Some deformations of T-varieties

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Introduction

From a very distant point of view, deformation theory deals with families of mathematical objects that vary "continuously" over some parameter space. Within algebraic geometry, an important special case is the theory of *deformations of singularities*, which deals with the existence and study of flat families of schemes near a prescribed singular fibre. Such a family is called a deformation of this fibre.

In general, the deformation theory of a given singularity is hard to understand. There are various specific classes of singularities that have seen particular attention, usually because they come with symmetries that make them more tractable.

One of these is the class of varieties with $\mathbb{C}^*$-action, whose deformation theory has been studied by Pinkham [Pin74]. This includes the class of cyclic quotient singularities, which arise as quotients of $\mathbb{C}^2$ by finite group actions. Beginning with Riemenschneider [Rie74], their deformation theory has seen a wealth of study over the past decades.

Cyclic quotient singularities have a second characterization: They are precisely the two-dimensional toric singularities. A *toric variety* X is a variety with an action by an embedded algebraic torus $T = (\mathbb{C}^*)^n$. Toric varieties have the desirable property of being described entirely in terms of convex geometry, namely through polyhedral cones in lattices.

Thus besides allowing all varieties with $\mathbb{C}^*$-action, the class of cyclic quotient singularities may also be enlarged by generalizing only to toric singularities. Following this approach, Altmann has studied the deformation theory of toric varieties by using the translation to discrete geometry, compare section 3.3.

The theory of toric varieties has been generalized to actions of smaller tori, including $\mathbb{C}^*$-actions, with the introduction of *p-divisors* [AH06]. These are certain divisors

$$\mathcal{D} = \sum_{P \subset Y} \Delta_P \otimes P$$

on varieties Y with polyhedral coefficients that fully describe affine *T-varieties* via a functor

$$\mathrm{TV}\colon \{\text{p-divisors}\} \to \{\text{affine T-varieties}\}.$$

T-varieties are normal varieties with effective torus action. The *complexity* of a T-variety is the codimension of the torus with respect to X, which equals the dimension of Y. Toric varieties are the case of complexity zero.

The general topic of this thesis is the study of deformations using p-divisors, both by deforming T-varieties directly, and by expressing known deformations of toric varieties using p-divisors. From an abstract point of view, I suggest that the right class of objects to study in an equivariant setting is that of equivariant deformations. In particular, for toric varieties, we should think of the total spaces not as toric varieties, but as T-varieties of the right complexity.

I will now present an overview of the text. As stated above, one of the central objects are p-divisors. Chapter 1 summarizes this theory, including the global variant using divisorial fans.

This sets up the ground for chapter 2, where we develop the basic technique of *upgrades* of p-divisors. Upgrades and downgrades are the polyhedral analogues to extensions and restrictions of torus actions. The main result here is the construction of an upgrade functor which maps p-divisors to p-divisors.

In chapter 3, we turn to deformation theory. After defining the basic notions and introducing the concept of an equivariant deformation, a large part of the chapter is devoted to a summary of Altmann's results on toric varieties. In particular, this includes Altmann's constructions of deformations associated with Minkowski decompositions both in negative and non-negative degrees.

After this, we turn to T-varieties that are described by p-divisors on $\mathbb{P}^1$; such varieties are rational varieties with a torus action of complexity one. The main result of chapter 4 is a construction of invariant deformations associated with Minkowski decompositions: Given a p-divisor $\mathcal{D} = \sum \Delta_P \otimes P$ and decompositions $\Delta_P = \sum_i \Delta_P^i$, we find a p-divisor $\mathcal{E} = \sum \Delta_P^i \otimes E(P,i)$ on $\mathbb{P}^1 \times \mathbb{A}^l$ that defines an invariant deformation $\mathrm{TV}(\mathcal{E})$ of $\mathrm{TV}(\mathcal{D})$. Intuitively, this means that we obtain a flat family if we move the supporting points of a p-divisor on $\mathbb{P}^1$, under conditions on the Minkowski sums that arise when points collide. Thus the divisor divisor $\mathcal{E}$ can be regarded as a family of p-divisors on $\mathbb{P}^1$, and we get our hands on the individual fibres of the deformation as T-varieties.

Chapter 5 then ties together the preceding results. First we interpret Altmann's deformations from chapter 3 as deformations of certain p-divisors on $\mathbb{P}^1$. This removes the distinction between negative and non-negative degrees. Combined with Altmann's results it shows that in the toric setting, the deformations of chapter 4 cover a significant part of the toric deformation theory. At this point, it should be noted that Mavlyutov's approach to toric deformations by way of the Cox construction also provides a unified view of these two classes of deformations [Mav09].

Subsequently, we apply the upgrade construction of chapter 2 to determine p-divisors that describe these deformations as equivariant deformations. The chapter ends with a brief outlook on p-divisors for deformations in mixed degrees.

The final chapter 6 deals with first-order deformations. For any singularity X, the deformations over the ring $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ of dual numbers form a vector space T^1_X. We begin with a conceptual proof of the grading on T^1_X associated with a T-variety X. This is followed by a new and somewhat unfinished description of T^1 for a toric variety. In particular, I provide a possible interpretation of first-order deformations of a toric variety by so-called deformations of polyhedra: A polyhedron defines a function $\Delta\colon M \to \mathbb{Z}$, and a deformation of this polyhedron is a concave map from M to $\mathbb{C} \times \mathbb{Z}$ that lifts the original Δ, where $\mathbb{C} \times \mathbb{Z}$ is endowed with a special order. The study of these deformations includes some criteria for counting non-trivial such deformations, and may eventually help to compute $\dim_{\mathbb{C}} T^1_X$ for the associated toric variety.

Aside from yielding a better understanding of some aspects of the theories of deformations and T-varieties, the results of this thesis have immediate applications. The upgrade functor may be used to determine p-divisors for cones and Cox rings [IV11a]. The result on invariant deformations of rational complexity one T-varieties has been used by Ilten to analyze the deformation theory of global toric varieties [IV11b, Ilt10].

The results also raise a number of questions. While chapter 2 can be seen to deal with the upgrade question conclusively, the inverse problem of downgrades has only been settled in complexity one [IV11a]. Chapter 3 only provides a brief glimpse at a possible abstract theory of equivariant deformations that encompasses more than T-varieties.

The immediate generalization of the result of chapter 4 would be to non-rational T-varieties in complexity one. A similar result should hold, but the proof needs to be modified. The next step which is much less clear involves raising the complexity: While the definition of a deformation of p-divisors generalizes easily to higher complexity, finding criteria for their existence is more difficult. Note that this necessarily includes the entire deformation theory of normal surface singularities, which are trivial T-varieties of complexity two.

One related topic that is not discussed in this text is how the results of Pinkham [Pin78] and Wahl [Wah76] fit into the language of p-divisors. In particular, Pinkham's study of $\mathbb{C}^*$-surfaces includes a description of the surface that is closely related to the p-divisor associated with the surface. The theme of their work is to study such deformations that are compatible with a resolution of the singularity, which are closely related to the contraction-free varieties of section 1.7.

Regarding the interpretation of toric deformations via p-divisors, one immediate question concerns the combination of deformations in different degrees as discussed in section 5.3. While this can be handled to some extent in the case of negative degrees, handling the general case by deforming the total space would involve invariant deformations of complexity two. It is expected that the resulting total spaces agree with the diptych varieties constructed by Brown and Reid [BR], which have immediate applications to the minimal model program.

Returning to the geometric approach to toric T^1, a proof of the stated conjecture is obviously missing. A straightforward next step lies in the generalization to non-negative degrees, that is to divisors on $\mathbb{P}^1$ that are supported on 0 and ∞. One would expect that in the general case of rational T-varieties of complexity one, the spaces of first-order deformations associated with the individual coefficient polyhedra should add up to form (part of) T^1. Another open question is how this description compares to Altmann's approach to T^1 [Alt94], and to descriptions of T^1 for cyclic quotient singularities.

Some of the results presented in this thesis have previously been published jointly with Nathan Ilten [IV11a, IV11b]. Apart from the proof of the properness of the upgraded divisor of Theorem 2.15 in sections 2.4 through 2.6, which is joint work with Ilten, all results are my own unless explicitly attributed to others.

Chapter 1

T-varieties and p-divisors

The purpose of this chapter is to summarize the theory of T-varieties to the extent that will be used in the rest of the text. The main point is to define the categories of p-divisors and divisorial fans as well as the functor of T-varieties that identifies these with the categories of affine T-varieties and general T-varieties, respectively. Throughout, we will be working over the field of complex numbers $\mathbb{C}$.

Most of the notions and results presented here may be found in the original articles on p-divisors by Altmann and Hausen [AH06] and divisorial fans by Altmann, Hausen and Süß [AHS08]. A more detailed summary of the theory may be found in the review article by Altmann et al. [AIP$^+$11].

What is new here to some extent is the definition of a category of divisorial fans, as well as the statement of Corollary 1.15. The notion of a contraction-free divisorial fan was developed while studying upgrades of T-varieties; it is the same as Petersen's toroidal fans [Pet11].

1.1 Convex geometry

Let M, N be dual *lattices*, that is free abelian groups of finite rank. We will denote the associated $\mathbb{Q}$-vector spaces by $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$, $N_{\mathbb{Q}}$. A *rational polyhedral cone* (briefly just *cone*) in the lattice N is a set of the form

$$\sigma = \operatorname{pos}\{v_1, \dots, v_k\} \subset N_{\mathbb{Q}};$$

its dual is

$$\sigma^\vee = \{u \in M_{\mathbb{Q}} \mid \langle v, u \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

A cone is *pointed* if it contains no linear subspace, equivalently if the dual is of full dimension. A cone ρ generated by one non-zero vector is called a *ray*; its primitive lattice generator is denoted v_ρ.

A *polyhedron* Δ in N is an intersection of a finite number of affine half-spaces in $N_{\mathbb{Q}}$. Unless stated otherwise, we will also allow the empty set. For two polyhedra Δ, Δ', their *Minkowski sum* is

$$\Delta + \Delta' = \{v + v' \mid v \in \Delta, v' \in \Delta'\}.$$

With a non-empty polyhedron, we can associate its *tail cone*

$$\operatorname{tail}(\Delta) = \{v \in N_{\mathbb{Q}} \mid \Delta + v \subset \Delta\}.$$

Dualizing a polyhedron $\Delta \neq \emptyset$ gives the piecewise affine concave function

$$\begin{aligned} \mathrm{tail}(\Delta)^\vee &\to \mathbb{Q} \\ u &\mapsto \min\langle \Delta, u\rangle := \min_{v\in\Delta}\langle v, u\rangle. \end{aligned}$$

Here as elsewhere, concave means that

$$\min\langle \Delta, u\rangle + \min\langle \Delta, v\rangle \leq \min\langle \Delta, u + v\rangle.$$

We will denote by $\mathrm{Pol}(N, \sigma)$ the set of polyhedra with tail cone σ together with the empty set. It forms the semigroup $\mathrm{Pol}(N, \sigma)$ of σ-polyhedra. Then by extending the above construction to include the empty set, any σ-polyhedron is dual to a map $\sigma^\vee \to \mathbb{Q} \cup \{\infty\}$, where the empty set corresponds to the constant map with value ∞. $\mathrm{Pol}(N, \sigma)$ is naturally a module over the non-negative rational numbers.

An important notion in the study of deformations is that of a Minkowski decomposition of a polyhedron.

Definition 1.1. An *r-parameter Minkowski decomposition* of a polyhedron $\Delta \in \mathrm{Pol}(N, \sigma)$ is a tuple of σ-polyhedra $\Delta^0, \ldots, \Delta^r$ that satisfy

$$\Delta = \Delta^0 + \ldots + \Delta^r.$$

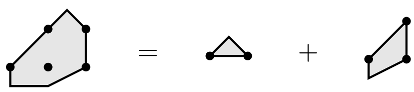
Such a decomposition is said to be *admissible* if it satisfies one of the following equivalent properties.

1. For each $u \in \sigma^\vee \cap M$, at most one of the faces $\mathrm{face}(\Delta^i, u)$ has no lattice vertices.

2. For each $u \in \sigma^\vee \cap M$, at most one of the evaluations $\min\langle \Delta^i, u\rangle$ is not an integer.

3. For each vertex $v \in \Delta$, at most one of the corresponding vertices of the Δ^i is not a lattice point.

For example,



is an admissible one-parameter decomposition of a non-lattice polyhedron with tail cone $\{0\}$.

1.2 Tori and T-varieties

An *algebraic group* is a variety G together with regular maps $m\colon G \times G \to G$ (multiplication), $i\colon G \to G$ (inverse) and $e\colon \bullet \to G$ that satisfy the appropriate laws. For example, the following diagram must commute, where $\bullet = \mathrm{Spec}\,\mathbb{C}$.

$$\begin{array}{ccc} G & \xrightarrow{(\mathrm{id}_G, i)} & G \times G \\ \downarrow & & \downarrow m \\ \bullet & \xrightarrow[e]{} & G \end{array}$$

An *action* of G on a scheme X is a regular map

$$\begin{aligned} \rho\colon G \times X &\to X \\ (g, x) &\mapsto gx \end{aligned}$$

(short: $G \circlearrowright X$) such that the diagram below commutes.

$$\begin{CD} G \times G \times X @>{\mathrm{id}_G \times \rho}>> G \times X \\ @V{m \times \mathrm{id}_X}VV @VV{\rho}V \\ G \times X @>>{\rho}> X \end{CD}$$

The *kernel* of an action ρ is the subgroup of G consisting of the $g \in G$ where $\rho_{|\{g\}\times X}$ induces the identity. The action is *effective* if the action has a trivial kernel.

Such a scheme X with a G-action will also be called a G-scheme. A morphism from a G-scheme X to an H-scheme Y is a pair (φ, f) of a morphism $\varphi\colon G \to H$ of algebraic groups and a regular map $f\colon X \to Y$ that commute with the actions as pictured below, forming the category **GSch**.

$$\begin{CD} G \times X @>{\rho_1}>> X \\ @V{\varphi \times f}VV @VV{f}V \\ H \times Y @>>{\rho_2}> Y \end{CD}$$

For a fixed group G, we have the (non-full) subcategory G-**Sch** of G-schemes, where morphisms involve the identity on G.

A G*-variety* is a G-scheme X such that X is a normal variety and such that the action by G is effective. A morphism of G-varieties $f\colon G \circlearrowright X \to H \circlearrowright Y$ is a morphism of G-schemes that is also *orbit dominating*, i.e., such that $H \cdot f(X)$ is dense in Y. Thus, G-varieties form a non-full subcategory **GVar** of **GSch**. Again, the varieties with fixed group G form a subcategory G-**Var**.

Remark 1.2. We restrict to orbit dominating morphisms since those are the morphisms that can be described by the maps of polyhedral divisors introduced below. In general, this means we exclude embeddings of orbit closures. An example of such a morphism is the embedding of the central fibre in chapter 4.

An *algebraic torus* of dimension n is an algebraic group $T \cong (\mathbb{C}^*)^n$. It comes with the dual lattices

$$\begin{aligned} M &= \mathrm{Hom}_{\text{alg.gp.}}(T, \mathbb{C}^*) \cong \mathbb{Z}^n \\ N &= \mathrm{Hom}_{\text{alg.gp.}}(\mathbb{C}^*, T) \cong \mathbb{Z}^n \end{aligned}$$

of characters and one-parameter subgroups, respectively. Then $T = \operatorname{Spec} \mathbb{C}[M]$, with $\mathbb{C}[M] \cong \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, or $T = N \otimes_{\mathbb{Z}} \mathbb{C}^*$.

We denote by **TSch** and **TVar** the subcategories of **GSch** and **GVar** of objects where the group is a torus. These are *T-schemes* or *T-varieties*.

The *complexity* of a T-variety is the minimal codimension of an orbit $Tx = \{tx \mid t \in T\}$ for points $x \in X$.

For an affine variety $X = \operatorname{Spec} A$, an action by T corresponds to an M-grading of $A = \bigoplus_{u \in M} A_u$. The *weight cone* ω of the action is the convex polyhedral cone in $M_{\mathbb{Q}}$ generated by the weights $\{u \in M | A_u \neq 0\}$; it is of full dimension if and only if the action is effective.

1.3 Polyhedral divisors

In the following sections, I will summarize how T-varieties can be described with so called polyhedral divisors on normal semiprojective varieties Y.

A variety Y is *semicomplete* or *semiprojective* if the affine contraction morphism $r\colon Y \to Y_0 = \operatorname{Spec} H^0(Y, \mathcal{O}_Y)$ is proper or projective, respectively.

Proposition 1.3. *A variety Y is semiprojective if and only if it is quasiprojective and semicomplete.*

Proof. If Y is semiprojective, it follows immediately that it is quasiprojective and semicomplete. Assume then that Y is quasiprojective and semicomplete. Thus Y is embedded in some $\mathbb{P}^m$ and we have a proper map $\theta\colon Y \to Z$ for some affine Z. The graph Γ of θ in $\mathbb{P}^m \times Z$ is isomorphic to Y; furthermore, we claim that it is closed. Indeed, we have maps $\Gamma \to \mathbb{P}^m \times Z \to Z$, with the first separated, and the second and the composition proper. Thus, the first is proper as well and the image of Γ in $\mathbb{P}^m \times Z$ is closed, making Y semiprojective. $\square$

From now on, Y will be a semiprojective normal variety. The group of Weil divisors on Y will be denoted by $\operatorname{WDiv}(Y)$; that of $\mathbb{Q}$-Weil divisors by $\operatorname{WDiv}_{\mathbb{Q}}(Y) = \operatorname{WDiv}(Y) \otimes \mathbb{Q}$. Similarly we have the ($\mathbb{Q}$-)Cartier divisors in $\operatorname{CDiv}(Y)$ and $\operatorname{CDiv}_{\mathbb{Q}}(Y)$.

For a $\mathbb{Q}$-Weil divisor D on Y, the associated sheaf $\mathcal{O}_Y(D)$ has sections

$$H^0(U, \mathcal{O}_Y(D)) = \{f \in \mathbb{C}(U) \mid \operatorname{div}(f) + D_{|U} \geq 0\}.$$

The vector space of global sections will be denoted by $L(D) := H^0(Y, \mathcal{O}_Y(D))$. Note that by definition, the sheaf of sections of a $\mathbb{Q}$-Weil divisor $D = \sum_P a_P \cdot P$ agrees with that of its *round-down* $\lfloor D \rfloor = \sum_P \lfloor a_P \rfloor \cdot P \in \operatorname{WDiv}(Y)$.

We will also consider divisors on Y with coefficients in $\mathbb{Q} \cup \{\infty\}$, where $\infty + a = \infty$ for all $a \in \mathbb{Q} \cup \{\infty\}$. We identify a divisor $D \in \operatorname{WDiv}_{\mathbb{Q} \cup \{\infty\}}(Y)$ with coefficients that are possibly infinite

$$D = \sum_{P \text{ prime}} a_P \cdot P$$

on Y with the conventional divisor

$$D = D_{|\operatorname{loc} D} = \sum_{a_P \neq \infty} a_P \cdot (P \cap \operatorname{loc} D)$$

on the *locus* $\operatorname{loc} D := Y \setminus \bigcup_{a_P = \infty} P$ of D.

Fix now a torus T with lattices M, N and a cone σ in N. A *polyhedral divisor* for these data is an element of $\operatorname{WDiv}(Y, \sigma) := \operatorname{WDiv}(Y, N, \sigma) := \operatorname{Pol}(N, \sigma) \otimes_{\mathbb{Q}_{\geq 0}} \operatorname{WDiv}_{\mathbb{Q}}(Y)$, that is, a finite linear combination

$$\mathcal{D} = \sum \Delta_i \otimes D_i,$$

where the Δ_i are σ-polyhedra and the D_i are divisors on Y. A polyhedral divisor may be expressed uniquely as a sum

$$\mathcal{D} = \sum \Delta_P \otimes P$$

ranging over all prime divisors P on Y. The coefficient of P in $\mathcal{D}$ will also be denoted by $\mathcal{D}_P := \Delta_P$.

1.4 The functor TV

Let $\mathcal{D} \in \mathrm{WDiv}(Y, \sigma)$ be a polyhedral divisor. Dualization of the coefficient polyhedra yields a dual concave piecewise linear map

$$\begin{aligned} \sigma^\vee &\to \mathrm{WDiv}_{\mathbb{Q}\cup\{\infty\}}(\mathcal{D}) \\ u &\mapsto \mathcal{D}(u) := \sum \min\langle \Delta_i, u\rangle D_i. \end{aligned}$$

Here, $\emptyset \in \mathrm{Pol}(N, \sigma)$ corresponds to $\infty \in \mathbb{Q} \cup \{\infty\}$ via $\min\langle \emptyset, u\rangle = \infty$. Thus, we define $\operatorname{loc}\mathcal{D} = Y \setminus \bigcup_{\mathcal{D}_P=\emptyset} P$. Then $\mathcal{D}$ restricts to a polyhedral divisor on $\operatorname{loc}\mathcal{D}$ that has no empty coefficients, yielding a map $\sigma^\vee \to \mathrm{WDiv}_{\mathbb{Q}}(\operatorname{loc}\mathcal{D})$.

Concavity just means that $\mathcal{D}(u) + \mathcal{D}(u') \leq \mathcal{D}(u + u')$, so we have multiplication maps $\mathcal{O}_{\operatorname{loc}\mathcal{D}}(\mathcal{D}(u)) \otimes \mathcal{O}_{\operatorname{loc}\mathcal{D}}(\mathcal{D}(u')) \to \mathcal{O}_{\operatorname{loc}\mathcal{D}}(\mathcal{D}(u + u'))$. Hence, $\mathcal{D}$ defines a sheaf of algebras $\mathcal{A}(\mathcal{D}) := \bigoplus_{u\in\sigma^\vee\cap M} \mathcal{O}_{\operatorname{loc}\mathcal{D}}(\mathcal{D}(u))$, and its algebra of global sections $A(\mathcal{D}) := \bigoplus_{u\in\sigma^\vee\cap M} A(\mathcal{D}(u))$. We can now define $\mathrm{TV}(\mathcal{D}) := \operatorname{Spec} A(\mathcal{D})$, though we will need to put some extra conditions on $\mathcal{D}$ to make the construction well-behaved. Note that the M-grading on $A(\mathcal{D})$ defines an action of T on $\mathrm{TV}(\mathcal{D})$.

In order to make the construction functorial, we need to define the notion of a morphism of polyhedral divisors. A *polyhedral Cartier divisor* is an element of $\mathrm{CDiv}(Y, \sigma) := \mathrm{Pol}(N, \sigma) \otimes_{\mathbb{Q}_{\geq 0}} \mathrm{CDiv}_{\mathbb{Q}}(Y) \subset \mathrm{WDiv}(Y, \sigma)$; a polyhedral Weil divisor is Cartier if and only if the evaluations $\mathcal{D}(u)$ are all $\mathbb{Q}$-Cartier divisors on $\operatorname{loc}\mathcal{D}$.

For a polyhedral Cartier divisor $\mathcal{D} \in \mathrm{CDiv}(Y, N, \sigma)$ and a dominant map $\psi\colon Y' \to Y$, the *pull-back* of $\mathcal{D}$ along ψ is

$$\begin{aligned} \psi^*\mathcal{D}\colon \sigma^\vee &\to \mathrm{CDiv}_{\mathbb{Q}}(Y) \\ u &\mapsto \psi^*(\mathcal{D}(u)). \end{aligned}$$

Secondly, the pull-back $F^{-1}\mathcal{D}$ of $\mathcal{D}$ along a lattice homomorphism $F\colon N' \to N$ has coefficients

$$F^{-1}(\mathcal{D})_P := F^{-1}(\mathcal{D}_P).$$

Note that $F^{-1}(\mathcal{D})$ need not be Cartier, compare Example 2.6. Finally, given a cone σ in N, the principal polyhedral divisor $\operatorname{div}(\mathfrak{f}) \in \mathrm{CDiv}(Y, \sigma)$ associated with a function $\mathfrak{f} = \sum_i v_i \otimes f_i \in N \otimes \mathbb{C}(Y)^*$ is defined by

$$\operatorname{div}(\mathfrak{f})(u) := \sum_i \langle v_i, u\rangle \operatorname{div}(f_i).$$

Now consider a triple $\varphi = (\psi, F, \mathfrak{f})$ of a dominant map $\psi\colon Y' \to Y$, a lattice homomorphism $F\colon N' \to N$ and a function $\mathfrak{f} \in N \otimes \mathbb{C}(Y')^*$. The pull-back of $\mathcal{D} \in \mathrm{CDiv}(Y, \sigma)$ under φ is

$$\varphi^{-1}(\mathcal{D}) := F^{-1}(\psi^*\mathcal{D} + \operatorname{div}(\mathfrak{f}));$$

it is an element of $\mathrm{WDiv}(Y', F^{-1}(\sigma))$.

A *morphism of polyhedral divisors* from $\mathcal{D}' \in \mathrm{CDiv}(Y', \sigma')$ to $\mathcal{D} \in \mathrm{CDiv}(Y, \sigma)$ is a triple φ as above such that for every prime divisor P on Y', we have $\mathcal{D}'_P \subset \varphi^{-1}(\mathcal{D})_P$. If $\varphi' = (\psi', F', \mathfrak{f}')\colon \mathcal{D}'' \to \mathcal{D}'$ is another morphism, their composition is defined as

$$\varphi \circ \varphi' := (\psi \circ \psi', F \circ F', F(\mathfrak{f}') \cdot \psi'^{*}\mathfrak{f}).$$

Thus, the Cartier polyhedral divisors form a category **PolDiv**. As with WDiv and CDiv, this can be partially applied to restrict to divisors with specified base or similar, for example $\mathbf{PolDiv}(Y, \sigma)$, $\mathbf{PolDiv}(N)$. We will refer to $\mathbf{PolDiv}(Y, \sigma)$ instead of $\mathrm{CDiv}(Y, \sigma)$ from now on.

A morphism $\varphi\colon \mathcal{D}' \to \mathcal{D}$ defines a graded homomorphism

$$\begin{aligned} &A(\mathcal{D}) \to A(\mathcal{D}') \\ &g \cdot \chi^u \mapsto \psi^* g \cdot \mathfrak{f}(u)^{-1} \cdot \chi^{F^* u}, \end{aligned}$$

which shows that $\mathrm{TV} = \mathrm{Spec} \circ A$ is a functor $\mathrm{TV}\colon \mathbf{PolDiv} \to \mathbf{TSch}$.

Example 1.4. The identity on $\mathcal{D}$ is $\mathrm{id}_{\mathcal{D}} = (\mathrm{id}_Y, \mathrm{id}_N, 1)$. If $\psi\colon Y' \to Y$ is proper and birational, then for $\mathcal{D} \in \mathbf{PolDiv}(Y, \sigma)$, the map

$$\mathrm{TV}((\psi, \mathrm{id}_N, 1))\colon\ \mathrm{TV}(\psi^* \mathcal{D}) \to \mathrm{TV}(\mathcal{D})$$

is an isomorphism.

Example 1.5. Let $\mathcal{D}$ be a polyhedral divisor on Y, and $f\colon Y \to S$ a dominant regular map to an affine scheme S. Then $(f, 0, 0)$ determines an invariant map $\mathrm{TV}(X) \to S$.

1.5 P-divisors

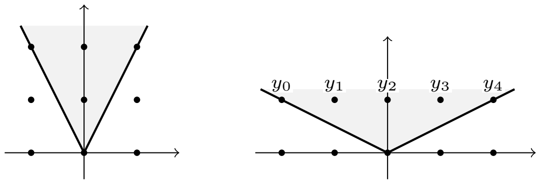
Let $\mathcal{D} \in \mathrm{CDiv}(Y, \sigma)$ be a polyhedral divisor. Then we say that $\mathcal{D}$ is a *p-divisor* (short for *proper polyhedral divisor*) if it satisfies the following two conditions.

1. $\mathrm{loc}\, \mathcal{D}$ is semi-projective.
2. $\mathcal{D}$ is *big*, that is, $\mathcal{D}(u)$ is semi-ample for all $u \in \sigma^\vee$ (so $\mathcal{D}$ is *semi-ample*), $\mathcal{D}(u)$ is big for $u \in \mathrm{relint}\, \sigma^\vee$ and $\sigma^\vee$ is of full dimension.

