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On the split closure of the periodic timetabling polytope

Niels Lindner[1] · **Berenike Masing**[2]

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Abstract

The Periodic Event Scheduling Problem (PESP) is the central mathematical tool for periodic timetable optimization in public transport. PESP can be formulated in several ways as a mixed-integer linear program with typically general integer variables. We investigate the split closure of these formulations and show that split inequalities are identical with the recently introduced flip inequalities. While split inequalities are a general mixed-integer programming technique, flip inequalities are defined in purely combinatorial terms, namely cycles and arc sets of the digraph underlying the PESP instance. It is known that flip inequalities can be separated in pseudo-polynomial time. We prove that this is best possible unless P = NP, but also observe that the complexity becomes linear-time if the cycle defining the flip inequality is fixed. Moreover, introducing mixed-integer-compatible maps, we compare the split closures of different formulations, and show that reformulation or binarization by subdivision do not lead to stronger split closures. Finally, we estimate computationally how much of the optimality gap of the instances of the benchmark library PESPlib can be closed exclusively by split cuts, and provide better dual bounds for five instances.

Keywords Periodic event scheduling problem · Periodic timetabling · Split closure · Mixed-integer programming

Mathematics Subject Classification 90C11 · 90C35 · 90B35 · 90B20

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✉ Niels Lindner
lindner@zib.de

Berenike Masing
masing@zib.de

1 Freie Universität Berlin, Berlin, Germany

2 Zuse Institute Berlin, Berlin, Germany

List of symbols

A	Set of arcs of graph G
$A_{\mathrm{C}}, A_{\mathrm{I}}$	Matrices to describe the continuous, resp. integral components of a general mixed-integer set S
B	Integral cycle basis of graph G
$\mathcal{C}$	Cycle space of graph G
F	Set of arcs used for a flip inequality
G	Directed graph $G = (V, A)$ of a PESP instance
I	PESP instance, $I = (G, T, \ell, u, w)$
ℓ	Lower bound vector in $\mathbb{Z}^A$
P	(Fractional) polyhedron associated to the general mixed-integer set S
$\mathcal{P}$	Fractional periodic timetabling polytope
P_{Ch}	Chvátal closure of the general mixed-integer set S
$\mathcal{P}_{\mathrm{Ch}}$	Chvátal closure for a PESP instance
P_{I}	Convex hull of the general mixed-integer set S
$\mathcal{P}_{\mathrm{I}}$	Integral periodic timetabling polytope
$\mathcal{P}_{\mathrm{flip}}$	Flip polytope
P_{split}	Split closure of the general mixed-integer set S
$\mathcal{P}_{\mathrm{split}}$	Split closure for a PESP instance
$P^{(\beta,\beta_0)}$	Polyhedron associated to the split (β, β_0)
p	Periodic offset, vector in $\mathbb{Z}^A$
S	General mixed-integer set generated by $(A_{\mathrm{C}}, A_{\mathrm{I}}, b)$
T	Period time of a PESP instance
u	Upper bound vector in $\mathbb{Z}^A$
V	Set of vertices of G
w	Arc weights, vector in $\mathbb{R}^A_{\geq 0}$
x	Periodic tension, vector in $\mathbb{R}^A$
z	Cycle offset, vector in $\mathbb{Z}^B$
$\alpha_{\gamma,F}$	Parameter for the definition of the flip inequality for (γ, F)
(β, β_0)	Split defining a split disjunction
(γ, F)	Pair of a cycle and a set of flipped arcs defining a flip inequality
Γ	Cycle matrix of an integral cycle basis B of G
γ	Cycle, element of the cycle space $\mathcal{C}$
γ_+, γ_-	Positive, negative parts of cycle γ
Λ	Set of multiplier vectors defining inequalities describing the split closure
λ	Multiplier vector defining a split inequality
μ	Cyclomatic number, rank of the cycle space $\mathcal{C}$
π	Periodic timetable, vector in $\mathbb{R}^V$
φ	Mixed-integer-compatible map
$[\cdot]_T$	Modulo T operator with values in $[0, T)$
$[\cdot]_1$	Fractional part in $[0, 1)$

1 Introduction

The timetable is the core of a public transportation system. It serves as a basis for cost-sensitive tasks such as vehicle and crew scheduling, and is required for accurate planning of passenger routes. A high-quality timetable is thus of utmost importance for a well-planned transportation system. Particularly in the context of urban traffic, a large number of transportation networks are operated with a periodic pattern, creating the demand to optimize periodic timetables. The standard mathematical model for this task is the *Periodic Event Scheduling Problem* (PESP) introduced by Serafini and Ukovich [40]. PESP is a combinatorial optimization problem on a digraph with respect to a certain period time, and it is notoriously hard: Deciding whether a feasible periodic timetable exists is NP-complete for any fixed period time $T \geq 3$ [36]. The feasibility problem remains NP-hard on graphs with bounded treewidth [28]. The difficulty of PESP is also reflected in the fact that since its establishment in 2012, none of the instances of the benchmark library PESPlib could be solved to proven optimality up to date [15]. Nevertheless, many primal heuristics have been developed [8, 16, 17, 27, 34, 38], and there are success stories concerning the implementation of mathematically optimized timetables in practice [20, 22].

PESP can be formulated as a mixed-integer linear program (MIP) in a multitude of ways, cf. Liebchen [21]. Several studies of the *periodic timetabling polytope* have been conducted, leading to the discovery of families of cutting planes, such as, e.g., Odijk's *cycle inequalities* [36], Nachtigall's *change-cycle inequalities* [32], and more recently, the *flip inequalities* by Lindner and Liebchen [26]. The separation of cycle and change-cycle inequalities is known to be NP-hard [7], and flip inequalities are a superset of both cycle and change-cycle inequalities [26]. A common theme that cycle, change-cycle and flip inequalities share as well with other families of cutting planes [30, 32] is that they are all described in purely combinatorial terms. For example, flip inequalities are determined by a cycle and a set of arcs of the underlying digraph of the PESP instance.

In this paper, we pursue a somewhat opposite strategy: Rather than starting with a combinatorial analysis, we investigate *split inequalities*, a general-purpose tool for treating MIPs introduced by Cook et al. [12] as an analogon to the Chvátal closure for pure integer programs. The *split closure* given by these inequalities has several nice properties: It is a polyhedron [10, 12], coincides with the closure given by mixed-integer rounding and Gomory mixed-integer cuts [13, 35], and leads to finite cutting plane algorithms for binary MIPs [4].

While the second Chvátal closure for a pure IP formulation has already been investigated by Liebchen and Swarat [25], we apply split closure techniques to proper mixed-integer formulations of PESP. Our first result is the following correspondence (Theorem 3.1): Every non-trivial split inequality is a non-trivial flip inequality, and vice versa. The split closure of the periodic timetabling polytope is therefore identical with the closure given by the flip inequalities. Moreover, the Chvátal inequalities coincide with Odijk's cycle inequalities (Theorem 3.3).

In general, the separation of split inequalities is NP-hard [9]. In the periodic timetabling situation, we show in Theorem 4.4 that it is weakly NP-hard to separate maximally violated split/flip inequalities. This is best possible unless P = NP, as Lindner and Liebchen [26] have already outlined a pseudo-polynomial-time algorithm. The separation problem can however by solved by a parametric IP in the spirit of Balas and Saxena [3] and Bonami [6], which in the special case of PESP boils down to a sequence of $\lfloor T/2 \rfloor - 1$ standard IPs (Theorem 4.5). In the event that the cycle defining a flip inequality is fixed, the separation becomes linear-time (Theorem 4.7).

So far, the results on the split closure of the periodic timetabling polytope apply for the cycle-based MIP formulation of PESP [24, 32]. Another popular formulation is the incidence-based formulation that is straightforward from the original problem definition [40]. In order to compare the split closures of two different MIP formulations, we introduce *mixed-integer-compatible maps*, i.e., affine maps that map mixed-integer points to mixed-integer points. These maps have the general property that they map split closures into split closures (Theorem 5.3). For PESP, the polytope defined by the cycle-based formulation turns out to be a mixed-integer-compatible projection of the polytope defined by the incidence-based formulation. However, we show that the restriction of this projection to split closures is surjective, so that there is no gain in information concerning split cuts when switching to a different formulation (Theorem 5.11). More results that can be proven using mixed-integer-compatible maps are the following: The split closure commutes with Cartesian products (Theorem 5.5). This enables us to show that the split closure of PESP instances on cactus graphs is exact (Corollary 5.6).

The behavior of split or lift-and-project closures with respect to binarizations, i.e., MIP reformulations with only binary integer variables, has received some attention lately [2, 14]. In the context of PESP, the incidence-based MIP formulation can be binarized in a combinatorial manner by subdivision of arcs. Although split closures of binary MIPs can computationally behave better [2], and have the theoretical advantage that the split rank is finite [4] in contrast to the situation with general integer variables [12], we prove by another application of mixed-integer-compatible maps that this binarization procedure does not lead to stronger split closures either (Theorem 5.15).

Finally, we evaluate split closures in practice. To this end, we consider the 22 PESPlib instances and some derived subinstances. We devise an algorithmic procedure to optimize over the split closure making use of our theoretical insights. Our separation algorithm consists of a heuristic and an exact part. The outcome is that although the split closure closes a significant part of the primal–dual gap, it is almost never exact. However, our separation method produces incumbent dual bounds for 5 of the PESPlib instances.

The paper is structured as follows: We summarize the relevant definitions and notions for PESP in Sect. 2. The correspondence between split and flip inequalities follows in Sect. 3. The subsequent Sect. 4 is devoted to separation of split/flip inequalities. Mixed-integer compatible maps and the results on comparing split closures of different formulations are presented in Sect. 5. Our computational results can be found in Sect. 6. We conclude the paper in Sect. 7.

A summary of the theoretical results of this paper is given in Table 1. An overview of important symbols and where they are defined is attached in Appendix A.

Table 1 Summary of the paper results

Section	Theorem	Description
Sect. 3.2	Theorem 3.1	Flip inequalities = split inequalities
	Corollary 3.2	Split closure is exact for cyclomatic number $\mu \leq 1$
Sect. 3.3	Theorem 3.3	Odijk's cycle inequalities = Chvátal inequalities
Sect. 4.1	Theorem 4.2	Flip inequalities for simple cycles determine the split closure
	Corollary 4.3	Odijk's cycle inequalities for simple cycles determine Chvátal closure
Sect. 4.2	Theorem 4.4	Separating flip inequalities is weakly NP-hard
	Theorem 4.5	Parametric IP for separating flip inequalities
Sect. 4.3	Theorem 4.7	Separation for fixed cycle is linear-time
Sect. 5.1	Theorem 5.3	Mixed-integer compatible maps descend to split closures in general
Sect. 5.2	Theorem 5.5	Split closure is compatible with Cartesian products in general
	Corollary 5.6	Split closure decomposes on blocks of PESP instances
Sect. 5.3	Theorem 5.11	Free augmentations do not give stronger split closures
Sect. 5.4	Theorem 5.15	Binarization by subdivision does not give stronger split closures

2 Periodic event scheduling

The Periodic Event Scheduling Problem has originally been introduced by Serafini and Ukovich [40], and has gained much attention ever since. In this chapter, we establish the basics, formally state the problem, introduce two equivalent model formulations and introduce our main object of interest, the periodic timetabling polytope.

2.1 Problem definition

An instance I of the *Periodic Event Scheduling Problem* (PESP) is given by a 5-tuple $I = (G, T, \ell, u, w)$, where

- $G = (V, A)$ is a directed graph,
- $T \in \mathbb{N}$, $T \geq 2$, is a *period time*,
- $\ell \in \mathbb{Z}^A$ is a vector of *lower bounds*,
- $u \in \mathbb{Z}^A$ is a vector of *upper bounds*,
- $w \in \mathbb{R}^A_{\geq 0}$ is a vector of non-negative *weights*.

A *periodic tension* is a vector $x \in \mathbb{R}^A$ with $\ell \leq x \leq u$ such that

$$\exists \pi \in [0, T)^V : \quad \forall a = (i, j) \in A : \quad x_a \equiv \pi_j - \pi_i \mod T. \tag{1}$$

In this case, the vector π is called a *periodic timetable*. In the context of periodic timetabling in public transport, the vertices of G typically correspond to arrival or

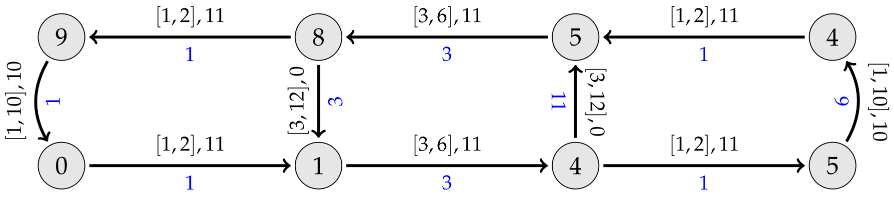

Fig. 1 A PESP instance on a digraph $G = (V, A)$ with $T = 10$. The upper label of an arc $a \in A$ is $[\ell_a, u_a], w_a$. The blue lower arc labels indicate a periodic tension x compatible with the periodic timetable π as given by the vertex labels

departure *events* of vehicles at some station. The arcs of G are *activities*; they model relations between the events such as, e.g., driving between two stations, dwelling at a station, or passenger transfers [23]. A periodic timetable π thus assigns timings in $[0, T)$ to each event, repeating periodically with period T. The periodic tension x collects the activity durations, which are supposed to lie within the feasible interval $[\ell, u]$. A typical source for the weight of an arc is the estimated number of passengers using the corresponding activity. A reasonable quality indicator of a periodic timetable is hence $w^\top x$, the total travel time of all passengers.

Definition 2.1 [40] Given $I = (G, T, \ell, u, w)$ as above, the *Periodic Event Scheduling Problem* is to find a periodic tension x such that $w^\top x$ is minimum, or to decide that none exists.

Example 2.2 Fig. 1 shows a small PESP instance together with an optimal periodic tension and a compatible periodic timetable.

Remark 2.3 As described, e.g., by Liebchen [21], any PESP instance can be preprocessed in such a way that G contains no loops and is weakly connected, i.e., G is connected when ignoring arc orientations, and furthermore $0 \leq \ell < T$ and $\ell \leq u < \ell + T$.

2.2 Mixed-integer programming formulations

PESP can be formulated as a mixed-integer linear program in several ways [21]. The *incidence-based* model is a straightforward interpretation of the problem definition, introducing auxiliary integer *periodic offsets* to resolve the modulo constraints (1):

$$\begin{array}{lll} \text{Minimize} & w^\top x & \\ \text{s.t.} & x_a = \pi_j - \pi_i + T p_a, & \forall a = (i, j) \in A, \\ & \ell_a \leq x_a \leq u_a, & \forall a \in A, \\ & 0 \leq \pi_i \leq T - 1, & \forall i \in V, \\ & p_a \text{ integer}, & \forall a \in A. \end{array} \tag{2}$$

When all periodic offsets p_a are fixed, (2) becomes a linear program with a totally unimodular constraint matrix. It is hence no restriction to assume that x and π are integral, so that the bound $\pi < T$ in (1) can safely be replaced with $\pi \leq T - 1$. For the purpose of this paper, we will however not treat (2) as a pure integer program, as was done by Liebchen and Swarat [25]. We will instead investigate proper mixed-integer formulations, where the periodic tension variables x and the periodic timetable variables π are considered as continuous variables.

An alternative MIP formulation for PESP is the *cycle-based* formulation, which has been reported to be computationally beneficial (see., e.g., [8, 22, 24, 37, 39]):

$$\begin{aligned} \text{Minimize} \quad & w^\top x \\ \text{s.t.} \quad & \Gamma x = T z, \\ & \ell \leq x \leq u, \\ & z \text{ integer.} \end{aligned} \tag{3}$$

In (3), x represents a periodic tension, and z is an integral *cycle offset*. A periodic timetable π can be recovered from x by a graph traversal.

To explain the further ingredients of the formulation (3), we will require more definitions about cycles, cycle spaces and cycle bases, see Kavitha et al. [19] for an overview. The *cycle space* $\mathcal{C}$ of G is the abelian group

$$\mathcal{C} := \left\{ \gamma \in \mathbb{Z}^A \;\middle|\; \forall i \in V : \sum_{a \in \delta^+(i)} \gamma_a = \sum_{a \in \delta^-(i)} \gamma_a \right\}.$$

In terms of linear algebra, $\mathcal{C}$ is the kernel over the integers of the incidence matrix of G; in the language of network flows, $\mathcal{C}$ is the space of all integer-valued (and arbitrarily signed) circulations in G. The rank of $\mathcal{C}$ is the *cyclomatic number* μ of G. We assume that G is weakly connected (Remark 2.3), so that $\mu = |A| - |V| + 1$.

A vector $\gamma \in \mathcal{C} \cap \{-1, 0, 1\}^A$ will be called an *oriented cycle*. When ignoring arc directions, the support $\{a \in A \mid \gamma_a \neq 0\}$ makes up a possibly non-simple cycle in G. We call arcs a with $\gamma_a > 0$ *forward* and those with $\gamma_a < 0$ *backward*. Any $\gamma \in \mathcal{C}$ can be decomposed into its positive resp. negative part γ_+ resp. γ_-, i.e., $\gamma_+ := \max(\gamma, 0)$ and $\gamma_- := \max(-\gamma, 0)$. The *length* of an oriented cycle γ is $|\gamma| := |\{a \in A \mid \gamma_a \neq 0\}|$.

