on projective spaces. We give the necessary definitions of linear algebraic groups and group schemes. We also explain stability conditions for these and explain how autodual vector bundles and Higgs bundles can be interpreted as principal bundles with decorations.

The second chapter is devoted to the study of autodual instanton bundles on projective space. We explain how instanton bundles of trivial splitting type can be constructed from ADHM-data. After that we investigate how the autoduality structure is reflected in the ADHM-datum and obtain an extended datum. For symplectic and orthogonal instanton bundles these extended data can be refined. Finally we take a look at the construction of examples of symplectic and orthogonal instanton bundles from an extended ADHM-datum.

In the last chapter, we investigate principal Higgs bundles on smooth projective curves. We start by introducing root systems and complementary polyhedra and explain how a connected reductive algebraic group equipped with a maximal torus defines a root system. We then explain Behrend's construction of the complementary polyhedron associated to a principal bundle and compute some examples. A section is devoted to the study of torus reductions. Then we give an original construction of a complementary polyhedron associated to a principal Higgs bundles. Finally we give consequences of the complementary Higgs polyhedron, i.e. the existence and uniqueness of a canonical Higgs reduction.

Zusammenfassung

Diese Dissertation untersucht dekorierte Prinzipalbündel auf glatten projektiven Schemata. Wir betrachten den Modulraum autodualer Instantonbündel auf projektiven Räumen sowie kanonische Reduktionen von Higgs-Prinzipalbündeln auf glatten projektiven Kurven. Das Aufkommen dieser Objekte in der algebraischen Geometrie wurde stark von theoretischer Physik beeinflusst.

Im ersten Kapitel führen wir die zugrundeliegenden Objekte ein, nämlich Vektor- und Prinzipalbündel auf projektiven Räumen. Wir definieren linear algebraische Gruppen und Gruppenschemata. Desweiteren erklären wir Stabilität dieser Bündel und zeigen wie autoduale Vektorbündel und Higgs-Bündel als dekorierte Prinzipalbündel aufgefasst werden können.

Im zweiten Kapitel widmen wir uns dem Studium von autodualen Instantonbündeln auf projektiven Räumen. Wir erklären wie Instantonbündel von trivialem Spaltungstyp aus den sogenannten ADHM-Daten konstruiert werden können. Danach untersuchen wir wie die Autodualität sich in diesen ADHM-Daten widerspiegelt und erhalten daraus ein erweitertes Datum. Im symplektischen und orthogonalen Fall kann dieses erweiterte Datum noch weiter verfeinert werden. Schießlich werfen wir einen Blick auf die Konstruktion von Beispielen von symplektischen und orthogonalen Instantonbündeln aus diesen erweiteren ADHM-Daten.

Im letzten Kapitel untersuchen wir Higgs-Prinzipalbündel auf glatten projektiven Kurven. Wir beginnen mit Wurzelsystemen und komplementären Polyedern und erklären wie eine zusammenhängende reduktive algebraische Gruppe zusammen mit einem maximalen Torus ein Wurzelsystem definiert. Dann erklären wir die Konstruktion von Behrend die einem Prinzipalbündel einen komplementären Polyeder zuordnet und geben zur Verdeutlichung einige Beispiele. Einen Abschnitt widmen wir außerdem den Torusreduktionen. Anschließend geben wir eine Konstruktion die einem Higgs-Prinzipalbündel einen komplementären Polyeder zuordnet. Zum Abschluß erklären wir die Konsequenzen des komplementären Higgs-Polyeder, nämlich die Existenz und Eindeutigkeit einer kanonischen Higgs-Reduktion.