P-divisors form the subcategory **PDiv** of **PolDiv**. Denote by $\mathbf{PDiv}_{\mathrm{loc}}$ the localization of **PDiv** at the morphisms of the form $(\psi, \mathrm{id}_N, 1)$ with ψ proper and birational, and denote by $\mathbf{TVar}_{\mathrm{aff}}$ the subcategory of affine T-varieties. Then we can state the main result on p-divisors.

Theorem 1.6 ([AH06, Theorem 8.6])**.** $\mathrm{TV}\colon \mathbf{PDiv}_{\mathrm{loc}} \to \mathbf{TVar}_{\mathrm{aff}}$ *is an equivalence of categories.*

Example 1.7. If Y is a point, all divisors are 0, and the only condition is that $\sigma^\vee$ is full-dimensional. Then $X = \mathrm{TV}(\bullet, \sigma, N) = \mathrm{TV}(\sigma) = \mathrm{Spec}\, \mathbb{C}[\sigma^\vee \cap M]$ is the affine toric variety associated with the pointed cone σ in N.



(a) σ (b) $\sigma^\vee$

Figure 1.1: Cones for Example 1.8

Example 1.8. Let $\sigma = \langle(-1,2), (1,2)\rangle \subseteq \mathbb{Q}^2$. This gives rise to the dual cone $\sigma^\vee = \langle[-2,1], [2,1]\rangle$; the semigroup $\sigma^\vee \cap M$ is generated by $E = \{[e,1] \mid -2 \leq e \leq 2\}$, as illustrated in Figure 1.1. The resulting toric variety $X = \mathrm{TV}(\sigma) \subseteq \mathbb{C}^5$ is the cone over the rational normal curve of degree 4, and its defining ideal is generated by the six minors expressing the inequality $\operatorname{rk}\left(\begin{smallmatrix} y_0 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{smallmatrix}\right) \leq 1$.

Example 1.9. Take X from Example 1.8 and consider the action by the subtorus $H = \mathbb{C}^* \hookrightarrow T = (\mathbb{C}^*)^2$ corresponding to $\mathbb{Z} \oplus 0 \subset N$. Then a p-divisor for X as an H-variety is

$$\mathcal{D} = \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \otimes \{0\}$$

on $Y = \mathbb{A}^1$. (Compare Example 2.1 for how to compute this "downgrade" of the toric variety X.)

1.6 Fans of p-divisors

The construction can be globalized by gluing T-varieties associated with p-divisors. For divisors $\mathcal{D}, \mathcal{D}' \in \mathrm{WDiv}(Y, N)$ which may have differing tail cones, we say that $\mathcal{D}$ is a *face* of $\mathcal{D}'$ if we have inclusions $\mathcal{D}_P \subset \mathcal{D}'_P$ for all prime divisors in Y, and if the associated map $\mathrm{TV}(\mathrm{id}_Y, \mathrm{id}_N, 1)$ is an open embedding. It follows that $\mathcal{D}_P$ is a face of $\mathcal{D}'_P$ for every P. The *intersection* of two such polyhedral divisors $\mathcal{D}$, $\mathcal{D}'$ has coefficients $(\mathcal{D} \cap \mathcal{D}')_P := \mathcal{D}_P \cap \mathcal{D}'_P$.

A *fan of p-divisors* on Y is a set $\mathcal{S}$ of p-divisors $\mathcal{D} \in \mathbf{PDiv}(Y, N)$ such that for any two divisors $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$, their intersection $\mathcal{D} \cap \mathcal{D}'$ also lies in $\mathcal{S}$, and is a face both of $\mathcal{D}$ and of $\mathcal{D}'$.

Let $\mathcal{S}$ be a fan of p-divisors on Y. The tail cones of the occurring p-divisors form the *tail fan* $\mathrm{tail}(\mathcal{S}) = \{\mathrm{tail}\,\mathcal{D} \mid \mathcal{D} \in \mathcal{S}\}$. If P is a prime divisor on Y, then the *slice* of $\mathcal{S}$ at P is the polyhedral decomposition $\mathcal{S}_P = \{\mathcal{D}_P \mid \mathcal{D} \in \mathcal{S}\}$ of $|\mathcal{S}_P| = \bigcup \mathcal{D}_P$.

Theorem 1.10 ([AHS08, Theorems 5.3 and 5.6])**.** *For a fan of p-divisors $\mathcal{S}$, the affine T-varieties $\{\mathrm{TV}(\mathcal{D}) \mid \mathcal{D} \in \mathcal{S}\}$ glue to form a T-scheme $\mathrm{TV}(\mathcal{S})$. Conversely, for any T-variety X, there exists a fan of p-divisors $\mathcal{S}$ on some projective variety Y such that X is equivariantly isomorphic to $\mathrm{TV}(\mathcal{S})$.*

We will call a fan of p-divisors such that $\mathrm{TV}(\mathcal{S})$ is separated (hence a T-variety) a *divisorial fan*. These form the category **PFan**, where morphisms consist of the same data as morphisms of polyhedral divisors, subject to an extra condition. Namely, for $\varphi = (\psi, F, \mathfrak{f})$ to define a morphism $\mathcal{S} \to \mathcal{S}'$ we require that for each $\mathcal{D} \in \mathcal{S}$, there exists a $\mathcal{D}' \in \mathcal{S}'$ with $\mathcal{D} \subset \varphi^{-1}(\mathcal{D}')$.

Example 1.11. A *refinement* of a divisorial fan $\mathcal{S}' \in \mathbf{PFan}(Y, N)$ is a divisorial fan $\mathcal{S} \in \mathbf{PFan}(Y, N)$ such that for each $\mathcal{D} \in \mathcal{S}$, there exists $\mathcal{D}' \in \mathcal{S}'$ such that $\mathcal{D}$ is a face of $\mathcal{D}'$, and such that the images of the induced open embeddings $\mathrm{TV}(\mathcal{D}) \to \mathrm{TV}(\mathcal{S}')$ cover $\mathrm{TV}(\mathcal{S}')$. Then $(id_Y, id_N, 1)$ is a map $\mathcal{S} \to \mathcal{S}'$ giving an isomorphism $\mathrm{TV}(S) \xrightarrow{\sim} \mathrm{TV}(S')$.

Remark 1.12. In general, φ defines a rational map $\Phi\colon Y \times T \to Y' \times T'$ which uniquely determines $\mathrm{TV}(\varphi)\colon \mathrm{TV}(\mathcal{S}) \to \mathrm{TV}(\mathcal{S}')$. Note that we exclude some φ where Φ extends to a regular map $\mathrm{TV}(\mathcal{S}) \to \mathrm{TV}(\mathcal{S}')$. Such regular maps can be described by refining $\mathcal{S}$ [Süß11, Satz 2.8].

Example 1.13. Take the refinement morphism $(id_Y, id_N, 1)\colon \mathcal{S} \to \mathcal{S}'$ from Example 1.11. Then in general, the triple $(id_Y, id_N, 1)$ is not a map $\mathcal{S}' \to \mathcal{S}$. To describe the inverse isomorphism $\mathrm{TV}(S') \to \mathrm{TV}(S)$, we need to refine $\mathcal{S}'$, for example to $\mathcal{S}$.

Example 1.14. Consider the triple $(f, 0, 0)$ of Example 1.5. If S is affine, then for any divisorial fan $\mathcal{S}$ on Y we get a map $\mathrm{TV}(\mathcal{S}) \to S$.

Let us drop the requirement that S be affine, taking for instance $S = Y$ and $f = \mathrm{id}_Y$. A divisorial fan for the trivial T-variety Y consists of an open affine cover of Y, with a trivial divisor on each. If $\operatorname{loc} \mathcal{D} \subset Y$ is affine but not contained in one of the elements of the open cover, then while the rational quotient map $\mathrm{TV}(\mathcal{D}) \to Y$ is actually regular, we disallow $(f, 0, 0)$. If on the other hand $\operatorname{loc} \mathcal{D} = Y$ is not affine, then the rational quotient map is not regular.

In summary, we obtain the following corollary to Theorem 1.10, noting that **PDiv** embeds in **PFan** naturally by sending $\mathcal{D}$ to the fan $\{\mathcal{D}\}$.

Corollary 1.15. *The functor* TV *extends to an essentially surjective functor* $\mathrm{TV}\colon \mathbf{PFan} \to \mathbf{TVar}$. *If we localize* **PFan** *by the morphisms corresponding to birational proper maps of the base and refinements of divisorial fans, this becomes an equivalence of categories.*

We will later use the following characterization of proper maps of divisorial fans. This relies on an extension of the notion of the slice of a fan $\mathcal{S}$ on Y to any point $y \in Y$:

$$\mathcal{D}_y := \sum_{P \ni y} \mathcal{D}_P \qquad\qquad \mathcal{S}_y := \{\mathcal{D}_y \mid \mathcal{D} \in \mathcal{S}\}$$

Theorem 1.16 ([Süß11, Satz 2.16])**.** *A map $\varphi\colon \mathcal{S} \to \mathcal{D}$ defines a proper morphism if and only if for all points $y \in Y$, it holds that $|\mathcal{S}_y| = \varphi^{-1}(\mathcal{D})_y$.*

1.7 Contraction-free T-varieties

For a p-divisor $\mathcal{D}$ on Y, besides the spectrum of global sections $\mathrm{TV}(\mathcal{D}) = \operatorname{Spec} A(\mathcal{D})$, the relative spectrum $\widetilde{\mathrm{TV}}(\mathcal{D}) = \operatorname{Spec}_Y \mathcal{A}(\mathcal{D})$ is also a T-variety, fitting into the following diagram.

$$\begin{array}{ccc} \widetilde{\mathrm{TV}}(\mathcal{D}) & \xrightarrow{\ r\ } & \mathrm{TV}(\mathcal{D}) \\ \downarrow & & \\ Y & & \end{array}$$

Note that if $\operatorname{loc}\mathcal{D}$ is affine, $\widetilde{\mathrm{TV}}(\mathcal{D})$ and $\mathrm{TV}(\mathcal{D})$ agree. For a divisorial fan $\mathcal{S}$, the $\widetilde{\mathrm{TV}}(\mathcal{D})$ for $\mathcal{D} \in \mathcal{S}$ glue to form a T-variety $\widetilde{\mathrm{TV}}(\mathcal{S})$; the affine contraction maps r glue to $r\colon \widetilde{\mathrm{TV}}(\mathcal{S}) \to \mathrm{TV}(\mathcal{S})$.

Example 1.17. The existence of the map $\mathrm{TV}(\mathcal{S}) \to S$ of Examples 1.5, 1.14 also follows from the universal property of affine contraction: We have a regular map $\widetilde{\mathrm{TV}}(\mathcal{S}) \to Y \to S$, which factors through the affine contraction $r\colon \widetilde{\mathrm{TV}}(\mathcal{S}) \to \mathrm{TV}(\mathcal{S})$.

Definition 1.18. A divisorial fan $\mathcal{S}$ is *contraction-free* if the following equivalent conditions hold.

1. The contraction map r is an isomorphism.
2. The rational quotient map $\pi\colon\ \mathrm{TV}(\mathcal{S}) \dashrightarrow Y$ is regular.
3. For all $\mathcal{D} \in \mathcal{S}$, the locus of $\mathcal{D}$ is affine.

Proposition 1.19. *Let $\mathcal{S}$ be a divisorial fan on Y. Then there is a contraction-free divisorial fan $\mathcal{S}'$ on Y such that $\mathcal{S}_P = \mathcal{S}'_P$ for all slices and there is a birational proper map $\mathrm{TV}(\mathcal{S}') \to \mathrm{TV}(\mathcal{S})$.*

Proof. The restrictions of the p-divisors $\mathcal{D} \in \mathcal{S}$ to an open affine cover form a divisorial fan $\mathcal{S}'$ with $\mathrm{TV}(\mathcal{S}') = \widetilde{\mathrm{TV}}(\mathcal{S})$ [Süß11, Bsp. 1.18]. The birational proper map $\mathrm{TV}(\mathcal{S}') \to \mathrm{TV}(\mathcal{S})$ is just the contraction map r. □

Remark 1.20. A stronger requirement is for a divisorial fan to be *toroidal*. Let $U \subset Y$ be the largest open subset of Y such that all restrictions $\mathcal{D}_{|U}$ for $\mathcal{D} \in \mathcal{S}$ are trivial. (U is the complement of the *support* of $\mathcal{S}$.) Then $\mathcal{S}$ is said to be toroidal if it is contraction-free and the embedding $U \subset S$ is a toroidal embedding in the sense of Kempf et al. [KKMSD73]. It follows that the embedding $U \times T \subset \mathrm{TV}(\mathcal{S})$ is toroidal.

Chapter 2

Upgrading polyhedral divisors

When working with polyhedral divisors, the problems of *upgrading* and *downgrading* come up naturally. These are polyhedral analogues to extensions and restrictions of torus actions on T-varieties. Downgrading is perhaps a little more straightforward to specify: Consider a polyhedral divisor $\mathcal{D}$, defining a T-variety X. If H is a subtorus of T, then by a downgrade of $\mathcal{D}$ we mean a polyhedral divisor $\mathcal{E}$ for X as an H-variety. We would like to be able to compute $\mathcal{E}$ from $\mathcal{D}$ directly.

Upgrading is the inverse problem: Given a p-divisor $\mathcal{E}$ for H that is invariant with respect to some T'-action, find a p-divisor $\mathcal{D}$ for the torus $T = T' \times H$ giving the same variety as $\mathcal{E}$.

The chapter starts by fixing precisely what we mean by up- and downgrades. Then, we proceed to construct an upgrade functor. The largest part of the chapter is devoted to proving that the upgrade of a p-divisor is indeed proper again; this is the result of joint work with Ilten [IV11a]. The main differences compared to that article are the functorial approach to upgrades, including the description of upgrades of maps of p-divisors, and the extraction of the upgrade result for polyhedral divisors.

2.1 Downgrades and upgrades

To define the notion of a downgrade more precisely, let $\iota\colon H \to T$ be the embedding of H, which induces the downgrade functor

$$\iota^*\colon T\text{-}\mathbf{Sch} \to H\text{-}\mathbf{Sch}.$$

Downgrading then involves providing explicitly a functor

$$\mathbf{d}\colon \mathbf{PolDiv}(T) \to \mathbf{PolDiv}(H)$$

such that $\iota^* \circ \mathrm{TV} = \mathrm{TV} \circ \mathbf{d}$. Even without functoriality, a mapping of objects that satisfies this equation will be called a downgrade.

Example 2.1 ([AH06, section 11]). Let $X = \mathrm{TV}(\bullet, \sigma, N)$ be a toric variety and $H \subset T$ a subtorus corresponding to a split exact sequence

$$0 \longrightarrow N_H \underset{r}{\overset{i}{\rightleftarrows}} N \overset{p}{\longrightarrow} N' \longrightarrow 0.$$

Let Σ be the fan in N' that is the coarsest common refinement of the cones $p(\tau)$ for all faces τ of σ, and Y the associated toric variety $\mathrm{TV}(\Sigma)$. Define a polyhedral divisor $\mathcal{D}$ on Y such that the coefficient for the invariant prime divisor D_ρ corresponding to a ray $\rho \in \Sigma$ is

$$\mathcal{D}_{D_\rho} = r(p^{-1}(v_\rho) \cap \sigma).$$

Then the p-divisor $\mathcal{D}$ is a downgrade of σ.

Upgrading is a sort of inverse to downgrading, except that $\mathbf{d}$ can't be inverted for obvious reasons: Like ι^*, it is not essentially surjective, as it doesn't hit H-varieties that don't admit an extended action. Furthermore, we lose the information of how the H-action should extend. We fix this by introducing the notion of an equivariant p-divisor.

Definition 2.2. Let H and T' be tori. A *T'-equivariant polyhedral divisor for H* consists of a semiprojective T'-variety Y together with a polyhedral divisor for H on Y taking values in $\mathrm{CDiv}_{\mathbb{Q}}(Y)^{T'}$.

We denote these by $\mathbf{PolDiv}(H)^{T'}$; there is a forgetful functor

$$\mathbf{f}\colon \mathbf{PolDiv}(H)^{T'} \to \mathbf{PolDiv}(H).$$

In the case of an embedding $H \hookrightarrow T$, we would take $T' = T/H$ (or take $T' = T$ and ask that T acts on Y with generic stabilizer H).

Note that T' acts naturally on the sections of a T'-invariant divisor, hence $H \times T'$ acts on the spectrum of global sections. Clearly, we can do the same for p-divisors, with the category $\mathbf{PDiv}(H)^{T'}$ of T'-invariant p-divisors.

Proposition 2.3. *The functor* TV *lifts to*

$$\mathrm{TV}'\colon \mathbf{PolDiv}(H)^{T'} \to (H \times T')\text{-}\mathbf{Sch},$$

yielding a commutative diagram

$$\begin{array}{ccc} \mathbf{PolDiv}(H)^{T'} & \xrightarrow{\mathrm{TV}'} & (H \times T')\text{-}\mathbf{Sch} \\ \downarrow \mathbf{f} & & \downarrow \iota^* \\ \mathbf{PolDiv}(H) & \xrightarrow{\mathrm{TV}} & H\text{-}\mathbf{Sch}. \end{array}$$

For p-divisors, we get $\mathrm{TV}'\colon \mathbf{PDiv}(H)^{T'} \to (H \times T')\text{-}\mathbf{Var}$*, with the corresponding commutative diagram.*

Proof. The only thing to check is that in the case of p-divisors, $H \times T'$ acts effectively. But that is clear since T' acts effectively on Y, and for $\mathcal{D} \in \mathbf{PDiv}(H, Y)$, $\mathrm{TV}(\mathcal{D})$ is H-equivariantly birational to $H \times Y$. □

Now the T'-variety Y may in turn be expressed by a divisorial fan, i.e., an object of $\mathbf{PFan}(T')$. Let $\mathbf{PolDiv}^2(T', H)$ consist of triples $(Z, \mathcal{S}, \mathcal{D})$, where $\mathcal{S}$ is a divisorial fan on Z for T', and $\mathcal{D}$ is a T'-equivariant polyhedral divisor for H on $\mathrm{TV}(\mathcal{S})$. Applying the functor TV to the first two arguments, we get

$$\mathrm{TV} \times \mathrm{id}\colon \mathbf{PolDiv}^2(T', H) \to \mathbf{PolDiv}(H)^{T'}.$$

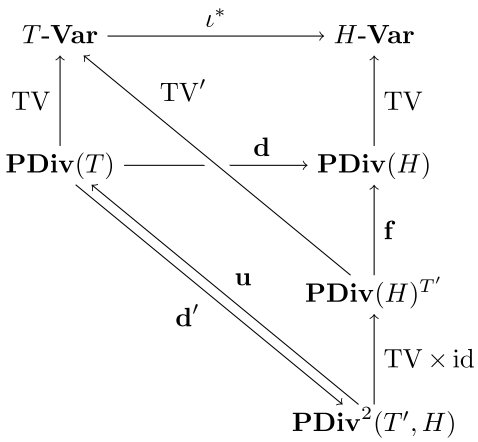


Figure 2.1: Upgrade and downgrade functors for $T = T' \times H$

With this, it becomes reasonable to ask our downgrade functor to factor as $\mathbf{d} = \mathbf{f} \circ (\mathrm{TV} \times \mathrm{id}) \circ \mathbf{d}'$. It is the functor $\mathbf{d}'$ that upgrading should invert. Again, we have the analogous notions for p-divisors. For the case of a split torus $T = T' \times H$ and p-divisors, the situation is illustrated in Figure 2.1.

Example 2.4. Suppose we have divisors $(Z, \mathcal{S}, \mathcal{D}) \in \mathbf{PolDiv}^2(T', H)$ and $(Z', \mathcal{E}) \in \mathbf{PolDiv}(T' \times H)$, $\mathcal{E}$ describing the T-variety X and $\mathcal{D}$ describing the downgraded H-variety X. So $\mathcal{D}$ is a downgrade of $\mathcal{E}$, and $\mathcal{E}$ is an upgrade of $\mathcal{D}$.

Then $\mathrm{id}_X : H \circlearrowright X \to T \circlearrowright X$ is a morphism of T-varieties. As argued below, id_X corresponds to the morphism of polyhedral divisors

$$\varphi = ((\mathrm{id}_Z, 0, 0), N_H \hookrightarrow N_T, \mathfrak{f} \colon (u_1, u_2) \mapsto \chi^{-u_1}),$$

after modifying the divisors involved and identifying Z and Z'.

Since id_X is proper, Theorem 1.16 implies that the slices of $\mathcal{S}$ are complete, and that $\mathcal{D}$ is equal to $\varphi^{-1}(\mathcal{E})$. In particular, if we have the downgraded base $Y = \mathrm{TV}(\mathcal{S})$, the polyhedral divisor $\mathcal{D}$ on Y is determined. Compare Example 2.10 below for the coefficients of $\varphi^{-1}(\mathcal{E})$.

Now to argue that id_X arises as claimed. Resolving the indeterminacy of the induced rational map of quotients $Y = \mathrm{TV}(\mathcal{S}) \dashrightarrow Z'$ by replacing $(Z, \mathcal{S})$ via some proper birational map $(Z'', \mathcal{S}'') \to (Z, \mathcal{S})$ and pulling back $\mathcal{D}$ to $\mathrm{TV}(\mathcal{S}'')$, the identity on X corresponds to $((\psi, 0, 0), N_H \hookrightarrow N_T, \mathfrak{f})$ with some $\mathfrak{f} \colon M_T \to \mathbb{C}(\mathrm{TV}(\mathcal{S}))^*$.

The divisors $(\mathcal{S}, \mathcal{D})$ and $\mathcal{E}$ determine rational isomorphisms of X with $Z \times T$ and $Z' \times T$, respectively; they can be made compatible by modifying $\mathcal{S}$, $\mathcal{D}$ and $\mathcal{E}$ with principal polyhedral divisors, making ψ identify $\mathbb{C}(Z)$ with $\mathbb{C}(Z')$. It follows that $\mathfrak{f}(u_1, u_2) = 1 \cdot \chi^{-u_1} \in \mathbb{C}(\mathrm{TV}(\mathcal{S})) = \mathbb{C}(Z \times H)$, so that for $g \cdot \chi^{(u_1, u_2)} \in \mathbb{C}(Z) \otimes \mathbb{C}(T') \otimes \mathbb{C}(H)$ we have

$$g \cdot \chi^{(u_1, u_2)} \mapsto \psi^*(g) \cdot \mathfrak{f}(u_1, u_2)^{-1} \cdot \chi^{u_2} = g \cdot \chi^{u_1} \cdot \chi^{u_2}.$$

By pulling back $\mathcal{E}$ along the proper map $\psi \colon Z \to Z'$, we can even assume that $Z = Z'$ and ψ is the identity.

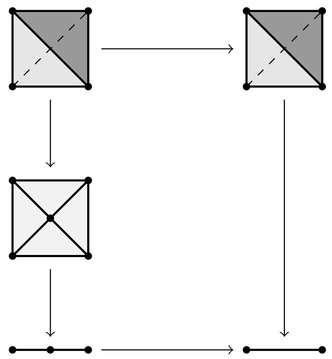


Figure 2.2: Fans of the toric varieties involved in Example 2.5 in height one

Example 2.5. We saw above that downgrading is easy if we have $Y = \mathrm{TV}(\mathcal{S})$. However, finding $\mathcal{S}$ is not always easy. In particular, given a p-divisor $\mathcal{E}$ on Z, the base Y of the downgraded divisor need not admit a description by a fan on Z.

As an example, take the $T' \times H$-action on $\mathbb{A}^4$ with weights $(0, 1)$, $(1, -1)$, $(0, -1)$, $(-1, 1)$. The divisors mentioned below may be obtained by straightforward toric downgrade calculations which I omit.

$$\mathcal{E} = \mathrm{conv}\{0, e_1\} \otimes D_1 + \mathrm{conv}\{0, e_1 + e_2\} \otimes D_2$$

is a p-divisor for X on $\mathbb{A}^2$, where D_i are the coordinate axes. On the other hand, the Chow quotient of X by H is by the toric variety Y that arises by subdividing the cone spanned by the square $\rho_1 = (1, 0, 0)$, $\rho_2 = (0, 1, 0)$, $\rho_3 = (1, 0, 1)$, $\rho_4 = (0, 1, 1)$ along the central ray $\rho_5 = (1, 1, 1)$. In this order, the coefficients of a minimal p-divisor $\mathcal{D}$ on Y are $\{0\}$, $\{0\}$, $\{0\}$, $\{1\}$ and $[0, 1]$.

Now Y doesn't admit a description by a divisorial fan on $Z = \mathbb{A}^2$: In order to downgrade $\mathcal{E}$, we first need to blow up the origin of $\mathbb{A}^2$ to Z'. The situation is illustrated in Figure 2.2. The left column shows X over Y over Z', while the right shows X over Z.

Example 2.6. In Example 2.5, let us choose Y to be the affine quotient of $\mathbb{A}^4$ by H. This is the toric variety $\mathrm{TV}(\mathrm{pos}\{\rho_1, \dots, \rho_4\})$, which admits a description by a p-divisor on $Z = \mathbb{A}^2$. Then the candidate downgrade divisor $\mathcal{D}$ of Example 2.4 has the same coefficients $\{0\}$,$\{0\}$,$\{0\}$ and $\{1\}$ as above. This divisor is not even $\mathbb{Q}$-Cartier.

The difficulty of needing to blow up does not arise in case $\mathcal{E}$ is a p-divisor on a curve. In that case, Y may be constructed by a so-called divisorial polyhedron on Z, leading to a downgrade result in complexity one [IV11a, Theorem 5.2]. We now turn to upgrades.

Proposition 2.7. *Consider* $(Z, \mathcal{S}, \mathcal{D}) \in \mathbf{PolDiv}^2(T', H)$ *(or in* $\mathbf{PDiv}^2(T', H)$*). Then there is an equivalent triple* $(Z', \mathcal{S}', \mathcal{D}')$ *such that* Z' *is smooth and* $\mathcal{S}'$ *is contraction-free.*

Proof. We choose for Z' a resolution of Z, and pull back the divisorial fan $\mathcal{S}$ to $\widetilde{\mathcal{S}}$ on Z'. Then $\mathrm{TV}(\mathcal{S}) = \mathrm{TV}(\widetilde{\mathcal{S}})$, so $(Z', \widetilde{\mathcal{S}}, \mathcal{D})$ is an element of $\mathbf{PolDiv}^2(T', H)$, equivalent to the original one. By Proposition 1.19, we can refine $\widetilde{\mathcal{S}}$ to a contraction-free $\mathcal{S}'$ on Z'. We pull back $\mathcal{D}$ along the birational proper map $\mathrm{TV}(\mathcal{S}') \to \mathrm{TV}(\widetilde{\mathcal{S}}) = \mathrm{TV}(\mathcal{S})$, giving the desired polyhedral divisor $\mathcal{D}'$. This is a p-divisor if we started out with one. ☐

Thus, in particular it is not a restriction to assume that Z is smooth. To avoid trouble with non-Cartier divisors later on, we restrict $\mathbf{PolDiv}^2$ to triples $(Z, \mathcal{S}, \mathcal{D})$ with Z smooth.

In the following, we will define an upgrade functor $\mathbf{u}\colon (Z, \mathcal{S}, \mathcal{D}) \mapsto (Z, \mathcal{E})$. In order to be able to define $\mathcal{E}$ as a polyhedral divisor, in section 2.2 we will start by investigating invariant divisors on T-varieties, that is, how to express the T'-invariant divisors $\mathcal{D}(u)$ on Y with respect to $\mathcal{S}$ and Z. Then, section 2.4 contains necessary results to show that if $\mathcal{D}$ is a p-divisor and $\mathcal{S}$ is contraction-free, $\mathcal{E}$ is actually a p-divisor.

2.2 Invariant divisors on T-varieties

Fix some T-variety X described by a divisorial fan $\mathcal{S}$ on Y and let $n = \dim T$. (X, Y and T here correspond to Y, Z and T' in the previous section; we will stick with this convention when there is only one torus action around.) Petersen and Süß describe all T-invariant prime Weil divisors on a T-variety [PS11]. These fit into two classes as follows.

