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Посвящается моей маме, Елизавете Горбуновой

Dedicated to my Mom, Elizaveta Gorbunova

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Introduction

This dissertation consists of four independent chapters in the field of contract theory. I study the influence of private information of different parties (be it static or dynamic information) on the optimal contracts depending on the context. *Chapters 1 and 2* deal with a crowdfunding model, which usually illustrates a static direct exchange between a seller and potential buyers contributing to the production of a good. *Chapters 3 and 4* are on delegated expertise, where a firm (a principal) delegates information acquisition about a quality of a risky project to an expert (an agent) before investing.

In *the first chapter* (**Crowdfunding platforms**, E. Gorbunova) I study crowdfunding platforms, which are becoming more present in numerous domains (education, the energy sector, creative work). In order to explain contribution schemes found on such platforms, I enrich the crowdfunding model by introducing a platform / a broker between the seller and the potential buyers. I then derive the optimal direct incentive-compatible mechanism which maximizes the platform's expected profit and offer an implementation of the optimal mechanism.

In *the second chapter* (**Sequential screening in the presence of fixed costs**, E. Gorbunova, based on joint work with P. Pillath) we consider a direct interaction between the seller and the buyers but introduce dynamic formation of the buyers' valuations for the good and study its influence on the crowdfunding procedures. Doing so allows us, in particular, to address the case where the seller faces the decision on whether or not to incur production costs and what pricing scheme to offer long before the buyers have full information about their demand for the good. We first characterize the optimal selling mechanism for this case and later address the case where the seller can postpone production until the buyers have learned their true valuations.

In *the third chapter* (**Dynamic information collection: two-sided tests**, E. Gorbunova, based on joint work with D. Gromb and F. de Vericourt) we consider an information collection problem with symmetric two-sided tests, which are informative about the project quality, with no false positives and no false negatives; whereas in *the forth chapter* (**Search order in delegated data analytics**, E. Gorbunova, based on joint work with D. Gromb and F. de Vericourt) we consider the tests with no false positives but allow for false negatives, and we tailor the model to the domain of data analytics specifically. Distinctiveness of these papers compared to the literature on delegated testing / delegated search lies in the research question itself (the optimal order of the available information sources) and in the fact that we consider heterogeneous information sources (two tests / data sets differing in the precision of their findings and the cost of analysis). We tackle two building blocks of delegated expertise: optimal compensation - characterizing optimal incentive contracts for each order of the tests under a combined moral hazard and adverse selection problem; and task design - finding the optimal order.

Crowdfunding Platforms*

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Abstract

This paper derives a platform-optimal selling procedure for a non-rival but excludable good. The good is yet to be produced and selling it is only possible via a platform. The seller of the good is privately informed about the fixed production costs, multiple potential buyers are privately informed about their valuations for the good before contracting. The platform designs the contract which specifies when the good will be produced (production rule), how much to pay the seller, which buyers get access to the good (allocation rule) and how much the buyers pay the platform. I derive the optimal direct incentive-compatible mechanism which maximizes the platform's expected profit. I then propose an implementation which resembles contractual features found on reward-based crowdfunding platforms such as Patreon.

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Introduction

Crowdfunding platforms currently emerge in many domains. Such platforms are wide-spread in education, e.g. DonorsChoose and RallyUp Schools; and even in the energy sector, e.g. Citizenergy, GreenCrowding and Bettervest. Prominent examples for the platforms accommodating creative work are Kickstarter, Artist-Share and Patreon (for more examples see Belleflamme *et al.* (2015), who provide an extensive review on different types of crowdfunding platforms and their business models). Very often in the case of reward-based crowdfunding platforms a so-called club good is created. This is a non-rival but excludable good: e.g. once created, a particular online comic book from an author on Patreon does not depreciate or become more scarce if many people read it, but it is only possible to get access to the comic if a reader has contributed to its creation through a pledge towards the crowdfunding campaign.

This paper takes a closer look at the selling mechanism for such a good on a crowdfunding platform, who acts as a broker between the seller and potential buyers. Although selling mechanisms are more intricate in reality, with this work I try to address the following two features which are present e.g. on Patreon: the plaform sets a platform-wide minimum pledge amount for the buyers[1] and asks the sellers for a percentage fee from the collected pledges[2]. More generally, I derive the optimal direct mechanism which maximizes the platform's expected profit in a case of one seller, who is privately informed about the (fixed) production costs, and multiple potential buyers, who are privately informed about their valuations for the good. The good is yet to be produced, and selling the good is only possible via the platform. I also propose an indirect implementation of the optimal direct mechanism.

The case of one seller and multiple buyers interacting directly, as well as the production costs being public information has been discussed in detail by Cornelli (1996). She explains why price dispersion may occur when it comes to club goods: the buyers are willing to contribute different amounts towards the same good out of fear of the good not being produced. She characterizes the seller-optimal direct mechanism and provides examples for implementation, which incorporate a minimum price for the buyers. Naturally, I make use of her derivation techniques in this paper, adjusting them to the setting with the platform and the seller's private information about the production costs[3].

Another paper, upon which this work expands, is Myerson & Satterthwaite (1983), who consider a case of one seller, one buyer and a broker. They derive the broker-optimal direct mechanism and provide examples for implementation, to which I relate when expanding the model by introducing a club good with multiple potential buyers, as the payment to the seller will depend on the buyers' payments. Hence, this paper can also be related to the literature which explains the seller fee structure on platforms often observed in practice: e.g. Wang & Wright (2017), who explain the optimality of ad valorem fees on the platforms through the fact that the value of a transaction is proportional to the costs.

Other papers which are related to this work are Barbieri & Malueg (2010), where the authors derive profit-maximizing selling procedures for discrete public goods in subscription games with two buyers; Ellman & Hurkens (2019), who derive seller-optimal ex-ante selling mechanisms with binary buyers' types, within which the seller decides on a minimum price and a funding threshold; Strausz (2017), who also considers direct

[1] See https://support.patreon.com/hc/en-us/articles/360044376211-Managing-members-with-custom-pledges.

[2] See https://support.patreon.com/hc/en-us/articles/11111747095181-Creator-fees-overview.

[3] The case of one seller and one buyer interacting directly when the seller offers the contract and also has private information when contracting is discussed in Maskin & Tirole (1990).

interaction between the seller and the buyers with binary types incorporating the seller's private information on the costs and moral hazard; and Loertscher & Niedermayer (2023), who model the seller-buyers interaction happening via a platform, where a small number of buyers arrives in each period, the seller chooses the price (English auction with reserve price), and the platform, which can be competitive, chooses the seller fee. This paper also captures the indirect interaction between the seller and the buyers via a crowdfunding platform, but it preserves the set-up with fixed costs and a non-rival but excludable good as in Cornelli (1996), which matches e.g. digital products quite well. I also study the case where the platform fully designs the contract, since in reality the platforms tend to have a say in who gets access to the good once it is produced, e.g. through a platform-wide minimum pledge amount as on e.g. Patreon. Hence, this paper is distinct from the literature in one or more of the following modeling choices.

In this model I consider a non-rival but excludable good (a club good), which is yet to be produced. The seller, who is privately informed about the continuous (fixed) production costs[4] before contracting, faces multiple potential buyers, who are privately informed about their continuous valuations for the good before contracting, on the platform. The platform is the designer, who maximizes expected profit and offers the seller and the buyers a contract, specifying when the good will be produced (call this production rule), how much to pay the seller, which buyers get access to the good (call this allocation rule) and how much the buyers pay the platform.

According to the optimal incentive-compatible direct mechanism, the optimal allocation rule is standard: only buyers with positive virtual valuations should obtain the good in case it is produced. Whether or not the good will be produced depends on whether the sum of positive virtual valuations surpasses the virtual costs. When it comes to implementation (indirect mechanism), there are many ways in which one could construct the actual transfers depending on the desired structure: one can consider e.g. contribution resp. subscription schemes, defined in Admati & Perry (1991) as games, in which players' contributions are not refunded if the project is not completed resp. those in which they are refunded. The actual transfers should satisfy the conditions for the expected transfers, which make the optimal direct mechanism incentive-compatible and individually rational.

I concentrate on the contribution schemes, for which there is no reimbursement in case the good is not produced, and propose the following implementation (illustrated first by a two-buyer case with uniform distributions, although a more general case with multiple buyers and general distributions is also considered). According to the indirect mechanism, the seller is to announce his production costs to the platform. The buyers are to choose a payment to the platform above a minimum price. The platform states that the good will be produced if the composition of the buyers' payments exceeds the seller's announced costs by the platform fee. The platform offers to pay the composition of the buyers' payments minus the platform fee to the seller for the production of the good. If the good is produced, the buyers who pay above the minimum price receive the good. If the good is not produced, the buyers do not receive their payments back.

The paper proceeds as follows. Section 1 presents the setup. Section 2 characterizes the optimal direct mechanism. Section 3 provides an implementation of the optimal direct mechanism for a two-buyer case with uniform distributions as well as a case with more than two buyers and general distributions. The last section concludes. The Appendix contains examples with uniform distributions from Myerson & Satterthwaite (1983) and Cornelli (1996).

[4] Assume zero marginal costs.

1 Setup

Consider a non-rival but excludable good, which is yet to be produced. This type of good is appropriate in the context of reward-based crowdfunding-platforms: once produced, such good does not depreciate or become more scarce if many buyers consume it, but not everyone will get access to the good (in contrast to a pure public good which is non-rival and non-excludable).

Consider a model with universal risk-neutrality. A seller of a non-rival but excludable good is privately informed about the fixed costs of production, denoted by M, which are distributed with a c.d.f. $G(M)$ over the interval $[\underline{M}, \overline{M}]$. Assume zero marginal costs[5]. Assume that selling the good, which is yet to be produced, is only possible via a platform. The platform decides whether the good will be produced, denote this production rule by $m \in \{0,1\}$[6]; and how much to pay the seller, denote this transfer by t_{ps}. Hence, the seller's ex-post utility is: $t_{ps} - M \cdot m$.

On the platform, the seller faces the set $N = \{1, ..., n\}$ of symmetric buyers with private information about the individual valuations for the good, denoted by θ_i, which are distributed with a c.d.f. $F(\theta_i)$ over the interval $[\underline{\theta}, \overline{\theta}]$. The buyers' valuations are uncorrelated and each buyer only knows his own valuation. For $\theta \equiv (\theta_j)_{j \in N}$, define $\bar{F}(\theta) \equiv \prod_{j \in N} F(\theta_j)$. Further define $\Theta \equiv [\underline{\theta}, \overline{\theta}]^N$. The platform decides whether each buyer receives the good once it is produced, denote this allocation probability by $p_i \in [0,1]$; and how much the buyer has to pay, denote this trasfer by $t_{b_i s}$. Hence, the buyer's ex-post utility is: $\theta_i \cdot p_i - t_{b_i s}$ and the platform's ex-post profit is $\sum_i t_{b_i p} - t_{ps}$.

The timing is as follows:

1. The seller privately learns M, each buyer privately learns θ_i.
2. The platform offers a contract specifying when the good is produced, what transfer the seller receives, whether each buyer gets access to the good and how much each buyer has to pay.

   For instance, in the case of a direct mechanism the platform makes $m(\theta', M')$, $t_{ps}(\theta', M')$, $p_i(\theta', M')$ and $t_{b_i p}(\theta', M')$ for all i dependent on the announced costs by the seller M' and the announced valuations by the buyers θ'[7].
3. The seller accepts or rejects the contract. Each buyer accepts or rejects the contract.
4. The seller produces the good according to the production rule m. Allocation payoffs are realized.

This paper characterizes the optimal direct mechanism maximizing the platform's expected profit and proposes an indirect implementation of the optimal direct mechanism.

[5] Not restrictive, the model could accommodate strictly positive marginal costs.
[6] Deterministic for simplicity, but not restrictive.
[7] Direct mechanism is defined in the next section.

2 Direct Mechanism

By the Revelation Principle we can concentrate on direct incentive-compatible mechanisms, without loss of generality. Consider a direct mechanism within which the seller reports her costs and each buyer reports his valuation simultaneously to the platform. Based on the announced costs and valuations, the platform follows the mechanism (full commitment), which states whether the good will be produced, how much to pay the seller, which buyers get access to the good and how much the buyers pay the platform. Hence, the direct mechanism is characterized by the following functions:

1. Production rule $m(\theta', M')$, which maps buyers' announced valuations and the seller's announced costs onto $\{0, 1\}$;
2. Transfer from the platform to the seller $t_{ps}(\theta', M')$, which maps buyers' announced valuations and the seller's announced costs onto $\mathbb{R}$;
3. Allocation rule $p_i(\theta', M')$, for all i, which maps buyers' announced valuations and the seller's announced costs onto $[0, 1]$, i.e. this is the probability with which buyer i gets the good;
4. Transfer from buyer i to the platform $t_{b_ip}(\theta', M')$, for all i, which maps buyers' announced valuations and the seller's announced costs onto $\mathbb{R}$.

Production and allocation rules are technically connected through a feasibility constraint, i.e. the good can be allocated to the buyers only if it is produced:

$$p_i(\theta', M') \leq m(\theta', M') \tag{FC}$$

The seller does not know the buyers' valuations when she decides whether to accept or reject the platform's mechanism. Assuming that the buyers report their valuations truthfully, her expected utility is:

$$U_s(M) = \int_{\Theta} [t_{ps}(\theta, M) - M \cdot m(\theta, M)] \, d\bar{F}(\theta),$$

Each buyer also does not know other buyers' valuations or the seller's production costs and builds an expectation. Assuming that other buyers report their valuations truthfully and that the seller reports her production costs truthfully, his expected utility is:

$$U_{b_i}(\theta_i) = \int_{\Theta_{-i}} \int_{\underline{M}}^{\overline{M}} [\theta_i \cdot p_i(\theta, M) - t_{b_ip}(\theta, M)] \, dG(M) d\bar{F}_{-i}(\theta_{-i}),$$

where $\Theta_{-i} \equiv [\underline{\theta}, \overline{\theta}]^{N-1}$, $\theta_{-i} \equiv (\theta_j)_{j \in N, j \neq i}$, and $\bar{F}_{-i}(\theta_{-i}) \equiv \prod_{j \in N, j \neq i} F(\theta_j)$.

When offering the mechanism, the platform makes sure that it is incentive-compatible (in the Bayesian sense) and individually rational. We start by laying out incentive-compatibility constraints.

The seller should report her production costs truthfully instead of misreporting them as M':

$$U_s(M) \geq \int_{\Theta} [t_{ps}(\theta, M') - M \cdot m(\theta, M')] \, d\bar{F}(\theta) \qquad \forall M, M' \in [\underline{M}, \overline{M}] \tag{IC$_s$}$$

Each buyer should report his valuation truthfully instead of misreporting it as θ_i':

$$U_{b_i}(\theta_i) \geq \int_{\Theta_{-i}} \int_{\underline{M}}^{\overline{M}} [\theta_i \cdot p_i(\theta_i', \theta_{-i}, M) - t_{b_i p}(\theta_i', \theta_{-i}, M)] \, dG(M) d\bar{F}_{-i}(\theta_{-i}) \qquad \forall \theta_i, \theta_i' \in [\underline{\theta}, \overline{\theta}] \qquad (\text{IC}_{b_i})$$

We turn to individual rationality constraints. The seller and each buyer should obtain non-negative expected utilities from accepting the mechanism:

$$U_s(M) \geq 0 \qquad \forall M \in [\underline{M}, \overline{M}] \qquad (\text{IR}_s)$$

$$U_{b_i}(\theta_i) \geq 0 \qquad \forall \theta_i \in [\underline{\theta}, \overline{\theta}] \qquad (\text{IR}_{b_i})$$

Thus, the platform chooses the production and allocation rules as well as the respective transfers as to maximize its expected profit, i.e. the sum of the expected transfers from the buyers minus the expected transfer to the seller, subject to feasibility, incentive-compatibility and individual rationality constraints:

$$\max_{\{m, t_{ps}, (p_i, t_{b_i p})_{i \in N}\}} \quad \int_{\Theta} \int_{\underline{M}}^{\overline{M}} \left[\sum_i t_{b_i p}(\theta, M) - t_{ps}(\theta, M) \right] dG(M) d\bar{F}(\theta)$$

$$\text{s.t.} \quad (\text{FC}) \ \& \ (\text{IC}_s) \ \& \ (\text{IC}_{b_i}) \ \forall i \ \& \ (\text{IR}_s) \ \& \ (\text{IR}_{b_i}) \ \forall i$$

We can rewrite the platform's optimization problem in terms of the buyers' virtual valuations and the seller's virtual costs, and state how to construct individually rational and incentive-compatible expected transfers.

Proposition 1. *The optimization problem is equivalent to:*

$$\max_{\{m, (p_i)_{i \in N}\}} \quad \int_{\Theta} \int_{\underline{M}}^{\overline{M}} \left[\sum_i C_{b_i}(\theta_i) \cdot p_i(\theta, M) - C_s(M) \cdot m(\theta, M) \right] dG(M) d\bar{F}(\theta) - U_s(\overline{M}) - \sum_i U_{b_i}(\underline{\theta})$$

$$\textit{s.t.} \quad \int_{\Theta_{-i}} \int_{\underline{M}}^{\overline{M}} p_i(\theta_i, \theta_{-i}.M) dG(M) d\bar{F}_{-i}(\theta_{-i}) \text{ - } \textit{increasing in } \theta_i$$

$$\textit{and} \quad \int_{\Theta} m(\theta, M) d\bar{F}(\theta) \text{ - } \textit{decreasing in } M,$$

where $C_{b_i}(\theta_i) \equiv \theta_i - \dfrac{1 - F(\theta_i)}{f(\theta_i)}$ *- is the virtual valuation; and* $C_s(M) \equiv M + \dfrac{G(M)}{g(M)}$ *- are the virtual costs.*

Optimally, $U_s(\overline{M}) = 0$ *and* $U_{b_i}(\underline{\theta}) = 0 \ \forall i$*. Expected transfers which satisfy incentive-compatibility are:*

$$\int_{\Theta} t_{ps}(\theta, M) d\bar{F}(\theta) = \int_{\Theta} \left(M \cdot m(\theta, M) + \int_{M}^{\overline{M}} m(\theta, x) dx \right) d\bar{F}(\theta), \qquad (1)$$

$$\int_{\Theta_{-i}} \int_{\underline{M}}^{\overline{M}} t_{b_i p}(\theta, M) dG(M) d\bar{F}_{-i}(\theta_{-i}) = \int_{\Theta_{-i}} \int_{\underline{M}}^{\overline{M}} \left(\theta_i \cdot p_i(\theta, M) - \int_{\underline{\theta}}^{\theta_i} p_i(x, \theta_{-i}, M) dx \right) dG(M) d\bar{F}_{-i}(\theta_{-i}). \qquad (2)$$

Proof: the proof follows Myerson (1981) and Myerson & Satterthwaite (1983) with appropriate adjustments. Define $m^{exp}(M) \equiv \int_\Theta m(\theta, M) d\bar{F}(\theta)$ and $p_i^{exp}(\theta_i) \equiv \int_{\Theta_{-i}} \int_{\underline{M}}^{\overline{M}} p_i(\theta_i, \theta_{-i}, M) dG(M) d\bar{F}_{-i}(\theta_{-i})$.

From (IC$_s$) for M and (IC$_s$) for M' we obtain: $(M'-M)\cdot m^{exp}(M) \geq U_s(M) - U_s(M') \geq (M'-M)\cdot m^{exp}(M')$. If $M' > M$, then $m^{exp}(M)$ is decreasing in M, hence, $U_s'(M) = -m^{exp}(M)$, and we have:

$$U_s(M) = U_s(\overline{M}) + \int_M^{\overline{M}} m^{exp}(x) dx. \tag{3}$$

From (IC$_{b_i}$) for θ_i and (IC$_{b_i}$) for θ_i' we obtain: $(\theta_i' - \theta_i) \cdot p_i^{exp}(\theta_i') \geq U_{b_i}(\theta_i') - U_{b_i}(\theta_i) \geq (\theta_i' - \theta_i) \cdot p_i^{exp}(\theta_i)$. If $\theta_i' > \theta_i$, then $p_i^{exp}(\theta_i)$ is increasing in θ_i, hence, $U_{b_i}'(\theta_i) = p_i^{exp}(\theta_i)$, and we have:

$$U_{b_i}(\theta_i) = U_{b_i}(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} p_i^{exp}(\theta_i) dx. \tag{4}$$

Make use of the fact that the expected gains from trade minus the platform's expected profit, i.e. $U_p \equiv \int_\Theta \int_{\underline{M}}^{\overline{M}} \left[\sum_i t_{b_i p}(\theta, M) - t_{ps}(\theta, M)\right] dG(M) d\bar{F}(\theta)$, equals to the expected gains to the seller and the buyers. Using standard techniques we obtain:

$$\begin{aligned}
U_p \quad &= \quad \int_\Theta \int_{\underline{M}}^{\overline{M}} \left(\sum_i \theta_i \cdot p_i(\theta, M) - M \cdot m(\theta, M) \right) dG(M) d\bar{F}(\theta) \\
&- \quad \int_{\underline{M}}^{\overline{M}} U_s(M) dG(M) - \sum_i \left(\int_{\underline{\theta}}^{\overline{\theta}} U_{b_i}(\theta_i) dF(\theta_i) \right) \\
&\overset{(3),(4)}{=} \quad \int_{\underline{\theta}}^{\overline{\theta}} \int_{\Theta_{-i}} \int_{\underline{M}}^{\overline{M}} \sum_i \theta_i \cdot p_i(\theta, M) dG(M) d\bar{F}_{-i}(\theta_{-i}) dF(\theta_i) - \int_\Theta \int_{\underline{M}}^{\overline{M}} M \cdot m(\theta, M) dG(M) d\bar{F}(\theta) \\
&- \quad U_s(\overline{M}) - \int_{\underline{M}}^{\overline{M}} \int_M^{\overline{M}} m^{exp}(x) dx dG(M) - \sum_i U_{b_i}(\underline{\theta}) - \sum_i \left(\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\theta_i} p_i^{exp}(x) dx dF(\theta_i) \right) \\
&= \quad \sum_i \left(\int_{\underline{\theta}}^{\overline{\theta}} \theta_i \int_{\Theta_{-i}} \int_{\underline{M}}^{\overline{M}} p_i(\theta_i, \theta_{-i}, M) dG(M) d\bar{F}_{-i}(\theta_{-i}) dF(\theta_i) \right) - \int_{\underline{M}}^{\overline{M}} M \int_\Theta m(\theta, M) d\bar{F}(\theta) dG(M) \\
&- \quad U_s(\overline{M}) - \int_{\underline{M}}^{\overline{M}} G(M) \cdot m^{exp}(M) dM - \sum_i U_{b_i}(\underline{\theta}) - \sum_i \left(\int_{\underline{\theta}}^{\overline{\theta}} (1 - F(\theta_i)) \cdot p_i^{exp}(\theta_i) d\theta_i \right) \\
&= \quad \sum_i \left(\int_{\underline{\theta}}^{\overline{\theta}} \theta_i \cdot p_i^{exp}(\theta_i) dF(\theta_i) \right) - \sum_i \left(\int_{\underline{\theta}}^{\overline{\theta}} (1 - F(\theta_i)) \cdot p_i^{exp}(\theta_i) d\theta_i \right) - \int_{\underline{M}}^{\overline{M}} M \cdot m^{exp}(M) dG(M) \\
&- \quad \int_{\underline{M}}^{\overline{M}} G(M) \cdot m^{exp}(M) dM - U_s(\overline{M}) - \sum_i U_{b_i}(\theta_i)
\end{aligned}$$

Applying definitions of $m^{exp}(M)$ and $p_i^{exp}(\theta_i)$ again and rearranging the terms gives us:

$$U_p = \int_\Theta \int_{\underline{M}}^{\overline{M}} \left[\sum_i \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) \cdot p_i(\theta, M) - \left(M + \frac{G(M)}{g(M)} \right) \cdot m(\theta, M) \right] dG(M) d\bar{F}(\theta) - U_s(\overline{M}) - \sum_i U_{b_i}(\underline{\theta}).$$

$U_s(M)$ decreases in M, so it suffices to ensure that (IR$_s$) is satisfied for $\overline{M}$. Optimally, (IR$_s$) for $\overline{M}$ will bind. Analogously, $U_{b_i}(\theta_i)$ increases in θ_i, need to ensure that (IR$_{b_i}$) is satisfied for $\underline{\theta}$. Optimally, (IR$_{b_i}$) $\forall i$ for $\underline{\theta}$ will bind. Then, (1) resp. (2) is implied by def. of $U_s(M)$ and (3) resp. def. of $U_{b_i}(\theta_i)$ and (4). ■

We can now derive production and allocation rules maximizing the platform's expected profit. For that, I impose a standard assumption on $F(\theta_i)$ and $G(M)$ that is satisfied for many distributions (see Myerson & Satterthwaite (1983)).

Assumption 1. *$C_{b_i}(\theta_i)$ and $C_s(M)$ are strictly increasing.*

Proposition 2. *The following production and allocation rules are optimal under assumption 1:*

$$m(\theta, M) = \begin{cases} 1 & \textit{if } \sum_{i \in I*} C_{b_i}(\theta_i) \geq C_s(M) \\ 0 & \textit{otherwise} \end{cases}$$

$$p_i(\theta, M) = \begin{cases} 1 & \textit{if } C_{b_i}(\theta_i) \geq 0 \textit{ and } \sum_{i \in I*} C_{b_i}(\theta_i) \geq C_s(M) \\ 0 & \textit{otherwise} \end{cases}$$

where $I^* \equiv \{i \in N : C_{b_i}(\theta_i) \geq 0\}$.

Proof: the proof follows Cornelli (1996) with appropriate adjustments.

Given that the good is produced, it is optimal to give access to the good to those buyers, who have a positive virtual valuation, i.e. for $m(\theta, M) = 1$ we must have:

$$p_i(\theta, M) = \begin{cases} 1 & \text{if } C_{b_i}(\theta_i) \geq 0 \text{ and } m(\theta, M) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Define the set of the buyers with positive virtual valuations as $I^* \equiv \{i \in N : C_{b_i}(\theta_i) \geq 0\}$. We can now show that it is optimal to produce the good if the sum of positive virtual valuations surpasses the virtual costs:

$$m(\theta, M) = \begin{cases} 1 & \text{if } \sum_{i \in I*} C_{b_i}(\theta_i) \geq C_s(M) \\ 0 & \text{otherwise} \end{cases}$$

Under assumption 1, as θ_i increases / M decreases, it becomes easier to satisfy $\sum_{i \in I*} C_{b_i}(\theta_i) \geq C_s(M)$ and $C_{b_i}(\theta_i) \geq 0$, meaning that monotonicity conditions for $m^{exp}(M)$ and $p_i^{exp}(\theta_i)$ from proposition 1 are satisfied. ■

There are many ways in which one could construct the actual transfers t_{ps} and $t_{b_i p}$ $\forall i$, depending on the desired structure: e.g. contribution or subscription schemes, see Admati & Perry (1991). As long as these transfers satisfy the expressions for the expected transfers (1) and (2) with the optimal $m(\theta, M)$ and $p_i(\theta, M)$, the direct mechanism will be incentive-compatible and individually rational.

3 Indirect Mechanism

The optimal direct mechanism described in Propositions 1 and 2 can be implemented via a crowdfunding scheme which combines features of Myerson & Satterthwaite (1983) and Cornelli (1996). I concentrate on the so-called contribution schemes (see Admati & Perry (1991)) for which there is no reimbursement in case the good is not produced: the buyers' actual transfers will correspond to their expected transfers.

I start with a two-buyer case and the seller's costs and each buyer's valuation being uniformly distributed over $[0, 1]$ to make a clearer point. The proposed selling mechanism can be conveniently compared to the examples of the indirect mechanisms described by Myerson & Satterthwaite (1983) and Cornelli (1996), see Appendix. After the proof, I also discuss this scheme in more detail. I then characterize a case of N buyers and general distributions $F(\theta_i)$ and $G(M)$, which nonetheless satisfy assumption 1.

Proposition 3. *The seller is to announce his production costs M to the platform. The buyers are to choose a payment to the platform T_i above the minimum price $T^* = 1/16$. The platform states that the good will be produced if the composition of the buyers' payments exceeds the seller's announced costs by the platform fee, namely, the following must hold:*

$$2 \sum_{i \in J_T} \left[2\left(T_i + 1/16\right)\right]^{1/2} - s \geq 2M,$$

where J_T is the set of buyers with payments above T^* and s is the number of buyers with payments above T^*. The platform offers to pay the composition of the buyers' payments minus the platform fee to the seller for the production of the good. If the good is produced, the buyers who pay at least T^* receive the good. If the good is not produced, the buyers receive neither the good not their payments back[8].

Proof: **The buyers:** the minimum price for the buyers $T^* = 1/16$ is derived similarly to Cornelli (1996), by using the optimal production and allocation rules from proposition 2:

$$m(\theta, M) = \begin{cases} 1 & \text{if } \sum_{i \in I*}(2\theta_i - 1) \geq 2M \\ 0 & \text{otherwise} \end{cases}$$

$$p_i(\theta, M) = \begin{cases} 1 & \text{if } \theta_i \geq 1/2 \text{ and } \sum_{i \in I*}(2\theta_i - 1) \geq 2M \\ 0 & \text{otherwise} \end{cases}$$

where $I^* \equiv \{i \in I : \theta_i \geq 1/2\}$; and the expression for the expected payment (2):

$$\int_{\Theta_{-i}} \int_M t_{b_i p}(\theta, M) dM d\theta_{-i} = \int_{\Theta_{-i}} \int_M \left(\theta_i \cdot p_i(\theta, M) - \int_{\underline{\theta}}^{\theta_i} p_i(x, \theta_{-i}, M) dx\right) dM d\theta_{-i}.$$

[8] A similar scheme but with reimbursement to the buyers in case the good is not produced can be constructed following a readjustment as in Cornelli (1996).

The buyers' actual payments T_i's correspond to their expected payments in the case of no reimbursement:

1. For buyers with $\theta_i < 1/2$ we have $p_i = 0$ and $T_i = 0$, i.e. these buyers never receive the good, since their virtual valuations are negative.

2. For buyers with $\theta_i \geq 1/2$ we have $p_i = 1$ if the good is produced.

   If buyer i is the only one with a positive virtual valuation, the good is produced as long as $\theta_i \geq M + 1/2$ ("one seller, one buyer and a broker" case as in Myerson & Satterthwaite (1983)). If both buyers have positive virtual valuations, the good is produced as long as $\theta_1 + \theta_2 \geq M + 1$. Meaning that the buyers' valuations have to exceed the seller's announced costs by the platform fee (in this case 1/2 per buyer).

   Hence, we have:

$$\begin{aligned} T_i &= \int_{\Theta_{-i}} \int_M t_{b_i p}(\theta, M) dM d\theta_{-i} \\ &= \int_0^{1/2} \int_0^{\theta_i - 1/2} \left(\theta_i \cdot 1 - \int_{1/2}^{\theta_i} p_i(x, \theta_{-i}, M) dx \right) dM d\theta_{-i} \\ &+ \int_{1/2}^{1} \int_0^{\theta_i + \theta_{-i} - 1} \left(\theta_i \cdot 1 - \int_{1/2}^{\theta_i} p_i(x, \theta_{-i}, M) dx \right) dM d\theta_{-i} \\ &= 1/2 \cdot \theta_i^2 - 1/16 \end{aligned}$$

   The minimum price is $T^* \equiv T_i(\theta_i = 1/2) = 1/16$.

T_i is strictly increasing and invertable:

$$\theta_i = \left[2\left(T_i + 1/16\right)\right]^{1/2}.$$

Thus, the optimal production rule as a function of the buyers' payments and the seller's announced costs is:

$$2 \sum_{i \in J_T} \left[2\left(T_i + 1/16\right)\right]^{1/2} - s \geq 2M,$$

where J_T is the set of buyers with payments above T^* and s is the number of buyers with payments above T^*.

The seller: the payment from the platform to the seller for producing the good is derived similarly to Myerson & Satterthwaite (1983): from a second-price auction with a reserve price (for a detailed derivation, see Appendix).

If only one buyer has a positive virtual valuation (pays above T^*), then the offered payment to the seller is $\theta_i - 1/2$ ("one seller, one buyer and a broker" case as in Myerson & Satterthwaite (1983)), or $\left[2\left(T_i + 1/16\right)\right]^{1/2} - 1/2$. If both buyers have positive virtual valuations (both pay above T^*), then the offered payment to the seller is $\theta_1 + \theta_2 - 1$ (arrive at this value by solving $\sum_{i \in I^*} C_{b_i}(\theta_i) = C_s(M)$ or in this case $(2\theta_1 - 1) + (2\theta_2 - 1) = 2M$ for M) or $\left[2\left(T_1 + 1/16\right)\right]^{1/2} + \left[2\left(T_2 + 1/16\right)\right]^{1/2} - 1$. Meaning that the platform offers to pay the composition of the buyers' payments minus the platform fee (in this case 1/2 per buyer) to the seller for the production of the good. In other words, the platform pays the seller the highest production costs she could have announced that would still lead to the production of the good, given the buyers' valuations (inverse functions of their payments T_i's). ■

The proposed scheme resembles contractual features which can be found on reward-based crowdfunding platforms, such as Patreon, i.e. a combination of a minimum pledge for the "patrons" and a percentage fee for the creators. To recognize that the payment to the seller from proposition 4 is set in relation to the buyers' contributions T_i's (although it has a more complex structure than a percentage-fee), let us differentiate between the case where only one buyer pays above the minimum price T^* and the case where both buyers contribute above T^*.

According to the contract, if the seller announces M and only one buyer pays T_i above T^*, the good is produced as long as $[2(T_i + 1/16)]^{1/2} \geq M + 1/2$, i.e. the *transformed* payment by the buyer exceeds the seller's announced costs by the platform fee of 1/2, which could be understood as a processing fee, for instance. The seller is offered a payment of $[2(T_i + 1/16)]^{1/2} - 1/2$ for the production of the good, i.e. she receives the transformed buyer's payment net of the platform fee of 1/2 (processing fee of 1/2).

If the seller announces M and both buyers pay T_i's above T^*, the good is produced as long as $[2(T_1 + 1/16)]^{1/2} + [2(T_2 + 1/16)]^{1/2} \geq M + 1$, i.e. the sum of the *transformed* payments by the buyers exceeds the seller's announced costs by the platform fee of 1/2 per buyer, which could be understood as a processing fee per transaction, for instance. The seller is offered a payment of $[2(T_1 + 1/16)]^{1/2} + [2(T_2 + 1/16)]^{1/2} - 1$ for the production of the good, i.e. she receives the sum of the transformed buyers' payments net of the platform fee of 1/2 per buyer (processing fee of 1/2 per transaction).

Finally, let us consider a more abstract case of N buyers with general distributions $F(\theta_i)$ and $G(M)$ (assumption 1 still holds).

Proposition 4. *The seller is to announce his production costs M to the platform. The buyers are to choose a payment T_i to the platform above the minimum price T^*. The platform states that the good will be produced if the the following condition is fulfilled:*

$$\sum_{i \in J_T} \psi_i(T_i) \geq \xi_s(M),$$

where J_T is the set of buyers with payments above T^*, $\psi_i(\cdot)$ and $\xi_s(\cdot)$ are defined in the proof.

The platform offers to pay the seller the highest production costs she could have announced that would still lead to the production of the good, given the buyers' payments. If the good is produced, the buyers who pay at least T^* receive the good. If the good is not produced, the buyers receive neither the good not their payments back.

Proof: **The buyers:** the minimum price for the buyers T^* is derived similarly to Cornelli (1996), by using the optimal production and allocation rules from proposition 2:

$$m(\theta, M) = \begin{cases} 1 & \text{if } \sum_{i \in I*} C_{b_i}(\theta_i) \geq C_s(M) \\ 0 & \text{otherwise} \end{cases}$$

$$p_i(\theta, M) = \begin{cases} 1 & \text{if } \theta_i \geq \theta^* \text{ and } \sum_{i \in I*} C_{b_i}(\theta_i) \geq C_s(M) \\ 0 & \text{otherwise} \end{cases}$$

where θ^* solves $C_{b_i}(\theta_i) = 0$, $I^* \equiv \{i \in N : \theta_i \geq \theta^*\}$; and the expression for the expected payment (2):

$$\int_{\Theta_{-i}} \int_{\underline{M}}^{\overline{M}} t_{b_i p}(\theta, M) dG(M) d\bar{F}_{-i}(\theta_{-i}) = \int_{\Theta_{-i}} \int_{\underline{M}}^{\overline{M}} \left(\theta_i \cdot p_i(\theta, M) - \int_{\underline{\theta}}^{\theta_i} p_i(x, \theta_{-i}, M) dx \right) dG(M) d\bar{F}_{-i}(\theta_{-i}).$$

The buyers' actual payments T_i's correspond to their expected payments in the case of no reimbursement:

1. For buyers with $\theta_i < \theta^*$ we have $p_i = 0$ and $T_i = 0$, i.e. these buyers never receive the good, since their virtual valuations are negative.

2. For buyers with $\theta_i = \theta^*$ we have $p_i = 1$ if the sum of the remaining positive virtual valuations surpasses virtual costs, and the good is produced.

   Define $\Delta_{-i} \equiv \{\theta_j | j \in I^*, j \neq i\}$ and let $M^*(\Delta_{-i}, \theta_i)$ be the solution to $\sum_{i \in I*} C_{b_i}(\theta_i) = C_s(M)$ when solving for M. Then, we obtain:

$$\begin{aligned} T^* &= \int_{\Theta_{-i}} \int_{\underline{M}}^{\overline{M}} t_{b_i p}(\theta, M) dG(M) d\bar{F}_{-i}(\theta_{-i}) \\ &= \int_{\Theta_{-i}} \int_{\underline{M}}^{M^*(\Delta_{-i}, \theta^*)} \theta^* \cdot 1 \; dG(M) d\bar{F}_{-i}(\theta_{-i}) = \theta^* \Pr \left(\sum_{j \in I*} C_{b_j}(\theta_j) \geq C_s(M) \right). \end{aligned}$$

3. For buyers with $\theta_i > \theta^*$, we obtain:

$$\begin{aligned} T_i &= \int_{\Theta_{-i}} \int_{\underline{M}}^{\overline{M}} t_{b_i p}(\theta, M) dG(M) d\bar{F}_{-i}(\theta_{-i}) \\ &= \int_{\Theta_{-i}} \int_{\underline{M}}^{M^*(\Delta_{-i}, \theta_i)} \left(\theta_i \cdot 1 - \int_{\theta_i^{min}}^{\theta_i} 1 dx \right) dG(M) d\bar{F}_{-i}(\theta_{-i}) \\ &= \int_{\Theta_{-i}} \int_{\underline{M}}^{M^*(\Delta_{-i}, \theta_i)} \theta_i^{min} dG(M) d\bar{F}_{-i}(\theta_{-i}) \equiv \phi_i(\theta_i), \end{aligned}$$

   where $\theta_i^{min} \geq \theta^*$ is the lowest value of θ_i, for which the good will be produced given the other buyers' valuations and the seller's costs, i.e. the lowest $\theta_i \geq \theta^*$ which solves $\sum_{i \in I*} C_{b_i}(\theta_i) \geq C_s(M)$.

Due to assumption 1, $\phi_i(\theta_i)$ is strictly increasing and invertable. Hence, $\theta_i = \phi_i^{-1}(T_i)$.

Define $\psi_i(T_i) \equiv \phi_i^{-1}(T_i) - \dfrac{1 - F(\phi^{-1}(T_i))}{f(\phi^{-1}(T_i))}$ (from $C_{b_i}(\theta_i)$), and $\xi_s(M) \equiv M + \dfrac{G(M)}{g(M)}$ (from $C_s(M)$).

Then the optimal production rule as a function of the buyers' payments and the seller's announced costs is: $\sum_{i \in J_T} \psi_i(T_i) \geq \xi_s(M)$, where J_T is the set of the buyers with the payments above T^*.

The seller: the payment from the platform to the seller for producing the good is derived similarly to Myerson & Satterthwaite (1983): from a second-price auction with a reserve price (for a detailed derivation, see Appendix). We solve $\sum_{i \in I*} C_{b_i}(\theta_i) = C_s(M)$ for M - these will be the highest production costs for which the good will be produced according to the optimal production rule given the buyers' valuations (inverse functions of their payments T_i's). ■

Conclusion

This paper derived the platform-optimal direct mechanism and proposed an implementation for this direct mechanism. The results combine features of Cornelli (1996) and Myerson & Satterthwaite (1983): in fact, for the number of buyers being equal to one, one arrives at Myerson & Satterthwaite (1983) exactly. Comparing the results with Cornelli (1996) requires a different setup. In this paper, it was the platform which designed the contract specifying the production and allocation rules. In reality, even on Patreon, the sellers can have more degree of freedom. It would be worthwhile to model the case where the seller decides upon the production and allocation rules, whilst the platform chooses the transfers (similar to Loertscher & Niedermayer (2023)), whilst still considering a club good.

One could enrich the model in other ways, i.e. by introducing dynamic valuation formation for the buyers and/or the seller; by considering platform competition; or by incorporating platform-specific benefits which might provide an even better match to the existing payment schemes.

A planned model extension is keeping the setup as it is and introducing multiple sellers, either to have competing sellers, each of whom could produce the good alone; or to be in a situation where multiple sellers are needed for the production of the good, and there are different constellations of the sellers which might work. Symmetrically to the buyers, the sellers will most likely need to choose the payment below a certain maximum price this time and not merely receive a payment which is equal to the highest costs they could have announced which would still lead to the production of the good given the buyers' valuations.

References

Admati, Anat R, & Perry, Motti. 1991. Joint projects without commitment. *The Review of Economic Studies*, **58**, 259–276.

Barbieri, Stefano, & Malueg, David. 2010. Profit-Maximizing Sale of a Discrete Public Good via the Subscription Game in Private-Information Environments. *The B.E. Journal of Theoretical Economics*, **10**, 1–31.

Belleflamme, Paul, Omrani, Nessrine, & Peitz, Martin. 2015. The economics of crowdfunding platforms. *Information Economics and Policy*, **33**, 11–28.

Cornelli, Francesca. 1996. Optimal selling procedures with fixed costs. *Journal of Economic Theory*, **71**(1), 1–30.

Ellman, Matthew, & Hurkens, Sjaak. 2019. Optimal crowdfunding design. *Journal of Economic Theory*, **184**(C).

Loertscher, Simon, & Niedermayer, Andras. 2023. An optimal pricing theory of transaction fees in thin markets. *Working paper, revisit and resubmit, RAND Journal of Economics.*

Maskin, Eric, & Tirole, Jean. 1990. The Principal-Agent Relationship with an Informed Principal: The Case of Private Values. *Econometrica*, **58**(2), 379–409.

Myerson, Roger, & Satterthwaite, Mark A. 1983. Efficient mechanisms for bilateral trading. *Journal of Economic Theory*, **29**(2), 265–281.

Myerson, Roger B. 1981. Optimal Auction Design. *Mathematics of Operations Research*, **6**(1), 58–73.