A set B of μ oriented cycles is called an *integral cycle basis* of G if B is a basis for $\mathcal{C}$ as an abelian group, i.e., if every element of the cycle space $\mathcal{C}$ can be written as a unique integral linear combination of the oriented cycles in B. A particular class of integral cycle bases are the *(strictly) fundamental cycle bases*: Let $\mathcal{T}$ be some spanning tree of G. Then the fundamental cycle induced by the co-tree arc a of $\mathcal{T}$ is the unique cycle γ obtained by adding a to $\mathcal{T}$ with the convention that $\gamma_a = 1$. A fundamental cycle basis is then given by the collection of μ fundamental cycles of $\mathcal{T}$. Arranging the oriented cycles of an integral cycle basis B as rows of a matrix, we obtain a *cycle matrix* $\Gamma \in \{-1, 0, 1\}^{B \times A}$.

Example 2.4 In the example from Fig. 1, we have $\mu = 3$. An integral cycle basis B is outlined in Fig. 2.

The following theorem shows that the MIP (3) is indeed a valid formulation of PESP.

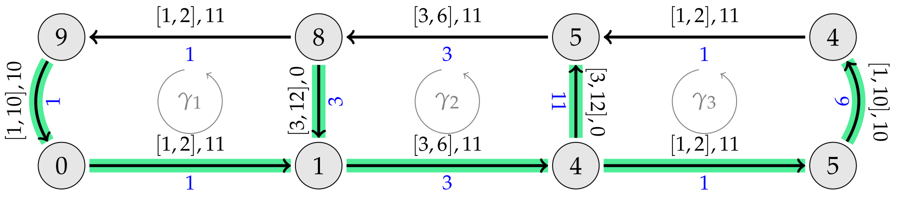

Fig. 2 In the instance from Fig. 1, the oriented cycles $\gamma_1, \gamma_2, \gamma_3$ constitute an integral cycle basis, as they are the fundamental cycles of the highlighted spanning tree. The cycle γ_2 uses only forward arcs, while γ_1 and γ_3 have both forward and backward arcs. The tension $\gamma_3^\top x$ along γ_3 is $1+1+9-11 = 0 \equiv 0 \bmod 10$, and $\gamma_1^\top x$ and $\gamma_2^\top x$ are integer multiples of $T = 10$ as well

Theorem 2.5 (Cycle periodicity property, Liebchen and Peeters [24]) *For a vector* $x \in \mathbb{R}^A$*, the following are equivalent:*

(a) *x satisfies condition* (1),
(b) $\gamma^\top x \equiv 0 \bmod T$ *for all* $\gamma \in \mathcal{C}$,
(c) $\Gamma x \equiv 0 \bmod T$ *for the cycle matrix* Γ *of an integral cycle basis of* G.

In the sequel, we will focus on the cycle-based formulation (3), which is justified by the following remark.

Remark 2.6 The incidence-based formulation (2) is a particular incarnation of the cycle-based formulation (3) in the following sense: Let $I = (G, T, \ell, u, w)$ be a PESP instance. We can augment I to an instance I' such that the incidence-based MIP formulation (2) for I coincides with the cycle-based MIP formulation (3) for I' for a certain integral cycle basis B with cycle matrix Γ. To this end, we add a new vertex s and connect it to every original vertex $i \in V$. Set $\ell_{si} := 0$, $u_{si} := T-1$, $w_{si} := 0$. The subgraph $\mathcal{T}$ on the arcs $\{(s,i) \mid i \in V\}$ is a spanning tree of the augmented graph. Each fundamental cycle has the vertex sequence (s, i, j, s) for some arc $a = (i, j) \in A$; we assume that the arcs (s, i) and (i, j) are forward, and that the arc (s, j) is backward. The constraint in (3) for the cycle (s, i, j, s) is then given by $x_{si} + x_a - x_{sj} = T z_a$. Relabeling x_{si} as π_i for $i \in V$ and z_a as p_a for $a \in A$, the formulation (3) for the augmented instance and the cycle matrix Γ given by the fundamental cycle basis with respect to $\mathcal{T}$ indeed turns out to be the same as the formulation (2) for the original instance I. In particular, the PESP instances I and I' can be considered equivalent.

Example 2.7 Fig. 3 shows the augmented instance I' obtained from the instance I from Fig. 1 according to Remark 2.6.

2.3 The periodic timetabling polytope

Before analyzing the split closure, we need to understand the geometric object behind the feasible region of a PESP instance, and also of its natural LP relaxation.

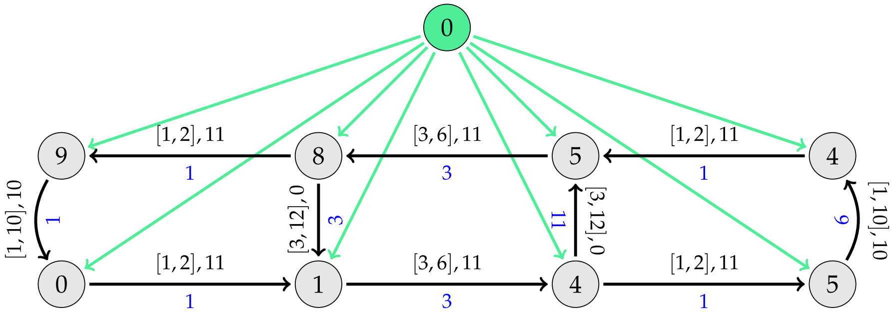

Fig. 3 Augmentation of the instance in Fig. 1 according to Remark 2.6. The new vertex s and the new arcs (s, i) are highlighted in green. The highlighted arcs form a spanning tree of the augmented instance. The periodic tension x_{si} of a highlighted arc (s, i) can be read off the timetable value π_i given as vertex label at the gray vertex i

Definition 2.8 For a PESP instance $I = (G, T, \ell, u, w)$ and a cycle matrix Γ of an integral cycle basis B, define

$$\mathcal{P} := \{(x, z) \in \mathbb{R}^A \times \mathbb{R}^B \mid \Gamma x = Tz, \ell \le x \le u\},$$
$$\mathcal{P}_\mathrm{I} := \mathrm{conv}\{(x, z) \in \mathbb{R}^A \times \mathbb{Z}^B \mid \Gamma x = Tz, \ell \le x \le u\}.$$

We will call $\mathcal{P}$ the *fractional periodic timetabling polytope* and $\mathcal{P}_I$ the *integer periodic timetabling polytope*.

The polytope $\mathcal{P}_\mathrm{I}$ is the convex hull of the feasible solutions to (3), and the fractional periodic timetabling polytope $\mathcal{P}$ is the polyhedron associated to the natural linear programming relaxation of (3). Observe that this relaxation is very weak, as $\mathcal{P}$ is combinatorially equivalent to the hyperrectangle $\prod_{a \in A}[\ell_a, u_a]$: An optimal vertex of the LP relaxation of (3) is then given by $(\ell, \Gamma\ell/T)$.

Remark 2.9 The choice of a cycle basis Γ is not essential for the definition of $\mathcal{P}$ and $\mathcal{P}_\mathrm{I}$: If Γ' is the cycle matrix of another integral cycle basis, then there is a unimodular matrix U such that $\Gamma' = U\Gamma$, and $(x, z) \mapsto (x, Uz)$ is a $\mathbb{Z}$-linear isomorphism.

Several classes of valid inequalities for $\mathcal{P}_\mathrm{I}$ are known [26, 30, 32, 33, 36]. We will focus on those that are defined in terms of elements of the cycle space $\mathcal{C}$. The cycle periodicity property (Theorem 2.5) immediately shows:

Theorem 2.10 (Odijk [36]) *Let $\gamma \in \mathcal{C}$. Then the following* cycle inequality *holds for all $(x, z) \in \mathcal{P}_\mathrm{I}$:*

$$\left\lceil \frac{\gamma_+^\top \ell - \gamma_-^\top u}{T} \right\rceil \le \frac{\gamma^\top x}{T} \le \left\lfloor \frac{\gamma_+^\top u - \gamma_-^\top \ell}{T} \right\rfloor. \tag{4}$$

Since the rows of the cycle matrix Γ are oriented cycles, Theorem 2.10 implies bounds on the z-variables in Definition 2.8 as well, so that $\mathcal{P}$ and $\mathcal{P}_\mathrm{I}$ are indeed polytopes.

Let $[\cdot]_T$ denote the modulo T operator with values in $[0, T)$. Another well-known class of inequalities is the following:

Theorem 2.11 (Nachtigall [32]) *Let* $\gamma \in \mathcal{C}$ *and* $\alpha_\gamma := [-\gamma^\top \ell]_T$*. Then the following* change-cycle inequality *holds for all* $(x, z) \in \mathcal{P}_\mathrm{I}$*:*

$$(T - \alpha_\gamma)\gamma_+^\top (x - \ell) + \alpha_\gamma \gamma_-^\top (x - \ell) \geq \alpha_\gamma (T - \alpha_\gamma). \tag{5}$$

A class generalizing both cycle and change-cycle inequalities are the *flip inequalities* introduced by Lindner and Liebchen [26]. Let I be a PESP instance and let $F \subseteq A$ be an arbitrary subset of arcs. We construct a new PESP instance I_F from I by "flipping" the arcs in F: We replace each arc $a = (i, j) \in F$ by an arc $\overline{a} = (j, i)$, and set $\ell_{\overline{a}} := -u_a$, $u_{\overline{a}} := -\ell_a$, and $w_{\overline{a}} := -w_a$. From any periodic tension x for I, we obtain a periodic tension x_F for I_F by defining $x_{F,a} := x_a$ for $a \in A \backslash F$, and $x_{F,\overline{a}} := -x_a$ for $a \in F$. In particular, I is feasible if and only if I_F is feasible, and in case of feasibility, both I and I_F have the same optimal objective value. Moreover, for any $\gamma \in \mathcal{C}$, we obtain an element γ_F in the cycle space of I_F by setting $\gamma_{F,a} := \gamma_a$ for $a \in A \setminus F$, and $\gamma_{F,\overline{a}} := -\gamma_a$ for $a \in F$. We can hence consider the change-cycle inequality for γ_F on I_F and transform it back to I:

Theorem 2.12 (Lindner and Liebchen [26]) *Let* $\gamma \in \mathcal{C}$ *and* $F \subseteq A$*. Set*

$$\alpha_{\gamma,F} := \left[- \sum_{a \in A \backslash F} \gamma_a \ell_a - \sum_{a \in F} \gamma_a u_a \right]_T.$$

Then the following flip inequality *holds for all* $(x, z) \in \mathcal{P}_\mathrm{I}$*:*

$$\begin{aligned}
&(T - \alpha_{\gamma,F}) \sum_{\substack{a \in A \backslash F: \\ \gamma_a > 0}} \gamma_a (x_a - \ell_a) + \alpha_{\gamma,F} \sum_{\substack{a \in A \backslash F: \\ \gamma_a < 0}} (-\gamma_a)(x_a - \ell_a) \\
&\quad + \alpha_{\gamma,F} \sum_{\substack{a \in F: \\ \gamma_a > 0}} \gamma_a (u_a - x_a) + (T - \alpha_{\gamma,F}) \sum_{\substack{a \in F: \\ \gamma_a < 0}} (-\gamma_a)(u_a - x_a) \\
&\quad \geq \alpha_{\gamma,F}(T - \alpha_{\gamma,F}).
\end{aligned} \tag{6}$$

Remark 2.13 The flip inequalities (6) for $F = \emptyset$ give exactly the change-cycle inequalities (5). Moreover, by flipping all backward resp. all forward arcs of some $\gamma \in \mathcal{C}$, we obtain Odijk's cycle inequalities (4). Since the left-hand side of (6) is always non-negative for $(x, z) \in \mathcal{P}$, flip inequalities with $\alpha_{\gamma,F} = 0$ are trivial. Due to symmetry reasons, the flip inequalities for (γ, F) and $(-\gamma, F)$ coincide, and $\alpha_{\gamma,F} = T - \alpha_{-\gamma,F}$ when $\alpha_{\gamma,F} \geq 1$.

Example 2.14 Fig. 4 illustrates the coefficients of (6) and how to realize cycle (4) and change-cycle inequalities (5) as flip inequalities for specific choices of the set F on an oriented cycle γ on five arcs with two backward arcs.

We briefly explain how to obtain one of Odijk's cycle inequalities (4) in this example by flipping the two backward arcs of γ: As indicated by the third picture, the flip

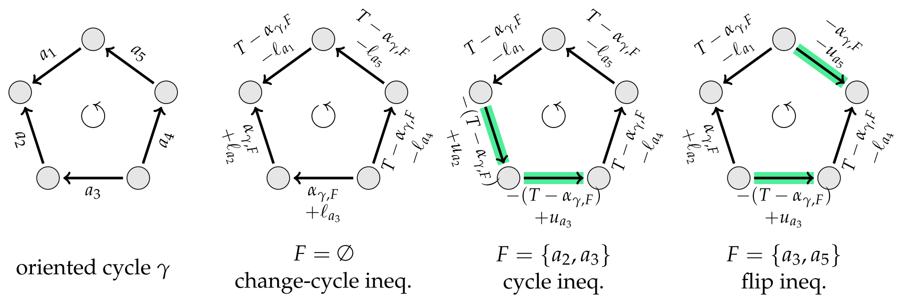

Fig. 4 Some flip inequalities for an oriented cycle γ: The arcs a are labeled with the coefficient of x_a and the contribution to $\alpha_{\gamma,F}$ according to Theorem 2.12. Flipped arcs $a \in F$ lead to nonpositive coefficients, unflipped arcs to nonnegative coefficients

inequality (6) for $F = \{a_2, a_3\}$ reads as

$$\begin{aligned}(T - \alpha_{\gamma,F}) \big((x_{a_1} - \ell_{a_1}) + (u_{a_2} - x_{a_2}) + (u_{a_3} - x_{a_3}) \\ +(x_{a_4} - \ell_{a_4}) + (x_{a_5} - \ell_{a_5})\big) \geq \alpha_{\gamma,F}(T - \alpha_{\gamma,F}),\end{aligned}$$

where

$$\alpha_{\gamma,F} = \left[-\ell_{a_1} + u_{a_2} + u_{a_3} - \ell_{a_4} - \ell_{a_5}\right]_T .$$

Since $\alpha_{\gamma,F} < T$, we can divide by $T - \alpha_{\gamma,F}$. Rearranging terms, the flip inequality is equivalent to

$$\begin{aligned}x_{a_1} - x_{a_2} - x_{a_3} + x_{a_4} + x_{a_5} \geq \ell_{a_1} - u_{a_2} - u_{a_3} + \ell_{a_4} + \ell_{a_5} \\ + \left[-\ell_{a_1} + u_{a_2} + u_{a_3} - \ell_{a_4} - \ell_{a_5}\right]_T .\end{aligned}$$

In vector notation, since $(\gamma_{a_1}, \gamma_{a_2}, \gamma_{a_3}, \gamma_{a_4}, \gamma_{a_5}) = (1, -1, -1, 1, 1)$, this becomes

$$\gamma^\top x \geq \gamma_+^\top \ell - \gamma_-^\top u + [-\gamma_+^\top \ell + \gamma_-^\top u]_T.$$

The right-hand side is 0 modulo T, and it is the smallest integer multiple of T that is greater than or equal to $\gamma_+^\top \ell - \gamma_-^\top u$. Dividing by T, we arrive at one of Odijk's cycle inequalities (4):

$$\frac{\gamma^\top x}{T} \geq \left\lceil \frac{\gamma_+^\top \ell - \gamma_-^\top u}{T} \right\rceil.$$

Definition 2.15 We define the *flip polytope* as

$$\mathcal{P}_{\text{flip}} := \{(x, z) \in \mathcal{P} \mid (x, z) \text{ satisfies the flip inequality for all } \gamma \in \mathcal{C} \text{ and } F \subseteq A\}.$$

Apart from the trivial relation $\mathcal{P}_\mathrm{I} \subseteq \mathcal{P}_\mathrm{flip} \subseteq \mathcal{P}$, the flip polytope has some interesting properties [26]: Every vertex of $\mathcal{P}_\mathrm{I}$ is a vertex of $\mathcal{P}_\mathrm{flip}$, but in general not a vertex of $\mathcal{P}$. Moreover, if G is a cactus graph, i.e., every arc is contained in at most one simple cycle, then $\mathcal{P}_\mathrm{flip} = \mathcal{P}_\mathrm{I}$. However, there are PESP instances with $\mu = 2$ and $\mathcal{P}_\mathrm{flip} \neq \mathcal{P}_\mathrm{I}$.

3 The split closure of the periodic timetabling polyhedron

The relation between the periodic timetabling polytope and the flip polytope seems close and deserves more attention. In fact, in this section, we will establish that the flip polytope can be identified with the split closure.