There are *vertical* invariant prime divisors, arising as the closure of a family of n-dimensional T-orbits. These are parametrized by certain pairs of prime divisors $P \subset Y$ and vertices v of $\mathcal{S}_P$. Given P, we denote the set of such vertices by $\mathbf{vert}_P(\mathcal{S})$, and the divisor associated with P and v by $D_{P,v}$.

Then there are *horizontal* invariant prime divisors, arising as the closure of a family of $(n-1)$-dimensional T-orbits. These are parametrized by certain rays ρ of $\mathrm{tail}(\mathcal{S})$. We denote the set of such rays of $\mathrm{tail}(\mathcal{S})$ by $\mathbf{ray}(\mathcal{S})$, and the divisor associated with ρ by D_ρ.

Remark 2.8. If $\mathcal{S}$ is contraction-free, then all vertices of $\mathcal{S}_P$ and all rays of $\mathrm{tail}(\mathcal{S})$ correspond to prime divisors. In general, $\mathbf{vert}_P(\mathcal{S})$ and $\mathbf{ray}(\mathcal{S})$ consist of those that are not contracted under the map $r\colon \widetilde{X} \to X$. These are characterized by Petersen and Süß [PS11, Proposition 3.13].

Lemma 2.9. *[cf. [PS11, Proposition 3.14]] Consider some $f \in \mathbb{C}(Y)$, $u \in M$. Then*

$$\mathrm{div}(f \cdot \chi^u) = \sum_{\rho} \langle v_\rho, u \rangle D_\rho + \sum_{P,v} \mu(v)(\langle v, u\rangle + \mathrm{ord}_P(f))D_{P,v}$$

where $\mu(v)$ is the smallest positive integer such that $\mu(v)v \in N$.

Example 2.10. Recall the polyhedral function $\mathfrak{f}(u_1, u_2) = \chi^{-u_1}$ of Example 2.4, and the T'-fan $\mathcal{S}$ for Y on Z. By Lemma 2.9, we get

$$\mathrm{div}(\mathfrak{f})(u_1, u_2) = \sum_{\rho} -\langle v_\rho, u_1\rangle D_\rho + \sum_{(P,v)} -\mu(v)\langle v, u_1\rangle D_{(P,v)},$$

so with tail cone σ, we have

$$\operatorname{div}(\mathfrak{f}) = \sum_\rho -(v_\rho + \sigma)D_\rho + \sum_\rho -(\mu(v)v + \sigma)D_{(P,v)}.$$

The pull-back of a prime divisor P in Z along $\pi = (\mathrm{id}_Z, 0, 0)$ is just

$$\sum_{v \in \mathbf{vert}_P(\mathcal{S})} \mu(v)D_{(P,v)},$$

so the downgrade $\mathcal{D} = \varphi^{-1}(\mathcal{E})$ has coefficients

$$\begin{aligned}\mathcal{D}_{D_{(P,v)}} &= \mu(v) \cdot ((\mathcal{E}_P - (v,0)) \cap N_{H,\mathbb{Q}})\\ \mathcal{D}_{D_\rho} &= (\operatorname{tail}(\mathcal{E}) - (v_\rho, 0)) \cap N_{H,\mathbb{Q}}.\end{aligned}$$

Now consider any T-invariant $\mathbb{Q}$-Weil divisor D on X, which by the above description we can write as

$$D = \sum_\rho a_\rho D_\rho + \sum_{P,v} \mu(v) b_{P,v} D_{P,v}.$$

With any such divisor D, we associate a polyhedron $\square^D \subset M_\mathbb{Q}$ and a piecewise-affine concave function $\Psi^D \colon \square^D \to \mathrm{WDiv}_{\mathbb{Q}\cup\{\infty\}} Y$ as follows.

$$\begin{aligned}\square^D &:= \{u \in M_\mathbb{Q} \mid \langle v_\rho, u\rangle + a_\rho \geq 0 \text{ for all } \rho \in \mathbf{ray}(\mathcal{S})\}\\ \Psi^D_P(u) &:= \inf_{v \in \mathbf{vert}_P(\mathcal{S})} (\langle v, u\rangle + b_{P,v}) \in \mathbb{Q} \cup \{\infty\}\\ \Psi^D(u) &:= \sum_P \Psi^D_P(u) P.\end{aligned}$$

This generalizes the function h^* defined by Petersen and Süß [PS11]. All $\square^D$ share the same tail cone ω, dual to the cone generated by the rays $\rho \in \mathbf{ray}(\mathcal{S})$. The set of such functions will be denoted by $\mathrm{PACF}(M, Y) = \mathrm{PACF}(M, Y, \omega)$; it forms an abelian group with addition defined as follows.

Definition 2.11. Consider two concave piecewise-affine maps

$$\Psi^i \colon \square^i \to \mathrm{WDiv}_{\mathbb{Q}\cup\{\infty\}} Y,$$

$i = 1, 2$. We define their sum to be

$$\begin{aligned}(\Psi^1 + \Psi^2) \colon (\square^1 + \square^2) &\to \mathrm{WDiv}_{\mathbb{Q}\cup\{\infty\}} Y\\ u &\mapsto \sum_P \max_{\substack{u_i \in \square^i\\ u_1 + u_2 = u}} (\Psi^1_P(u_1) + \Psi^2_P(u_2))P.\end{aligned}$$

Note that the map

$$\begin{aligned}\mathrm{WDiv}_\mathbb{Q}(X)^T &\to \mathrm{PACF}(M, Y)\\ D &\mapsto (\Psi^D \colon \square^D \to \mathrm{WDiv}_{\mathbb{Q}\cup\{\infty\}} Y)\end{aligned}$$

is not a group homomorphism, but just subadditive. As we will see later, it commutes with addition when restricted to semiample divisors.

Proposition 2.12. *For D as above, we have*

$$L(D) = \bigoplus_{u \in \square^D \cap M} L(\Psi^D(u)) \cdot \chi^u.$$

Proof. The proof is similar to that of [PS11], Proposition 3.23. Consider some $u \in M$ and $f \in \mathbb{C}(Y)$. Then by Lemma 2.9, $f \cdot \chi^u \in L(D)$ if and only if

$$\sum_\rho \langle v_\rho, u\rangle D_\rho + \sum_{P,v} \mu(v)(\langle v, u\rangle + \mathrm{ord}_P(f))D_{P,v} + D \geq 0.$$

This is equivalent to satisfying the following inequalities:

$$\begin{aligned} \langle v_\rho, u\rangle + a_\rho \geq 0; \\ \langle v, u\rangle + \mathrm{ord}_P(f) + b_{P,v} \geq 0. \end{aligned}$$

The first line of inequalities is equivalent to $u \in \square^D$. The second is equivalent to $f \in L(\Psi^D(u))$. □

We now recall some general facts about T-invariant divisors that we will need later.

Remark 2.13. Suppose that D is any T-invariant $\mathbb{Q}$-Cartier divisor on X. If $\mathcal{O}_X(D)$ is globally generated, these generators can be taken to be T-invariant. Indeed, $H^0(X, \mathcal{O}_X(D))$ is generated as an $H^0(X, \mathcal{O}_X)$-module by T-invariant sections $s_1, \ldots, s_k$, which will then globally generate $\mathcal{O}_X(D)$.

Lemma 2.14 (cf. [Ful93, page 61])**.** *Suppose that $X = \mathrm{TV}(\mathcal{D})$ is an affine T-variety and consider any T-invariant Cartier divisor D on X. Then there is a T-invariant covering of X on which D is principal and defined by invariant functions.*

Proof (communicated by H. Süß). It is sufficient to consider the case D effective. Thus, D corresponds to an ideal I of $A := H^0(X, \mathcal{O}_X)$ which is M-homogeneous since D is T-invariant. Let $f_1, \ldots, f_k$ be homogeneous generators of I. Consider some prime $\mathfrak{p} \in \operatorname{Spec} A$. Then $I_\mathfrak{p} \subset A_\mathfrak{p}$ is generated by some f_j. Indeed, some f_j doesn't lie in $\mathfrak{p} \cdot I$, otherwise the f_i can't generate I. Since $I_\mathfrak{p}$ is principal, by Nakayama's lemma this f_j then generates $I_\mathfrak{p}$.

It follows that the f_i locally define D, say on some open sets U_i. Now, we can even find a T-invariant cover on which the f_i define D. Indeed, let U_i' be the complement of all prime divisors where f_i doesn't define D. Since f_i and D are T-invariant, then U_i' is as well. Furthermore, the U_i' cover X since $U_i \subset U_i'$. □

2.3 Upgrading a torus action

We can now state the main result on upgrades. Take a triple $(Z, \mathcal{S}, \mathcal{D}) \in \mathbf{PolDiv}^2(T', H)$, and let M' and M be the character lattices of T' and H, respectively. Somewhat informally, this can be stated as follows. The composition

$$\omega \cap M \xrightarrow{\mathcal{D}} \mathrm{WDiv}_{\mathbb{Q}\cup\{\infty\}}(Y)^{T'} \xrightarrow{\Psi^\bullet} \mathrm{PACF}(M', Z)$$

defines a partial function $M \to [M' \to \mathrm{WDiv}_{\mathbb{Q}} Z]$, and correspondingly a partial function $M \oplus M' \to \mathrm{WDiv}_{\mathbb{Q}} Z$. This is our candidate for the upgraded divisor $\mathcal{E}$ for the torus $T = H \times T'$.

Precisely, we define

$$\widetilde{\omega} := \{(u, u') \in M_{\mathbb{Q}} \oplus M'_{\mathbb{Q}} \mid u \in \omega,\ u' \in \Box^{\mathcal{D}(u)}\}$$

and set

$$\begin{aligned} \mathcal{E} \colon \widetilde{\omega} &\to \mathrm{WDiv}_{\mathbb{Q} \cup \{\infty\}} Z \\ (u, u') &\mapsto \Psi^{\mathcal{D}(u)}(u'). \end{aligned}$$

Theorem 2.15. *This construction defines an upgrade functor*

$$\begin{aligned} \mathbf{u} \colon \mathbf{PolDiv}^2(Z, T', H) &\to \mathbf{PolDiv}(Z, T \times H) \\ (Z, \mathcal{S}, \mathcal{D}) &\mapsto \mathcal{E} \end{aligned}$$

This functor maps objects $(Z, \mathcal{S}, \mathcal{D})$ of $\mathbf{PDiv}^2(Z, T', H)$ such that $\mathcal{S}$ is contraction-free to $\mathbf{PDiv}(Z, T \times H)$.

Remark 2.16. As defined above, it is not clear that $\mathcal{E}$ should be Cartier if we don't require Z to be smooth. Regardless, we have $\mathrm{TV}(\mathcal{E}) = \mathrm{TV}(\mathcal{D})$, but we can't speak of functoriality in that case.

Remark 2.17. The theorem can also be used to upgrade non-affine T-varieties. Indeed, if $\mathcal{S}$ is a contraction-free divisorial fan on smooth Y, and Ξ is a divisorial fan on $\mathrm{TV}(\mathcal{S})$ consisting of invariant p-divisors, then the upgraded p-divisors $\{\mathbf{d}(\mathcal{D})\}_{\mathcal{D} \in \Xi}$ form a divisorial fan describing $\mathrm{TV}(\Xi)$ with the upgraded torus action.

It will be useful to have a dual view of this upgrade construction, both for proving convexity properties and to know $\mathbf{u}(\mathcal{D})$ more explicitly in applications. $\mathcal{D}$ can be expressed as a divisor with polyhedral coefficients

$$\mathcal{D} = \sum_{\rho \in \mathbf{ray}(\mathcal{S})} \Delta_\rho \otimes D_\rho + \sum_{\substack{P \subset Z \\ v \in \mathbf{vert}_P(\mathcal{S})}} \Delta_{P,v} \otimes \mu(v) D_{P,v},$$

where the coefficients Δ have the common tail cone $\sigma = \omega^\vee$.

Proposition 2.18. *The dual of the upgraded weight cone $\widetilde{\omega}$ is*

$$\widetilde{\sigma} = \mathrm{pos}\big\{(\sigma \times \{0\}) \cup \bigcup_{\rho \in \mathbf{ray}(\mathcal{S})} (\Delta_\rho \times \{v_\rho\})\big\}.$$

The upgraded divisor $\mathcal{E}$ can be expressed dually as $\mathcal{E} = \sum \Delta_P \otimes P$ with

$$\Delta_P = \mathrm{conv}\big\{\Delta_{P,v} \times \{v\} \mid v \in \mathbf{vert}_P(\mathcal{S})\big\} + \widetilde{\sigma}.$$

Proof. For (u, u') to be an element of $\widetilde{\omega}$, first u must be in ω, equivalently $\langle (v, 0), (u, u') \rangle \geq 0$ for all $v \in \sigma$. Then, we need $u' \in \Box^{\mathcal{D}(u)}$, equivalently

$\langle v_\rho, u'\rangle + a_\rho \geq 0$ for all occurring rays ρ, with a_ρ the coefficient of D_ρ in $\mathcal{D}(u)$. Thus, $a_\rho = \min\langle \Delta_\rho, u\rangle$, and the condition becomes

$$\langle v_\rho, u'\rangle + \min_{x\in\Delta_\rho}\langle x, u\rangle = \min_{x\in\Delta_\rho}\langle (x, v_\rho), (u, u')\rangle \geq 0.$$

This proves the claim about $\widetilde{\sigma}$.

For the second claim, consider a prime divisor P on Z. By definition of $\mathcal{E}$, the coefficient of P in $\mathcal{E}(u, u')$ can be calculated as follows.

$$\begin{aligned}
\Psi_P^{\mathcal{D}(u)}(u') &= \min_{v\in\mathbf{vert}_P(\mathcal{S})}\langle v, u'\rangle + b_{P,v} \\
&= \min_{v\in\mathbf{vert}_P(\mathcal{S})}\langle v, u'\rangle + \min_{x\in\Delta_{P,v}}\langle x, u\rangle \\
&= \min_{v\in\mathbf{vert}_P(\mathcal{S})}\min_{x\in\Delta_{P,v}}\langle (x, v), (u, u')\rangle \\
&= \min_{(x,v)\in\Delta_P}\langle (x, v), (u, u')\rangle
\end{aligned}$$

This is just the evaluation of the polyhedral divisor $\sum \Delta_P \otimes P$ at (u, u'). □

To show that $\mathbf{u}$ is a functor, we will have a quick look at morphisms in $\mathbf{PolDiv}^2$ and how those upgrade. Recall the notion of a morphism of polyhedral divisors from section 1.4.

Definition 2.19. Let $\mathcal{D}_i = (Z_i, \mathcal{S}_i, \mathcal{D}_i)$ be objects of $\mathbf{PolDiv}^2$, for $i = 1, 2$. A morphism $\Phi\colon \mathcal{D}_1 \to \mathcal{D}_2$ is a tuple $(\varphi = (\pi, F, \mathfrak{f}), G, \mathfrak{g})$ where φ is a morphism $(Z_1, \mathcal{S}_1) \to (Z_2, \mathcal{S}_2)$, and $(\mathrm{TV}(\varphi), G, \mathfrak{g})$ is a morphism $(\mathrm{TV}(\mathcal{S}_1), \mathcal{D}_1) \to (\mathrm{TV}(\mathcal{S}_2), \mathcal{D}_2)$.

Furthermore, the upgrade $\mathbf{u}(\Phi)\colon \mathbf{u}(\mathcal{D}_1) \to \mathbf{u}(\mathcal{D}_2)$ of such a morphism is defined to be the triple $(\pi, F \oplus G, \mathfrak{f} \oplus \mathfrak{g})$.

Proof of Theorem 2.15. For the moment, we will only prove the first statement. The second statement will be proved at the end of section 2.6.

First, Z is smooth by assumption, so $\mathcal{E}$ is Cartier. By Proposition 2.18, $\mathcal{E}$ is a polyhedral divisor. For $\mathbf{u}$ to be an upgrade functor on objects just means that $\mathrm{TV}(\mathcal{E})$ and $\mathrm{TV}'(\mathcal{D}, \mathrm{TV}(\mathcal{S}))$ should agree as $T' \times H$-varieties; this follows from Proposition 2.12. It is then a straightforward calculation to show that $\mathbf{u}$ is a functor with the definition of upgraded morphisms above. This just involves applying the definition of the functor TV for morphisms. □

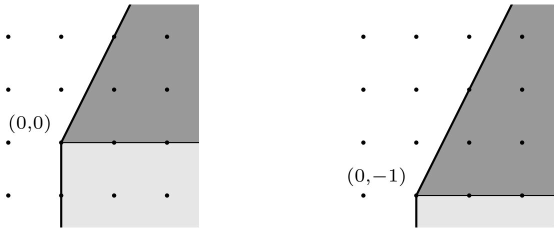
Example 2.20 (Affine cones)**.** Let $Y = \mathrm{TV}(\mathcal{S})$ be a T'-variety, with $\mathcal{S}$ a contraction-free divisorial fan on some smooth Z, and let $D = \sum a_\rho D_\rho + \sum b_{P,v}\mu(v) D_{P.v}$ be some very ample invariant Cartier divisor on Y giving a projectively normal embedding $Y \subset \mathbb{P}^m$. Then the cone $C(Y) \subset \mathbb{A}^{m+1}$ over Y is a $\mathbb{C}^*$-variety given by the p-divisor $\mathcal{D} = [1, \infty) \otimes D$. By the above theorem, $C(Y)$ is also a $T = \mathbb{C}^* \times T'$-variety given by the p-divisor

$$\mathcal{E} = \sum_P (\mathrm{conv}\{\{b_{P,v}\} \times \{v\}\} + \widetilde{\sigma}) \otimes P$$

where $\widetilde{\sigma} = \mathbb{Q}_{\geq 0} \times \{0\} + \mathrm{pos}\{\{a_\rho\} \times \{v_\rho\}\}$. This generalizes Proposition 4.1 of [IS11].

(a) $\mathrm{tail}(\mathcal{S})$ (b) $\mathcal{S}_\infty$

Figure 2.3: A divisorial fan for $\mathbb{P}^2$



(a) $\widetilde{\sigma}$ (b) Δ_∞

Figure 2.4: Upgrading with non-contraction-free $\mathcal{S}$

Example 2.21 (A non-contraction-free $\mathbb{P}^2$)**.** Consider some divisorial fan $\mathcal{S}$ on $Z = \mathbb{P}^1$ with tail fan and single nontrivial slice $\mathcal{S}_\infty$ as pictured in Figure 2.3, where all polyhedra with tail cone ρ_2 belong to the same polyhedral divisor, but those with tail cone ρ_1 don't. The resulting T'-variety $Y = \mathrm{TV}(\mathcal{S})$ is in fact $\mathbb{P}^2$, but $\mathcal{S}$ is not contraction-free. Here, $\mathbf{ray}(\mathcal{S})$ consists of ρ_1. Now consider the p-divisor $\mathcal{D} = \Delta_{\rho_1} \otimes D_{\rho_1}$, where $\Delta_{\rho_1} = [\frac{1}{2}, \infty) \subset N_\mathbb{Q} = \mathbb{Z}_\mathbb{Q}$.

In the upgraded lattice $N \oplus N' = \mathbb{Z}^2$, the upgraded tail cone $\widetilde{\sigma}$ is generated by $(1,0)$ and $(1,2)$, see the darkly shaded region of Figure 2.4(a). Likewise, the upgraded polyhedral divisor $\mathcal{E}$ on $\mathbb{P}^1$ is $\Delta_\infty \otimes \{\infty\}$, where $\Delta_\infty = (0,-1) + \widetilde{\sigma}$, see the darkly shaded region of Figure 2.4. $\mathcal{E}$ is clearly not proper.

However, we can replace $\mathcal{S}$ with the contraction-free $\mathcal{S}'$ that we get by requiring that not all polyhedra with tail cone ρ_2 belong to the same polyhedral divisor. Now, $\mathbf{ray}(\mathcal{S})$ consists of ρ_1 and ρ_2. The pullback of $\mathcal{D}$ to $\mathrm{TV}(\mathcal{S}')$ is still $\Delta_{\rho_1} \otimes D_{\rho_1}$, but the upgraded tail cone and coefficient now encompass the lightly and darkly shaded regions of Figure 2.4. In particular, the resulting polyhedral divisor is a p-divisor.

2.4 Semiample invariant divisors

Before we treat positivity of invariant divisors, we associate some additional data with a T-invariant $\mathbb{Q}$-Cartier divisor

$$D = \sum_\rho a_\rho D_\rho + \sum_{P,v} \mu(v) b_{P,v} D_{P,v}$$

on $X = \mathrm{TV}(\mathcal{S})$.

Lemma 2.22. *$D \in \mathrm{CDiv}_\mathbb{Q}(X)$ uniquely determines piecewise-affine functions $h_P^D \colon |\mathcal{S}_P| \to \mathbb{Q}$ for each prime divisor $P \subset Y$, satisfying the following conditions.*

1. *h^D_P is affine on each $\Delta \in \mathcal{S}_P$.*

2. *For any $\rho \in \mathbf{ray}(\mathcal{S})$, $\mathrm{slope}_\rho(h_P) = -a_\rho$.*

3. *For any $v \in \mathbf{vert}_P(\mathcal{S})$, $h^D_P(v) = -b_{P,v}$.*

Here, $\mathrm{slope}_\rho(h_P)$ is the slope of h_P along any one-dimensional polyhedron in $\mathcal{S}_P$ with tail cone ρ.

Proof. Note that h^D_P is determined by its values at the vertices and along the rays of $|\mathcal{S}_P|$. Since $\mathcal{S}$ is contraction-free, these values are prescribed, so uniqueness is immediate, and it remains to show that affine functions with these values exist for each $\Delta \in \mathcal{S}_P$.

We can assume that D is Cartier, since if the statement is true for lD, it will be true for D. Since D is T-invariant, by Lemma 2.14, it is defined by some open affine invariant cover $\{U_i\}$ of X together with invariant functions $g_i = f_i \cdot \chi^{u_i}$, where $f_i \in \mathbb{C}(Y)$. We consider the divisorial fan induced by the open invariant subsets $\mathrm{TV}(\mathcal{D}) \cap U_i$ for $\mathcal{D} \in \mathcal{S}$; this has the same slices as $\mathcal{S}$. Now any $\Delta \in \mathcal{S}_P$ which is maximal with respect to inclusion appears as the P-coefficient for the p-divisor of some $\mathrm{TV}(\mathcal{D}) \cap U_i$. We define an affine function $h\colon \Delta \to \mathbb{Q}$ by $h(v) = -\langle v, u\rangle - \mathrm{ord}_P f_i$. Lemma 2.9 shows that for each vertex v of Δ and for each ray ρ in the tail cone of Δ, we have

$$\begin{aligned} b_{P,v} &= \langle v, u\rangle + \mathrm{ord}_P f_i = -h(v) \\ a_\rho &= \langle v_\rho, u\rangle = -(h(v + v_\rho) - h(v)), \end{aligned}$$

completing the proof. □

These functions are related to the support functions occurring in [PS11]. Note that h^D_P determines the function Ψ^D_P introduced earlier. The converse is not true in general, but the following characterization shows that it does hold for base-point free divisors. For any point $y \in Y$, let $\mathcal{P}(y)$ be the set of all prime divisors passing through y.

Theorem 2.23. *Consider a T-invariant Cartier divisor D on the T-variety $\mathrm{TV}(\mathcal{S})$ with $\mathcal{S}$ contraction-free. Then D is base-point free if and only if for every $\mathcal{D} \in \mathcal{S}$ and $y \in \mathrm{loc}\,\mathcal{D}$ there exist $u \in \square^D \cap M$ and $s \in L(\Psi^D(u))$ satisfying*

1. *$\Psi^D_P(u) + \mathrm{ord}_P\, s = 0$ for all $P \in \mathcal{P}(y)$;*

2. *$h^D_P(v) = \langle v, u\rangle - \Psi^D_P(u)$ for all $P \in \mathcal{P}(y)$ and $v \in \mathcal{D}_P$.*

Proof. Note that by Remark 2.13, a point $x \in \mathrm{TV}(\mathcal{S})$ is a base-point of D if and only if for every homogeneous section $s \cdot \chi^u \in L(D)$, x lies in the support of the associated effective invariant divisor $\mathrm{div}(s \cdot \chi^u) + D$.

Suppose D is base-point free, and consider any $\mathcal{D} \in \mathcal{S}$ and $y \in \mathrm{loc}\,\mathcal{D}$. The torus T acts on the fibre $X_y \subset \mathrm{TV}(\mathcal{D})$, and the orbits of this action correspond to the faces of $\mathcal{D}_y = \sum_{P \in \mathcal{P}(y)} \mathcal{D}_P$, see [AH06, section 7]. We choose $x \in X_y$ in a closed orbit of this action, corresponding to the maximal face of $\mathcal{D}_y$. This point is contained in every vertical Weil divisor $D_{P,v}$ for $P \in \mathcal{P}(y)$, as well as every horizontal divisor D_ρ for any ray ρ in the tail cone of $\mathcal{D}$.

Now x is not a base-point of D, so we find $s \cdot \chi^u \in L(D)$ such that $x \notin \operatorname{supp}\operatorname{div}(s \cdot \chi^u) + D$, hence the coefficients of the invariant divisors through x must cancel out. By Lemma 2.9, this means that

$$\langle v_\rho, u\rangle + a_\rho = 0 \qquad\qquad \langle v, u\rangle + \operatorname{ord}_P(s) + b_{P,v} = 0.$$

for all rays and vertices in the polyhedra $\mathcal{D}_P$. Applying Lemma 2.22, it follows that

$$h_P^D(v) = \langle v, u\rangle + \operatorname{ord}_P(s)$$

on $\mathcal{D}_P$ for all $P \in \mathcal{P}(y)$.

By Proposition 2.12, we know that s is a section of $\Psi^D(u)$, so $\Psi_P^D(u) + \operatorname{ord}_P(s) \geq 0$ for all P. On the other hand, by definition of Ψ_P^D, we have

$$\Psi_P^D(u) \leq \langle v, u\rangle - h_P^D(v) = -\operatorname{ord}_P(s)$$

for all $P \in \mathcal{P}(y)$ and $v \in \mathcal{D}_P$. It follows that $\Psi_P^D(u) = -\operatorname{ord}_P(s)$, so points one and two of the theorem are satisfied.

Conversely, let us assume that D satisfies the conditions of the theorem, and show that D is base-point free. Take any $x \in X$ mapping to $y \in Y$, lying in $\operatorname{TV}(\mathcal{D})$ for some $\mathcal{D} \in \mathcal{S}$. With u and s as given by the hypothesis, it suffices to show that $\operatorname{div}(s \cdot \chi^u) + D$ doesn't meet X_y. But this follows by inverting the application of Lemmas 2.9 and 2.22 in the first part of the proof, since we know that $h_P^D(v) = \langle v, u\rangle + \operatorname{ord}_P(s)$. □

We now draw some consequences from the above theorem. Consider any concave piecewise-affine function $\Psi\colon \square \to \operatorname{WDiv}_{\mathbb{Q}\cup\{\infty\}} Y$, that is the minimum of only finitely many affine functions. The *lineality space* of Ψ_P is the largest subspace $L_P \subset M_{\mathbb{Q}}$ such that Ψ_P and $\square$ are invariant under translation by elements of L_P. Thus, we can consider Ψ_P to be defined on $\square/L_P$. The *vertices* of Ψ_P are those $\overline{u} \in \square/L_P$ such that $(\overline{u}, \Psi_P(\overline{u}))$ is a vertex of the graph of Ψ_P.

Definition 2.24. We say that Ψ is *asymptotically sharp* if for any prime divisor $P \subset Y$ with $\Psi_P \not\equiv \infty$, the following holds for all vertices $\overline{u}$ of Ψ_P:

$$\begin{array}{l}\text{There exists } k > 0 \text{ and } u \in \square \cap (\overline{u} + L_P) \text{ such that there is some}\\ s \in L(k\Psi(u)) \text{ satisfying } \operatorname{ord}_P s + k\Psi_P(u) = 0.\end{array} \qquad (**)$$

We say that Ψ is *sharp* if we can always take $k = 1$ and $u \in M$.