Strausz, Roland. 2017. A Theory of Crowdfunding: A Mechanism Design Approach with Demand Uncertainty and Moral Hazard. *The American Economic Review*, **107**(6), 1430–1476.

Wang, Zhu, & Wright, Julian. 2017. Ad valorem platform fees, indirect taxes, and efficient price discrimination. *The RAND Journal of Economics*, **48**(2), 467–484.

Appendix

Myerson & Satterthwaite (1983)

This paper considers the case of **one seller, one buyer and a broker**, the good has already been produced. Broker's optimal direct mechanism is derived. In case of the seller's and the buyer's valuations being uniformly distributed over $[0, 1]$, it is implemented via the following selling procedure which resembles a **second-price auction**:

1. The seller and the buyer are to submit sealed bids. The broker states that the good will be transferred from the seller to the buyer if the buyer's bid exceeds the seller's bid by $1/2$ (this could be interpreted as *the broker's fee*). If trade occurs, the broker charges the buyer the seller's bid + $1/2$; and offers to pay the buyer's bid - $1/2$ to the seller.
2. The seller and the buyer submit their bids.
3. Payments are realized (if there is no trade, the payments are 0).

Derivation: the payments can be derived from a second-price auction with a reserve price.

The buyer: in the standard case (no broker) and with only one potential buyer, whose valuation is distributed with the c.d.f. $F(\cdot)$, the optimal reserve price r set by the seller is derived from:

$$\max_r \qquad (1 - F(r)) \cdot r + F(r) \cdot v_S$$

$$r^* = \frac{1 - F(r^*)}{f(r^*)} + v_S,$$

where v_S is the value of the good to the seller.

The value of the good to a broker, however, is *the virtual valuation of the seller*:

$$v_S + \frac{G(v_S)}{g(v_S)} = 2v_S$$

in case of uniform $G(\cdot)$. The broker charges the buyer the following reserve price (uniform $F(\cdot)$ over $[0, 1]$):

$$r^* = \frac{1 - F(r^*)}{f(r^*)} + 2v_S$$
$$r^* = 1 - r^* + 2v_S$$
$$r^* = v_S + 1/2.$$

The seller: in the standard case (no broker) and with only one potential seller, whose valuation is distributed with the c.d.f. $G(\cdot)$, the optimal reserve price r set by the buyer is derived from:

$$\max_r \qquad G(r) \cdot (v_B - r) + (1 - G(r)) \cdot 0$$

$$r^* = v_B - \frac{G(r^*)}{g(r^*)},$$

where v_B is the value of the good to the buyer.

The value of the good to a broker, however, is *the virtual valuation of the buyer*:

$$v_B - \frac{1 - F(v_B)}{f(v_B)} = 2v_B - 1$$

in case of uniform $F(\cdot)$ over $[0, 1]$. The broker charges the seller the following reserve price (uniform $G(\cdot)$):

$$r^* = (2v_B - 1) - \frac{G(r^*)}{g(r^*)}$$
$$r^* = (2v_B - 1) - r^*$$
$$r^* = v_B - 1/2$$

Cornelli (1996)

This paper considers the case of **one seller and N buyers interacting directly, without a broker**, the good is yet to be produced. Seller's optimal direct mechanism is derived. In a two-buyer case with each buyer's valuation uniformly distributed over $[0, 1]$ and the seller's costs of M (public information), it is implemented by the following **pay-what-you-want** scheme, with a **minimum price**:

1. The seller announces a minimum price $T^* = 1/4 - M/4$ and asks the buyers to pay an amount above the minimum price.
2. The buyers who pay at least T^* will receive the good if it is produced. The seller commits to producing the good if the composition of the buyers' payments T_i's exceeds the costs, namely:

$$2 \sum_{i \in J_T} \left[2\left(T_i - 1/8 + M/4\right)\right]^{1/2} - s \geq M,$$

   where J_T is the set of buyers with payments above T^* and s is the number of buyers with payments above T^*.
3. The buyers choose how much to pay.
4. If the good is produced, the buyers receive the good. If the good is not produced, the buyers receive neither the good nor their payments back.

Sequential Screening in the Presence of Fixed Costs*

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Abstract

This paper characterizes optimal production and selling procedures when fixed costs are present and buyers' valuations for the good are learned over time. We study situations in which many potential buyers contribute to the production of a non-rival but excludable good (e.g. private members club facilities, art exhibitions). We formally decouple the production and allocation decisions of a monopolistic seller. We stress the role of dynamic information in such settings: oftentimes, the buyers' true valuations for the good are revealed after the contracts are signed and after the seller has to produce the good. Additionally, we address the case where the seller can postpone production until the buyers have learned their true valuations. We derive profit maximizing sales mechanisms for both cases. These could be implemented as buy-option and "contribute-option" contracts.

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Introduction

Fixed costs of production often tend to be presented as neglectable in the literature. In this paper we take a closer look at the situations where a seller has to incur fixed costs and multiple potential buyers can contribute to the production of a good by agreeing to certain pricing schemes (memberships, subscriptions, etc.).

In particular, we consider goods which are characterised by the following aspects: first, the cost of production is not tied to individual consumption, i.e. once produced, such goods can be provided to buyers at no marginal cost. Second, these goods are non-rival, i.e. their supply is unaffected by consumption. Third, exclusion from consumption is possible (for example, through entry tickets). This is the case in a variety of contexts: gym members agreeing to membership fees as to make sure that the gym continues to operate and possibly offers new courses / equipment; theater enthusiasts buying membership cards to support potential theater plays over the course of the season; or supporters subscribing to the channels of their favourite artists and, thus, contributing to the production of their future content.

Oftentimes, however, the seller faces the decision on whether or not to incur production costs and what pricing scheme to offer long before the buyers have full information about their demand for the good. Think of a gym owner deciding on prolonging the opening hours: schedule for the staff has to be planned before customers will know for sure if the new training hours work for them. A museum has to plan a temporary exhibition while visitors might not yet know for sure if they will be interested or able to visit.

In other situations production and the buyers' valuation formation might be even more intertwined. Consider again a gym owner deciding on buying new equipment for the gym. Only by seeing a demonstration of the new equipment might customers learn how much they will value it. Imagine buyers who want to take part in a crowd-funding campaign for which a prototype has to be produced first for the buyers to understand how much they might value the new product.

Producing and selling such goods is a risky endeavour, potentially leading to misinvestments. Deciding on the optimal procedure is a difficult task. Using a mechanism design approach we find the optimal direct mechanism for a monopolist who is selling a non-rival but excludable good to multiple buyers at a point in time when the buyers only observe an imprecise signal about their true valuations for the good. We thereby contribute to the literature on optimal sales mechanisms in the presence of fixed costs as in Cornelli (1996) by incorporating sequential screening as in Courty & Li (2000).

We find that when it comes to the allocation decision, i.e. who gets access to the good, the standard result holds: optimally, buyers with positive virtual valuations[1] will get access to the good, since

[1] Virtual valuations expressions are of dynamic nature, see Courty & Li (2000).

marginal costs are zero. As the production decision has to take place before the allocation decision, it will be optimal to produce the good if the sum of expected positive virtual valuations surpasses fixed costs of production.

Furthermore, we show that the optimal production and selling procedure can be implemented by the so-called buy-option contracts. In fact, the type of contracts we characterise is often found in practice: at private members clubs (e.g. the Soho House) and museums (e.g. the Metropolitan Museum of Arts). They offer different membership tiers which require varying membership fees and are associated with different sets of perks and prices for their facilities and events. We show that membership fees serve a dual purpose in such contracts: the higher the membership fee, the higher is the possibility that existing facilities are maintained and future events take place; but also the lower are the entrance fees.

Additionally, we offer a potential explanation for why in some cases such membership fees are strictly positive and without paying them it is impossible to get access to facilities (e.g. the Soho House); and in other cases membership fees can be zero and access to the facilities can be granted if mere entrance fees are paid (e.g. the Metropolitan Museum of Arts). We find that if it is unlikely that fixed costs are covered through the collected membership fees (e.g. fixed costs are particularly high or the number of buyers is low), then the minimum membership fee is likely to be zero.

Further contributions of this paper are twofold. Firstly, we analyse the importance of observability of production. In case production is observable, the buyers might gain additional information which intensifies incentive issues and complicates revenue extraction. We show that this is indeed not the case and that the optimal direct mechanism coincides with the one where production is not observable.

Secondly, we extend our main model by allowing the seller to wait until the buyers have learned their true valuations before making the production decision. We find that the allocation decision remains the same but the production decision takes into account actual and not expected positive virtual valuations. Optimal transfers are also altered by flexible production and a possible implementation which we offer can be described as "contribute-option" contracts. After collecting initial fees from the buyers (e.g. membership fees), the seller now offers an option to contribute to the production of the good, once the buyers learn their true valuations. In particular, it is optimal for the seller to ask the buyers to pay as much as they want when the production decision takes place.

The paper proceeds as follows. Section 1 is dedicated to literature on selling procedures for non-rival but excludable goods and contribution and subscription games in particular. Section 2 presents the setup. Section 3 defines the dynamic direct mechanism for our setting. The optimal mechanism is characterized in Section 4. Section 5 studies a particular contribution scheme implementing the optimal direct mechanism. Section 6 extends the main model by allowing for a more flexible production, where the seller can postpone production till the buyers' true valuations are realized. The last section concludes. The appendix contains all the proofs.

1 Related Literature

Non-rival but excludable goods have been well-studied in the literature, starting with the efficiency analysis in case of the monopolist being informed about the buyers' valuations, as in Brennan & Walsh (1985); and moving to the case of private information, with the most prominent contribution by Cornelli (1996). She consideres multiple buyers with private valuations who need to contribute to production of a good first before they can consume. In particular, Cornelli explains why we might observe price dispersion in practice: buyers are willing to contribute different amounts to consume the same good in the end, out of fear of the good not being produced otherwise.

Schmitz (1997) also investigates profit-maximizing provision of a non-rival but excludable good to multiple buyers whose valuations for the good are static private information. In addition to production costs, he considers distribution costs. Ellman & Hurkens (2019) illustrate the crucial role of pre-production sales and the threat of not producing the good as a rent-extracting instrument of the seller by considering discrete buyers' types.

In contrast to other contributions, this paper enriches Cornelli's setup by introducing dynamic private information: we consider buyers whose valuations are revealed to them over time. There are different approaches to this dynamic nature of information in the literature such as Battaglini (2005) which uses Markov, Pavan *et al.* (2014) or Eső & Szentes (2007). We model dynamic information by following Courty & Li (2000), who derive optimal contracts under dynamic private information and argue why offering a contract before true valuations are realized is optimal. We adapt their definition of dynamic direct mechanism to our setting with fixed costs.

We are also interested in possible implementations of the optimal direct mechanism that we find. There are many papers on different selling procedures and pricing structures. Commonly found schemes are the so-called contribution and subscription games, see Admati & Perry (1991). They define contribution games (e.g. public radio, TV fundraising campaigns) as such in which players' contributions are not refunded if the project is not completed; and subscription games as such where contributions are refunded. Many authors make use of this terminology and analyse optimal contracts for pre-specified contribution and subscription schemes.

Vega-Redondo (1995) studies reputation formation / bargaining in multistage binary contribution games, where players with continuous valuations contribute either a pre-specified irreversible amount or nothing at each stage. Menezes *et al.* (2001) analyzes efficient private provision of discrete public goods in form of contribution and subscription games with continuous contributions. Barbieri & Malueg (2010) derive profit-maximizing selling mechanisms for public goods in subscription games with two buyers, where the seller states a contribution threshold which is different than the cost of production. In contrast to all these papers, ours does not impose any particular payment structure (binary or continuous contributions) but rather derives profit-maximizing contribution schemes.

2 Setup

Consider a seller (she), who can produce a non-rival but excludable good. Her production decision is denoted by $m \in \{0, 1\}$[2]. She can sell the good to a set of buyers indexed by $i \in I = \{1, ..., N\}$ who will get access to the good with probabilities $q_i \in [0, 1]$. The transaction also involves a payment t_i from each buyer to the seller. Hence, the economic allocation is defined by: $\{m, (q_i, t_i)_{i \in I}\} \in \{0, 1\} \times [0, 1]^N \times \mathbb{R}^N$.

Buyers: Prior to meeting the seller, symmetric buyers do not know their exact valuations for the good. They do, however, observe a private signal τ_i, their so called "ex-ante type". This signal is distributed with the cumulative distribution function $G(\tau_i)$ and density $g(\tau_i) > 0$ on the support $[\underline{\tau}, \overline{\tau}]$. It is informative about their true valuation for the good θ_i, their so called "ex-post type", in the following sense: conditional on τ_i, the ex-post type θ_i is distributed with the cumulative distribution function $F(\theta_i|\tau_i)$ and density $f(\theta_i|\tau_i) > 0$ on the support $[\underline{\theta}, \overline{\theta}]$. Hence, the utility of buyer i is:

$$\theta_i \cdot q_i - t_i$$

Note that the utility does not depend *directly* on whether the good was produced. The utility of the buyers is solely determined by whether they get access to the good, although production and allocation are connected in a sense that the good can only be accessed in case it was produced.

We assume first-order stochastic dominance (FOSD) for $F(\cdot|\tau_i)$, $\tau_i \in [\underline{\tau}, \overline{\tau}]$: $\partial F(\theta_i|\tau_i)/\partial \tau_i < 0 \ \forall \ \theta_i \in (\underline{\theta}, \overline{\theta})$. This captures the idea that a higher ex-ante type indicates a higher ex-post type. We denote $\tau = (\tau_1, ..., \tau_N)$ and $\mathrm{T} = [\underline{\tau}, \overline{\tau}]^N$. Analogously, $\theta = (\theta_1, ..., \theta_N)$ and $\Theta = [\underline{\theta}, \overline{\theta}]^N$. For the products of distributions we write $\bar{G}(\tau) = \prod_i G(\tau_i)$ for the ex-ante types and $\bar{F}(\theta|\tau) = \prod_i F(\theta_i|\tau_i)$ for the ex-post types.

Seller: The seller incurs fixed costs C in case she decides to produce the good regardless of how many buyers will access the good. There are no marginal costs of providing the good to a buyer. Thus, her payoff is:

$$\sum_i t_i - C \cdot m$$

Moreover, there is no constraint on how many people can consume the good. This is in contrast to a unit private good setting which would be constrained by the fact that the good can't be allocated to more than one person, i.e. $\sum_i q_i \leq 1$.

An important characteristic of the good is whether or not its production is observable to the buyers. We start by assuming that production is not observable, e.g. it is not easy for museum attendees to know whether preparations for a new exhibition are taking place. In the analysis we later argue that whether production is observable or not will not change the optimal solution for the seller.

[2] Deterministic for simplicity, but not restrictive.

Timing: Before stating the timing of the entire game, it is important to discuss when production decision takes place. We first assume that the seller has to decide whether or not to produce the good *before* the buyers learn their true valuations for the good. The seller cannot produce after true valuations are learned. This can be thought of in different but analytically identical ways. It could be the case that production is needed for the true valuations to be learned, e.g. via prototypes or trailers; or buyers could learn their valuations regardless but there is no way for the good to be produced anymore, e.g. organizing art exhibitions. In section 6 we then assume that the seller can wait until after the buyers learn their ex-post types to decide whether to produce or not.[3]

Hence, *the timing of the game* is as follows:

1. The buyers privately learn their own ex-ante types τ_i
2. The seller offers each buyer i a contract: $(m, (q_i, t_i))$
3. Each buyer i decides whether to accept or reject
4. The seller produces the good (and incurs C) or not according to the terms of the contract
5. The buyers learn their own ex-post types θ_i and do not observe whether the good was produced or not
6. Allocation payoffs are realized

We will refer to the stage between the buyers learning their τ_i and θ_i as *interim*, and the stage after the buyers learn their θ_i as *ex-post*.

Goal: The goal of this paper is to find a contract which maximizes expected profit for the seller. We begin by narrowing the class of games we need to look at in the next section.

[3] In fact, if the timing of the production decision was endogenous, this case is desirable for the seller as it leads to higher expected profits: it is always worthwhile to wait for more information about the buyers' valuations to be revealed before producing the good.

3 Dynamic Direct Mechanism

In this section we formulate the family of mechanisms that we can focus on in our search for the optimal contract. We proceed in the standard way: first, we define the dynamic direct mechanism in our setting; second, we define incentive compatibility and individual rationality to make use of the Revelation Principle.

As the production decision has to be taken at the interim stage, the seller can only make the decision at a point in time when the buyers only know their ex ante types τ_i. She cannot wait until buyers have learned their ex-post types. This is captured by m mapping only the possible ex-ante types space T onto a decision, i.e. 0 or 1.

Mechanism 1 (Dynamic Direct Mechanism)**.** *The dynamic direct mechanism consists of functions m, q_i and t_i for all i, where*

$$\begin{aligned} m &: \mathrm{T} && \rightarrow \{0,1\} \\ q_i &: \mathrm{T} \times \Theta && \rightarrow [0,1] \\ t_i &: \mathrm{T} \times \Theta && \rightarrow \mathbb{R} \end{aligned}$$

$m(\tau)$*: discrete decision of producing the good*

$q_i(\tau, \theta)$*: probability that buyer i gets access to the good*

$t_i(\tau, \theta)$*: monetary transfer of buyer i to the seller*

Since the good can only be sold if it was produced we have the following feasibility constraint.

Definition 1 (Feasibility)**.** *A mechanism is "feasible" if*

$$q_i(\tau, \theta) \leq m(\tau) \quad \forall \tau \in \mathrm{T}, \theta \in \Theta \quad \forall i \tag{FC}$$

Following the literature, we adjust the Revelation Principle by Myerson (1981) to our dynamic setting with multiple buyers which allows us to focus on dynamic direct mechanisms which induce truthful reporting by the buyers for both of their types when searching for the expected profit maximizing contract.[4] In order to define incentive compatibility (IC) and individual rationality (IR) we introduce some additional notations regarding the utility and expectations of the buyers.

The utility of the buyers is solely determined by whether they receive access to the good or not. They do not derive any direct utility from the good being produced in itself.

[4]The full formulation of the Revelation Principle in our setting can be found in the appendix.

Therefore, we define their utility as follows:

$$\theta_i \cdot q_i(\tau, \theta) - t_i(\tau, \theta)$$

Note that this utility depends on all reported τ and θ. At a point when a buyer has to report his ex-ante type, he does not know the types of other buyers. Assuming that other buyers report truthfully, he will form expectations about his allocation probability and transfer, denoted by:

$$Q_i(\tau_i, \theta_i) = \int_{\mathrm{T}_{-i}} \int_{\Theta_{-i}} q_i(\tau_i, \theta_i, \tau_{-i}, \theta_{-i}) d\bar{F}_{-i}(\theta_{-i}|\tau_{-i}) d\bar{G}_{-i}(\tau_{-i})$$

$$T_i(\tau_i, \theta_i) = \int_{\mathrm{T}_{-i}} \int_{\Theta_{-i}} t_i(\tau_i, \theta_i, \tau_{-i}, \theta_{-i}) d\bar{F}_{-i}(\theta_{-i}|\tau_{-i}) d\bar{G}_{-i}(\tau_{-i})$$

where $\bar{G}_{-i}(\tau_{-i}) = \prod_{j \in I \setminus \{i\}} G(\tau_j)$ and $\bar{F}_{-i}(\theta_{-i}|\tau_{-i}) = \prod_{j \in I \setminus \{i\}} F(\theta_j|\tau_j)$.

Using these expectations we define the following on-path expected utilities at the interim stage, i.e. between the first and the second reporting stages: $u_i(\tau_i, \theta_i) = \theta_i \cdot Q_i(\tau_i, \theta_i) - T_i(\tau_i, \theta_i)$; and at the ex-ante stage, i.e. before the first reporting stage: $U_i(\tau_i) = \int_{\underline{\theta}}^{\bar{\theta}} u_i(\tau_i, \hat{\theta}) dF(\hat{\theta}|\tau_i)$. Note that our definition at the ex-ante stage implicitly assumes truth-telling of the ex-post type. This formulation will be needed for our analysis.

We now turn to the formulation of incentive compatibility (IC) and individual rationality (IR).

Incentive Compatibility: The Revelation Principle allows us to focus on mechanisms where truth-telling for each agent about both types is optimal. However, it does not specify anything about the reporting strategies off the equilibrium path. This is reflected by our definition of incentive compatibility, which is divided into two parts:

1. At the second reporting stage, reporting the true θ_i must be optimal when the true τ_i has been reported in stage one;

2. At the first reporting stage, reporting the true τ_i must be optimal for any possible combination of alternative reports about τ_i and θ_i (possibly combining lies).

Definition 2. *A direct mechanism is "incentive-compatible" if:*

With respect to ex-post type θ_i:

$$u_i(\tau_i, \theta_i) \geq \theta_i \cdot Q_i(\tau_i, \theta_i') - T_i(\tau_i, \theta_i') \quad \forall \theta_i, \theta_i' \tag{θ-IC}$$

With respect to ex-ante type τ_i:

$$U_i(\tau_i) \geq \int_{\underline{\theta}}^{\overline{\theta}} \hat{\theta} \cdot Q_i(\tau_i', \theta_i^r(\hat{\theta})) - T_i(\tau_i', \theta_i^r(\hat{\theta})) dF(\hat{\theta}|\tau_i) \tag{τ-IC}$$

$\forall \tau_i, \tau_i', \forall \theta_i^r : [\underline{\theta}, \overline{\theta}] \to [\underline{\theta}, \overline{\theta}]$

If a direct mechanism satisfies this definition of incentive compatibility, truth-telling is indeed optimal, even though in general truth-telling might not be optimal off the equilibrium path, i.e. it might be optimal to lie about θ_i in case τ_i was already misreported. In our model truth-telling off the equilibrium path still holds true due to the agent's ex-post type being equal to their payoff type. This insight will be useful for our technical analysis.

Observability of production: An important aspect of the model is whether the production decision taken by the seller is observable to the buyers at the interim stage, namely before the second reporting stage. Observability or its absence could potentially lead to different optimal mechanisms: assuming a buyer can observe whether the good was produced or not *before* they have to report their ex-post types, they would update their belief about the other buyers' ex-ante types. This would need to be reflected in the incentive constraints. However, we prove that this has no effect on the optimal solution. We prove this in section 4.3 by solving the problem assuming unobservability and then showing that the solution also solves the problem assuming observability.

Individual Rationality: Since the contract is offered at a point in time when the buyers do not yet know their true valuations, we require their expected utility of participating in the game based only on their ex-ante types to be non-negative, motivating the following notion of individual rationality.

Definition 3. *A direct mechanism is "individual rational" if*

$$U_i(\tau_i) \geq 0 \quad \forall \tau_i \in [\underline{\tau}, \overline{\tau}] \tag{IR}$$

Having defined the game we now turn to the profit-maximization problem of the seller in the next section.

4 Optimal Mechanism

We begin by stating the seller's maximization problem. We then solve the problem and finish this section by turning to the problem of observability.

4.1 The Seller's Optimization Problem

The seller maximizes her expected revenues (expected payments collected from the buyers) net of expected costs (expected costs of producing the good) by choosing $\{m, (q_i, t_i)_{i \in I}\}$. She builds expectations about the buyers' ex-ante types τ_i and ex-post types θ_i, which are related through $F(\theta_i|\tau_i)$. The seller must set $\{m, (q_i, t_i)_{i \in I}\}$ such that incentive-compatibility and individual rationality (IR) constraints are satisfied. As discussed in section 3, due to dynamic information we introduce two types of incentive-compatibility constraints: (τ-IC) and (θ-IC). Moreover, the seller has to satisfy feasibility constraint (FC), as the good can only be offered access to if it was produced.

Thus, the optimization problem is as follows:

$$\begin{aligned} \max_{\{m,(q_i,t_i)_{i\in I}\}} \quad & \sum_i \int_{\mathrm{T}} \int_{\Theta} [t_i(\tau_i, \theta_i) - C \cdot m(\tau)] \, d\bar{F}(\theta|\tau) \, d\bar{G}(\tau) \\ \text{s.t.} \quad & (\tau\text{-IC}) \ \& \ (\theta\text{-IC}) \ \& \ (\text{IR}) \ \& \ (\text{FC}) \end{aligned} \tag{OP}$$

By following Courty & Li (2000) and adjusting the expressions for our setting with fixed costs and multiple buyers we can arrive at a reformulated problem. In particular, necessary and sufficient conditions for (θ-IC) are standard and are presented in lemma 1 (the proof is omitted).

Lemma 1. (θ-IC) *are satisfied if and only if:*

1. $\partial u_i(\tau_i, \theta_i)/\partial \theta_i = Q_i(\tau_i, \theta_i)$,
2. Q_i *is increasing in* θ_i *for every* τ_i.

From lemma 1 and the definition of $u_i(\tau_i, \theta_i)$, for every τ_i and every θ_i we have:

$$T_i(\tau_i, \theta_i) = T_i(\tau_i, \underline{\theta}) + (\theta_i Q_i(\tau_i, \theta_i) - \underline{\theta} Q_i(\tau_i, \underline{\theta})) - \int_{\underline{\theta}}^{\theta_i} Q_i(\tau_i, \hat{\theta}) d\hat{\theta} \tag{1}$$

Using analogous properties w.r.t. τ_i, we cannot find necessary *and* sufficient conditions for (τ-IC). This happens due to the the multi-dimentionality of our problem: a buyer's utility at the first reporting stage includes expectation over his utility at the second reporting stage. Nevertheless, we are able to obtain the following:

Lemma 2. *If* (θ-IC) *and* (τ-IC) *are satisfied, then we have:*

$$\partial U_i(\tau_i)/\partial \tau_i = -\int_{\underline{\theta}}^{\overline{\theta}} Q_i(\tau_i, \hat{\theta}) \frac{\partial F(\hat{\theta}|\tau_i)}{\partial \tau_i} d\hat{\theta}.$$

From lemma 2 and the definition of $U_i(\tau_i)$, for every τ_i we have:

$$\int_{\underline{\theta}}^{\overline{\theta}} T_i(\tau_i, \hat{\theta}) f(\hat{\theta}|\tau_i) d\hat{\theta} = \int_{\underline{\theta}}^{\overline{\theta}} \hat{\theta} Q_i(\tau_i, \hat{\theta}) f(\hat{\theta}|\tau_i) d\hat{\theta} + \int_{\underline{\theta}}^{\overline{\theta}} (T_i(\underline{\tau}, \hat{\theta}) - \hat{\theta} Q_i(\underline{\tau}, \hat{\theta})) f(\hat{\theta}|\underline{\tau}) d\hat{\theta} \tag{2}$$

$$+ \int_{\underline{\tau}}^{\tau_i} \int_{\underline{\theta}}^{\hat{\theta}} Q_i(\hat{\tau}, \hat{\theta}) \frac{\partial F(\hat{\theta}|\hat{\tau})}{\partial \tau_i} d\hat{\theta} d\hat{\tau}$$

In the case of FOSD, however, Q_i increasing in τ_i for every θ_i is sufficient for (τ-IC).

Lemma 3. *If Q_i is increasing in θ_i and τ_i, expected transfers $T_i(\tau_i, \theta_i)$ can be found such that* (θ-IC) *and* (τ-IC) *are satisfied.*

Due to our assumption of FOSD, $U_i(\tau_i)$ is increasing in τ_i by lemma (2), and so (IR) are satisfied if and only if $U_i(\underline{\tau}) \geq 0$.

Using standard techniques we can now restate the seller's expected profits as a function of $(m, q_i)_{i \in I}$ only:

$$\int_{\mathrm{T}} \left[\int_{\Theta} \sum_i \left[\theta_i + \frac{1 - G(\tau_i)}{g(\tau_i)} \frac{\partial F(\theta_i|\tau_i)/\partial \tau_i}{f(\theta_i|\tau_i)} \right] q_i(\tau, \theta) \, d\bar{F}(\theta|\tau) - C \cdot m(\tau) - \sum_{i \in I} U_i(\underline{\tau}) \right] d\bar{G}(\tau)$$

We then optimally set $U_i(\underline{\tau}) = 0$, as the reformulated objective is decreasing in these terms and (IR) will bind.

Proposition 1. *The objective of the optimization problem* (OP) *can be reformulated as*

$$\int_{\mathrm{T}} \left[\int_{\Theta} \sum_i \Psi(\tau_i, \theta_i) q_i(\tau, \theta) \, d\bar{F}(\theta|\tau) - C \cdot m(\tau) \right] d\bar{G}(\tau)$$

where $\Psi(\tau_i, \theta_i) = \left[\theta_i + \frac{1 - G(\tau_i)}{g(\tau_i)} \frac{\partial F(\theta_i|\tau_i)/\partial \tau_i}{f(\theta_i|\tau_i)} \right]$ *is the virtual valuation.*

We can find expected transfers such that DDM is incentive-compatible:

$$T_i(\tau_i, \theta_i) = T_{i0}(\tau_i) + \theta_i Q_i(\tau_i, \theta_i) - \int_{\underline{\theta}}^{\theta_i} Q_i(\tau_i, x) dx$$

where

$$T_{i0}(\tau_i) = T_i(\underline{\tau}, \underline{\theta}) - \underline{\theta} Q_i(\underline{\tau}, \underline{\theta}) + \int_{\underline{\tau}}^{\tau_i} \int_{\underline{\theta}}^{\overline{\theta}} Q_i(\hat{\tau}, \hat{\theta}) \frac{\partial F(\hat{\theta}|\hat{\tau})}{\partial \tau_i} d\hat{\theta} d\hat{\tau}$$

$$+ \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\hat{\theta}} [Q_i(\tau_i, x) f(\hat{\theta}|\tau_i) - Q_i(\underline{\tau}, x) f(\hat{\theta}|\underline{\tau})] dx d\hat{\theta}$$

and $T_i(\underline{\tau}, \underline{\theta})$ *is pinned down by* $U_i(\underline{\tau}) = 0$.

Note how the virtual valuation's expression reflects "informativeness measure" (the last term in the second summand) as introduced by Courty & Li (2000), since the seller has to not only extract information about the buyers' ex-ante types but also ex-post types.

4.2 Optimal Mechanism

To derive the solution of the maximization problem (OP), it is instructive to understand:

1. *Who should get the good in case it is produced:* by pointwise maximization all buyers with a positive virtual valuation ($\Psi(\tau_i, \theta_i) \geq 0$) should get the good in order to maximize the expected profit. Let I^* be the set of all buyers with a positive virtual valuation: $I^* \equiv \{i \in I : \Psi(\theta_i, \tau_i) \geq 0\}$.
2. *When should the good be produced:* produce, i.e. $m = 1$, iff

$$\int_{\Theta} \sum_{j \in I^*} \Psi(\tau_j, \theta_j) d\bar{F}(\theta|\tau) \geq C \qquad (\star)$$

As stated earlier, T_i makes the allocation rule q_i incentive-compatible if Q_i is increasing in θ_i and τ_i. Given the optimal allocation rule we derive, we ensure that Q_i is indeed increasing in both arguments by introducing the following assumption.

Assumption 1. $\Psi(\tau_i, \theta_i)$ *is increasing in* τ_i *and* θ_i.

The solution to the (OP) is as follows:

Proposition 2. *The mechanism* $\{m, (q_i, t_i)_{i \in I}\}$ *is optimal if and only if:*

$$m(\tau) = \begin{cases} 1 & \textit{if } \int_{\Theta} \sum_{j \in I*} \Psi(\tau_j, \theta_j) d\bar{F}(\theta|\tau) \geq C \\ 0 & \textit{otherwise} \end{cases}$$

$$q_i(\tau, \theta) = \begin{cases} 1 & \textit{if } \Psi(\tau_i, \theta_i) \geq 0 \textit{ and } \int_{\Theta} \sum_{j \in I^*} \Psi(\tau_j, \theta_j) d\bar{F}(\theta|\tau) \geq C \\ 0 & \textit{otherwise} \end{cases}$$

where $I^* \equiv \{i \in I : \Psi(\tau_i, \theta_i) \geq 0\}$, *and*

$$T_i(\tau_i, \theta_i) = \begin{cases} 0 & \textit{if } \tau_i < \tilde{\tau}_i^{min} \\ T_{i0}(\tau_i) & \textit{if } \tilde{\tau}_i^{min} \leq \tau_i \textit{ and } \theta_i < v(\tau_i) \\ T_{i0}(\tau_i) + v(\tau_i)H(\tau_i) & \textit{if } \tilde{\tau}_i^{min} \leq \tau_i \textit{ and } \theta_i \geq v(\tau_i) \end{cases}$$

where $v(\tau_i) = \{\theta_i | \Psi(\theta_i, \tau_i) = 0\}$ *and* $H(\tau_i)$, *defined in the proof, is the probability that given* τ_i *the ex-ante types of other buyers are great enough for* ($\star$) *to hold (i.e. for the good to be produced).*

Let $\tilde{\tau}_i(\tau_{-i})$ *be defined for a given* τ_{-i}, *as the value of* τ_i *such that the expression* ($\star$) *holds with equality, so that* $\tilde{\tau}_i^{min} \equiv max\{\underline{\tau}, \tilde{\tau}_i(\overline{\tau}...\overline{\tau})\}$ *is the minimum* τ_i *for which expression* ($\star$) *holds with equality.*

The optimal allocation rule q states that only the buyers with a positive virtual valuation should receive acces to the good. Since this is optimal *once the good is produced*, the decision of producing the good (before the buyers' true valuations are realized) depends on whether the sum of expected positive virtual valuations surpasses production costs or not.

Those potential buyers, whose ex-ante types are so low, i.e. $\tau_i < \tilde{\tau}_i^{min}$, that no matter the composition of other buyers' ex-ante types the good would not be produced (the sum of expected positive virtual valuations of all buyers would not cover production costs), pay nothing.

If a potential buyer's ex-ante type is high enough, i.e. $\tau_i \geq \tilde{\tau}_i^{min}$, so that there is a *chance* that given other buyers' ex-ante types the good could be produced, he pays $T_{i0}(\tau_i)$, when his ex-post type is too low, i.e. $\theta_i < v(\tau_i)$, with $v(\tau_i)$ being his ex-post type leading to a positive virtual valuation; or $T_{i0}(\tau_i) + v(\tau_i)H(\tau_i)$, when his ex-post type is high enough, i.e. $\theta_i \geq v(\tau_i)$, with $H(\tau_i)$ being the probability of the good being produced. Note that $T_{i0}(\tilde{\tau}_i{}^{min})$ is 0 if $\tilde{\tau}_i(\overline{\tau}...\overline{\tau}) \geq \underline{\tau}$ (see definition of T_{i0} in the proof).

Since it will be useful for the implementation in section 5, we further discuss the precise value of $T_{i0}(\tau_i)$, which depends on the buyer i's expected probability of receiving the good (see proposition 1).

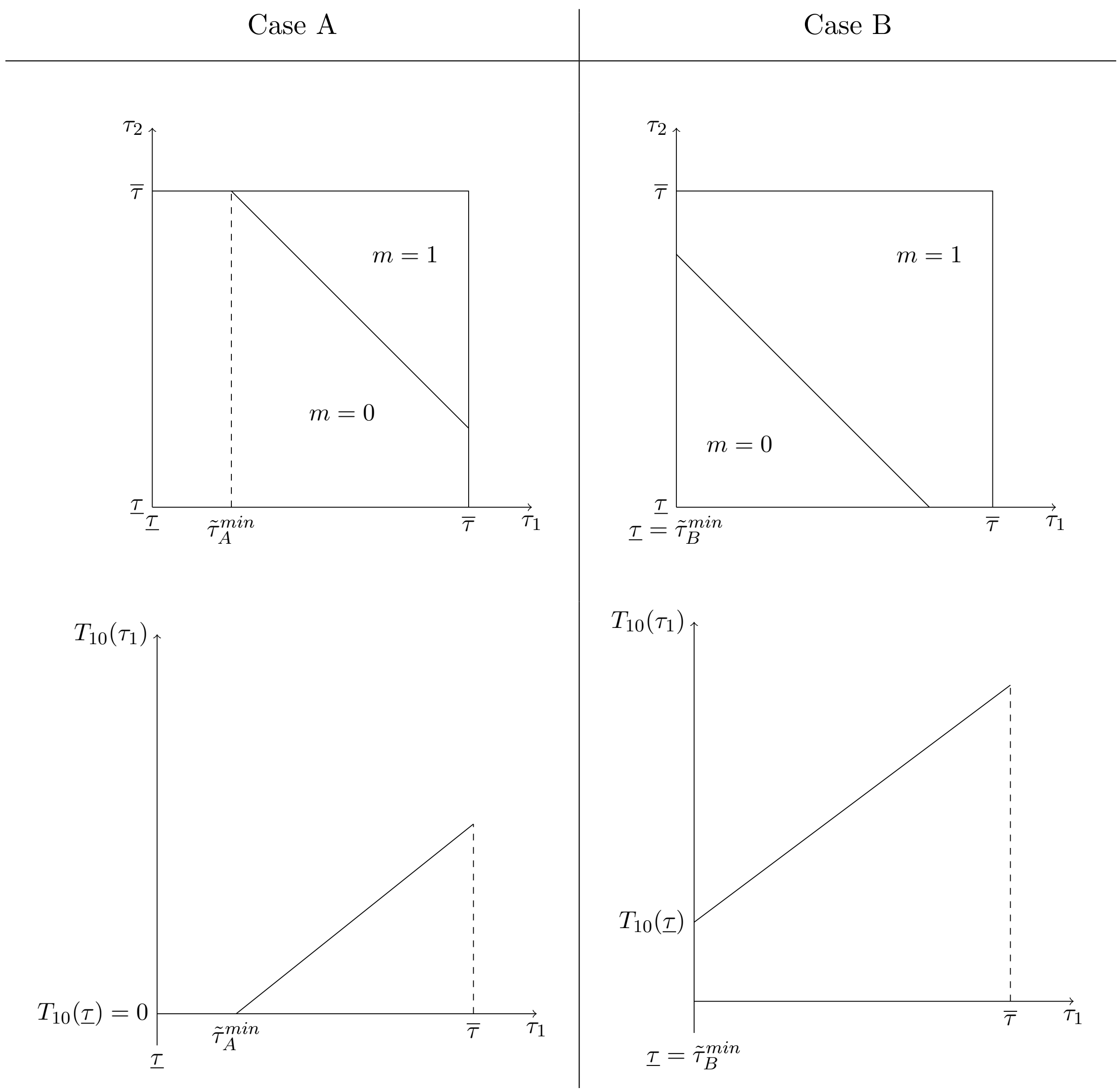

Figure 1: Comparison sketch of $T_{i0}(\tau_i)$ for a two-buyer case

We consider a two-buyer case to illustrate possible $T_{i0}(\tau_i)$ values (see figure 1):

- Case A: $T_{i0}(\tilde{\tau_i}^{min})$ is 0.
  - This happens when there are values of τ_i for which the good will never be produced (i.e. the other buyer's τ_{-i} might even be $\overline{\tau}$ and this still will not suffice for the good to be produced according to the optimal m).
  - The threshold type $\tilde{\tau_A}^{min}$ will have $T_{i0}(\tau_i) = 0$.
- Case B: $T_{i0}(\tilde{\tau_i}^{min})$ is strictly positive.
  - This happens when for all values of τ_i the good will be produced with a positive probability (i.e. the other buyer's τ_{-i} need not be $\overline{\tau}$).
  - Even the lowest type $\tilde{\tau_B}^{min} = \underline{\tau}$ will have $T_{i0}(\tau_i) > 0$

There are multiple factors determining which case will be present. One of them is the cost of production C. The higher the cost, the lower the probability that for a given τ_i the good will be produced, making case A more likely. The lower the cost, the higher is the probability that the good will be produced, making case B more likely.

So far, we have derived expected payments T_i's although the optimal DDM requires the actual payments t_i's. In fact, many different actual payments could be found (depending on the desired structure: e.g. contribution or subscription schemes, see Admati & Perry (1991)). As long as the expected payments' expressions are satisfied they all solve our optimizaion problem. We talk about this in more detail in section 5 on indirect mechanisms.

Next we show that even if the production decision is observable the same optimal mechanism as defined above applies.

4.3 Observability

As mentioned above the analytical difference between observable and unobservable production lies in the potential updating of beliefs of the buyers. In case the buyers can observe the production decision and know the production rule m, they will potentially change their belief over the other buyers' ex-ante type at the second reporting stage. As we are looking at Bayesian Nash equilibrium changing beliefs of a buyer over the other buyers' true type might lead to a different optimal solution for the seller. To illustrate the problem, we give a short example in binary types.

Example 1. *Assume binary ex-ante type $\{l, h\}$. Let the production decision be:*

$$m(\tau_1, \tau_2) = \begin{cases} 1 & \textit{if } (\tau_1, \tau_2) = (h, h) \textit{ or } (\tau_1, \tau_2) = (l, l) \\ 0 & \textit{otherwise} \end{cases}$$

If production is:

- *unobservable: buyer 1's belief stays equal to his prior after production as he cannot observe the production decision m.*
- *observable: buyer 1 knows:*
  - *if the good is produced: $\tau_1 = \tau_2$*
  - *if the good is not produced: $\tau_1 \neq \tau_2$*

Since the expectation of the true type is contingent on τ buyer 1 now has a different expectation over buyer 2's true valuation.

Impact: Second stage beliefs change, second stage IC changes and, therefore, expectations in first period change.

The incentive constraints have to account for this, leading to different constraints of the optimization problem. We denote the original problem without observability P_{unobs} and the problem with observability P_{obs}. Proposition 3 states that we do not have to worry about this aspect of production as the optimal solution is the same in both cases.

Proposition 3. *Solution to P_{unobs} is optimal for P_{obs}.*

We give a short sketch of the two-step proof (the full proof can be found in the Appendix).

First, we show that the problem with observability is more restrictive for the seller as the buyers have weakly more information at the second reporting stage. And so every mechanism that that satisfies the constraints of P_{obs} also satisfies P_{unobs}.

Second, we show that the mechanism from proposition 2, optimal under P_{unobs}, also satisfies the constraints of P_{obs}. This is intuitive by looking at the mechanism's second reporting stage where allocation and transfer rules depend only the buyer's own type and not on the types of others. Additional information about the buyers' ex-ante type does not benefit a buyer at this stage. He could gain complete information about the other buyers' type at the second reporting stage and it would have no effect on his behaviour. Hence, the mechanism from proposition 2 can be implemented in the P_{obs} and will be optimal.

5 Indirect Mechanisms / Implementation

The optimal mechanism found above only defined expected payments / ex-ante transfer rule. Given the expected payments T_i's, it is possible to come up with many distinctive actual payments schemes t_i's. As long as the conditions for the optimal mechanism from above are satisfied / the ex-ante transfer rule is satisfied, all transfer rules will yield the same expected profit to the seller.

We focus on the case where consumers can acquire buy-option contracts without reimbursement in case the good is not produced. The seller offers a menu of price pairs, consisting of an upfront payment and an exercise price. If enough money is collected through the upfront payments, the good will be produced. In case it is produced, everyone who payed an upfront payment gets the *option* to buy the good at the corresponding prespecified exercise price. There is no reimbursement of the upfront payment in case the good is not produced (similar to contribution schemes, see Admati & Perry (1991)).