3.1 Preliminaries

We will now recall the definition of split inequalities, split disjunctions, and the split closure, following the treatment by Conforti et al. [11]. To two matrices $A_C \in \mathbb{Q}^{m \times n}$, $A_I \in \mathbb{Q}^{m \times p}$ and a vector $b \in \mathbb{Q}^p$, we associate the mixed-integer set

$$S := \{(x, z) \in \mathbb{R}^n \times \mathbb{Z}^p \mid A_C x + A_I z \leq b\},$$

and the two polyhedra

$$P := \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^p \mid A_C x + A_I z \leq b\}, \quad P_\mathrm{I} := \operatorname{conv}(S).$$

A *split* is a pair $(\beta, \beta_0) \in \mathbb{Z}^p \times \mathbb{Z}$. The disjunction

$$\beta^\top z \leq \beta_0 \vee \beta^\top z \geq \beta_0 + 1$$

is satisfied for all $(x, z) \in \operatorname{conv}(S)$ and is called a *split disjunction*. In particular, the polyhedron

$$P^{(\beta,\beta_0)} := \operatorname{conv}(\{(x, z) \in P \mid \beta^\top z \leq \beta_0\} \cup \{(x, z) \in P \mid \beta^\top z \geq \beta_0 + 1\})$$

contains $\operatorname{conv}(S)$. The *split closure* is now defined [12] as

$$P_\mathrm{split} := \bigcap_{(\beta,\beta_0) \in \mathbb{Z}^p \times \mathbb{Z}} P^{(\beta,\beta_0)} = \bigcap_{\beta \in \mathbb{Z}^p} \operatorname{conv}(\{(x, z) \in P \mid \beta^\top z \in \mathbb{Z}\}). \tag{7}$$

The split closure P_split is a polyhedron [12] with the property that $\operatorname{conv}(S) \subseteq P_\mathrm{split} \subseteq P$. It is identical to the closure given by Gomory's mixed-integer (GMI) cuts or mixed-integer rounding (MIR) cuts [35]. On the downside, optimization and hence separation are in general NP-hard [9].

The split closure can be described as well in terms of defining inequalities: Define

$$\Lambda := \left\{ \lambda \in \mathbb{R}^m \;\middle|\; \begin{array}{c} \lambda^\top A_C = 0, \lambda^\top A_I \in \mathbb{Z}^p, \lambda^\top b \notin \mathbb{Z}, \text{and the rows of } (A_C \; A_I) \\ \text{corresponding to non-zero entries of } \lambda \text{ are linearly independent} \end{array} \right\}.$$

Each multiplier vector $\lambda \in \Lambda$ defines the *split inequality*

$$\frac{\lambda_+^\top(b - A_C x - A_I z)}{[\lambda^\top b]_1} + \frac{\lambda_-^\top(b - A_C x - A_I z)}{1 - [\lambda^\top b]_1} \geq 1. \tag{8}$$

Here, we decompose λ into its positive part λ_+ and negative part λ_-, and denote by $[\cdot]_1$ the fractional part in $[0, 1)$. The split inequality (8) for $\lambda \in \Lambda$ is valid for $P^{(\lambda^\top A_I, \lfloor \lambda^\top b \rfloor)}$. Conversely, any facet of $P^{(\beta,\beta_0)}$ is defined by an inequality of the form (8) for some $\lambda \in \Lambda$ with $\lambda^\top A_I = \beta$ and $\lfloor \lambda^\top b \rfloor = \beta_0$. Consequently,

$$P_{\text{split}} = \{(x, z) \in P \mid (x, z) \text{ satisfies the split inequality (8) for all } \lambda \in \Lambda\}.$$

3.2 Flipping is splitting for periodic timetabling

We investigate now the split closure for the cycle-based MIP formulation (3) for the Periodic Event Scheduling Problem. Thus let $I = (G, T, \ell, u, w)$ be a PESP instance, and let B be an integral cycle basis of G with cycle matrix Γ. Rewriting in the form $A_C x + A_I z \leq b$, the fractional periodic timetabling polytope $\mathcal{P}$ is defined by

$$\underbrace{\begin{pmatrix} \Gamma \\ -\Gamma \\ I \\ -I \end{pmatrix}}_{A_C}\!\!\!\!\underbrace{\begin{matrix} -TI \\ TI \\ 0 \\ 0 \end{matrix}}_{A_I} \begin{pmatrix} x \\ z \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ u \\ -\ell \end{pmatrix}, \tag{9}$$

where I denotes the identity matrix. We will write a multiplier vector $\lambda \in \Lambda$ as

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^B \times \mathbb{R}^B \times \mathbb{R}^A \times \mathbb{R}^A$$

corresponding to the four row blocks in (9).

Theorem 3.1 *Every flip inequality with $\alpha_{\gamma,F} \neq 0$ is a split inequality for the cycle-based MIP formulation of PESP (3) and vice versa. In particular, $\mathcal{P}_{\text{split}} = \mathcal{P}_{\text{flip}}$.*

Proof We analyze the set Λ for the MIP (3). For $\lambda \in \Lambda$, we have

$$\lambda^\top A_C = (\lambda_1 - \lambda_2)^\top \Gamma + (\lambda_3 - \lambda_4)^\top \quad \text{and} \quad \lambda^\top A_I = -T(\lambda_1 - \lambda_2)^\top.$$

As $\lambda^T A_I$ is integer, we find that $\gamma := T(\lambda_1 - \lambda_2)^\top \Gamma$ is an integer linear combination of the rows of the cycle matrix Γ, so that $\gamma \in \mathcal{C}$. From $\lambda^T A_C = 0$ we infer that $\lambda_3 - \lambda_4 = -\gamma/T$. By the linear independence condition, for an arc $a \in A$, not both of $\lambda_{3,a}$ and $\lambda_{4,a}$ can be non-zero. Hence, when we set $F := \{a \in A \mid \lambda_{3,a} \neq 0\}$, we have

$$\lambda_{3,a} = \begin{cases} -\gamma_a/T & \text{if } a \in F, \\ 0 & \text{if } a \in A \setminus F, \end{cases} \quad \text{and} \quad \lambda_{4,a} = \begin{cases} \gamma_a/T & \text{if } a \in A \setminus F, \\ 0 & \text{if } a \in F. \end{cases} \tag{10}$$

With that, the fractional part $[\lambda^\top b]_1$ evaluates to

$$[\lambda^\top b]_1 = [\lambda_3^\top u - \lambda_4^\top \ell]_1 = \left[-\sum_{a \in F} \frac{\gamma_a u_a}{T} - \sum_{a \in A \setminus F} \frac{\gamma_a \ell_a}{T} \right]_1 .$$

Observe that for any $y \in \mathbb{R}$, we have

$$T\left[\frac{y}{T}\right]_1 = T\left(\frac{y}{T} - \left\lfloor \frac{y}{T} \right\rfloor\right) = y - T\left\lfloor \frac{y}{T} \right\rfloor = [y]_T,$$

so that

$$T[\lambda^\top b]_1 = T[\lambda_3^\top u - \lambda_4^\top \ell]_1 = \alpha_{\gamma,F}, \tag{11}$$

where $\alpha_{\gamma,F}$ is as in Theorem 2.12, and we have $\alpha_{\gamma,F} \neq 0$ because $\lambda^\top b \notin \mathbb{Z}$.

We now consider the expressions $\lambda_\pm^\top(b - A_C x - A_I z)$. Since for $(x, z) \in \mathcal{P}$,

$$b - A_C x - A_I z = (-\Gamma x + Tz, \Gamma x - Tz, u - x, x - \ell)^\top = (0, 0, u - x, x - \ell)^\top,$$

we have

$$\begin{aligned} \lambda_+^\top(b - A_C x - A_I z) &= \frac{1}{T} \sum_{\substack{a \in F \\ \gamma_a < 0}} \gamma_a (u_a - x_a) + \frac{1}{T} \sum_{\substack{a \in A \setminus F \\ \gamma_a > 0}} \gamma_a (x_a - \ell_a), \\ \lambda_-^\top(b - A_C x - A_I z) &= \frac{1}{T} \sum_{\substack{a \in F \\ \gamma_a > 0}} \gamma_a (u_a - x_a) + \frac{1}{T} \sum_{\substack{a \in A \setminus F \\ \gamma_a < 0}} \gamma_a (x_a - \ell_a). \end{aligned} \tag{12}$$

It is now evident from (11) and (12) that multiplying the split inequality (8) for λ with $\alpha_{\gamma,F}(T - \alpha_{\gamma,F})$ yields the flip inequality (6) for (γ, F).

To prove the converse, starting from $\gamma \in \mathcal{C}$ and $F \subseteq A$ with $\alpha_{\gamma,F} \neq 0$, we define λ_3 and λ_4 as in (10), so that $\lambda_3 - \lambda_4 = -\gamma/T$. Moreover, as (11) holds, we have that $\lambda^\top b$ is not an integer. Since $\gamma \in \mathcal{C}$, there is an integral vector $\eta \in \mathbb{Z}^B$ with $\eta^\top \Gamma = \gamma$. Then $\lambda := (\eta/T, 0, \lambda_3, \lambda_4) \in \Lambda$, and the split inequality for λ is equivalent to the flip inequality for (γ, F). □

Lindner and Liebchen [26] proved that $\mathcal{P}_{\text{flip}} = \mathcal{P}_{\text{I}}$ if the cyclomatic number μ of G is at most one by analyzing the combinatorial structure of $\mathcal{P}_{\text{flip}}$. In terms of the split closure, this result becomes almost trivial:

Corollary 3.2 *Suppose that* $\mu \leq 1$. *Then* $\mathcal{P}_{\text{split}} = \mathcal{P}_{\text{I}}$.

Proof. This is clear for $\mu = |B| = 0$, as

$$\mathcal{P} = \mathcal{P}_{\text{split}} = \mathcal{P}_{\text{I}} = \{(x, z) \in \mathbb{R}^A \mid \ell \leq x \leq u\}.$$

For $\mu = |B| = 1$, there is only a single integer variable z, and by virtue of (7),

$$\mathcal{P}_{\text{split}} = \bigcap_{\beta \in \mathbb{Z}} \operatorname{conv}\{(x,z) \in \mathcal{P} \mid \beta z \in \mathbb{Z}\} = \operatorname{conv}\{(x,z) \in \mathcal{P} \mid z \in \mathbb{Z}\} = \mathcal{P}_{\text{I}}. \qquad \square$$

3.3 Chvátal closure

For any mixed-integer set S defined by (A_C, A_I, b) with associated polyhedron P, one can define the *Chvátal closure* as a "one-side split closure" by

$$P_{\text{Ch}} := \bigcap \left\{ P^{(\beta,\beta_0)} \middle| (\beta, \beta_0) \in \mathbb{Z}^p \times \mathbb{Z} \text{ s.t. } P \cap \{\beta^\top z \le \beta_0\} = \emptyset \right.$$
$$\left. \text{or} \quad P \cap \{\beta^\top z \ge \beta_0 + 1\} = \emptyset \right\},$$

see, e.g., [10, 11]. It is clear that $P_{\text{split}} \subseteq P_{\text{Ch}} \subseteq P$. For Periodic Event Scheduling, we find:

Theorem 3.3 *The Chvátal closure of the MIP* (3) *is given by*

$$\mathcal{P}_{\text{Ch}} = \{(x,z) \in \mathcal{P} \mid (x,z) \textit{ satisfies the cycle inequality } (4) \textit{ for all } \gamma \in \mathcal{C}\}.$$

Proof We need to determine those $(\beta, \beta_0) \in \mathbb{Z}^B \times \mathbb{Z}$ for which one of $\mathcal{P} \cap \{\beta^\top z \le \beta_0\}$ or $\mathcal{P} \cap \{\beta^\top z \ge \beta_0 + 1\}$ is empty. Since $\Gamma x = Tz$ holds for all $(x,z) \in \mathcal{P}$, we have $\beta^\top z = \frac{\gamma^\top x}{T}$ for $\gamma := \beta^\top \Gamma \in \mathcal{C}$ for an arbitrary choice of $\beta \in \mathbb{Z}^B$. Let

$$k_1 := \left\lceil \min \left\{ \frac{\gamma^\top x}{T} \middle| (x,z) \in \mathcal{P} \right\} \right\rceil = \left\lceil \frac{\gamma_+^\top \ell - \gamma_-^\top u}{T} \right\rceil,$$
$$k_2 := \left\lfloor \max \left\{ \frac{\gamma^\top x}{T} \middle| (x,z) \in \mathcal{P} \right\} \right\rfloor = \left\lfloor \frac{\gamma_+^\top u - \gamma_-^\top \ell}{T} \right\rfloor.$$

Then $\mathcal{P} \cap \{\frac{\gamma^\top x}{T} \le \beta_0\} = \emptyset$ for all $\beta_0 \le k_1 - 1$ and $\mathcal{P} \cap \{\frac{\gamma^\top x}{T} \ge \beta_0 + 1\} = \emptyset$ for $\beta_0 \ge k_2$. If $k_1 \ge k_2 + 1$, then $\mathcal{P}_{\text{Ch}} = \emptyset$, and no $(x,z) \in \mathcal{P}$ satisfies the cycle inequality (4) for γ. Otherwise, both polyhedra $\mathcal{P} \cap \{\frac{\gamma^\top x}{T} \ge k_1\}$ and $\mathcal{P} \cap \{\frac{\gamma^\top x}{T} \le k_2\}$ are non-empty, and they are defined by $\mathcal{P}$ and Odijk's cycle inequalities (4).

Moreover, since $\mathcal{P} \cap \{\frac{\gamma^\top x}{T} \ge k_1\} \subseteq \mathcal{P} \cap \{\frac{\gamma^\top x}{T} \ge \beta_0\}$ for any $\beta_0 \le k_1$ and $\mathcal{P} \cap \{\frac{\gamma^\top x}{T} \le k_2\} \subseteq \mathcal{P} \cap \{\frac{\gamma^\top x}{T} \le \beta_0\}$ for $\beta_0 \ge k_2$, we can conclude that

$$\bigcap_{\beta_0 \le k_1 - 1} P^{(\beta,\beta_0)} = P^{(\beta, k_1 - 1)} \quad \text{and} \quad \bigcap_{\beta_0 \ge k_2} P^{(\beta,\beta_0)} = P^{(\beta, k_2)}.$$

We conclude that for each $\beta \in \mathbb{Z}^B$ and k_1 and k_2 as above,

$$\bigcap \left\{ P^{(\beta,\beta_0)} \middle| \beta_0 \in \mathbb{Z} \text{ s.t. } P \cap \{\beta^\top z \leq \beta_0\} = \emptyset \text{ or } P \cap \{\beta^\top z \geq \beta_0 + 1\} = \emptyset \right\}$$
$$= P^{(\beta,k_1-1)} \cap P^{(\beta,k_2)},$$

from which the claim follows. □

4 Separation of split cuts

From a practical point of view, the split closure can be a valuable tool to provide dual bounds for mixed-integer programs. Of course, this requires efficient separation methods. As we have established that the split closure is of a specific form in the case of periodic timetabling, we can make use of the combinatorial structure behind flip inequalities to separate cuts.

4.1 Simple cycles

We show at first that for separating split/flip inequalities, it suffices to consider *simple* oriented cycles, i.e., oriented cycles $\gamma \in \mathcal{C} \cap \{-1, 0, 1\}^A$ that yield a simple cycle on the underlying undirected graph of G.

Lemma 4.1 (Orientation-preserving cycle decomposition) *Let $\gamma \in \mathcal{C}$. Then there are simple oriented cycles $\delta_1, \dots, \delta_r \in \mathcal{C}$ such that $\gamma = \sum_{k=1}^r \delta_k$, $\gamma_+ = \sum_{k=1}^r \delta_{k,+}$, and $\gamma_- = \sum_{k=1}^r \delta_{k,-}$.*

Proof Let $A^+ := \{a \in A \mid \gamma_a > 0\}$ and $A^- := \{a \in A \mid \gamma_a < 0\}$ be the set of forward and backward arcs of γ, respectively. Construct a digraph G_γ, whose set of arcs A_γ is given by

$$A_\gamma := A^+ \cup \{(j, i) \mid (i, j) \in A^-\}.$$

Define $g_{ij} := \gamma_{ij}$ if $(i, j) \in A^+$ and $g_{ji} := -\gamma_{ij}$ if $(i, j) \in A^-$. Then $g \geq 0$ is a circulation in G_γ, so that it decomposes into simple directed cycles $d_1, \dots, d_r$. Finally set $\delta_{k,ij} := d_{k,ij}$ if $(i, j) \in A^+$ and $\delta_{k,ij} := -d_{k,ji}$ if $(i, j) \in A^-$. □

Theorem 4.2 *Let $F \subseteq A$ and $(x, z) \in \mathcal{P}$. If (x, z) satisfies all flip inequalities w.r.t. F and all simple oriented cycles γ, then it satisfies all flip inequalities w.r.t. F and all $\gamma \in \mathcal{C}$. In particular,*

$$\mathcal{P}_{\text{split}} = \mathcal{P}_{\text{flip}} = \left\{ (x, z) \in \mathcal{P} \,\middle|\, \begin{array}{c} (x, z) \textit{ satisfies the flip inequality} \\ \textit{for all simple oriented cycles } \gamma \in \mathcal{C} \textit{ and all } F \subseteq A \end{array} \right\}.$$

Proof The inclusion ($\subseteq$) is clear, it remains to show ($\supseteq$). Suppose that (x, z) satisfies the flip inequalities w.r.t. F and all simple oriented cycles. Moving to the flipped instance I_F as in Sect. 2.3, we can assume that $F = \emptyset$, so that it suffices to consider the change-cycle inequality (5) for an arbitrary $\gamma \in \mathcal{C}$. Let $\gamma = \delta_1 + \cdots + \delta_r$ be an orientation-preserving decomposition as in Lemma 4.1. We proceed by induction on r.