Corollary 2.25. *Let D be a $\mathbb{Q}$-Cartier divisor on a T-variety $\operatorname{TV}(\mathcal{S})$, where $\mathcal{S}$ is contraction-free and $|\mathcal{S}_P|$ is convex for all prime divisors P. Then if D is base-point free/semiample, it follows that*

1. *Ψ^D is sharp/asymptotically sharp; and*
2. *For any $P \subset Y$ prime, h_P^D is concave.*

Proof. We will prove the statement for the base-point free case. The semiample case follows immediately by passing to a sufficient multiple of D. We first remark that h_P^D is concave if and only if for all $\Delta \in \mathcal{S}_P$, there is $u \in \square^D$ such that $\Psi^D(u) = \langle v, u\rangle - h_P^D(v)$ for any $v \in \Delta$.

Now, consider any prime divisor $P \subset Y$. Then by the remark above and Theorem 2.23, h_P^D must be concave. For any maximal dimensional $\Delta \in \mathcal{S}_P$,

$h_P^D(v)$ restricted to Δ is of the form $\langle v, \overline{u} \rangle + \mathrm{ord}_P\, s$ for some uniquely determined $\overline{u} \in M/L_P$, which must be a vertex of Ψ_P^D. Furthermore, all vertices of Ψ_P^D arise in this way. This together with the above theorem implies the sharpness condition. $\square$

We can draw another consequence of this characterization of base-point free divisors.

Proposition 2.26. *Let D and E be semiample T-invariant $\mathbb{Q}$-Cartier divisors on X. Then $\square^D + \square^E = \square^{D+E}$ and $\Psi^D + \Psi^E = \Psi^{D+E}$.*

Proof. This is a consequence of Corollary 2.25. Indeed, since h_P^D and h_P^E are concave, the claim follows from general facts about convexity. $\square$

2.5 Semicompleteness

As before, let $X = \mathrm{TV}(\mathcal{S})$ be a T-variety, with $\mathcal{S}$ a divisorial fan on Y. It turns out that semiampleness of the divisors $\Psi^0(u) \in \mathrm{WDiv}_{\mathbb{Q}\cup\{\infty\}} Y$ associated with the trivial divisor on X is related to semicompleteness of X.

The affine contraction $X_0 = \mathrm{Spec}\, H^0(X, \mathcal{O}_X)$ retains an action by T. This action need not be effective, but there is an effective residual action by a quotient torus T', corresponding to the sublattice M' of M generated by the weight cone of the T-action. Based on the characterization of maps of T-varieties of sections 1.4 and 1.6, we can express the equivariant morphism $X \to X_0$ as follows.

Lemma 2.27. *Assume that $\mathcal{S}$ is contraction-free with tail fan Σ.*

1. *The function $\Psi^0 \colon \square^0 \to \mathrm{WDiv}_{\mathbb{Q}\cup\{\infty\}}(Y)$ associated with the trivial divisor on X is a polyhedral divisor with tail cone $(\square^0)^\vee = \sigma := \mathrm{conv}\,|\Sigma|$ and coefficients $\Psi_P^0 = \mathrm{conv}\,|\mathcal{S}_P|$.*

2. *The algebra of global sections is the coordinate ring of X_0, that is, $X_0 = \mathrm{TV}(\Psi^0)$.*

3. *$r \colon X \to X_0$ is defined by the map $(\mathrm{id}_Y, \mathrm{id}_N, 1) \colon \mathcal{S} \to \Psi^0$.*

Proof. For the first claim, note that by definition, $\square^0$ is a polyhedral cone and Ψ^0 is concave and piecewise-linear. The rest follows by dualizing the definitions of $\square^0$ and Ψ^0 and using that $\mathcal{S}$ is contraction-free.

The second claim is a direct consequence of Proposition 2.12. The third follows from the definition of the morphism associated with a map of polyhedral divisors. $\square$

In general, Ψ^0 is not a p-divisor for X_0. But if X is semicomplete, we will see that it is semiample.

Proposition 2.28. *If $X = \mathrm{TV}(\mathcal{S})$ is semicomplete for a contraction-free $\mathcal{S}$ and Ψ^0 is $\mathbb{Q}$-Cartier, then Ψ^0 is semiample, and for all prime divisors P on Y, $|\mathcal{S}_P|$ is convex. Furthermore, $\mathrm{loc}\,\Psi^0$ is semicomplete.*

Proof. Let the p-divisor $\mathcal{D}$ on Y' describe the T'-variety X_0. By Corollary 1.15, the equivariant morphism $X \to X_0$ corresponds to a map $\mathcal{S} \to \mathcal{D}$, after blowing up Y if necessary. (Refinements of $\mathcal{S}$ are not required since we are mapping to an affine variety.)

Unless Y is unchanged, the pull-back of $\mathcal{S}$ is not contraction-free anymore, so we replace it by $\mathcal{S}'$ as in Proposition 1.19. Then the affine contractions of $\mathrm{TV}(\mathcal{S})$ and $\mathrm{TV}(\mathcal{S}')$ agree, and the map $r\colon \mathrm{TV}(\mathcal{S}') \to X_0$ corresponds to a map $\varphi\colon \mathcal{S}' \to \mathcal{D}$.

Since X is semicomplete, r is proper, and so by Theorem 1.16 applied to the generic points of prime divisors, we get $|\mathcal{S}_P| = |\mathcal{S}'_P| = \varphi^{-1}(\mathcal{D}')_P$. This shows the convexity claim. Now by Lemma 2.27, we have that $|\mathcal{S}_P| = \Psi^0_P$. Thus, the pullback of Ψ^0 is the pullback of a p-divisor (which is in particular semiample), so it must be semiample itself.

To show the semicompleteness of $\operatorname{loc}\Psi^0$, we first note that the map $\operatorname{loc}\mathcal{S}' \to \operatorname{loc}\mathcal{D}$ is proper; this is a consequence of Theorem 1.16. From the semiprojectivity of $\operatorname{loc}\mathcal{D}$, we thus have a proper map $\theta'\colon \operatorname{loc}\mathcal{S}' \to Z$, where Z is affine. Furthermore, θ' factors through $\operatorname{loc}\mathcal{S}$, since the regular functions on $\operatorname{loc}\mathcal{S}$ and $\operatorname{loc}\mathcal{S}'$ are equal, and Z is affine. Let θ denote this map from $\operatorname{loc}\mathcal{S}$ to Z. We claim that θ is proper. Indeed, this follows from the separatedness of θ, the surjectivity of $\operatorname{loc}\mathcal{S}' \to \operatorname{loc}\mathcal{S}$, and the properness of θ', see [Gro61, Corollary 5.4.3]. □

Example 2.29. Let Y be the blowup of $\mathbb{A}^2$ at the origin; let D_1, D_2 be the strict transforms of the coordinate axes and E the exceptional divisor. Consider the divisorial fan $\mathcal{S}$ generated by

$$\begin{aligned}\mathcal{D}^1 &= [0,\infty)\otimes D_1 + [1,\infty)\otimes E + \emptyset\otimes D_2\\ \mathcal{D}^2 &= \emptyset\otimes D_1 + [1,\infty)\otimes E + [0,\infty)\otimes D_2\,.\end{aligned}$$

Then X is not semicomplete, and Ψ^0 is big but not semiample:

$$\Psi^0 = [0,\infty)\otimes D_1 + [1,\infty)\otimes E + [0,\infty)\otimes D_2,$$

so $\Psi^0(1) = E$. A p-divisor for $X_0 = \mathbb{A}^3$ is

$$\mathcal{D}^0 = [0,\infty)\otimes D_1 + [0,\infty)\otimes D_2,$$

which may be defined on $Y_0 = \operatorname{Spec} H^0(Y,\mathcal{O}_Y) = \mathbb{A}^2$.

Example 2.30. Consider the same example as above, but modify D_1 and D_2 so that E has coefficient $[-1,\infty)$. Then Ψ^0 is a p-divisor, and X is semicomplete. In contrast to the previous example, X_0 can not be expressed by a p-divisor on Y_0.

Example 2.31. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, with a complexity one action through one factor. The divisorial fan for X is defined on $Y = \mathbb{P}^1$. Then Ψ^0 is a semiample divisor on $\mathbb{P}^1$ for the point X_0, but it is not big.

2.6 Semiample decomposition

Let $\mathrm{TV}(\mathcal{S})$ be some T-variety, and D an invariant divisor on $\mathrm{TV}(\mathcal{S})$. Even if D is semiample, it does not hold in general that $\Psi^D(u)$ is semiample for all

$u \in \square^D$. But we will see that it is true provided that $\mathcal{S}$ is contraction-free and $\mathrm{TV}(\mathcal{S})$ is semicomplete. We first need the following lemma:

Lemma 2.32. *Let $\mathcal{S}$ be contraction-free with $\mathcal{S}_P$ convex and $\operatorname{loc}\mathcal{S}$ semiprojective. We consider the variety $X = \mathrm{TV}(\mathcal{S})$ and assume that $\Psi^0(u)$ is semiample for all $u \in \square^0$. Consider a semiample invariant divisor D on X. If there is a concave semiample-valued $\Psi' \colon \square^D \to \mathrm{WDiv}_{\mathbb{Q}\cup\{\infty\}} Y$ with semiprojective locus such that $L(k\Psi^D(u)) = L(k\Psi'(u))$ for all $u \in \square^D \cap \frac{1}{k}M$ and all $k \in \mathbb{Z}_{\geq 0}$, then $\Psi^D(u)$ is semiample for all $u \in \square^D$.*

Proof. Let Ψ' be as in the hypothesis. It follows from Lemma 9.1 of [AH06] that $\Psi^D_P \geq \Psi'_P$ as long as $\Psi'_P \not\equiv \infty$. Note that the proof of the lemma does not require the second divisor to be semiample. Furthermore, if $\Psi'_P \equiv \infty$, then we must have $\Psi^D_P \equiv \infty$ as well. Indeed, the complement of $\operatorname{loc}\Psi'$ is some semiample divisor C. Thus, any nonempty $L(k\Psi'(u))$ has a section with an arbitrarily large pole along C. Similarly, if $\Psi^D_P \equiv \infty$, then $\Psi'_P \equiv \infty$ due to the semiprojectivity of $\operatorname{loc}\mathcal{S}$.

Fix now some prime P with $\mathcal{S}_P \neq \emptyset$. Let $\overline{\square}_P \subset \square^D/L_P$ be the convex hull of the vertices of Ψ_P and $\square_P$ its inverse image in $\square^D$. It follows from asymptotic sharpness that for any vertex $\overline{u}$ of some Ψ^D_P, there is some $u \in \square^P$ mapping to $\overline{u}$ with $\Psi^D_P(u) = \Psi'_P(u)$. Consider any $w \in L_P$. Then we even have $\Psi^D_P(u+w) = \Psi'_P(u+w)$. Indeed, $\Psi^D(u+w) \geq \Psi^D(u) + \Psi^0(w)$, and some multiple k of the left hand side thus has a section vanishing along P of order

$$k(\Psi^D_P(u) + \Psi^0_P(w)) = k\Psi^D_P(u+w)$$

by choice of u and the semiampleness of $\Psi^0(w)$. It even follows by convexity that $\Psi^D_P(u) = \Psi'_P(u)$ for any $u \in \square_P$.

Now consider any $u \in \square_P$ and $w \in \operatorname{tail}(\square^D)$. Then there exists $l \gg 0$ such that for $\lambda \geq 0$

$$\begin{aligned}\Psi^D(u + (l+\lambda)w) &= \Psi^D(u+lw) + \Psi^0(\lambda w);\\ \Psi'(u + (l+\lambda)w) &\leq \Psi'(u+lw) + \Psi^0(\lambda w).\end{aligned}$$

Since the right hand side of the second line above is semiample, we must actually have equality again by Lemma 9.1 of [AH06]. The concavity of Ψ' and equality of Ψ^D_P and Ψ'_P on $\overline{\square}_P$ together with the above imply

$$2\Psi'_P(u+lw) \geq \Psi'_P(u+2lw) + \Psi'_P(u) = \Psi'_P(u+lw) + \Psi^0_P(lw) + \Psi^D_P(u).$$

We can thus conclude that $\Psi'_P(u+lw) \geq \Psi^0_P(lw) + \Psi^D_P(u) = \Psi^D_P(u+lw)$. From the concavity of Ψ' we then get $\Psi^D_P(u+\lambda w) = \Psi'_P(u+\lambda w)$ for any $\lambda \geq 0$. But any $u' \in \square^D$ can be written as such a sum $u + \lambda w$. Thus, $\Psi^D = \Psi'$, so Ψ^D is semiample-valued. □

Theorem 2.33. *Let $\mathcal{S}$ be a contraction-free divisorial fan on some smooth Y such that $X = \mathrm{TV}(\mathcal{S})$ is semicomplete. Consider a semiample divisor D on X. Then $\Psi^D(u)$ is semiample for all $u \in \square^D$. Furthermore, if D is big, then $\Psi^D(u)$ is big for all $u \in \operatorname{relint}\square^D$.*

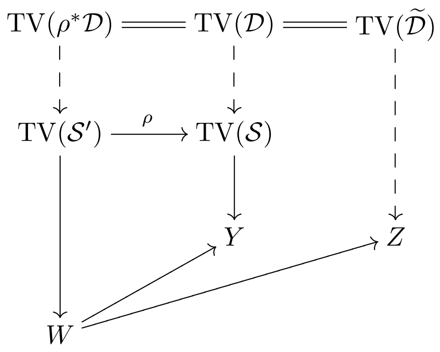


Figure 2.5: Situation in proof of Theorem 2.33

Proof. Without loss of generality, we can assume that Y is complete. Indeed, we can complete Y and pull back the divisorial fan while retaining contraction-freeness.

Now, let $\mathcal{D}$ be a T-invariant p-divisor on X with some weight cone $\omega \subset \mathbb{Q}^2$, such that for some $w_0 \in \omega$, $\mathcal{D}(w_0) = D$; such a $\mathcal{D}$ always exists. If D is big, we can even require $w_0 \in \operatorname{relint} \omega$. Then $\mathrm{TV}(\mathcal{D})$ is a $(\mathbb{C}^*)^2$-variety. But it also inherits the T-action of $\mathrm{TV}(\mathcal{S})$, so it is in fact a $\widetilde{T}$ variety, where $\widetilde{T} = \mathbb{C}^* \otimes (N \oplus \mathbb{Z}^2)$.

Let Z be a projective completion for the normalization of the special component of the Chow quotient of $\mathrm{TV}(\mathcal{D})$ by the action of $\widetilde{T}$, see [AH06, section 6]. Let $\widetilde{\mathcal{D}}$ be the corresponding p-divisor on Z with $\mathrm{TV}(\widetilde{\mathcal{D}}) = \mathrm{TV}(\mathcal{D})$. Then Z and Y are birational, so we find a projective variety W mapping properly and birationally to both Y and Z. We now pull back $\mathcal{S}$ to W, possibly blowing up to make it contraction-free again, giving us a new divisorial fan $\mathcal{S}'$ and a map $\rho\colon \mathrm{TV}(\mathcal{S}') \to \mathrm{TV}(\mathcal{S})$. The situation thus far is pictured in Figure 2.5.

Now, since X is semicomplete, Ψ^0 is semiample, $\operatorname{loc} \mathcal{S}$ is semicomplete, and $\mathcal{S}_P$ is convex by Proposition 2.28. Furthermore, since $\operatorname{loc}(\mathcal{S})$ was semicomplete, $\operatorname{loc}(\mathcal{S}')$ will be semicomplete as well, and thus semiprojective by Proposition 1.3.

Note that for any invariant Cartier divisor E on $\mathrm{TV}(\mathcal{S})$, $\Psi^{\rho^* E}$ is simply the pullback of Ψ^E to W. We now pull back $\widetilde{\mathcal{D}}$ to W, and after possibly correcting with a principal polyhedral divisor, we have $L(\Psi^{\mathcal{D}(w_0)}(u)) = L(\widetilde{\mathcal{D}}(u, w_0))$ for all $u \in \square^{\mathcal{D}(w_0)} \cap M$ by Proposition 2.12. Setting $\Psi'(u) = \widetilde{\mathcal{D}}(u, w_0)$, we can apply the above lemma and conclude that the pullback of $\Psi^D(u)$ to W must have been semiample. Thus, $\Psi^D(u)$ must have been semiample as well. Furthermore, from the proof of the above lemma, we actually have that the pullback of $\Psi^{\mathcal{D}(w_0)}(u)$ is the pullback of $\Psi'(u)$, which is big for $u \in \operatorname{relint} \square^D$ if $w_0 \in \operatorname{relint} \omega$. Thus, if D is big, $\Psi^{\mathcal{D}(w_0)}(u)$ is big as well. □

Remark 2.34. As we can see by Example 2.29, the above theorem does not necessarily hold if $\mathrm{TV}(\mathcal{S})$ is not semicomplete.

We are now in a position to prove the properness part of the upgrade theorem.

Proof of Theorem 2.15, part 2. It remains to show that $\mathcal{E} = \mathbf{u}(Z, \mathcal{S}, \mathcal{D})$ is a p-divisor, given that $\mathcal{S}$ is contraction-free. Since $Y = \mathrm{TV}(\mathcal{S})$ is semiprojective,

$\operatorname{loc}\mathcal{E}$ is semiprojective by Propositions 1.3 and 2.28. Furthermore, we can apply Theorem 2.33 to prove that $\mathcal{E}$ is semiample, and big on the interior of $\widetilde{\omega}$. Finally, note that the cone $\widetilde{\omega}$ must be full-dimensional since $\dim X = \dim \widetilde{\omega} + \dim Z$. $\square$

Chapter 3

Equivariant deformation theory

In this chapter, we will review some general notions of deformation theory in both the general and equivariant setting, before focussing on Altmann's results on deformations of toric varieties.

The aim of section 3.1 is to provide a brief definition of the relevant terms of deformation theory. Then, section 3.2 summarizes some general results on the deformations of G-varieties before introducing the notion of a category of equivariant deformations together with a possible definition of a general equivariant T^1. Since later parts of this thesis don't include general statements on the deformation theory of T-varieties, I will not go into too much detail here.

Finally, section 3.3 contains a summary of relevant results of Altmann on deformations of toric varieties. In particular, we will define the l-parameter deformation of $\mathrm{TV}(\delta)$ associated with a Minkowski decomposition

$$\delta \cap [r = 1] = \Delta_0 + \cdots + \Delta_l,$$

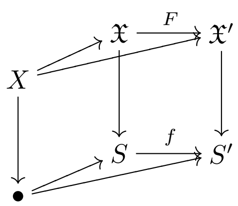
which will be revisited in the following chapters.

3.1 Deformation theory

Denote by $\bullet = \operatorname{Spec} \mathbb{C}$ the point, and let X be an affine variety. A deformation of X is a Cartesian square (in the category of schemes over $\mathbb{C}$)

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \bullet & \longrightarrow & S \end{array}$$

where π is a flat map. That is, $\pi\colon \mathfrak{X} \to S$ is a flat family with special fibre X. A morphism of deformations of X is a pair of maps $f\colon S \to S'$ and $F\colon \mathfrak{X} \to \mathfrak{X}'$ that makes the following diagram commute.



In this way, the deformations of X form a category $\operatorname{Def}(X)$. Usually, the base scheme S is taken to be from some subcategory $\mathcal{C}$ of $\mathbb{C}$-schemes, say the category of Artinian $\mathbb{C}$-algebras. $\operatorname{Def}(X)$ comes with a functor $\mathbf{b}\colon \operatorname{Def}(X) \to \mathcal{C}$, mapping a deformation to its base.

The classical approach of Schlessinger [Sch68] studies the induced deformation functor

$$\begin{aligned} D\colon \mathcal{C} &\to \mathrm{Set} \\ S &\mapsto \text{isomorphism classes of } \mathbf{b}^{-1}(S); \end{aligned}$$

a more general approach is to treat $\operatorname{Def}(X)$ over $\mathcal{C}$ as a homogeneous fibred category, compare Rim's treatment of equivariant deformations [Rim80]. Here, homogeneity is a limited existence of fibred coproducts.

Denoting by $\mathbb{C}[\varepsilon] := \mathbb{C}[\varepsilon]/(\varepsilon^2)$ the ring of dual numbers, the set of isomorphism classes of *first-order deformations*

$$T^1_X := D(\operatorname{Spec}\mathbb{C}[\varepsilon])$$

admits a vector space structure, compare again [Rim80]. This is the tangent space to the deformation functor D. The construction of the vector space $T^1_{\operatorname{Spec} A}$ is the same as that of $\operatorname{Ex}_{\mathbb{C}}(A, A)$ of section 6.1.

In general, it is too much to hope for a *universal deformation*, that is, a deformation of $\mathfrak{X} \to S$ of X such that every deformation arises uniquely as the pull-back $\mathfrak{X} \times_S S'$ along some map $S' \to S$. But if certain conditions are satisfied, one can find a *versal deformation*, minimal with respect to (not necessarily unique) existence of pull-backs. Then the tangent space to the origin in the base space of a versal deformation is isomorphic to the vector space T^1. Note that some authors don't require minimality in the definition of versality, and call a minimal versal deformation *miniversal* or *semi-universal*.

Example 3.1 (cf. [Ste03])**.** Suppose the (germ of) an affine scheme X is given by the ideal $I = (f_1, \ldots, f_k) \subset P = \mathbb{C}[y_1, \ldots, y_n]$, so $X = \operatorname{Spec} B$ with $B = P/I$. Then the exact sequence

$$I/I^2 \xrightarrow{d} \Omega_{P/\mathbb{C}} \otimes B \longrightarrow \Omega_{B/\mathbb{C}} \longrightarrow 0,$$

dualizes to

$$\operatorname{Hom}_B(\Omega_{P/\mathbb{C}} \otimes B, B) \longrightarrow \operatorname{Hom}_B(I/I^2, B) \longrightarrow T^1 \longrightarrow 0.$$

Generators and relations of I yield an exact sequence

$$P^l \xrightarrow{R} P^k \xrightarrow{F} P \longrightarrow B \longrightarrow 0,$$

which allows expressing elements of $\operatorname{Hom}_B(I/I^2, B) = \operatorname{Hom}_P(I, B)$ and therefore first order deformations as vectors $A \in P^k$ such that entries of $R^*(A)$ lie in I. Then the trivial deformations are just those coming from the partial derivatives of the f_i.

The total space of the deformation associated with such a vector A is given by the ideal $J \subset P[\varepsilon]$ with generators $F_i = f_i + \varepsilon A_i$.

3.2 Equivariant deformations

Suppose now that X comes with an action by an algebraic group G. Then one would expect T^1_X and a hypothetical versal deformation to be G-equivariant. This in not true in general: Siebert provides an example of an action by a non-reductive group that does not extend to the versal deformation [Sie00].

However, there are positive results under certain restrictions. Pinkham shows explicitly that for the case of a good $\mathbb{C}^*$-action on X, T^1 admits a $\mathbb{C}^*$ action, as does the versal deformation [Pin74, chapter 2]. Thus, T^1 is graded by the character lattice $\mathbb{Z}$. Altmann shows that for a toric variety X, T^1 admits an action by the embedded torus as a graded Ext-module over the graded module of differentials [Alt94]. In the case of isolated singularities, Rim shows that T^1 as well as the versal deformation admit a G-action for any reductive group G [Rim80].

Proposition 3.2. *If X is an affine T-variety, then $T^1_X = \bigoplus_{u \in M} T^1_X(u)$ is naturally graded by the character lattice M of T.*

Proof. By embedding X equivariantly in some $\mathbb{A}^n$ (compare the proof of Proposition 6.5), the existence of a grading is a direct consequence of the explicit description of T^1_X of Example 3.1. A more conceptual proof that shows naturality is provided in section 6.1. □

As a general setting for discussing deformations of G-varieties, I suggest the category of equivariant deformations.

Definition 3.3. A *G-equivariant deformation* of X is a deformation of X such that G acts on the entire diagram. That is, G acts on $\mathfrak{X}$, G acts on $\bullet$ (trivially) and G acts on S, such that all maps are equivariant. The category of G-equivariant deformations will be denoted by $\mathrm{Def}^G(X)$.

We say that a deformation is *G-invariant* if the action on the base S is trivial.

In particular, for an equivariant deformation, the G-action on $\mathfrak{X}$ restricts to the given G-action on the central fibre X. Note that only in the case of an invariant deformation does X deform as a G-variety: In this case, every fibre of $\mathfrak{X} \to S$ admits an action by G.

Remark 3.4. Consider a G-equivariant deformation $\mathfrak{X}$ of X, and let H be the kernel of the G-action on the base S. Then $\mathfrak{X}$ is also an H-invariant deformation of X.

Let us now try to carry over the definition of T^1 to Def^G. Note first that the group of automorphisms of $\mathbb{C}[\varepsilon]$ that leaves (ε) fixed is $\mathbb{C}^* \cong \{\varepsilon \mapsto \lambda\varepsilon \mid \lambda \in \mathbb{C}^*\}$, so any G-equivariant first-order deformation determines a character $r\colon G \to \mathbb{C}^*$.

The definition of addition on T^1 involves pull-back along the "addition" map

$$\operatorname{Spec} \mathbb{C}[\varepsilon] \to \operatorname{Spec} \mathbb{C}[\varepsilon] \times \operatorname{Spec} \mathbb{C}[\varepsilon],$$

which is not equivariant unless G acts on both copies of $\operatorname{Spec} \mathbb{C}[\varepsilon]$ through the same character. Thus we are led to define the following.

Definition 3.5. For a G-variety X and a character $\chi^r : G \to \mathbb{C}^*$, we set $T^1_X(-r)$ to be the set of isomorphism classes of equivariant first order deformations of X where G acts on $\operatorname{Spec} \mathbb{C}[\varepsilon]$ through χ^r. We say that such a deformation is *homogeneous* of degree $-r$.

Note that the choice of sign is implied by the natural module structure on T^1, compare section 6.1.

Remark 3.6. The notion of a homogeneous first-order deformation extends naturally to one-parameter deformations of higher order, say to families over the toric variety $\mathbb{C}^1$.

By the results of section 6.1, $T^1_X(r)$ is a $\mathbb{C}$-vector space in the case of a T-variety. Assuming that the same is true for the group G, we can define an equivariant version of T^1.

Definition 3.7. The *equivariant space of first-order deformations* of a G-variety X is the vector space $T^1_X(G) = \bigoplus_r T^1_X(r)$, where the sum ranges over the characters of G.

Then we know that for a T-variety, we have $T^1_X(T) = T^1_X$, while in general and assuming its existence, we have $T^1_X(G) \subset T^1_X$.

Remark 3.8. For the case of a toric variety X (say with embedded torus T), Altmann introduces the notion of a *toric deformation* of X. Here, the total space should not just admit a T-action, but in fact be a toric variety in itself, with some larger embedded torus T'. It turns out that such deformations don't cover all of T^1. In particular, the deformations of Theorem 3.12 below are not toric in general.

3.3 Toric deformations

I will now provide a brief overview of Altmann's results on deformations of toric varieties. Fix some torus T, with associated lattices M and N, and an affine toric variety $X = \mathrm{TV}(\delta)$ for a cone δ in N. We fix some primitive degree $r_0 \in M$ and consider a positive multiple $r = kr_0$. For simplicity of notation, we choose a splitting $N = N_{r_0} \oplus \mathbb{Z}$, where $N_{r_0} = N \cap r_0^\perp$.

Using the toric description of Ω_X [Dan78], δ determines a complex that allows computing $T^1_X(-r)$ and, if X is non-singular in codimension 2, $T^2_X(-r)$ [Alt94, Alt97a].

$T^1_X(-r)$ may also be described as a vector space of Minkowski summands of the polyhedron $\delta_r := \delta \cap [r = 1]$: the set of scalar multiples of Minkowski summands of a polyhedron Δ forms a cone $C(\Delta)$, with Grothendieck group $V(\Delta) := C(\Delta) - C(\Delta)$. For example, if Δ is a parallelogram, $C(\Delta)$ is two-dimensional, spanned by the edges of Δ. Then $T^1_X(-r)$ may be described by augmenting $V(\delta_r)$ with information about possible non-lattice vertices of δ_r [Alt00, Theorem 2.5].