Focusing on contribution schemes (no reimbursement) has two advantages. First, understanding how the expected payments can be translated into actual payments is quite straightforward in this case. Second, it is a procedure that resembles payment schemes observed in practice, as we will illustrate with two examples below. Proposition 4 formally defines such a contribution scheme.As it can be noticed, it resembles Cornelli (1996) result but is adjusted to the dynamic nature of our setup.

Proposition 4. *The following selling procedure is optimal assuming that $G(\tau_i)$ and $F(\theta_i|\tau_i)$ are such that we obtain strictly increasing T_i.*[5] *The seller submits a menu of possible price pairs $\{(T_i, p_i)\}$ with a minimum price T^* for T_i's. The buyers who choose not to pay anything never obtain the good. Those who choose a price pair from the menu pay T_i while only knowing their τ_i. In case the good is not produced, i.e. $\sum \xi_i(T_i) < C$ (where $\xi_i(T_i)$ is defined in the proof) they will not get the good and will not be reimbursed. In case the good is produced they will get the option to buy the good for the prespecified price p_i. If they choose not to buy, they will not get the upfront payment reimbursed.*

The price menu in this proposition describes combinations of two payments which we will refer to as the upfront payments (T) and the exercise prices (p). For the upfront payment, there exists a non-negative minimum price T^*. This means that some ex-ante types might be excluded from consumption even though there is a positive probability that they end up with a high true valuation.

The buyers also understand that choosing their upfront payments impacts the probability of whether the good will be produced. In fact, this probability not only depends on the sum of the paid upfront payments but also on their composition. By making the ξ function concave resp. convex, the seller can put more importance on lower resp. higher upfront payments in order to make lower resp. higher

[5] This assumption is satisfied for many distributions (see proof of proposition 4 for examples).

ex-ante types feel more pivotal in the production decision. This could be done by announcing that contributions below resp. above a certain threshold will be matched by a third party. The optimal ξ will be determined by the distributions G and F.

If a buyer chooses to pay the upfront payment, he will get the option of buying the good in case it is produced for the price p_i. This exercise price is determined as the threshold valuation that will lead to a non-negative virtual valuation given the ex-ante type τ_i. Note that a higher τ_i leads to a lower threshold type θ_i (due to assumption 1) and consequently lowers the exercise price p_i.

Understanding this makes it clear that the upfront payment in our setting reflects two dynamics: the higher the chosen T_i, the higher is the probability of production; and the lower is the exercise price for buyer i. These two roles of the upfront payment are not separable.

Before turning to two examples, we note the following about the minimum price T^*. As analysed in section 4.2, the probability that the expected sum of virtual valuations covers the cost is crucial for the determination of $T_{i0}(\tilde{\tau_i}^{min})$, which is exactly T^* (see the proof of propostion 4). This probability is, for instance, decreasing in C and increasing in N. Therefore, the harder it is to cover the cost (high C, low N), the lower is the minimum price T^* (the minimum price can even be equal 0). We now discuss the two examples: private members clubs and museum memberships.

Example 1: Private members clubs

Private members clubs offer exclusive access to restaurants, hotels and events to their members. In particular, if the membership fee is not paid, there is no opportunity to access facilities later on, similar to the minimum upfront payment being strictly positive. Provision of such club goods inherently comes with fixed costs: building and maintaining new facilities, hiring the staff in advance, etc.

There are many different clubs with different monetization schemes. We consider the Soho House which is a private members club with 30 hotels worldwide. As of writing this paper the Soho house offers three types of membership. Each tier offers benefits as the preceding one plus additional benefits. The tiers are the following: "Soho Friends", "Cities Without Houses", and "Soho House". If you want to enter one of the Soho House hotels you need to be in tiers "Cities Without Houses" or "Soho House". If you want to enter as a "Soho Friend", you need to book a bedroom for the night. Non-members cannot enter at all[6].

Meaning, the membership fee in this context could be interpreted using the two dynamics explained above: first, those who pay higher membership fees (higher upfront payments) might value club facilities more and are interested in the maintenance and possible expansion to new locations (want to increase the probability of the good being produced); and second, they might want to lower the prices for additional perks (want to decrease their exercise prices).

[6]See: https://www.sohohouse.com/terms-and-policies/soho-friends-terms-and-conditions

Example 2: Museum memberships

We hereby consider membership possibilities at the Metropolitan Museum of Art (the Met). Similar to the Soho House, the Met offers different membership tracks: "Membership", "Patron Circles", "The Apollo Circle" and "The Met Family", all associated with different sets of benefits, such as free admissions to the upcoming exhibitions and certain events[7]. Organizing exhibitions comes at high fixed costs and so supporters would like to make sure that the offers always take place by contributing more via their membership fees. On top of this, higher membership fees come with lower exercise prices for when exhibitions actually take place (sometimes in form of free admissions). Hence, high upfront payment serves a dual purpose in this example as well.

The only difference to the Soho House scheme is that in case of the Met the minimum upfront payment (membership fee) could be zero: one does not need to be a member or an official supporter at the Met to attend their exhibitions, as these are open to the public. In that case, however, one has to pay high entrance fees. This could be explained by how the minimum price is determined as mentioned earlier: if it is harder to cover the costs of production, there might not be a positive minimum upfront payment.

[7]See: https://www.metmuseum.org/join-and-give/membership

6 Extension: Ex post production

In this section we will examine the case where the production decision can be taken after the buyers have learned their true valuations, at the ex-post stage. For this we will need to introduce a new timing, adapt some definitions and solve the new optimization problem. We will interpret and find an implementation for the new optimal direct mechanism and compare it to the one found above in the interim timing.

The only difference to the timing in the main model is that the production decision now takes place after the buyers have learned their ex-post types. This timing can again be thought of as exogenously given but as mentioned above it is also the optimal one if the seller *can* choose the timing herself. The following is timing of the game with an ex-post production:

1. The buyers privately learn their own ex-ante types τ_i
2. The seller offers a contract: $\{m, (q_i, t_i)_{i \in I}\}$
3. Each buyer i decides whether to accept or reject
4. The buyers learn their ex-post types θ_i
5. The seller produces the good (and incurs C) or not according to the terms of the contract
6. Allocation payoffs are realized

As the production decision now depends on both reporting stages we define a direct mechanism for this timing as follows.

Mechanism 2 (Ex post production)**.** *The dynamic direct mechanism in the ex-post production timing consists of functions m, q_i and t_i for all i, where*

$$\begin{aligned} m :& \mathrm{T} \times \Theta && \rightarrow \{0, 1\} \\ q_i :& \mathrm{T} \times \Theta && \rightarrow [0, 1] \\ t_i :& \mathrm{T} \times \Theta && \rightarrow \mathbb{R} \end{aligned}$$

$m(\tau, \theta)$*: discrete decision of producing the good*

$q_i(\tau, \theta)$*: probability that buyer i gets the good*

$t_i(\tau, \theta)$*: monetary transfer of buyer i*

Note that the m function now also takes Θ as an input to determine the production decision. Reflecting this idea we define the feasibility constraint for this timing as follows.

Definition 4 (Ex post feasibility)**.** *A mechanism in the ex-post production timing is "feasible" if*

$$q_i(\tau,\theta) \leq m(\tau,\theta) \quad \forall \tau \in \mathrm{T}, \theta \in \Theta \tag{FC$^{\text{exp}}$}$$

Observability is not relevant in this context since production happens after the last reporting stage of the agents. Whether or not the production is observable cannot have any impact on the buyers reporting strategies.

The optimization problem for the seller is the same as in the main model except for the changes in the m function and the feasibility constraint. Using standard techniques, we can again rewrite seller's expected profits in terms of the virtual valuations of the buyers. Since the incentive compatibility and individual rationality constraints stay the same, we have the same expression for the virtual valuation and the expected transfer rule (the proof is omitted).

Proposition 5. *The objective for the optimization problem for the ex-post production timing can be reformulated as*

$$\max_{\{m,(q_i)_{i\in I}\}} \quad \int_{\mathrm{T}}\int_{\Theta}\left[\sum_i \Psi(\tau_i,\theta_i)q_i(\tau,\theta) - C\cdot m(\tau,\theta)\right] d\bar{F}(\theta|\tau)\, d\bar{G}(\tau)$$

where $\Psi(\tau_i,\theta_i) = \left[\theta_i + \frac{1-G(\tau_i)}{g(\tau_i)}\frac{\partial F(\theta_i|\tau_i)/\partial\tau_i}{f(\theta_i|\tau_i)}\right]$ *is the virtual valuation.*

We can find expected transfers such that DDM is incentive-compatible:

$$T_i(\tau_i,\theta_i) = T_{i0}(\tau_i) + \theta_i Q_i(\tau_i,\theta_i) - \int_{\underline{\theta}}^{\theta_i} Q_i(\tau_i,x)dx$$

where

$$T_{i0}(\tau_i) = T_i(\underline{\tau},\underline{\theta}) - \underline{\theta}Q_i(\underline{\tau},\underline{\theta}) + \int_{\underline{\tau}}^{\tau_i}\int_{\underline{\theta}}^{\overline{\theta}} Q_i(\hat{\tau},\hat{\theta})\frac{\partial F(\hat{\theta}|\hat{\tau})}{\partial \tau_i}d\hat{\theta}d\hat{\tau}$$
$$+ \int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\theta}}^{\hat{\theta}}[Q_i(\tau_i,x)f(\hat{\theta}|\tau_i) - Q_i(\underline{\tau},x)f(\hat{\theta}|\underline{\tau})]dxd\hat{\theta}$$

and $T_i(\underline{\tau},\underline{\theta})$ *is pinned down by* $U_i(\underline{\tau}) = 0$.

We arrive at the optimal mechanism by deciding who gets access to the good in case it is produced (by pointwise maximization, all buyers with a positive virtual valuation, i.e. $\Psi(\tau_i, \theta_i) \geq 0$, should get the good in order to maximize the expected profit); and when the good should be produced ($m = 1$ iff $\sum_{j \in I^*} \Psi(\tau_j, \theta_j) \geq C$, where $I^* \equiv \{i \in I : \Psi(\theta_i, \tau_i) \geq 0\}$). Assumption 1 applies.

Proposition 6. *For the timing of ex-post production, the mechansim* $\{m, (q_i, t_i)_{i \in I}\}$ *is optimal if and only if:*

$$m(\tau, \theta) = \begin{cases} 1 & \text{if } \sum_{i \in I^*} \Psi(\tau_i, \theta_i) \geq C \\ 0 & \text{otherwise} \end{cases}$$

$$q_i(\tau, \theta) = \begin{cases} 1 & \text{if } \Psi(\tau_i, \theta_i) \geq 0 \text{ and } \sum_{j \in I^*} \Psi(\tau_j, \theta_j) \geq C \\ 0 & \text{otherwise} \end{cases}$$

where $I^* \equiv \{i \in I : \Psi(\tau_i, \theta_i) \geq 0\}$ *and*

$$T_i(\tau_i, \theta_i) = \begin{cases} T_{i0}(\tau_i) & \text{if } \theta_i < v(\tau_i) \\ T_{i0}(\tau_i) + v(\tau_i) + y(\tau_i, \theta_i) & \text{if } v(\tau_i) \leq \theta_i < V(\tau_i) \\ T_{i0}(\tau_i) + v(\tau_i) + Y(\tau_i) & \text{if } \theta_i \geq V(\tau_i) \end{cases}$$

where $T_{i0}(\tau_i)$*,* $y(\tau_i, \theta_i)$ *and* $Y(\tau_i)$ *are defined in the proof,* $v(\tau_i) = \{\theta_i | \Psi(\theta_i, \tau_i) = 0\}$ *and* $V(\tau_i) = \{\theta_i | \Psi(\theta_i, \tau_i) = C\}$*.*

Since it is optimal to give access to the good to the buyers with positive virtual valuations once the good is produced, the decision of producing the good depends on whether the sum of positive virtual valuations surpasses production costs or not. This is the main difference to the interim production case, where the sum of *expected* positive virtual valuations mattered for the production decision.

Those potential buyers, whose ex-post types turn out to be too low, i.e. $\theta_i < v(\tau_i)$, pay $T_{i0}(\tau_i)$. This payment is strictly positive, except for when $\tau_i = \underline{\tau}$, then it is 0 (see T_{i0} definition in the proof).

If a potential buyer's ex-post type turns out to be high enough, i.e. $v(\tau_i) \leq \theta_i < V(\tau_i)$, so that he should get access to the good in case it is produced, but not high enough for him to cover the cost alone, he pays $T_{i0}(\tau_i) + v(\tau_i) + y(\tau_i, \theta_i)$, where $y(\tau_i, \theta_i)$ is increasing in θ_i. If a potential buyer's ex-post type turns out to so high, that this buyer's contribution can cover production costs alone, i.e. $\theta_i \geq V(\tau_i)$, he pays $T_{i0}(\tau_i) + v(\tau_i) + Y(\tau_i)$. In contrast with the previous case, this buyer's payment no longer depends on θ_i: every $\theta_i \geq V(\tau_i)$ will pay the same expected amount.

With this the optimal direct mechanism for ex-post production is defined (DDM requires actual payments t_i's, many such payments could be found as long as expected payments' expressions T_i's we derived are satisfied). We now turn to implementation. Again we focus on the case where consumers do not get reimbursed in case the good is not produced (similar to contribution schemes, see Admati & Perry (1991)), however, proposed payment scheme differs from the one in propostion 4 substantially due to the difference in the timing of the game.

We focus on the case where consumers can acquire "contribute-option" contracts with the possibility to contribute to the production of the good once their true valuations are realized and get access to the good if it is produced. The seller offers a menu of price triplets, consisting of an upfront payment and a suggested interval for contributions. If enough money is collected through the upfront payments and the contributions, the good will be produced. In case it is produced, everyone who has contributed gets access to the good. There is no reimbursement of either the upfront payment or the contribution in case the good is not produced.

Proposition 7. *The following selling procedure is optimal assuming that $G(\tau_i)$ and $F(\theta_i|\tau_i)$ are such that we obtain strictly increasing T_i.*[8] *The monopolist submits a menu of possible price triplets $\{(T_i, p_i^{min}, p_i^{max})\}$. The consumers who choose not to pay anything never obtain the good. Those who choose a price triplet from the menu pay T_i while only knowing their τ. Then they learn their θ_i and can choose to contribute any amount p_i between p_i^{min} and p_i^{max}, if they paid T_i. In case $\sum \chi_i(T_i, p_i) < C$ (where $\chi_i(T_i, p_i)$ is defined in the proof) the good is not produced, no consumer receives the good and there are no reimbursements. In case $\sum \chi_i(T_i, p_i) \geq C$ the good is produced and every consumer who contributed at least p_i^{min} receives the good.*

The price menu in this proposition describes combinations of two payments which we will refer to as the upfront payments (T) and contributions (p), with the latter falling within the intervals (p^{min}, p^{max}). If a buyer chooses to pay the upfront payment, he will get the option to contribute p_i after he learns his ex-post type. He will need to contribute at least p_i^{min} and will not want to contribute more than p_i^{max}: p_i^{max} is defined for the case that the buyer's virtual valuation is so high, that he can cover production costs alone, contributing above p_i^{max} does not increase the chance of the good being produced. Moreover, a higher upfront payment T_i allows to lower the minimum required contribution p_i^{min} after θ_i is learned.

The buyers understand that choosing their upfront payments and contributions impacts the probability of whether the good will be produced. In fact, this probability not only depends on the sum of the paid upfront payments and contributions but also on their composition. The optimal χ will be determined by the distributions G and F.

The proposed payment scheme includes the "pay-as-much-as-you-want" stage: after the upfront payments are collected, those buyers who have paid are asked to contribute as much as they want

[8] This assumption is satisfied for many distributions (see proof of proposition 7 for examples).

within a prespecified interval. This happens after they have learned their true valuations, which coincides with the timing of the production decision. The more they contribute at this stage, the higher are the chances of production. In contrast, if we recall the interim production case, we can see that these were only the higher upfront payments (see proposition 4) which lead to a higher chance of producing the good.

Concluding Remarks

We have derived the optimal dynamic direct mechanisms for interim and ex post production timings. We characterized payment schemes (buy-option and "contribute-option" contracts) implementing the optimal production and selling procedures and provided examples for interim production. We discussed how production timing influences profit-maximizing pricing schemes.

Further aspects could make our setting even more realistic but at the same time might complicate the analysis. When it comes to non-rival but excludable goods, the monopolist might naturally face a capacity constraint (Ely *et al.* (2017) or Gale & Holmes (1993)), from which we have abstracted so far. If not all potential buyers can be served, one could derive mechanisms which e.g. define access priority to the good for the buyers based on their contributions, thus combining peak-load pricing literature with sequential screening.

We have also assumed away any cash constraints by the seller. In case of interim production timing, however, it could be interesting to see how the optimal mechanism changes if we want to make sure that the seller does not make losses when in expectation fixed costs are covered, and so the good is produced, but actual valuations in the next stage turn out to be too low.

Finally, one could question our set-up choices at the beginning and, for instance, consider alternative ways to model dynamic information and / or signal structure. Especially in the section where we discuss the influence of observability of production on the optimal direct mechanism, one could consider correlated valuations of the buyers and study whether or not obersavability of production remains irrelevant as we claim in the case of uncorrelated valuations.

References

Admati, Anat R, & Perry, Motti. 1991. Joint projects without commitment. *The Review of Economic Studies*, **58**, 259–276.

Barbieri, Stefano, & Malueg, David. 2010. Profit-Maximizing Sale of a Discrete Public Good via the Subscription Game in Private-Information Environments. *The B.E. Journal of Theoretical Economics*, **10**, 1–31.

Battaglini, Marco. 2005. Long-term contracting with Markovian consumers. *American Economic Review*, **95**(3), 637–658.

Brennan, Geoffrey, & Walsh, Cliff. 1985. Private Markets in (Excludable) Public Goods: A Reexamination. *The Quarterly Journal of Economics*, **100**, 811–819.

Cornelli, Francesca. 1996. Optimal selling procedures with fixed costs. *Journal of Economic Theory*, **71**(1), 1–30.

Courty, Pascal, & Li, Hao. 2000. Sequential Screening. *The Review of Economic Studies*, **67**, 697–717.

Ellman, Matthew, & Hurkens, Sjaak. 2019. Optimal crowdfunding design. *Journal of Economic Theory*, **184**, 104939.

Ely, Jeffrey C, Garrett, Daniel F, & Hinnosaar, Toomas. 2017. Overbooking. *Journal of the European Economic Association*, **15**(6), 1258–1301.

Eső, Péter, & Szentes, Balazs. 2007. Optimal information disclosure in auctions and the handicap auction. *The Review of Economic Studies*, **74**(3), 705–731.

Gale, Ian L, & Holmes, Thomas J. 1993. Advance-purchase discounts and monopoly allocation of capacity. *The American Economic Review*, 135–146.

Menezes, Flavio M, Monteiro, Paulo K., & Temimi, Akram. 2001. Private provision of discrete public goods with incomplete information. *Journal of Mathematical Economics*, **35**, 493–514.

Myerson, Roger B. 1981. Optimal auction design. *Mathematics of operations research*, **6**(1), 58–73.

Pavan, Alessandro, Segal, Ilya, & Toikka, Juuso. 2014. Dynamic mechanism design: A myersonian approach. *Econometrica*, **82**(2), 601–653.

Schmitz, Patrick W. 1997. Monopolistic provision of excludable public goods under private information. *Public Finance*, **52**(1), 89–101.

Vega-Redondo, Fernando. 1995. Public subscription and private contributions. *Social Choice and Welfare*, **12**, 193–212.

Appendix / Proofs

Revelation Principle

In our setting a dynamic mechanism is any arbitrary game Γ in which for each buyer i the strategy γ_i is the mapping from his type in the action space defined by Γ. The game could be designed arbitrarily complex and have arbitrarily many reporting stages. In contrast, a *direct* dynamic mechanism in our setting induces only two reporting stages: first each buyer reports his ex-ante type. After the buyers have learned their true valuations, there is a second reporting stage where where given his true ex-ante type, true ex-post type and reported ex-ante type, each buyer reports his ex-post type. We denote the respective strategies with $\sigma_i : \mathrm{T}_i \to \mathrm{T}_i$ and $\Sigma_i : \mathrm{T}_i \times \Theta_i \times \mathrm{T}_i \to \Theta_i$.[9]

Revelation Principle. *For every dynamic mechanism Γ and Bayesian Nash equilibrium γ of Γ, there exists a direct mechanism Γ' and Bayesian Nash equilibrium $\gamma' = (\sigma, \Sigma)$ such that*

(i) For every τ_i and every θ_i, the strategy vector γ' satisfies

$\sigma_i(\tau_i) = \tau_i$
$\Sigma_i(\tau_i, \theta_i, \tau_i) = \theta_i$

that is, γ' prescribes telling the truth about τ_i and, after the buyer reported τ_i truthfully, telling the truth about θ_i

(ii) For every type vector (τ, θ), the distribution over outcomes that result under Γ if the agents play γ is the same as the distribution over outcomes that result under Γ' if the agents play γ', and the expected value of the transfer payments that result under Γ if the agents play γ is the same as the transfer payments that result under Γ' if the agents play γ'.

Proof of Lemma 2.

First, we show that $U_i(\tau_i)$ is increasing in τ_i if DDM is incentive-compatible. For that, we can use lemma 1 and apply intergration by parts:

$$\hat{U}_i(\tau_i'|\tau_i) \equiv \int_{\underline{\theta}}^{\overline{\theta}} u_i(\tau_i', \hat{\theta}) f(\hat{\theta}|\tau_i) d\hat{\theta} = \int_{\underline{\theta}}^{\overline{\theta}} Q_i(\tau_i', \hat{\theta}) d\hat{\theta} - \int_{\underline{\theta}}^{\overline{\theta}} Q_i(\tau_i', \hat{\theta}) F(\hat{\theta}|\tau_i) d\hat{\theta}$$

$$= \int_{\underline{\theta}}^{\overline{\theta}} Q_i(\tau_i', \hat{\theta}) \left[1 - F(\hat{\theta}|\tau_i)\right] d\hat{\theta}$$

[9] In case the production is observable we would need Σ_i to also depend on the observation into the report, i.e. $\Sigma_i : \mathrm{T}_i \times \Theta_i \times \mathrm{T}_i \times \{0,1\} \to \Theta_i$. The presented revelation principle would need to account for this. The argument, however, still holds.

For $\tau_i' > \tau_i$ we have:

$$U_i(\tau_i') - U_i(\tau_i) \overset{(\tau\text{-IC})}{\geq} \hat{U}_i(\tau_i|\tau_i') - \hat{U}_i(\tau_i|\tau_i) = \int_{\underline{\theta}}^{\overline{\theta}} Q_i(\tau_i, \hat{\theta}) \left[F(\hat{\theta}|\tau_i) - F(\hat{\theta}|\tau_i')\right] d\hat{\theta} \overset{\text{FOSD}}{\geq} 0$$

Hence, we can differentiate w.r.t. τ_i and by exchanging the order of differentiation and integration we get:

$$\frac{\partial \hat{U}_i(\tau_i'|\tau_i)}{\partial \tau_i} = \frac{\partial}{\partial \tau_i} \int_{\underline{\theta}}^{\overline{\theta}} Q_i(\tau_i', \hat{\theta}) \left[1 - F(\hat{\theta}|\tau_i)\right] d\hat{\theta} = -\int_{\underline{\theta}}^{\overline{\theta}} Q_i(\tau_i, \hat{\theta}) \frac{\partial F(\hat{\theta}|\tau_i)}{\partial \tau_i} d\hat{\theta}$$

□

Proof of Lemma 3.

We first show that if (θ-IC) and

$$U_i(\tau_i) \geq \hat{U}_i(\tau_i'|\tau_i) \text{ for all } \tau_i, \tau_i' \in [\underline{\tau}, \overline{\tau}]$$

are satisfied, then so are (τ-IC).

From (θ-IC), for any $\theta_i^r : [\underline{\theta}, \overline{\theta}] \to [\underline{\theta}, \overline{\theta}]$ we have:

$$\int_{\underline{\theta}}^{\overline{\theta}} \hat{\theta} \cdot Q_i(\tau_i', \hat{\theta}) - T_i(\tau_i', \hat{\theta}) dF(\hat{\theta}|\tau_i) \geq \int_{\underline{\theta}}^{\overline{\theta}} \hat{\theta} \cdot Q_i(\tau_i', \theta_i^r(\hat{\theta})) - T_i(\tau_i', \theta_i^r(\hat{\theta})) dF(\hat{\theta}|\tau_i)$$

Thus, together with $U_i(\tau_i) \geq \hat{U}_i(\tau_i'|\tau_i) = \int_{\underline{\theta}}^{\overline{\theta}} \hat{\theta} \cdot Q_i(\tau_i', \hat{\theta}) - T_i(\tau_i', \hat{\theta}) dF(\hat{\theta}|\tau_i)$, we obtain ($\tau$-IC).

We can now show that if $Q_i(\tau_i, \theta_i)$ is increasing in both arguments, then we can find $T_i(\tau_i, \theta_i)$ such that DDM is incentive-compatible.

We can define $T_i(\tau_i, \theta_i)$ using equations (1) and (2) together, and by substituting out relevant terms, we obtain:

$$T_i(\tau_i, \theta_i) = T_{i0}(\tau_i) + \theta_i Q_i(\tau_i, \theta_i) - \int_{\underline{\theta}}^{\theta_i} Q_i(\tau_i, x) dx$$

where

$$T_{i0}(\tau_i) = T_i(\underline{\tau}, \underline{\theta}) - \underline{\theta} Q_i(\underline{\tau}, \underline{\theta}) + \int_{\underline{\tau}}^{\tau_i} \int_{\underline{\theta}}^{\overline{\theta}} Q_i(\hat{\tau}, \hat{\theta}) \frac{\partial F(\hat{\theta}|\hat{\tau})}{\partial \tau_i} d\hat{\theta} d\hat{\tau}$$
$$+ \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\hat{\theta}} [Q_i(\tau_i, x) f(\hat{\theta}|\tau_i) - Q_i(\underline{\tau}, x) f(\hat{\theta}|\underline{\tau})] dx d\hat{\theta}$$

With this $T_i(\tau_i, \theta_i)$ and $Q_i(\tau_i, \theta_i)$ increasing in θ_i, by lemma 1 (θ-IC) holds. It remains to verify that $U_i(\tau_i) \geq \hat{U}_i(\tau_i'|\tau_i)$:

$$U_i(\tau_i) - \hat{U}_i(\tau_i'|\tau_i) = \int_{\tau_i'}^{\tau_i} \left[\frac{\partial U_i(\hat{\tau})}{\partial \tau_i} - \frac{\partial \hat{U}_i(\tau_i'|\hat{\tau})}{\partial \tau_i} \right] d\hat{\tau} \overset{\text{lemma 2}}{=} \int_{\tau_i'}^{\tau_i} \int_{\underline{\theta}}^{\overline{\theta}} \left[Q_i(\tau_i', \hat{\theta}) - Q_i(\hat{\tau}, \hat{\theta}) \right] \frac{\partial F(\hat{\theta}|\hat{\tau})}{\partial \tau_i} d\hat{\theta} d\hat{\tau}$$

This expression is non-negative due to FOSD and $Q_i(\tau_i, \theta_i)$ being increasing in τ_i.

□

Proof of Proposition 1.

This follows immediately from equations (1) and (2), leaving one degree of freedom, $T_i(\underline{\tau}, \underline{\theta})$. This expected payment is optimally pinned down by $U_i(\underline{\tau})$.

□

Proof of Proposition 2.

Allocation rule: Given that the good is produced it is easy to see that only buyers with a positive virtual valuation should receive the good. Hence:

$$q_i(\tau, \theta) = \begin{cases} 1 & \text{if } \Psi(\tau_i, \theta_i) \geq 0 \text{ and } m(\tau) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus whether the good is produced or not should be decided upon whether the expected sum of the positive virtual valuations surpasses the cost of production:

$$m(\tau) = \begin{cases} 1 & \text{if } \int_{\Theta} \sum_{i \in I} \Psi(\tau_i, \theta_i) q_i(\tau, \theta) d\bar{F}(\theta|\tau) \geq C \\ 0 & \text{otherwise} \end{cases}$$

Transfer Rule: By proposition 1 we know it must hold that:

$$T_i(\tau_i, \theta_i) = T_{i0}(\tau_i) + \theta_i Q_i(\tau_i, \theta_i) - \int_{\underline{\theta}}^{\theta_i} Q_i(\tau_i, x) dx$$

where

$$T_{i0}(\tau_i) = T_i(\underline{\tau}, \underline{\theta}) - \underline{\theta} Q_i(\underline{\tau}, \underline{\theta}) + \int_{\underline{\tau}}^{\tau_i} \int_{\underline{\theta}}^{\overline{\theta}} Q_i(\hat{\tau}, \hat{\theta}) \frac{\partial F(\hat{\theta}|\hat{\tau})}{\partial \tau_i} d\hat{\theta} d\hat{\tau} + \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\hat{\theta}} [Q_i(\tau_i, x) f(\hat{\theta}|\tau_i) - Q_i(\underline{\tau}, x) f(\hat{\theta}|\underline{\tau})] dx d\hat{\theta}$$

and $T_i(\underline{\tau}, \underline{\theta})$ is pinned down by $U_i(\underline{\tau}) = 0$.

Insert the optimal allocation into the expression for $T_i(\tau_i, \theta_i)$. We have:

1. For $\tau_i < \tilde{\tau}_i^{min}$, $q_i(\tau, \theta) = 0 \ \forall \theta_i$, hence:

$$T_i(\tau_i, \theta_i) = T_{i0}(\tau_i) \overset{q_i(\tau,\theta)=0 \ \forall\theta_i}{=} T_i(\underline{\tau}, \underline{\theta}) \overset{U_i(\underline{\tau})=0}{\equiv} 0$$

2. For $\tau_i \geq \tilde{\tau}_i^{min}$ and $\theta_i < v(\tau_i)$, $q_i(\tau, \theta) = 0$, hence:

$$T_i(\tau_i, \theta_i) = T_{i0}(\tau_i)$$

3. $\tau_i \geq \tilde{\tau}_i^{min}$ and $\theta_i \geq v(\tau_i)$, $q_i(\tau, \theta) = 1$: For this case we let $\tilde{\tau}_N(\tau_{-N})$ be the minimum τ of buyer N given τ_{-N} for which the good is produced. We first define the expectation of buyer i to receive the good if he is of type (τ_i, θ_i):

$$\begin{aligned}
Q_i(\tau_i, \theta_i) &= \int_{\mathrm{T}_{-i}} \int_{\Theta_{-i}} q_i(\tau, \theta) d\bar{F}_{-i}(\theta_{-i}|\tau_{-i}) d\bar{G}_{-i}(\tau_{-i}) \\
&= \int_{\mathrm{T}_{-i-N}} \int_{\underline{\tau}}^{\overline{\tau}} \int_{\Theta_{-i}} q_i(\tau, \theta) d\bar{F}_{-i}(\theta_{-i}|\tau_{-i}) dG(\tau_N) d\bar{G}_{-i-N}(\tau_{-i-N}) \\
&= \int_{\mathrm{T}_{-i-N}} \int_{\tilde{\tau}_N(\tau_{-N})}^{\overline{\tau}} 1 \ dG(\tau_N) d\bar{G}_{-i-N}(\tau_{-i-N}) \\
&= \int_{\mathrm{T}_{-i-N}} (1 - G(\tilde{\tau}_N(\tau_{-N}))) d\bar{G}_{-i-N}(\tau_{-i-N}) \equiv H(\tau_i)
\end{aligned}$$

   We can now define the expected payment for this type by using the expected allocation from above.

$$\begin{aligned}
T_i(\tau_i, \theta_i) &= T_{i0}(\tau_i) + \theta_i Q_i(\tau_i, \theta_i) - \int_{\underline{\theta}}^{\theta_i} Q_i(\tau_i, x) dx \\
&= T_{i0}(\tau_i) + \theta_i H(\tau_i) - \int_{v(\tau_i)}^{\theta_i} H(\tau_i) dx \\
&= T_{i0}(\tau_i) + v(\tau_i) H(\tau_i)
\end{aligned}$$

$\square$

Proof of Proposition 3.

We begin the proof by defining $\mathbf{P_{unobs}}$ and $\mathbf{P_{obs}}$.

Let $\mathbf{P_{unobs}}$ be the maximization problem under unobservability as above:

$$\max_{q,m} \quad \sum_i \int_{\mathrm{T}} \int_{\Theta} t_i(\tau_i, \theta_i)\, d\bar{F}(\theta|\tau)\, d\bar{G}(\tau) - C \int_{\mathrm{T}} \int_{\Theta} m(\tau)\, d\bar{F}(\theta|\tau)\, d\bar{G}(\tau)$$

$$\begin{aligned}
\text{s.t.} \quad & U_i(\tau_i) \geq \int_{\underline{\theta}}^{\bar{\theta}} \hat{\theta} \cdot Q_i(\tau_i', \theta_i^r(\hat{\theta})) - T_i(\tau_i', \theta_i^r(\hat{\theta})) dF(\hat{\theta}|\tau_i) && \tau\text{-IC}^{unobs} \\
& u_i(\tau_i, \theta_i) \geq \theta_i \cdot Q_i(\tau_i, \theta_i') - T_i(\tau_i, \theta_i') && \theta\text{-IC}^{unobs} \\
& U_i(\tau_i) \geq 0 && \text{IR} \\
& q_i(\tau, \theta) \leq m(\tau) && \text{FC}
\end{aligned}$$

In the optimization problem with observability the incentive constraints change because the buyers observe the production decision before the second reporting stage. Hence:

- At the first reporting stage the buyers form expectations over whether the good will be produced or not. We call this probability:

$$\alpha(\tau_i) = \Pr(m(\tau) = 1|\tau_i)$$

- At the second reporting stage the buyers know whether the good was produced or not. Hence we have two ICs at the θ-stage.

Let $\mathbf{P_{obs}}$ be the maximization problem under observability:

$$\max_{q,m} \quad \sum_i \int_{\mathrm{T}} \int_{\Theta} t_i(\tau_i, \theta_i)\, d\bar{F}(\theta|\tau)\, d\bar{G}(\tau) - C \int_{\mathrm{T}} \int_{\Theta} m(\tau)\, d\bar{F}(\theta|\tau)\, d\bar{G}(\tau)$$

$$\begin{aligned}
\text{s.t.} \quad & U_i(\tau_i) \geq \alpha(\tau_i) \int_{\underline{\theta}}^{\bar{\theta}} [\hat{\theta} \cdot Q_i(\tau_i', \theta_{i1}^r(\hat{\theta})|m = 1) - T_i(\tau_i', \theta_{i1}^r(\hat{\theta})|m = 1)] dF(\hat{\theta}|\tau_i) + \\
& \quad + (1 - \alpha(\tau_i)) \int_{\underline{\theta}}^{\bar{\theta}} [\hat{\theta} \cdot Q_i(\tau_i', \theta_{i0}^r(\hat{\theta})|m = 0) - T_i(\tau_i', \theta_{i0}^r(\hat{\theta})|m = 0)] dF(\hat{\theta}|\tau_i) && \tau\text{-IC}^{obs} \\
& u_i(\tau_i, \theta_i|m = 1) \geq \theta_i \cdot Q_i(\tau_i, \theta_i'|m = 1) - T_i(\tau_i, \theta_i'|m = 1) && \theta\text{-IC}^{obs}_{m=1} \\
& u_i(\tau_i, \theta_i|m = 0) \geq \theta_i \cdot Q_i(\tau_i, \theta_i'|m = 0) - T_i(\tau_i, \theta_i'|m = 0) && \theta\text{-IC}^{obs}_{m=0} \\
& U_i(\tau_i) \geq 0 && \text{IR} \\
& q_i(\tau, \theta) \leq m(\tau) && \text{FC}
\end{aligned}$$

Having defined the two problems we continue in two steps. First we show that every mechanism which is implementable under P_{obs} is also implementable under P_{unobs}. Second, we show that the optimal mechanism of the P_{unobs} is implementable under P_{obs}, i.e. θ-IC in the P_{obs}:

First part:

- Start from P_{unobs}
  - Take any mechanism $\gamma' = \{m, q, t\}$
  - Then γ' is θ-IC^{unobs} iff:

$$u(\tau_i, \theta_i) = \int_{\underline{\theta}}^{\theta_i} Q(\tau_i, \tilde{\theta}) d\tilde{\theta} + u(\tau_i, \underline{\theta}) \tag{$\star$}$$

  - Call $\beta(\tau_i) = \Pr(m = 1|\tau_i)$, according to γ'
  - Note that $Q(\tau_i, \theta_i) = \beta(\tau_i) Q(\tau_i, \theta_i|m = 1)\ \forall \tau_i, \theta_i$ so we can rewrite equation ($\star$) as:

$$u(\tau_i, \theta_i) = \beta(\tau_i) \left[\int_{\theta_i} Q(\tau_i, \theta_i|m = 1) d\theta_i + u(\tau_i, \underline{\theta}|m = 1) \right] + (1 - \beta(\tau_i)) u(\tau_i, \underline{\theta}|m = 0)) \tag{A}$$

- Now consider the P_{obs}
  - Take any mechanism γ
  - Call $\alpha(\tau_i) = \Pr(m = 1|\tau_i)$ according to γ
  - γ is θ-IC iff $\forall(\tau_i, \theta_i)$

$$u(\tau_i, \theta_i|m = 1) = \int_{\underline{\theta}}^{\theta_i} Q(\tau_i, \tilde{\theta}|m = 1) d\tilde{\theta} + u(\tau_i, \underline{\theta}|m = 1) \tag{B}$$

  - Meaning that, here, the IC must hold conditionally on $m = 1$ being visible
  - Write $\forall(\tau_i, \theta_i)$,

$$u(\tau_i, \theta_i) = \alpha(\tau_i) u(\tau_i, \theta_i|m = 1) + [1 - \alpha(\tau_i)] u(\tau_i, \underline{\theta}|m = 0) \tag{C}$$

- If $u(\tau_i, \theta_i|m = 1)$ respects B then C becomes identical to A
  - Namely, mechanism γ is also θ-IC $\forall(\tau_i, \theta_i)$ in the unobservable case
- It can be shown that the relation extends to the whole set of incentive constraints
  - Then P_{obs} is weakly more constraint then P_{unobs}

Second part: show that the optimal mechanism of the P_{unobs} is θ-IC in the P_{obs}:

- If $m = 1$:
  - For all θ_i such that $Q_i(\tau_i, \theta_i, \tau_{-i}, \theta_{-i}) = 0$ (since $\theta_i < v(\tau_i)$) we have:
    $u_i = 0 - T_{i0}(\tau_i) = -T_{i0}(\tau_i)$
    Misreporting θ_i as θ_i' such that $Q_i(\tau_i, \theta_i', \tau_{-i}, \theta_{-i}) = 1$ gives:
    $\theta_i - (T_{i0}(\tau_i) + v(\tau_i)) < -T_{i0}(\tau_i)$
  - For all θ_i such that $Q_i(\tau_i, \theta_i, \tau_{-i}, \theta_{-i}) = 1$ (since $\theta_i \geq v(\tau_i)$) we have:
    $u_i = \theta_i - (T_{i0}(\tau_i) + v(\tau_i))$
    Misreporting θ_i as θ_i' such that $Q_i(\tau_i, \theta_i', \tau_{-i}, \theta_{-i}) = 0$ gives:
    $-T_{i0}(\tau_i) \leq \theta_i - (T_{i0}(\tau_i) + v(\tau_i))$
  - θ_i, θ_i' both such that $q = 0$: misreporting does not change utility
  - θ_i, θ_i' both such that $q = 1$: misreporting does not change utility
- If $m = 0$: $q_i = 0$ for all i - misreporting θ_i does not change utility.

□

Proof of Proposition 4.

First, let the probability for the good to be produced for a given τ_i be denoted as

$$\Pr(\tau_i) \equiv \int_{\Delta \mathrm{T}_{-i}(\tau_i)} d\bar{G}_{-i}(\tau_{-i})$$

where $\Delta T_{-i}(\tau_i)$ is the set of the ex-ante types of all other players that allow the good to be produced:

$$\Delta T_{-i}(\tau_i) \equiv \{\tau_j | j \neq i, \int_{\Theta} \sum_{j \in J^*} \Psi(\tau_j, \theta_j) d\bar{F}(\theta|\tau) \geq C - \int_{\Theta_i} \Psi(\tau_i, \theta_i) d\bar{F}(\theta_i|\tau_i)\}$$

with $J^* \equiv \{j | \Psi(\tau_j, \theta_j) \geq 0\}$ the set of positive virtual valuations.

We rewrite $T_{i0}(\tau_i)$ using the optimal allocation rule from proposition 2 and $\Pr(\tau_i)$:

$$
\begin{aligned}
T_{i0}(\tau_i) =& T_i(\underline{\tau},\underline{\theta}) - \underline{\theta} Q_i(\underline{\tau},\underline{\theta}) + \int_{\underline{\tau}}^{\tau_i} \int_{\underline{\theta}}^{\overline{\theta}} Q_i(\hat{\tau},\hat{\theta}) \frac{\partial F(\hat{\theta}|\hat{\tau})}{\partial \tau_i} d\hat{\theta} d\hat{\tau} + \\
&+ \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\hat{\theta}} [Q_i(\tau_i, x) f(\hat{\theta}|\tau_i) - Q_i(\underline{\tau}, x) f(\hat{\theta}|\underline{\tau})] dx d\hat{\theta} \\
=& T_i(\underline{\tau},\underline{\theta}) - \underline{\theta} Q_i(\underline{\tau},\underline{\theta}) + \int_{\tilde{\tau}_i^{min}}^{\tau_i} \int_{v(\hat{\tau})}^{\overline{\theta}} \Pr(\hat{\tau}) \frac{\partial F(\hat{\theta}|\hat{\tau})}{\partial \tau_i} d\hat{\theta} d\hat{\tau} \\
&+ \Pr(\tau_i) \int_{v(\tau_i)}^{\overline{\theta}} f(\hat{\theta}|\tau_i)(\hat{\theta} - v(\tau_i)) d\hat{\theta} + \Pr(\underline{\tau}) \int_{v(\underline{\tau})}^{\overline{\theta}} f(\hat{\theta}|\underline{\tau})(\hat{\theta} - v(\underline{\tau})) d\hat{\theta}
\end{aligned}
$$

Using these expressions we now begin to define the price menu (T_i,p_i) and the minimum price T^*. Let $\tilde{\tau}_i^{min}$ be defined as in proposition 2.