If $r \leq 1$, then $\gamma = 0$ or γ is simple, and there is nothing to show.

Now assume $r \geq 2$. There is nothing to show if $\alpha_\gamma = 0$, as the left-hand side of the change-cycle inequality is always non-negative. We hence assume $\alpha_\gamma > 0$.

By induction hypothesis, (x, z) satisfies the change-cycle inequality for the cycles $\delta := \delta_1$ and $\varepsilon := \delta_2 + \cdots + \delta_r$. If $\alpha_\delta = 0$ resp. $\alpha_\varepsilon = 0$, then we have $\alpha_\gamma = \alpha_\varepsilon$ resp. $\alpha_\gamma = \alpha_\delta$, as $\alpha_\gamma = [\alpha_\delta + \alpha_\varepsilon]_T$. The validity of the change-cycle inequality for γ then follows immediately because the right-hand side equals the one for ε resp. δ, while the left-hand side can only become larger.

We are hence left with the case $\alpha_\gamma, \alpha_\delta, \alpha_\varepsilon > 0$. In this case we can rewrite the change-cycle inequality (5) so that it is of the form

$$\frac{\gamma_+^\top (x - \ell)}{\alpha_\gamma} + \frac{\gamma_-^\top (x - \ell)}{T - \alpha_\gamma} \geq 1. \tag{13}$$

With $y := x - \ell$, using that $\gamma_+ = \delta_+ + \varepsilon_+$ and $\gamma_- = \delta_- + \varepsilon_-$, we find

$$\begin{aligned}\frac{\gamma_+^\top y}{\alpha_\gamma} + \frac{\gamma_-^\top y}{T - \alpha_\gamma} &= \frac{\delta_+^\top y}{\alpha_\gamma} + \frac{\delta_-^\top y}{T - \alpha_\gamma} + \frac{\varepsilon_+^\top y}{\alpha_\gamma} + \frac{\varepsilon_-^\top y}{T - \alpha_\gamma} \\ &= \frac{\alpha_\delta}{\alpha_\gamma}\frac{\delta_+^\top y}{\alpha_\delta} + \frac{T - \alpha_\delta}{T - \alpha_\gamma}\frac{\delta_-^\top y}{T - \alpha_\delta} + \frac{\alpha_\varepsilon}{\alpha_\gamma}\frac{\varepsilon_+^\top y}{\alpha_\varepsilon} + \frac{T - \alpha_\varepsilon}{T - \alpha_\gamma}\frac{\varepsilon_-^\top y}{T - \alpha_\varepsilon}.\end{aligned}$$

If we define

$$\kappa_\delta := \min\left\{\frac{\alpha_\delta}{\alpha_\gamma}, \frac{T - \alpha_\delta}{T - \alpha_\gamma}\right\} \quad \text{and} \quad \kappa_\varepsilon := \min\left\{\frac{\alpha_\varepsilon}{\alpha_\gamma}, \frac{T - \alpha_\varepsilon}{T - \alpha_\gamma}\right\},$$

then

$$\frac{\gamma_+^\top y}{\alpha_\gamma} + \frac{\gamma_-^\top y}{T - \alpha_\gamma} \geq \kappa_\delta \left(\frac{\delta_+^\top y}{\alpha_\delta} + \frac{\delta_-^\top y}{T - \alpha_\delta}\right) + \kappa_\varepsilon \left(\frac{\varepsilon_+^\top y}{\alpha_\varepsilon} + \frac{\varepsilon_-^\top y}{T - \alpha_\varepsilon}\right).$$

Since (x, z) satisfies the change-cycle inequality (13) w.r.t. both δ and ε,

$$\frac{\gamma_+^\top y}{\alpha_\gamma} + \frac{\gamma_-^\top y}{T - \alpha_\gamma} \geq \kappa_\delta + \kappa_\varepsilon.$$

Claim $\kappa_\delta + \kappa_\varepsilon = 1.$

If the claim holds, then the change-cycle inequality (13) w.r.t. γ holds for (x, z), and we are done. Recall that $\alpha_\gamma = [\alpha_\delta + \alpha_\varepsilon]_T$, so that

$$\alpha_\gamma = \begin{cases} \alpha_\delta + \alpha_\varepsilon & \text{if} \quad \alpha_\delta + \alpha_\varepsilon < T, \\ \alpha_\delta + \alpha_\varepsilon - T & \text{otherwise.} \end{cases}$$

If $\alpha_\delta + \alpha_\varepsilon < T$, then $\alpha_\gamma = \alpha_\delta + \alpha_\varepsilon$, hence $\alpha_\delta \leq \alpha_\gamma$ and $T - \alpha_\gamma = T - \alpha_\delta - \alpha_\varepsilon \leq T - \alpha_\delta$, so that $\kappa_\delta = \alpha_\delta/\alpha_\gamma$. Analogously, $\kappa_\varepsilon = \alpha_\varepsilon/\alpha_\gamma$, so that $\kappa_\delta + \kappa_\varepsilon = 1$.

In the other case, we have $\alpha_\gamma = \alpha_\delta + \alpha_\varepsilon - T$. From this, we infer $\alpha_\delta = \alpha_\gamma + T - \alpha_\varepsilon \geq \alpha_\gamma$ and $T - \alpha_\gamma = 2T - \alpha_\delta - \alpha_\varepsilon \geq (T - \alpha_\delta) + (T - \alpha_\varepsilon) \geq T - \alpha_\delta$, so that $\kappa_\delta = (T - \alpha_\delta)/(T - \alpha_\gamma)$. Analogously, $\kappa_\varepsilon = (T - \alpha_\varepsilon)/(T - \alpha_\gamma)$, so that again $\kappa_\delta + \kappa_\varepsilon = 1$. □

Using Theorem 3.3 and Remark 2.13, Theorem 4.2 implies the analogous result for the Chvátal split closure and the cycle inequalities:

Corollary 4.3 *Let $(x, z) \in \mathcal{P}$. If (x, z) satisfies all cycle inequalities w.r.t. all simple oriented cycles γ, then it satisfies all cycle inequalities for all $\gamma \in \mathcal{C}$. In particular,*

$$\mathcal{P}_{\mathrm{Ch}} = \left\{ (x, z) \in \mathcal{P} \;\middle|\; \begin{array}{c} (x, z) \text{ satisfies the cycle inequality} \\ \text{for all simple oriented cycles } \gamma \in \mathcal{C} \end{array} \right\}.$$

4.2 Separation hardness

Lindner and Liebchen [26] outline a pseudo-polynomial time algorithm based on the dynamic program by Borndörfer et al. [7] that finds a maximally violated flip inequality (if there is any), i.e., a simple cycle γ and a set $F \subseteq A$ such that the difference of the right-hand and left-hand sides of (6) is maximum. We prove here that pseudo-polynomial time is best possible unless P = NP:

Theorem 4.4 *Given $(x, z) \in \mathcal{P}$ and $M \geq 0$, it is weakly NP-hard to decide whether there exist a simple cycle γ and a subset $F \subseteq A$ such that (x, z) violates the flip inequality for (γ, F) by at least M.*

Proof We reduce the weakly NP-hard Ternary Partition Problem [7]: Given $m \in \mathbb{N}$ and $c \in \mathbb{N}^m$, is there $a \in \{-1, 0, 1\}^m$ such that $\sum_{i=1}^m a_i c_i = \pm\frac{1}{2}\sum_{i=1}^m c_i$? For a Ternary Partition intance (m, c), we define a PESP instance $I = (G, T, \ell, u, w)$ as follows: The digraph $G = (V, A)$ is given by a complete directed graph on the vertex set

$$V := \{1^+, 1^-, 2^+, 2^-, \ldots, m^+, m^-\},$$

where we delete the arcs $(1^-, 1^+), (2^-, 2^+), \ldots, (m^-, m^+)$. We set $T := \sum_{i=1}^m c_i$ and

$$\ell_{i^+i^-} := c_i, \quad u_{i^+i^-} := T, \quad w_{i^+i^-} := 1 \quad \text{for all} \quad i \in \{1, \ldots, m\}.$$

For all other arcs a, we set $\ell_a := u_a := w_a := 0$. As for any PESP instance, the optimal solution to the LP relaxation of (3) is given by $x^* = \ell$.

Suppose now that $x^* = \ell$ violates some flip inequality (6) for some simple oriented cycle γ and some $F \subseteq A$ by at least $M := \frac{T^2}{4}$. Since $x^* = \ell$, only arcs in F contribute non-trivially to the left-hand side of (6), moreover, these arcs are all of the form (i^+, i^-). We hence obtain

$$\alpha_{\gamma,F} \sum_{(i^+,i^-)\in F, \gamma_{i^+i^-}=1} (T - c_i) + (T - \alpha_{\gamma,F}) \sum_{(i^+,i^-)\in F, \gamma_{i^+i^-}=-1} (T - c_i) \leq \alpha_{\gamma,F}(T - \alpha_{\gamma,F}) - M, \tag{14}$$

where $\alpha_{\gamma,F} = [-\sum_{(i^+,i^-)\notin F} \gamma_{i^+i^-} c_i]_T$. As the left-hand side of (6) is non-negative, we have that $\alpha_{\gamma,F}(T - \alpha_{\gamma,F}) \geq M = T^2/4$, which implies $\alpha_{\gamma,F} = T/2$. Set $a_i := -\gamma_{i^+,i^-}$ for all $i \in \{1,\dots,m\}$. Then $[\sum_{i=1}^m a_i c_i]_T = \alpha_{\gamma,F} = T/2$, and as $-T \leq \sum_{i=1}^m a_i c_i \leq T$, we find that $\sum_{i=1}^m a_i c_i = \pm T/2$. In particular, a violated flip inequality leads to a positive answer to the Ternary Partition instance.

Conversely, suppose that there is $a \in \{-1,0,1\}^m$ such that $\sum_{i=1}^m a_i c_i = \pm T/2$. Construct a simple oriented cycle γ with $\gamma_{i^+i^-} := -a_i$ for all $i \in \{1,\dots,m\}$. Then $\alpha_{\gamma,\emptyset} = T/2$, and the flip inequality for γ and $F = \emptyset$ (i.e., the change-cycle inequality for γ) is violated by at least $T^2/4 = M$, because the left-hand side of (14) vanishes. □

In practice, the dynamic program indicated in [26] consumes too much memory. It is therefore advantageous to switch to a cut-generating MIP. Balas and Saxena [3] describe a parametric MIP with a single parameter $\theta \in [0,1]$ for this purpose. We are however in a better situation: Translating to periodic timetabling via Theorem 3.1, the parameter θ essentially corresponds to $\alpha_{\gamma,F}$, which is always an integer between 0 and $T-1$. This means that the parametric MIP can be replaced by a finite sequence of standard IPs for each such integer $\alpha_{\gamma,F}$. The formulation of the IP (15) is straightforward from the definition (6) of flip inequalities:

Theorem 4.5 *Let $(x,z) \in \mathcal{P}\setminus\mathcal{P}_{\text{flip}}$ and $\alpha \in \{1,\dots,T\}$. Then a maximally violated flip inequality w.r.t. (x,z) with $\alpha_{\gamma,F} = \alpha$ among all oriented cycles γ and all $F \subseteq A$ is found by the following integer program:*

$$\begin{aligned}
\textit{Minimize} \quad & (T-\alpha)\sum_{a\in A}(x_a - \ell_a)y_a^+ + \alpha\sum_{a\in A}(x_a-\ell_a)y_a^- \\
& +\alpha\sum_{a\in A}(x_a-\ell_a)f_a^+ + (T-\alpha)\sum_{a\in A}(u_a - x_a)f_a^- \\
\textit{s.t.} \quad & \sum_{a\in A}\ell_a(y_a^- - y_a^+) + \sum_{a\in A} u_a(f_a^- - f_a^+) + kT = \alpha, \\
& \sum_{a\in\delta^+(v)}\gamma_a - \sum_{a\in\delta^-(v)}\gamma_a = 0, \qquad \forall v \in V, \\
& f^+ - f^- + y^+ - y^- = \gamma, \\
& 0 \leq f^+ + f^- + y^+ + y^- \leq 1, \\
& f^+, f^-, y^+, y^- \in \{0,1\}^A, \\
& \gamma \in \{-1,0,1\}^A, \\
& k \in \mathbb{Z}.
\end{aligned} \tag{15}$$

Any feasible solution of (15) with objective value less than $\alpha(T-\alpha)$ will produce a violated flip inequality. Recall from Remark 2.13 that flip inequalities with $\alpha_{\gamma,F}=0$ are trivial and cannot be violated, and that due to symmetry, it is not necessary to consider the IP (15) for $\alpha \geq T/2$.

4.3 Separation for a fixed cycle

We discuss now how to find a maximally violated flip inequality in linear time when the cycle γ is already fixed. To this end, we take the perspective of split cuts. Consider again a mixed-integer set defined by (A_C, A_I, b) and the associated polyhedron $P = \{A_C x + A_I z \leq b\}$. When a split (β, β_0) is fixed, then the separation problem on $P^{(\beta,\beta_0)}$ can be solved as follows [4, 6, 11]: Given $(x, z) \in P$, check whether $\beta^\top z \leq \beta_0$ or $\beta^\top z \geq \beta_0 + 1$. If yes, then $(x, z) \in P^{(\beta,\beta_0)}$. Otherwise, solve the linear program

$$\begin{aligned} \text{Minimize} \quad & (s-t)^\top b + \frac{1}{\beta^\top z - \beta_0} \cdot t^\top (b - A_C x - A_I z) \\ \text{s.t.} \quad & (s-t)^\top A_C = 0, \\ & (s-t)^\top A_I = \beta^\top, \\ & s, t \geq 0. \end{aligned} \tag{16}$$

If the value of (16) is at least $\beta_0 + 1$, then $(x, z) \in P^{(\beta,\beta_0)}$, otherwise it is not. In the latter case, if we take a basic optimal solution (s^*, t^*), then (x, z) is separated by the split inequality w.r.t. $s^* - t^*$. This cut-generating LP (16) finds a maximally violated split inequality in the following sense:

Lemma 4.6 *Suppose that $(x, z) \in P \backslash P^{(\beta,\beta_0)}$. Let (s^*, t^*) be an optimal basic solution of (16), $\lambda^* := s^* - t^*$. Then*

$$\begin{aligned} & [\lambda^{*\top} b]_1 (1 - [\lambda^{*\top} b]_1) - (1 - [\lambda^{*\top} b]_1) \lambda_+^{*\top} (b - A_C x - A_I z) \\ & - [\lambda^{*\top} b]_1 \lambda_-^{*\top} (b - A_C x - A_I z) \end{aligned} \tag{17}$$

is maximum among all $\lambda = s - t$ such that (s, t) is feasible for (16) and $\lambda^\top b \in [\beta_0, \beta_0 + 1)$.

Proof Since (s^*, t^*) is basic, we have $\lambda_+^* = s$ and $\lambda_-^* = t$. As $(x, z) \in P \setminus P^{(\beta,\beta_0)}$, $\beta^\top z - \beta_0 > 0$. Then λ^* maximizes

$$-(\beta^\top z - \beta_0) \lambda^\top b - \lambda_-^\top (b - A_C x - A_I z).$$

Adding a constant term, λ^* also maximizes

$$\begin{aligned} & (\beta^\top z - \beta_0)(\beta_0 + 1) - (\beta^\top z - \beta_0) \lambda^\top b - \lambda_-^\top (b - A_C x - A_I z) \\ & = (\beta^\top z - \beta_0)(\beta_0 + 1 - \lambda^\top b) - \lambda_-^\top (b - A_C x - A_I z). \end{aligned}$$

Observing that $\lambda^\top(b - A_C x - A_I z) = \lambda^\top b - \beta^\top z$, this is the same as

$$\begin{aligned}&(\lambda^\top b - \lambda^\top(b - A_C x - A_I z) - \beta_0)(\beta_0 + 1 - \lambda^\top b) - \lambda_-^\top(b - A_C x - A_I z)\\ &\quad = (\lambda^\top b - \beta_0)(\beta_0 + 1 - \lambda^\top b) - (\beta_0 + 1 - \lambda^\top b)\lambda^\top(b - A_C x - A_I z)\\ &\qquad - \lambda_-^\top(b - A_C x - A_I z).\end{aligned}$$

With $\lambda = \lambda_+ - \lambda_-$, this equals

$$\begin{aligned}&(\lambda^\top b - \beta_0)(\beta_0 + 1 - \lambda^\top b) - (\beta_0 + 1 - \lambda^\top b)\lambda_+^\top(b - A_C x - A_I z)\\ &\quad - (\lambda^\top b - \beta_0)\lambda_-^\top(b - A_C x - A_I z).\end{aligned}$$

Since $\lambda^\top b \in [\beta_0, \beta_0 + 1)$, $[\lambda^\top b]_1 = \lambda^\top b - \beta_0$ and $1 - [\lambda^\top b]_1 = \beta_0 + 1 - \lambda^\top b$, and we arrive at (17). □

Note that the condition $\lambda^\top b \in [\beta_0, \beta_0 + 1)$ in Lemma 4.6 is no restriction, since it suffices to consider λ for which $\lfloor \lambda^\top b \rfloor = \beta_0$ (cf. Sect. 3). We obtain the following in the context of periodic timetabling:

Theorem 4.7 *Let $\mathcal{P}$ be a fractional periodic timetabling polytope. Let $(x, z) \in \mathcal{P}$, $\gamma \in \mathcal{C}$ with $\gamma^\top x \notin T\mathbb{Z}$, and set $g := T/[-\gamma^\top x]_T$. Then the flip inequality w.r.t. γ and*

$$\begin{aligned}F &:= \{a \in A \mid \gamma_a > 0 \text{ and } u_a - \ell_a \geq g(u_a - x_a)\}\\ &\quad \cup \{a \in A \mid \gamma_a < 0 \text{ and } u_a - \ell_a \leq g(x_a - \ell_a)\}\end{aligned}$$

is maximally violated by (x, z) among the flip inequalities w.r.t. γ. In particular, a maximally violated flip inequality w.r.t. γ can be found in $O(|\gamma|)$ time.