To construct deformations of X, fix an admissible decomposition (compare Definition 1.1)

$$\delta_r = (\Delta_0, \tfrac{1}{k}) + (\Delta_1, 0) + \cdots + (\Delta_l, 0).$$

The coefficients have tail cone $\sigma := \delta \cap [r = 0]$, which we consider as a cone in M_{r_0}. Note that admissibility can be checked with degrees $[u, 0] \in M_{r_0} \oplus \mathbb{Z}$.

Remark 3.9. For non-primitive r, that is if $k > 1$, $(\Delta_0, \frac{1}{k})$ doesn't contain lattice points, so the polyhedra $\Delta_1, \dots, \Delta_l$ must be lattice polyhedra. Note that this decomposition induces an admissible decomposition

$$\delta_{r_0} = (k\Delta_0, 1) + (k\Delta_1, 0) + \cdots + (k\Delta_l, 0)$$

in the primitive degree r_0. Proposition 5.3 shows how the associated deformations can be related geometrically.

We define the lattice $\widetilde{N} := N_{r_0} \oplus \mathbb{Z}^{l+1}$ together with an embedding

$$\begin{aligned} \iota\colon N &\to \widetilde{N} \\ (v, a) &\mapsto (v, a, ka, \dots, ka) \end{aligned}$$

and the cone $\widetilde{\delta}$ in $\widetilde{N}$ generated by $\sigma \times \{0\}$, $k\Delta_0 \times \{e_0\}$ and $\Delta_i \times \{e_i\}$, $0 \le i \le l$.

Theorem 3.10 ([Alt00, Theorem 3.2])**.** *Assume that r is positive on δ, i.e., $r \in \delta^\vee$. Set $\mathfrak{X} = \mathrm{TV}(\widetilde{\delta})$, and consider the closed embedding $\mathrm{TV}(\iota)\colon X \hookrightarrow \mathfrak{X}$ and the map $\pi\colon \mathfrak{X} \to \mathbb{C}^l$ defined by the binomials $\chi^{[0,ke^0]} - \chi^{[0,e^i]}$.*

1. *$\mathfrak{X} \to \mathbb{C}^l$ is a toric deformation of X.*
2. *The corresponding Kodaira-Spencer map $\mathbb{C}^l \to T^1_X(-r)$ maps e_i to the class of the Minkowski summand $\Delta_i \in C(\sigma_r) \subset V(\delta_r)$.*

Example 3.11. We consider deformations of the cone over the rational normal curve from Example 1.8 with $r = [0, 1]$. The non-trivial Minkowski decompositions of $\delta_r = \mathrm{conv}\{(-\frac{1}{2}, 1), (\frac{1}{2}, 1)\}$ correspond to the decompositions of the interval $[-\frac{1}{2}, \frac{1}{2}] = [-\frac{1}{2}, 0] + [0, \frac{1}{2}] = \{-\frac{1}{2}\} + [0, 1]$. For the first decomposition, we get the cones (generated by the columns of)

$$\widetilde{\sigma} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \widetilde{\sigma}^\vee = \begin{pmatrix} -2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

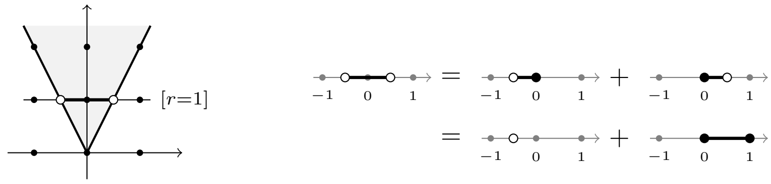
with Hilbert basis $E = \{[e, 0, 1] \mid -2 \le e \le 0\} \cup \{[e, 1, 0] \mid 0 \le e \le 2\}$. For the second decomposition, we get the cones

$$\overline{\sigma} = \begin{pmatrix} -\frac{1}{2} & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \overline{\sigma}^\vee = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

with Hilbert basis $E = \{[-2, 1, 2]\} \cup \{[e, 1, 1] \mid -1 \le e \le 0\} \cup \{[e, 1, 0] \mid 0 \le e \le 2\}$. The equations for $\mathrm{TV}(\widetilde{\sigma})$ and $\mathrm{TV}(\overline{\sigma})$ are

$$\mathrm{rk}\begin{pmatrix} y_0 & y_1 & y_2' & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \le 1 \qquad \mathrm{rk}\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2' & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix} \le 1,$$

yielding two one-parameter deformations with deformation parameters $s = y_2' - y_2$ of degree r. These two one-parameter deformations generate $T^1(-r)$. This is Pinkham's famous example of a singularity whose versal base space consists of two irreducible components [Pin74, chapter 8]; we see curves from both components via these toric deformations.



(a) σ_r (b) Two decompositions.

Figure 3.1: Toric deformations of the cone over the rational normal curve

Consider now some degree $r \notin \delta^\vee$, and an admissible decomposition of δ_r as above. I will summarize Altmann's construction of an associated l-parameter deformation of $\mathrm{TV}(\sigma)$ [Alt00, section 3.5]. Setting $\tau = \delta \cap [r \geq 0]$, we get an induced admissible decomposition of τ_r, so Theorem 3.10 yields a toric deformation $\mathrm{TV}(\widetilde{\tau}) \to \mathbb{C}^l$ of $\mathrm{TV}(\tau)$. To construct the deformation of $\mathrm{TV}(\delta)$, set $\widetilde{\delta} = \widetilde{\tau} + \iota(\delta)$ and define the ring B with

$$A := \mathbb{C}[\widetilde{\delta}^\vee \cap \widetilde{M}] \subset B := A[\chi^{[0,e^i]} - \chi^{[0,e^0]} \mid i \geq 1] \subset \mathbb{C}[\widetilde{\tau}^\vee \cap \widetilde{M}].$$

Theorem 3.12. *[Alt00, Theorem 3.4] Let $\mathfrak{X} := \operatorname{Spec} B$, and define the map $\pi\colon \mathfrak{X} \to \mathbb{C}^l$ by the binomials $\chi^{[0,ke^0]} - \chi^{[0,e^i]}$. Then $\mathfrak{X} \to \mathbb{C}^l$ is a deformation of X.*

Note that the generators of B are homogeneous with respect to the M-grading induced by ι^*, so $\mathfrak{X}$ admits a T-action. See chapter 4 for a simpler construction of the deformations of Theorem 3.12, putting these on an equal footing to those of Theorem 3.10.

Example 3.13. Consider the the cone $X = C\mathbb{P}(1,2,3)$ over weighted projective space $\mathbb{P}(1,2,3)$. As a toric variety, it is given by the cone δ over the lattice polytope $\Pi = \operatorname{conv}\{(-1,-1),(2,-1),(-1,1)\}$; this means that X is Gorenstein. The dual cone is the cone over the polytope $\Pi^\vee = \operatorname{conv}\{[1,0],[0,1],[-2,-3]\}$, illustrating that X is a cone.

Take the degree $r = [-1,0,1]$. Then the compact part of δ_r is

$$\tfrac{1}{2}\operatorname{conv}\{(-1,-1,1),(-1,1,1)\} \subset \tfrac{1}{2}\cdot(\Pi \times \{1\}),$$

which decomposes as

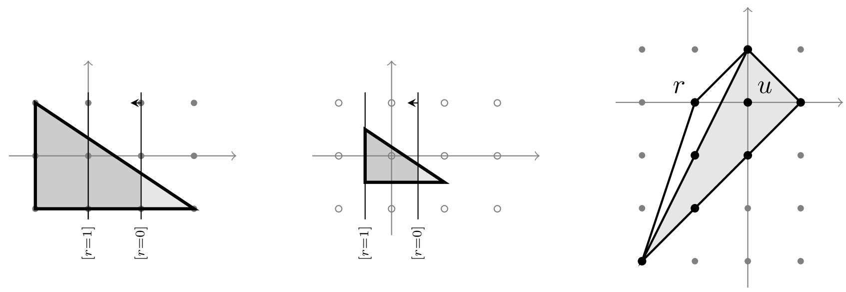
$$\{\tfrac{1}{2}\cdot(-1,1,1)\} + \operatorname{conv}\{(0,0,0),(0,-1,0)\}.$$

This gives a one-parameter deformation $\mathfrak{X}$ by Theorem 3.12.

The cone τ involved in the construction of $\mathfrak{X}$ is the cone over $\Pi \cap [r \geq 0]$. The dual cone $\tau^\vee$, which is the weight cone of the T-action on $\mathfrak{X}$, is the cone over $\operatorname{conv}(\Pi^\vee \cup \{[-1,0]\})$.

The situation is illustrated in Figure 3.2. Part (a) shows Π; the darker shaded part is the intersection with τ. The left edge of the triangle in part (b) is the (non-lattice) compact edge of δ_r that is decomposed. Part (c) shows $\Pi^\vee$; its lattice points generate the coordinate ring of X. The deformation degree r lies outside $\delta^\vee$, but is a generator of the weight cone $\tau^\vee$ of $\mathfrak{X}$.

For computing the deformation, we will stay with the splitting of M with respect to u, so the embedding $N \hookrightarrow N \oplus \mathbb{Z}$ maps $v \in N$ to $(v, r(v))$. The cone



(a) δ in height one   (b) δ in height one half   (c) $\delta^\vee$ in height one

Figure 3.2: Cross-cuts of the cone of $C\mathbb{P}(1, 2, 3)$ with respect to the Gorenstein degree u

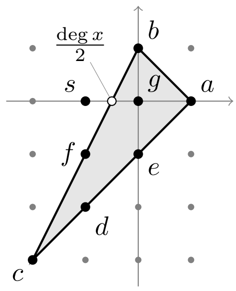


Figure 3.3: Weights of generators of the coordinate ring of $\mathfrak{X}$

$\widetilde{\delta}$ is generated by the columns of the first matrix below. Columns 1 and 2,3 come from the summands of δ_r, column 4 comes from the ray of δ that is cut off by $[r \geq 0]$. The generators of $\iota(\tau)$ are redundant. The second matrix lists a Hilbert basis of the dual cone, the first four elements of which are generators of the cone.

$$\begin{pmatrix} -1 & 0 & 0 & 2 \\ 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix} \qquad \begin{matrix} a & b & c & x & d & e & f & g \end{matrix} \atop \begin{pmatrix} 1 & 1 & -2 & -1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 & -2 & -1 & -1 & 0 \\ 1 & 0 & 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The deformation parameter of Theorem 3.12 translates to the binomial $s = \chi^{[0,0,0,1]} - \chi^{[-1,0,1,0]}$, and the coordinate ring B of $\mathfrak{X}$ is generated by s over $\mathbb{C}[a, b, c, d, e, f, g, x] \subset \mathbb{C}[M \oplus \mathbb{Z}]$. The weights of the homogeneous generators of B are illustrated in Figure 3.3, For the elements of the Hilbert basis of $\widetilde{\delta}^\vee$, these are just the projections under ι^*; note that $\iota^*(\deg x)$ lies in height 2.

To end the review of Altmann's results, note that if X is non-singular in codimension 2, it is even possible to describe the homogeneous parts of the versal deformation: The *universal Minkowski summand* $\widetilde{C}(\delta_r)$ is a cone lying over $C(\delta_r)$. Through the associated map of toric varieties $\mathrm{TV}(\widetilde{C}(\delta_r)) \to \mathrm{TV}(C(\delta_r))$, we can obtain a family $\mathfrak{X} \to \overline{\mathcal{M}}$; the scheme $\overline{\mathcal{M}}$ is determined by $C(\delta_r) \subset V(\delta_r)$.

Theorem 3.14 ([Alt97b],[AK]). *$\mathfrak{X} \to \overline{\mathcal{M}}$ is the versal deformation of X in degree $-r$. If X is an isolated Gorenstein singularity, then T^1 is concentrated in the Gorenstein degree r_0, so $\mathfrak{X}$ is the versal deformation.*

Chapter 4

Invariant deformations of T-varieties

Suppose now that X is some affine T-variety. If $r_0 \in M$ is some primitive degree in the character lattice, we define the subtorus $T_{r_0} := \ker(\chi^{r_0} \colon T \to \mathbb{C}^*)$ with respect to the associated character. Then the study of homogeneous deformations in degree $r \in \mathbb{Z}r_0$ may be reduced to the study of invariant deformations of the T_{r_0}-variety X. Thus, in the following, we replace T by T_{r_0}, and consider equivariant deformations in degree 0. That is, we consider families where the torus T acts trivially on the base, and acts on every fibre.

The main result is that admissible Minkowski decompositions of the coefficient polyhedra of a p-divisor $\mathcal{D}$ on $\mathbb{P}^1$ result in an invariant deformation of $\mathrm{TV}(\mathcal{D})$; this has been published in a joint article with Ilten [IV11b, section 2]. Here, the result is packaged in a slightly different manner, introducing the notion of a *deformation of p-divisors*.

Intuitively, the result can be summarized as follows. Take a p-divisor $\mathcal{D}$ on $\mathbb{P}^1$ and move the supporting points. Provided that whenever such points meet, the Minkowski sums that arise are admissible, we get a flat family.

4.1 Deformations of p-divisors

Fix an affine T-variety X, given by a p-divisor $\mathcal{D}$ on Z. We will study invariant deformations of X that arise by varying the p-divisor that describes the T-variety X. To fix the meaning of "varying" a p-divisor, we introduce the notion of a deformation of p-divisors.

Definition 4.1. A *deformation* over S of a p-divisor $\mathcal{D}$ on Z consists of:

1. a deformation $Y \to S$ of Z over the affine scheme S
2. a p-divisor $\mathcal{E}$ on Y, with coefficients in the same lattice N as $\mathcal{D}$

such that

1. for each $s \in S$, $\mathcal{E}$ restricts to a p-divisors $\mathcal{E}_s = \mathcal{E}_{|Y_s}$ on Y_s
2. the induced map $\mathrm{TV}(\mathcal{E}) \to S$ is flat
3. $\mathcal{E}_0 = \mathcal{D}$ when we identify Y_0 with Z

4. for each $s \in S$, the embedding $Y_s \hookrightarrow Y$ induces a closed embedding $\mathrm{TV}(\mathcal{E}_s) \hookrightarrow \mathrm{TV}(\mathcal{E})$ of $\mathrm{TV}(\mathcal{E}_s)$ as the fibre $\mathrm{TV}(\mathcal{E})_s$

Since S is affine, the existence of the structure map $\mathrm{TV}(\mathcal{E}) \to S$ is immediate, compare Example 1.17.

Remark 4.2. This is a somewhat deficient definition for a variety of reasons. For one, it would be preferable to not involve the T-varieties explicitly: The right conditions on the family $Y \to S$ and the p-divisor $\mathcal{E}$ should imply the properties we ask for. Some of the results of the following sections hint at a better definition. Certainly, some relativity assumption on the support of $\mathcal{E}$ is required in order to be able to even define the restrictions $\mathcal{E}_s$. Furthermore, we will see that lattice conditions on the coefficients of $\mathcal{E}$ are necessary to satisfy the closed embedding condition, compare Corollary 4.6.

Secondly, the choice of working within the category of T-varieties limits the options of applying general deformation theory: To be able to work with the total spaces as T-varieties, we require Y to be a semiprojective normal variety. This means for instance that infinitesimal deformations don't fit in, and deformations of p-divisors don't form a useful abstract deformation theory in the sense of section 3.1.

The remainder of this chapter is devoted to showing that the class of deformations of p-divisors is not trivial: We will construct such objects for the case of rational T-varieties of complexity one. As discussed in section 5.1, these include the well-known deformations of toric varieties, which in turn cover the deformation theory of toric varieties.

4.2 Decompositions of p-divisors on curves

Let Z be a smooth projective curve and let $\mathcal{D}$ be a p-divisor on Z with $\delta = \mathrm{tail}(\mathcal{D})$. We describe how to construct candidates for deformations of $\mathcal{D}$ by decomposing a coefficient $\mathcal{D}_P$ of $\mathcal{D}$ as a sum of polyhedra. Recall the notion of an admissible decomposition of a polyhedron from Definition 1.1.

Let $\mathcal{P} \subset Z$ be a finite set of points in Z, including all those points P with nontrivial coefficient $\mathcal{D}_P$. Suppose now that for each $P \in \mathcal{P}$ we have Minkowski decompositions $\mathcal{D}_P = \sum_{s=0}^{l_P} \mathcal{D}_P^s$. We call such data a *decomposition* of the polyhedral divisor $\mathcal{D}$; it is *admissible* if each decomposition of the coefficients is admissible. Let $l = \sum_{P \in \mathcal{P}} l_P$ be the total number of parameters; this is finite since $\mathcal{P}$ is a finite set.

Remark 4.3. We allow empty coefficients. A decomposition of the empty set is any sum of δ-polyhedra that includes an empty summand. It is admissible if the non-empty summands form an admissible decomposition of their sum. Compare Remark 4.15 for the (in)significance of such decompositions.

Consider some smooth affine variety S with special point 0 cut out by a regular sequence $t_1, \ldots, t_k$. For $0 \le j \le k$, let S_j be the subvariety cut out by $t_{j+1}, \ldots, t_k$. Now consider some family $\gamma\colon Y \to S$ with Y smooth such that $Y_j := V(t_{j+1}, \ldots, t_k) \subset Y$ is equal to $\gamma^{-1}(S_j)$, and $\gamma^{-1}(0) = Y_0 = Z$. Furthermore, let $E(P, i)$ be a collection of pairwise different prime divisors on

Y intersecting the Y_j properly, such that $E(P, i)$ restricts to P in Z. From this information, we define the polyhedral divisors

$$\mathcal{E} = \sum_{P,i} \mathcal{D}_P^i \otimes E(P, i).$$

Note that since we required the $E(P, i)$ to restrict to P in Z, we have $\mathcal{E}_{|Z} = \mathcal{D}$, since for each P, the coefficients of the $E(P, i)$ sum up to $\mathcal{D}_P$. In particularly, $\mathcal{E}(u)_{|Z} = \mathcal{D}(u)$ for all u.

We assume for the moment that all $\mathcal{E}_{|Y_i}$ are proper polyhedral divisors. Let $\mathfrak{X} = \mathrm{TV}(\mathcal{E})$, and consider the map $\pi\colon \mathfrak{X} \to S$. We want the special fibre of π to be X, i.e. $\pi^{-1}(0) = X = \mathrm{TV}(\mathcal{D})$.

Proposition 4.4. *The map of T-varieties $X \to \mathfrak{X}$ induced by $Z \hookrightarrow Y$ embeds X as the special fibre $\pi^{-1}(0)$ if, for each $u \in \delta^\vee \cap M$, the following two conditions hold:*

1. $\lfloor \mathcal{E}(u) \rfloor_{|Z} = \lfloor \mathcal{E}(u)_{|Z} \rfloor$
2. *With $D = \lfloor \mathcal{E}(u) \rfloor$, the natural morphisms*

$$H^0(Y_i, D_{|Y_i}) \to H^0(Y_{i-1}, D_{|Y_{i-1}})$$

are surjective for $1 \le i \le k$.

Proof. The claim is equivalent to the exactness of

$$0 \longrightarrow I \cdot H^0(Y, \mathcal{E}(u)) \longrightarrow H^0(Y, \mathcal{E}(u)) \xrightarrow{\ \nu\ } H^0(Z, \mathcal{D}(u)) \longrightarrow 0$$

for each $u \in \delta^\vee \cap M$, where $I = \langle t_1, \dots, t_k \rangle$. The map ν arises as follows (compare section 8 of [AH06]):

$$\begin{array}{ccccc}
H^0(Y, \mathcal{E}(u)) & \longrightarrow & & \longrightarrow & H^0(Z, \mathcal{D}(u)) \\
\| & & & & \| \\
H^0(Y, \lfloor \mathcal{E}(u) \rfloor) & \xrightarrow{\ \varphi\ } & H^0(Z, \lfloor \mathcal{E}(u) \rfloor_{|Z}) & \overset{\psi}{\hookrightarrow} & H^0(Z, \lfloor \mathcal{D}(u) \rfloor)
\end{array}$$

Since $\mathcal{E}(u)_{|Z} = \mathcal{D}(u)$, surjectivity of ψ follows from condition 1. Surjectivity of φ follows from condition 2. Thus, ν is surjective (and $X \to \mathfrak{X}$ is a closed embedding).

We must still check that the kernel of ν is correct; an easy calculation shows that it contains $I \cdot H^0(Y, \mathcal{E}(u))$. Choose some open affine $U \subset Y$ such that $U \cap Z \ne \emptyset$ and U is disjoint from the support of $D = \lfloor \mathcal{E}(u) \rfloor$. We can expand the above sequence to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I \cdot H^0(Y, D) & \longrightarrow & H^0(Y, D) & \longrightarrow & H^0(Z, D_{|Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I \cdot H^0(U, \mathcal{O}_U) & \longrightarrow & H^0(U, \mathcal{O}_U) & \longrightarrow & H^0(Z \cap U, \mathcal{O}_{Z \cap U}) & \longrightarrow & 0
\end{array}$$

where the vertical arrows are inclusions. If we can show that $I \cdot H^0(U, \mathcal{O}_U) \cap H^0(Y, D) = I \cdot H^0(Y, D)$, we are done by the exactness of the second row.

Assume that $k = 1$ and take $s \in I \cdot H^0(U, \mathcal{O}_U) \cap H^0(Y, D)$; we can thus write $s = t_1 g$ for $g \in H^0(U, \mathcal{O}_U)$. Furthermore,

$$\operatorname{div}(t_1 g) + D \geq 0.$$

But $\operatorname{div}(t_1 g) = \operatorname{div}(t_1) + \operatorname{div}(g)$ and the order of the components of D along $\operatorname{div}(t_1) = Y$ are zero, so $g \in H^0(Y, D)$. Thus $I \cdot H^0(U, \mathcal{O}_U) \cap H^0(Y, D) = I \cdot H^0(Y, D)$.

Assume that $k > 1$. After slight adjustment, the above arguments show that

$$0 \to t_i \cdot H^0(Y_i, D_{|Y_i}) \to H^0(Y_i, D_{|Y_i}) \to H^0(Y_{i-1}, D_{|Y_{i-1}}) \to 0 \tag{4.1}$$

is exact for $1 \leq i \leq k$. Now, consider also the sequence

$$0 \to \langle t_1, \ldots, t_j \rangle \cdot H^0(Y_j, D_{|Y_j}) \to H^0(Y_j, D_{|Y_j}) \to H^0(Z, D_{|Z}) \to 0 \tag{4.2}$$

and assume that this is exact for some $j = m$, $1 \leq m < k$. A straightforward diagram chase shows that the exactness of (4.1) for $i = m + 1$ and exactness of (4.2) for $j = m$ gives the exactness of (4.2) for $j = m + 1$. Induction on m completes the proof. □

Condition 1 is where admissibility comes into play:

Lemma 4.5. *Suppose $D = \sum a_P^i E(P, i)$ is a $\mathbb{Q}$-divisor on Y. Then $\lfloor nD \rfloor_{|Z} = \lfloor (nD)_{|Z} \rfloor$ for all integers $n \geq 0$ if and only if, for each $P \in Z$, at most one of the coefficients a_P^i is not an integer.*

Proof. Due to our choice of divisors $E(P, s)$, this follows from the following fact: Let $p, q \in \mathbb{Q} \setminus \mathbb{Z}$, $p, q \geq 0$. Then there exists an integer $n \geq 0$ such that $\lfloor np + nq \rfloor > \lfloor np \rfloor + \lfloor nq \rfloor$. □

Corollary 4.6. *Condition 1 of Proposition 4.4 holds for each $u \in \delta^\vee \cap M$ if and only if the Minkowski decompositions underlying $\mathcal{E}$ are admissible.*

Example 4.7 (A non-admissible decomposition)**.** Take the trivial family $Y = \mathbb{A}^1 \times \mathbb{A}^1 \to S = \mathbb{A}^1$, where S has coordinate y and Y has coordinates x, y. Set $D_0 = V(x)$ and $D_1 = V(y - x)$, and define the p-divisor

$$\mathcal{E} = [\tfrac{1}{2}, \infty) \otimes D_0 + [\tfrac{1}{2}, \infty) \otimes D_1$$

on Y. This evaluates as $\mathcal{E}(k) = \frac{k}{2}(D_0 + D_1)$ for $k \in M = \mathbb{Z}$, so $A(\mathcal{E})$ is generated as a k-algebra by

$$x, y \in L(\mathcal{E}(0)) \qquad t \in L(\mathcal{E}(1)) \qquad \frac{t^2}{x(x-y)} \in L(\mathcal{E}(2)),$$

where $t = \chi^1$ is the coordinate on $\mathbb{C}^*$. Setting $v = t^2 x^{-1}(x - y)^{-1}$ for the generator in degree 2, $\mathfrak{X} = \mathrm{TV}(\mathcal{E})$ is embedded in $\mathbb{A}^4$ with equation $t^2 = x(x - y)v$.

The special fibre $\mathfrak{X}_0$ is the non-normal toric variety $\operatorname{Spec} \mathbb{C}[x, t, \frac{t^2}{x^2}]$. But $\mathcal{E}_0 = [1, \infty) \otimes \{0\}$ on $\mathbb{A}^1$ gives $\mathrm{TV}(\mathcal{E}_0) = \mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x, \frac{t}{x}]$, and the induced map $\mathrm{TV}(\mathcal{E}_0) \to \mathfrak{X}$ factors through $\mathfrak{X}_0$ as the normalization.

4.3 Deformations of p-divisors on $\mathbb{P}^1$

From now on, we assume that our base curve Z is $\mathbb{P}^1$. For each $P \in \mathcal{P}$, let $y_P \in \mathbb{C}(Z)$ be a rational function with its sole zero at P. Let $t_{P,1}, \ldots, t_{P,l_P}$ be coordinates on $\mathbb{A}^{l_P}$ for $P \in \mathcal{P}$, and set $t_{P,0} = 0$. Let S be any open affine neighbourhood of the origin in $\prod_{P \in \mathcal{P}} \mathbb{A}^{l_P}$ such that a divisor on $\mathbb{P}^1 \times S$ of the form $V(y_P - t_{P,i})$ doesn't intersect any divisor of the form $V(y_Q - t_{Q,j})$ for $P \neq Q$ and $P, Q \in \mathcal{P}$.

We consider the trivial family $Y = \mathbb{P}^1 \times S$ with the prime divisors $E(P,i) = V(y_P - t_{P,i})$; these clearly restrict as desired to P.

Theorem 4.8. *The polyhedral divisor $\mathcal{E}$ on $\mathbb{P}^1 \times S$ associated with an admissible decomposition of $\mathcal{D}$ is a deformation of $\mathcal{D}$.*

In the remainder of this section, we will prove that the various conditions required of a deformation of p-divisors are satisfied by $\mathcal{E}$. An example of such a family is pictured in Figure 4.1 for $Y = \mathbb{P}^1 \times \mathbb{A}^1$, with $l_0 = 1$ and $l_\infty = 0$.

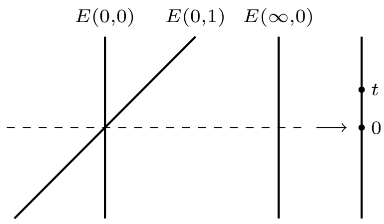


Figure 4.1: A family of prime divisors $E(P,i)$ on $\mathbb{P}^1 \times \mathbb{A}^1$

As before, for a point $s \in S$, we denote by $\mathcal{E}_s$ the restriction of $\mathcal{E}$ to the fibre Y_s of Y over s. By identifying Y_s with $Z = \mathbb{P}^1$, we can view $\mathcal{E}_s$ as a polyhedral divisor on Z, which we can describe explicitly. Suppose s is given by the equations $t_{P,i} = s_{P,i}$, and set $s_{P,0} = 0$ for each $P \in \mathcal{P}$. For $0 \leq i \leq l_P$, let $D_s(P,i)$ be the divisor on Z given by the vanishing of $y_P - s_{P,i}$. Then the polyhedral divisor $\mathcal{E}_s$ is given by

$$\mathcal{E}_s = \sum_{\substack{P \in \mathcal{P} \\ 0 \leq i \leq l_P}} \mathcal{D}_P^i \otimes D_s(P,i),$$

where the coefficients in front of prime divisors appearing multiple times are added via Minkowski sums. Note that prime divisors appear multiple times whenever $s_{P,i} = s_{P,j}$ for $i \neq j$.