- T_i and T^*: To define these we look at the following three cases.
  - **Case 1:** $\tau_i < \tilde{\tau}_i^{min}$**:** then $q_i = 0$ (because $m = 0$) for any τ_{-i} and therefore the corresponding T_i is null. Therefore, if a buyer does not pay anything he will never receive the good.
  - **Case 2:** $\tau_i = \tilde{\tau}_i^{min}$**:**

    Since we know that $U(\underline{\tau}) = 0$ it follows that:

    $$T^* \equiv T_{i0}(\tilde{\tau}_i^{min}) = \int_{v(\tilde{\tau}_i^{min})}^{\overline{\theta}} [\hat{\theta} - v(\tilde{\tau}_i^{min})] f(\hat{\theta}|\tilde{\tau}_i^{min}) d\hat{\theta} \times \Pr(\tilde{\tau}_i^{min})$$

    Note:
    * If $\tilde{\tau}_i^{min} = \tilde{\tau}_i(\overline{\tau}...\overline{\tau}) \geq \underline{\tau}$ then the last term is zero ($Pr(\tilde{\tau}_i^{min}) = 0$) and hence:

      $$T^* = 0$$

    * If for $\underline{\tau}$ there is a positive probability that the good will be produced then the lowest type $\underline{\tau}$ will pay :

      $$T^* > 0$$

  - **Case 3:** $\tau_i > \tilde{\tau}_i^{min}$**:**
    * Define:

      $$T_i = T_{i0}(\tau_i) \equiv \phi_i(\tau_i)$$

* And let $\xi_i(T_i)$:

$$\int_{\underline{\theta}}^{\overline{\theta}} \Psi(\tau_i, \hat{\theta}) dF(\hat{\theta}|\tau_i) = \int_{\underline{\theta}}^{\overline{\theta}} \Psi(\phi_i^{-1}(T_i), \hat{\theta}) dF(\hat{\theta}|\phi_i^{-1}(T_i)) \equiv \xi_i(T_i)$$

* Note: $\phi_i(\tau_i)$ needs to be strictly increasing and invertable. We can verify that this is indeed the case for many distributions. We chose two examples:

$$i = 1, 2 \quad \tau_i,\, \theta_i \in [0, 1] \quad M = 1 \quad G(\tau_i) = \tau_i$$

$$F_A(\theta_i|\tau_i) = \theta_i + \tau_i \left((\theta_i - 1/2)^2 - 1/4 \right)$$

$$F_B(\theta_i|\tau_i) = 1/2 - 1/4 \cdot \tau_i + \theta_i(1/2 + 1/4 \cdot \tau_i)$$

Whilst $F_A(\theta_i|\tau_i)$ generates positive virtual valuations only, negative valuations are possible with $F_B(\theta_i|\tau_i)$, which assigns a strictly positive, τ-dependent probability mass to $\theta_i = 0$ values. In fact, many distributions with the properties similar to $F_B(\theta_i|\tau_i)$ can be found.

- Let $p_i = v(\tau_i)$ from proposition 2.
- What is left to show is that the menu (T_i, p_i) in combination with the minimum price T^* implements the optimal allocation: at the second stage where the buyer has to decide whether to buy the good or not (given the good was produced) he will do so only in the case that his valuation exceeds the price, which is equivalent to his virtual valuation being positive. At the first stage when choosing the the price pair the buyers know the function $\xi_i(T_i)$ and hence know how the payment of T_i will influence the probability of production. Choosing the T_i which corresponds to their true τ_i is optimal as it gives the same expected utility as reporting the true τ_i under the optimal direct mechanism.

□

Proof of Proposition 6.

Optimal allocation Given that the good is produced it is easy to see that only buyers with a positive virtual valuation should receive the good. Hence:

$$q_i(\tau, \theta) = \begin{cases} 1 & \text{if } \Psi(\tau_i, \theta_i) > 0 \text{ and } m(\tau, \theta) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus whether the good is produced or not should be decided upon whether the sum of the positive virtual valuations surpasses the cost of production:

$$m(\tau,\theta) = \begin{cases} 1 & \text{if } \sum_{i\in I} \Psi(\tau_i,\theta_i) q_i(\tau,\theta) > C \\ 0 & \text{otherwise} \end{cases}$$

Optimal transfer: By standard procedure we know it must hold that:

$$T_i(\tau_i,\theta_i) = T_{i0}(\tau_i) + \theta_i Q_i(\tau_i,\theta_i) - \int_{\underline{\theta}}^{\theta_i} Q_i(\tau_i,x)dx$$

where

$$T_{i0}(\tau_i) = T_i(\underline{\tau},\underline{\theta}) - \underline{\theta} Q_i(\underline{\tau},\underline{\theta}) + \int_{\underline{\tau}}^{\tau_i}\int_{\underline{\theta}}^{\overline{\theta}} Q_i(\hat{\tau},\hat{\theta}) \frac{\partial F(\hat{\theta}|\hat{\tau})}{\partial \tau_i} d\hat{\theta} d\hat{\tau} + \int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\theta}}^{\hat{\theta}} [Q_i(\tau_i,x) f(\hat{\theta}|\tau_i) - Q_i(\underline{\tau},x) f(\hat{\theta}|\underline{\tau})] dx d\hat{\theta}$$

and $T_i(\underline{\tau},\underline{\theta})$ is pinned down by $U_i(\underline{\tau}) = 0$.

Insert the optimal allocation into the expression for $T_i(\tau_i,\theta_i)$. We have:

- For $\theta_i < v(\tau_i)$, $q_i(\tau,\theta) = 0$, hence:

$$T_i(\tau_i,\theta_i) = T_{i0}(\tau_i)$$

- For $v(\theta_i) \le \theta_i < V(\tau_i)$, first define $\tilde{\theta}_N(\theta_{-N}|\tau)$ as follows: for a given τ and θ_{-N}, it is the minimum value of θ_N such that $\sum_{i\in I^*} \Psi(\tau_i,\theta_i) = C$ holds. We start by defining $Q_i(\tau_i,\theta_i)$ for this case:

$$\begin{aligned} Q_i(\tau_i,\theta_i) &= \int_{\mathrm{T}_{-i}}\int_{\Theta_{-i}} q(\tau,\theta)\ d\bar{F}_{-i}(\theta_{-i}|\tau_{-i}) d\bar{G}_{-i}(\tau_{-i}) \\ &= \int_{\mathrm{T}_{-i}}\int_{\Theta_{-i-N}}\int_{\underline{\theta}}^{\overline{\theta}} q(\tau,\theta)\ dF(\theta_N|\tau_N) d\bar{F}_{-i-N}(\theta_{-i-N}|\tau_{-i-N}) d\bar{G}_{-i}(\tau_{-i}) \\ &= \int_{\mathrm{T}_{-i}}\int_{\Theta_{-i-N}}\int_{\tilde{\theta}_N(\theta_{-N}|\tau)}^{\overline{\theta}} 1\ dF(\theta_N|\tau_N) d\bar{F}_{-i-N}(\theta_{-i-N}|\tau_{-i-N}) d\bar{G}_{-i}(\tau_{-i}) \\ &= \int_{\mathrm{T}_{-i}}\int_{\Theta_{-i-N}} (1 - F(\tilde{\theta}_N(\theta_{-N}|\tau))) d\bar{F}_{-i-N}(\theta_{-i-N}|\tau_{-i-N}) d\bar{G}_{-i}(\tau_{-i}) \qquad \equiv H(\tau_i,\theta_i) \end{aligned}$$

$$
\begin{aligned}
T_i(\tau_i,\theta_i) &= T_{i0}(\tau_i) + \theta_i Q_i(\tau_i,\theta_i) - \int_{\underline{\theta}}^{\theta_i} Q_i(\tau_i,x)dx \\
&= T_{i0}(\tau_i) + \theta_i[1-(1-H(\tau_i,\theta_i))] - \int_{v(\tau_i)}^{\theta_i}[1-(1-H(\tau_i,x))]dx \\
&= T_{i0}(\tau_i) + v(\tau_i) \underbrace{-\theta_i[1-H(\tau_i,\theta_i)] + \int_{v(\tau_i)}^{\theta_i}[1-H(\tau_i,x)]dx}_{\equiv y(\tau_i,\theta_i)}
\end{aligned}
$$

- For $\theta_i \geq V(\tau_i)$, $q_i(\tau,\theta) = 1$

$$
\begin{aligned}
T_i(\tau_i,\theta_i) &= T_{i0}(\tau_i) + \theta_i - \int_{v(\tau_i)}^{V(\tau_i)} Q_i(\tau,x)dx - \int_{V(\tau_i)}^{\theta_i} 1\ dx \\
&= T_{i0}(\tau_i) + V(\tau_i) - \int_{v(\tau_i)}^{V(\tau_i)} H(\tau_i,x)dx \\
&= T_{i0}(\tau_i) + v(\tau_i) + (V(\tau_i) - v(\tau_i)) - \int_{v(\tau_i)}^{V(\tau_i)} H(\tau_i,x)dx \\
&= T_{i0}(\tau_i) + v(\tau_i) + \underbrace{\int_{v(\tau_i)}^{V(\tau_i)} (1-H(\tau_i,x))dx}_{\equiv Y(\tau_i)}
\end{aligned}
$$

□

Proof of Proposition 7.

First, let the probability for the good to be produced for a given (τ_i,θ_i) be denoted as

$$
\Pr(\tau_i,\theta_i) \equiv \int_{T_{-i}} \int_{\Delta^{ex}\Theta(\boldsymbol{\tau},\theta_i)} d\bar{F}_{-i}(\theta_{-i}|\tau_{-i}) d\bar{G}_{-i}(\tau_{-i})
$$

and $\Delta^{ex}\Theta(\boldsymbol{\tau},\theta_i)$ is the set of ex-post types of all other players for which the good will be produced depending on the given ex-ante types:

$$
\Delta^{ex}\Theta(\boldsymbol{\tau},\theta_i) \equiv \{\theta_j | j \neq i, \sum_{j\in J^*} \Psi(\tau_j,\theta_j)) \geq C - \Psi(\tau_i,\theta_i)\}
$$

with $J^* \equiv \{j|\Psi(\tau_j,\theta_j) \geq 0\}$ the set of positive virtual valuations.

- Let T_i be:

$$T_i = \int_{\underline{\tau}}^{\tau_i} \int_{v(\tau_i)}^{\overline{\theta}} \Pr(\hat{\tau}, \hat{\theta}) \frac{\partial F(\hat{\theta}, \hat{\tau})}{\partial \tau_i} d\hat{\theta} d\hat{\tau} + \int_{v(\tau_i)}^{\overline{\theta}} \int_{v(\tau_i)}^{\overline{\theta}} \Pr(\tau_i, x) f(\hat{\theta}|\tau_i) dx d\hat{\theta}$$

$$- \int_{v(\underline{\tau})}^{\overline{\theta}} \int_{v(\underline{\tau})}^{\overline{\theta}} \Pr(\underline{\tau}, x) f(\hat{\theta}|\underline{\tau}) dx d\hat{\theta} \equiv \phi_i(\tau_i)$$

and $v(\tau_i)$ as in proposition 6.

- In order to define $p_i^{min}, p_i, p_i^{max}$ and the function $\chi_i(T_i, p_i)$ we look at three cases for θ_i

  - For $\theta_i = v(\tau_i)$:

$$p_i^{min} = v(\tau_i) H(\tau_i, \theta_i)$$

    with $H(\tau_i, \theta_i)$ as in the proof of proposition 6.

  - For $v(\tau_i) < \theta_i < V(\tau_i)$:

$$p_i = v(\tau_i) - \theta_i [1 - H(\tau_i, \theta_i)] + \int_{v(\tau_i)}^{\theta_i} [1 - H(\tau_i, x)] dx \equiv \kappa_i(\tau_i, \theta_i)$$

  - For $\theta_i \geq V(\tau_i)$:

$$p_i^{max} = v(\tau_i) + \int_{v(\tau_i)}^{V(\tau_i)} (1 - H(\tau_i, x)) dx$$

    with $V(\tau_i)$ as in proposition 6.

- For a given τ_i, $\kappa_i(\tau_i, \theta_i)$ is strictly increasing and invertable in θ_i.

  $\phi_i(\tau_i)$ needs to be strictly increasing and invertable. We can verify that this is indeed the case for many distributions. We chose the following example:

$$i = 1, 2 \quad \tau_i, \theta_i \in [0, 1] \quad M = 1 \quad G(\tau_i) = \tau_i \quad F(\theta_i|\tau_i) = 1/2 - 1/4 \cdot \tau_i + \theta_i (1/2 + 1/4 \cdot \tau_i)$$

  $F(\theta_i|\tau_i)$ assigns a strictly positive, τ-dependent probability mass to $\theta_i = 0$ values. In fact, many distributions with the properties similar to $F(\theta_i|\tau_i)$ can be found.

- Let $\chi_i(T_i, p_i) \equiv \Psi(\phi_i^{-1}(T_i), \kappa_i^{-1}(p_i|T_i))$.

- What is left to show is that the menu $(T_i, p_i^{min}, p_i^{max})$ implements the optimal allocation from the direct mechanism: at the second stage a buyer has to decide which price to pay from the interval $[p_i^{min}, p_i^{max}]$. Since we assume no reimbursement a buyer with $\theta_i < v(\tau_i)$

will never buy as the minimum price is $v(\tau_i)H(\tau_i, \theta_i)$ and he will only get the good if it is actually produced which happens with probability $H(\tau_i, \theta_i)$. For $\theta_i > v(\tau_i)$ the knows that his payment will directly affect the probability of production through the function $\kappa_i^{-1}(p_i|T_i)$ and $\chi_i(T_i, p_i)$ given his first stage payment T_i. He will choose the p_i which maximizes his expected utility which is equivalent to reporting his true θ_i under the direct mechanism. If his valuation exceeds $V(\tau_i)$ he will always pay p_i^{max} as production is then guaranteed and he has no incentive to pay more.

At the first stage when choosing a triple from the price menu the buyers know the functions $\kappa_i(\tau_i, \theta_i)$ and $\phi_i(\tau_i)$ and hence know how the payment T_i will influence the probability of production at the second stage. Choosing the T_i which corresponds to their true τ_i is optimal as it gives the same expected utility as reporting the true τ_i under the optimal direct mechanism.

□

Dynamic Information Collection: Two-Sided Tests*

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Abstract

Principal-agent framework is used to model delegation of information collection. Before deciding whether to invest in a project, a firm can learn about its quality by running up to two tests differing in efficiency defined as precision-to-cost ratio. These tests generate no false positives and no false negatives, so the firm can stop after one conclusive result to make an informed investment decision and save the cost of the second test. To do so, however, the firm must hire and incentivize an expert to run the tests and report their outcomes. We characterize incentive contracts and find the optimal order in which the tests should be performed. We find that the optimal order under agency differs from that in the first-best: while it is first-best optimal to start with the most efficient test, agency considerations imply that it is always optimal to start with the least efficient test.

*Based on joint work with Denis Gromb (HEC Paris) and Francis de Vericourt (ESMT Berlin).
In memory of Denis Gromb, whose continuous dedication, optimism and competence made this project possible.
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Introduction

Decision-makers can sequentially collect costly information about a risky project, namely, by running tests, before committing to an investment. Information collection problems such as this correspond to a variety of tasks in organisations, including new product development (Schlapp & Schumacher (2022), Nikpayam *et al.* (2023)), innovation and research project management (McCardle *et al.* (2016)), and technology adoptions (McCardle (1985); Ulu & Smith (2009)).

Due to the lack of in-house capabilities, firms typically rely on experts to run the tests and interpret their results: in case testing requires extensive data analytics, for instance, firms make use of a vast crowdsourcing market with platforms such as Twine AI, Appen, Scale AI. Delegating expertise, however, gives rise to incentive issues (Feldman *et al.* (2017)). In particular, experts may not complete the task they have been assigned (moral hazard) or misreport results of the tests they have performed (adverse selection).[1]

Several approaches have been proposed to reduce these agency frictions, including financial incentives (Haefner & Taylor (2022)), the use of multiple experts (Gromb & Martimort (2007), Schlapp & Schumacher (2022)), the use of deadlines (Gerardi & Maestri (2012)), or scoring rules (Choi & Han (2020)), or simply performing the tests internally - provided the firm has required expertise (Zorc *et al.* (2022)). The dynamic nature of information collection and its influence on incentive provision have been addressed in the literature as well (Bergemann & Hege (1998), Halac *et al.* (2016)).

This paper studies another crucial feature of the information collection problem, which can be used to mitigate incentive issues via an appropriate task design: the order in which information is elicited. Distinctiveness of our paper lies in the fact that we study tests which differ in precision and cost. This is a realistic scenario: research departments, for instance, can often develop and test several prototypes using various technologies and designs of different quality (Erat & Kavadias (2008)), which in turn provide information of different accuracy and come at different costs.

The order in which these different tests are performed affects the expert's incentives. For instance, if a test with sufficiently high precision is performed early on in the information collection process, the expert can have an incentive to extrapolate its results onto the subsequent tests and not run them to avoid further cost, leaving the firm without refined information about the project. In contrast, if a test with low precision and/or high cost is offered early on, the expert may prefer not to run it and wait for a subsequent test with better precision and/or lower cost.

Our model is as follows. We consider a principal, who, before deciding whether to invest in a project, can learn about its quality from two tests differing in precision and cost. Each of these tests generates a signal that is either conclusive (positive or negative[2]) or inconclusive. We assume no false positives and no false negatives. Hence, a positive signal can be interpreted as “good news” about the project and a negative signal as “bad news”, which is indicative of project failure. Thus, it is optimal to run the tests sequentially: the firm can stop after one conclusive result, make an informed investment decision and avoid the cost of the second test. If the first test generates an inconclusive result, we assume that it is still worthwhile to run the second test.

[1] The more decision-makers lack the skills to run the tests, the more they need to rely on external expertise, and the stronger the agency frictions are since decision-makers are less able to verify the work of experts.

[2] We consider symmetric two-sided tests.

The first-best optimal order of the tests (if the principal were to run the tests himself) is to start with the most efficient test, i.e. the one with the highest precision-to-cost ratio. In particular, starting with a more precise test is attractive because more precise tests are more likely to generate a conclusive signal, which in turn helps to avoid the cost of the second test. Starting with the least costly test is also attractive because it opens a possibility of avoiding the high cost of the second test.

The principal, however, must hire an expert who has to be incentivized to run the tests (moral hazard) and report the signals they generate truthfully (adverse selection). This paper tackles two building blocks of delegated expertise: optimal compensation - characterizing optimal incentive contracts for each order of the tests under a combined moral hazard and adverse selection problem; and task design - finding the optimal order.

We find that the optimal order of the two tests under agency considerations differs from that in the first-best: agency frictions imply that it is optimal to start with the least efficient test, i.e. the one with the lowest precision-to-cost ratio. When the agent has to be motivated to actually run the tests and to report the generated signals truthfully with the help of corresponding transfers from the principal, starting with the least efficient test results in a cheaper incentive provision. Namely, the interplay of dynamic incentive constraints implies that it is cheaper to make the expert run the least efficient test first than to ensure that he does not skip the least efficient test in the second round if he starts with the most efficient test.

The paper proceeds as follows. Section 1 presents the model, finds the first-best, and discusses agency frictions under delegated expertise. Section 2 characterizes optimal incentive contracts and agency rents they imply. Section 3 finds the optimal order of the tests under agency frictions. The last section concludes.

1 Model

We consider a model with universal risk-neutrality and no discounting.

1.1 Decision Problem

A principal must decide whether to undertake a risky project at cost $I > 0$. The project yields a payoff $X > 0$ if it succeeds and 0 if it fails. The project's quality Q can be good ($Q = G$), in which case it succeeds, or bad ($Q = B$), in which case it fails. The prior probability that the project is good is ν and on that basis, the principal would reject it:

$$\nu X - I < 0.$$

However, before deciding, the principal can hire an agent to assess the project's quality. The agent is unbiased, i.e. *a priori* indifferent about whether to undertake the project. His reservation utility is zero, i.e. he accepts employment by the principal provided his utility is non-negative. The agent can run up to two tests, A and B, generating signals, σ_A and σ_B, for which he incurs a cost, ψ_A and ψ_B.[3] For test $i = A, B$, signal σ_i can be either conclusive: $\overline{\sigma}_i$ (positive) / $\underline{\sigma}_i$ (negative); or inconclusive, $\emptyset_i$. We consider symmetric two-sided tests and define precision as:

$$\theta_i \equiv \Pr[\sigma_i = \overline{\sigma}_i | Q = G] = \Pr[\sigma_i = \underline{\sigma}_i | Q = B] > 1/2,$$

with no false positives, i.e.

$$\Pr[\sigma_i = \overline{\sigma}_i | Q = B] = 0,$$

and no false negatives, i.e.

$$\Pr[\sigma_i = \underline{\sigma}_i | Q = G] = 0.$$

Thus, both tests are informative about the project's quality, and $\sigma_i = \overline{\sigma}_i$ resp. $\sigma_i = \underline{\sigma}_i$ constitute positive resp. negative news.

Let $p(\omega)$ be the probability of an event ω and $\nu(\omega)$ the probability of success conditional on the event ω. For instance, $p(\overline{\sigma}_A) \equiv \Pr[\sigma_A = \overline{\sigma}_A] = \nu\theta_A + (1-\nu)0 = \nu\theta_A$ and $\nu(\emptyset_B) \equiv \Pr[Q = G | \sigma_B = \emptyset_B] = \nu(1-\theta_B)/[\nu(1-\theta_B) + (1-\nu)(1-\theta_B)] = \nu$. Due to symmetry, we have $\nu(\emptyset_A) = \nu(\emptyset_B) = \nu(\emptyset_A, \emptyset_B) = \nu$.

We assume that running up to two tests dominates running only one test or none, irrespective of the order of the tests.

Assumption 1.

$$\min_{i=A,B;\, i\neq j} \{p(\overline{\sigma}_i \vee (\emptyset_i, \overline{\sigma}_j))(X - I) - \psi_i - p(\emptyset_i)\psi_j\} \geq \max\{\max_{i=A,B} \{p(\overline{\sigma}_i)(X - I) - \psi_i\}, 0\}$$

[3] Since there is no discounting, it is (at least weakly) optimal to collect the signals sequentially.

Let σ_1 and σ_2 denote the first and second test. For $(\sigma_1, \sigma_2) \in \{(\sigma_A, \sigma_B), (\sigma_B, \sigma_A)\}$, the efficient decision rule is as follows: run test 1; if $\sigma_1 = \overline{\sigma}_1$, undertake the project and do not run test 2; if $\sigma_1 = \underline{\sigma}_1$ reject the project and do not run test 2; otherwise, run test 2 and undertake the project if and only if $\sigma_2 = \overline{\sigma}_2$.

1.2 First-Best

The principal maximizes the project's expected payoff net of the expected cost of running up to two tests:

$$p(\overline{\sigma}_1 \vee (\emptyset_1, \overline{\sigma}_2))(X - I) - \psi_i - p(\emptyset_1)\psi_2$$

$$= [\nu\theta_1 + \nu(1 - \theta_1)\theta_2](X - I) - \psi_1 - (1 - \theta_1)\psi_2,$$

which can be written as:

$$[\nu\theta_1 + \nu\theta_2 - \nu\theta_1\theta_2](X - I) - (\psi_1 + \psi_2) + \theta_1\psi_2.$$

Note that only the last term depends on the order of the tests. Indeed, the order matters only in that a conclusive first signal, which occurs with probability θ_1, allows the principal to avoid the cost ψ_2 of test 2. Thus, starting with σ_i, $i = A, B$, is optimal if and only if:

$$\theta_i\psi_j \geq \theta_j\psi_i.$$

Definition 1. *Efficiency of test i is defined as its precision-to-cost ratio θ_i/ψ_i.*

Proposition 1. *The first-best is to start with the most efficient test.*

In particular, starting with a more precise test is attractive because more precise tests are more likely to generate a conclusive signal, which in turn helps to avoid the cost of the second test. Starting with the least costly test is also attractive because it opens a possibility of avoiding the high cost of the second test.

Without loss of generality, we assume that test A is more efficient than test B:

Assumption 2. $(\theta_A/\psi_A)/(\theta_B/\psi_B) \geq 1$.

1.3 Delegation

The principal and the expert are in an agency relationship: the expert enters a contract with the principal to run the tests and receive transfers in return, under some constraints:

Assumption 3. *Principal-agent contracts are subject to the following constraints:*

- *Moral hazard: whether the agent runs a test or not (information collection) is not observable.*
- *Limited liability: transfers from the principal to the agent must be non-negative.*
- *Asymmetric information: the signals generated by the tests are only observable by the agent.*

The first point implies that the agent must be provided with incentives to run the tests. The second point implies that incentive provision is costly for the principal. Indeed, without limited liability, the agent could be made residual claimant to the project. In particular, if the project failed, the agent would have to pay the principal. Instead, under limited liability, that is not possible and incentive provision requires leaving an information rent to the agent. The third point implies that the agent has private information. In particular, he can claim having run a test and report a signal even if he has not run the test. Therefore, the contract must resolve a combined moral hazard and adverse selection problem.

We study the optimal order of the tests as implied by agency costs and, in particular, whether and when the optimal order might deviate from the first-best. To do so, we first characterize the optimal incentive contract implementing the efficient decision rule for a given order of the tests, as well as and the agency cost it implies. Second, we find the optimal order of the tests taking agency costs into account.

2 Incentive Contracts

For a given order of signals, $(\sigma_1, \sigma_2) \in \{(\sigma_A, \sigma_B), (\sigma_B, \sigma_A)\}$, we characterize the optimal contract implementing the efficient decision rule, i.e. run test 1; if $\sigma_1 = \overline{\sigma}_1$, undertake the project and do not run test 2; if $\sigma_1 = \underline{\sigma}_1$ reject the project and do not run test 2; otherwise, run test 2 and undertake the project if and only if $\sigma_2 = \overline{\sigma}_2$.

By the Revelation Principle, we can concentrate on direct incentive-compatible mechanisms. A direct mechanism consists of transfers from the principal to the agent conditional on his reports and, if the project is implemented, on its outcome. Since the project's outcome is observed only if it is implemented, a contract consists of the following transfers $\{t_1^s,\ t_1^f,\ t_1,\ t_2^s,\ t_2^f,\ t_2,\ t\}$: if the agent reports $\sigma_1 = \overline{\sigma}_1$, the project is implemented and the agent is paid t_1^s if it succeeds or t_1^f if it fails; if he reports $\sigma_1 = \underline{\sigma}_1$, the project is rejected and the agent is paid t_1; if the signal is inconclusive, $\sigma_1 = \emptyset_1$, the agent runs test 2; if the agent reports $\sigma_2 = \overline{\sigma}_2$, the project is implemented and the agent is paid t_2^s if it succeeds or t_2^f if it fails; if he reports $\sigma_2 = \underline{\sigma}_2$, the project is rejected and the agent is paid t_2; is the signal is inconclusive, $\sigma_2 = \emptyset_2$, the project is rejected and the agent gets t. Incentive-compatible mechanisms ensure that the agent has the incentive to run the tests as per the efficient decision rule and to report any signal generated truthfully.

2.1 Optimization Problem

The principal's expected payoff equals the project's expected payoff net of the expected transfer to the agent. Since we focus on contracts implementing the efficient decision rule, the project's expected payoff is a constant equal to

$$p(\overline{\sigma}_1 \vee (\emptyset_1, \overline{\sigma}_2))(X - I) = [\nu\theta_1 + \nu\theta_2 - \nu\theta_1\theta_2]\,(X - I),$$

maximizing the principal's expected payoff amounts to minimizing the expected transfer to the agent $E[\tilde{t}]$:

$$\min_{\{t_1^s,\ t_1^f,\ t_1,\ t_2^s,\ t_2^f,\ t_2,\ t\}} \quad p(\overline{\sigma}_1)t_1^s + p(\underline{\sigma}_1)t_1 + p(\emptyset_1, \overline{\sigma}_2)t_2^s + p(\emptyset_1, \underline{\sigma}_2)t_2 + p(\emptyset_1, \emptyset_2)t$$

Indeed, if the first test reveals the project to be good, which occurs with probability $p(\overline{\sigma}_1)$, it is implemented. Good project succeeds and the agent receives t_1^s. If the first test reveals the project to be bad, which occurs with probability $p(\underline{\sigma}_1)$, it is rejected and the agent receives t_1. If the first test is inconclusive but the second test reveals the project to be good, which occurs with probability $p(\emptyset_1, \overline{\sigma}_2)$, the project is implemented after the second test. The project succeeds and the agent receives t_2^s. If the first test is inconclusive but the second test reveals the project to be bad, which occurs with probability $p(\emptyset_1, \underline{\sigma}_2)$, the project is rejected and the agent receives t_2. If both tests are inconclusive, which happens with probability $p(\emptyset_1, \emptyset_2)$, the project is rejected and the agent receives t.

The objective, i.e. the expected transfer to the agent $E[\tilde{t}]$, can be written as:

$$\min_{\{t_1^s,\ t_1^f,\ t_1,\ t_2^s,\ t_2^f,\ t_2,\ t\}} \quad \theta_1[\nu t_1^s + (1-\nu)t_1] + (1-\theta_1)A_2 + (1-\theta_1)\psi_2, \tag{1}$$

with $A_2 \equiv \theta_2[\nu t_2^s + (1-\nu)t_2] + (1-\theta_2)t - \psi_2$, the agent's payoff from test 2 after the first inconclusive signal.

This optimization is subject to a number of constraints – the agent's participation constraint, the constraints that limited liability imposes on the contract, as well as incentive compatibility constraints – which we now detail.

The agent must agree to the contract, i.e. his **participation constraint** must be satisfied:

$$P_a \geq 0, \tag{2}$$

with $P_a \equiv E[\tilde{t}] - \psi_1 - (1-\theta_1)\psi_2 = \theta_1[\nu t_1^s + (1-\nu)t_1] + (1-\theta_1)A_2 - \psi_1$, which is the agent's payoff from running up to two tests and reporting the signals truthfully.

Limited liability requires that all transfers are non-negative:

$$\textit{(i)} \quad t_1^s \geq 0, \quad \textit{(ii)} \quad t_1^f \geq 0, \quad \textit{(iii)} \quad t_1 \geq 0, \quad \textit{(iv)} \quad t_2^s \geq 0, \quad \textit{(v)} \quad t_2^f \geq 0, \quad \textit{(vi)} \quad t_2 \geq 0, \quad \textit{(vii)} \quad t \geq 0. \tag{3}$$

The contract must also satisfy several incentive compatibility constraints, i.e. motivate the agent to actually run the tests and to report the generated signals truthfully. We start by laying out **adverse selection** incentive constraints.

After test 1 is performed, the agent should not postpone revealing conclusive signals he finds:

$$t_1^s \geq t_2^s, \tag{4}$$

$$t_1 \geq t_2. \tag{5}$$

Having observed the second signal the agent should report truthfully:

$$t_2 \geq t, \tag{6}$$

$$t \geq t_2. \tag{7}$$

(6) and (7) together imply $t = t_2$. Indeed, a second negative signal or a second inconclusive signal both result in the project being rejected. We use $t = t_2$ to simplify the objective and the constraints further.

$$t_2^s \geq t = t_2, \tag{8}$$

$$t \geq \nu t_2^s + (1-\nu)t_2^f. \tag{9}$$

Having observed the first signal, the agent should report truthfully:

$$t_1^s \geq t_1, \tag{10}$$

$$t_1^s \geq t = t_2. \tag{11}$$

Note that (8) and (4) together imply (11). Indeed, making sure that the second positive signal is reported truthfully and that the first positive signal is not postponed also ensures that the first positive signal is not misreported as a second negative signal.

We now proceed with laying out **moral hazard** incentive constraints.

The agent should not skip both tests and bet:

$$P_a \geq t_1, \tag{12}$$

$$P_a \geq t = t_2, \tag{13}$$

$$P_a \geq \nu t_1^s + (1-\nu)t_1^f, \tag{14}$$

$$P_a \geq \nu t_2^s + (1-\nu)t_2^f. \tag{15}$$

Note that (13) is implied by (12) and (5). Indeed, ensuring that a negative signal from test 1 is not postponed requires the respective payment after test 1 to be higher than that after test 2.

The agent should not skip test 1 by blindly reporting an inconclusive signal and run test 2 only:

$$P_a \geq A_2. \tag{16}$$

Having run test 1 and found an inconclusive signal, the agent should not skip test 2 and bet:

$$A_2 \geq \nu t_1^s + (1-\nu)t_1^f, \tag{17}$$

$$A_2 \geq \nu t_2^s + (1-\nu)t_2^f, \tag{18}$$

$$A_2 \geq t_1, \tag{19}$$

$$A_2 \geq t = t_2. \tag{20}$$

Note that (20) is implied by (19) and (5). Indeed, ensuring that a negative signal from test 1 is not postponed requires the respective payment after test 1 to be higher than that after test 2. Moreover, (12) resp. (14) is implied by (16) and (19) resp. (16) and (17). By making sure that test 1 and subsequently test 2 are performed, the agent is ensured to run up to two tests and report the signals instead of simply betting.

Finally, (8) resp. (10) is implied by (20) resp. (16) and (19): if the agent anticipates misreporting a signal, he will not incur the cost of running the respective test but would simply bet instead. To arrive at the former, rewrite (20) as: $t_2^s \geq t_2 + \psi_2/(\nu\theta_2) > t_2$. To arrive at the latter, rewrite (16) as: $\nu t_1^s + (1-\nu)t_1 - \psi_1/\theta_1 \geq A_2 \overset{(19)}{\geq} t_1$, hence, $t_1^s \geq t_1 + \psi_1/(\nu\theta_1) > t_1$.

As it can be seen, the payments in case a positive signal is reported and the project is implemented and fails, i.e, t_1^f and t_2^f, do not appear in the objective (under the "no-false-positives" assumption and because a good project always succeeds) but only on the RHS of incentive constraints (9), (14), (15), (17) and (18). Reducing t_1^f and t_2^f relaxes these constraints, so optimally we set $t_1^f = t_2^f = 0$. Hence, it is optimal not to reward the agent for advising to undertake a project which eventually fails. A contract should incentivize the agent with rewards when the project's outcome aligns with the agent's advice. Rewarding the agent otherwise would be useless at best and increase the cost of incentive provision at worst.

With $t_1^f = t_2^f = 0$ we are able to simplify constraints further: (15) is implied by (14) and (4); (18) is implied by (17) and (4). Indeed, ensuring that a positive signal from test 1 is not postponed requires the respective payment after test 1 to be higher than that after test 2.

Lemma 1. *Optimally,* $t = t_2$ *and* $t_1^f = t_2^f = 0$*. The simplified optimization problem is as follows:*

$$\min_{\{t_1^s,\ t_1,\ t_2^s,\ t_2\}} \quad \theta_1[\nu t_1^s + (1-\nu)t_1] + (1-\theta_1)A_2 + (1-\theta_1)\psi_2 \ \ s.t.$$

$$t_1^s \geq t_2^s \tag{4}$$

$$t_1 \geq t_2 \tag{5}$$

$$t_2 \geq \nu t_2^s \tag{9}$$

$$\nu t_1^s + (1-\nu)t_1 - \psi_1/\theta_1 \geq A_2 \tag{16}$$

$$A_2 \geq \nu t_1^s \tag{17}$$

$$A_2 \geq t_1 \tag{19}$$

With $A_2 \equiv \nu\theta_2 t_2^s + (1-\nu\theta_2)t_2 - \psi_2$*, which is the agent's payoff from running test 2 given that the first signal is inconclusive.*

2.2 Optimal Contracts

First, consider the case that only one test is performed. We characterize the optimal contract and show that the least efficient test – test B (assumption 2) – requires higher transfers to the agent.

2.2.1 Single Test

For a given test $i = \{A, B\}$, the principal minimizes the expected transfer to the agent subject to participation constraint, limited liability constraints, moral hazard and adverse selection constraints. The contract consists of the following payments: if a positive signal is reported, the project is implemented and the agent receives t_i^s if it succeeds and t_i^f if it fails; if a negative signal is reported, the project is rejected and the agent receives t_i; if a signal is inconclusive, the project is rejected and the agent receives $t_i^{\emptyset}$.

The objective (the expected transfer to the agent) is as follows:

$$\min_{t_i^s, t_i^f, t_i, t_i^{\emptyset}} \theta_i[\nu t_i^s + (1-\nu)t_i] + (1-\theta_i)t_i^{\emptyset}$$

Adverse selection constraints ensure that the signals found are reported truthfully:

$$t_i \geq t_i^{\emptyset}, \tag{21}$$

$$t_i^{\emptyset} \geq t_i. \tag{22}$$

Hence, optimally, $t_i^{\emptyset} = t_i$. We use this to simplify the objective:

$$\min_{t_i^s, t_i^f, t_i} \nu\theta_i t_i^s + (1-\nu\theta_i)t_i$$

and constraints further:

$$t_i^s \geq t_i^{\emptyset} = t_i, \tag{23}$$

$$t_i^{\emptyset} \geq \nu t_i^s + (1-\nu)t_i^f. \tag{24}$$

Moral hazard constraints ensure that the agent runs the test instead of simply betting:

$$\nu\theta_i t_i^s + (1-\nu\theta_i)t_i - \psi_i \geq \nu t_i^s + (1-\nu)t_i^f, \tag{25}$$

$$\nu\theta_i t_i^s + (1-\nu\theta_i)t_i - \psi_i \geq t_i^{\emptyset} = t_i. \tag{26}$$

The transfer t_i^f only appears on the RHS of (24) and (26), optimally $t_i^f = 0$. We need only find the optimal t_i^s and t_i.

Proposition 2. *The optimal contract for a given single test $i = A, B$ is unique and it is defined by* (24), (25) *and* (26) *binding. The optimal transfers are as follows:*

$$\begin{aligned} t_i^s &= \frac{\psi_i}{\nu(1-\nu)\theta_i}, \\ t_i^f &= 0, \\ t_i = t_i^{\emptyset} &= \frac{\psi_i}{(1-\nu)\theta_i}. \end{aligned}$$

The rent to the agent is t_i.

Proof: with $t_i = t_i^{\emptyset}$ and $t_i^f = 0$, the four remaining constraints simplify to:

$$t_i^s \geq t_i \tag{23}$$

$$t_i \geq \nu t_i^s \tag{24}$$

$$(1-\nu\theta)t_i \geq \nu(1-\theta)t_i^s + \psi_i \tag{25}$$

$$t_i^s \geq t_i + \frac{\psi_i}{\nu\theta_i} > t_i^s \tag{26}$$

(26) binds and (23) is slack; then (24) and (25) are equivalent and bind also as the only constraints with t_i on the LHS. Participation and limited liability constraints are satisfied. The rent to the agent is defined by (26) binding - the agent could report a negative or an inconclusive signal without running the test and secure a payment t_i. ■

Corollary 1. *Under assumption 2, transfers to the agent and his rent for $i = B$ are higher than for $i = A$. If the optimal transfers for $i = B$ are used together with test A, the agent's payoff is higher than his rent with test B.*

2.2.2 Most Efficient Test First

We proceed with the case of running up to two tests. If the order of collected signals is $(\sigma_1, \sigma_2) = (\sigma_A, \sigma_B)$, the most efficient test comes first, and we refer to this order as AB. If the order of collected signals is $(\sigma_1, \sigma_2) = (\sigma_B, \sigma_A)$, the least efficient test comes first, and we refer to this order as BA.

At first, one might try to simply have the optimal contract from proposition 2 for signal 1 followed by the optimal contract for signal 2. However, three additional constraints must be satisfied. Constraint (4): $t_1^s \geq t_2^s$ needs to hold – otherwise, postponing a positive signal is attractive; constraint (5): $t_1 \geq t_2$ as well – otherwise, postponing a negative signal is attractive; and, finally, constraint (19): $A_2 \geq t_1$ – otherwise the agent would rather report a negative signal than an inconclusive signal after test 1 and run test 2.

Consider **the AB order**. Test 2 is the least efficient test and it is the test that is most problematic in terms of incentive provision. Incentives to skip test 2 are high and respective moral hazard constraints (17) and (19) will bind, resulting in high transfers after test 2: t_2^s and t_2. Since these transfers are high, incentives to postpone conclusive signals found after test 1 are intensified and respective adverse selection constraints (4) and (5) will bind, resulting in high transfers after test 1: t_1^s and t_1. With transfers this high, the agent is more that ensured not to skip test 1 – moral hazard constraint (16) is slack.

Proposition 3. *The optimal contract for the AB order is unique and it is defined by* (4)*,* (5)*,* (9)*,* (17) *and* (19) *binding. The optimal transfers are as follows:*

$$\begin{aligned} t_1^s = t_2^s &= \frac{\psi_B}{\nu(1-\nu)\theta_B}, \\ t_1^f = t_2^f &= 0, \\ t_1 = t_2 = t &= \frac{\psi_B}{(1-\nu)\theta_B}. \end{aligned}$$

The expected transfer to the agent is $E[\tilde{t}] = \dfrac{\psi_B}{\theta_B}\left(\theta_A + \theta_B - \theta_A\theta_B + \dfrac{1}{(1-\nu)}\right)$.

Proof: consider lemma 1 which states the simplified optimization problem of the principal. First, ignore (9) and (16), check these afterwards. Constraint (4) resp. (5) will bind as the only constraint with t_1^s resp. t_1 on the LHS. Then, (19) resp. (17) binds as the only constraint with t_1^s resp. t_1 on the LHS. Hence, $\nu t_1^s = t_1$ and (9) binds. Check (16):

$$\nu t_1^s + (1-\nu)t_1 - \psi_1/\theta_1 \geq A_2 \overset{(19)}{=} t_1$$

With $t_1^s = t_1 + \psi_2/(\nu\theta_2)$ from (4), (5) and (19) binding, (16) is satisfied for $\psi_2/\theta_2 \geq \psi_1/\theta_1$, which is true for the AB order under assumption 2. Moreover, (16) is slack for $\psi_2/\theta_2 > \psi_1/\theta_1$. ■

Corollary 2. *All optimal transfers for the AB order correspond to the respective optimal transfers for a single test* $i = B$*.*

Test B is test 2, its optimal contract, i.e. t_2^s and t_2, is, in fact, the optimal contract for a single test $i = B$. Test B is less efficient than test A, so the agent extracts more rent from this contract than from the optimal contract with the single test being test A. If the principal would start with the single test contract $i = A$ for test 1, the agent has an incentive to move to test 2 irrespective of what he actually observes. As a result, the principal has no choice but to offer the same payments for test 1 as for test 2. Thus, the contract for test 1, i.e. t_1^s and t_1, is also the single test contract for test B.

2.2.3 Least Efficient Test First

Now consider **the BA order**. Test 1 is the least efficient test and it is the test that is most problematic in terms of incentive provision. Incentives to skip test 1 are high and the respective moral hazard constraint (16) will bind, resulting in high transfers after test 1: t_1^s and t_1. Since these transfers are high, incentives to skip test 2 and bet are intensified and respective moral hazard constraints (17) and (19) will bind. There is, however, little incentive to postpone a conclusive signal from test 1, the respective adverse selection constraint (4) will be slack and we assume that (5) will bind – this represents one possible solution.