Proof We first write down the cut-generating LP (16) for the PESP situation (9):

$$\begin{aligned}\text{Minimize} \quad & s_3^\top u - t_3^\top u - s_4^\top \ell + t_4^\top \ell + \frac{1}{\beta^\top z - \beta_0} \cdot (t_3^\top(u - x) + t_4^\top(x - \ell))\\ \text{s.t.} \quad & (s_1 - t_1 - s_2 + t_2)^\top \Gamma + s_3^\top - t_3^\top - s_4^\top + t_4^\top = 0,\\ & -s_1 + t_1 + s_2 - t_2 = \frac{\beta}{T},\\ & s_1, s_2, s_3, s_4, t_1, t_2, t_3, t_4 \geq 0.\end{aligned}$$

Recall from Theorem 3.1 that a flip inequality w.r.t. γ corresponds to a split inequality derived from $P^{(\beta,\beta_0)}$ with $\beta^\top\Gamma = -\gamma$. Since $(x, z) \in \mathcal{P}$, we have $\beta_0 = \lfloor \beta^\top z \rfloor = \lfloor -\gamma^\top x/T \rfloor$. Eliminating the variables s_1, s_2, t_1, t_2, and setting

$$g := \frac{1}{\beta^\top z - \beta_0} = \frac{1}{[\beta^\top z]_1} = \frac{1}{[\beta^\top \Gamma x/T]_1} = \frac{1}{[-\gamma^\top x/T]_1} = \frac{T}{[-\gamma^\top x]_T},$$

this becomes

$$\begin{aligned} \text{Minimize} \quad & s_3^\top u - t_3^\top u - s_4^\top \ell + t_4^\top \ell + g \cdot (t_3^\top (u - x) + t_4^\top (x - \ell)) \\ \text{s.t.} \quad & s_3 - t_3 - s_4 + t_4 = -\frac{\gamma}{T}, \\ & s_3, s_4, t_3, t_4 \geq 0. \end{aligned}$$

This linear program is trivial to solve: In each basic solution, for each arc $a \in A$ at most one of $s_{3,a}, s_{4,a}, t_{3,a}, t_{4,a}$ will be non-zero, and $s_{3,a} = s_{4,a} = t_{3,a} = t_{4,a} = 0$ for all $a \in A$ with $\gamma_a = 0$. We examine the contribution to the objective for each arc a in γ in such a basic solution:

If $\gamma_a > 0$, then either $t_{3,a} > 0$ or $s_{4,a} > 0$. In the first case, the contribution to the objective is $\gamma_a(g(u_a - x_a) - u_a)/T$, otherwise $-\gamma_a \ell_a / T$. Otherwise, if $\gamma_a < 0$, then either $s_{3,a} > 0$ or $t_{4,a} > 0$, the contribution being $-\gamma_a u_a / T$ resp. $-\gamma_a(\ell_a + g(x_a - \ell_a))/T$. In particular, an optimal solution is given by

$$\begin{aligned} t_{3,a} &:= \frac{\gamma_a}{T} \quad \text{for all } a \text{ s.t. } \gamma_a > 0 \text{ and } -\ell_a \geq g(u_a - x_a) - u_a, \\ s_{4,a} &:= \frac{\gamma_a}{T} \quad \text{for all } a \text{ s.t. } \gamma_a > 0 \text{ and } -\ell_a < g(u_a - x_a) - u_a, \\ s_{3,a} &:= -\frac{\gamma_a}{T} \quad \text{for all } a \text{ s.t. } \gamma_a < 0 \text{ and } u_a \leq g(x_a - \ell_a) + \ell_a, \\ t_{4,a} &:= -\frac{\gamma_a}{T} \quad \text{for all } a \text{ s.t. } \gamma_a < 0 \text{ and } u_a > g(x_a - \ell_a) + \ell_a, \end{aligned}$$

and $s_{3,a} := s_{4,a} := t_{3,a} := t_{4,a} := 0$ otherwise. The cut derived from this solution is the split inequality for $\lambda = s - t$, which by Theorem 3.1 corresponds to the flip inequality for γ and

$$\begin{aligned} F &= \{a \in A \mid \lambda_{3,a} \neq 0\} \\ &= \{a \in A \mid \gamma_a > 0 \text{ and } u_a - \ell_a \geq g(u_a - x_a)\} \\ &\quad \cup \{a \in A \mid \gamma_a < 0 \text{ and } u_a - \ell_a \leq g(x_a - \ell_a)\}. \end{aligned}$$

Observe that by (11), $T[\lambda^\top b]_1 = \alpha_{\gamma,F}$. Using (12) and multiplying (17) with T^2 therefore yields the violation of the flip inequality w.r.t. (γ, F). By Lemma 4.6, we conclude that the violation is indeed maximal. □

5 Comparing split closures

Recall that the Periodic Event Scheduling Problem can be formulated in two ways as a MIP, where the incidence-based formulation (2) is essentially a special case of the cycle-based formulation (3) by virtue of Remark 2.6. The methods of Sect. 3 therefore apply to both formulations, and the question arises whether one of the two split closures is stronger. We will show that both closures are in fact of the same strength in Sect. 5.3.

Typically, the integer variables in both formulations are general. However, under certain circumstances, the periodic offset variables p_a in (2) can be assumed to be binary [21]. We will discuss Sect. 5.4 how to achieve binary variables by a subdivision procedure. We will show that this binarization approach does not lead to a stronger split closure.

We show in Sect. 5.2 that split closures commute with Cartesian products, which means in the PESP situation that the split closures can be considered on blocks of G individually.

However, to be able to compare split closures of different polyhedra, we need to develop a few technicalities first in Sect. 5.1.

5.1 Mixed-integer-compatible maps

We begin with two mixed-integer sets

$$S_i := \{(x, z) \in \mathbb{R}^{n_i} \times \mathbb{Z}^{p_i} \mid A^i_C x + A^i_I z \leq b^i\}, \quad i \in \{1, 2\},$$

and the associated polyhedra

$$P_i := \{(x, z) \in \mathbb{R}^{n_i} \times \mathbb{R}^{p_i} \mid A^i_C x + A^i_I z \leq b^i\}, \quad (P_i)_\mathrm{I} := \mathrm{conv}(S_i), \quad i \in \{1, 2\}.$$

Definition 5.1 A map $\varphi : \mathbb{R}^{n_1} \times \mathbb{R}^{p_1} \to \mathbb{R}^{n_2} \times \mathbb{R}^{p_2}$ is *mixed-integer-compatible* if φ is affine and $\varphi(\mathbb{R}^{n_1} \times \mathbb{Z}^{p_1}) \subseteq \mathbb{R}^{n_2} \times \mathbb{Z}^{p_2}$.

In particular, if $\varphi(P_1) \subseteq P_2$ for a mixed-integer-compatible map φ, then $\varphi(S_1) \subseteq S_2$ and $\varphi((P_1)_\mathrm{I}) \subseteq (P_2)_\mathrm{I}$.

Lemma 5.2 *Let $\psi : \mathbb{R}^{n_1} \times \mathbb{R}^{p_1} \to \mathbb{R}^{n_2} \times \mathbb{R}^{p_2}$ be a linear map and let $\psi^* : \mathbb{R}^{n_2} \times \mathbb{R}^{p_2} \to \mathbb{R}^{n_1} \times \mathbb{R}^{p_1}$ be the corresponding dual linear map, identifying dual vector spaces choosing standard bases. Then the following are equivalent:*

(a) *ψ is mixed-integer-compatible.*
(b) *$\psi(\mathbb{R}^{n_1} \times \{0\}) \subseteq \mathbb{R}^{n_2} \times \{0\}$ and $\psi^*(\{0\} \times \mathbb{Z}^{p_2}) \subseteq \{0\} \times \mathbb{Z}^{p_1}$.*

Proof (a) $\Rightarrow$ (b): For the first statement consider for $i \in [n_1]$ the i-th standard basis vector $e_i \in \mathbb{R}^{n_1}$. Then $\psi(e_i, 0) = (x, z)$ for some $x \in \mathbb{R}^{n_2}$ and $z \in \mathbb{R}^{p_2}$. But as ψ is linear and mixed-integer-compatible, $\psi(\lambda e_i, 0) = (\lambda x, \lambda z)$ with $\lambda z \in \mathbb{Z}^{p_2}$ for all $\lambda \in \mathbb{R}$, so that $z = 0$.

For the second statement, consider for $j \in [p_2]$ the j-th standard basis vector e_j. Then for $i \in [n_1]$, the i-th coordinate of $\psi^*(0, e_j)$ is given by $(0, e_j)^\top \psi(e_i, 0) = 0$ by the first statement. For $i \in [p_1]$, the $(n_1 + i)$-th coordinate of $\psi^*(0, e_j)$ is given by $(0, e_j)^\top \psi(0, e_i)$, which is integral as ψ is mixed-integer-compatible.

(b) $\Rightarrow$ (a): Let $(x, z) \in \mathbb{R}^{n_1} \times \mathbb{Z}^{p_1}$. Then $\psi(x, z) = \psi(x, 0) + \psi(0, z)$, so using linearity and the first statement in (b), it suffices to consider $\psi(0, e_i)$ for $i \in [p_1]$. But now for $j \in [p_2]$, the $(n_1 + i)$-th coordinate of $\psi^*(0, e_j)$ is integral by the second statement in (b), and since it is given by $(0, e_j)^\top \psi(0, e_i)$, we conclude that the $(n_2 + j)$-th coordinate of $\psi(0, e_i)$ is integer. Consequently, ψ must be mixed-integer-compatible. □

The following is a generalization of Theorem 1 in [14].

Theorem 5.3 *Let φ be a mixed-integer-compatible map with $\varphi(P_1) \subseteq P_2$. Then $\varphi((P_1)_{\text{split}}) \subseteq (P_2)_{\text{split}}$.*

Proof Consider $(x_1, z_1) \in (P_1)_{\text{split}}$ and $\beta_2 \in \mathbb{Z}^{p_2}$. We need to show that $\varphi(x_1, z_1)$ is a convex combination of points $(x_2^i, z_2^i) \in P_2$ with $\beta_2^\top z_2^i$ integral. Since φ is mixed-integer-compatible, the last p_2 entries of $\varphi(0, 0)$ are integral, and so the linear map $\psi := \varphi - \varphi(0, 0)$ is mixed-integer-compatible as well. By Lemma 5.2, $\psi^*(0, \beta_2) = (0, \beta_1)$ for some $\beta_1 \in \mathbb{Z}^{p_1}$. Since $(x_1, z_1) \in (P_1)_{\text{split}}$, it is a convex combination of $(x_1^i, z_1^i) \in P_1$ with $\beta_1^\top z_1^i \in \mathbb{Z}$. Write

$$(x_2^i, z_2^i) := \varphi(x_1^i, z_1^i) = \psi(x_1^i, z_1^i) + \varphi(0, 0) \in P_2.$$

Then

$$\begin{aligned}
\beta_2^\top z_2^i = (0, \beta_2)^\top (x_2^i, z_2^i) &= (0, \beta_2)^\top \psi(x_1^i, z_1^i) + (0, \beta_2)^\top \varphi(0, 0) \\
&= \psi^*(0, \beta_2)^\top (x_1^i, z_1^i) + (0, \beta_2)^\top \varphi(0, 0) \\
&= (0, \beta_1)^\top (x_1^i, z_1^i) + (0, \beta_2)^\top \varphi(0, 0) \\
&= \beta_1^\top z_1^i + (0, \beta_2)^\top \varphi(0, 0) \\
&\in \mathbb{Z}.
\end{aligned}$$

As φ is affine and hence preserve convex combinations, $\varphi(x_1, z_1)$ is a convex combination of the $(x_2^i, z_2^i) \in P_2$. □

Example 5.4 An example for a mixed-integer-compatible map is provided by the change of the cycle basis in the context of periodic timetabling. Let $I = (G, T, \ell, u, w)$ be a PESP instance and let Γ, Γ' be two cycle matrices of integral cycle bases of G. As in Remark 2.9, there is an unimodular matrix U such that $\Gamma' = U\Gamma$. The map $\varphi : (x, z) \mapsto (x, Uz)$ maps the fractional periodic timetabling polytope $\mathcal{P}_1$ defined by Γ to the fractional periodic timetabling polytope $\mathcal{P}_2$ defined by Γ'. The map φ is clearly linear and maps mixed-integer points to mixed-integer points, so that φ is mixed-integer-compatible by definition. We conclude that $\varphi((\mathcal{P}_1)_{\text{split}}) \subseteq (P_2)_{\text{split}}$. Since U is unimodular, φ has a mixed-integer compatible inverse, so that φ provides a "mixed-integer" isomorphism of $(P_1)_{\text{split}}$ with $(P_2)_{\text{split}}$.

5.2 Split closure of cartesian products

As first application of mixed-integer-compatible maps, we prove that split closures are compatible with Cartesian products.

Theorem 5.5 *Consider two mixed-integer sets*

$$S_i = \{(x, z) \in \mathbb{R}^{n_i} \times \mathbb{Z}^{p_i} \mid A_C^i x + A_I^i z \leq b^i\}, \quad \forall i \in \{1, 2\},$$

and the associated polyhedra

$$P_i := \{(x, z) \in \mathbb{R}^{n_i} \times \mathbb{R}^{p_i} \mid A_C^i x + A_I^i z \leq b^i\}, \quad \forall i \in \{1, 2\}.$$

Then $(P_1 \times P_2)_{\text{split}} = (P_1)_{\text{split}} \times (P_2)_{\text{split}}$.

Proof We first prove $(P_1)_{\text{split}} \times (P_2)_{\text{split}} \subseteq (P_1 \times P_2)_{\text{split}}$ using the characterization (7):

$$\begin{aligned}
&(P_1)_{\text{split}} \times (P_2)_{\text{split}} \\
&= \left(\bigcap_{\beta_1 \in \mathbb{Z}^{p_1}} \operatorname{conv}(\{(x_1, z_1) \in P_1 \mid \beta_1^\top z_1 \in \mathbb{Z}\}) \right) \\
&\quad \times \left(\bigcap_{\beta_2 \in \mathbb{Z}^{p_2}} \operatorname{conv}(\{(x_2, z_2) \in P_2 \mid \beta_2^\top z_2 \in \mathbb{Z}\}) \right) \\
&= \bigcap_{\beta_1 \in \mathbb{Z}^{p_1}} \bigcap_{\beta_2 \in \mathbb{Z}^{p_2}} \Big(\operatorname{conv}(\{(x_1, z_1) \in P_1 \mid \beta_1^\top z_1 \in \mathbb{Z}\}) \\
&\quad \times \operatorname{conv}(\{(x_2, z_2) \in P_2 \mid \beta_2^\top z_2 \in \mathbb{Z}\}) \Big) \\
&= \bigcap_{(\beta_1, \beta_2) \in \mathbb{Z}^{p_1} \times \mathbb{Z}^{p_2}} \operatorname{conv}(\{(x_1, z_1) \in P_1 \mid \beta_1^\top z_1 \in \mathbb{Z}\} \times \{(x_2, z_2) \in P_2 \mid \beta_2^\top z_2 \in \mathbb{Z}\}) \\
&= \bigcap_{(\beta_1, \beta_2) \in \mathbb{Z}^{p_1} \times \mathbb{Z}^{p_2}} \operatorname{conv}(\{(x_1, z_1, x_2, z_2) \in P_1 \times P_2 \mid \beta_1^\top z_1 \in \mathbb{Z}, \beta_2^\top z_2 \in \mathbb{Z}\}) \\
&\subseteq \bigcap_{(\beta_1, \beta_2) \in \mathbb{Z}^{p_1} \times \mathbb{Z}^{p_2}} \operatorname{conv}(\{(x_1, z_1, x_2, z_2) \in P_1 \times P_2 \mid (\beta_1, \beta_2)^\top (z_1, z_2) \in \mathbb{Z}\}) \\
&= (P_1 \times P_2)_{\text{split}}.
\end{aligned}$$

To show the reverse inclusion, we consider the natural projections $\varphi_i : P_1 \times P_2 \to P_i$ for $i \in \{1, 2\}$. Both φ_i are mixed-integer-compatible, so that by Theorem 5.3, $\varphi_i((P_1 \times P_2)_{\text{split}}) \subseteq (P_i)_{\text{split}}$. In particular, the map (φ_1, φ_2), which is the identity map, maps $(P_1 \times P_2)_{\text{split}}$ into $(P_1)_{\text{split}} \times (P_2)_{\text{split}}$. □

We apply now Theorem 5.5 to periodic timetabling. Consider for an arbitrary digraph G its decomposition into blocks. Since each cycle is part of a unique block, the cycle space of G decomposes into the direct sum of the cycle spaces of its blocks. This has the consequence that any cycle matrix Γ of G has a block structure as well, so that the fractional periodic timetabling polytope $\mathcal{P}$ is the Cartesian product of the fractional periodic timetabling polytopes associated to the subinstances of each block.