Lemma 4.9. *$\mathcal{E}$ is a p-divisor on $Y = \mathbb{P}^1 \times S$. Likewise, $\mathcal{E}_s$ is a proper polyhedral divisor on $\mathbb{P}^1$.*

Proof. For any $u \in \delta^\vee \cap M$, consider the $\mathbb{Q}$-divisor $\mathcal{E}(u)$. One easily checks that $a \cdot \mathcal{E}(u) \sim a \cdot \mathcal{D}(u) \times S$ for some large $a \in \mathbb{N}$, since for each $P \in Y$ and $i \leq l_P$ we have $V(y_P - t_{P,i}) \sim V(y_P)$. Thus, $\mathcal{E}(u)$ is semiample or big exactly when $\mathcal{D}(u)$ is semiample or big, so the properness of $\mathcal{E}$ follows from the properness of $\mathcal{D}$. Similarly, properness of $\mathcal{E}_s$ follows from that of $\mathcal{E}$. $\square$

Proposition 4.10. *If the decomposition of $\mathcal{D}$ is admissible, the map* $\mathrm{TV}(\mathcal{D}) \to \mathrm{TV}(\mathcal{E})$ *is a closed embedding given by the ideal generated by all $t_{P,i}$ for $P \in \mathcal{P}$, $1 \leq i \leq l_P$.*

Proof. Let $D = \lfloor \mathcal{E}(u) \rfloor$ for some $u \in \delta^\vee \cap M$. We twist the short exact sequence for the embedding $Y_{i-1} \hookrightarrow Y_i$

$$0 \to \mathcal{I}_{Y_{i-1}} \to \mathcal{O}_{Y_i} \to \mathcal{O}_{Y_{i-1}} \to 0$$

by the locally free sheaf $\mathcal{O}_Y(D)_{|Y_i}$. Consider the associated long exact sequence in cohomology

$$H^0(Y_i, D_{|Y_i}) \to H^0(Y_{i-1}, D_{|Y_{i-1}}) \to H^1(Y_i, \mathcal{I}_{Y_{i-1}}(D)).$$

Assume that $H^0(Y_{i-1}, D_{|Y_{i-1}})$ is not zero. We claim that $H^1(Y_i, \mathcal{I}_{Y_{i-1}}(D))$ vanishes, which proves the statement by Proposition 4.4.

Indeed, since $\mathcal{I}_{Y_{i-1}} = t_i \cdot \mathcal{O}_{Y_i}$, we have $H^1(Y_i, \mathcal{I}_{Y_{i-1}}(D)) = H^1(Y_i, D_{|Y_i})$. But this vanishes, since $D_{|Y_i}$ is an effective semiample divisor on the product of $\mathbb{P}^1$ with some subset of affine space. $\square$

Proof of theorem 4.8. By choice of $S \subset \mathbb{A}^l$, the admissible decomposition of $\mathcal{D}$ also induces admissible decompositions of $\mathcal{E}_s$ which result in the same divisor $\mathcal{E}$ on Y. Thus, after coordinate change in S we can apply Proposition 4.10 and get that for any $s \in S$, $\mathrm{TV}(\mathcal{E}_s) \cong \pi^{-1}(s)$. Furthermore, $\mathrm{TV}(\mathcal{E}_s) \hookrightarrow \mathfrak{X}$ is cut out by a regular sequence so π is flat. $\square$

Remark 4.11. Suppose a Minkowski summand $\mathcal{D}_P^i$ is a multiple $k\Delta$ of a lattice polyhedron Δ. Then replacing $E(P,i) = V(y_P - t_{P,i})$ by $E(kP,i) := V(y_P^k - t_{P,i})$ and $\mathcal{D}_P^i$ by Δ in $\mathcal{E}$ also gives a deformation of X, after changing S accordingly. Indeed, since $\Delta \otimes E(kP,i)$ restricts to $\Delta \otimes kP$, $\mathcal{E}$ restricts to $\mathcal{D}$ as before. The change doesn't affect the integrality considerations since Δ is a lattice polyhedron. The rest of the arguments carry through unchanged.

We end this section with a corollary to Theorem 4.8.

Corollary 4.12. *Let $\mathcal{D}$ be a proper polyhedral divisor on $\mathbb{P}^1$ with affine locus. Consider some admissible decomposition of $\mathcal{D}$. The general fibre of the corresponding deformation π has exactly the analytic singularities* $\mathrm{TV}(\mathrm{Cone}(\mathcal{D}_P^i \times \{1\}))$ *for $P \in \mathcal{P}$ and $0 \leq i \leq l_P$, where* $\mathrm{TV}(\mathrm{Cone}(\mathcal{D}_P^i \times \{1\}))$ *is the toric singularity corresponding to the cone over the polyhedron $\mathcal{D}_P^i$.*

Proof. This follows from the description of the general fibre from Theorem 4.8 coupled with [LS10], Theorem 5.3. $\square$

Remark 4.13. Ilten used explicit equations to calculate the singularities in the general fibre for toric deformations of cyclic quotient singularities [Ilt09, section 4]. Combining this with the description of affine toric deformations in chapter 5, the above corollary provides a way of doing this without using the equations. Furthermore, the above corollary can be applied to see whether a toric deformation, or more generally, a deformation of p-divisors, is a smoothing. Note that if $\mathcal{D}$ has complete locus and $\mathrm{TV}(\mathcal{D})$ is singular, no deformation of p-divisors can be a smoothing (see [LS10, Proposition 5.1]).

Example 4.14. The Minkowski decompositions of Example 3.11 induce T_r-equivariant deformations with divisors

$$\begin{aligned} \mathcal{E}^1 &= [-\tfrac{1}{2}, 0] \otimes D^0 + [0, \tfrac{1}{2}] \otimes D^1 \\ \mathcal{E}^2 &= \quad -\tfrac{1}{2} \otimes D^0 + [0, 1] \otimes D^1 \end{aligned}$$

on $\mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x, y]$, with $D^0 = V(x)$ and $D^1 = V(y - x)$, where y is the deformation parameter. Theorem 5.1 in the following chapter shows that these deformations agree with the corresponding deformations of Example 3.11.

Remark 4.15. From the point of view of deforming affine singularities, families that arise from decompositions of empty set coefficients are trivial: Since the locus is necessarily affine, the deformation only affects fibres of $X \to Y$ near a point of $\mathbb{P}^1 \setminus Y$, where X splits as a product $Y \times \mathrm{TV}(\operatorname{tail} \mathcal{D})$. The singularities of X are unaffected.

On the other hand, these deformations are crucial when gluing deformations of global T-varieties. For $X = \mathrm{TV}(\mathcal{S})$, Ilten shows how to glue compatible deformations of the p-divisors $\mathcal{D} \in \mathcal{S}$ to a deformation of X, and goes on to prove that for complete smooth toric varieties, these deformations cover T^1. [IV11b, section 4].

Chapter 5

Toric deformations as T-varieties

In this chapter, we will investigate how to express the deformations of toric varieties of section 3.3 with p-divisors. Throughout, X will be a toric variety with embedded torus T, given by a polyhedral cone δ in the lattice N. Then a decomposition of $\delta_r = \delta \cap [r = 1]$ in the degree $r \in M$ determines a deformation $\mathfrak{X} \to \mathbb{A}^l$ by the theorems of section 3.3.

In section 5.1, we will see that $\mathfrak{X}$ is the same deformation as that constructed in chapter 4: The decomposition of δ_r induces a decomposition of the p-divisor that expresses X as a T_{r_0}-variety, where T_{r_0} is the kernel of $r_0 \colon T \to \mathbb{C}^*$. This was previously treated with less detail in section 3 of [IV11b].

But $\mathfrak{X}$ is not just a T_{r_0}-invariant deformation, it is also naturally T-equivariant. In section 5.2, we will apply the results of chapter 2 to upgrade the divisor $\mathcal{E}$ for $\mathfrak{X}$ as a T_{r_0}-variety to a divisor $\mathcal{F}$ that describes $\mathfrak{X}$ as a T-variety.

Finally, section 5.3 contains some examples illustrating a possible next step in understanding deformations of toric varieties as T-varieties: In general, a single reduced component of the versal deformation of a toric variety may be made up of deformations in different degrees. These so-called deformations in mixed degree still admit a T-action, so determining their p-divisors is a natural goal.

5.1 Toric deformations as deformations of p-divisors

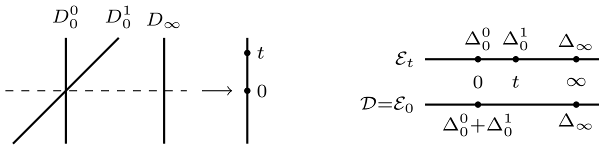
To begin, we need to determine a p-divisor for X as a T_{r_0}-variety. Once again, we chose a splitting $N = N_{r_0} \oplus \mathbb{Z}$, so that $r_0 = [0, 1]$. The dual of the weight cone of the T_{r_0}-action on X is $\sigma := \delta \cap (N_{r_0,\mathbb{Q}}, 0)$, and with

$$(\Delta^+, 1) := \delta_{r_0} = \delta \cap (N_{r_0,\mathbb{Q}}, 1)$$
$$(\Delta^-, -1) := \delta_{-r_0} = \delta \cap (N_{r_0,\mathbb{Q}}, -1),$$

the toric downgrade procedure shows that X has the p-divisor $\mathcal{D} = \Delta^+ \otimes 0 + \Delta^- \otimes \infty$ on $\mathbb{P}^1$. Note that Δ^- is the empty set for $r \in \delta^\vee$, so in that case $\mathcal{D}$ is effectively a p-divisor on $\mathbb{A}^1$.

Now consider an admissible Minkowski decomposition

$$\delta_r = (\Delta_0, \tfrac{1}{k}) + (\Delta_1, 0) + \cdots + (\Delta_l, 0),$$



(a) $Y = \mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ (b) Restrictions of $\mathcal{E}$

Figure 5.1: One-parameter toric deformation as deformation of p-divisors

giving rise to the deformation $\mathfrak{X}$ of section 3.3. In the case $k = 1$, this immediately defines a deformation of $\mathcal{D}$ by Theorem 4.8. For larger k, the associated decomposition $k\Delta_0 + k\Delta_1 + \cdots + k\Delta_l$ of δ_{r_0} induces also a deformation of $\mathcal{D}$. However, the one we require here is that of Remark 4.11.

Theorem 5.1. *Let $\mathfrak{X} \to \mathbb{A}^l$ be the deformation of X associated with an admissible decomposition of δ_r. Let $Y = \mathbb{P}^1 \times \mathbb{A}^l$ and consider the divisors $D_0 = V(x_1)$, $D_i = V(x_1^k - y_i x_0^k)$ for $1 \le i \le l$, and $D_\infty = V(x_0)$. Then $\mathfrak{X}$ is given by the p-divisor*

$$\mathcal{E} = k\Delta_0 \otimes D_0 + \Delta_1 \otimes D_1 + \cdots + \Delta_l \otimes D_l + \Delta^- \otimes D_\infty$$

on Y. The structure map $\mathfrak{X} \to \mathbb{A}^l$ of the family corresponds to the projection $Y \to \mathbb{A}^l$.

The situation is summarized in Figure 5.1 for the case $k = l = 1$.

Proof. We will first treat the case of $r \in \delta^\vee$, so the total space is the toric variety $\mathrm{TV}(\widetilde{\delta})$. As a cone in $N_r \oplus \mathbb{Z}^{l+1}$, $\widetilde{\delta}$ is generated by $(\sigma, 0)$, $(\Delta_0, \frac{1}{k}e_0)$ and (Δ_i, e_i).

We apply the toric downgrade of Example 2.1. The projection of $\widetilde{\delta}$ to $\mathbb{Z}^{l+1}$ is the positive orthant; the faces of $\widetilde{\delta}$ correspond to the faces of the Δ_i and all map to faces of the positive orthant. Thus, the downgraded p-divisor lives on the toric variety $Y' = \mathbb{A}^{l+1}$. The coefficient for the i-th coordinate hyperplane $V(\chi^{e^i})$ is the fibre of $\widetilde{\delta}$ over e_i, which is Δ_i for $i \ge 1$, and $k\Delta_0$ for $i = 0$ since $(\Delta_0, \frac{1}{k}e_0)$ scales to $(k\Delta_0, e_0)$.

The structure map of the family $\mathrm{TV}(\widetilde{\delta}) \to \mathbb{A}^l$ is given by the binomials $y_i := \chi^{ke^0} - \chi^{e^i}$. The corresponding coordinate change leaves $D_0 := V(\chi^{e^0})$ unchanged and translates the other hyperplanes to $D_i := V(y_i - \chi^{ke^0})$. Note that $\Delta^- = \emptyset$, which means $\operatorname{loc}\mathcal{E} = \mathbb{A}^1 \times \mathbb{A}^l \subset \mathbb{P}^1 \times \mathbb{A}^l$, so we are done for the case of positive r.

In the general case, we have the cone $\tau = \delta \cap [r \ge 0]$ and the associated cone $\widetilde{\tau}$ generated by $(\sigma, 0)$, $(\Delta_0, \frac{1}{k}e_0)$ and (Δ_i, e_i) as before. Furthermore, we have the cone $\widetilde{\delta} = \widetilde{\tau} + \iota(\delta)$. Note that as a cone in $N_r \oplus \mathbb{Z}$, δ is generated by $(\Delta^+, 1)$, $(\sigma, 0)$ and $(\Delta^-, -1)$, so that $\widetilde{\delta}$ is generated by $\widetilde{\tau}$ and $(\Delta^-, -e_0 - \cdots - e_l)$. $((\Delta^+, 1) = (k\Delta_0, 1) + (\Delta_1, 0) + \cdots + (\Delta_l, 0)$ is redundant.)

We apply the toric downgrade to $\mathrm{TV}(\widetilde{\delta})$. Compared to the result before, the addition of $(\Delta^-, -e_0 - \cdots - e_l)$ gives the extra ray $\mathbb{Q}_{\ge 0} \cdot (-1, \ldots, -1)$ in $\mathbb{Z}^{l+1}$, so the toric quotient is $Y'' = \mathbb{P}^{l+1}$, and the p-divisor picks up the coefficient Δ^- for the new hyperplane at infinity. Denoting the toric divisors by $E_0, \ldots, E_l$ and E_∞ for the hyperplane at infinity we thus have the p-divisor

$$\mathcal{F} = k\Delta_0 E_0 + \Delta_1 E_1 + \cdots + \Delta_l E_l + \Delta^- E_\infty.$$

Set $f_i := \chi^{e^i}$ for $i \geq 0$; we then have

$$\operatorname{div}(f_i) = E_0 - E_\infty$$

and the global sections of a divisor $\mathcal{F}(u) = \sum a_i E_i + bE_\infty$ are generated over $H^0(\mathbb{P}^{l+1}, \mathcal{O}_{\mathbb{P}^{l+1}}) = \mathbb{C}$ by the rational functions

$$\{ \prod f_i^{\alpha_i} \mid \alpha_i \geq -a_i, i \geq 0; \sum \alpha_i \leq -b \}.$$

Now on $Y = \mathbb{P}^1 \times \mathbb{A}^l$, the same rational functions $f_i = \chi^{e^i}$ have principal divisors

$$\operatorname{div}(f_i) = D_i - D_\infty,$$

and the global sections of a divisor $\sum a_i D_i + bD_\infty$ are once again generated over $H^0(Y, \mathcal{O}_Y) = \mathbb{C}[y_1, \ldots, y_l]$ by

$$\{ \prod f_i^{\alpha_i} \mid \alpha_i \geq -a_i, i \geq 0; \sum \alpha_i \leq -b \}.$$

Thus, passing from the ring of sections of the p-divisor $\mathcal{F}$ on $\mathbb{P}^l$ to $\mathcal{E}$ on $Y = \mathbb{P}^1 \times \mathbb{A}^l$ just involves extending by the regular functions on Y, which are precisely the functions $y_i = \chi^{ke^0} - \chi^{e^i}$ that are added in the definition of the coordinate ring of $\mathfrak{X}$. □

Example 5.2 (An A_1-singularity)**.** Consider the quadric cone $X = V(uw - v^2) \subset \mathbb{A}^3$. As a toric variety, this is given by $\delta = \operatorname{pos}\{(1,1), (-1,1)\}$, and the variables u, v, w have weights $[1,1], [0,1], [-1,1]$. We can deform X in degree $r = [0,2]$ to $\mathfrak{X} = V(uw - v^2 - y_1) \cong \mathbb{A}^3$, corresponding to the Minkowski decomposition

$$\delta_r = \left([-\tfrac{1}{2}, \tfrac{1}{2}], \tfrac{1}{2}\right) = (\Delta_0, \tfrac{1}{2}) + (\Delta_1, 0)$$

with $\Delta_0 = \{-1/2\}$ and $\Delta_1 = [0,1]$, see Figure 5.2. Since $\Delta^- = \emptyset$, we may take $Y = \mathbb{A}^1 \times \mathbb{A}^1$ with coordinates x and y. Then as a T-variety, $\mathfrak{X}$ is given by the p-divisor $\mathcal{E}$ on Y, where $D_0 = V(x)$ has coefficient $2\Delta_0 = \{-1\}$, and $D_1 = V(y - x^2)$ has coefficient $\Delta_1 = [0,1]$.

Base extension with $\alpha\colon \mathbb{A}^1 \to \mathbb{A}^1$, $z \mapsto z^2$ induces a deformation $\mathfrak{X}' = \alpha^*\mathfrak{X} \to \mathbb{A}^1$ which is homogeneous of degree $r_0 = [0,1]$. Pulling back $\mathcal{E}$ to $Y' = \alpha^* Y = \mathbb{A}^1 \times \mathbb{A}^1$ (coordinates x and y') gives a p-divisor $\mathcal{E}'$ for $\mathfrak{X}'$; D_0 pulls back to $D_0' = V(x)$, D_1 to $D_{\pm 1}' = V(x \pm y')$, so

$$\mathcal{E}' = \{-1\} \otimes D_0' + [0,1] \otimes D_{+1}' + [0,1] \otimes D_{-1}'.$$

This is illustrated in Figure 5.3. (Note that while $Y' \to Y$ is proper, it is not birational, so this doesn't contradict Example 1.4.)

Since $\{-1\} \otimes D_0'$ can be moved into one of the two other summands by adding a principal divisor, $\mathcal{E}'$ is linearly equivalent to the p-divisor $\mathcal{E}'' = [-1,0] \otimes D_0 + [0,1] \otimes D_1$, corresponding to the decomposition $[-1,1] = [-1,0] + [0,1]$ of δ_{r_0}.

The previous example generalizes as follows. The proof is straightforward.

Proposition 5.3. *Consider a Minkowski decomposition and the associated p-divisor $\mathcal{E}$ of Theorem 5.1. Take the map $\alpha\colon \mathbb{A}^l \to \mathbb{A}^l$ that raises each coordinate to its kth power. Then $\mathcal{E}$ pulls back to*

$$\mathcal{E}' = k\Delta_0 \otimes D_0 + \Delta_1 \otimes (D_1^1 + \cdots + D_1^k) + \cdots + \Delta^- \otimes D_\infty$$

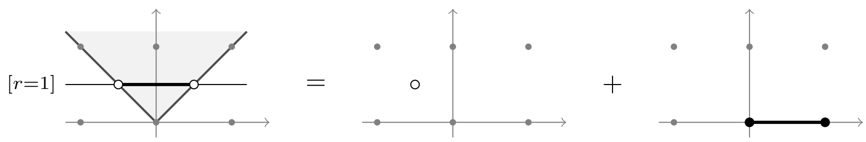


Figure 5.2: A Minkowski decomposition for the quadric cone

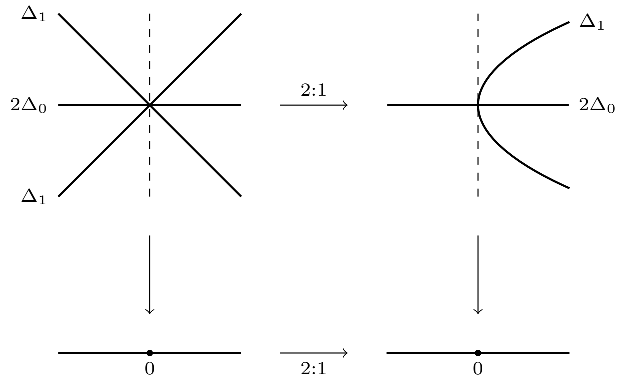


Figure 5.3: Pulling back a non-primitive deformation

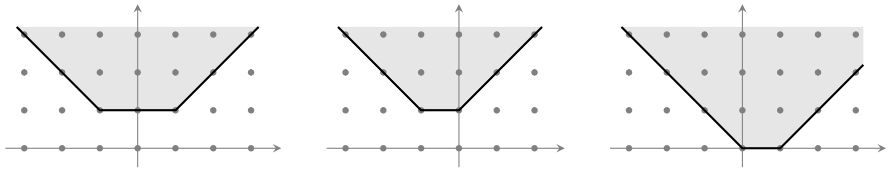
on $\alpha^* Y \cong Y$, where $\alpha^* D_i = V(x^k - y_i^k)$ decomposes into a sum of the degree 1 *divisors D_i^j.*

On the other hand, take the induced $k \cdot l$-parameter decomposition

$$\delta_{r_0} = (\Delta_0, 1) + k \cdot (\Delta_1, 0) + \cdots + k \cdot (\Delta_l, 0)$$

in degree r_0. The associated $k \cdot l$-parameter deformation corresponds to a p-divisor $\widetilde{\mathcal{E}}$ on $\widetilde{Y} = \mathbb{P}^1 \times \mathbb{A}^{kl}$. Then there is a linear embedding $\mathbb{A}^l \hookrightarrow \mathbb{A}^{kl}$ such that $\widetilde{\mathcal{E}}$ restricts to $\mathcal{E}'$ via the associated embedding $Y' \hookrightarrow \widetilde{Y}$. □

Example 5.4. We consider an example of a toric threefold with deformations in non-negative degrees. Let $N' = \mathbb{Z}^3$ with standard basis e_1, e_2, e_3 and σ generated by the four rays $(\pm 1, 1, \pm 1)$. $X = \mathrm{TV}(\sigma)$ is the cone over the singular projective Fano surface X' corresponding to the Fano polytope $\mathrm{conv}\{\pm e^1, \pm e^2\}$ in $M = \mathbb{Z}^2 \oplus 0 \subset M'$.



(a) Δ (b) Δ^0 (c) Δ^1

Figure 5.4: Minkowski decomposition for an affine threefold singularity

As a T-variety with respect to the subtorus $T = \mathbb{C}^* \otimes N$, X corresponds to the p-divisor $\mathcal{D} = \Delta \otimes \{0\} + \Delta \otimes \{\infty\}$ on $Y = \mathbb{P}^1$, with Δ as in Figure 5.4(a). The Minkowski decompositions $\mathcal{D}_0 = \Delta^0 + \Delta^1$ and $\mathcal{D}_\infty = \Delta^0 + \Delta^1$ induce a two-parameter deformation π of X. Restricting to the coordinate axes of the base space gives homogeneous deformations in degrees $-e^3$ and e^3, neither of which lie in $\sigma^\vee$.

5.2 Upgrades of toric deformations

We now show how to upgrade the T_{r_0}-action on $\mathfrak{X}$ to a T-action. The splitting $N = N_{r_0} \oplus \mathbb{Z}$ corresponds to a splitting $T = T_{r_0} \times T'$, with $T' = \mathbb{C}^*$.

Theorem 5.5. *Let $\mathfrak{X} \to \mathbb{A}^l$ be the deformation of X associated with an admissible decomposition of δ_r. Then the T-action on X extends to $\mathfrak{X}$, with weight cone dual to*

$$\widetilde{\sigma} = \delta \cap [r_0 \geq 0].$$

To describe $\mathfrak{X}$ as a T-variety, consider $\mathbb{P}^l = \operatorname{Proj} \mathbb{C}[y_0, \dots, y_l]$, and let Z be the blowup of $\mathbb{P}^l$ at the point $O = (1 : 0 : \dots : 0)$, with the divisors $P_0 = V(y_0)$, $P_i =$ strict transform of $V(y_0 - y_i)$ and the exceptional divisor Q. Then $\mathfrak{X}$ is described by the p-divisor on Z with tail cone $\widetilde{\sigma}$ and coefficients

$$\Delta_{P_0} = (\Delta_0, \tfrac{1}{k}) + \widetilde{\sigma} \quad \Delta_{P_i} = (\Delta_i, 0) + \widetilde{\sigma} \quad \Delta_Q = \operatorname{conv}\{(\tfrac{1}{k}\Delta^-, -\tfrac{1}{k}), (0,0)\} + \widetilde{\sigma}.$$

Note that if $l = 1$, we have $Z = \mathbb{P}^1$ and $Q = O = V(y_1)$.

Remark 5.6. The coefficients of the P_i provide a Minkowski decomposition

$$\Delta_{P_0} + \dots + \Delta_{P_r} = \delta \cap [r \geq 1],$$

while $\Delta_Q = \delta \cap [r \geq -1]$.

Proof. By Proposition 5.1, we can describe $\mathfrak{X}$ by the p-divisor

$$\mathcal{E} = k\Delta_0 \otimes D_0 + \Delta_1 \otimes D_1 + \dots + \Delta_l \otimes D_l + \Delta^- \otimes D_\infty$$

on $Y = \mathbb{P}^1 \times \mathbb{A}^l$. In order to upgrade the T_{r_0}-action, we will first need to express $\mathcal{E}$ as an invariant p-divisor with respect to an appropriate action of T' on Y.

Since $\mathfrak{X} \to \mathbb{A}^l$ should be T-equivariant if T acts on $\mathbb{A}^l$ with weights r, it follows that $\deg y_i = r \in M$. For $D_i = V((\frac{x_1}{x_0})^k - y_i)$ to be T'-invariant, we want $\deg \frac{x_1}{x_0} = r_0$. Thus, we consider the T'-action on Y given by the weights $(1, k, \dots, k)$.

Now by Proposition 5.7 below, $\mathcal{E}$ pulls back to a T'-invariant p-divisor $\mathcal{E}'$ on a contraction-free T'-variety $\widetilde{Y} = \operatorname{TV}(\mathcal{S})$ over Z, where $\mathcal{S}$ has slices

$$\mathcal{S}_{P_0} = [\tfrac{1}{k}, \infty) \qquad \mathcal{S}_{P_i} = [0, \infty) \qquad \mathcal{S}_Q = [-\tfrac{1}{k}, 0] \cup [0, \infty).$$

The positive half-line ρ is the only ray in $\operatorname{tail} \mathcal{S}$, and expressing $\mathcal{E}'$ in the form required by Proposition 2.18, we get

$$\begin{aligned} \mathcal{E}' = &\ \Delta^+ \otimes D_\rho \\ &+ \Delta_0 \otimes kD_{(P_0, \frac{1}{k})} \\ &+ \Delta_1 \otimes D_{(P_1,0)} + \ldots + \Delta_l \otimes D_{(P_l,0)} \\ &+ \tfrac{1}{k}\Delta^- \otimes kD_{(Q,-\frac{1}{k})}. \end{aligned}$$

An application of Theorem 2.15 in conjunction with Proposition 2.18 completes the proof. □

Proposition 5.7. *Let $\mathcal{E}$ on $Y = \mathbb{P}^1 \times \mathbb{A}^l$ be as in the proof of Theorem 5.5. Recall that Y has coordinates x_0, x_1 and $y_1, \dots, y_l$. Let $Z = \mathrm{Bl}_0 \mathbb{P}^l$ be the blowup of $\mathbb{P}^l$ at the origin O, where $\mathbb{P}^l$ has homogeneous coordinates $y_0 = (\frac{x_1}{x_0})^k$ and y_i, $1 \le i \le l$.*

1. *Denote by $P_0 = V(y_0)$ the hyperplane at infinity, and let Q be the exceptional divisor. Then as a T'-variety, Y is given by the divisorial fan $\mathcal{S}$ on Z generated by*

$$\mathcal{D}^+ = [\tfrac{1}{k}, \infty) \otimes P_0$$
$$\mathcal{D}^- = \emptyset \otimes P_0 + [-\tfrac{1}{k}, 0] \otimes Q.$$

2. *Let $P_i \subset Z$ be the strict transform of the image $V(y_0 - y_i)$ of D_i in $\mathbb{P}^l$ for $i \ge 1$. The prime divisors D_0, D_i and D_∞ on Y are T'-invariant with*

$$D_0 = D_{(P_0, \frac{1}{k})} \qquad D_i = D_{(P_i, 0)} \qquad D_\infty = D_{(Q, -\frac{1}{k})}.$$

   In particular, $\mathcal{E}$ is a T'-invariant p-divisor.