Proposition 4. *The optimal contract for the BA order is not unique but it can be defined by* (5)*,* (16)*,* (17) *and* (19) *binding. The optimal transfers are as follows:*

$$\begin{aligned} t_1^s &= \frac{\psi_B}{\nu(1-\nu)\theta_B}, \\ t_2^s &= \frac{\psi_B}{(1-\nu)\theta_B} + \frac{\psi_A}{\nu\theta_A} < t_1^s, \\ t_1^f = t_2^f &= 0, \\ t_1 = t_2 = t &= \frac{\psi_B}{(1-\nu)\theta_B}. \end{aligned}$$

The expected transfer to the agent is $E[\tilde{t}] = \dfrac{\psi_B}{\theta_B}\left(\theta_B + \dfrac{1}{(1-\nu)}\right) + (1-\theta_B)\psi_A.$

Proof: consider lemma 1 which states the simplified optimization problem of the principal. First, ignore (4) and (9), check these afterwards. Constraint (16) binds as the only constraint with t_1^s on the LHS. We can find such a contract where (5) is binding: we can increase t_2 till (5) binds, whilst decreasing t_2^s as to keep A_2 constant. Then, (19) resp. (17) binds as the only constraint with t_1^s resp. t_1 on the LHS. Constraint (4) is satisfied for $\psi_1/\theta_1 \geq \psi_2/\theta_2$, which is true for the BA order. Moreover, (4) is slack for $\psi_1/\theta_1 > \psi_2/\theta_2$. With $\nu t_1^s = t_1$ and (4) slack, constraint (9) is satisfied and slack. ■

Corollary 3. *All optimal transfers for the BA order but* t_2^s *correspond to the respective optimal transfers for a single test* $i = B$*.*

This time, test B is test 1, its optimal contract, i.e. t_1^s and t_1, is the optimal contract for a single test $i = B$. The agent does not want to move to test 2 because the rent is lower if we use the single test contract for test A. The agent does not have an incentive to postpone any conclusive signals either but the agent may claim to have found a negative signal in test 1 when the first signal was inconclusive. Hence, the principal needs to increase the agent's rent for test 2, and there are many ways to do it. One possible way is to increase t_2 up to t_1, which is the optimal single test payment for a negative signal from test B. Notice, however, that we need not increase t_2^s all the way up to t_1^s, the latter being equal to the optimal single test payment for a positive signal from test B.

3 Optimal Order of the Tests

Having characterized the optimal contracts for both test orders: for the most efficient test first (AB) in proposition 3 and the least efficient test first (BA) in proposition 4, we find the optimal order of the tests.

The principal's problem is to find the order of the test which maximizes the project's expected payoff net of the expected transfer to the agent:

$$p(\overline{\sigma}_1 \vee (\emptyset_1, \overline{\sigma}_2))(X - I) - E[\tilde{t}]$$

$$= [\nu\theta_1 + \nu(1 - \theta_1)\theta_2](X - I) - E[\tilde{t}],$$

which can be written as:

$$[\nu\theta_1 + \nu\theta_2 - \nu\theta_1\theta_2](X - I) - E[\tilde{t}].$$

Note that only the last term (the expected transfer to the agent) differs across test orders. Indeed, whether the project is undertaken and the likelihood of its success do not depend on the order of the tests. Hence, the optimal solution minimizes the expected transfer $E[\tilde{t}]$.

Theorem 1. *While it is first-best optimal to start with the most efficient test, agency considerations imply that it is optimal to start with the least efficient test – BA is optimal.*

Proof: consider propositions 3 and 4, and calculate the difference between the optimal expected transfers to the agent under the AB order and under the BA order:

$$\begin{aligned} \Delta &\equiv \frac{\psi_B}{\theta_B}\left(\theta_A + \theta_B - \theta_A\theta_B + \frac{1}{(1-\nu)}\right) - \left[\frac{\psi_B}{\theta_B}\left(\theta_B + \frac{1}{(1-\nu)}\right) + (1-\theta_B)\psi_A\right] \\ &= \theta_A(1-\theta_B)\frac{\psi_B}{\theta_B} - \theta_A(1-\theta_B)\frac{\psi_A}{\theta_A} \\ &= \theta_A(1-\theta_B)\left(\frac{\psi_B}{\theta_B} - \frac{\psi_A}{\theta_A}\right) \geq 0 \end{aligned}$$

The inequality holds under assumption 2. ■

Resolving a combined moral hazard and adverse selection problem when delegating running up to two tests to the expert leads to a deviation from the first-best optimal order, where the principal's objective was merely to avoid the cost of test 2. When the agent has to be motivated to actually run the tests and to report the generated signals truthfully with the help of corresponding transfers from the principal, starting with the least efficient test results in a cheaper incentive provision. In fact, all transfers under the BA order are less-or-equal to those under the AB order, hence, the expected transfer is always lower.

Conclusion

If the principal runs up to two tests himself, it is first-best optimal to start with the most efficient test in order to avoid the cost of the second test. However, if running up to two tests is delegated to an expert, we show that the optimal test order is reverse: due to agency frictions, it is optimal to start with the least efficient test. The interplay of dynamic incentive constraints implies that it is cheaper to make the expert run the least efficient test first than to ensure that he does not skip the least efficient test in the second round if he starts with the most efficient test.

We arrive at this result by first defining optimal compensation to the agent such that he is incentivized to actually run the tests and report the generated signals truthfully under both test orders. Despite having to take care of multiple possible deviations from the part of the agent, we are able to substantially simplify this combined moral hazard and adverse selection problem. When the optimal expected transfers are defined for both test orders, we compare the two and find which test order results in the cheapest incentive provision.

It is worthwhile to study the optimal number of tests, since so far we have assumed that running up to two tests is indeed desirable: it could be the case that incentive provision becomes so costly, that the principal would be better off by delegating to run one test only (the one that results in the highest expected payoff net of the expected transfer to the agent).

Richer signal structures could be interesting, too, although according to our attempts, extending the model to a general case seemes to be intractable. Nonetheless, we believe that our model provides a suitable framework to study neighbouring questions with regards to delegated expertise: introducing multiple agents to study collusion or thinking about the set-up costs for the testing procedures, which are born by the experts, to name a couple.

References

Bergemann, Dirk, & Hege, Ulrich. 1998. Venture capital financing, moral hazard, and learning. *Journal of Banking Finance*, **22**(6), 703–735.

Choi, Jin Hyuk, & Han, Kookyoung. 2020. Optimal contract for outsourcing information acquisition. *Economics Letters*, **195**(C), S0165176520302755.

Erat, Sanjiv, & Kavadias, Stylianos. 2008. Sequential testing of product designs: Implications for learning. *Management Science*, **54**(5), 956–968.

Feldman, Michael, Juldaschewa, Frida, & Bernstein, Abraham. 2017. Data Analytics on Online Labor Markets: Opportunities and Challenges. *arXiv*, **170701790**.

Gerardi, Dino, & Maestri, Lucas. 2012. A principal–agent model of sequential testing. *Theoretical Economics*, **7**(5), 425–463.

Gromb, Denis, & Martimort, David. 2007. Collusion and the organization of delegated expertise. *Journal of economic theory*, **137**(1), 271–299.

Haefner, Samuel, & Taylor, Curtis R. 2022. On young Turks and yes men: optimal contracting for advice. *RAND Journal of Economics*, **53**(1), 63–94.

Halac, Marina, Kartik, Navin, & Liu, Qingmin. 2016. Optimal Contracts for Experimentation. *The Review of Economic Studies*, **83**(3), 1040–1091.

McCardle, Kevin F. 1985. Information Acquisition and the Adoption of New Technology. *Management Science*, **31**(11), 1372–1389.

McCardle, Kevin F., Tsetlin, Ilia, & Winkler, Robert L. 2016. When to Abandon a Research Project and Search for a New One. *INSEAD Working Paper*, **2016/10/DSC**.

Nikpayam, Hossein, Schlapp, Jochen, & Fleischmann, Moritz. 2023. Composing New Product Development Portfolios with Internal and External Projects. *SSRN*.

Schlapp, Jochen, & Schumacher, Gerrit. 2022. Delegated Concept Testing in New Product Development. *Operations Research*, **70**(5), 2732–2748.

Ulu, Canan, & Smith, James E. 2009. Uncertainty, Information Acquisition, and Technology Adoption. *Operations Research*, **57**(3), 740–752.

Zorc, Sasa, Tsetlin, Ilia, Hasija, Sameer, & Chick, Stephen E. 2022. The When and How of Delegated Search. *INSEAD Working Paper*, **2022/13/DSC/TOM**.

Search Order in Delegated Data Analytics*

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Abstract

A firm faces the following search problem: before deciding whether to invest in a project, it can find some evidence for the project's potential success by analyzing up to two data sets differing in the cost of analysis and the precision of their findings. The firm can stop the search after a positive result. We characterize the order of the data sets in which the search should be performed. The first-best is to start with the data set that has the highest precision-to-cost-of-analysis ratio. However, if the firm delegates analyses to a data scientist, we find that starting with the data set that has the lowest precision-to-cost-of-analysis ratio is optimal when the project is a priori sufficiently likely to be of bad quality. This result holds even if analyses findings are correlated.

*Based on joint work with Denis Gromb (HEC Paris) and Francis de Vericourt (ESMT Berlin).
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Introduction

Data analytics is used extensively when it comes to assessing the risks of potential investments. Before introducing a new drug, its safety and efficacy are tested, very often this happens in multiple facilities (as a part of multi-site clinical trials[1]); before scaling up a new marketing strategy, its effect is studied on the basis of a small sample of stores[2]; even policy experiments take place on a small scale before a larger scale roll-out takes place[3].

Data resulting from such studies can vary across the sites from which it was collected: data size might differ (e.g. the number of participating patients in a new drug testing), diverse control variables might have been chosen for a particular study (e.g. when assessing a new policy various data on the participants' background may have been collected via questionnaires); data may be encoded using different software. Nonetheless, it is to be expected that the results coming from different sources still reveal similar causalities, i.e. the results are correlated.

Analyzing these heterogeneous data sets can be challenging and requires a certain set of technical skills, especially in the case of fine-grained data (George *et al.* (2014), De Mauro *et al.* (2018)), that is why statisticians or data scientists are often involved. Be it in-house expertise or an outside consultancy, the work of the experts and their interpretation of the results are often not easily or cheaply verifiable. In fact, when data analytics is delegated, both moral hazard and adverse selection incentive issues seem to arise (Feldman *et al.* (2017)). This paper addresses the peculiarities of delegated data analytics mentioned above and frames them as a part of a search problem.

Search problems and their applications in general have been vastly studied (see e.g. McCall (1970), Weizman (1979), McCardle (1985)). Delegated search in particular seems to be a frequent topic in the more recent literature as well. Bergemann & Hege (2005) derive optimal contracts for relationship and arm's-length financing, where the entrepreneur controls the allocation of funds over time (these investments can be understood as experiments providing information on the project's quality). Lewis & Ottaviani (2008) characterize a sequence of short-term contracts in both a monitored and a delegated search for the best alternative. Lewis (2012) characterized optimal contract under delegated search for a better alternative which generates additional surplus to the principal. Xiao *et al.* (2022) find the optimal information policy for a delegated search when only the principal observes search outcomes.

Ulbricht (2016) derives optimal compensation for a combined moral hazard and adverse selection problem with an expert who is ex-ante better informed about search prospects than the principal. Gerardi & Maestri (2012) model dynamic information acquisition by the agent, using symmetric tests for each period and characterizing the optimal length of the information acquisition process and the appropriate optimal contacts. Choi & Han (2023) consider a continuous time setting and study delegated information acquisition with both moral hazard and adverse selection as well but they introduce noisy signals. Distinctiveness of our paper lies in the fact that we study the optimal order of the data sets which are offered for the analysis as means to resolve incentive issues. Additionally, we focus on heterogeneous data sets.

[1]Defined by the NIH as: https://www.nhlbi.nih.gov/grants-and-training/funding-opportunities/foa-Investigator-Initiated-Multi-Site-Clinical-Trials-FAQ

[2]E.g. Uniqlo trying out their print service at a limited number of stores: https://www.uniqlo.com/us/en/special-feature/utme?srsltid=AfmBOoq-YlJkAK8$_Q 6ypXDK013wZ1TH9qjD5l49GCeyGR4LTuUpL8riJ$

[3]See policy experimentation examples on https://peep.pt/policy-experimentation/

We model two-period delegated search in the following fashion: a risk-neutral principal offers up to two data sets to a risk-neutral agent for sequential analysis. He does so in attempt to learn about a risky project's quality before investing. The data sets differ in the cost of analysis and in the precision of their findings, i.e. in how likely are the data sets to reveal that the project is of good quality. Even good quality projects can, however, fail with an exogenous probability, i.e. it is not possible to assure project's success through data analytics alone. The principal must decide which data set to offer first. At the beginning of each period, the agent then decides whether or not to run the analysis, skip it or stop the search altogether. In particular, once a positive result is found, the search can terminate. In the baseline model we consider analyses findings which are independent conditional on the project's quality. In the extended model we introduce positive correlation.

Absent of incentive provision issues, it is first-best optimal to always start with the data set which has the highest precision-to-cost-of-analysis ratio: this way it is possible and potentially cheap (in terms of the cost of analysis) to encounter a positive signal early on and stop the search as to avoid running the second analysis. However, due to the delegation of data analytics, the agent must be incentivized to run the analyses and report the findings truthfully. In order to find the optimal search order in the presence of this combined moral hazard and adverse selection problem, we first characterize appropriate compensation schemes for both data set sequences, and then find the order which results in the cheapest incentive provision.

We find that it is optimal to start with the data set that has the lowest precision-to-cost-of-analysis ratio if the project is a priori sufficiently likely to be of bad quality. This result holds even if analyses findings are positively correlated. It is optimal to deviate from the first-best because if the data set with the highest precision-to-cost-of-analysis ratio is offered early during the search process, the incentive to skip the second analysis is very high, since the project potentially being of bad quality is not likely to generate a positive result during the second analysis if it hasn't generated one during the first analysis. Making sure that the second analysis in not skipped requires high payments for analysing the second data set. In this case, however, adverse selection issues intensify: the agent might want to postpone announcing having found a positive result during the first analysis. Altogether, incentive provision becomes too expensive under the first-best search order if the project is a priori sufficiently likely to be of bad quality.

The paper proceeds as follows. Section 1 defines the set-up and characterizes heterogeneous data sets. Section 2 finds the optimal search order in the absence of incentive provision issues. Section 3 characterizes optimal compensation for the two possible data set sequences in the presence of moral hazard and adverse selection. Section 4 finds the optimal search order. Section 5 discusses the extended model when analyses findings are positively correlated. The last section concludes. The appendix contains all the proofs.

1 Model

A risk-neutral principal must decide whether to undertake a risky investment requiring an outlay $I > 0$. The project can either succeed or fail: it generates a single payoff $\tilde{X}$ equal to $X > 0$ if it succeeds and 0 if it fails. The project's quality Q can be good ($Q = G$), in which case it succeeds with probability q, or bad ($Q = B$), in which case it fails with certainty. Based on the prior probability that the project is good ν, the principal rejects the project, i.e. $\nu q X - I < 0$.

Before making a decision, the principal can delegate the assessment of the project's quality to an agent. The agent is risk-neutral and unbiased. The agent can search for evidence that the project is good from different sources of information, i.e. data sets. Specifically, we assume that the agent can analyze up to two data sets A resp. B at a cost ψ_A resp. ψ_B (depending on data size or how the data is encoded). The data sets are imperfect such that if the project is good, data set $S \in \{A, B\}$ reveals this with probability $\theta_S \in (1/2, 1)$, referred to as analysis precision (which could, for instance, in case of regression represent the explanatory power of the model or the quality and quantity of control variables).

Denote "good news" from data set S by $\sigma_S = \overline{\sigma}_S$ and "bad news" (no positive signal) by $\sigma_S = \underline{\sigma}_S$. We have:

$$\theta_S \equiv \Pr[\sigma_S = \overline{\sigma}_S | Q = G] > 1/2,$$

and we assume no false positives, i.e.

$$\Pr[\sigma_S = \overline{\sigma}_S | Q = B] = 0.$$

In contrast with the previous paper "Dynamic information collection: Two-sided tests", we allow for false negatives, i.e. $\Pr[\sigma_S = \underline{\sigma}_S | Q = G] = 1 - \theta_S$. Thus, if a negative result is observed, the firm does not know for sure whether a bad quality project generated it or a good quality project, so the firm rejects the project based on soft information:

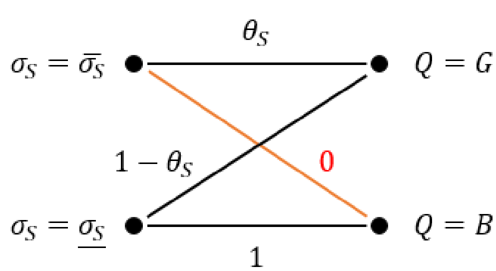


Figure 1: Signal structure.

In the baseline model, the findings are independent conditional on the project's quality, and without loss of generality, we assume that data set A has a higher precision-to-cost-of-analysis ratio than data set B:

Assumption 1. $\theta_A/\psi_A \geq \theta_B/\psi_B$.

The principle chooses the order in which to analyze the data sets sequentially,[4] and we refer to the sequence AB resp. BA as the sequence in which the principal starts with data set A resp. B. Given a sequence, the agent essentially faces a two period search problem. At the beginning of each period, the agent decides whether or not to run the analysis, skip it or stop the search altogether. The agent updates his belief about the quality of the project according to Bayes' rule as the analyses fail to find evidence that the project is good.

Specifically, for $i \in \{1,2\}$ we denote by p_i the probability that the i^{th} data set reveals the project to be good, given that no previous data set has revealed it thus far. Similarly, we denote by ν_i the agent's belief that the project is good, given that no previous data set has revealed it thus far[5]. We have $p_i = \theta_i \nu_{i-1}$ and $\nu_i = \dfrac{\nu_{i-1}(1-\theta_i)}{[\nu_{i-1}(1-\theta_i)+1-\nu_{i-1}]}$ for $i \in \{1,2\}$, where $\nu_0 = \nu$ and $\theta_i \in \{\theta_A, \theta_B\}$ depending on the sequence.

[4] There is no discounting, hence, it is (at least weakly) optimal to search sequentially.

[5] As the firm conducts analyses sequentially and fails to find positive results, it gets more pessimistic about the project quality (in line with the literature on search models).

2 First-Best Search Order

At first best, the objective of the principal is to maximize the project's expected payoff net of search costs. To focus on the more interesting situations, we assume that both analyses are valuable:

Assumption 2.

$$p_2(qX - I) - \psi_2 \geq 0$$

This means in particular that if the first data set fails to reveal that the project is of good quality, analyzing the second one is valuable irrespective of the sequence. The findings being conditionally independent, assumption 2 also implies that analyzing the first data set is valuable.[6]

Given a sequence, the optimal decision rule corresponds to a classical two-period optimal search policy: the principal continues the search as long as analysis fails, but stops the search altogether and implements the project if an analysis finds the project to be of good quality. If all analyses fail at the end of the search, the principal rejects the project.

Now, consider the optimal search order. The principal maximizes the project's expected payoff net of search cost:

$$[p_1 + (1 - p_1)p_2] \cdot (qX - I) - \psi_1 - (1 - p_1)\psi_2,$$

which can be written as:

$$[1 - (1 - p_1)(1 - p_2)] \cdot (qX - I) - (\psi_1 + \psi_2) + p_1\psi_2.$$

Note that only the last term depends on the order in which the data sets are analyzed. Indeed, the order matters only in that the first successful analysis saves the cost ψ_2 of running the second analysis if it reveals the project to be good which occurs with probability p_1. Thus, starting with data set A is better if and only if $\nu\theta_A\psi_B \geq \nu\theta_B\psi_A$.

We therefore obtain the following first-best:

Proposition 1. *The first-best search order is AB: starting with the data set that has the highest precision-to-cost-of-analysis ratio.*

[6] The latter implication only requires that the findings are not negatively correlated so $p_2 \leq \nu\theta_2$ for both sequences.

3 Optimal Contracts

The agent enters a contract with the principal to analyze the data sets sequentially and receive transfers in return. The agent must be provided with incentives to run the analyses (moral hazard) and report the findings truthfully (adverse selection), since we assume that neither the analyses themselves, nor the findings are observable to the principal. We assume limited liability as well: in this case the agent can no longer be made a residual claimant to the project and pay to the principal in case the project is implemented and fails.

We begin by characterizing the optimal incentive contract for a given sequence, i.e. AB and BA. The efficient decision rule is as follows: analyze data set 1; if it reveals the project to be good, stop the search, undertake the project and do not analyze data set 2; otherwise, run analysis 2 and undertake the project only if it reveals the project to be good.

By the Revelation Principle, we can concentrate on direct incentive-compatible mechanisms. A direct mechanism consists of transfers from the principal to the agent conditional on his reports and, if the project is implemented, on its outcome. Since the project's outcome is observed only if it is implemented, a contract consists of five transfers $\{t_1^s,\ t_1^f,\ t_2^s, t_2^f, t\}$: if the agent reports that data set 1 revealed the project to be good, the search stops, the project is implemented and the agent is paid t_1^s if the project succeeds or t_1^f if it fails. Otherwise, the agent continues the search and analyses data set 2. If he then reports that analysis 2 revealed the project to be good, the project is implemented and the agent is paid t_2^s if the project succeeds or t_2^f if it fails. Otherwise, the project is rejected and the agent is paid t. Incentive-compatibility ensures that the agent has the incentive to run the analyses as per the efficient decision rule and to report findings truthfully.

The principal's expected payoff equals the project's expected payoff net of the expected transfer to the agent. Since we focus on contracts implementing the efficient decision rule, the project's expected payoff is a constant equal to

$$[p_1 + (1-p_1)p_2] \cdot (qX - I) = (1 - (1-p_1)(1-p_2)) \cdot (qX - I),$$

maximizing the principal's expected payoff amounts to minimizing the expected transfer to the agent:

$$\min_{\{t_1^s,\ t_1^f,\ t_2^s, t_2^f, t\}} \quad p_1[qt_1^s + (1-q)t_1^f] + (1-p_1)p_2[qt_2^s + (1-q)t_2^f] + (1-p_1)(1-p_2)t. \tag{1}$$

Indeed, if the first data set reveals the project to be good, which occurs with probability p_1, the project is implemented. Good project succeeds with probability q, in which case the agent receives t_1^s. Otherwise, the project fails and the agent receives t_1^f. If the first analysis does not reveal the project to be good, but the second does, which occurs with probability $(1-p_1)p_2$, the project is implemented after the second analysis. The project succeeds with probability q, in which case the agent receives t_2^s. Otherwise, the project fails and the agent receives t_2^f. If neither analysis revealed the project to be good, which happens with probability $(1-p_1)(1-p_2)$ the project is rejected and the agent receives t.

This optimization is subject to a number of constraints – the agent's participation constraint, the constraints that limited liability imposes on the contract, as well as incentive compatibility constraints – which we detail.

The agent must agree to the contract, i.e., his participation constraint must be satisfied:

$$p_1[qt_1^s + (1-q)t_1^f] + (1-p_1)p_2[qt_2^s + (1-q)t_2^f] + (1-p_1)(1-p_2)t - \psi_1 - (1-p_1)\psi_2 \geq 0. \tag{2}$$

Limited liability requires that all transfers be non-negative:

$$\textit{(i)} \quad t_1^s \geq 0, \qquad \textit{(ii)} \quad t_1^f \geq 0, \qquad \textit{(iii)} \quad t_2^s \geq 0, \qquad \textit{(iv)} \quad t_2^f \geq 0, \qquad \textit{(v)} \quad t \geq 0. \tag{3}$$

The contract must also satisfy several incentive compatibility constraints, i.e., motivate the agent to actually run the analyses and to report the findings truthfully.

Having run analysis 1, which did not reveal the project to be good, and having then run analysis 2, the agent should report the second finding truthfully, be it "good news", i.e. the second analysis revealed the project to be good, or "bad news", i.e. the second analysis did not reveal the project to be good:

$$qt_2^s + (1-q)t_2^f \geq t, \tag{4}$$

$$t \geq \nu_2 qt_2^s + (1-\nu_2 q)t_2^f. \tag{5}$$

Having run analysis 1, which did not reveal the project to be good, the agent should run analysis 2 instead of making an uninformed second report, be it "good news" or "bad news":

$$-\psi_2 + p_2[qt_2^s + (1-q)t_2^f] + (1-p_2)t \geq \nu_1 qt_2^s + (1-\nu_1 q)t_2^f, \tag{6}$$

$$-\psi_2 + p_2[qt_2^s + (1-q)t_2^f] + (1-p_2)t \geq t. \tag{7}$$

Having run analysis 1, which did not reveal the project to be good, the agent should report it truthfully and run analysis 2, instead of misreporting the first finding as "good news":

$$-\psi_2 + p_2[qt_2^s + (1-q)t_2^f] + (1-p_2)t \geq \nu_1 qt_1^s + (1-\nu_1 q)t_1^f. \tag{8}$$

Having run analysis 1, which revealed the project to be good, the agent should not misreport it as "bad news", followed by an uninformed second report, be it "good news" or "bad news"[7]:

$$qt_1^s + (1-q)t_1^f \geq qt_2^s + (1-q)t_2^f, \tag{9}$$

$$qt_1^s + (1-q)t_1^f \geq t. \tag{10}$$

[7]The agent has nothing more to learn about the project following a positive finding. Hence, we do not need to consider the possibility that he runs analysis 2.

The agent should not skip analysis 1, make an uninformed "bad news" report, to run analysis 2 and make a recommendation based only on the second finding:

$$\begin{aligned} p_1[qt_1^s + (1-q)t_1^f] + (1-p_1)p_2[qt_2^s + (1-q)t_2^f] + (1-p_1)(1-p_2)t - \psi_1 - (1-p_1)\psi_2 \\ \geq -\psi_2 + \nu\theta_2[qt_2^s + (1-q)t_2^f] + (1-\nu\theta_2)t. \end{aligned} \tag{11}$$

The agent should not skip both analyses and make an uninformed report, be it "good news" coming from the first analysis, "bad news" from analysis 1 and "good news" from analysis 2 or "bad news" from both analyses:

$$\begin{aligned} p_1[qt_1^s + (1-q)t_1^f] + (1-p_1)p_2[qt_2^s + (1-q)t_2^f] + (1-p_1)(1-p_2)t - \psi_1 - (1-p_1)\psi_2 \\ \geq \nu qt_1^s + (1-\nu q)t_1^f, \end{aligned} \tag{12}$$

$$\begin{aligned} p_1[qt_1^s + (1-q)t_1^f] + (1-p_1)p_2[qt_2^s + (1-q)t_2^f] + (1-p_1)(1-p_2)t - \psi_1 - (1-p_1)\psi_2 \\ \geq \nu qt_2^s + (1-\nu q)t_2^f, \end{aligned} \tag{13}$$

$$p_1[qt_1^s + (1-q)t_1^f] + (1-p_1)p_2[qt_2^s + (1-q)t_2^f] + (1-p_1)(1-p_2)t - \psi_1 - (1-p_1)\psi_2 \geq t. \tag{14}$$

We first note that it is (weakly) optimal not to reward the agent for having advised to undertake a project which eventually failed. Indeed, a contract incentivizes the agent with rewards when the project's outcome aligns with his advice.

Lemma 1. *An optimal contract exists such that* $t_1^f = t_2^f = 0$.

A number of the other constraints can be shown to be slack. The agent's participation constraint (2) is implied by the limited liability constraints ($3.v$) and, for instance, incentive compatibility constraint (14). Indeed, irrespective of the contract, the agent can always opt to incur no cost and collect non-negative transfers, thus ensuring himself a non-negative expected payoff. Constraints (4), (5) and (8) ensuring the agent does not misreport his findings are implied by constraints (6), (7) and (12) ensuring he runs the corresponding analysis. Indeed, if the agent intends to misreport a finding, he will not incur the cost of running the analysis in the first place.

The transfer for reporting "good news" after analysis 1 must exceed that for reporting "good news" after analysis 2 to avoid postponement of the report (constraint (9)). Together with the fact that the latter has to be non-negative (constraint ($3.iii$)), this implies that the former is non-negative (constraint ($3.i$)). Moreover, if the agent runs no analysis, making an uninformed report about "good news" after analysis 1 is more tempting than making an uninformed report about "good news" after analysis 2: constraints (9) and (12) imply constraint (13).

The transfer t_2^s for advising to implement the project after "good news" from analysis 2 must exceed that for rejecting the project - otherwise the agent would always reject the project (constraint (4)). Together with the fact that the latter has to be non-negative (constraint ($3.v$)), this implies that the former is non-negative (constraint ($3.iii$)). Moreover, constraint (4) together with constraint (9) imply constraint (10).

Finally, if the agent anticipates skipping analysis 2, reporting "bad news" and rejecting the project, he will not incur the cost of running any analyses and instead advise against the project - constraint (14) is slack.

Lemma 2. *Constraints* (2), *(3.i), (3.iii), (3.v),* (4)*-to-*(6), (8), (10), (13) *and* (14) *are slack.*

The problem simplifies to:

$$\min_{\{t_1^s,\ t_2^s,\ t\}} p_1 q t_1^s + (1-p_1)p_2 q t_2^s + (1-p_1)(1-p_2)t$$

subject to

$$(1-p_1)p_2 q t_2^s \geq (1-p_1)\psi_2 + (1-p_1)(1-p_2)t \tag{7}$$

$$q t_1^s \geq q t_2^s \tag{9}$$

$$p_1 q t_1^s + (1-p_1)p_2 q t_2^s + (1-p_1)(1-p_2)t - \psi_1 - (1-p_1)\psi_2 \geq -\psi_2 + \nu\theta_2 q t_2^s + (1-\nu\theta_2)t \tag{11}$$

$$(1-p_1)p_2 q t_2^s + (1-p_1)(1-p_2)t \geq \psi_1 + (1-p_1)\psi_2 + (1-p_1)\nu_1 q t_1^s \tag{12}$$

The remaining constraints are as follows: constraint (7) ensures the agent does not skip analysis 2, which requires t_2^s to be large enough; constraint (9) ensures he does not postpone reporting "good news" after analysis 1 as "good news" after analysis 2, which requires t_1^s to exceed t_2^s; constraint (11) ensures he does not skip analysis 1, which requires t_1^s to be large enough; constraint (12) ensures he does not skip both analyses making an uninformed report about "good news" after analysis 1, which requires t to be large enough.

Proposition 2. *The optimal contract is such that constraints* (7) *and* (12) *bind. Moreover:*

If $\dfrac{\theta_2/\psi_2}{\theta_1/\psi_1} \leq \nu + (1-\nu)\dfrac{(1-\theta_2)}{(1-\theta_1)}$, *constraint* (9) *binds, the optimal transfers are:*

$$\begin{aligned}
t_1^{s\star} &= t_2^{s\star} = \frac{1}{q}\cdot\left[\frac{\psi_1}{\theta_1}\cdot\frac{\theta_1}{\nu(1-\nu)} + \frac{\psi_2}{\theta_2}\cdot\frac{(1-\nu\theta_1)^2}{\nu(1-\nu)(1-\theta_1)}\right],\\
t_1^{f\star} &= t_2^{f\star} = 0,\\
t^\star &= \frac{\psi_1}{\theta_1}\cdot\frac{\theta_1}{(1-\nu)} + \frac{\psi_2}{\theta_2}\cdot\frac{1-\nu\theta_1}{1-\nu},
\end{aligned}$$

and the expected transfer is:

$$E\left[\tilde{t}^\star\right] = [\psi_1 + (1-p_1)\psi_2] + \nu q t_1^{s\star}.$$

Otherwise, constraint (11) *binds, the optimal transfers are:*

$$\begin{aligned}
t_1^{s\star} &= \frac{1}{q}\cdot\left[\frac{\psi_1}{\theta_1}\cdot\frac{1}{\nu(1-\nu)} + \frac{\psi_2}{\theta_2}\cdot\frac{\theta_2(1-\nu\theta_1)}{\nu(1-\theta_1)}\right],\\
t_2^{s\star} &= \frac{1}{q}\cdot\left[\frac{\psi_1}{\theta_1}\cdot\frac{1}{(1-\nu)} + \frac{\psi_2}{\theta_2}\cdot\frac{1-\nu\theta_1+\nu\theta_2(1-\theta_1)}{\nu(1-\theta_1)}\right],\\
t_1^{f\star} &= t_2^{f\star} = 0,\\
t^\star &= \frac{\psi_1}{\theta_1}\cdot\frac{1}{(1-\nu)} + \frac{\psi_2}{\theta_2}\cdot\theta_2,
\end{aligned}$$

and the expected transfer is:

$$E\left[\tilde{t}^\star\right] = [\psi_1 + (1-p_1)\psi_2] + \nu q t_1^{s\star}.$$

Note that the exogenous probability of a good project succeeding, i.e. q, is only relevant as a scaling factor of the transfers t_1^s and t_2^s. Morevoer, the expected transfer consists of two parts. The terms $[\psi_1 + (1-p_1)\psi_2]$ compensate the agent for the cost of running the analyses. The other term is the agency rent, i.e., an additional payoff due to incentive issues, which ensures that the agent gets a strictly positive expected payoff, at a cost to the principal. Since it stems from incentive problems, its expression depends on which constraints bind.

4 Optimal Search Order

We have characterized the optimal contract for both possible sequences, i.e. AB and BA. We now characterize the optimal search order, and, in particular, whether and when it deviates from the first-best.

The optimal sequence of data sets maximizes:

$$[p_1 + (1-p_1)p_2] \cdot (qX - I) - E[\tilde{t}^\star] = (1 - (1-p_1)(1-p_2)) \cdot (qX - I) - E[\tilde{t}^\star].$$

Note that only the expected transfer differs across sequences. Indeed, whether the project is undertaken and the likelihood of its success do not depend on the order of the data sets. Hence, the optimal order is that which minimizes the expected transfer $E[\tilde{t}^\star]$.

Theorem 1. *A unique threshold $\nu^\star \in [0,1)$ exists such that sequence BA is optimal if and only if $\nu \leq \nu^\star$. Further, $\nu^\star = 0$ if and only if $\frac{\psi_B/\theta_B}{\psi_A/\theta_A} < \frac{(1-\theta_A)}{(1-\theta_B)}$ and $\nu^\star$ solves the following equation otherwise:*

$$(1-\nu)\left(\frac{\frac{\psi_B/\theta_B}{\psi_A/\theta_A}}{(1-\theta_A)} - \frac{1}{(1-\theta_B)}\right) - \theta_A \nu \left(\frac{1}{(1-\nu)} - (1-\theta_B)\right)\left(\frac{\psi_B/\theta_B}{\psi_A/\theta_A} - 1\right) = 0.$$

When the agent has to be motivated to actually run the analyses and report the findings truthfully, starting with the data set with the lowest precision-to-cost-of-analysis ratio is optimal provided that the project is a priori sufficiently likely to be of bad quality.

Corollary 1. *If $\nu \leq \nu^\star$, constraint* (9) *binds for sequence AB, whilst constraint* (11) *binds for sequence BA.*

When deviating from the first-best is optimal, transfer $t_1^{s\star}$ is defined by constraint (9) for sequence AB: it ensures that the agent does not postpone reporting "good news" from analysis 1; and by constraint (11) for sequence BA: it ensures that the agent does not skip analysis 1 to run analysis 2.

For sequence AB, data set 2 is the one with the lowest precision-to-cost-of-analysis ratio, and so the temptation for the agent to extrapolate from data set 1 and not run analysis 2 is high. For sequence BA, data set 2 is the one with the highest precision-to-cost-of-analysis ratio, and so the temptation for the agent to skip analysis 1 and run analysis 2 only is high. Ensuring that the former deviation is not pursued by the agent for the AB sequence is more costly for the principal than ensuring that the latter deviation is not pursued by the agent for the BA sequence, if we expect the project to be a priori of bad quality.

Corollary 2. *If $\nu \leq \nu^\star$, transfer $t_1^{s\star}$ is higher for sequence AB than for sequence BA.*

If the project is a priori likely to be of bad quality, i.e. if ν is low, "good news" are unlikely. Consider sequence AB and assume that the agent has analyzed data set 1 and it did not reveal the project to be good. The agent's temptation to skip analysis 2 is high for two reasons: first, data set 2 has the lowest precision-to-cost-of-analysis ratio; and second, analysis 2 is unlikely to reveal that the project is good.

Thus, the agent must be incentivized to run analysis 2, which, in particular, requires a high $t_2^{s\star}$ for potentially reporting "good news". At the same time, $t_1^{s\star}$ needs to match $t_2^{s\star}$ to satisfy constraint (9), i.e. for the agent not to postpone reporting "good news" after analysis 1. The resulting transfer $t_1^{s\star}$ for the AB sequence is higher than what would have been needed to incentivize the agent to run analysis 1, and it is also higher than the value of $t_1^{s\star}$ required for the BA sequence to incentivize the agent not to skip analysis 1.

In fact, not only is the transfer $t_1^{s\star}$ is higher for sequence AB than for sequence BA, the difference between these transfers surpasses the difference in the expected cost of analysis, i.e. $\psi_1 + (1-p_1)\psi_2$, between sequence AB and sequence BA. The latter is always negative since data set A has a higher precision-to-cost-of-analysis ratio than B. The difference in the expected cost of analysis alone would always lead to optimality of the AB sequence - as in the first-best.

5 Extension: Correlated Findings

We now consider the case where the findings from analyses can be correlated, which is a likely scenario in the context of data analytics. Even if the data sets are heterogeneous, they sometimes reveal similar results (be it two data sets coming from two different hospitals testing the efficacy of the same drug; or two districts testing the efficacy of the same communal policy).

We model correlation in the following fashion: for $\theta_j \geq \theta_i$ (data set j is more likely to reveal the project to be good than data set i), define $\chi \in [\theta_j, 1]$ as the conditional probability that data set j reveals the project to be good given that data set i has revealed the project to be good. For $\chi = \theta_j$, analyses findings are conditionally independent (as in the baseline model). For $\chi = 1$, the data set which is least likely to reveal the project to be good is redundant.

As the findings from the analyses are now interdependent, we need to adjust the expressions for p_2 (probability that data set 2 reveals the project to be good, given that data set 1 has not revealed it) and ν_2 (the agent's belief that the project is good, given that neither of the analyses have revealed it thus far) in the constrained optimization from the baseline model, accounting for the role of the correlation parameter.

Expressions for p_1, ν_1 and the first-best remain unchanged; no additional incentive constraints appear due to the correlated analyses findings. We redefine p_2 and ν_2 to account for correlation:

$$p_2 = \frac{\nu\theta_2\left(1 - \chi \cdot \dfrac{min\,[\theta_1, \theta_2]}{\theta_2}\right)}{1 - p_1},$$

$$\nu_2 = \frac{\nu(1-\theta_1)(1-\theta_2)}{1 - p_1 - \nu\theta_2\left(1 - \chi \cdot \dfrac{min\,[\theta_1, \theta_2]}{\theta_2}\right)}.$$

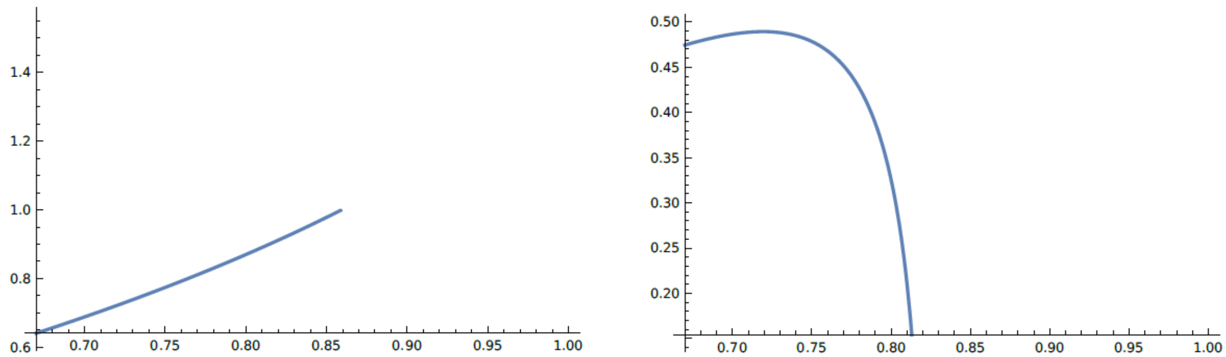
The optimal search order is similar to the baseline model although its derivation becomes more tedious:

Theorem 2. *Threshold $\nu^\star \in [0,1)$ exists such that: if $\nu \leq \nu^\star$, sequence BA is optimal; otherwise, sequence AB is optimal.*

$$\nu^\star \;\; solves \begin{cases} \frac{(1-\nu\theta_B)(\chi-\nu\theta_A)\theta_A}{(1-\nu)(\theta_A-\theta_B\chi)}\frac{\psi_A}{\theta_A} + \frac{1}{(1-\nu)}\frac{\psi_B}{\theta_B} - \nu\theta_B\psi_A \\ -\left(\frac{\nu}{(1-\nu)}\psi_A + \frac{(1-\nu\theta_A)^2}{(1-\chi)(1-\nu)}\frac{\psi_B}{\theta_B} - \nu\theta_A\psi_B\right) = 0 & if\ \theta_A \geq \theta_B \\ \frac{\theta_A(1-\nu\theta_B)}{(1-\chi)(1-\nu)\theta_B}\left[(1-\nu\theta_B) - (1-\chi)\right]\frac{\psi_A}{\theta_A} + \frac{1}{(1-\nu)}\frac{\psi_B}{\theta_B} - \nu\theta_A\theta_B\frac{\psi_A}{\theta_A} \\ -\left(\frac{\nu\theta_A}{(1-\nu)}\frac{\psi_A}{\theta_A} + \frac{(1-\nu\theta_A)^2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)(1-\nu)}\frac{\psi_B}{\theta_B} - \nu\theta_A\theta_B\frac{\psi_B}{\theta_B}\right) = 0 & if\ \theta_A < \theta_B \end{cases}$$

To study how the threshold $\nu^\star$ changes if we increase the correlation between the two analyses findings from $\chi = \max\{\theta_A, \theta_B\}$ to $\chi = 1$, we distinguish between two cases: data set A being more precise than B and vice versa. Data set A's precision-to-cost-of-analysis ratio still exceeds that of B per assumption 1.

Conjecture 1. *For $\theta_A \geq \theta_B$, ν^* is increasing in χ. For $\theta_A < \theta_B$, ν^* is non-monotone in χ.*



(a) $\theta_A = 0.65 \geq \theta_B = 0.55$ (b) $\theta_A = 0.55 < \theta_B = 0.65$

Figure 2: Effect of χ on $\nu^\star$.

If data set B is less precise than A, as the correlation between the findings increases and data set B becomes more redundant, incentive provision issues, which made the principal deviate from the first-best order in the baseline model, intensify. Starting with the data set with the lowest precision-to-cost-of-analysis ratio becomes more likely compared to the scenario with conditionally independent analyses findings when minimizing the expected transfer to the agent.

If data set B is more precise than A, then the two opposing effects are at play. On the one hand, as in the case above, as the correlation between the findings increases and the second analysis becomes more redundant, incentive provision issues intensify. On the other hand, offering the AB sequence may not be as problematic in terms of incentives because data set B is now more precise and it is likely to provide positive evidence for the project's good quality. The overall effect is unambiguous.