Corollary 5.6 (cf. [26]) *If* $G_1, \ldots, G_k$ *are the blocks of* G *and* $\mathcal{P}_1, \ldots, \mathcal{P}_k$ *are the fractional periodic timetabling polytopes of the subinstances of* $G_1, \ldots, G_k$*, respectively, then*

$$\mathcal{P}_{\text{split}} = (\mathcal{P}_1)_{\text{split}} \times \cdots \times (\mathcal{P}_k)_{\text{split}}.$$

In particular, if G is a cactus graph, then $\mathcal{P}_{\text{split}} = \mathcal{P}_{\text{I}}$.

Proof By the above discussion, this is a direct consequence of Theorem 5.5. If G is a cactus graph, then each block satisfies $\mu \leq 1$. It remains to apply Corollary 3.2. □

5.3 Incidence-based versus cycle-based formulation

Recall from Remark 2.6 that the incidence-based formulation (2) of a PESP instance is identical to a particular cycle-based formulation (3) of an augmented instance, where the augmentation consists in successively adding arcs a with $\ell_a = 0$ and $u_a = T - 1$. Such arcs with $u_a - \ell_a = T - 1$ are sometimes called *free* (e.g., by Goerigk and Liebchen [16]), as they do not impact the feasibility of a PESP instance. The augmentation procedure in Remark 2.6 hence decomposes as a sequence of $|V|$ *simple free augmentations*, which we formally define as follows:

Definition 5.7 Let $I = (G, T, \ell, u, w)$ be a PESP instance. Let $I' = (G', T, \ell', u', w')$ be a PESP instance such that I arises from I' by deleting a *free* arc $\overline{a}$, i.e., $u'_{\overline{a}} - \ell'_{\overline{a}} = T - 1$. We say that I' is a *simple free augmentation* of I by $\overline{a}$.

We will first investigate a trivial case of a simple augmentation I' of I by $\overline{a}$: If $\overline{a}$ is a bridge, then $\overline{a}$ constitutes a block of G', so that we conclude by 5.6 that

$$\mathcal{P}'_{\text{split}} = \mathcal{P}_{\text{split}} \times [\ell'_{\overline{a}}, u'_{\overline{a}}]_{\text{split}} = \mathcal{P}_{\text{split}} \times [\ell'_{\overline{a}}, u'_{\overline{a}}], \tag{18}$$

where $\mathcal{P}$ and $\mathcal{P}'$ are the fractional periodic timetabling polytopes of I and I', respectively. Since $\overline{a}$ is a bridge, any cycle basis for G is a cycle basis for G', so that the choice of any integral cycle basis yields a natural projection $\mathcal{P}'_{\text{split}} \to \mathcal{P}_{\text{split}}$, $(x, x_{\overline{a}}, z) \mapsto (x, z)$, which is well-defined and surjective by (18). Thus any split inequality for I' is trivially a split inequality for I and vice versa.

We will hence turn our interest to the more interesting case that $\overline{a}$ is not a bridge. We start with an observation about cycle bases:

Lemma 5.8 *Let I' be a simple free augmentation of I by $\overline{a}$ such that $\overline{a}$ is not a bridge of G'. Then there is an integral cycle basis B of G and an oriented cycle $\overline{\gamma}$ such that $B' := B \cup \{\overline{\gamma}\}$ is an integral cycle basis of G' and $\overline{a} \in \overline{\gamma}$.*

Proof Since $\overline{a}$ is not a bridge, G and G' have the same set of nodes, so that any spanning tree of G is a spanning tree of G'. Hence, if B is any fundamental cycle basis of G, we can augment B by the fundamental cycle $\overline{\gamma}$ induced by $\overline{a}$ in G'. □

Choose cycle bases B, B', and an oriented cycle $\overline{\gamma}$ as in Lemma 5.8. We assume that $\mathcal{P}$ is defined using the cycle matrix Γ of B, and that $\mathcal{P}'$ is defined using the cycle matrix Γ' of B', so that Γ' arises from Γ by appending the row $\overline{\gamma}^\top$.

Lemma 5.9 *Let I' be a simple free augmentation of I by $\overline{a}$ such that $\overline{a}$ is not a bridge of G'. The natural projection $\varphi : \mathcal{P}' \to \mathcal{P}$, $(x, x_{\overline{a}}, z, z_{\overline{\gamma}}) \mapsto (x, z)$ is mixed-integer-compatible. In particular, $\varphi(\mathcal{P}'_{\text{split}}) \subseteq \mathcal{P}_{\text{split}}$.*

Proof The map φ is linear and maps mixed-integer points to mixed-integer points. That φ descends to split closures follows from Theorem 5.3. □

In view of Lemma 5.9, the split closure of the simple free augmentation is hence never worse, but could provide a potentially tighter relaxation by additional "projected split inequalities". We show now that this is not the case.

Lemma 5.10 *Let $\varphi : \mathcal{P}' \to \mathcal{P}$ denote the natural projection as in Lemma 5.9. Then $\varphi(\mathcal{P}'_{\text{split}}) = \mathcal{P}_{\text{split}}$.*

Proof Since I' is an augmentation of I by $\overline{a}$, we note at first, using the interpretation of split inequalities as flip inequalities from Theorem 3.1, that the set of defining inequalities of $\mathcal{P}'_{\text{split}}$ can be partitioned into the set of defining inequalities of $\mathcal{P}_{\text{split}}$, which cannot contain the variable $x_{\overline{a}}$, and a remaining set of inequalities, which do all contain $x_{\overline{a}}$. The image of $\varphi(\mathcal{P}'_{\text{split}})$ can be described by Fourier–Motzkin elimination of the variable $x_{\overline{a}}$. It is therefore sufficient to show that all inequalities generated by the Fourier–Motzkin procedure are redundant for $\mathcal{P}_{\text{split}}$. Since the redundancy is clear for those inequalities that do not contain $x_{\overline{a}}$, we will hence consider only the remaining inequalities where $x_{\overline{a}}$ has a non-zero coefficient.

Among the defining inequalities of $\mathcal{P}'_{\text{split}}$, $x_{\overline{a}}$ occurs precisely in the bound inequalities $x_{\overline{a}} \geq \ell'_{\overline{a}}$ and $x_{\overline{a}} \leq u'_{\overline{a}}$, and in the flip inequalities of simple cycles containing $\overline{a}$. Fourier–Motzkin considers pairs of these inequalities, one of them giving a lower bound, and the other an upper bound on $x_{\overline{a}}$. That is, the following types of pairs have to be considered:

(a) $x_{\overline{a}} \geq \ell'_{\overline{a}}$ and $x_{\overline{a}} \leq u'_{\overline{a}}$,
(b) $x_{\overline{a}} \geq \ell'_{\overline{a}}$ and a flip inequality for (γ, F) with $\overline{a} \in \gamma$ and $\overline{a} \in F$,
(c) $x_{\overline{a}} \leq u'_{\overline{a}}$ and a flip inequality for (γ, F) with $\overline{a} \in \gamma$ and $\overline{a} \notin F$,
(d) two flip inequalities for (γ, F_γ) and (δ, F_δ) with $\overline{a} \in \gamma, \overline{a} \notin F_\gamma, \overline{a} \in \delta, \overline{a} \in F_\delta$.

In all those flip inequalities, we can assume that the cycles are simple and that the parameter α is at least 1. Moreover, using the symmetry in Remark 2.13, we can without loss of generality fix the direction of $\overline{a}$ as forward or backward, replacing γ by $-\gamma$ if necessary. Let us proceed with Fourier–Motzkin:

(a) Elimination yields $\ell'_{\overline{a}} \leq u'_{\overline{a}}$, which is trivially true.
(b) Assume that $\overline{a}$ is forward in γ. Then we can write the flip inequality (6) for (γ, F) with $\alpha := \alpha_{\gamma,F} \geq 1$ as

$$\alpha(u'_{\overline{a}} - x_{\overline{a}}) + f(x) \geq \alpha(T - \alpha),$$

where $f(x) \geq 0$ for all $(x, z) \in \mathcal{P}$. Fourier–Motzkin elimination with $x_{\overline{a}} \geq \ell'_{\overline{a}}$ yields

$$\alpha u'_{\overline{a}} + f(x) \geq \alpha(T - \alpha) + \alpha\ell'_{\overline{a}},$$

or equivalently, recalling that $u'_{\overline{a}} - \ell'_{\overline{a}} = T - 1$,

$$f(x) \geq \alpha(T - \alpha - u'_{\overline{a}} + \ell'_{\overline{a}}) = \alpha(1 - \alpha),$$

but this is redundant for $\mathcal{P}_{\text{split}}$, since $(x, z) \in \mathcal{P}$ and $\alpha \geq 1$ imply $f(x) \geq 0 \geq \alpha(1-\alpha)$.

(c) is analogous to (b).

(d) This is the most tedious part. We assume without loss of generality that $\overline{a}$ is backward in γ and forward in δ. We will show that the Fourier–Motzkin inequality is valid for all points $(x, z) \in \mathcal{P}$ with $(\gamma+\delta)^\top x \in T\mathbb{Z}$. Since $\gamma+\delta$ is an element of the cycle space $\mathcal{C}$ of G, the Fourier–Motzkin inequality is hence valid for the convex hull of those points and in particular for $\mathcal{P}_{\text{split}}$ by virtue of (7).
We first write down the flip inequalities, omitting F_γ and F_δ in the subscripts of α:

$$\alpha_\gamma(x_{\overline{a}} - \ell'_{\overline{a}}) + f(x) \geq \alpha_\gamma(T - \alpha_\gamma),$$
$$\alpha_\delta(u'_{\overline{a}} - x_{\overline{a}}) + g(x) \geq \alpha_\delta(T - \alpha_\delta),$$

where $f(x), g(x) \geq 0$ for all $(x, z) \in \mathcal{P}$. Elimination produces

$$\begin{aligned}\alpha_\delta f(x) + \alpha_\gamma g(x) &\geq \alpha_\gamma\alpha_\delta(2T - \alpha_\gamma - \alpha_\delta - u'_{\overline{a}} + \ell'_{\overline{a}})\\ &= \alpha_\gamma\alpha_\delta(T + 1 - \alpha_\gamma - \alpha_\delta).\end{aligned} \tag{19}$$

The inequality (19) is trivially redundant if $\alpha_\gamma + \alpha_\delta \geq T + 1$. We hence assume from now on $\alpha_\gamma + \alpha_\delta \leq T$. Let $(x, z) \in \mathcal{P}$ with $(\gamma+\delta)^\top x \in T\mathbb{Z}$. Then

$$\sum_{a \in A \setminus F_\gamma} \gamma_a x_a + \sum_{a \in F_\gamma} \gamma_a x_a + \sum_{a \in A \setminus F_\delta} \delta_a x_a + \sum_{a \in F_\delta} \delta_a x_a \equiv 0 \mod T.$$

Since $\gamma_{\overline{a}} + \delta_{\overline{a}} = 0$, $\overline{a} \notin F_\gamma$, $\overline{a} \in F_\delta$, this implies

$$\sum_{a \in A \setminus (F_\gamma \cup \{\overline{a}\})} \gamma_a x_a + \sum_{a \in F_\gamma} \gamma_a x_a + \sum_{a \in A \setminus F_\delta} \delta_a x_a + \sum_{a \in F_\delta \setminus \{\overline{a}\}} \delta_a x_a \equiv 0 \mod T,$$

so that, using the definition of $\alpha_\gamma, \alpha_\delta$ (cf. Theorem 2.12),

$$\begin{aligned}&\sum_{a \in A \setminus (F_\gamma \cup \{\overline{a}\})} \gamma_a(x_a - \ell_a) - \sum_{a \in F_\gamma} \gamma_a(u_a - x_a)\\ &\quad + \sum_{a \in A \setminus F_\delta} \delta_a(x_a - \ell_a) - \sum_{a \in F_\delta \setminus \{\overline{a}\}} \delta_a(u_a - x_a)\\ &\equiv -\sum_{a \in A \setminus (F_\gamma \cup \{\overline{a}\})} \gamma_a \ell_a - \sum_{a \in F_\gamma} \gamma_a u_a - \sum_{a \in A \setminus F_\delta} \delta_a \ell_a - \sum_{a \in F_\delta \setminus \{\overline{a}\}} \delta_a u_a \mod T\\ &\equiv \alpha_\gamma + \alpha_\delta + u'_{\overline{a}} - \ell'_{\overline{a}} \mod T\\ &\equiv \alpha_\gamma + \alpha_\delta - 1 \mod T\end{aligned}$$

As we can assume $\alpha_\gamma, \alpha_\delta \geq 1$, we have that $\alpha := \alpha_\gamma + \alpha_\delta - 1 \geq 0$. This implies that

$$\begin{aligned} D := &\sum_{a \in A \setminus (F_\gamma \cup \{\overline{a}\})} \gamma_a (x_a - \ell_a) - \sum_{a \in F_\gamma} \gamma_a (u_a - x_a) \\ &+ \sum_{a \in A \setminus F_\delta} \delta_a (x_a - \ell_a) - \sum_{a \in F_\delta \setminus \{\overline{a}\}} \delta_a (u_a - x_a) \end{aligned}$$

is either $\leq \alpha - T$ (i) or $\geq \alpha$ (ii). Before showing that (19) is redundant in both cases, we write down the left-hand side of (19) explicitly:

$$\begin{aligned} &\alpha_\delta f(x) + \alpha_\gamma g(x) \\ &= \alpha_\delta (T - \alpha_\gamma) \sum_{a \in A \setminus (F_\gamma \cup \{\overline{a}\}), \gamma_a = 1} (x_a - \ell_a) + \alpha_\delta \alpha_\gamma \sum_{a \in A \setminus (F_\gamma \cup \{\overline{a}\}), \gamma_a = -1} (x_a - \ell_a) \\ &\quad + \alpha_\delta \alpha_\gamma \sum_{a \in F_\gamma, \gamma_a = 1} (u_a - x_a) + \alpha_\delta (T - \alpha_\gamma) \sum_{a \in F_\gamma, \gamma_a = -1} (u_a - x_a) \\ &\quad + \alpha_\gamma (T - \alpha_\delta) \sum_{a \in A \setminus F_\delta, \delta_a = 1} (x_a - \ell_a) + \alpha_\gamma \alpha_\delta \sum_{a \in A \setminus F_\delta, \delta_a = -1} (x_a - \ell_a) \\ &\quad + \alpha_\gamma \alpha_\delta \sum_{a \in F_\delta \setminus \{\overline{a}\}, \delta_a = 1} (u_a - x_a) + \alpha_\gamma (T - \alpha_\delta) \sum_{a \in F_\delta \setminus \{\overline{a}\}, \delta_a = -1} (u_a - x_a). \end{aligned} \tag{20}$$

(i) Expanding $\alpha_\delta(T - \alpha_\gamma)$ and $\alpha_\gamma(T - \alpha_\delta)$ in (20), and then bounding all summands with T as a factor by 0 from below, we obtain

$$\alpha_\delta f(x) + \alpha_\gamma g(x) \geq -\alpha_\gamma \alpha_\delta D \geq -\alpha_\gamma \alpha_\delta (\alpha - T) = \alpha\gamma \alpha_\delta (T + 1 - \alpha_\gamma - \alpha_\delta).$$

Hence, (19) holds in the case that $D \leq \alpha - T$.

(ii) Let $\nu := \min(\alpha_\gamma, \alpha_\delta)$. Expanding $\alpha_\delta(T - \alpha_\gamma)$ and $\alpha_\gamma(T - \alpha_\delta)$ in (20), bounding $T\alpha_\gamma, T\alpha_\delta$ from below by $T\nu$, we find

$$\alpha_\delta f(x) + \alpha_\gamma g(x) \geq (T\nu - \alpha_\gamma \alpha_\delta) D.$$

Since ν is one of $\alpha_\gamma, \alpha_\delta$ and $\alpha_\gamma, \alpha_\delta \leq T - 1$, we have $T\nu - \alpha_\gamma \alpha_\delta \geq 0$. This implies with $D \geq \alpha$ that

$$\begin{aligned} &\alpha_\delta f(x) + \alpha_\gamma g(x) \\ &\geq (T\nu - \alpha_\gamma \alpha_\delta)\alpha \\ &= (T\nu - \alpha_\gamma \alpha_\delta)(\alpha_\gamma + \alpha_\delta - 1) \\ &= T\nu(\alpha_\gamma + \alpha_\delta - 1) + \alpha_\gamma \alpha_\delta (1 - \alpha_\gamma - \alpha_\gamma) \\ &= T\nu(\alpha_\gamma + \alpha_\delta - 1) - T\alpha_\gamma \alpha_\delta + \alpha_\gamma \alpha_\delta (T + 1 - \alpha_\gamma - \alpha_\gamma). \end{aligned}$$

It remains to show that $T\nu(\alpha_\gamma + \alpha_\delta - 1) - T\alpha_\gamma\alpha_\delta \geq 0$. This is true since ν is one of $\alpha_\gamma, \alpha_\delta \geq 1$.