3. *$Y = \mathbb{P}^1 \times \mathbb{A}^l$ becomes contraction-free over Z by blowing up to $\widetilde{Y}$ at the origin and along (the strict transform of) $\mathbb{P}^1 \times \{0\}$. This introduces two (exceptional) divisors $E = D_\rho$, $\rho = \mathbb{Q}_{\ge 0}$ and $D_{(Q,0)}$. The divisors D_i pull back to*

$$\widetilde{D}_0 = D_{(P_0, \frac{1}{k})} + E \qquad \widetilde{D}_i = D_{(P_i, 0)} + E \qquad \widetilde{D}_\infty = D_{(Q, -\frac{1}{k})}.$$

Proof. The first statement is a straightforward application of the toric downgrade procedure. The second follows directly from the characterization of Weil divisors on T-varieties. For the last claim, we just need to check how the divisors D_i intersect the centres of the blowups. The origin of $\mathbb{P}^l$ is contained in all D_i but not in D_∞, while none of the interesting divisors contain $\mathbb{P}^1 \times \{0\}$. □

We can also describe the structure map of the family. To this end, we will first describe the map $\widetilde{Y} \to \mathbb{A}^l$ as a map of polyhedral divisors; the proof is straightforward.

Proposition 5.8. *Consider $\mathbb{A}^l$ as a $\mathbb{C}^*$-variety with p-divisor $[1, \infty) \otimes H$ on $\mathbb{P}^{l-1} = \mathbb{P}(\mathbb{A}^l)$, where H is any hyperplane in $\mathbb{P}^{l-1}$. The maps $Y, \widetilde{Y} \to \mathbb{A}^l$ are equivariant with respect to the homomorphism $T' \to \mathbb{C}^*$ corresponding to $F\colon N' = \mathbb{Z} \to \mathbb{Z}$, $v \mapsto kv$. These equivariant morphisms correspond to the triple $(\pi, F, \mathfrak{f})$, where $\pi\colon Z \to \mathbb{P}^{l-1}$ resolves the locus of indeterminacy of the projection $\mathbb{P}^l \dashrightarrow \mathbb{P}^{l-1}$ from the point $O \in \mathbb{P}^l$, and $\mathfrak{f}(1)$ is the principal divisor $P_0 - \pi^* H - Q$ on Z.* □

Proposition 5.9. *To describe the T-equivariant structure map $\mathfrak{X} \to \mathbb{A}^l$, let $[1, \infty) \otimes H$ on $\mathbb{P}^{l-1}$ be a p-divisor for $\mathbb{A}^l$ as a $\mathbb{C}^*$-variety, where H is any hyperplane in $\mathbb{P}^{l-1}$. Then if $\pi\colon Z \to \mathbb{P}^{l-1}$ is the projection from the (blown-up) origin O, the structure map corresponds to the triple $(\pi, r, \mathfrak{f})$, where $\mathfrak{f}(1)$ is the principal divisor $P_0 - \pi^* H - Q$ on Z.*

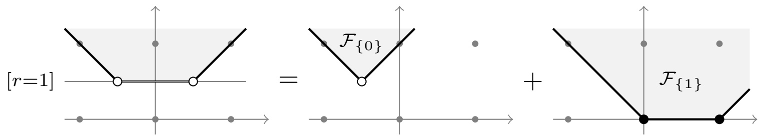


Figure 5.5: Upgraded Minkowski decomposition

Proof. If we consider $\mathfrak{X}$ as a T_{r_0}-variety, the structure map $\mathfrak{X} \to \mathbb{A}^l$ corresponds to the triple $(Y \to \mathbb{A}^l, 0\colon N \to 0, 0\colon 0 \to K(Y)^*)$. We can apply the upgrade functor to the result of Proposition 5.8, which yields the desired result by Definition 2.19. □

Example 5.10 (Toric total spaces)**.** If Δ^- is empty, the coefficient of Q is trivial, and the upgraded divisor may be linearly translated to one supported in the coordinate hyperplanes $P_0', \dots, P_r'$. This is a configuration of invariant divisors on the toric variety Z, which shows that in this case $\mathfrak{X}$ is toric itself.

Example 5.11 (An A_1-singularity, continued)**.** Upgrading the deformation of the quadric cone discussed in Example 5.2 gives

$$\mathcal{F} = \big((\Delta_0, \tfrac{1}{2}) + \delta\big) \otimes \{0\} + \big((\Delta_1, 0) + \delta\big) \otimes \{1\}$$

on $\mathbb{P}^1$, see Figure 5.5.

Example 5.12 ($C\mathbb{P}(1,2,3)$)**.** Take the one-parameter deformation $\mathfrak{X}$ of $X = C\mathbb{P}(1,2,3)$ from Example 3.13. By Theorem 5.5, a $\mathfrak{X}$ is given by the p-divisor $\mathcal{D}$ on $\mathbb{P}^1$ with coefficients

$$\begin{aligned}\Delta_0 &= \tfrac{1}{2}\cdot(-1,-1,1) + \tau\\ \Delta_1 &= \operatorname{conv}\{(0,0,0),(0,1,0)\} + \tau\\ \Delta_\infty &= \operatorname{conv}\{(0,0,0),(2,-1,1)\} + \tau,\end{aligned}$$

where denote $\tau = \delta \cap [r \geq 0]$ as before. Let us verify the theorem directly in this case.

The coefficients Δ_P induce piecewise linear functions on $\tau^\vee$, which are determined by their values in the cross-cut of τ in height one, as displayed in Figure 5.6.

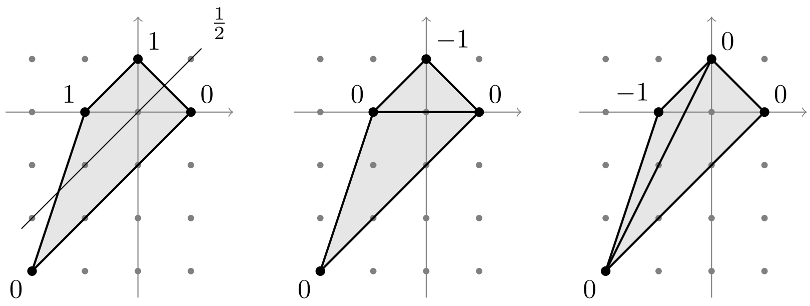
Thus, for $u \in \tau^\vee \cap M$ in height one, we have only two non-zero evaluations $\lfloor \mathcal{D}(u) \rfloor$:

$$\mathcal{D}([0,1,1]) = \{0\} - \{1\} \qquad\qquad \mathcal{D}([-1,0,1]) = \{0\} - \{\infty\}$$

(For the two interior lattice points, $\frac{1}{2}\{0\}$ rounds down to 0.) Since all of these divisors are of degree 0 on $\mathbb{P}^1$, we get one global section each, namely (with z the usual coordinate on $\mathbb{P}^1$) $z^{-1}\cdot\chi^{[-1,0,1]}$, $(1-z^{-1})\cdot\chi^{[0,1,1]}$ and $1\cdot\chi^u$ for the remaining degrees u.

It is easy to check that these generate $A(\mathcal{D})$. For instance, for $u = [-1,0,2]$, we have $\mathcal{D}(u) = \frac{3}{2}\cdot\{0\}$ with sections

$$\begin{aligned}z^{-1}\chi^{[-1,0,2]} &= 1\chi^{[0,0,1]}\cdot z^{-1}\chi^{[-1,0,1]}\\ 1\chi^{[-1,0,2]} &= 1\chi^{[-1,-1,1]}\cdot(1-z^{-1})\chi^{[0,1,1]} + z^{-1}\chi^{[-1,0,2]}.\end{aligned}$$



(a) Δ_0 (b) Δ_1 (c) Δ_∞

Figure 5.6: Piecewise linear functions of $\mathcal{D} = \sum \Delta_P \otimes \{P\}$

Now consider the ring homomorphism

$$\begin{aligned}\varphi\colon \mathbb{C}[M \oplus \mathbb{Z}] &\to \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}(z)\\ \chi^{[u,0]} &\mapsto 1 \cdot \chi^u\\ \chi^{[0,1]} &\mapsto (1 - z^{-1}) \cdot \chi^{[-1,0,1]}.\end{aligned}$$

Then φ identifies the generators of B from Example 3.13 with the generators of $A(\mathcal{D})$. In particular, we have

$$\begin{aligned}\varphi(s) &= \varphi(\chi^{[-1,0,1,0]} - \chi^{[0,0,0,1]}) = z^{-1} \cdot \chi^{[-1,0,1]}\\ \varphi(b) &= \varphi(\chi^{[1,1,0,1]}) = \chi^{[1,1,0]} \cdot (1 - z^{-1})\chi^{[-1,0,1]} = (1 - z^{-1}) \cdot \chi^{[0,1,1]}.\end{aligned}$$

Thus, B and $A(\mathcal{D})$ are isomorphic as M-graded algebras, so $\mathcal{D}$ is a p-divisor for $\mathfrak{X}$ as claimed.

Remark 5.13. The same approach allows for the description of certain mixed deformations, where different multiples of the same primitive degree r_0 occur. Namely, consider an admissible decomposition

$$\Delta^+ = \Delta_0 + k_1\Delta_1 + \cdots + k_l\Delta_l$$

where the multiplicities k_i are part of the data, i.e. $\Delta_0 + 2\Delta_1$ differs from $\Delta_0 + \Delta_1 + \Delta_1$. This determines a T_{r_0}-invariant deformation of X by Theorem 4.8 and Remark 4.11. It has the p-divisor

$$\mathcal{E} = \Delta_0 \otimes D_0 + \Delta_1 \otimes D_1 + \ldots + \Delta_l \otimes D_l + \Delta^- \otimes D_\infty$$

on $Y = \mathbb{P}^1 \times \mathbb{A}^l$, where D_0 and D_∞ are as above and $D_i = V(x_1^{k_i} - y_i x_0^{k_i})$ are adapted to varying multiplicities. T' acts on Y with weights $(1, k_1, \ldots, k_l)$.

If k is the greatest common divisor of the k_i, and $k_i = h_i k$, then we can upgrade $\mathcal{E}$ to a p-divisor on the blowup of the weighted projective space $\mathbb{P}(1, h_1, \ldots, h_l)$ at the origin. With $P_i = V(y_0^{h_i} - y_i)$, this divisor has coefficients

$$\Delta_{P_0} = (\Delta_0, \tfrac{1}{k}) + \widetilde{\sigma} \quad \Delta_{P_i} = (\Delta_i, 0) + \widetilde{\sigma} \quad \Delta_Q = \operatorname{conv}\{(\tfrac{1}{k}\Delta^-, -\tfrac{1}{k}), (0,0)\} + \widetilde{\sigma}.$$

5.3 Multi-parameter deformations in mixed degrees

To end this chapter, I want to turn to a more general class of deformations of toric varieties, namely to so-called deformations in *mixed degrees*. The aim of this section is to illustrate the issues that are involved in some examples.

Suppose $X = \mathrm{TV}(\delta)$ is a toric variety, and we are given two admissible Minkowski decompositions in degrees r and s. By the construction of section 3.3, these define one-parameter deformations $\mathfrak{X}_1 \to S_1 = \mathbb{A}^1$, $\mathfrak{X}_2 \to S_2 = \mathbb{A}^1$ where T acts on S_1 with character r and on S_2 with character s.

Question 5.14. When do the families $\mathfrak{X}_1, \mathfrak{X}_2$ combine to form a (T-equivariant) deformation $\mathfrak{X} \to S_1 \times S_2$? What is a p-divisor for $\mathfrak{X}$ as a T-variety?

Example 5.15. Consider the cone over the rational normal curve from Example 3.11. That example presents two decompositions in degree $r = [0, 1]$; there are two further decompositions in degrees $[-1, 1]$ and $[1, 1]$ whose total spaces have equations

$$\mathrm{rk}\begin{pmatrix} y_0 & y_1' & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \leq 1 \qquad \mathrm{rk}\begin{pmatrix} y_0 & y_1 & y_2 & y_3' \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \leq 1.$$

Together with the similar deformation in degree $[0, 1]$, these combine to form a 3-parameter deformation with toric total space. On the other hand, the second deformation in degree $[0, 1]$ is not compatible with with the others; it corresponds to the other irreducible component of the versal base space.

Consider first the case that r and s are linearly dependent. For $r = s$, two deformations clearly combine if their Minkowski decompositions can be refined. It seems likely that this is also a necessary condition. The situation is similar if r and s lie on the same ray, compare Remark 5.13. On the other hand, if r and s lie on opposite rays, the decompositions are independent, and the deformations combine, compare Example 5.4.

In general, a promising approach to constructing combined deformations is to take the deformation $\mathfrak{X}_1$ in degree r, and deform the total space in degree s, compatibly with the given deformation of X in degree s. This works particularly well in the non-negative toric case, as there the total space is itself a toric variety. In general, the complexity rises by one with each deformation, so the results of chapter 4 don't suffice: For the second step, we need to understand also non-invariant deformations of T-varieties on curves, or equivalently deformations of p-divisors on surfaces.

Even when $\mathfrak{X}$ exists as a toric variety, its p-divisor as a T-variety is something of mystery. Compare the following examples.

Example 5.16. As a simplest case where two deformations in different degree combine, take the cone X over the rational normal curve in degree 3. As a toric variety, this corresponds to the positive quadrant δ in the lattice $N = \mathbb{Z}^2 + \frac{1}{3}(1, 1)\cdot\mathbb{Z}$, illustrated in Figure 5.7. (The same description works for the cone of Example 3.11 with $\frac{1}{4}(1, 1)$.) The dual lattice is $M = \{[a, b] \in \mathbb{Z}^2 \mid a+b \in 3\mathbb{Z}\}$. The equations of X and the total spaces of the deformations occur again as

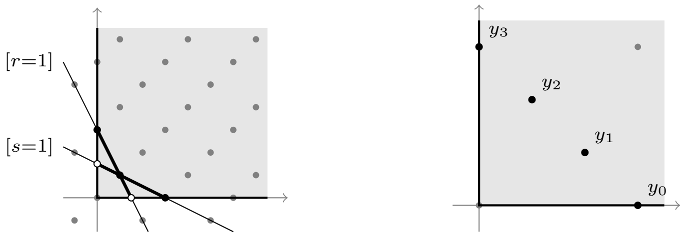


Figure 5.7: Cones for Example 5.16

minors of matrices:

$$\operatorname{rk}\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \end{pmatrix} \le 1 \qquad \operatorname{rk}\begin{pmatrix} y_0 & y_1' & y_2 \\ y_1 & y_2 & y_3 \end{pmatrix} \le 1 \qquad \operatorname{rk}\begin{pmatrix} y_0 & y_1 & y_2' \\ y_1 & y_2 & y_3 \end{pmatrix} \le 1$$

The two one-parameter deformations correspond to Minkowski decompositions in degrees $r = \deg y_1 = [2,1]$ and $s = \deg y_2 = [1,2]$. The total space of the combined deformation $\mathfrak{X}$, which is the versal deformation here, embeds in $\mathbb{A}^6$ with equations

$$\operatorname{rk}\begin{pmatrix} y_0 & y_1' & y_2' \\ y_1 & y_2 & y_3 \end{pmatrix} \le 1.$$

$\mathfrak{X}$ admits an action by $T = \operatorname{Spec}\mathbb{C}[M \oplus \mathbb{Z}^2]$ as follows. The weights in $M \oplus \mathbb{Z}^2$ are the columns of the first matrix; these generate the dual cone to the cone σ in $N \oplus \mathbb{Z}^2$ generated by the columns of the second matrix.

$$\begin{matrix} y_0 & y_1 & y_1' & y_2 & y_2' & y_3 \end{matrix}$$
$$\begin{pmatrix} 3 & 2 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix} \qquad \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{matrix} \begin{pmatrix} -\frac{1}{3} & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 & 1 \end{pmatrix}$$

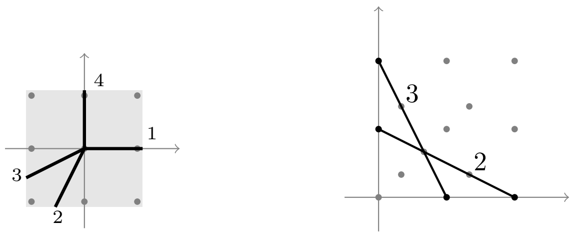
Note that σ intersects $N_{\mathbb{Q}}$ in δ as expected, corresponding to the embedding of the central fibre. Computing a p-divisor $\mathcal{D}$ for the natural T-action on $\mathfrak{X}$ amounts to a toric downgrade. That is, we determine the toric quotient Y by projecting σ to $\mathbb{Z}^2$, yielding the fan Σ generated by the projections of the v_i. The coefficients are the projections to N of the cross-cuts of δ in over these projections. Σ and the compact parts of the coefficients of $\mathcal{D}$ are pictured in Figure 5.8. Ray 5 has coefficient $\frac{1}{3}(1,1)+\sigma$, while the other coefficients all have two vertices.

Example 5.17. Given that the base Y in Example 5.16 is somewhat mysterious, maybe at least it just depends on the deformation degrees? To show that this is not the case, take X as above, and consider the *trivial deformation* in degrees $r = [2,1]$ and $s = [1,2]$. Namely, take $\mathfrak{X}' = X \times \mathbb{A}^2$, where T acts on the coordinates of $\mathbb{A}^2$ with weights r and s. The analogous matrices to the previous

(a) The fan of Y

(b) Coefficients of the p-divisor

Figure 5.8: A p-divisor for the versal deformation of $\frac{1}{3}(1,1)$



(a) The fan of Y'

(b) Coefficients of the p-divisor

Figure 5.9: A p-divisor for a trivial deformation of $\frac{1}{3}(1,1)$

example are

$$\begin{matrix} y_0 & y_1 & y_1' & y_2 & y_2' & y_3 \end{matrix} \qquad \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix}$$
$$\begin{pmatrix} 3 & 2 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & -2 & 0 \\ 0 & -2 & -1 & 1 \end{pmatrix}$$

The fan of the quotient Y' as well as the compact parts of the non-trivial coefficients are pictured in Figure 5.9. Rays 1 and 4 have trivial coefficients.

This construction generalizes: Take a cone σ in N, and two degrees r and s in M. Then the trivial two-parameter family in these degrees is given by a p-divisor on a toric surface Y. If $v_1, \ldots, v_k$ are the rays of σ, then the fan of Y has rays through $(1,0)$, $(0,1)$ and $w_i = -(r(v_i), s(v_i))$. The coefficients of the first two divisors are trivial, while the coefficient of a ray $w_i = -(a,b)$ is $\sigma \cap [r \geq a, s \geq b]$. In particular, v_i is a vertex of the coefficient associated with w_i.

Chapter 6

First-order deformations

Finally, let us return to the vector space T^1 in the equivariant setting. First, section 6.1 provides a conceptual proof of Proposition 3.2. We show that for any T-variety X, the module T^1_X is naturally graded by the character lattice M of T. We approach this by defining the notion of homogeneity for ring extensions, which ties in naturally with the definition of the category $\mathrm{Def}^T(X)$ of equivariant deformations of X, which was defined in section 3.2.

Then, we turn to first-order deformations of toric varieties. Section 6.2 introduces the notion of a first-order deformation of a p-divisor, which includes a study of Cartier divisors on the non-reduced scheme $\mathbb{A}^1 \times \operatorname{Spec} \mathbb{C}[\varepsilon]$. We will see by example that this allows expressing obstructed deformations of toric varieties. Then, section 6.3 provides a translation of the notion of a first-order deformation of a toric variety into terms of convex geometry, by defining first-order deformations of polyhedra. An analysis of generators and relations of such deformations as well as the subspace of trivial deformations leads to a new approach to computing toric $T^1(-r)$.

6.1 Grading T^1

The aim of this section is to prove that for a finitely generated M-graded A-algebra R, the module of A-extensions of R by a graded module I is graded. The treatment closely follows section 1.1.2 of Sernesi's book [Ser06]. Recall that an extension (R', φ) of R by I over A is an exact sequence

$$0 \longrightarrow I \longrightarrow R' \xrightarrow{\ \varphi\ } R \longrightarrow 0$$

of A-modules where R' is an A-algebra, φ is a homomorphism of A-algebras, and the kernel I is an ideal that satisfies $I^2 = 0$. Given an A-algebra R and an R-module I, the R-module $\mathrm{Ex}_A(R, I)$ consists of isomorphism classes of extensions of R by I. Then since the spectrum of a $\mathbb{C}$-extension of a ring R by itself is a deformation over $\mathbb{C}[\varepsilon]$, the module T^1_X associated with a variety $X = \operatorname{Spec} R$ over $\mathbb{C}$ is $\mathrm{Ex}_{\mathbb{C}}(R, R)$.

Suppose now that $R = \bigoplus_{u \in M} R_u$ is an M-graded A-algebra, and that I is a graded R-module. Recall that for graded R-modules I and J, a module homomorphism $f\colon I \to J$ which satisfies

$$f(I_u) \subset J_{u+v} \qquad \text{for all } u \in M$$

is called homogeneous of degree v. Thus $\operatorname{Hom}_R(I, J)$ is a graded R-module.

Definition 6.1. A *homogeneous extension* of R by I of degree $u \in M$ is an extension

$$0 \longrightarrow I \xrightarrow{\ \alpha\ } R' \xrightarrow{\ \varphi\ } R \longrightarrow 0$$

such that R' is also M-graded, φ is a graded homorphism of A-algebras, and α is homogeneous of degree $-u$.

We define $\operatorname{Ex}_A(R, I)_u$ to be the set of isomorphism classes of homogeneous extensions of degree u.

Lemma 6.2. *If (R', φ) is a homogeneous extension of R by I of degree u, and $\lambda\colon I \to J$ is homogeneous of degree v, then $\lambda_*(R', \varphi)$ is homogeneous of degree $u + v$.*

Proof. Following Sernesi [Ser06, section 1.1.2], the push-forward is the bottom row in the following commutative diagram, where the homogeneous R-module homomorphisms are shifted to degree 0 by shifting the modules I and J.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I(u) & \xrightarrow{\ \alpha\ } & R' & \xrightarrow{\ \varphi\ } & R & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda} & & \downarrow & & \| & & \\
0 & \longrightarrow & J(u+v) & \longrightarrow & R' \coprod_I J & \longrightarrow & R & \longrightarrow & 0
\end{array}$$

Here $R' \coprod_I J$ is defined to be the quotient of $S = R' \oplus J$ by the ideal $L = ((\alpha(i), -\lambda(i)) \mid i \in I)$, where multiplication on S is defined by $(r, i) \cdot (s, j) = (rs, rj + si)$. There are multiple ways to extend the M-grading on R' to S; we set $S_a = R_a \oplus J_{u+v+a}$. Then $R' \to S$ is a graded homomorphism of A-algebras and L is a homogeneous ideal, so S/L is graded, and $J \to S/L$ is homogeneous of degree $-(u + v)$ as claimed. □

Two extensions (R', φ) and (R'', ψ) induce an extension $(R' \times_R R'', \zeta)$ of R by $I \oplus I$. The push-forward of this extension along the addition homomorphism $\delta\colon I \oplus I \to I$ is used to define addition on $\operatorname{Ex}_A(R, I)$.

Proposition 6.3. *For $r \in R_u$ and a homogeneous extension (R', φ) of degree v, the extension $r_*(R', \varphi)$ is homogeneous of degree $u + v$. If (R'', ψ) is another homogeneous extension of degree u, then their sum $\delta_*(R' \times_R R'', \zeta)$ is also homogeneous of degree u.*

Proof. The first claim follows immediately from Lemma 6.2 applied to the homogeneous multiplication homomorphism $r\colon I \to I$. For the second claim, note that

$$R' \times_R R'' = \{(r, s) \in R' \times R'' \mid \varphi(r) = \psi(s)\} = \bigoplus_{u \in M} R'_u \times_{R_u} R''_u$$

is graded, and the product homomorphism $I \oplus I \to R' \times_R R''$ is homogeneous of degree $-u$. Since δ is homogeneous of degree 0, this extension pushes forward to an extension of degree u by Lemma 6.2. □

Corollary 6.4. *$\mathrm{Ex}_A(R, I)_u$ is an A-submodule of $\mathrm{Ex}_A(R, I)$. The direct sum $\bigoplus_{u\in M} \mathrm{Ex}_A(R, I)_u$ is a submodule of $\mathrm{Ex}_A(R, I)$ that is also a graded R-module.*

Proposition 6.5. *Suppose R is finitely generated as an A-algebra. Then*

$$\mathrm{Ex}_A(R, I) = \bigoplus_{u\in M} \mathrm{Ex}_A(R, I)_u$$

is a graded R-module.

Proof. The claim amounts to showing that homogeneous extensions generate $\mathrm{Ex}_A(R, I)$. By choosing a homogeneous set of generators (f_i) of R, we get a surjection $P = A[x_1, \dots, x_r] \to R$ that is homogeneous if we set $\deg x_i = \deg f_i \in M$. Then the kernel J is M-graded, and so is the module $\mathrm{Hom}_R(J/J^2, I)$.

Now consider the following extension η of R by J/J^2.

$$0 \longrightarrow J/J^2 \longrightarrow P/J^2 \longrightarrow R \longrightarrow 0$$

This is homogeneous of degree 0, so by Lemma 6.2, its push-forward $\lambda_*(\eta)$ is homogeneous of degree u if λ is homogeneous of degree u.

But $\lambda \mapsto \lambda_*(\eta)$ is surjective, compare [Ser06, 1.1.7]. □

6.2 P-divisors for first-order deformations

As discussed in the previous section and section 3.2, the elements of the vector space $T^1(-r)$ are equivalence classes of equivariant deformations over $\mathbb{C}[\varepsilon]$, where ε is of degree r. In this section, I will present a geometric approach to $T^1(-r)$ for toric varieties. While the toric T^1 is already well understood by Altmann's work [Alt94], this description may lend itself better to a generalization to T-varieties of higher complexity.

The approach involves a generalization of p-divisors to a non-reduced setting, which allows a description of first-order deformations of toric varieties in positive degrees.

The following section 6.3 is dedicated to applying this description in order to compute the dimension of $T^1(-r)$, by characterizing such deformations in terms of convex geometry, and by determining what divisors yield trivial deformations.

Definition 6.6. A *generalized polyhedral divisor* on the scheme Y with tail cone σ in N is a concave map

$$\mathcal{E} \colon \sigma^\vee \cap M \to \mathrm{CDiv}(Y),$$

where for two Cartier divisors $D, D' \in \mathrm{CDiv}(Y)$, we define

$$D \le D' \iff H^0(Y, D) \subseteq H^0(Y, D')$$

as subsets of the global sections of the total quotient sheaf of Y. In addition, we require that $\mathcal{E}(0)$ is the zero divisor.

Vanishing of $\mathcal{E}(0)$ ensures that $\mathcal{A}(\mathcal{E})_0 = \mathcal{O}_Y(\mathcal{E}(0)) = \mathcal{O}_Y$, so $\mathcal{A}(\mathcal{E})$ is an $\mathcal{O}_Y$-algebra, and we can define $\mathrm{TV}(\mathcal{E}) = \mathrm{Spec} \bigoplus H^0(Y, \mathcal{E}(u))$ with a T-action as in section 1.4.

On its own, this definition is not too useful. For example, we have no way to ensure finite generation. Things get better when we relate $\mathcal{E}$ with a p-divisor on $Z = V(\varepsilon)$, see Definition 6.8. First, some analysis of Cartier divisors on Y is in order.