Conclusion

Resolving a combined moral hazard and adverse selection problem with dynamic interdependencies results in the deviation from the first-best optimal search order if the project is a priori sufficiently likely to be of bad quality. This "threshold" result holds even if analyses findings are positively correlated. Based on numerical studies we were able to show that changing the correlation parameter had a non-monotone effect on the optimal threshold. The next step would be to arrive at this result analytically and provide deeper intuition for its nature.

Another natural variation of our model is considering the search for a negative result, after which the search can terminate, instead of a positive one. We solved this specification as well (with and without correlation) and, as expected, the result mirrors that of the current paper, i.e. deviating from the first-best becomes optimal if the project is a priori sufficiently likely to be of good quality. A more general framework which combines imperfect positive and negative results seems to be intractable according to our attempts.

We believe that our model could be enriched in a number of ways. One could consider delegated search with a higher number of periods (although a two-period search with heterogeneous data sets already results in an intricate optimization problem). One could also study the optimal number of data sets to analyze (one vs. up-to-two data sets vs. up-two-three data sets), which would basically depend on the investment costs and the project's reward. Lastly, one could discuss the set-up costs born by the data scientist, as data preparation plays an essential role in data analytics, to see how the results of our model would change.

References

Bergemann, Dirk, & Hege, Ulrich. 2005. The Financing of Innovation: Learning and Stopping. *The RAND Journal of Economics*, **36**(4), 719—-752.

Choi, Jin Hyuk, & Han, Kookyoung. 2023. Delegation of information acquisition, information asymmetry, and outside option. *International Journal of Game Theory*, **52**(04), 1–28.

De Mauro, Andrea, Greco, Marco, Grimaldi, Michele, & Ritala, Paavo. 2018. Human resources for Big Data professions: A systematic classification of job roles and required skill sets. *Information Processing Management*, **54**(5), 807–817. In (Big) Data we trust: Value creation in knowledge organizations.

Feldman, Michael, Juldaschewa, Frida, & Bernstein, Abraham. 2017. Data Analytics on Online Labor Markets: Opportunities and Challenges. *arXiv*, **170701790**.

George, Gerard, Haas, Martine R., & Pentland, Alex. 2014. Big data and management: From the Editors. *Academy of Management Journal*, **57**(2), 321–326.

Gerardi, Dino, & Maestri, Lucas. 2012. A principal–agent model of sequential testing. *Theoretical Economics*, **7**(5), 425–463.

Lewis, Tracy R. 2012. A theory of delegated search for the best alternative. *The RAND Journal of Economics*, **43**(3), 391–416.

Lewis, Tracy R., & Ottaviani, Marco. 2008. Search agency. *Working paper*.

McCall, J.J. 1970. Economics of Information and Job Search. *The Quarterly Journal of Economics*, **84**(1), 113—-126.

McCardle, Kevin F. 1985. Information Acquisition and the Adoption of New Technology. *Management Science*, **31**(11), 1372–1389.

Ulbricht, Robert. 2016. Optimal delegated search with adverse selection and moral hazard. *Theoretical Economics*, **11**(1), 253—-278.

Weizman, Martin L. 1979. Optimal Search for the Best Alternative. *Econometrica*, **47**(3), 641—-654.

Xiao, Yangge, Hu, Zhenyu, & Wang, Shouqiang. 2022. Information Design of a Delegated Search. *SSRN*.

Appendix

Proof of Lemma 1

For $i = 1, 2$, make the change of variable $T_i = qt_i^s + (1-q)t_i^f$ and eliminate t_1^s and t_2^s in the optimization problem. Objective (1) is rewritten as follows:

$$\min_{\{T_1, T_2, t_1^f, t_2^f, t\}} \quad p_1 T_1 + (1-p_1)p_2 T_2 + (1-p_1)(1-p_2)t$$

Because t_1^f and t_2^f do not appear in the objective, the proof consists in showing that they only appear, if at all, on the RHS of the constraints. This holds true for the limited liability constraints (3) which are rewritten as:

$$T_1 \geq (1-q)t_1^f, \quad t_1^f \geq 0, \quad T_2 \geq (1-q)t_2^f, \quad t_2^f \geq 0, \quad t \geq 0$$

and for the agent's participation constraint (2) which is rewritten as:

$$p_1 T_1 + (1-p_1)p_2 T_2 + (1-p_1)(1-p_2)t - (\psi_1 + (1-p_1)\psi_2) \geq 0$$

This also holds for constraints (12), (13), and (14) which are rewritten as:

$$p_1 T_1 + (1-p_1)p_2 T_2 + (1-p_1)(1-p_2)t - (\psi_1 + (1-p_1)\psi_2) \geq \nu T_1 + (1-\nu)t_1^f$$

$$p_1 T_1 + (1-p_1)p_2 T_2 + (1-p_1)(1-p_2)t - (\psi_1 + (1-p_1)\psi_2) \geq \nu T_2 + (1-\nu)t_2^f$$

$$p_1 T_1 + (1-p_1)p_2 T_2 + (1-p_1)(1-p_2)t - (\psi_1 + (1-p_1)\psi_2) \geq t$$

Constraints (7) and (11) are rewritten as:

$$(1-p_1)p_2 T_2 + (1-p_1)(1-p_2)t - (1-p_1)\psi_2 \geq (1-p_1)t$$

$$p_1 T_1 + (1-p_1)p_2 T_2 + (1-p_1)(1-p_2)t - (\psi_1 + (1-p_1)\psi_2) \geq \nu\theta_2 T_2 + (1-\nu\theta_2)t - \psi_2$$

Constraints (8), (9), (10), and (6) are rewritten as:

$$(1-p_1)p_2 T_2 + (1-p_1)(1-p_2)t - (1-p_1)\psi_2 \geq (1-p_1)[\nu_1 T_1 + (1-\nu_1)t_1^f]$$

$$T_1 \geq T_2$$

$$T_1 \geq t$$

$$(1-p_1)p_2 T_2 + (1-p_1)(1-p_2)t - (1-p_1)\psi_2 \geq (1-p_1)[\nu_1 T_2 + (1-\nu_1)t_2^f]$$

And, finally, constraints (4) and (5) are rewritten as:

$$T_2 \geq t$$

$$t \geq \nu_2 T_2 + (1-\nu_2)t_2^f$$

Proof of Lemma 2

(10) is implied by (9) and (4) and can, therefore, be eliminated. (4) is implied by (7) and can also be eliminated. Following Lemma 1, we can simplify constraints further.

The limited liability constraints (3) can be rewritten as:

$$T_1 \geq 0, \quad t_1^f \geq 0, \quad T_2 \geq 0, \quad t_2^f \geq 0, \quad t \geq 0$$

Constraints (12) and (13) can be rewritten as:

$$p_1 T_1 + (1-p_1)p_2 T_2 + (1-p_1)(1-p_2)t - (\psi_1 + (1-p_1)\psi_2) \geq \nu T_1$$

$$p_1 T_1 + (1-p_1)p_2 T_2 + (1-p_1)(1-p_2)t - (\psi_1 + (1-p_1)\psi_2) \geq \nu T_2$$

Note that (13) is implied by (9) and (12) and, therefore, can be eliminated.

Constraints (8) and (6) can be rewritten as:

$$(1-p_1)p_2 T_2 + (1-p_1)(1-p_2)t - (1-p_1)\psi_2 \geq (1-p_1)\nu_1 T_1$$

$$(1-p_1)p_2 T_2 + (1-p_1)(1-p_2)t - (1-p_1)\psi_2 \geq (1-p_1)\nu_1 T_2$$

Note that (6) is implied by (9) and (8) and, therefore, can be eliminated.

Constraints (5) can be rewritten as:

$$t \geq \nu_2 T_2$$

Note that (5) is implied by (6) and can be eliminated.

We can now lay out still remaining constraints:

$$p_1 T_1 + (1-p_1)p_2 T_2 + (1-p_1)(1-p_2)t - (\psi_1 + (1-p_1)\psi_2) \geq \nu T_1 \tag{12}$$

$$p_1 T_1 + (1-p_1)p_2 T_2 + (1-p_1)(1-p_2)t - (\psi_1 + (1-p_1)\psi_2) \geq t \tag{14}$$

$$(1-p_1)p_2[T_2 - t] \geq (1-p_1)\psi_2 \tag{7}$$

$$p_1 T_1 + (1-(1-p_1)p_2)T_2 - \nu\theta_1(1-\theta_2)t \geq \psi_1 - p_1\psi_2 \tag{11}$$

$$(1-p_1)p_2 T_2 + (1-p_1)(1-p_2)t - (1-p_1)\psi_2 \geq (1-p_1)\nu_1 T_1 \tag{8}$$

$$T_1 \geq T_2 \tag{9}$$

Note that the participation constraint (2) is implied by (3) and (14). Furthermore, (7) and (5) imply $t_2^s \geq 0$; (9) implies $t \geq 0$; (5) implies $t_1^s \geq 0$.

Constraint (8) is implied by (12):

$$(1-p_1)p_2T_2 + (1-p_1)(1-p_2)t \geq (\nu - p_1)T_1 + (\psi_1 + (1-p_1)\psi_2) > (1-p_1)\nu_1 T_1 + (1-p_1)\psi_2$$

Constraint (14) is always slack due to (7) and (11): the LHS's of (14) and (11) are the same. The RHS of (11) is, however, larger than that of (14):

$$\nu\theta_2 T_2 + (1-\nu\theta_2)t - \psi_2 \geq \nu\theta_2[t + \frac{(1-p_1)}{(1-p_1)p_2}\psi_2] + (1-\nu\theta_2)t - \psi_2 +$$

$$= t + \psi_2 \cdot [\frac{(1-p_1)\nu\theta_2 - (1-p_1)p_2}{(1-p_1)p_2}] > t$$

Thus, the limited liability constraints, the participation constraint and a number of incentive constraints are all implied by the following four constraints:

$$p_1T_1 + (1-p_1)p_2T_2 + (1-p_1)(1-p_2)t - (\psi_1 + (1-p_1)\psi_2) \geq \nu T_1 \tag{12}$$

$$(1-p_1)p_2[T_2 - t] \geq (1-p_1)\psi_2 \tag{7}$$

$$p_1T_1 - \nu\theta_1\theta_2T_2 - \nu\theta_1(1-\theta_2)t \geq \psi_1 - p_1\psi_2 \tag{11}$$

$$T_1 \geq T_2 \tag{9}$$

Proof of Proposition 2

Following Lemma 1, we can rewrite the principal's objective as:

$$\min_{\{t_1^s,\ t_2^s,\ t\}} \quad p_1qt_1^s + (1-p_1)p_2qt_2^s + (1-p_1)(1-p_2)t$$

Define $\delta_i \equiv qt_i^s - t$ for $i = 1, 2$. Then the principal's problem can be rewritten as:

$$\min_{\{\delta_1,\ \delta_2,\ t\}} \quad p_1\delta_1 + (1-p_1)p_2\delta_2 + t$$

subject to

$$p_1\delta_1 + (1-p_1)p_2\delta_2 + t - (\psi_1 + (1-p_1)\psi_2) \geq \nu qt_1^s \tag{12}$$

$$(1-p_1)p_2\delta_2 \geq (1-p_1)\psi_2 \tag{7}$$

$$p_1\delta_1 \geq \nu\theta_1\theta_2\delta_2 + (\psi_1 + (1-p_1)\psi_2) - \psi_2 \tag{11}$$

$$\delta_1 \geq \delta_2 \tag{9}$$

Note that constraint (12) binds as the only constraint with t on the LHS. Constraint (7) then binds as the only constraint with δ_2 on the LHS. Hence, we can rewrite constraints (11) and (9) as:

$$\begin{aligned}
\nu\left(1-\theta_1\right)\delta_1 &\geq \left(1-\theta_1\right)\frac{\psi_1}{\theta_1}+\left(1-\nu\right)\theta_2\frac{\psi_2}{\theta_2} \\
\nu\left(1-\theta_1\right)\delta_1 &\geq \left[\left(1-\nu\right)+\nu\left(1-\theta_1\right)\right]\frac{\psi_2}{\theta_2}
\end{aligned}$$

The difference in the RHS's between (9) and (11) is

$$\begin{aligned}
\Delta_{12} &= \left[\left(1-\nu\right)+\nu\left(1-\theta_1\right)\right]\frac{\psi_2}{\theta_2}-\left(1-\theta_1\right)\frac{\psi_1}{\theta_1}-\left(1-\nu\right)\theta_2\frac{\psi_2}{\theta_2} \\
&= \left[\left(1-\nu\right)\left(1-\theta_2\right)+\nu\left(1-\theta_1\right)\right]\frac{\psi_2}{\theta_2}-\left(1-\theta_1\right)\frac{\psi_1}{\theta_1} \\
&= \left(1-\nu\right)\left(\theta_1-\theta_2\right)\frac{\psi_2}{\theta_2}-\left(1-\theta_1\right)\left(\frac{\psi_1}{\theta_1}-\frac{\psi_2}{\theta_2}\right)
\end{aligned}$$

(9) binds and (11) is slack iff:

$$\Delta_{12}\equiv(1-\nu)(\theta_1-\theta_2)\frac{\psi_2}{\theta_2}-(1-\theta_1)(\frac{\psi_1}{\theta_1}-\frac{\psi_2}{\theta_2})\geq 0$$

$$\begin{aligned}
&\overset{x_{12}\equiv\frac{\psi_2/\theta_2}{\psi_1/\theta_1}}{\Leftrightarrow} (1-\nu\theta_1-(1-\nu)\theta_2)x_{12}\geq 1-\theta_1 \\
&\Leftrightarrow x_{12}\geq\frac{1-\theta_1}{\nu(1-\theta_1)+(1-\nu)(1-\theta_2)}
\end{aligned}$$

If constraints (12), (7) and (9) are binding, we obtain the following contract:

$$\begin{aligned}
\delta_1 &= \frac{\left[\left(1-\nu\right)+\nu\left(1-\theta_1\right)\right]}{\nu\left(1-\theta_1\right)}\frac{\psi_2}{\theta_2} \\
\delta_2 &= \frac{\left[\left(1-\nu\right)+\nu\left(1-\theta_1\right)\right]}{\nu\left(1-\theta_1\right)}\frac{\psi_2}{\theta_2} \\
t &= \frac{\left[\left(1-\nu\right)+\nu\left(1-\theta_1\right)\right]}{\left(1-\nu\right)}\frac{\psi_2}{\theta_2}+\frac{\psi_1}{\left(1-\nu\right)}
\end{aligned}$$

The expected transfer then is:

$$\mathrm{E}\left[t(9)\right] = \left[\left(1-\nu\right)+\nu\left(1-\theta_1\right)\right]\left(\frac{\left[\theta_1+\left(1-\theta_1\right)\theta_2\right]}{\left(1-\theta_1\right)}+\frac{1}{\left(1-\nu\right)}\right)\frac{\psi_2}{\theta_2}+\frac{\theta_1}{\left(1-\nu\right)}\frac{\psi_1}{\theta_1}$$

If constraints (12), (7) and (11) are binding, we obtain the following contract:

$$\begin{aligned}
\delta_1 &= \frac{1}{\nu}\frac{\psi_1}{\theta_1}+\frac{\left(1-\nu\right)\theta_2}{\nu\left(1-\theta_1\right)}\frac{\psi_2}{\theta_2} \\
\delta_2 &= \frac{\left[\left(1-\nu\right)+\nu\left(1-\theta_1\right)\right]}{\nu\left(1-\theta_1\right)}\frac{\psi_2}{\theta_2} \\
t &= \frac{\psi_1}{\left(1-\nu\right)\theta_1}+\theta_2\frac{\psi_2}{\theta_2}
\end{aligned}$$

The expected transfer then is:

$$\mathrm{E}\left[t(11)\right] = \left(\theta_1 + \frac{1}{(1-\nu)}\right)\frac{\psi_1}{\theta_1} + \left(1 + \frac{(1-\nu)}{(1-\theta_1)} + \nu\left(1-\theta_1\right)\right)\theta_2\frac{\psi_2}{\theta_2}$$

Proof of Theorem 1

Denote $x \equiv \dfrac{\psi_B/\theta_B}{\psi_A/\theta_A}$, $x \geq 1$ by assumption 1.
For AB, (9) binds and (11) is slack iff:

$$x \geq \frac{1-\theta_A}{\nu(1-\theta_A)+(1-\nu)(1-\theta_B)} \qquad (*)$$

For BA, (9) binds and (11) is slack iff:

$$x \leq \frac{\nu(1-\theta_B)+(1-\nu)(1-\theta_A)}{1-\theta_B} \qquad (**)$$

We show that $RHS(*) \leq RHS(**)$ for all ν: firstly, $RHS(*) = RHS(**)$ for $\nu = 0$ and $\nu = 1$; secondly, $RHS(*) - RHS(**)$ is convex in ν:

$$\frac{\partial(RHS(*) - RHS(**))}{\partial\nu} = (\theta_A - \theta_B)^2(2\nu - 1).$$

We proceed by going through all possible cases of binding constraints for the AB and BA contracts. For each case we evaluate the sign of the difference between the expected transfers for AB and BA. If the difference is negative, AB order is optimal. Otherwise, BA is optimal.

1. If $\theta_A \geq \theta_B$, (9) binds for AB and (11) binds for BA:

$$\begin{aligned}\Delta &\equiv E[t(9)^{AB}] - E[t(11)^{BA}] = \\ &= \theta_A\left(1-\nu\right)\left(\frac{1}{(1-\theta_A)}\frac{\psi_B}{\theta_B} - \frac{1}{(1-\theta_B)}\frac{\psi_A}{\theta_A}\right) - \theta_A\nu\left(\frac{1}{(1-\nu)} - (1-\theta_B)\right)\left(\frac{\psi_B}{\theta_B} - \frac{\psi_A}{\theta_A}\right) = \\ &= (1-\nu)\left[\frac{1}{(1-\theta_A)}x - \frac{1}{(1-\theta_B)}\right] - \nu\left[\frac{1}{(1-\nu)} - (1-\theta_B)\right](x-1)\end{aligned}$$

   Note that Δ is decreasing in ν; for $\nu = 0$, $\Delta = \left[\frac{1}{(1-\theta_A)}x - \frac{1}{(1-\theta_B)}\right] > 0$; for $\nu \to 1$, $\Delta \to -\left[\frac{1}{(1-\nu)} - (1-\theta_B)\right](x-1) < 0$.

2. If $\theta_A < \theta_B$, we need to consider further subcases.
   If $x \geq \dfrac{(1-\theta_A)}{(1-\theta_B)}$, (9) binds for AB and (11) binds for BA:
   Δ is decreasing in ν; for $\nu = 0$, $\Delta = \left[\frac{1}{(1-\theta_A)}x - \frac{1}{(1-\theta_B)}\right] > 0$; for $\nu \to 1$, $\Delta \to -\left[\frac{1}{(1-\nu)} - (1-\theta_B)\right](x-1) < 0$.

If $x < \dfrac{(1-\theta_A)}{(1-\theta_B)}$, $\exists\ \hat{\nu}, \hat{\hat{\nu}}$ with $\hat{\nu} < \hat{\hat{\nu}}$ such that:

- If $\nu \leq \hat{\nu}$, (11) binds for AB and (9) binds for BA:
  $E[t(11)^{AB}] - E[t(9)^{BA}] = -(E[t(9)^{BA}] - E[t(11)^{AB}]) =$
  $= (1-\nu)\left[\dfrac{1}{1-\theta_A} - \dfrac{1}{1-\theta_B} \cdot \dfrac{1}{x}\right] - \nu\left[\dfrac{1}{1-\nu} - (1-\theta_A)\right](1-1/x) < 0.$
- If $\nu \in [\hat{\nu}, \hat{\hat{\nu}}]$, (9) binds for both AB and BA. For BA, if (9) is removed, (11) binds and the optimum improves: $E[t(11)^{BA}] < E[t(9)^{BA}]$.
  Since $\Delta = (1-\nu)\left[\dfrac{1}{(1-\theta_A)}x - \dfrac{1}{(1-\theta_B)}\right] - \nu\left[\dfrac{1}{(1-\nu)} - (1-\theta_B)\right](x-1) < 0$, $E[t(9)^{AB}] < E[t(11)^{BA}] < E[t(9)^{BA}]$.
- If $\nu > \hat{\hat{\nu}}$, (9) binds for AB and (11) binds for BA: $\Delta < 0$.

Proof of Theorem 2

We start the proof by defining precision again (as in the Model section). For this we denote "good news" from data set $i = 1, 2$ by $\overline{\sigma}_i$ and "bad news" (no positive signal) by $\underline{\sigma}_i$. Probability of the project being of good quality is $\Pr[Q = G] = \nu$, and $\Pr[\sigma_i = \overline{\sigma}_i \mid Q = G] \equiv \theta_i$. For $\theta_j \geq \theta_i$, we define correlation as $\Pr[\sigma_j = \overline{\sigma}_j \mid Q = G \cap \sigma_i = \overline{\sigma}_i] = \Pr[\sigma_j = \overline{\sigma}_j \mid \sigma_i = \overline{\sigma}_i] \equiv \chi \in [\theta_j, 1]$. Let $p(\omega)$ be the probability of an event ω and $\nu(\omega)$ the belief about the project being good conditional on the event ω.

We further **outline and simplify incentive constraints**. This proof contains own constraint enumeration. We are facing the following optimization problem, where $T_i \equiv q t_i^s + (1-q) t_i^f$, $i = 1, 2$:

$$\min p(\overline{\sigma}_1) T_1 + p(\underline{\sigma}_1, \overline{\sigma}_2) T_2 + p(\underline{\sigma}_1, \underline{\sigma}_2) t$$

subject to a participation constraint and limited liability constraints:

$$p(\overline{\sigma}_1) T_1 + p(\underline{\sigma}_1, \overline{\sigma}_2) T_2 + p(\underline{\sigma}_1, \underline{\sigma}_2) t - \psi_1 - p(\underline{\sigma}_1)\psi_2 \geq 0 \tag{A1}$$

$$t_1^s \geq 0 \tag{A2}$$

$$t_1^f \geq 0 \tag{A3}$$

$$t_2^s \geq 0 \tag{A4}$$

$$t_2^f \geq 0 \tag{A5}$$

$$t \geq 0 \tag{A6}$$

If $(\underline{\sigma}_1, \underline{\sigma}_2)$, report truthfully:

$$t \geq \nu(\underline{\sigma}_1, \underline{\sigma}_2) T_2 + (1 - \nu(\underline{\sigma}_1, \underline{\sigma}_2)) t_2^f \tag{A7}$$

If $(\underline{\sigma}_1, \bar{\sigma}_2)$, report truthfully:

$$T_2 \geq t \tag{A8}$$

If $\underline{\sigma}_1$, report and acquire σ_2:

$$-\psi_2 + p(\bar{\sigma}_2|\underline{\sigma}_1)T_2 + p(\underline{\sigma}_2|\underline{\sigma}_1)t \geq t \tag{A9}$$

$$-\psi_2 + p(\bar{\sigma}_2|\underline{\sigma}_1)T_2 + p(\underline{\sigma}_2|\underline{\sigma}_1)t \geq \nu(\underline{\sigma}_1)T_2 + (1-\nu(\underline{\sigma}_1))t_2^f \tag{A10}$$

$$-\psi_2 + p(\bar{\sigma}_2|\underline{\sigma}_1)T_2 + p(\underline{\sigma}_2|\underline{\sigma}_1)t \geq \nu(\underline{\sigma}_1)T_1 + (1-\nu(\underline{\sigma}_1))t_1^f \tag{A11}$$

If $\bar{\sigma}_1$, then report:

$$T_1 \geq t \tag{A12}$$

$$T_1 \geq T_2 \tag{A13}$$

Acquire σ_1:

$$p\left(\overline{\sigma}_1\right)T_1 + p\left(\underline{\sigma}_1, \overline{\sigma}_2\right)T_2 + p\left(\underline{\sigma}_1, \underline{\sigma}_2\right)t - \psi_1 - p(\underline{\sigma}_1)\psi_2 \geq t \tag{A14}$$

$$p\left(\overline{\sigma}_1\right)T_1 + p\left(\underline{\sigma}_1, \overline{\sigma}_2\right)T_2 + p\left(\underline{\sigma}_1, \underline{\sigma}_2\right)t - \psi_1 - p(\underline{\sigma}_1)\psi_2 \geq \nu T_1 + (1-\nu)t_1^f \tag{A15}$$

$$p\left(\overline{\sigma}_1\right)T_1 + p\left(\underline{\sigma}_1, \overline{\sigma}_2\right)T_2 + p\left(\underline{\sigma}_1, \underline{\sigma}_2\right)t - \psi_1 - p(\underline{\sigma}_1)\psi_2 \geq \nu T_2 + (1-\nu)t_2^f \tag{A16}$$

$$p\left(\overline{\sigma}_1\right)T_1 + p\left(\underline{\sigma}_1, \overline{\sigma}_2\right)T_2 + p\left(\underline{\sigma}_1, \underline{\sigma}_2\right)t - \psi_1 - p(\underline{\sigma}_1)\psi_2 \geq -\psi_2 + p(\bar{\sigma}_2)T_2 + p(\underline{\sigma}_2)t \tag{A17}$$

Elimination of constraints is similar to the baseline model and is as follows: (A3) and (A5) bind (weakly) because they are on the RHS of all constraints; (A1) is implied by (A6) and (A14); (A2) is implied by (A6) and (A12); (A4) is implied by (A6) and (A8); (A6) is implied by (A7) and (A9); (A7) is implied by (A10); (A8) is implied by (A9); (A10) is implied by (A11) and (A13); (A11) is implied by (A15); (A12) is implied by (A9) and (A13); (A14) is implied by (A9) and (A17); and (A16) is implied by (A13) and (A15).

Thus, the problem boils down to:

$$\min p\left(\overline{\sigma}_1\right)T_1 + p\left(\underline{\sigma}_1, \overline{\sigma}_2\right)T_2 + p\left(\underline{\sigma}_1, \underline{\sigma}_2\right)t$$

subject to

$$p\left(\underline{\sigma}_1, \overline{\sigma}_2\right)T_2 \geq p\left(\underline{\sigma}_1\right)\psi_2 + p\left(\underline{\sigma}_1, \overline{\sigma}_2\right)t \tag{A9}$$

$$T_1 \geq T_2 \tag{A13}$$

$$p\left(\underline{\sigma}_1, \overline{\sigma}_2\right)T_2 + p\left(\underline{\sigma}_1, \underline{\sigma}_2\right)t \geq \psi_1 + p\left(\underline{\sigma}_1\right)\psi_2 + p\left(\underline{\sigma}_1\right)\nu\left(\underline{\sigma}_1\right)T_1 \tag{A15}$$

$$p\left(\overline{\sigma}_1\right)T_1 + p\left(\underline{\sigma}_1, \overline{\sigma}_2\right)T_2 + p\left(\underline{\sigma}_1, \underline{\sigma}_2\right)t - \psi_1 - p\left(\underline{\sigma}_1\right)\psi_2 \geq -\psi_2 + p\left(\overline{\sigma}_2\right)T_2 + p\left(\underline{\sigma}_2\right)t \tag{A17}$$

(A15) is the only constraint on t (because $p\left(\underline{\sigma}_2\right) \geq p\left(\underline{\sigma}_1, \underline{\sigma}_2\right)$ in (A17)). Therefore, (A15) is binding. Eliminating t, we can rewrite the problem as:

$$\min \nu T_1 + \psi_1 + p(\underline{\sigma}_1)\psi_2$$

subject to

$$p(\underline{\sigma}_1, \underline{\sigma}_2) p(\underline{\sigma}_1, \overline{\sigma}_2) T_2 \geq p(\underline{\sigma}_1, \underline{\sigma}_2) p(\underline{\sigma}_1)\psi_2 + p(\underline{\sigma}_1, \overline{\sigma}_2)[\psi_1 + p(\underline{\sigma}_1)\psi_2 + p(\underline{\sigma}_1)\nu(\underline{\sigma}_1) T_1 - p(\underline{\sigma}_1, \overline{\sigma}_2) T_2] \tag{A9}$$

$$T_1 \geq T_2 \tag{A13}$$

$$p(\underline{\sigma}_1, \underline{\sigma}_2)\nu T_1 \geq -p(\underline{\sigma}_1, \underline{\sigma}_2)\psi_2 + p(\underline{\sigma}_1, \underline{\sigma}_2) p(\overline{\sigma}_2) T_2 + p(\underline{\sigma}_2)\begin{bmatrix} \psi_1 + p(\underline{\sigma}_1)\psi_2 \\ +p(\underline{\sigma}_1)\nu(\underline{\sigma}_1) T_1 - p(\underline{\sigma}_1, \overline{\sigma}_2) T_2 \end{bmatrix} \tag{A17}$$

or, after simplification:

$$\min \nu T_1 + \psi_1 + p(\underline{\sigma}_1)\psi_2$$

subject to

$$p(\underline{\sigma}_1) p(\underline{\sigma}_1, \overline{\sigma}_2) T_2 \geq p(\underline{\sigma}_1, \underline{\sigma}_2) p(\underline{\sigma}_1)\psi_2 + p(\underline{\sigma}_1, \overline{\sigma}_2)\psi_1 + p(\underline{\sigma}_1) p(\underline{\sigma}_1, \overline{\sigma}_2)\psi_2 + p(\underline{\sigma}_1)\nu(\underline{\sigma}_1) p(\underline{\sigma}_1, \overline{\sigma}_2) T_1 \tag{A9}$$

$$T_1 \geq T_2 \tag{A13}$$

$$\begin{bmatrix} p(\underline{\sigma}_1, \underline{\sigma}_2)\nu \\ -p(\underline{\sigma}_1)\nu(\underline{\sigma}_1) p(\underline{\sigma}_2) \end{bmatrix} T_1 + \begin{bmatrix} p(\underline{\sigma}_1, \underline{\sigma}_2) \\ -p(\underline{\sigma}_1) p(\underline{\sigma}_2) \end{bmatrix}\psi_2 \geq p(\underline{\sigma}_2)\psi_1 + \begin{bmatrix} p(\underline{\sigma}_1, \underline{\sigma}_2) p(\overline{\sigma}_2) \\ -p(\underline{\sigma}_1, \overline{\sigma}_2) p(\underline{\sigma}_2) \end{bmatrix} T_2 \tag{A17}$$

(A9) is the only constraint on T_2 (because no-negative correlation implies $p(\overline{\sigma}_2) \geq p(\underline{\sigma}_1, \overline{\sigma}_2)$ on the RHS of (A17)). Therefore, (A9) is binding. Eliminating T_2, we can rewrite the problem as:

$$\min \nu T_1 + \psi_1 + p(\underline{\sigma}_1)\psi_2$$

subject to

$$p(\underline{\sigma}_1) p(\underline{\sigma}_1, \overline{\sigma}_2) T_1 \geq p(\underline{\sigma}_1, \underline{\sigma}_2) p(\underline{\sigma}_1)\psi_2 + p(\underline{\sigma}_1, \overline{\sigma}_2)\psi_1 + p(\underline{\sigma}_1) p(\underline{\sigma}_1, \overline{\sigma}_2)\psi_2 + p(\underline{\sigma}_1)\nu(\underline{\sigma}_1) p(\underline{\sigma}_1, \overline{\sigma}_2) T_1 \tag{A13}$$

$$\begin{aligned} & p(\underline{\sigma}_1) p(\underline{\sigma}_1, \overline{\sigma}_2)[p(\underline{\sigma}_1, \underline{\sigma}_2)\nu - p(\underline{\sigma}_1)\nu(\underline{\sigma}_1) p(\underline{\sigma}_2)] T_1 + p(\underline{\sigma}_1) p(\underline{\sigma}_1, \overline{\sigma}_2)[p(\underline{\sigma}_1, \underline{\sigma}_2) - p(\underline{\sigma}_1) p(\underline{\sigma}_2)]\psi_2 \\ & \geq p(\underline{\sigma}_1) p(\underline{\sigma}_1, \overline{\sigma}_2) p(\underline{\sigma}_2)\psi_1 + \begin{bmatrix} p(\underline{\sigma}_1, \underline{\sigma}_2) p(\overline{\sigma}_2) \\ -p(\underline{\sigma}_1, \overline{\sigma}_2) p(\underline{\sigma}_2) \end{bmatrix}\begin{bmatrix} p(\underline{\sigma}_1, \underline{\sigma}_2) p(\underline{\sigma}_1)\psi_2 + p(\underline{\sigma}_1, \overline{\sigma}_2)\psi_1 \\ +p(\underline{\sigma}_1) p(\underline{\sigma}_1, \overline{\sigma}_2)\psi_2 + p(\underline{\sigma}_1)\nu(\underline{\sigma}_1) p(\underline{\sigma}_1, \overline{\sigma}_2) T_1 \end{bmatrix} \end{aligned} \tag{A17}$$

or, after simplification:

$$\min \nu T_1 + \psi_1 + p(\underline{\sigma}_1)\psi_2$$

subject to

$$p(\underline{\sigma}_1, \overline{\sigma}_2)(1-\nu) T_1 \geq p(\underline{\sigma}_1, \overline{\sigma}_2)\psi_1 + p(\underline{\sigma}_1)^2\psi_2 \tag{A13}$$

$$p(\underline{\sigma}_1, \overline{\sigma}_2)(1-\nu) p(\underline{\sigma}_1, \underline{\sigma}_2) p(\overline{\sigma}_1) T_1 \geq p(\underline{\sigma}_1) p(\underline{\sigma}_1, \underline{\sigma}_2)[p(\underline{\sigma}_1) p(\overline{\sigma}_2) - p(\underline{\sigma}_1, \overline{\sigma}_2)]\psi_2 + p(\underline{\sigma}_1, \overline{\sigma}_2) p(\underline{\sigma}_1, \underline{\sigma}_2)\psi_1 \tag{A17}$$

Thus, (A13) is binding iff

$$p\left(\underline{\sigma}_1,\overline{\sigma}_2\right)p\left(\underline{\sigma}_1,\underline{\sigma}_2\right)p\left(\overline{\sigma}_1\right)\psi_1+p\left(\underline{\sigma}_1\right)^2p\left(\underline{\sigma}_1,\underline{\sigma}_2\right)p\left(\overline{\sigma}_1\right)\psi_2$$
$$\geq p\left(\underline{\sigma}_1\right)p\left(\underline{\sigma}_1,\underline{\sigma}_2\right)\left[p\left(\underline{\sigma}_1\right)p\left(\overline{\sigma}_2\right)-p\left(\underline{\sigma}_1,\overline{\sigma}_2\right)\right]\psi_2+p\left(\underline{\sigma}_1,\overline{\sigma}_2\right)p\left(\underline{\sigma}_1,\underline{\sigma}_2\right)\psi_1$$

or

$$p\left(\underline{\sigma}_1\right)p\left(\overline{\sigma}_1\right)\psi_2\geq\left[p\left(\underline{\sigma}_1\right)p\left(\overline{\sigma}_2\right)-p\left(\underline{\sigma}_1,\overline{\sigma}_2\right)\right]\psi_2+p\left(\underline{\sigma}_1,\overline{\sigma}_2\right)\psi_1$$

or

$$\frac{\left[p\left(\underline{\sigma}_1\right)p\left(\overline{\sigma}_1\right)+p\left(\underline{\sigma}_1,\overline{\sigma}_2\right)-p\left(\underline{\sigma}_1\right)p\left(\overline{\sigma}_2\right)\right]p\left(\overline{\sigma}_2\right)}{p\left(\underline{\sigma}_1,\overline{\sigma}_2\right)p\left(\overline{\sigma}_1\right)}\geq\frac{\psi_1/p\left(\overline{\sigma}_1\right)}{\psi_2/p\left(\overline{\sigma}_2\right)}$$

or expressing everything with $p\left(\overline{\sigma}_i\right)$:

$$\frac{\left[p\left(\overline{\sigma}_1\right)+p\left(\overline{\sigma}_1\right)p\left(\overline{\sigma}_2\right)-p\left(\overline{\sigma}_1\right)^2-p\left(\overline{\sigma}_1,\overline{\sigma}_2\right)\right]p\left(\overline{\sigma}_2\right)}{\left[p\left(\overline{\sigma}_2\right)-p\left(\overline{\sigma}_1,\overline{\sigma}_2\right)\right]p\left(\overline{\sigma}_1\right)}\geq\frac{\psi_1/p\left(\overline{\sigma}_1\right)}{\psi_2/p\left(\overline{\sigma}_2\right)}$$

- If $\theta_1\geq\theta_2$: (A13) is binding iff

$$\frac{\left[\nu\theta_1+\nu\theta_1\nu\theta_2-\nu^2\theta_1^2-\nu\theta_2\chi\right]\nu\theta_2}{\left[\nu\theta_2-\nu\theta_2\chi\right]\nu\theta_1}\geq\frac{\psi_1/p\left(\overline{\sigma}_1\right)}{\psi_2/p\left(\overline{\sigma}_2\right)}$$

  or

$$\frac{\left[1-\frac{\theta_2}{\theta_1}\chi-\nu\left(\theta_1-\theta_2\right)\right]}{(1-\chi)}\geq\frac{\psi_1/\theta_1}{\psi_2/\theta_2}$$

- If $\theta_1\leq\theta_2$: (A13) is binding iff

$$\frac{\left[\nu\theta_1+\nu\theta_1\nu\theta_2-\nu^2\theta_1^2-\nu\theta_1\chi\right]\nu\theta_2}{\left[\nu\theta_2-\nu\theta_1\chi\right]\nu\theta_1}\geq\frac{\psi_1/p\left(\overline{\sigma}_1\right)}{\psi_2/p\left(\overline{\sigma}_2\right)}$$

  or

$$\frac{\left[1-\chi+\nu\left(\theta_2-\theta_1\right)\right]}{\left[1-\frac{\theta_1}{\theta_2}\chi\right]}\geq\frac{\psi_1/\theta_1}{\psi_2/\theta_2}$$

As in the baseline model, we proceed by **considering all possible cases for binding constraints for the *AB* and *BA* sequences**. We **then consider the difference in the expected transfers across *AB* and *BA* to define the optimal order**. Be begin with the case where data set A has a higher precision.