We conclude that the image $\varphi(\mathcal{P}'_{\text{split}})$ is fully described by the flip inequalities of cycles not containing $\overline{a}$ and the variable bounds for all arcs except $\overline{a}$. Hence, $\varphi(\mathcal{P}'_{\text{split}}) = \mathcal{P}_{\text{split}}$. □

As a corollary to Lemma 5.10, we obtain the following result.

Theorem 5.11 *Let I and I' be PESP instances with fractional periodic timetabling polyhedra $\mathcal{P}$ and $\mathcal{P}'$, respectively. Suppose that I' arises from I by a sequence of simple free augmentations. If $\varphi : \mathcal{P}' \to \mathcal{P}$ denotes the natural projection, then $\varphi(\mathcal{P}'_{\text{split}}) = \mathcal{P}_{\text{split}}$.*

In particular, recalling Remark 2.6, the incidence-based formulation (2) is not stronger than the cycle-based formulation (3) in terms of split closures. Consequently, it is of no use to develop methods which augment an instance by a free arc, obtain a flip/split inequality and project down again, as this will not lead to information which cannot already be obtained from the split closure of the original instance.

5.4 Binarization by subdivision

A reformulation of a MIP general variables into one with binary variables can exhibit stronger split closures [14] or lift-and-project closures [2]. For the application of periodic timetabling, there is a combinatorial binarization method: Let $I = (G, T, \ell, u, w)$ be a PESP instance, $G = (V, A)$. We assume that $0 \leq \ell < T$ and $\ell \leq u < \ell + T$ by preprocessing (see Remark 2.3), so that the integer periodic offset variables p_a in the incidence-based formulation (2) of PESP can only take values in $\{0, 1, 2\}$. Moreover, if $u_a \leq T$ for some $a \in A$, then $p_a \in \{0, 1\}$ for any integer feasible solution (x, π, p) [21].

Definition 5.12 Let $I' = (G', T, \ell', u', w')$ be a PESP instance that arises from I by subdividing an arc $\overline{a} \in A$ with $\ell_{\overline{a}} < u_{\overline{a}}$ into two new arcs a_1, a_2 such that:

$$
\begin{aligned}
&0 \leq \ell'_{a_1} \leq u'_{a_1},\\
&0 \leq \ell'_{a_2} \leq u'_{a_2},\\
&\ell'_{a_1} + \ell'_{a_2} = \ell_{\overline{a}},\\
&u'_{a_1} + u'_{a_2} = u_{\overline{a}},\\
&w'_{a_1} = w'_{a_2} = w_{\overline{a}}.
\end{aligned}
$$

We call I' a *simple subdivision* of I at $\overline{a}$.

Observe that if the bounds on the arc $\overline{a}$ are such that $u_{\overline{a}} > T$, one can always construct a simple subdivision I' of I at $\overline{a}$ such that $u'_{a_i} - \ell'_{a_i} > 0$ and $u'_{a_i} \leq T$ for $i \in \{1, 2\}$, due to the assumption that $\ell < T$ and $u < \ell + T$. As a result, for the instance I' arising from subdividing each arc $\overline{a}$ with $u_{\overline{a}} > T$ as above, the incidence-based MIP formulation (2) will then have exclusively binary variables.

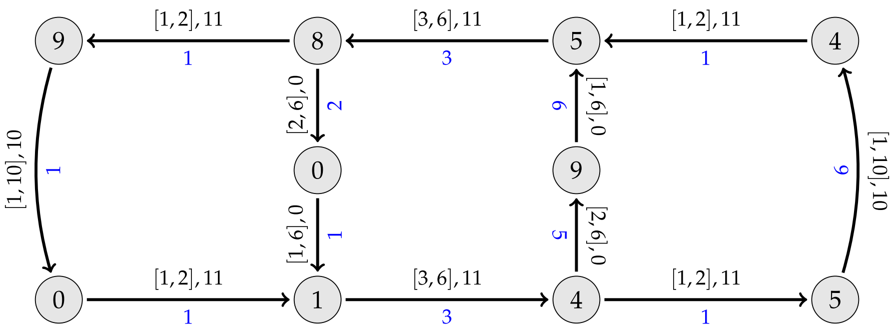

Fig. 5 Subdivision of the instance in Fig. 1 obtained from two simple subdivisions such that $u_a \leq T$ for all arcs a

Example 5.13 Fig. 5 shows the instance obtained from the instance I from Fig. 1 by subdividing every arc with $u_a > T$.

Let I' be a simple subdivision of a PESP instance I at $\overline{a}$, introducing new arcs a_1 and a_2. The cycle spaces of G and G' are isomorphic: If γ is an element of the cycle space of G, then γ' with

$$\gamma'_a := \begin{cases} \gamma_{\overline{a}} & \text{if} \quad a \in \{a_1, a_2\}, \\ \gamma_a & \text{if} \quad a \notin \{a_1, a_2\} \end{cases}$$

defines an element of the cycle space of G', and the whole cycle space of G' arises this way. We can therefore associate to an integral cycle basis $B = \{\gamma_1, \dots, \gamma_\mu\}$ of G the integral cycle basis $B' = \{\gamma'_1, \dots, \gamma'_\mu\}$. Then any cycle offset z in (3) w.r.t. B defines a cycle offset z' w.r.t. B' by $z'_{\gamma'} := z_\gamma$, so that cycle offsets are essentially the same. We will use Γ and Γ' to define $\mathcal{P}$ and $\mathcal{P}'$, the fractional periodic timetabling polytopes of I' and I, respectively.

Lemma 5.14 *Consider a simple subdivision I' of I at an arc $\overline{a}$ with notation as above.*

(a) *The map $\rho : \mathcal{P}' \to \mathcal{P}, (x, x_{a_1}, x_{a_2}, z) \mapsto (x, x_{a_1} + x_{a_2}, z)$ is well-defined and mixed-integer-compatible.*
(b) *The map $s : \mathcal{P} \to \mathcal{P}', (x, x_{\overline{a}}, z) \mapsto \left(x, \ell'_{a_1} + \frac{u'_{a_1} - \ell'_{a_1}}{u_{\overline{a}} - \ell_{\overline{a}}}(x_{\overline{a}} - \ell_{\overline{a}}), \ell'_{a_2} + \frac{u'_{a_2} - \ell'_{a_2}}{u_{\overline{a}} - \ell_{\overline{a}}}(x_{\overline{a}} - \ell_{\overline{a}}), z\right)$ is well-defined and mixed-integer-compatible.*
(c) *$\rho \circ s : \mathcal{P} \to \mathcal{P}$ is the identity map.*
(d) *$\rho(\mathcal{P}'_{split}) = \mathcal{P}_{\text{split}}$.*

Proof (a) The map is well-defined: The hypothesis $\ell'_{a_1} + \ell'_{a_2} = \ell_{\overline{a}}$ and $u'_{a_1} + u'_{a_2} = u_{\overline{a}}$ implies that $\ell_{\overline{a}} \leq x_{a_1} + x_{a_2} \leq u_{\overline{a}}$ holds for all $(x, x_{a_1}, x_{a_2}, z) \in \mathcal{P}'$. As ρ is linear and does not affect the integrality of z, it is mixed-integer-compatible.

(b) The map is well defined: Due to the assumption of subdividing arcs with $u_{\overline{a}} > T$ only, we have $u_{\overline{a}} - \ell_{\overline{a}} > 0$ and $u'_{a_1} - \ell'_{a_1} \geq 0$. Since $x_{\overline{a}} - \ell_{\overline{a}} \geq 0$ for all

$(x, x_{\overline{a}}, z) \in \mathcal{P}$, we conclude

$$\ell'_{a_1} = \ell'_{a_1} + \frac{u'_{a_1} - \ell'_{a_1}}{u_{\overline{a}} - \ell_{\overline{a}}}(\ell_{\overline{a}} - \ell_{\overline{a}})$$
$$\leq \ell'_{a_1} + \frac{u'_{a_1} - \ell'_{a_1}}{u_{\overline{a}} - \ell_{\overline{a}}}(x_{\overline{a}} - \ell_{\overline{a}}) \leq \ell'_{a_1} + \frac{u'_{a_1} - \ell'_{a_1}}{u_{\overline{a}} - \ell_{\overline{a}}}(u_{\overline{a}} - \ell_{\overline{a}}) = u'_{a_1}.$$

The argument for the x_{a_2} entry is analogous. We note that s is affine and maps point with integral z to points with integral z, so that s is mixed-integer-compatible.

(c) This follows since

$$\ell'_{a_1} + \frac{u'_{a_1} - \ell'_{a_1}}{u_{\overline{a}} - \ell_{\overline{a}}}(x_{\overline{a}} - \ell_{\overline{a}}) + \ell'_{a_2} + \frac{u'_{a_2} - \ell'_{a_2}}{u_{\overline{a}} - \ell_{\overline{a}}}(x_{\overline{a}} - \ell_{\overline{a}})$$
$$= \ell_{\overline{a}} + \frac{u_{\overline{a}} - \ell_{\overline{a}}}{u_{\overline{a}} - \ell_{\overline{a}}}(x_{\overline{a}} - \ell_{\overline{a}}) = x_{\overline{a}}.$$

(d) Since ρ and s are mixed-integer compatible, $\rho(\mathcal{P}'_{\text{split}}) \subseteq \mathcal{P}_{\text{split}}$ and $s(\mathcal{P}_{\text{split}}) \subseteq \mathcal{P}'_{\text{split}}$. The composition $\rho|_{\mathcal{P}'_{\text{split}}} \circ s|_{\mathcal{P}_{\text{split}}}$ of the restrictions to split closures is hence well-defined, and by (3), it is the identity map on $\mathcal{P}_{\text{split}}$. We conclude that $\rho|_{\mathcal{P}'_{\text{split}}}$ is surjective. $\square$

A repeated application of Lemma 5.14 together with Theorem 5.11 yields:

Theorem 5.15 *Let I and I' be PESP instances with fractional periodic timetabling polyhedra $\mathcal{P}$ and $\mathcal{P}'$, respectively. Suppose that I' arises from I by a sequence of simple subdivisions and simple free augmentations. If $\psi : \mathcal{P}' \to \mathcal{P}$ denotes the composition of the summation maps ρ in Lemma 5.14 (1) for the subdivisions and the projection maps φ in Lemma 5.9 for the free augmentations, then $\psi(\mathcal{P}'_{\text{split}}) = \mathcal{P}_{\text{split}}$.*

In particular, when we binarize the MIP (3) by first performing simple subdivisions and then move to the formulation (2), we gain no further insight about split inequalities.

6 Computational experiments

We want to assess how useful the split closure is for obtaining dual bounds for PESP in practice. To that end we introduce a procedure, which exploits Theorem 4.7 in a heuristic way, and proceeds to find cuts systematically once the heuristic fails, such that we optimize over the entire split closure by means of Theorem 4.5. We will also examine the performance of the heuristic in comparison to the systematic exploration.

6.1 Separation procedure

Our goal is to optimize over the entire split closure. We do so with our custom separator which proceeds as illustrated by the flowchart in Fig. 6.

At first, it tries to heuristically generate violated flip inequalities (highlighted in blue in the chart): We compute a minimum spanning tree with respect to the periodic slack $x - \ell$ of the current LP solution $(x, z) \in \mathcal{P}$, and determine a most violated flip inequality for the fundamental cycles of that tree by Theorem 4.7. When no more heuristic cuts are found, the parametric IP (15) as in Theorem 4.5 is solved. During the solution process of the IP, a callback retrieves intermediate incumbent solutions and generates the corresponding cuts. The procedure terminates when no more violated cuts can be found, or the time limit is hit.

Since the amount of cuts found by the heuristic is rather larger in the beginning, we apply the filtering mechanisms of SCIP to detect effective cuts. However, cuts found by the parametric IP will always be enforced, so that the whole procedure is correct up to numerical tolerances: If the procedure terminates because no more violated cuts can be detected, then the optimal solution over the split closure has been found.

6.2 Methodology

To conduct our computational experiments, we use the benchmark library PESPlib [15], whose instances are derived from real-world scenarios. Although significant process has been made in the past, no instance could be solved to proven optimality up to date.

Since the PESPlib instances are computationally very hard, we consider not only the full instances, but also two subinstances per instance whose cyclomatic number μ has been restricted to 25, and 100, respectively. Note that μ is the number of integer variables in (3). The restriction procedure for an instance $I = (G, T, \ell, u, w)$ works by iteratively removing arcs, deleting in each step one arc a with highest span $u_a - \ell_a$ and breaking ties by preferring lowest weight w_a (cf. Goerigk and Liebchen [16]). In contrast to the full PESPlib instances, these restricted variants can be solved to optimality within a reasonable amount of time.

We first preprocess each instance, so that in particular the assumptions as in Remark 2.3 hold. For each instance $I = (G, T, \ell, u, w)$, we consider the cycle-based formulation (3) using an integral cycle basis B that minimizes $\sum_{\gamma \in B} \sum_{a \in \gamma} (u_a - \ell_a)$. This choice of cycle basis is motivated by its good performance for computing dual bounds [8, 31]. By Theorem 5.11, the cycle-based formulation is not weaker than the incidence-based formulation, and by Remark 2.6, it is more compact. We then invoke the branch-cut-and-price framework SCIP [1]. The advantage of using SCIP is that it is highly customizable and we can disable everything that does not come from split cuts: We disable the built-in presolving, branching, heuristics, propagators separators and merely call a custom separation callback during the cutting loop at the root node.

In all experiments, we use SCIP 8.0.3 [5] with Gurobi 9.5.2 [18] as LP solver. We also use Gurobi to solve the parametric IP from Theorem 4.5. Gurobi is allowed to use 6 threads on an Intel Xeon E3-1270 v6 CPU running at 3.8 GHz with 32 GB RAM. The time limit has been set to 4 h wall time for each instance.

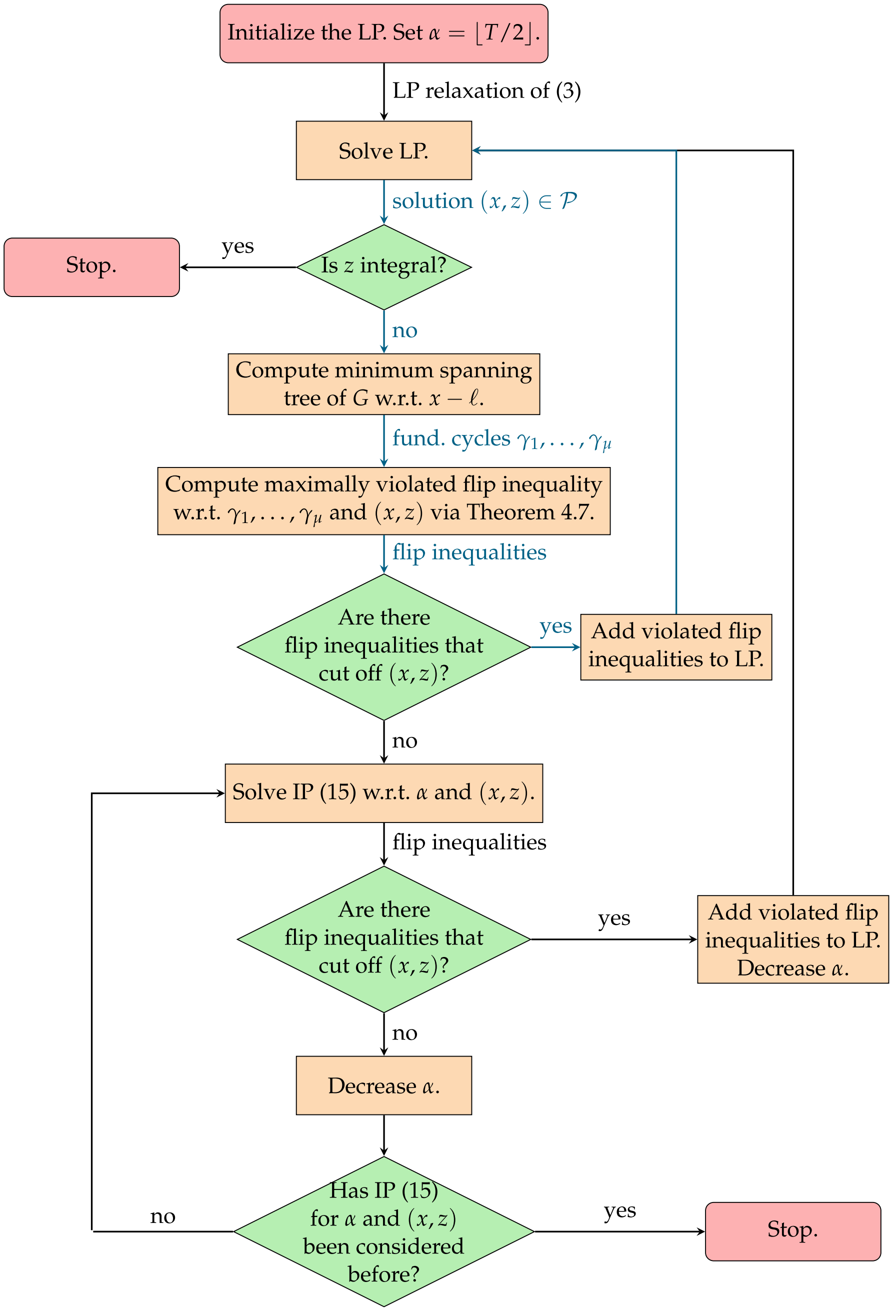

Fig. 6 Flowchart of the split cut generation procedure. "Decrease α" means to set $\alpha := \alpha - 1$ if $\alpha \geq 2$ and $\alpha := \lfloor T/2 \rfloor$ otherwise. We start with $\alpha = \lfloor T/2 \rfloor$, as this is likely to produce cuts with large violation