• $\boldsymbol{\theta_A\geq\theta_B}$

Lemma 3. *If $\theta_A\geq\theta_B$, constraint (A13) is binding for all ν for the AB sequence.*

Proof: (A13) is binding iff

$$\frac{\left[1+\nu\theta_B-\nu\theta_A-\theta_B\frac{\chi}{\theta_A}\right]}{(1-\chi)}\geq\frac{\psi_A/\theta_A}{\psi_B/\theta_B}$$

or

$$x \equiv \frac{\psi_B/\theta_B}{\psi_A/\theta_A} \geq \frac{(1-\chi)}{\left[1+\nu\theta_B-\nu\theta_A-\theta_B\frac{\chi}{\theta_A}\right]}.$$

The RHS of this condition increases with ν. Moreover:

$$\begin{aligned} 1-RHS(1) &= 1-\frac{(1-\chi)}{\left[1+\theta_B-\theta_A-\theta_B\frac{\chi}{\theta_A}\right]} = \frac{1+\theta_B-\theta_A-\theta_B\frac{\chi}{\theta_A}-(1-\chi)}{1+\theta_B-\theta_A-\theta_B\frac{\chi}{\theta_A}} \\ &= \frac{(\chi-\theta_A)(\theta_A-\theta_B)}{\theta_A\left(1+\theta_B-\theta_A-\theta_B\frac{\chi}{\theta_A}\right)} = \frac{(\chi-\theta_A)(\theta_A-\theta_B)}{((1-\theta_A)(\theta_A-\theta_B)+\theta_B(1-\chi))} > 0 \end{aligned}$$

Hence, $RHS(\nu) < 1 < x$ for all ν. Hence, (A13) is binding for all ν. ■

Lemma 4. *If $\theta_A \geq \theta_B$, constraint (A17) is binding for all ν for the BA sequence.*

Proof: (A13) is binding iff

$$\frac{[1+\nu\theta_A-\nu\theta_B-\chi]}{\left[1-\theta_B\frac{\chi}{\theta_A}\right]} \geq x$$

The RHS of this condition increases with ν. Moreover:

$$\begin{aligned} 1-RHS(1) &= 1-\frac{[1+\theta_A-\theta_B-\chi]}{\left[1-\theta_B\frac{\chi}{\theta_A}\right]} = \frac{-\theta_B\frac{\chi}{\theta_A}-\theta_A+\theta_B+\chi}{\left[1-\theta_B\frac{\chi}{\theta_A}\right]} \\ &= \frac{(\chi-\theta_A)(\theta_A-\theta_B)}{\theta_A\left[1-\theta_B\frac{\chi}{\theta_A}\right]} > 0 \end{aligned}$$

Hence, $RHS(\nu) < 1 < x$ for all ν. Hence, (A13) is slack and (A17) is binding for all ν. ■

Lemma 5. *Comparing expected transfers across AB and BA amounts to comparing $T_1 - \theta_1\psi_2$.*

Proof: Expected transfers are equal to

$$\nu T_1+\psi_1+p(\underline{\sigma}_1)\psi_2 = \nu T_1+\psi_1+(1-p(\overline{\sigma}_1))\psi_2 = \nu T_1+\psi_1+(1-\nu\theta_1)\psi_2 = \nu(T_1-\theta_1\psi_2)+(\psi_1+\psi_2).$$

The terms $(\psi_1+\psi_2)$ cancel out in the comparison. ■

For BA, T_1^{BA} is given by (A17) binding:

$$p(\underline{\sigma}_B,\overline{\sigma}_A)(1-\nu)p(\underline{\sigma}_B,\underline{\sigma}_A)p(\overline{\sigma}_B)T_1^{BA} = p(\underline{\sigma}_B)p(\underline{\sigma}_B,\underline{\sigma}_A)\begin{bmatrix} p(\underline{\sigma}_B)p(\overline{\sigma}_A) \\ -p(\underline{\sigma}_B,\overline{\sigma}_A)\end{bmatrix}\psi_A+p(\underline{\sigma}_B,\overline{\sigma}_A)p(\underline{\sigma}_B,\underline{\sigma}_A)\psi_B \tag{A17}$$

$$\begin{aligned} T_1^{BA} &= \frac{p(\underline{\sigma}_B)}{p(\underline{\sigma}_B,\overline{\sigma}_A)(1-\nu)p(\overline{\sigma}_B)}[p(\underline{\sigma}_B)p(\overline{\sigma}_A)-p(\underline{\sigma}_B,\overline{\sigma}_A)]\psi_A+\frac{1}{(1-\nu)p(\overline{\sigma}_B)}\psi_B \\ &= \frac{(1-\nu\theta_B)}{[\nu\theta_A-\nu\theta_B\chi](1-\nu)\nu\theta_B}[(1-\nu\theta_B)\nu\theta_A-(\nu\theta_A-\nu\theta_B\chi)]\psi_A+\frac{1}{(1-\nu)\nu\theta_B}\psi_B \\ &= \frac{(1-\nu\theta_B)(\chi-\nu\theta_A)\theta_A}{\nu(1-\nu)(\theta_A-\theta_B\chi)}\frac{\psi_A}{\theta_A}+\frac{1}{(1-\nu)\nu}\frac{\psi_B}{\theta_B} \end{aligned}$$

$$\begin{aligned} E\left[t_{BA}(A17)\right] &= \nu T_1^{BA} + \psi_B + p\left(\underline{\sigma}_B\right)\psi_A = \nu\left(T_1^{BA} - \theta_B\psi_A\right) + \left(\psi_A + \psi_B\right) \\ &= \frac{\left(1-\nu\theta_B\right)\left(\chi-\nu\theta_A\right)\theta_A}{\left(1-\nu\right)\left(\theta_A-\theta_B\chi\right)}\frac{\psi_A}{\theta_A} + \frac{1}{\left(1-\nu\right)}\frac{\psi_B}{\theta_B} - \nu\theta_B\psi_A + \left(\psi_A+\psi_B\right) \end{aligned}$$

For AB, T_1^{AB} is given by (A13) binding:

$$p\left(\underline{\sigma}_A, \overline{\sigma}_B\right)\left(1-\nu\right)T_1^{AB} = p\left(\underline{\sigma}_A, \overline{\sigma}_B\right)\psi_A + p\left(\underline{\sigma}_A\right)^2\psi_B \tag{A13}$$

$$\begin{aligned} T_1^{AB} &= \frac{1}{\left(1-\nu\right)}\psi_A + \frac{p\left(\underline{\sigma}_A\right)^2}{p\left(\underline{\sigma}_A, \overline{\sigma}_B\right)\left(1-\nu\right)}\psi_B \\ &= \frac{1}{\left(1-\nu\right)}\psi_A + \frac{\left(1-\nu\theta_A\right)^2}{\nu\left(1-\chi\right)\left(1-\nu\right)}\frac{\psi_B}{\theta_B} \end{aligned}$$

$$\begin{aligned} E\left[t_{AB}(A13)\right] &= \nu T_1^{AB} + \psi_A + p\left(\underline{\sigma}_A\right)\psi_B = \nu\left(T_1^{AB} - \theta_A\psi_B\right) + \left(\psi_A+\psi_B\right) \\ &= \frac{\nu}{\left(1-\nu\right)}\psi_A + \frac{\left(1-\nu\theta_A\right)^2}{\left(1-\chi\right)\left(1-\nu\right)}\frac{\psi_B}{\theta_B} - \nu\theta_A\psi_B + \left(\psi_A+\psi_B\right) \end{aligned}$$

Define $\Delta_1 \equiv E\left[t_{BA}(A17)\right] - E\left[t_{AB}(A13)\right]$. BA is optimal iff $\Delta_1 < 0$.

$$\Delta_1 = \frac{\left(1-\nu\theta_B\right)\left(\chi-\nu\theta_A\right)\theta_A}{\left(1-\nu\right)\left(\theta_A-\theta_B\chi\right)}\frac{\psi_A}{\theta_A} + \frac{1}{\left(1-\nu\right)}\frac{\psi_B}{\theta_B} - \nu\theta_B\psi_A - \left(\frac{\nu}{\left(1-\nu\right)}\psi_A + \frac{\left(1-\nu\theta_A\right)^2}{\left(1-\chi\right)\left(1-\nu\right)}\frac{\psi_B}{\theta_B} - \nu\theta_A\psi_B\right)$$

Δ_1 has the same sign as $\frac{(1-\nu)\Delta_1}{\nu} =$

$$\begin{aligned} &= \frac{\left(1-\nu\theta_B\right)\left(\chi-\nu\theta_A\right)\theta_A}{\nu\left(\theta_A-\theta_B\chi\right)}\frac{\psi_A}{\theta_A} + \frac{1}{\nu}\frac{\psi_B}{\theta_B} - \left(1-\nu\right)\theta_B\psi_A - \left(\psi_A + \frac{\left(1-\nu\theta_A\right)^2}{\nu\left(1-\chi\right)}\frac{\psi_B}{\theta_B} - \left(1-\nu\right)\theta_A\psi_B\right) \\ &= \frac{\left(1-\nu\theta_B\right)\left(\chi-\nu\theta_A\right)\theta_A}{\nu\left(\theta_A-\theta_B\chi\right)}\frac{\psi_A}{\theta_A} + \frac{1}{\nu}\frac{\psi_B}{\theta_B} - \psi_A - \frac{\left(1-\nu\theta_A\right)^2}{\nu\left(1-\chi\right)}\frac{\psi_B}{\theta_B} + \left(1-\nu\right)\theta_A\theta_B\left(\frac{\psi_B}{\theta_B} - \frac{\psi_A}{\theta_A}\right) \\ &= \frac{\left(1-\nu\theta_B\right)\left(\chi-\nu\theta_A\right) - \nu\left(\theta_A-\theta_B\chi\right)}{\nu\left(\theta_A-\theta_B\chi\right)}\psi_A + \frac{\left(1-\chi\right) - \left(1-\nu\theta_A\right)^2}{\nu\left(1-\chi\right)}\frac{\psi_B}{\theta_B} + \left(1-\nu\right)\theta_A\theta_B\left(\frac{\psi_B}{\theta_B} - \frac{\psi_A}{\theta_A}\right) \\ &= \frac{\chi - 2\nu\theta_A + \nu^2\theta_A\theta_B}{\nu\left(\theta_A-\theta_B\chi\right)}\psi_A + \frac{\left(1-\chi\right) - \left(1-\nu\theta_A\right)^2}{\nu\left(1-\chi\right)}\frac{\psi_B}{\theta_B} + \left(1-\nu\right)\theta_A\theta_B\left(\frac{\psi_B}{\theta_B} - \frac{\psi_A}{\theta_A}\right) \\ &= \frac{\chi - 1 + \left(1-\nu\theta_A\right)^2 - \nu^2\theta_A\left(\theta_A-\theta_B\right)}{\nu\left(\theta_A-\theta_B\chi\right)}\psi_A + \frac{\left(1-\chi\right) - \left(1-\nu\theta_A\right)^2}{\nu\left(1-\chi\right)}\frac{\psi_B}{\theta_B} + \left(1-\nu\right)\theta_A\theta_B\left(\frac{\psi_B}{\theta_B} - \frac{\psi_A}{\theta_A}\right) \\ &= -\frac{\nu^2\theta_A\left(\theta_A-\theta_B\right)}{\nu\left(\theta_A-\theta_B\chi\right)}\psi_A - \frac{\left(1-\chi\right) - \left(1-\nu\theta_A\right)^2}{\nu\left(1-\frac{\theta_B}{\theta_A}\chi\right)}\frac{\psi_A}{\theta_A} + \frac{\left(1-\chi\right) - \left(1-\nu\theta_A\right)^2}{\nu\left(1-\chi\right)}\frac{\psi_B}{\theta_B} + \left(1-\nu\right)\theta_A\theta_B\left(\frac{\psi_B}{\theta_B} - \frac{\psi_A}{\theta_A}\right) \\ &= -\frac{\nu\theta_A\left(\theta_A-\theta_B\right)}{\left(\theta_A-\theta_B\chi\right)}\psi_A + \frac{\left(1-\chi\right) - \left(1-\nu\theta_A\right)^2}{\nu}\left[\frac{1}{\left(1-\chi\right)}\frac{\psi_B}{\theta_B} - \frac{1}{\left(1-\frac{\theta_B}{\theta_A}\chi\right)}\frac{\psi_A}{\theta_A}\right] + \left(1-\nu\right)\theta_A\theta_B\left(\frac{\psi_B}{\theta_B} - \frac{\psi_A}{\theta_A}\right) \\ &= -\frac{\nu\theta_A\left(\theta_A-\theta_B\right)}{\left(1-\frac{\theta_B}{\theta_A}\chi\right)}\frac{\psi_A}{\theta_A} - \frac{\left(1-\nu\theta_A\right)^2 - \left(1-\chi\right)}{\nu}\left[\frac{1}{\left(1-\chi\right)}\frac{\psi_B}{\theta_B} - \frac{1}{\left(1-\frac{\theta_B}{\theta_A}\chi\right)}\frac{\psi_A}{\theta_A}\right] + \left(1-\nu\right)\theta_A\theta_B\left(\frac{\psi_B}{\theta_B} - \frac{\psi_A}{\theta_A}\right) \end{aligned}$$

When $\nu \to 0$:

$$\frac{(1-\nu)\,\Delta_1}{\nu} \to -\frac{\chi}{\nu}\left[\frac{1}{(1-\chi)}\frac{\psi_B}{\theta_B} - \frac{1}{\left(1-\frac{\theta_B}{\theta_A}\chi\right)}\frac{\psi_A}{\theta_A}\right] < 0 \qquad \Rightarrow BA \text{ is optimal for all } \chi$$

When $\nu \to 1$:

$$\frac{(1-\nu)\,\Delta_1}{\nu} \to -\frac{\theta_A\,(\theta_A-\theta_B)}{(\theta_A-\theta_B\chi)}\psi_A + \left((1-\chi)-(1-\theta_A)^2\right)\left[\frac{1}{(1-\chi)}\frac{\psi_B}{\theta_B} - \frac{1}{\left(1-\frac{\theta_B}{\theta_A}\chi\right)}\frac{\psi_A}{\theta_A}\right]$$

when $\chi = \theta_A$ (baseline model):

$$\begin{aligned}\frac{(1-\nu)\,\Delta_1}{\nu} &\to -\frac{(\theta_A-\theta_B)}{(1-\theta_B)}\psi_A + \theta_A\,(1-\theta_A)\left[\frac{1}{(1-\theta_A)}\frac{\psi_B}{\theta_B} - \frac{1}{(1-\theta_B)}\frac{\psi_A}{\theta_A}\right]\\ &\to \theta_A\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right) > 0 \qquad \Rightarrow AB \text{ is optimal}\end{aligned}$$

when $\chi \to 1$:

$$\frac{(1-\nu)\,\Delta_1}{\nu} \to -\frac{(1-\theta_A)^2}{(1-\chi)}\frac{\psi_B}{\theta_B} < 0 \qquad \Rightarrow BA \text{ is optimal}$$

$$\begin{aligned}\frac{\partial}{\partial\nu}\left(\frac{(1-\nu)\,\Delta_1}{\nu}\right) &= \frac{\left(\chi-\nu^2\theta_A^2\right)}{\nu^2}\left[\frac{1}{(1-\chi)}\frac{\psi_B}{\theta_B} - \frac{1}{\left(1-\frac{\theta_B}{\theta_A}\chi\right)}\frac{\psi_A}{\theta_A}\right] - \theta_A\theta_B\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right) - \frac{\theta_A\,(\theta_A-\theta_B)}{(\theta_A-\theta_B\chi)}\psi_A\\ &= \underbrace{\frac{1}{\nu^2}\left[\frac{\left(\chi-\nu^2\theta_A^2\right)}{(1-\chi)}\frac{\psi_B}{\theta_B} - \frac{\left(\chi-\nu^2\theta_A\theta_B\right)}{\left(1-\frac{\theta_B}{\theta_A}\chi\right)}\frac{\psi_A}{\theta_A}\right]}_{>0} - \theta_A\theta_B\underbrace{\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)}_{>0}\end{aligned}$$

The term in squared brackets is positive:

$$\begin{aligned}\frac{\left(\chi-\nu^2\theta_A^2\right)}{(1-\chi)} - \frac{\left(\chi-\nu^2\theta_A\theta_B\right)}{\left(1-\frac{\theta_B}{\theta_A}\chi\right)} &\propto \frac{\left(\chi-\nu^2\theta_A^2\right)}{(\chi-\nu^2\theta_A\theta_B)} - \frac{(1-\chi)}{\left(1-\frac{\theta_B}{\theta_A}\chi\right)}\\ &\propto \frac{\left(\chi-\nu^2\theta_A\theta_B+\nu^2\theta_A\theta_B-\nu^2\theta_A^2\right)}{(\chi-\nu^2\theta_A\theta_B)} - \frac{\left(1-\frac{\theta_B}{\theta_A}\chi+\frac{\theta_B}{\theta_A}\chi-\chi\right)}{\left(1-\frac{\theta_B}{\theta_A}\chi\right)}\\ &\propto -\frac{\nu^2\theta_A\,(\theta_A-\theta_B)}{(\chi-\nu^2\theta_A\theta_B)} + \frac{\chi\,(\theta_A-\theta_B)}{(\theta_A-\theta_B\chi)}\\ &\propto -\frac{\nu^2\theta_A}{(\chi-\nu^2\theta_A\theta_B)} + \frac{\chi}{(\theta_A-\theta_B\chi)} \propto -\nu^2\theta_A\,(\theta_A-\theta_B\chi) + \chi\left(\chi-\nu^2\theta_A\theta_B\right)\\ &\propto -\nu^2\theta_A^2 + \nu^2\theta_A\theta_B\chi + \chi^2 - \chi\nu^2\theta_A\theta_B\\ &\propto \chi^2 - \nu^2\theta_A^2 > 0\end{aligned}$$

$$\frac{\partial^2}{\partial\nu^2}\left(\frac{(1-\nu)\,\Delta_1}{\nu}\right) = -\frac{2\chi}{\nu^3}\left[\frac{1}{(1-\chi)}\frac{\psi_B}{\theta_B} - \frac{1}{\left(1-\frac{\theta_B}{\theta_A}\chi\right)}\frac{\psi_A}{\theta_A}\right] < 0$$

For $\nu = 1$, we have

$$\begin{aligned}
\frac{\partial}{\partial \nu}\left(\frac{(1-\nu)\,\Delta_1}{\nu}\right) &= \left[\frac{\left(\chi-\theta_A^2\right)}{(1-\chi)}\frac{\psi_B}{\theta_B}-\frac{(\chi-\theta_A\theta_B)}{\left(1-\frac{\theta_B}{\theta_A}\chi\right)}\frac{\psi_A}{\theta_A}\right]-\theta_A\theta_B\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)\\
&= \frac{\left(\chi-\theta_A^2\right)}{(1-\chi)}\frac{\psi_B}{\theta_B}-\frac{\left(\chi-\theta_A^2\right)}{(1-\chi)}\frac{\psi_A}{\theta_A}+\frac{\left(\chi-\theta_A^2\right)}{(1-\chi)}\frac{\psi_A}{\theta_A}-\frac{(\chi-\theta_A\theta_B)}{\left(1-\frac{\theta_B}{\theta_A}\chi\right)}\frac{\psi_A}{\theta_A}-\theta_A\theta_B\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)\\
&= \left[\frac{\left(\chi-\theta_A^2\right)}{(1-\chi)}-\frac{(\chi-\theta_A\theta_B)}{\left(1-\frac{\theta_B}{\theta_A}\chi\right)}\right]\frac{\psi_A}{\theta_A}+\left[\frac{\left(\chi-\theta_A^2\right)}{(1-\chi)}-\theta_A\theta_B\right]\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)
\end{aligned}$$

We know the first bracket to be positive because we have shown it to be positive for all ν and thus it is positive for $\nu = 1$. The second bracket is also positive because

$$\frac{\left(\chi-\theta_A^2\right)}{(1-\chi)}-\theta_A\theta_B$$

increases with χ and is thus minimum for $\chi = \theta_A$, where it equals

$$\frac{\left(\theta_A-\theta_A^2\right)}{(1-\theta_A)}-\theta_A\theta_B=\theta_A\,(1-\theta_B)>0$$

Therefore, $\frac{\partial}{\partial\nu}\left(\frac{(1-\nu)\Delta_1}{\nu}\right) > 0$ for $\nu = 1$. Since we have shown that $\frac{\partial^2}{\partial\nu^2}\left(\frac{(1-\nu)\Delta_1}{\nu}\right) < 0$, we have $\frac{\partial}{\partial\nu}\left(\frac{(1-\nu)\Delta_1}{\nu}\right) > 0$ for all ν.

Lemma 6. *For $\theta_A \geq \theta_B$, there exists $\nu^* \in (0,1]$ such that BA is optimal iff $\nu \leq \nu^*$.*

Proof: We have shown that $\Delta_1\,(\nu = 0) < 0$ and $\frac{\partial}{\partial\nu}\left(\frac{(1-\nu)\Delta_1}{\nu}\right) > 0$. ■

We now turn to the case where data set B has the higher precision. This is the more intricate case.

• $\boldsymbol{\theta_A < \theta_B}$

Sequence AB

(A13) is binding iff

$$\frac{[1-\chi+\nu\,(\theta_B-\theta_A)]}{\left[1-\frac{\theta_A}{\theta_B}\chi\right]}\geq\frac{\psi_A/\theta_A}{\psi_B/\theta_B}$$

or

$$x\geq\frac{1-\frac{\theta_A}{\theta_B}\chi}{[(1-\chi)+\nu\,(\theta_B-\theta_A)]}$$

We have

$$\begin{aligned}
RHS\,(\nu=0) &= \frac{1-\frac{\theta_A}{\theta_B}\chi}{[1-\chi]}>1\\
RHS\,(\nu=1) &= \frac{1-\frac{\theta_A}{\theta_B}\chi}{[1-\chi+(\theta_B-\theta_A)]}>1
\end{aligned}$$

Indeed,

$$\begin{aligned}
\frac{1-\frac{\theta_A}{\theta_B}\chi}{[1-\chi+(\theta_B-\theta_A)]}-1 &\propto \left(1-\frac{\theta_A}{\theta_B}\chi\right)-[1-\chi+(\theta_B-\theta_A)] \\
&\propto \left(-\frac{\theta_A}{\theta_B}\chi\right)-[-\chi+(\theta_B-\theta_A)] \\
&\propto (-\theta_A\chi)-\theta_B[-\chi+(\theta_B-\theta_A)] \\
&\propto (\chi-\theta_B)(\theta_B-\theta_A)>0
\end{aligned}$$

Moreover, $\frac{\partial}{\partial\nu}RHS<0$ hence:

- If $x\geq RHS(\nu=0)=\frac{1-\frac{\theta_A}{\theta_B}\chi}{[1-\chi]}$ then (A13) is binding for all ν.
- If $x\leq RHS(\nu=1)=\frac{1-\frac{\theta_A}{\theta_B}\chi}{[(1-\chi)+(\theta_B-\theta_A)]}$ then (A17) is binding for all ν.
- Otherwise, $\exists\hat{\nu}\in(0,1)$, such that $RHS(\nu=\hat{\nu})=x$ and (A13) is binding iff $\nu\geq\hat{\nu}$.

Sequence BA

(A13) is binding iff

$$\frac{\left[1+\nu\theta_A-\nu\theta_B-\frac{\theta_A}{\theta_B}\chi\right]}{(1-\chi)}\geq\frac{\psi_B/\theta_B}{\psi_A/\theta_A}$$

$$\frac{\left[\left(1-\frac{\theta_A}{\theta_B}\chi\right)-\nu(\theta_B-\theta_A)\right]}{(1-\chi)}\geq x$$

We have

$$\begin{aligned}
LHS(\nu=0) &= \frac{\left[1-\frac{\theta_A}{\theta_B}\chi\right]}{(1-\chi)}>1 \\
LHS(\nu=1) &= \frac{\left[1-\frac{\theta_A}{\theta_B}\chi-(\theta_B-\theta_A)\right]}{(1-\chi)}>1
\end{aligned}$$

Indeed,

$$\begin{aligned}
\frac{\left[1-\frac{\theta_A}{\theta_B}\chi-(\theta_B-\theta_A)\right]}{(1-\chi)}-1 &\propto \left[1-\frac{\theta_A}{\theta_B}\chi-(\theta_B-\theta_A)\right]-(1-\chi) \\
&\propto \left[-\frac{\theta_A}{\theta_B}\chi-(\theta_B-\theta_A)\right]+\chi \\
&\propto [-\theta_A\chi-\theta_B(\theta_B-\theta_A)]+\chi\theta_B \\
&\propto (\chi-\theta_B)(\theta_B-\theta_A)>0
\end{aligned}$$

Moreover, $\frac{\partial}{\partial\nu}LHS<0$ hence:

- If $x\geq LHS(\nu=0)=\frac{\left[1-\frac{\theta_A}{\theta_B}\chi\right]}{(1-\chi)}$ then (A17) is binding for all ν.

- If $x \leq LHS\left(\nu=1\right) = \frac{\left[1-\frac{\theta_A}{\theta_B}\chi-\left(\theta_B-\theta_A\right)\right]}{(1-\chi)}$ then (A13) is binding for all ν.
- Otherwise, $\exists\widehat{\nu} \in (0,1)$, such that $LHS\left(\nu=\widehat{\nu}\right) = x$ and (A13) is binding iff $\nu \leq \widehat{\nu}$.

We have

$$\begin{aligned}
LHS - RHS &= \frac{\left[\left(1-\frac{\theta_A}{\theta_B}\chi\right)-\nu\left(\theta_B-\theta_A\right)\right]}{(1-\chi)} - \frac{1-\frac{\theta_A}{\theta_B}\chi}{\left[(1-\chi)+\nu\left(\theta_B-\theta_A\right)\right]} \\
&\propto \left[\left(1-\frac{\theta_A}{\theta_B}\chi\right)-\nu\left(\theta_B-\theta_A\right)\right]\left[(1-\chi)+\nu\left(\theta_B-\theta_A\right)\right]-(1-\chi)\left(1-\frac{\theta_A}{\theta_B}\chi\right) \\
&\propto -(1-\chi)+\left(1-\frac{\theta_A}{\theta_B}\chi\right)-\nu\left(\theta_B-\theta_A\right) \\
&\propto \left(\chi-\nu\theta_B\right)\left(\theta_B-\theta_A\right) > 0
\end{aligned}$$

Because by definition $RHS\left(\nu=\hat{\nu}\right) = LHS\left(\nu=\widehat{\nu}\right) = x$, and because LHS decreases with ν, $LHS \geq RHS$ implies that $\widehat{\nu} \geq \hat{\nu}$.

- If $x \geq LHS\left(\nu=0\right) = RHS\left(\nu=0\right) = \frac{\left[1-\frac{\theta_A}{\theta_B}\chi\right]}{(1-\chi)}$ then
  - AB: (A13) is binding
  - BA: (A17) is binding
- If $LHS\left(\nu=0\right) = \frac{\left[1-\frac{\theta_A}{\theta_B}\chi\right]}{(1-\chi)} \geq x \geq LHS\left(\nu=1\right) = \frac{\left[1-\frac{\theta_A}{\theta_B}\chi-\left(\theta_B-\theta_A\right)\right]}{(1-\chi)}$ then
  - AB: $\exists\hat{\nu} \in (0,1)$, such that $RHS\left(\nu=\hat{\nu}\right) = x$ and (A13) is binding iff $\nu \geq \hat{\nu}$.
  - BA: $\exists\widehat{\nu} \in (0,1)$, such that $LHS\left(\nu=\widehat{\nu}\right) = x$ and (A13) is binding iff $\nu \leq \widehat{\nu}$.
- If $LHS\left(\nu=1\right) = \frac{\left[1-\frac{\theta_A}{\theta_B}\chi-\left(\theta_B-\theta_A\right)\right]}{(1-\chi)} \geq x \geq RHS\left(\nu=1\right) = \frac{1-\frac{\theta_A}{\theta_B}\chi}{\left[1-\chi+\left(\theta_B-\theta_A\right)\right]}$ then
  - AB: $\exists\hat{\nu} \in (0,1)$, such that $RHS\left(\nu=\hat{\nu}\right) = x$ and (A13) is binding iff $\nu \geq \hat{\nu}$.
  - BA: (A13) is binding (i.e. $\widehat{\nu} = 1$)
- If $RHS\left(\nu=1\right) = \frac{1-\frac{\theta_A}{\theta_B}\chi}{\left[1-\chi+\left(\theta_B-\theta_A\right)\right]} \geq x$ then
  - AB: (A17) is binding
  - BA: (A13) is binding

We further study all of these cases.

- $\boldsymbol{x \geq \frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}}$

If $x \geq LHS\,(\nu = 0) = RHS\,(\nu = 0) = \frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}$ then

- AB: (A13) is binding
- BA: (A17) is binding

Lemma 7. *If $x \geq LHS\,(\nu = 0) = RHS\,(\nu = 0) = \frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}$, we have BA optimal for $\nu \leq \nu^*$.*

Proof: for BA, T_1^{BA} is given by (A17) binding:

$$p\left(\underline{\sigma}_B, \overline{\sigma}_A\right)(1-\nu)\,p\left(\underline{\sigma}_B, \underline{\sigma}_A\right)p\left(\overline{\sigma}_B\right)T_1^{BA} = p\left(\underline{\sigma}_B\right)p\left(\underline{\sigma}_B, \underline{\sigma}_A\right)\begin{bmatrix} p\left(\underline{\sigma}_B\right)p\left(\overline{\sigma}_A\right) \\ -p\left(\underline{\sigma}_B, \overline{\sigma}_A\right)\end{bmatrix}\psi_A + p\left(\underline{\sigma}_B, \overline{\sigma}_A\right)p\left(\underline{\sigma}_B, \underline{\sigma}_A\right)\psi_B \tag{A17}$$

$$\begin{aligned} T_1^{BA} &= \frac{p\left(\underline{\sigma}_B\right)}{p\left(\underline{\sigma}_B, \overline{\sigma}_A\right)(1-\nu)\,p\left(\overline{\sigma}_B\right)}\left[p\left(\underline{\sigma}_B\right)p\left(\overline{\sigma}_A\right) - p\left(\underline{\sigma}_B, \overline{\sigma}_A\right)\right]\psi_A + \frac{1}{(1-\nu)\,p\left(\overline{\sigma}_B\right)}\psi_B \\ &= \frac{(1-\nu\theta_B)}{(\nu\theta_A - \nu\theta_A\chi)(1-\nu)\,\nu\theta_B}\left[(1-\nu\theta_B)\,\nu\theta_A - (\nu\theta_A - \nu\theta_A\chi)\right]\psi_A + \frac{1}{(1-\nu)\,\nu\theta_B}\psi_B \\ &= \frac{(1-\nu\theta_B)}{(1-\chi)(1-\nu)\,\nu\theta_B}\left[(1-\nu\theta_B) - (1-\chi)\right]\psi_A + \frac{1}{(1-\nu)\,\nu}\frac{\psi_B}{\theta_B} \end{aligned}$$

$$\begin{aligned} E\left[t_{BA}(A17)\right] &= \nu T_1^{BA} + \psi_B + p\left(\underline{\sigma}_B\right)\psi_A = \nu\left(T_1^{BA} - \theta_B\psi_A\right) + (\psi_A + \psi_B) \\ &= \frac{\theta_A(1-\nu\theta_B)}{(1-\chi)(1-\nu)\,\theta_B}\left[(1-\nu\theta_B) - (1-\chi)\right]\frac{\psi_A}{\theta_A} + \frac{1}{(1-\nu)}\frac{\psi_B}{\theta_B} - \nu\theta_A\theta_B\frac{\psi_A}{\theta_A} + (\psi_A + \psi_B) \end{aligned}$$

For AB, T_1^{AB} is given by (A13) binding:

$$p\left(\underline{\sigma}_A, \overline{\sigma}_B\right)(1-\nu)\,T_1^{AB} = p\left(\underline{\sigma}_A, \overline{\sigma}_B\right)\psi_A + p\left(\underline{\sigma}_A\right)^2\psi_B \tag{A13}$$

$$\begin{aligned} T_1^{AB} &= \frac{1}{(1-\nu)}\psi_A + \frac{p\left(\underline{\sigma}_A\right)^2}{p\left(\underline{\sigma}_A, \overline{\sigma}_B\right)(1-\nu)}\psi_B \\ &= \frac{\theta_A}{(1-\nu)}\frac{\psi_A}{\theta_A} + \frac{(1-\nu\theta_A)^2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)\nu\,(1-\nu)}\frac{\psi_B}{\theta_B} \end{aligned}$$

$$\begin{aligned} E\left[t_{AB}(A13)\right] &= \nu T_1^{AB} + \psi_A + p\left(\underline{\sigma}_A\right)\psi_B = \nu\left(T_1^{AB} - \theta_A\psi_B\right) + (\psi_A + \psi_B) \\ &= \frac{\nu\theta_A}{(1-\nu)}\frac{\psi_A}{\theta_A} + \frac{(1-\nu\theta_A)^2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)(1-\nu)}\frac{\psi_B}{\theta_B} - \nu\theta_A\theta_B\frac{\psi_B}{\theta_B} + (\psi_A + \psi_B) \end{aligned}$$

Define $\Delta_2 \equiv E\left[t_{BA}(A17)\right] - E\left[t_{AB}(A13)\right]$. BA is optimal iff $\Delta_2 < 0$.

$$
\begin{aligned}
\Delta_2 &= \frac{\theta_A(1-\nu\theta_B)}{(1-\chi)(1-\nu)\theta_B}\left[(1-\nu\theta_B)-(1-\chi)\right]\frac{\psi_A}{\theta_A}+\frac{1}{(1-\nu)}\frac{\psi_B}{\theta_B}-\nu\theta_A\theta_B\frac{\psi_A}{\theta_A}\\
&- \left(\frac{\nu\theta_A}{(1-\nu)}\frac{\psi_A}{\theta_A}+\frac{(1-\nu\theta_A)^2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)(1-\nu)}\frac{\psi_B}{\theta_B}-\nu\theta_A\theta_B\frac{\psi_B}{\theta_B}\right)\\
&= \frac{\theta_A(1-\nu\theta_B)}{(1-\chi)(1-\nu)\theta_B}\left[(1-\nu\theta_B)-(1-\chi)\right]\frac{\psi_A}{\theta_A}+\frac{1-\nu\theta_A}{(1-\nu)}\frac{\psi_A}{\theta_A}\\
&- \frac{(1-\nu\theta_A)^2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)(1-\nu)}\frac{\psi_B}{\theta_B}+\left(\nu\theta_A\theta_B+\frac{1}{(1-\nu)}\right)\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)\\
&= \frac{1}{(1-\chi)(1-\nu)\theta_B}\left[\theta_A(1-\nu\theta_B)(1-\nu\theta_B)-\theta_A(1-\nu\theta_B)(1-\chi)+(1-\nu\theta_A)(1-\chi)\theta_B\right]\frac{\psi_A}{\theta_A}\\
&- \frac{(1-\nu\theta_A)^2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)(1-\nu)}\frac{\psi_B}{\theta_B}+\left(\nu\theta_A\theta_B+\frac{1}{(1-\nu)}\right)\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)\\
&= \frac{1}{(1-\chi)(1-\nu)\theta_B}\left[\theta_A(1-\nu\theta_B)^2+(\theta_B-\theta_A)(1-\chi)\right]\frac{\psi_A}{\theta_A}\\
&- \frac{\theta_B(1-\nu\theta_A)^2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)(1-\nu)\theta_B}\frac{\psi_B}{\theta_B}+\left(\nu\theta_A\theta_B+\frac{1}{(1-\nu)}\right)\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)
\end{aligned}
$$

$$
\begin{aligned}
\Delta_2 &= -\frac{1}{(1-\nu)\theta_B}\left[\frac{\theta_B(1-\nu\theta_A)^2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{\theta_A(1-\nu\theta_B)^2}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]\\
&+ \frac{(\theta_B-\theta_A)}{(1-\nu)\theta_B}\frac{\psi_A}{\theta_A}+\left(\nu\theta_A\theta_B+\frac{1}{(1-\nu)}\right)\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)
\end{aligned}
$$

Δ_2 has the same sign as $\frac{(1-\nu)\Delta_2}{\nu}=$

$$
\begin{aligned}
&= -\frac{1}{\nu\theta_B}\left[\frac{\theta_B(1-\nu\theta_A)^2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{\theta_A(1-\nu\theta_B)^2}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]+\frac{(\theta_B-\theta_A)}{\nu\theta_B}\frac{\psi_A}{\theta_A}+\left((1-\nu)\theta_A\theta_B+\frac{1}{\nu}\right)\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)\\
&= -\frac{1}{\nu\theta_B}\left[\frac{\theta_B(1-\nu\theta_A)}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{\nu\theta_A\theta_B(1-\nu\theta_A)}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{\theta_A(1-\nu\theta_B)}{(1-\chi)}\frac{\psi_A}{\theta_A}+\frac{\nu\theta_A\theta_B(1-\nu\theta_B)}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]\\
&+ \frac{(\theta_B-\theta_A)}{\nu\theta_B}\frac{\psi_A}{\theta_A}+\left((1-\nu)\theta_A\theta_B+\frac{1}{\nu}\right)\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)\\
&= -\frac{1}{\nu\theta_B}\left[\frac{\theta_B(1-\nu\theta_A)}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{\theta_A(1-\nu\theta_B)}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]+\theta_A\left[\frac{(1-\nu\theta_A)}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{(1-\nu\theta_B)}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]\\
&+ \frac{(\theta_B-\theta_A)}{\nu\theta_B}\frac{\psi_A}{\theta_A}+\left((1-\nu)\theta_A\theta_B+\frac{1}{\nu}\right)\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\theta_A(\chi-\nu\theta_B)}{\nu\theta_B\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\left[\frac{\psi_B}{\theta_B}-\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]+\frac{\theta_A}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\left[(1-\nu\theta_A)\frac{\psi_B}{\theta_B}-(1-\nu\theta_B)\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]\\
&+ (1-\nu)\,\theta_A\theta_B\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)\\
&= \left(\frac{\nu\theta_B\theta_A-\theta_A(\chi-\nu\theta_B)}{\nu\theta_B\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\right)\left[\frac{\psi_B}{\theta_B}-\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]\\
&- \frac{\nu\theta_A\theta_B}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\left[\frac{\theta_A}{\theta_B}\frac{\psi_B}{\theta_B}-\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]+(1-\nu)\,\theta_A\theta_B\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial\nu}\frac{(1-\nu)\,\Delta_2}{\nu} &= \frac{\partial}{\partial\nu}-\frac{1}{\nu\theta_B}\left[\frac{\theta_B}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{\theta_A}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]-\theta_A\left[\frac{\nu\theta_A}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{\nu\theta_B}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]\\
&\quad+\frac{(\theta_B-\theta_A)}{\nu\theta_B}\frac{\psi_A}{\theta_A}+\left((1-\nu)\,\theta_A\theta_B+\frac{1}{\nu}\right)\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)\\
&= \frac{1}{\nu^2\theta_B}\left[\frac{\theta_B}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{\theta_A}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]-\theta_A\left[\frac{\theta_A}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{\theta_B}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]\\
&\quad-\frac{(\theta_B-\theta_A)}{\nu^2\theta_B}\frac{\psi_A}{\theta_A}-\left(\theta_A\theta_B+\frac{1}{\nu^2}\right)\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial\nu^2}\frac{(1-\nu)\,\Delta_2}{\nu} &= \frac{\partial}{\partial\nu}\frac{1}{\nu^2\theta_B}\left[\frac{\theta_B}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{\theta_A}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]-\frac{(\theta_B-\theta_A)}{\nu^2\theta_B}\frac{\psi_A}{\theta_A}-\frac{1}{\nu^2}\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)\\
&= \frac{\partial}{\partial\nu}\frac{1}{\nu^2\theta_B}\left[\frac{\theta_B}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{\chi\theta_A+(1-\chi)\,\theta_B}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]-\frac{1}{\nu^2}\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)\\
&\propto \left[-\frac{1}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}+\frac{\chi\frac{\theta_A}{\theta_B}+(1-\chi)}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]+\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)\\
&\propto \frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)-1}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}+\frac{\chi\frac{\theta_A}{\theta_B}}{(1-\chi)}\frac{\psi_A}{\theta_A}\\
&\propto \frac{-\frac{\theta_A}{\theta_B}\chi}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}+\frac{\chi\frac{\theta_A}{\theta_B}}{(1-\chi)}\frac{\psi_A}{\theta_A}\\
&\propto -\frac{\theta_A}{\theta_B\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\chi\frac{\psi_A}{\theta_A}\left[x-\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}\right]\\
&< 0 \qquad \text{because } x\geq\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}
\end{aligned}
$$

We now study the sign of $\frac{\partial}{\partial \nu}\frac{(1-\nu)\Delta_2}{\nu}(\nu=1)=$

$$\begin{aligned}
&= \left[\frac{1}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{\frac{\theta_A}{\theta_B}}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]-\theta_A\theta_B\left[\frac{\frac{\theta_A}{\theta_B}}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{1}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]\\
&+ \frac{\frac{\theta_A}{\theta_B}(1-\chi)}{(1-\chi)}\frac{\psi_A}{\theta_A}-\left(\frac{\psi_B}{\theta_B}\right)-\theta_A\theta_B\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)\\
&= \left[\frac{\frac{\theta_A}{\theta_B}\chi}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{\frac{\theta_A}{\theta_B}\chi}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]-\theta_A\theta_B\left[\frac{\frac{\theta_A}{\theta_B}}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{1}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]-\theta_A\theta_B\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)\\
&\propto \left[\frac{\frac{\theta_A}{\theta_B}\chi}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}x-\frac{\frac{\theta_A}{\theta_B}\chi}{(1-\chi)}\right]-\theta_A\theta_B\left[\frac{\frac{\theta_A}{\theta_B}}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}x-\frac{1}{(1-\chi)}\right]-\theta_A\theta_B(x-1)\\
&\propto \frac{\frac{\theta_A}{\theta_B}\chi}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\left[x-\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}\right]-\theta_A\theta_B\left[\frac{\frac{\theta_A}{\theta_B}}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}x+x-\frac{1}{(1-\chi)}\right]+\theta_A\theta_B\\
&\propto \frac{\frac{\theta_A}{\theta_B}\chi}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\left[x-\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}\right]-\frac{\theta_A\theta_B}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\left[\left(\frac{\theta_A}{\theta_B}+\left(1-\frac{\theta_A}{\theta_B}\chi\right)\right)x-\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}\right]+\theta_A\theta_B\\
&\propto \frac{\chi}{\theta_B}\left[x-\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}\right]-\theta_B\left[\left(\frac{\theta_A}{\theta_B}+\left(1-\frac{\theta_A}{\theta_B}\chi\right)\right)x-\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}\right]+\theta_B\left(1-\frac{\theta_A}{\theta_B}\chi\right)\\
&\propto \left[\left(\frac{\chi}{\theta_B}-\theta_B\left(\frac{\theta_A}{\theta_B}+\left(1-\frac{\theta_A}{\theta_B}\chi\right)\right)\right)x-\left(\frac{\chi}{\theta_B}-\theta_B\right)\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}\right]+\theta_B\left(1-\frac{\theta_A}{\theta_B}\chi\right)\\
&\propto \left[\left(\frac{\chi}{\theta_B}-\theta_A-\theta_B+\theta_A\chi\right)x-\left(\frac{\chi}{\theta_B}-\theta_B\right)\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}\right]+\theta_B\left(1-\frac{\theta_A}{\theta_B}\chi\right)\\
&\propto \left(\frac{\chi}{\theta_B}-\theta_B\right)\left[x-\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}\right]+\theta_A(\chi-1)x+\theta_B\left(1-\frac{\theta_A}{\theta_B}\chi\right)\\
&> 0 \qquad \text{because } x\geq\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}
\end{aligned}$$

This implies that $\frac{\partial}{\partial \nu}\frac{(1-\nu)\Delta_2}{\nu}>0$ for all ν. As $\nu\to 0$:

$$\begin{aligned}
\frac{(1-\nu)\Delta_2}{\nu} &\to -\frac{1}{\nu\theta_B}\left[\frac{\theta_B}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{\theta_A}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]+\frac{(\theta_B-\theta_A)}{\nu\theta_B}\frac{\psi_A}{\theta_A}+\frac{1}{\nu}\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)\\
&\to -\frac{1}{\nu\theta_B}\left[\frac{\theta_B}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{\theta_A}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]-\frac{\theta_A}{\nu\theta_B}\frac{\psi_A}{\theta_A}+\frac{\theta_B}{\nu\theta_B}\frac{\psi_B}{\theta_B}\\
&\to -\frac{\theta_A\chi}{\nu\theta_B\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_A}{\theta_A}\left[x-\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}\right]<0
\end{aligned}$$

As $\nu \to 1$:

$$\begin{aligned}
\frac{(1-\nu)\,\Delta_2}{\nu} &\to -\frac{1}{\theta_B}\left[\frac{\theta_B\,(1-\theta_A)^2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{\theta_A\,(1-\theta_B)^2}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]+\frac{(\theta_B-\theta_A)}{\theta_B}\frac{\psi_A}{\theta_A}+\left(\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right)\\
&\to -\left[\frac{(1-\theta_A)^2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\frac{\psi_B}{\theta_B}-\frac{\frac{\theta_A}{\theta_B}\,(1-\theta_B)^2}{(1-\chi)}\frac{\psi_A}{\theta_A}\right]-\frac{\theta_A}{\theta_B}\frac{\psi_A}{\theta_A}+\frac{\psi_B}{\theta_B}\\
&\to -\left[\left(\frac{(1-\theta_A)^2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}-1\right)\frac{\psi_B}{\theta_B}-\frac{\theta_A}{\theta_B}\left(\frac{(1-\theta_B)^2}{(1-\chi)}-1\right)\frac{\psi_A}{\theta_A}\right]\\
&\to \left[\left(1-\frac{(1-\theta_A)^2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\right)\frac{\psi_B}{\theta_B}-\frac{\theta_A}{\theta_B}\left(1-\frac{(1-\theta_B)^2}{(1-\chi)}\right)\frac{\psi_A}{\theta_A}\right]
\end{aligned}$$

If $\chi \to 1$: $\Delta_2 > 0$. If $\chi \to \theta_B$: $\Delta_2 > 0$

$$\begin{aligned}
\frac{(1-\nu)\,\Delta_2}{\nu} &\to -\left[\left(\frac{(1-\theta_A)^2}{(1-\theta_A)}-1\right)\frac{\psi_B}{\theta_B}-\frac{\theta_A}{\theta_B}\left(\frac{(1-\theta_B)^2}{(1-\theta_B)}-1\right)\frac{\psi_A}{\theta_A}\right]\\
&\to \theta_A\left[\frac{\psi_B}{\theta_B}-\frac{\psi_A}{\theta_A}\right]>0
\end{aligned}$$

■

- $\boldsymbol{x \leq \frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}}$

First, we focus on:

- ***AB*: (A17) is binding**
- ***BA*: (A13) is binding**

For BA, T_1^{BA} is given by (A13) binding:

$$p\left(\underline{\sigma}_B,\overline{\sigma}_A\right)(1-\nu)\,T_1^{BA}=p\left(\underline{\sigma}_B,\overline{\sigma}_A\right)\psi_B+p\left(\underline{\sigma}_B\right)^2\psi_A \qquad \text{(A13)}$$

$$\begin{aligned}
T_1^{BA} &= \frac{1}{(1-\nu)}\psi_B+\frac{p\left(\underline{\sigma}_B\right)^2}{p\left(\underline{\sigma}_B,\overline{\sigma}_A\right)(1-\nu)}\psi_A\\
&= \frac{1}{(1-\nu)}\psi_B+\frac{(1-\nu\theta_B)^2}{(\nu\theta_A-\nu\theta_A\chi)(1-\nu)}\psi_A
\end{aligned}$$

$$\begin{aligned}
E\left[t_{BA}(A13)\right] &= \nu T_1^{BA}+\psi_B+p\left(\underline{\sigma}_B\right)\psi_A=\nu\left(T_1^{BA}-\theta_B\psi_A\right)+(\psi_A+\psi_B)\\
&= \frac{\nu}{(1-\nu)}\psi_B+\frac{(1-\nu\theta_B)^2}{(1-\chi)(1-\nu)}\frac{\psi_A}{\theta_A}-\nu\theta_B\psi_A+(\psi_A+\psi_B)
\end{aligned}$$