Table 2 Results for the PESPlib instances restricted to $\mu = 25$

Instance	μ	Opt. Val. ($\mathcal{P}_{\mathrm{I}}$)	Dual Bd. ($\mathcal{P}_{\mathrm{split}}$)	Gap (%)	Cuts	IP Cuts	Time [s]
BL1	25	479,501	455,492	5.01	114	33	98
BL2	25	582,203	529,247	9.10	128	32	116
BL3	25	614,544	513,344	16.47	122	28	197
BL4	25	581,688	507,168	12.81	176	65	106
R1L1	25	1,469,763	1,314,105	10.59	284	123	747
R1L2	25	1,271,066	1,235,774	2.78	226	96	857
R1L3	25	1,704,349	1,693,441	0.64	238	114	1281
R1L4	25	1,543,182	1,429,795	7.35	294	118	936
R2L1	25	2,598,725	2,171,855	16.43	212	83	255
R2L2	25	2,726,109	2,471,181	9.35	238	75	335
R2L3	25	1,698,794	1,661,074	2.22	116	12	91
R2L4	25	2,417,447	2,325,110	3.82	244	64	119
R3L1	25	1,110,721	1,055,499	4.97	170	83	513
R3L2	25	1,283,884	1,148,551	10.54	152	67	201
R3L3	25	1,617,501	1,478,034	8.62	196	58	389
R3L4	25	1,063,438	987,067	7.18	143	56	399
R4L1	25	1,053,623	1,053,623	0.00	102	14	205
R4L2	25	1,394,526	1,313,700	5.80	136	39	231
R4L3	25	1,718,591	1,648,388	4.08	148	47	213
R4L4	25	498,913	488,043	2.18	171	60	701
R1L1v	25	1,741,592	1,645,779	5.50	128	58	14,400
R4L4v	25	3,660,000	3,660,000	0.00	0	0	0

The table lists the optimal objective value of the MIP (3) in terms of weighted slack $w^\top(x - \ell)$, the best dual bound obtained by split cuts, the primal–dual gap, the total number of applied split cuts, the number of cuts provided by the parametric IP (15), and the running time in seconds

6.3 Results

6.3.1 Restriction to $\mu = 25$

Table 2 shows the results for the restrictions of the PESPlib instances to the cyclomatic number $\mu = 25$. For all but one instance, the cut generation procedure of Sect. 6.2 terminates within 22 min, only R1L1v hits the time limit due to a hard parametric IP (15). Optimizing over the split closure is exact for R4L1 and R4L4v, but R4L4v is trivial in the sense that $x = \ell$ is an optimal solution. The average relative optimality gap with respect to the optimal objective value in terms of weighted slack $w^\top(x - \ell)$ and the best bound obtained by split cuts, taken over all 22 instances, is 6.61%.

Table 3 Results for the PESPlib instances restricted to $\mu = 100$

Instance	μ	Opt. Val. ($\mathcal{P}_I$)	Dual Bd. ($\mathcal{P}_{split}$)	Gap (%)	Cuts	IP Cuts	Time [s]
BL1	100	1,341,151	1,216,355	9.31	1092	357	954
BL2	100	1,733,429	1,451,049	16.29	910	307	1287
BL3	100	1,747,063	1,461,798	16.33	922	304	1864
BL4	100	1,605,968	1,427,228	11.13	975	349	1399
R1L1	100	5,481,154	4,582,018	16.40	1300	493	6903
R1L2	100	4,873,559	3,952,695	18.90	1138	348	7453
R1L3	100	6,256,521	5,151,095	17.67	998	324	4742
R1L4	100	5,008,640	4,202,959	16.09	1407	415	6184
R2L1	100	8,284,107	6,881,776	16.93	1021	294	2453
R2L2	100	7,099,578	6,244,993	12.04	1366	406	4648
R2L3	100	6,722,776	5,982,798	11.01	1102	342	6038
R2L4	100	5,516,243	4,996,368	9.42	1217	317	3242
R3L1	100	4,366,123	3,770,709	13.64	927	355	7180
R3L2	100	4,666,798	3,796,483	18.65	764	253	8554
R3L3	100	4,719,345	3,890,774	17.56	921	301	7492
R3L4	100	2,950,612	2,730,898	7.45	885	348	11,242
R4L1	100	4,428,800	3,715,032	16.12	717	179	2947
R4L2	100	4,101,438	3,492,759	14.84	789	236	14,400
R4L3	100	4,302,565	3,740,673	13.06	875	226	8785
R4L4	100	1,994,572	1,676,547	15.94	607	184	7448
R1L1v	100	10,253,906	9,715,723	5.25	191	0	14,400
R4L4v	100	14,880,000	14,880,000	0.00	2	0	1

The table lists the optimal objective value of the MIP (3) in terms of weighted slack $w^\top(x - \ell)$, the best dual bound obtained by split cuts, the primal–dual gap, the total number of applied split cuts, the number of cuts provided by the parametric IP (15), and the running time in seconds

6.3.2 Restriction to $\mu = 100$

The results for the restriction to $\mu = 100$ are summarized in Table 3. Again, we can determine the optimal solution of (3) for all these restricted instances. The cut generation procedure of Sect. 6.2 terminates within the time limit for 20 out of 22 instances. R4L4v is again almost trivial to solve, because two cuts suffice to produce an integral solution. The second smallest gap is at R1L1v, although the time limit is hit. The average optimality gap is 13.37%, which is about twice as much as in the case $\mu = 25$.

6.3.3 Full instances

Finally, the results for the full PESPlib instances are given in Table 4 in comparison to the best known primal bounds and in Table 5 in comparison to the best known dual bounds. All instances hit the time limit. Compared to the restricted instances, relatively few cuts are generated by the IP (15), which is both due to the large supply of heuristically generated cuts, and the difficulty of the IP. The time limit is not sufficient to unfold the power of the IP, on the other hand, increasing the time limit to 8 or 24 h empirically produced only marginal improvements. This effect is also illustrated in Fig. 7: The plot shows an exemplary progression of the dual bound and the number of applied cuts for the instance R2L1 with a logarithmic time axis. The heuristic separation procedure finds no more cuts for the first time after roughly 45 min (about $10^{3.43}$ s), and then the parametric IP takes over, causing a sudden and persisting drop in performance. We can observe that once the parametric IP came into effect, the heuristic stage provides only few further cuts. This could be due to the initial high quality results provided by the heuristic, such that the improvement through a cut from the parametric IP results in only a marginal change in the new solution. The subsequent spanning tree in the following heuristic stage could then be similar to the previous one, such that from this point on, only little to no improvement is found in the heuristic stage; and the costly parametric IP is the main contributor.

With respect to all instances, the average optimality gap is 40.83%. As expected, the quality of the results obtained by our method is dependent on the problem size. In particular for the 16 RiLj instances there is a strong correlation between the size of μ and the optimality gap. This is also evidenced by the Pearson correlation coefficient, which is approximately 95%.

On the dual side, the split closure provides at least 91.10% of the currently best known dual bound. This underlines the good performance of our method – most of the incumbent dual bounds have been obtained by longer computation times, and in contrast to our study, neither branching nor other types of cutting planes apart from split cuts have been forbidden. Despite being at a disadvantage in this regard, our method provides better dual bounds for five out of the 22 instances, with improvements up to 25%. Other bounds have been obtained with the help of heuristically separated flip inequalities as well, by, e.g., [8, 26, 29, 31], such that our procedure can be seen as an advancement of previous methods in the sense that our heuristic unlocks more potential due to exploiting Theorem 4.7.

6.4 Insights

From our experiments we have gained the following main insights:

- Our procedure is useful in computing qualitative dual bounds in practice:
  We were able to improve five instances of the benchmarking library PESPlib significantly. But also for the other instances, some of which have been treated excessively in the past, a high percentage of the bound could be reached in comparably little time by our procedure.
- The split closure is an essential contributor to qualitative dual bounds:
  The instances where the optimal solution could be obtained and our procedure terminates give us some indication of how well suited the split closure is for dual bounds in the context of PESP. Here, we were able to observe that the split closure provided fairly low optimality gaps on average, and even certified optimality in

Table 4 Results for the full PESPlib instances

Instance	μ	Primal Bd. ($\mathcal{P}_I$)	Dual Bd. ($\mathcal{P}_{split}$)	Gap (%)	Cuts	IP Cuts	Time [s]
BL1	5298	6,333,641	4,252,778	32.85	42,927	15	14,400
BL2	4880	6,799,331	4,299,517	36.77	37,498	84	14,400
BL3	6265	6,675,098	4,290,946	35.72	58,628	20	14,400
BL4	9684	6,562,147	3,923,974	40.20	88,640	265	14,400
R1L1	2722	29,894,745	19,041,890	36.30	21,965	60	14,400
R1L2	2876	30,507,180	19,059,669	37.52	23,767	45	14,400
R1L3	2848	29,319,593	18,193,974	37.95	23,468	61	14,400
R1L4	3769	26,516,727	16,441,121	38.00	30,460	18	14,400
R2L1	3206	42,422,038	24,806,675	41.52	27,739	163	14,400
R2L2	3360	40,642,186	24,464,467	39.81	28,842	159	14,400
R2L3	3239	38,558,371	22,645,939	41.27	28,816	95	14,400
R2L4	5514	32,483,894	19,102,410	41.19	47,958	0	14,400
R3L1	4630	43,271,824	25,343,534	41.43	38,725	17	14,400
R3L2	4800	45,220,083	25,963,773	42.58	41,951	19	14,400
R3L3	5446	40,483,617	22,273,090	44.98	46,099	6	14,400
R3L4	7478	33,335,852	17,027,192	48.92	46,773	0	14,400
R4L1	5331	49,426,919	27,938,824	43.47	42,505	6	14,400
R4L2	5688	48,764,793	27,585,028	43.43	45,946	7	14,400
R4L3	6871	45,493,081	23,849,465	47.58	46,277	0	14,400
R4L4	9371	36,703,391	16,488,684	55.08	42,579	0	14,400
R1L1v	2832	42,591,141	28,544,123	32.98	20,326	22	14,400
R4L4v	9637	61,968,380	38,307,814	38.18	45,916	0	14,400

The table lists the best known primal bound for the MIP (3) in terms of weighted slack $w^\top(x-\ell)$ according to [15], the best dual bound obtained by split cuts, the primal–dual gap, the total number of applied split cuts, the number of cuts provided by the parametric IP (15), and the running time in seconds

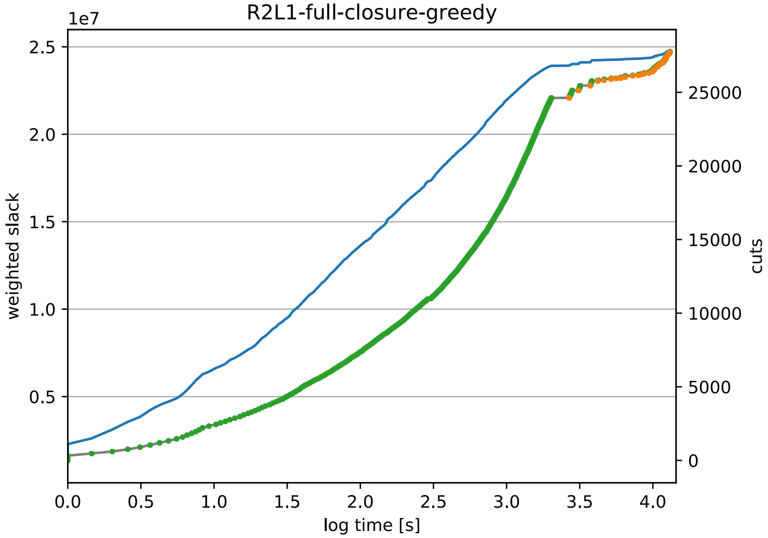

Fig. 7 Evolution of the dual bound in terms of weighted slack $w^\top(x - \ell)$ (blue, left axis) and the number of applied split cuts (grey, right axis) for the instance R2L1. Green markers correspond to cuts obtained from the heuristic, orange to cuts from the parametric IP. The time axis is logarithmic

three cases (see Tables 2 and 3). This means that the split closure is in many practical cases able to provide strong dual bounds.

- The primal–dual gap cannot be closed solely with the split closure:
  Consider, e.g., the worst case, namely for R1L2 with $\mu = 100$. There is a gap of 18.9% between the optimal dual bound of the split closure and the optimal solution. Thus, in such cases, to close the primal–dual gap entirely, further methods, e.g., higher rank split cuts will have to be applied. This is a practical manifestation of the fact that $\mathcal{P}_I \subsetneq \mathcal{P}_{\text{split}}$ in general [26].
- Our heuristic approach to generate split cuts is effective:
  The question arises whether it is worth it to explore the split closure to its full extent: After all, it will in general not be able to close the primal–dual gap, so that the fast, heuristic section of our procedure could already give a decent bound in practical applications. For an indication, we analyzed the instances where our procedure terminated before the time limit was reached: We found that the best bound before the parametric IP came into effect reached at least 89.1%, and on average even 95.6% of the final dual bound. We conclude that indeed the heuristic approach of separating flip inequalities is quite effective, as it is able to cover the majority of the dual bound that can be obtained from the split closure quickly. In our case, the addition of the parametric IP in the procedure was essential for the assessment of the split closure and might be helpful to find new cuts, so that the heuristic can produce effective cuts again. However for practical purposes, particularly when other methods aimed at improving the dual bounds are used in parallel, the time-consuming parametric IP might be too costly.

Table 5 Comparison of dual bounds for the full PESPlib instances

Instance	μ	Dual Bd. ($\mathcal{P}_I$)	Dual Bd. ($\mathcal{P}_{split}$)	Gap (%)	Dual Bd. Source
BL1	5298	3,668,148	4,252,778	– 15.94	[8]
BL2	4880	3,943,811	4,299,517	– 9.02	[8]
BL3	6265	3,571,976	4,290,946	– 20.13	[8]
BL4	9684	3,131,491	3,923,974	– 25.31	[8]
R1L1	2722	20,901,883	19,041,890	8.90	[29]
R1L2	2876	19,886,799	19,059,669	4.16	[31]
R1L3	2848	19,323,821	18,193,974	5.85	[31]
R1L4	3769	17,283,850	16,441,121	4.88	[31]
R2L1	3206	25,929,643	24,806,675	4.33	[31]
R2L2	3360	25,642,692	24,464,467	4.59	[31]
R2L3	3239	23,941,492	22,645,939	5.41	[31]
R2L4	5514	19,793,447	19,102,410	3.49	[31]
R3L1	4630	26,825,864	25,343,534	5.53	[31]
R3L2	4800	27,178,406	25,963,773	4.47	[31]
R3L3	5446	23,007,043	22,273,090	3.19	[31]
R3L4	7478	17,432,725	17,027,192	2.33	[31]
R4L1	5331	29,174,444	27,938,824	4.24	[31]
R4L2	5688	28,664,399	27,585,028	3.77	[31]
R4L3	6871	24,293,621	23,849,465	1.83	[31]
R4L4	9371	17,961,400	16,488,684	8.20	[26]
R1L1v	2832	29,620,775	28,544,123	3.63	[15]
R4L4v	9637	32,296,041	38,307,814	– 18.61	[15]

The table lists the best known dual bound for the MIP (3) in terms of weighted slack $w^\top(x - \ell)$ according to the source in the last column, the best dual bound obtained by split cuts, and the primal–dual gap

7 Conclusion

We have shown that in the context of periodic timetabling, the split closure can be expressed in combinatorial terms, namely via flip inequalities with respect to simple cycles. Consequently, this means that a dual bound obtained from flip inequalities is as good as from split cuts. However, flip inequalities are – in a way – easier to grasp: We show that for a fixed cycle, a separating flip inequality can be found in linear time. This can be used to obtain a heuristic, which turned out to be powerful in practice. In combination with a systematic exploration of violated flip inequalities, we were able to improve the dual bounds of five instances of the benchmark library PESPlib—proving both the effectiveness of our approach, but also of the benefit of the split closure in the context of PESP. One of our main contributions is also in the insight that the split closures of various equivalent PESP formulations are all equivalent as well, meaning that neither the specific MIP formulation, nor any amount of subdivision or augmentation will lead to a stronger split closure.

Our computational experiments also indicate that even with a full exploration of the flip polytope, a certain gap will remain. To close the primal–dual gap entirely, further research into stronger cuts is needed, which will have to be different from first-order split cuts.

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