For AB, T_1^{AB} is given by (A17) binding:

$$\begin{aligned}
T_1^{AB} &= \frac{p\left(\underline{\sigma}_A\right)}{p\left(\underline{\sigma}_A, \overline{\sigma}_B\right)(1-\nu)\, p\left(\overline{\sigma}_A\right)} \left[p\left(\underline{\sigma}_A\right) p\left(\overline{\sigma}_B\right) - p\left(\underline{\sigma}_A, \overline{\sigma}_B\right)\right] \psi_B + \frac{1}{(1-\nu)\, p\left(\overline{\sigma}_A\right)} \psi_A \\
&= \frac{(1-\nu\theta_A)}{(\nu\theta_B - \nu\theta_A\chi)(1-\nu)\,\nu\theta_A} \left[(1-\nu\theta_A)\,\nu\theta_B - (\nu\theta_B - \nu\theta_A\chi)\right]\psi_B + \frac{1}{(1-\nu)\,\nu\theta_A}\psi_A \\
&= \frac{(1-\nu\theta_A)}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)(1-\nu)\,\nu} \left[\chi - \nu\theta_B\right] \frac{\psi_B}{\theta_B} + \frac{1}{(1-\nu)\,\nu} \frac{\psi_A}{\theta_A}
\end{aligned}$$

$$\begin{aligned}
E\left[t_{AB}(A17)\right] &= \nu T_1^{AB} + \psi_A + p\left(\underline{\sigma}_A\right)\psi_B = \nu\left(T_1^{AB} - \theta_A\psi_B\right) + (\psi_A + \psi_B) \\
&= \frac{(1-\nu\theta_A)}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)(1-\nu)} \left[\chi - \nu\theta_B\right] \frac{\psi_B}{\theta_B} + \frac{1}{(1-\nu)} \frac{\psi_A}{\theta_A} - \nu\theta_A\psi_B + (\psi_A + \psi_B)
\end{aligned}$$

Define $\Delta_3 \equiv E\left[t_{BA}(A13)\right] - E\left[t_{AB}(A17)\right]$. BA is optimal iff $\Delta_3 < 0$. We have

$$\begin{aligned}
\Delta_3 &= \frac{\nu}{(1-\nu)}\psi_B + \frac{(1-\nu\theta_B)^2}{(1-\chi)(1-\nu)} \frac{\psi_A}{\theta_A} - \nu\theta_B\psi_A \\
&- \left(\frac{(1-\nu\theta_A)}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)(1-\nu)} \left[\chi - \nu\theta_B\right] \frac{\psi_B}{\theta_B} + \frac{1}{(1-\nu)} \frac{\psi_A}{\theta_A} - \nu\theta_A\psi_B \right) \\
&= \frac{\nu}{(1-\nu)}\psi_B + \frac{(1-\nu\theta_B)^2}{(1-\chi)(1-\nu)} \frac{\psi_A}{\theta_A} - \frac{(1-\nu\theta_A)}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)(1-\nu)} \left[\chi - \nu\theta_B\right] \frac{\psi_B}{\theta_B} \\
&- \frac{1}{(1-\nu)} \frac{\psi_A}{\theta_A} + \nu\theta_A\theta_B \left(\frac{\psi_B}{\theta_B} - \frac{\psi_A}{\theta_A} \right)
\end{aligned}$$

Lemma 8. *Define* $\Delta_3 \equiv E\left[t_{BA}(13)\right] - E\left[t_{AB}(17)\right]$*. We have*

$$\frac{(1-\nu)\,\Delta_3}{\psi_A/\theta_A} \equiv \alpha\nu^2 + \beta\nu + \gamma$$

with $\alpha \geq 0$*,* $\gamma \geq 0$ *and*

$$\beta \propto x - \frac{\left(\frac{2}{(1-\chi)} + \theta_A\right)}{\left(\frac{2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)} + \theta_A\right)}$$

Proof: From previous calculations we have:

$$\Delta_3 = \frac{\nu}{(1-\nu)}\psi_B + \frac{(1-\nu\theta_B)^2}{(1-\chi)(1-\nu)} \frac{\psi_A}{\theta_A} - \frac{(1-\nu\theta_A)}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)(1-\nu)} \left[\chi - \nu\theta_B\right] \frac{\psi_B}{\theta_B} - \frac{1}{(1-\nu)} \frac{\psi_A}{\theta_A} + \nu\theta_A\theta_B \left(\frac{\psi_B}{\theta_B} - \frac{\psi_A}{\theta_A} \right)$$

$$
\begin{aligned}
\frac{(1-\nu)\,\Delta_3}{\psi_A/\theta_A} &= x\theta_B\nu+\frac{(1-\nu\theta_B)^2}{(1-\chi)}-\frac{(1-\nu\theta_A)}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\left[\chi-\nu\theta_B\right]x-1+\nu\,(1-\nu)\,\theta_A\theta_B\,(x-1)\\
&= x\theta_B\nu+\frac{1-2\nu\theta_B+\theta_B^2\nu^2}{(1-\chi)}-\frac{(1-\nu\theta_A)}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\left[\chi-\nu\theta_B\right]x-1+\left(\nu-\nu^2\right)\theta_A\theta_B\,(x-1)\\
&= \left[\frac{\theta_B^2}{(1-\chi)}-\frac{\theta_A\theta_B x}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}-\theta_A\theta_B\,(x-1)\right]\nu^2\\
&+ \left[x\theta_B-\frac{2\theta_B}{(1-\chi)}+\frac{(\theta_B+\chi\theta_A)}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}x+\theta_A\theta_B\,(x-1)\right]\nu\\
&+ \left[\frac{1}{(1-\chi)}-\frac{\chi x}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}-1\right]\\
&= \frac{\theta_B}{(1-\chi)}\left[\theta_B-\frac{(1-\chi)}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\theta_A x-(1-\chi)\,\theta_A\,(x-1)\right]\nu^2\\
&+ \left[\theta_B x-\frac{2\theta_B}{(1-\chi)}+\frac{\left(1+\frac{\theta_A}{\theta_B}\chi\right)}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\theta_B x+\theta_A\theta_B\,(x-1)\right]\nu\\
&+ \frac{\chi}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\left[\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}-x\right]\\
&= \frac{\theta_B}{(1-\chi)}\left[\theta_B+(1-\chi)\,\theta_A-\left(\frac{1}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}+1\right)(1-\chi)\,\theta_A x\right]\nu^2\\
&+ \theta_B\left[x\left(1+\frac{\left(1+\frac{\theta_A}{\theta_B}\chi\right)}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}+\theta_A\right)-\left(\frac{2}{(1-\chi)}+\theta_A\right)\right]\nu\\
&+ \frac{\chi}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\left[\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}-x\right]\\
&= \frac{\theta_B}{(1-\chi)}\left[\theta_B+(1-\chi)\,\theta_A-\left(\frac{1}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}+1\right)(1-\chi)\,\theta_A x\right]\nu^2\\
&+ \theta_B\left[x\left(\frac{2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}+\theta_A\right)-\left(\frac{2}{(1-\chi)}+\theta_A\right)\right]\nu\\
&+ \frac{\chi}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\left[\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}-x\right]
\end{aligned}
$$

Therefore:

$$\begin{aligned}
\frac{\alpha}{\frac{\theta_B}{(1-\chi)}} &\equiv \theta_B + (1-\chi)\,\theta_A - \left(\frac{1}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)} + 1\right)(1-\chi)\,\theta_A x \\
&\geq \theta_B + (1-\chi)\,\theta_A - \left(\frac{1}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)} + 1\right)(1-\chi)\,\theta_A \frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)} \\
&\geq \theta_B + (1-\chi)\,\theta_A - \left(1 + \left(1-\frac{\theta_A}{\theta_B}\chi\right)\right)\theta_A \\
&\geq \theta_B - \chi\frac{\theta_A}{\theta_B}\theta_B - \left(\theta_A - \frac{\theta_A}{\theta_B}\chi\theta_A\right) \\
&\geq \left(1-\chi\frac{\theta_A}{\theta_B}\right)(\theta_B - \theta_A) \\
&\geq 0
\end{aligned}$$

$$\gamma \equiv \frac{\chi}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}\left[\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)} - x\right] \geq 0$$

$$\begin{aligned}
\beta &\equiv \theta_B\left[x\left(\frac{2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)} + \theta_A\right) - \left(\frac{2}{(1-\chi)} + \theta_A\right)\right] \\
&\propto x - \frac{\left(\frac{2}{(1-\chi)} + \theta_A\right)}{\left(\frac{2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)} + \theta_A\right)}
\end{aligned}$$

■

Lemma 9. *Recall that we are in the case* $\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)} \geq x \geq 1$*: We have*

$$\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)} \geq \frac{\left(\frac{2}{(1-\chi)} + \theta_A\right)}{\left(\frac{2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)} + \theta_A\right)} \geq 1$$

Hence, as x *decreases from* $\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}$ *to 1,* β *decreases from positive to negative.*

Proof:

$$\begin{aligned}
\Omega &= \frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)} - \frac{\left(\frac{2}{(1-\chi)}+\theta_A\right)}{\left(\frac{2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}+\theta_A\right)} \\
&\propto \left(1-\frac{\theta_A}{\theta_B}\chi\right)\left(\frac{2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}+\theta_A\right) - \left(\frac{2}{(1-\chi)}+\theta_A\right)(1-\chi) \\
&\propto \left(2+\theta_A\left(1-\frac{\theta_A}{\theta_B}\chi\right)\right) - \left(2+\theta_A\,(1-\chi)\right) \\
&\propto \chi\left(1-\frac{\theta_A}{\theta_B}\right) \geq 0
\end{aligned}$$

Because $\frac{\theta_A}{\theta_B} \leq 1$, we have

$$\frac{\left(\frac{2}{(1-\chi)}+\theta_A\right)}{\left(\frac{2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}+\theta_A\right)} \geq 1$$

■

Lemma 10. *For x such that $\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)} \geq x \geq \frac{\left(\frac{2}{(1-\chi)}+\theta_A\right)}{\left(\frac{2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}+\theta_A\right)}$, we have $\alpha \geq 0$, $\beta \geq 0$ and $\gamma \geq 0$.*

Hence, both roots are negative and $\Delta_3 > 0$. If (A17) binds for AB and (A13) binds for BA, then AB is optimal.

Proof: $\alpha \geq 0$ and $\beta \geq 0$ imply that the sum of the roots is negative and $\gamma \geq 0$ implies that their product is positive. Hence, both roots are negative. Therefore, $\Delta_3 > 0$ for $\nu \in [0,1]$. ■

Lemma 11. *$\beta^2 - 4\alpha\gamma$ is a convex function of x.*

Proof: The second derivative of $\beta^2 - 4\alpha\gamma$ w.r.t. x is $2\left(\frac{\partial\beta}{\partial x}\right)^2 - 8\left(\frac{\partial\alpha}{\partial x}\frac{\partial\gamma}{\partial x}\right) =$

$$\begin{aligned}
&= 2\theta_B^2\left(\frac{2}{(1-\frac{\theta_A}{\theta_B}\chi)}+\theta_A\right)^2 - 8\frac{\theta_B}{(1-\chi)}\left[-\left(\frac{1}{(1-\frac{\theta_A}{\theta_B}\chi)}+1\right)(1-\chi)\theta_A\right]\left[-\frac{\chi}{(1-\frac{\theta_A}{\theta_B}\chi)}\right] \\
&\propto \left(2+\theta_A(1-\frac{\theta_A}{\theta_B}\chi)\right)^2 - 4\frac{\theta_A}{\theta_B}\chi\left(2-\frac{\theta_A}{\theta_B}\chi\right) \\
&\propto 4+4\theta_A(1-\frac{\theta_A}{\theta_B}\chi)+\theta_A^2(1-\frac{\theta_A}{\theta_B}\chi)^2 - 4\frac{\theta_A}{\theta_B}\chi(1-\frac{\theta_A}{\theta_B}\chi) - 4\frac{\theta_A}{\theta_B}\chi \\
&\propto (1-\frac{\theta_A}{\theta_B}\chi)\cdot\left[4+4\theta_A+\theta_A^2(1-\frac{\theta_A}{\theta_B}\chi) - 4\frac{\theta_A}{\theta_B}\chi\right] \\
&\propto (1-\frac{\theta_A}{\theta_B}\chi)\cdot\left[4\theta_A+(1-\frac{\theta_A}{\theta_B}\chi)(4+\theta_A^2)\right] \geq 0
\end{aligned}$$

■

Lemma 12. *There exists a* x^* *with* $1 \leq x \leq \dfrac{1-\frac{\theta_A}{\theta_B}\chi}{(1-\chi)}$ *such that* $\beta = 0$ *for* $x = x^*$.

Proof: By inspection, β is increasing in x and equal to zero for $x = \dfrac{(\frac{2}{(1-\chi)}+\theta_A)}{(\frac{2}{(1-\frac{\theta_A}{\theta_B}\chi)}+\theta_A)}$ ■

Lemma 13. $\beta^2 - 4\alpha\gamma$ *is negative over* $[1, x^*]$.

Proof: For $x = x^*$, $\beta = 0$ so $\beta^2 - 4\alpha\gamma \leq 0$ because $\alpha \geq 0$ and $\gamma \geq 0$. For $x = 1$, we have:

$$\begin{aligned}
\beta^2 - 4\alpha\gamma[x \to 1] &= \theta_B^2 \left[\left(\frac{2}{(1-\frac{\theta_A}{\theta_B}\chi)} + \theta_A\right) - \left(\frac{2}{(1-\chi)} + \theta_A\right)\right]^2 \\
&\quad -4\frac{\theta_B}{(1-\chi)}\left[\theta_B + (1-\chi)\theta_A - \left(\frac{1}{1-\frac{\theta_A}{\theta_B}\chi} + 1\right)(1-\chi)\theta_A\right] \frac{\chi}{(1-\frac{\theta_A}{\theta_B}\chi)} \left[\frac{(1-\frac{\theta_A}{\theta_B}\chi)}{(1-\chi)} - 1\right] \\
&= \frac{4\theta_B^2\chi^2}{(1-\frac{\theta_A}{\theta_B}\chi)^2(1-\chi)^2}\left[1-\frac{\theta_A}{\theta_B}\right]^2 - \frac{4\theta_B^2\chi^2}{(1-\frac{\theta_A}{\theta_B}\chi)^2(1-\chi)^2}\left[1-\frac{\theta_A}{\theta_B}\right]^2 = 0
\end{aligned}$$

Since $\beta^2 - 4\alpha\gamma \leq 0$ for $x = 1$ and $x = x^*$, and it is convex over $[1, x^*]$, it is negative over $[1, x^*]$. ■

Conclusion: Either (i) $\beta \geq 0$ or (ii) $\beta < 0$ and there are no roots. Either way, Δ_3 is positive.

Now we focus on:

- ***AB*: (A13) is binding**
- ***BA*: (A17) is binding**

Lemma 14. *For* $\Delta_2 \equiv E\left[t_{BA}(A17)\right] - E\left[t_{AB}(A13)\right]$, *we have* $\dfrac{(1-\nu)\Delta_2}{\psi_A/\theta_A} = \alpha_1\nu^2 + \beta_1\nu + \gamma_1$, *with* $\alpha_1 \geq 0$ *and* $\gamma_1 \geq 0$.

Proof:

$$\begin{aligned}
\alpha_1 &\equiv \theta_A\left(-\frac{\theta_A}{1-\frac{\theta_A}{\theta_B}\chi}x + \frac{\theta_B}{1-\chi} - (x-1)\,\theta_B\right) \\
&\geq \theta_A\left(-\frac{\theta_A}{1-\frac{\theta_A}{\theta_B}\chi}\frac{1-\frac{\theta_A}{\theta_B}\chi}{1-\chi} + \frac{\theta_B}{1-\chi} - \left(\frac{1-\frac{\theta_A}{\theta_B}\chi}{1-\chi} - 1\right)\theta_B\right) \\
&\geq \theta_A\left(-\frac{\theta_A}{1-\chi} + \frac{\theta_B}{1-\chi} - \frac{\left(1-\frac{\theta_A}{\theta_B}\right)\chi}{1-\chi}\theta_B\right) \\
&\geq \theta_A\theta_B\left(1-\frac{\theta_A}{\theta_B}\right)
\end{aligned}$$

$$
\begin{aligned}
\gamma_1 &\equiv x-1+\left(1-\frac{\theta_A}{\theta_B}\right)+\frac{\frac{\theta_A}{\theta_B}}{1-\chi}-x\frac{1}{1-\frac{\theta_A}{\theta_B}\chi} \\
&\geq \frac{1-\frac{\theta_A}{\theta_B}\chi}{1-\chi}-1+\left(1-\frac{\theta_A}{\theta_B}\right)+\frac{\frac{\theta_A}{\theta_B}}{1-\chi}-\frac{1-\frac{\theta_A}{\theta_B}\chi}{1-\chi}\frac{1}{1-\frac{\theta_A}{\theta_B}\chi} \\
&\geq \frac{1-\frac{\theta_A}{\theta_B}\chi}{1-\chi}-1+\left(1-\frac{\theta_A}{\theta_B}\right)+\frac{\frac{\theta_A}{\theta_B}}{1-\chi}-\frac{1}{1-\chi} \\
&\geq \frac{-\frac{\theta_A}{\theta_B}\chi}{1-\chi}-\frac{\theta_A}{\theta_B}+\frac{\frac{\theta_A}{\theta_B}}{1-\chi} \\
&\geq \frac{-\frac{\theta_A}{\theta_B}\chi-\frac{\theta_A}{\theta_B}(1-\chi)+\frac{\theta_A}{\theta_B}}{1-\chi} \\
&\geq 0
\end{aligned}
$$

$$
\beta_1 \equiv \frac{2\theta_A}{(1-\frac{\theta_A}{\theta_B}\chi)}x-\frac{2\theta_A}{(1-\chi)}+\theta_A\theta_B(x-1)
$$

■

Lemma 15. *Recall that we are in the case* $\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)} \geq x \geq 1$*: We have*

$$
\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)} \geq \frac{\left(\frac{2}{(1-\chi)}+\theta_B\right)}{\left(\frac{2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}+\theta_B\right)} \geq 1
$$

Hence, as x *decreases from* $\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}$ *to* 1, β_1 *decreases from positive to negative.*

Proof:

$$
\begin{aligned}
\Omega_1 &= \frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)}-\frac{\left(\frac{2}{(1-\chi)}+\theta_B\right)}{\left(\frac{2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}+\theta_B\right)} \\
&\propto \left(1-\frac{\theta_A}{\theta_B}\chi\right)\left(\frac{2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}+\theta_B\right)-\left(\frac{2}{(1-\chi)}+\theta_B\right)(1-\chi) \\
&\propto \left(2+\theta_B\left(1-\frac{\theta_A}{\theta_B}\chi\right)\right)-\left(2+\theta_B\left(1-\chi\right)\right) \\
&\propto \chi\left(1-\frac{\theta_A}{\theta_B}\right)\geq 0
\end{aligned}
$$

Because $\frac{\theta_A}{\theta_B} \leq 1$, we have

$$
\frac{\left(\frac{2}{(1-\chi)}+\theta_B\right)}{\left(\frac{2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}+\theta_B\right)} \geq 1
$$

■

Lemma 16. *For x such that $\frac{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}{(1-\chi)} \geq x \geq \frac{\left(\frac{2}{(1-\chi)}+\theta_B\right)}{\left(\frac{2}{\left(1-\frac{\theta_A}{\theta_B}\chi\right)}+\theta_B\right)}$, we have $\alpha_1 \geq 0$, $\beta_1 \geq 0$ and $\gamma_1 \geq 0$.*

Hence, both roots are negative and $\Delta_2 > 0$. If (A13) binds for AB and (A17) binds for BA, then AB is optimal.

Proof: $\alpha_1 \geq 0$ and $\beta_1 \geq 0$ imply that the sum of the roots is negative and $\gamma_1 \geq 0$ implies that their product is positive. Hence, both roots are negative. Therefore, $\Delta_2 > 0$ for $\nu \in [0,1]$. ■

Lemma 17. *$\beta_1^2 - 4\alpha_1\gamma_1$ is a convex function of x.*

Proof:

$$
\begin{aligned}
\Omega_2 &= \left[\frac{2\theta_A}{(1-\frac{\theta_A}{\theta_B}\chi)}x - \frac{2\theta_A}{(1-\chi)} + \theta_A\theta_B(x-1)\right]^2 \\
&\quad -4\theta_A\theta_B\left[\frac{-\frac{\theta_A}{\theta_B}}{(1-\frac{\theta_A}{\theta_B}\chi)} + \frac{1}{(1-\chi)} - (x-1)\right]\frac{\frac{\theta_A}{\theta_B}\chi}{(1-\frac{\theta_A}{\theta_B}\chi)}\left(\frac{(1-\frac{\theta_A}{\theta_B}\chi)}{(1-\chi)} - x\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\Omega_2}{\partial x^2} &= 2\left(\frac{2\theta_A}{(1-\frac{\theta_A}{\theta_B}\chi)} + \theta_A\theta_B\right)^2 - 8\theta_A\theta_B\left[\frac{\frac{\theta_A}{\theta_B}}{(1-\frac{\theta_A}{\theta_B}\chi)} + 1\right]\frac{\frac{\theta_A}{\theta_B}\chi}{(1-\frac{\theta_A}{\theta_B}\chi)} \\
&= \frac{8\theta_A^2}{(1-\frac{\theta_A}{\theta_B}\chi)^2}\left(1+\frac{\theta_B}{2}(1-\frac{\theta_A}{\theta_B}\chi)\right)^2 - \frac{8\theta_A^2}{(1-\frac{\theta_A}{\theta_B}\chi)^2}\chi\left[\frac{\theta_A}{\theta_B} + (1-\frac{\theta_A}{\theta_B}\chi)\right] \\
&\propto \left(1+\frac{\theta_B}{2}(1-\frac{\theta_A}{\theta_B}\chi)\right)^2 - \chi\left[1+\frac{\theta_A}{\theta_B}(1-\chi)\right] \\
&\propto \geq \left(1+\frac{\theta_B}{2}(1-\chi)\right)^2 - \chi\left[1+(1-\chi)\right] \\
&\propto (1+\theta_B(1-\chi) + \frac{\theta_B^2}{4}(1-\chi)^2) - \chi\left[1+(1-\chi)\right] \\
&\propto (1-\chi) + \theta_B(1-\chi) + \frac{\theta_B^2}{4}(1-\chi)^2 - \chi(1-\chi) \\
&\propto (1-\chi) + \theta_B + \frac{\theta_B^2}{4}(1-\chi) \geq 0
\end{aligned}
$$

■

Lemma 18. *There exists a x^* with $1 \leq x \leq \frac{1-\frac{\theta_A}{\theta_B}\chi}{(1-\chi)}$ such that $\beta_1 = 0$ for $x = x^*$.*

Proof: By inspection, β_1 is increasing in x and equal to zero for $x = \dfrac{(\dfrac{2}{(1-\chi)} + \theta_B)}{(\dfrac{2}{(1-\dfrac{\theta_A}{\theta_B}\chi)} + \theta_B)}$. ■

Lemma 19. *$\beta_1^2 - 4\alpha_1\gamma_1$ is negative over $[1, x^*]$.*

Proof: For $x = x^*$, $\beta_1 = 0$ so $\beta_1^2 - 4\alpha_1\gamma_1 \le 0$ because $\alpha_1 \ge 0$ and $\gamma_1 \ge 0$. For $x = 1$, we have $\beta_1^2 - 4\alpha_1\gamma_1[x \to 1] =$

$$\begin{aligned}
&= \left[\frac{2\theta_A}{(1-\dfrac{\theta_A}{\theta_B}\chi)} - \frac{2\theta_A}{(1-\chi)}\right]^2 - 4\theta_A\theta_B\left[\frac{-\dfrac{\theta_A}{\theta_B}}{(1-\dfrac{\theta_A}{\theta_B}\chi)} + \frac{1}{(1-\chi)}\right]\frac{\dfrac{\theta_A}{\theta_B}\chi}{(1-\dfrac{\theta_A}{\theta_B}\chi)}\left(\frac{(1-\dfrac{\theta_A}{\theta_B}\chi)}{(1-\chi)} - 1\right) \\
&= \frac{4\theta_A^2\chi^2}{(1-\dfrac{\theta_A}{\theta_B}\chi)^2(1-\chi)^2}\left[1-\frac{\theta_A}{\theta_B}\right]^2 - \frac{4\theta_A^2\chi^2}{(1-\dfrac{\theta_A}{\theta_B}\chi)^2(1-\chi)^2}\left[1-\frac{\theta_A}{\theta_B}\right]^2 = 0
\end{aligned}$$

Since $\beta_1^2 - 4\alpha_1\gamma_1 \le 0$ for $x = 1$ and $x = x^*$, and it is convex over $[1, x^*]$, it is negative over $[1, x^*]$. ■

Conclusion: Either (i) $\beta_1 \ge 0$ or (ii) $\beta_1 < 0$ and there are no roots. Either way, Δ_2 is positive.

Lemma 20. *If (A13) binds for both AB and BA, then AB is optimal.*

Proof: For BA, if (A13) is removed, (A17) binds and the optimum improves, $E[t_{BA}(A17)] < E[t_{BA}(A13)]$. As shown above, $\Delta_2 > 0$, hence, $E[t_{BA}(A13)] > E[t_{BA}(A17)] > E[t_{AB}(A13)]$. ■

Summary

In the first chapter (**Crowdfunding platforms**, E. Gorbunova) I derive a platform-optimal selling procedure for a non-rival but excludable good, which is yet to be produced. The seller, who is privately informed about fixed production costs, faces multiple potential buyers, who are privately informed about their valuations for the good. Selling the good is only possible on the platform. The platform designs the contract specifying when the good will be produced (call this production rule), how much to pay the seller, which buyers get access to the good (call this allocation rule) and how much the buyers pay the platform. I derive the optimal direct incentive-compatible mechanism which maximizes the platform's expected profit. The optimal allocation rule is standard: only buyers with positive virtual valuations should obtain the good in case it is produced. Whether or not the good will be produced depends on whether the sum of positive virtual valuations surpasses the virtual costs.

When it comes to implementation, there are many ways to construct the actual transfers depending on the desired structure: I concentrate on the so-called contribution schemes, for which there is no reimbursement to the buyers in case the good is not produced. I then propose a scheme which combines features of the implementation of the optimal mechanism in *Cornelli's "Optimal selling procedures with fixed costs" (1996)* and *Myerson and Satterthwaite's "Efficient mechanisms for bilateral trading" (1983)*, and indeed resembles contractual features found on reward-based crowdfunding platforms such as Patreon.

The second chapter (**Sequential screening in the presence of fixed costs**, E. Gorbunova) is based on joint work with P. Pillath where we combine techniques used in *Cornelli's "Optimal selling procedures with fixed costs" (1996)* and *Courty and Li's "Sequential screening" (2000).* We describe how a monopolist optimally sells a club good if buyers learn their valuations for the good over time. We first characterize the optimal selling mechanism when the seller has to produce before the buyers have perfectly learned their valuations for the good. The optimal selling mechanism can be implemented as a "buy-option" contract: the seller first collects upfront payments and decides whether to produce; the buyers then learn their valuations perfectly and if they made an upfront payment and if the good was produced, they have an option to buy the good for a pre-specified exercise price. We show that the upfront payments serve a dual purpose: the higher the upfront payment, the higher the probability that a good will be produced, but also the lower the exercise price. Additionally, the optimal selling mechanism when the seller can produce after the buyers have perfectly learned their valuations for the good is derived. The optimal mechanism can be implemented as a "contribute-option" contract: after collecting upfront payments from the buyers, the seller now offers an option to contribute to the production of the good, once the buyers learn their true valuations. In particular, it is optimal for the seller to ask buyers to pay as much as they want when the production decision takes place.

The third chapter (**Dynamic information collection: two-sided tests**, E. Gorbunova) is based on joint work with D. Gromb and F. de Vericourt where we considers a firm which can learn about a risky project's quality before investing by running up to two tests differing in efficiency defined as precision-to-cost ratio. These tests generate no false positives and no false negatives, in other words, it is possible for the firm to accept or reject the project based on *hard information / evidence*. The signal structure is such that if a positive signal is observed, the firm knows for sure that a good quality project generated it (no false positives); if a negative signal is observed, the firm knows for sure that a bad quality project generated it (no false negatives). The firm collects signals sequentially and can stop the analysis after one conclusive

result to save the cost of the second test. If the first test generates an inconclusive result, we assume that it is still worthwhile to run the second test. Since we consider symmetric tests (it is as likely for a good quality project to generate a positive signal as it is for a bad quality project to generate a negative signal), encountering inconclusive results does not update the prior belief about the project's success. It is first-best optimal to start with the most efficient test.

The principal, however, must hire an expert who has to be incentivized to run the tests (moral hazard) and report the signals they generate truthfully (adverse selection). In this paper we tackle two building blocks of delegated expertise: optimal compensation - characterizing optimal incentive contracts for each order of the tests under a combined moral hazard and adverse selection problem; and task design - finding the optimal order. If running up to two tests is delegated to an expert, we show that the optimal test order is reverse: it is optimal to start with the least efficient test. The interplay of dynamic incentive constraints implies that it is cheaper to make the expert run the least efficient test first than to ensure that he does not skip the least efficient test in the second round if he starts with the most efficient test.

The forth chapter (**Search order in delegated data analytics**, E. Gorbunova) is based on joint work with D. Gromb and F. de Vericourt where we provide a narrower context for dynamic information collection by focusing on delegated data analytics. We tailor the model accordingly, first, by allowing good quality projects to fail with an exogenous probability (i.e. it is not possible to assure project's success though data analytics alone), and, second, by allowing analyses results to be correlated (even if the data sets are heterogeneous, they sometimes reveal similar results / causalities). We keep the no false positives assumption but allow for false negatives, i.e. this time, the firm can accept the project based on hard evidence but it rejects the project based on *soft information.* If a positive result is observed, the firm knows for sure that a good quality project generated it (no false positives); if a negative result is observed, the firm does not know for sure whether a bad quality project or a good quality project generated it (false negatives). As the firm conducts analyses sequentially and fails to find positive results, it gets more pessimistic about the project quality. This signal structure corresponds to a standard search problem.

Due to signal asymmetry, the result obtained in the forth chapter differs from that in the third chapter; exogenous probability of a good project failing does not play a significant role. We find that the deviation from the first-best optimal search order happens if the project is a priori sufficiently likely to be of bad quality. This "threshold" result holds even if analyses findings are positively correlated. A natural extension of this model is a sister problem with no false negatives and false positives, where the firm gets more optimistic about the project quality as it fails to find negative results, so it accepts the project based on soft information. This specification was also solved by us but it is outside of the scope of this thesis; its result, however, is briefly mentioned in the conclusion to the forth chapter.

Zusammenfassung

Im ersten Kapitel (**Crowdfunding platforms**, E. Gorbunova) leite ich ein plattformoptimales Verkaufsverfahren für ein nicht-rivales, aber exklusives Gut her, das noch zu produzieren ist. Der Verkäufer, der über private Informationen zu den fixen Produktionskosten verfügt, steht mehreren potenziellen Käufern gegenüber, die über private Informationen zu ihrer eigenen Wertschätzung des Gutes jeweils verfügen. Der Verkauf des Gutes ist ausschließlich über die Plattform möglich. Die Plattform gestaltet den Vertrag, der festlegt, wann das Gut produziert wird (Produktionsregel), wie viel dem Verkäufer für die Produktion zu zahlen ist, welche Käufer Zugang zum Gut erhalten (Allokationsregel) und wie viel die Käufer an die Plattform zahlen. Ich leite den optimalen direkten anreizkompatiblen Mechanismus her, der den erwarteten Gewinn der Plattform maximiert. Die optimale Allokationsregel ist standardmäßig: Nur die Käufer mit positiven virtuellen Wertschätzungen ("virtual valuations") sollten das Gut im Falle einer Produktion erhalten. Ob das Gut produziert wird, hängt davon ab, ob die Summe der positiven virtuellen Wertschätzungen die virtuellen Kosten ("virtual costs") übersteigt.

Bei der Implementierung gibt es viele Möglichkeiten, die tatsächlichen Zahlungen je nach gewünschter Struktur zu konstruieren: Ich konzentriere mich auf die sogenannten Beitragsschemata ("contribution schemes"), bei denen es keine Rückerstattung an die Käufer gibt, falls das Gut nicht produziert wird. Anschließend schlage ich ein Schema vor, das Merkmale der Implementierung des optimalen direkten Mechanismus bei *Cornellis "Optimal selling procedures with fixed costs" (1996)* und *Myersons und Satterthwaites "Efficient mechanisms for bilateral trading" (1983)* kombiniert und dabei den Crowdfunding-Verfahren auf Plattformen wie Patreon nahekommt

Das zweite Kapitel (**Sequential screening in the presence of fixed costs**, E. Gorbunova) basiert auf einer gemeinsamen Arbeit mit P. Pillath, in der wir Techniken aus *Cornellis "Optimal selling procedures with fixed costs" (1996)* und *Courtys und Lis "Sequential screening" (2000)* kombinieren. In diesem Kapitel beschreiben wir, wie ein Monopolist ein Clubgut ("club good") optimal verkauft, wenn die Käufer ihre Wertschätzung für das Gut im Laufe der Zeit erfahren. Zunächst charakterisieren wir den optimalen Verkaufsmechanismus, wenn der Verkäufer produzieren muss, bevor die Käufer ihre Wertschätzungen für das Gut vollständig gelernt haben. Der optimale direkte Verkaufsmechanismus kann als ein Vertrag mit einer "Kauf-Option" ("buy-option" contract) implementiert werden: Der Verkäufer erhebt zunächst Vorauszahlungen und entscheidet, ob er produziert; die Käufer lernen dann ihre Wertschätzungen vollständig kennen, und wenn sie eine Vorauszahlung geleistet haben und das Gut produziert wurde, haben sie eine Möglichkeit, das Gut zu einem vorher festgelegten Preis ("exercise price") zu kaufen. Wir zeigen, dass die Vorauszahlungen in diesem Fall einen doppelten Zweck erfüllen: Je höher die Vorauszahlung, desto höher die Wahrscheinlichkeit, dass ein Gut produziert wird, aber auch desto niedriger der Preis ("exercise price").

Zusätzlich wird der optimale Verkaufsmechanismus hergeleitet, wenn der Verkäufer produzieren kann nachdem die Käufer ihre Wertschätzungen für das Gut vollständig gelernt haben. Der optimale direkte Mechanismus kann als ein Vertrag mit einer "Beitrags-Option" ("contribute-option" contract) implementiert werden: Nach Erhalt von Vorauszahlungen von den Käufern bietet der Verkäufer nun eine Option an, zur Produktion des Gutes beizutragen, sobald die Käufer ihre wahre Wertschätzung kennen. Insbesondere ist es für den Verkäufer optimal, den Käufer anzubieten, zum Zeitpunkt der Produktionsentscheidung so viel beizutragen, wie sie möchten.

Das dritte Kapitel (**Dynamic information collection: two-sided tests**, E. Gorbunova) basiert auf eiener gemeinsamen Arbeit mit D. Gromb und F. de Vericourt und befasst sich mit einem Unternehmen, das die Qualität eines risikobehafteten Projekts vor der Investition durch Durchführung von bis zu zwei Tests lernen kann, die sich in ihrer Effizienz, definiert als Präzisions-Kosten-Verhältnis, unterscheiden. Diese Tests generieren keine falsch-positiven und keine falsch-negativen Ergebnisse, das heißt, es ist dem Unternehmen möglich, das Projekt basierend auf *harten Informationen ("hard information") / Beweisen* zu akzeptieren oder abzulehnen. Die Signalstruktur ist derart, dass bei Beobachtung eines positiven Signals das Unternehmen sicher weiß, dass ein Projekt guter Qualität es generiert hat (keine falsch-positiven Ergebnisse); bei Beobachtung eines negativen Signals weiß das Unternehmen sicher, dass ein Projekt schlechter Qualität es generiert hat (keine falsch-negativen Ergebnisse). Das Unternehmen sammelt Signale sukzessive und kann die Analyse nach einem eindeutigen Ergebnis stoppen, um die Kosten des zweiten Tests zu sparen. Wenn der erste Test ein unklares Ergebnis liefert, nehmen wir an, dass es sich trotzdem lohnt, den zweiten Test durchzuführen. Da wir symmetrische Tests betrachten (ein Projekt guter Qualität generiert mit gleicher Wahrscheinlichkeit ein positives Signal wie ein Projekt schlechter Qualität ein negatives Signal generiert), führen unklare Ergebnisse zu keiner Aktualisierung der Vorhersage ("belief") zum Projekterfolg. Es ist "first-best"-optimal, mit dem effizientesten Test anzufangen.

Der Prinzipal muss jedoch für die Analyse einen Experten beauftragen. Für ihn muss ein Anreiz bestehen, die Tests durchzuführen (moralisches Risiko) und die von ihnen erzeugten Signale wahrheitsgemäß zu melden (adverse Selektion). Diese Arbeit behandelt zwei Bausteine der delegierten Expertise: optimale Vergütung (Charakterisierung optimaler Anreizverträge für jede Reihenfolge der Tests unter der Kombination des moralischen Risikos und der adversen Selektion) und Aufgabengestaltung (Ermittlung der optimalen Reihenfolge der Tests). Wenn die Durchführung von bis zu zwei Tests an einen Experten delegiert wird, zeigen wir, dass die optimale Testreihenfolge umgekehrt ist: Es ist optimal, mit dem am wenigsten effizienten Test anzufangen. Das Zusammenspiel dynamischer Anreizrestriktionen ("incentive constraints") impliziert, dass es günstiger ist, den Experten zuerst den am wenigsten effizienten Test durchführen zu lassen, als sicherzustellen, dass er den am wenigsten effizienten Test in der zweiten Runde nicht auslässt, wenn er mit dem effizientesten Test beginnt.

Das vierte Kapitel (**Search order in delegated data analytics**, E. Gorbunova) basiert auf einer gemeinsamen Arbeit mit D. Gromb und F. de Vericourt, in der wir einen engeren Kontext für die dynamische Informationssammlung bieten, indem wir uns auf delegierte Datenanalyse konzentrieren. Wir passen das Modell entsprechend an: erstens lassen wir zu, dass Projekte guter Qualität mit einer exogenen Wahrscheinlichkeit scheitern können (d.h. es ist nicht möglich, den Projekterfolg allein durch Datenanalyse zu gewährleisten), und zweitens lassen wir positive Korrelation der Analyseergebnisse zu (selbst wenn die Datensätze heterogen sind, offenbaren sie manchmal ähnliche Ergebnisse / Kausalitäten). Wir behalten die Annahme "keine falsch-positiven Ergebnisse" bei, erlauben jedoch falsch-negative Ergebnisse, d.h. diesmal kann das Unternehmen das Projekt auf Basis harter Beweise akzeptieren, es lehnt das Projekt jedoch auf Basis *weicher Informationen ("soft information")* ab. Wird ein positives Ergebnis beobachtet, weiß das Unternehmen sicher, dass ein Projekt guter Qualität es generiert hat (keine falsch-positiven Ergebnisse); wird ein negatives Ergebnis beobachtet, weiß das Unternehmen nicht sicher, ob ein Projekt schlechter Qualität es generiert hat oder ein Projekt guter Qualität (falsch-negative Ergebnisse). Während das Unternehmen Analysen sukzessive durchführt und keine positiven Ergebnisse findet, wird es pessimistischer hinsichtlich der Projektqualität. Diese Signalstruktur entspricht einem standardmäßigen "search"-Problem.

Aufgrund der Signalasymmetrie unterscheidet sich das Ergebnis im vierten Kapitel von jenem im dritten Kapitel. Die exogene Wahrscheinlichkeit des Scheiterns eines guten Projekts spielt dabei keine signifikante Rolle. Wir stellen fest, dass die Abweichung von der "first-best"-optimalen Reihenfolge der Datensätze auftritt, wenn das Projekt a priori hinreichend wahrscheinlich von schlechter Qualität ist. Dieses Schwellenwert-Ergebnis ("threshold result") gilt auch bei positiv korrelierten Analyseergebnissen. Eine natürliche Erweiterung dieses Modells ist ein Schwestermodell ohne falsch-negative Ergebnisse aber mit falsch-positiven Ergebnissen, bei dem das Unternehmen optimistischer über die Projektqualität wird, wenn es keine negativen Ergebnisse findet, sodass es das Projekt auf Basis weicher Informationen ("soft information") akzeptiert. Diese Spezifikation wurde ebenfalls von uns gelöst, liegt jedoch außerhalb des Umfangs dieser Doktorarbeit; ihr Ergebnis wird jedoch kurz in der Schlussfolgerung des vierten Kapitels erwähnt.

Ort, Datum: Berlin, 18.12.2024

<u>Erklärung gem. § 10 Abs. 3</u>

Hiermit erkläre ich, dass ich für die Dissertation folgende Hilfsmittel und Hilfen verwendet habe:

Die Literaturquellen, die im Kapitel 1 benutzt wurden, sind auf der Seite 20 angegeben.
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Die Literaturquellen, die im Kapitel 2 benutzt wurden, sind auf der Seite 48 angegeben.
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Die Literaturquellen, die im Kapitel 3 benutzt wurden, sind auf der Seite 77 angegeben.
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Die Literaturquellen, die im Kapitel 4 benutzt wurden, sind auf der Seite 94 angegeben.
……………………………………………………………………………………………...

Auf dieser Grundlage habe ich die Arbeit selbstständig verfasst.

Unterschrift: …………………… (Ekaterina Gorbunova)

Dissertationskapitel (geschrieben von Ekaterina Gorbunova)	Basiert auf einer gemeinsamen Arbeit mit	Anteile am gemeinsamen Projekt
Crowdfunding platforms	Ekaterina Gorbunova (FU Berlin)	1
Sequential screening in the presence of fixed costs	Ekaterina Gorbunova (FU Berlin), Pascal Pillath (HU Berlin)	1/2 – 1/2 (gleicher Beitrag aller Autoren)
Dynamic information collection: two-sided tests	Ekaterina Gorbunova (FU Berlin), Denis Gromb (HEC Paris), Francis de Vericourt (ESMT Berlin)	1/3 – 1/3 – 1/3 (gleicher Beitrag aller Autoren)
Search order in delegated data analytics	Ekaterina Gorbunova (FU Berlin), Denis Gromb (HEC Paris), Francis de Vericourt (ESMT Berlin)	1/3 – 1/3 – 1/3 (gleicher Beitrag aller Autoren)