We denote by $\mathrm{CDiv}_0(Z) := \mathbb{Z}\cdot\{0\} \subset \mathrm{CDiv}(Z)$ the Cartier divisors supported in $\{0\}$ on $Z = \mathbb{A}^1$. These correspond to the functions x^k under the identification $\mathrm{CDiv}(Z) = \mathbb{C}(x)^*/\mathbb{C}^* \cong \{f \in \mathbb{C}(x)^* \mid f \text{ monic }\}$. The divisor associated with such f is written $\mathrm{div}(f)$.

Cartier divisors on Y correspond to elements of $\mathcal{K}(Y)^*/\mathbb{C}^*$, so we can express them as

$$\mathrm{CDiv}(Y) \cong \{f + f'\varepsilon \mid f \in \mathbb{C}(x)^* \text{ monic}, f' \in \mathbb{C}(x)\}.$$

We have a well-defined pull-back along the embedding $i\colon Z \to Y$ with $i^*(\mathrm{div}(f + f'\varepsilon)) = \mathrm{div}(f) \in \mathrm{CDiv}(Z)$. Thus, all Cartier divisors on Y are *relative* with respect to the structure map $Y \to \operatorname{Spec}\mathbb{C}[\varepsilon]$. We will write $\mathrm{CDiv}_0(Y) := (i^*)^{-1}(\mathrm{CDiv}_0(Z))$ for the divisors whose restriction to Z is supported in $\{0\}$.

Now recall that $Z = \mathbb{A}^1$ is a toric variety, with $\mathbb{C}^*$-action given by $\deg x = 1 \in \mathbb{Z}$. Given an integer $k \geq 1$, this action extends to an action on Y by setting $\deg \varepsilon = k$. The invariant Cartier divisors with respect to this action will be denoted by $\mathrm{CDiv}^k(Y)$. These are of the form $\mathrm{div}(x^l + \lambda x^{l-k}\varepsilon)$ for $\lambda \in \mathbb{C}$. In particular, we have $\mathrm{CDiv}^k(Y) \subset \mathrm{CDiv}_0(Y)$ for all k.

Write $D_0 := \mathrm{div}(x)$ and $D^k_\lambda := \mathrm{div}(x^k - \lambda\varepsilon)$. Then D_0 is in $\mathrm{CDiv}^k(Y)$ for any k, and D^k_λ is in $\mathrm{CDiv}^k(Y)$.

Proposition 6.7. *The group* $\mathrm{CDiv}^k(Y)$ *is isomorphic to the additive group* $\mathbb{C} \times \mathbb{Z}$ *under the map*

$$(\lambda, l) \mapsto D^k(\lambda, l) := D^k_\lambda + (l - k)D_0.$$

The induced order on $\mathbb{C} \times \mathbb{Z}$ *is such that*

$$(\lambda, l) \leq (\lambda', l') \iff l + k \leq l' \textit{ or } (\lambda = \lambda' \textit{ and } l \leq l').$$

This order will be referred to as the *k-order* on $\mathbb{C} \times \mathbb{Z}$. In the case $k = 1$, it simplifies as

$$(\lambda, l) \leq (\lambda', l') \iff l < l' \text{ or } (\lambda, l) = (\lambda', l').$$

Proof. $D^k(\lambda, l) + D^k(\lambda', l')$ is the divisor associated with

$$\begin{aligned}\left(x^{l-k} \cdot (x^k - \lambda\varepsilon)\right) \cdot \left(x^{l'-k} \cdot (x^k - \lambda'\varepsilon)\right) &= x^{(l+l')-2k} \cdot (x^{2k} - (\lambda + \lambda')x^k\varepsilon + 0)\\ &= x^{(l+l')-k} \cdot (x^k - (\lambda + \lambda')\varepsilon),\end{aligned}$$

so the map is a group homomorphism. From the characterization of elements of $\mathrm{CDiv}^k(Y)$ above, it is clearly bijective.

The claim on the order follows by checking when $D^k(\lambda, l)$ is effective:

$$\begin{aligned}D^k(\lambda, l) \geq 0 &\iff x^{l-k} \cdot (x^k - \lambda\varepsilon) \text{ regular}\\ &\iff l \geq k \text{ or } \lambda = 0 \text{ and } l - k + k \geq 0.\end{aligned}$$ ☐

Let us turn now to first oder deformations of toric varieties. The setup is as follows. Let $X = \mathrm{TV}(\delta)$ be a toric variety, given by the p-divisor $\mathcal{D} = \Delta \otimes \{0\}$ on $Z = \mathbb{A}^1$. We have $\delta_r = \delta \cap [r = 1]$ and $\delta_{r_0} = k\delta_r = \Delta \times \{1\}$ the positive degree $r = kr_0 = [0,k]$, and we have $\operatorname{tail}(\Delta) = \sigma = \delta \cap [r=0]$. Then we aim to describe first order deformations in degree $-r$ by generalized p-divisors on $Y = \mathbb{A}^1 \times \operatorname{Spec} \mathbb{C}[\varepsilon]$, in analogy with the approach of chapter 4.

Definition 6.8. A *homogeneous first-order deformation of multiplicity* k of $\mathcal{D}$ is a generalized polyhedral divisor

$$\mathcal{E}\colon \sigma^\vee \cap M_{r_0} \to \operatorname{CDiv}^k(Y)$$

such that $\mathcal{E}$ restricts to $\mathcal{D}$ on the central fibre $Z = V(\varepsilon) \subset Y$.

Let $\mathcal{E}$ be such a deformation, and take some degree $u \in M_{r_0}$. If $\lfloor \mathcal{D}(u) \rfloor = l \cdot \{0\}$, it follows that $\mathcal{E}(u)$ is of the form $D^k(\lambda, l)$.

Proposition 6.9. *For such a deformation* $\mathcal{E}$*,* $\operatorname{TV}(\mathcal{E})$ *is a first order deformation of* $\operatorname{TV}(\mathcal{D})$*.*

Proof. First note that $i^*(D^k(\lambda,l)) = l \cdot \{0\}$, so if we have $\lfloor \mathcal{D}(u) \rfloor = l \cdot \{0\}$ for $u \in M_{r_0}$, then $\mathcal{E}(u)$ is necessarily of the form $D^k(\lambda, l)$. Now consider the sequence

$$0 \longrightarrow \varepsilon \cdot H^0(\mathcal{E}(u)) \longrightarrow H^0(\mathcal{E}(u)) \longrightarrow H^0(\mathcal{D}(u)) \longrightarrow 0.$$

$H^0(\mathcal{E}(u))$ is generated over $\mathbb{C}[x,\varepsilon]$ by $f = x^{-l-k} \cdot (x^k + \lambda\varepsilon)$, which projects to the generator x^{-l} of $H^0(\mathcal{D}(u))$. The kernel of this map is generated by $\varepsilon f = \varepsilon x^{-l}$, so it is equal to $\varepsilon H^0(\mathcal{D}(u))$. Summing up these exact sequences yields an extension

$$0 \longrightarrow A(\mathcal{D}) \longrightarrow A(\mathcal{E}) \longrightarrow A(\mathcal{D}) \longrightarrow 0.$$

of $A(\mathcal{D})$ by itself. □

By the results of chapters 4 and 5, admissible decompositions of $\Delta = \Delta_0 + \Delta_1$ yield unobstructed deformations of $\mathcal{D}$ with p-divisor $\mathcal{E} = \Delta_0 \otimes \operatorname{div}(x) + \Delta_1 \otimes \operatorname{div}(y-x)$ on $\mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x,y]$. Modulo y^2, these induce the corresponding first order deformations of $\mathcal{D}$: If $\mathcal{D}(u) = a_0 \operatorname{div}(x) + a_1 \operatorname{div}(y-x)$, then the associated generalized divisor maps u to $\lfloor a_0 \rfloor \operatorname{div}(x) + \lfloor a_1 \rfloor \operatorname{div}(\varepsilon - x)$ on $Y = \mathbb{A}^1 \times \operatorname{Spec} \mathbb{C}[\varepsilon]$.

Example 6.10. Recall the two unobstructed one-parameter deformations of the cone over the rational normal curve of degree 4 from Example 3.11, given by p-divisors

$$\mathcal{D}_\alpha = [-\tfrac{1}{2}, 0] \otimes D_0 + [0, \tfrac{1}{2}] \otimes D_1$$
$$\mathcal{D}_\beta = \{-\tfrac{1}{2}\} \otimes D_0 + [0,1] \otimes D_1$$

on $\mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x,y]$ with $D_0 = V(x)$, $D_1 = V(x-y)$. These two deformations span the two-dimensional vector space $T^1(-r)$, but their proper linear combinations are obstructed.

The two unobstructed deformations as well as their sum as obtained by perturbing the equations are listed in Table 6.1; the columns are just the vectors A of Example 3.1. Generators for their coordinate rings as subalgebras of $\mathbb{C}(x)[\varepsilon][M_r]$ are listed in Table 6.2 next to the monomials generating the coordinate ring of X, where $t = \chi^1 \in \mathbb{C}[M_r]$. For α and β, these may be read

f	α	β	$\alpha+\beta$
$y_0y_2 - y_1^2$	0	$+y_0$	$+y_0$
$y_1y_3 - y_2^2$	$-y_2$	$-y_2$	$-2y_2$
$y_2y_4 - y_3^2$	$+y_4$	$+y_4$	$+2y_4$
$y_0y_3 - y_1y_2$	$-y_1$	0	$-y_1$
$y_1y_4 - y_2y_3$	0	0	0
$y_0y_4 - y_1y_3$	0	$+y_2$	$+y_2$
$y_0y_4 - y_2^2$	$-y_2$	0	$-y_2$

Table 6.1: Three first-order deformations

	α	β	$\alpha+\beta$
$y_0 = x\cdot t^{-2}$	$(x-\varepsilon)\cdot t^{-2}$	$(x-2\varepsilon)\cdot t^{-2}$	$(x-3\varepsilon)\cdot t^{-2}$
$y_1 = x\cdot t^{-1}$	$(x-\varepsilon)\cdot t^{-1}$	$(x-\varepsilon)\cdot t^{-1}$	$(x-2\varepsilon)\cdot t^{-1}$
$y_2 = x\cdot t^{0}$	$(x-\varepsilon)\cdot t^{0}, x\cdot t^0$	$(x-\varepsilon)\cdot t^{0}, x\cdot t^0$	$(x-2\varepsilon)\cdot t^{0}, (x-\varepsilon)\cdot t^0$
$y_3 = x\cdot t^{1}$	$x\cdot t^1$	$x\cdot t^1$	$x\cdot t^1$
$y_4 = x\cdot t^{2}$	$x\cdot t^2$	$x\cdot t^2$	$x\cdot t^2$

Table 6.2: Generators for the coordinate rings (Example 6.10)

off from the polyhedral divisors. For example $\lfloor \mathcal{D}_\alpha(-2)\rfloor = -D_1$ has the section $x-\varepsilon$, yielding $(x-\varepsilon)\cdot t^{-2}$, while $\lfloor \mathcal{D}_\beta(-2)\rfloor = D_0 - 2D_1$ has the section $\frac{(x-\varepsilon)^2}{x} = x-\varepsilon$. The generators listed in degree 0 are y_2 and $y_2' = y_2+\varepsilon$; we might as well take x and ε.

It is straightforward to show that the generators listed for $\alpha+\beta$ do indeed yield the right ring. For example, we can check that

$$y_0y_3 - y_1y_2 = ((x-3\varepsilon)x - (x-2\varepsilon)^2)\cdot t^{-1} = \varepsilon x\cdot t^{-1} = y_1\varepsilon$$
$$y_2y_4 - y_3^2 = ((x-2\varepsilon)x - x^2)\cdot t^{1} = -2\varepsilon x\cdot t^{2} = -2y_4\varepsilon.$$

Now from these generators, we can read off generalized p-divisors. For example, $x-3\varepsilon$ gives us the divisor $-D_3 = -D^1(3,1) = -\operatorname{div}(x-3\varepsilon)$. The values for $-2\le u\le 2$ are listed in Table 6.3. See Proposition 6.20 below for why these are enough to determine the divisors.

Remark 6.11. The homogeneous Cartier divisors $\operatorname{CDiv}^k(Y) = \mathbb{C}\times\mathbb{Z}$ admits a $\mathbb{C}$-vector space structure through the first factor, that is, the $\mathbb{C}$-algebra structure of $\mathbb{C}[a^{\pm 1}] \cong \mathbb{C}\times\mathbb{Z}$. Since the degree $l\in\mathbb{Z}$ is not affected, sums and scalar multiples of first order deformations of $\mathcal{D}$ remain such, hence we have an induced $\mathbb{C}$-vector space structure on the set of homogeneous first order deformations of a given $\mathcal{D}$.

It can be shown that this agrees with the operations on T^1 as described for example in section 6.1.

Example 6.12. The sum $\alpha+\beta$ of Example 6.10 agrees with this description of addition in T^1: For example, $\mathcal{E}_{\alpha+\beta}(-2) = D(-3,-1) = D((-1)+(-2),-1)$ with $\mathcal{E}_\alpha(-2) = D(-1,-1)$ and $\mathcal{E}_\beta(-2) = D(-2,-1)$.

degree	α	β	$\alpha+\beta$
-2	$-D_1$	$-D_2 = D_0 - 2D_1$	$-D_3$
-1	$-D_1$	$-D_1$	$-D_2$
0	0	0	0
1	$-D_0$	$-D_0$	$-D_0$
2	$-D_0$	$-D_0$	$-D_0$

Table 6.3: Generalized divisors for three first-order deformations

6.3 First-order deformations of polyhedra

Consider $\mathcal{D} = \Delta \otimes \{0\}$ as in the previous section. If the description of first order deformations of $\mathrm{TV}(\mathcal{D})$ by generalized p-divisors is to yield a better understanding of T^1, we need some idea of how many deformations exist, and we need to know which deformations are trivial. We will start by investigating trivial deformations, which come in two sets.

First, equivariant automorphisms of the trivial deformation

$$\pi\colon Y = \mathbb{A}^1 \times \operatorname{Spec} \mathbb{C}[\varepsilon] \to \operatorname{Spec} \mathbb{C}[\varepsilon]$$

induce isomorphic deformations. We denote by $\mathrm{Aut}(\pi, k)$ the group of such automorphisms, where $\mathbb{C}^*$ acts on Y with weights $(1, k)$.

Lemma 6.13. $\mathrm{Aut}(\pi, 1)$ *consists of the maps*

$$\begin{aligned} x &\mapsto x - \mu\varepsilon \\ \varepsilon &\mapsto \varepsilon \end{aligned}$$

for $\mu \in \mathbb{C}$, so $\mathrm{Aut}(\pi, 1) \cong \mathbb{C}$*. For $k > 1$,* $\mathrm{Aut}(\pi, k)$ *is trivial.*

Proof. We are looking for isomorphisms $f\colon Y \to Y$ with $\pi \circ f = \pi$ and $f \circ i = i$, where $i\colon \mathbb{A}^1 \to Y$ is the embedding of the special fibre. This implies $f^*x - x \in (\varepsilon)$, and $f^*\varepsilon = \varepsilon$. Equivariance means that f^* is graded, and since x is homogeneous of degree 1, the same is required of f^*x. □

The automorphism $x \mapsto x - \mu\varepsilon$ maps $D^1(\lambda, l)$ to $D^1(\lambda + l\mu, l)$. In particular, setting $\mathcal{E}(u) = D^1(\mu l, l)$ where $\lfloor \mathcal{D}(u) \rfloor = l \cdot \{0\}$ yields a trivial deformation $\mathrm{TV}(\mathcal{E})$.

Secondly, linear equivalence of generalized p-divisors yields isomorphic total spaces. Linear equivalence works the same here as for normal p-divisors, in that a function $\mathfrak{f} \in N \otimes \mathbb{C}(Y)^*$ determines an isomorphism $(\mathrm{id}_Y, \mathrm{id}_N, \mathfrak{f})\colon \mathcal{E} + \mathrm{div}(\mathfrak{f}) \to \mathcal{E}$. Here, compatibility with the embedding of X restricts to $\mathfrak{f} \in N \otimes (\mathbb{C}(x)[\varepsilon]^*)_0$, where the functions on Y are homogeneous of degree 0. This just means that for a linear map $\varphi\colon M \to \mathbb{C}$ and a deformation $\mathcal{E}$ of $\mathcal{D}$ of multiplicity k, the divisor $\mathcal{E}'(u) = \mathcal{E}(u) + D^k(\varphi(u), 0)$ defines an isomorphic deformation.

At this point, it is unclear how large the space of first-order deformations of $\mathcal{D}$ can get. To address this issue, consider the following alternative description of homogeneous first order deformations that does away with the geometric content.

Let $\Delta \subset N_{\mathbb{Q}}$ be a σ-polyhedron. This determines a concave function

$$\begin{aligned} \Delta\colon \sigma^\vee \cap M &\to \mathbb{Z} \\ u &\mapsto \Delta(u) := \lfloor \min\langle \Delta, u\rangle \rfloor. \end{aligned}$$

Note that the function $u \mapsto \Delta(u)$ determines Δ.

Lemma 6.14. *Let $\delta = \operatorname{pos}\{\Delta \times \{1\}, \operatorname{tail}(\Delta) \times \{0\}\}$ be the cone over Δ. Then for $[u, a] \in M \oplus \mathbb{Z}$, we have*

$$[u, a] \in \delta^\vee \iff a \geq -\Delta(u). \qquad \square$$

Definition 6.15. A *first-order deformation of multiplicity* k of Δ is a function $F\colon \sigma^\vee \cap M \to \mathbb{C} \times \mathbb{Z}$ such that $\mathrm{pr}_2 \circ F = \Delta$ and such that F is concave with respect to the k-order on $\mathbb{C} \times \mathbb{Z}$. In addition, we require that $F(0) = (0, 0)$.

For each multiplicity k, the first-order deformations of Δ form a $\mathbb{C}$-vector space $\operatorname{Def}^k(\Delta)$, where the vector space structure is induced through the first factor of $\mathbb{C} \times \mathbb{Z}$ as before. Given a deformation F, we write $F' = \mathrm{pr}_1 \circ F$ for the first component, so that $F = (F', \Delta)$.

By the earlier discussion, some of these deformations should be thought of as trivial. Thus, we denote by $\operatorname{Triv}^k(\Delta)$ the subset of $\operatorname{Inf}^k(\Delta)$ generated by deformations $F = (F', \Delta)$ with

- $F'(u) = \Delta(u)$ (only in case $k = 1$)
- $F'(u) = \psi(u)$ where $\psi\colon M \to \mathbb{Q}$ is linear.

Then we call the vector space of isomorphism classes of first-order deformations

$$T^1(\Delta, k) := \operatorname{Inf}^k(\Delta) / \operatorname{Triv}^k(\Delta).$$

Conjecture 6.16. Let $X = \mathrm{TV}(\delta) = \mathrm{TV}(\Delta \otimes \{0\})$ be the corresponding toric variety. Then

$$T^1_X(-[0, k]) = T^1(\Delta, k).$$

As one extra piece of notation, we write F_Δ for the map $u \mapsto (\Delta(u), \Delta(u))$, so that F_Δ is a trivial deformation of Δ in multiplicity 1.

Lemma 6.17. *F is concave if and only if, for all pairs u, v in $\sigma^\vee \cap M$ with $\Delta(u + v) < \Delta(u) + \Delta(v) + k$, we have $F'(u) + F'(v) = F'(u + v)$.*

Proof. This is immediate from the definition of the k-order on $\mathbb{C} \times \mathbb{Z}$: If $\Delta(u + v) \geq \Delta(u) + \Delta(v) + k$, then concavity of F for u and v is free, while in the other case it implies equality for F'. $\square$

Note that for $k = 1$, the condition is equality: $\Delta(u + v) = \Delta(u) + \Delta(v)$.

This condition has a useful translation into generators of the semigroup $S = \delta^\vee \cap M \oplus \mathbb{Z}$. For $w = [u, a] \in M \oplus \mathbb{Z}$, we call $h(w) = a + \Delta(u)$ the height of w, so S consists of the elements of non-negative height.

Lemma 6.18. *Let*

$$w = \sum b_i w_i = \sum b'_j w'_j$$

be a relation such that all for all i and j, the coefficients b_i and b'_j are positive integers, and we have $h(w_i) = 0 = h(w'_j)$. Then if $h(w) < k$, this induces a relation

$$\sum b_i F'(\mathrm{pr}_1(w_i)) = \sum b'_j F'(\mathrm{pr}_1(w'_j))$$

among the values of any multiplicity-k deformation F' of Δ.

In particular, suppose that δ (and Δ) are of full dimension, and let E be the *Hilbert basis* of S, i.e., the set of all irreducible elements in the semigroup.

Lemma 6.19. *Apart from $[0, 1]$, which may or may not be an element of E, all Hilbert basis elements are of height 0. In particular, the projection $\mathrm{pr}_1 : E \to M$ is injective.*

Proof. Let $[u, a] \in \delta^\vee \cap M$ with $u \neq 0$. Then if $a > -\Delta(u)$, $[u, a] = [u, a - 1] + [0, 1]$ is reducible, and thus not part of E. □

Let $\overline{E} := \pi(E) \setminus \{0\}$ be the non-zero projections of Hilbert basis elements.

Lemma 6.20. *A deformation F of Δ is determined by its values on $\overline{E}$.*

Proof. Consider $u \in \sigma^\vee \cap M \setminus \mathrm{pr}_1(E)$, and let $w = [u, -\Delta(u)]$ be the corresponding minimal element of S. Then w is a positive integral combination of Hilbert basis elements that doesn't involve $[0, 1]$. By Lemma 6.18, this implies that $F'(u)$ is determined by the values of F' at the projections of the Hilbert basis elements involved in the relation. □

Note that there may still be relations among Hilbert basis elements, as the examples below show. But at least, we have limited the dimension of the vector space of deformations of Δ to $\#\overline{E}$.

Example 6.21. Let $\delta = \mathrm{pos}\{(1, 1), (-1, 1)\}$, so $\delta^\vee = \mathrm{pos}\{[-1, 1], [1, 1]\}$ has Hilbert basis $E = \{[u, 1] \mid -1 \leq u \leq 1\}$. There is one relation $[-1, 1] + [1, 1] = 2 \cdot [0, 1]$, so $\mathrm{TV}(\delta) = V(y_0 y_2 - y_1^2) \subset \mathbb{A}^3$ is the quadric cone.

The p-divisor corresponding to $r_0 = [0, 1]$ is $\mathcal{D} = \Delta \times \{0\}$ with $\Delta = [-1, 1]$. This has the piecewise linear rounded-down evaluation function $u \mapsto -|u|$. We have $\overline{E} = \{-1, 1\}$, so

$$\begin{aligned} \varphi\colon \mathrm{Def}^k(\Delta) &\hookrightarrow \mathbb{C}^2 \\ F &\mapsto (F'(-1), F'(1)) \end{aligned}$$

embeds $\mathrm{Def}^k(\Delta)$ in $\mathbb{C}^2$ for every $k > 0$.

The height of the only relation is $h([0, 2]) = 2$, so for $k \geq 3$, it follows from Lemma 6.18 that $F'(-1) + F'(1) = 0$, so $\mathrm{Def}^k(\Delta)$ is one-dimensional. For $k = 1, 2$, the embedding φ is an isomorphism.

For any k, the subspace $\mathrm{Triv}^k(\Delta)$ contains linear maps F', so its image in $\mathbb{C}^2$ contains $\mathbb{C} \cdot (-1, 1)$. For $k \geq 2$ this is all, while for $k = 1$, $\mathbb{C} \cdot (-1, -1)$ is also trivial. Thus we see that $T^1(\Delta, k)$ is zero unless $k = 2$, in which case it is one-dimensional.

The example is illustrated in Figure 6.1(a). The Hilbert basis elements and the degree of the relation are emphasized.

Example 6.22. We continue with Example 6.21. We just saw that for $k \le 2$, $\mathrm{Inf}^k(\Delta)$ is two-dimensional, generated by $(1, 0) \in \mathbb{C}^2$ modulo linear functions.

For $k = 1$, this corresponds to the deformation $\mathcal{E}$ of $\mathcal{D}$ with $\mathcal{E}(-1) = D^1(1, -1)$ and $\mathcal{E}(1) = D^1(0, -1)$, whose global sections are generated by $x + \varepsilon$ and x, respectively. Thus the coordinate ring of the deformation is

$$A(\mathcal{E}) = \mathbb{C}[(x + \varepsilon) \cdot t^{-1}, x, \varepsilon, x \cdot t^1] \cong \mathbb{C}[y_0, y_1, y_1 + \varepsilon, y_2]/(y_0 y_2 - y_1(y_1 + \varepsilon)].$$

This deformation of the defining equation arises by deforming matrix entries as in Example 3.11: The central fibre is given by $\mathrm{rk}\left(\begin{smallmatrix} y_0 & y_1 \\ y_1 & y_2 \end{smallmatrix}\right) \le 1$, the deformation by $\mathrm{rk}\left(\begin{smallmatrix} y_1 & y_1' \\ y_1 & y_2 \end{smallmatrix}\right) \le 1$.

The admissible Minkowski decomposition $\Delta = \Delta_1 + \Delta_2 = [-1, 0] + [0, 1]$ induces a decomposition $\Delta(u) = \Delta_1(u) + \Delta_2(u)$. Then for the deformation above, we have $F = 0 \cdot F_{[-1,0]} \oplus (-1) \cdot F_{[0,1]}$, where $\oplus$ denotes addition in both factors of $\mathbb{C} \times \mathbb{Z}$, corresponding to addition of divisors, while scalar multiplication is the vector space structure through the first factor.

To see that this is indeed a trivial first-order deformation, note that $y_0 y_2 - y_1(y_1 + \varepsilon) = y_0 y_2 - (y_1 + \frac{1}{2}\varepsilon)^2$. Compare Example 5.2, where the deformation over $\mathbb{A}^1$ corresponding to this decomposition was shown to be the pull-back along $s \mapsto s^2$.

For $k = 2$, the deformation is $\mathcal{F}$ with $\mathcal{F}(-1) = D^2(1, -1)$ and $\mathcal{F}(1) = D^2(0, -1)$, which gives generators

$$y_0 = x^{-1}(x^2 + \varepsilon) \cdot t^{-1} \qquad y_1 = x, \varepsilon \qquad y_2 = x \cdot t^1$$

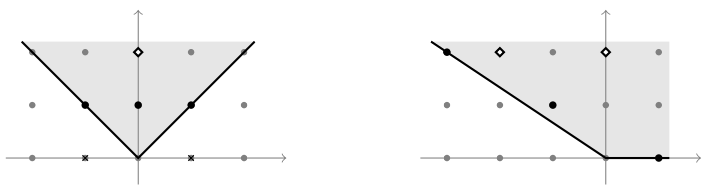
satisfying $y_0 y_2 - y_1^2 = \varepsilon$. This is the deformation corresponding to the decomposition $\{-\frac{1}{2}, \frac{1}{2}\} + [0, 1]$ of Example 5.2.

Example 6.23. We stay with the quadric cone of Example 6.21, but have a look at the degree $r = [1, 2]$. This is illustrated in the second part Figure 6.1(b). Using a suitable lattice isomorphism that moves the primitive degree r to $[0, 1]$, the cones become

$$\delta = \mathrm{pos}\{(2, 3), (0, 1)\} \qquad \delta^\vee = \mathrm{pos}\{[-3, -2], [1, 0]\};$$

the Hilbert basis is $E = \{[-3, -2], [-1, 1], [1, 0]\}$, the polytope in height one is the interval $\Delta = [0, \frac{2}{3}]$. Then the relation $[-3, -2] + [1, 0] = 2 \cdot [-1, 1] = [-2, 2]$ has height 0, so it counts for all k, so deformations are determined by their values at ± 1. The relation $[1, 0] + [-1, 1] = [0, 1]$ has height 1, counting for $k \ge 2$. In total, we have

$$\dim_{\mathbb{C}} T^1(\Delta, 1) = \dim_{\mathbb{C}} \mathrm{Def}^1(\Delta) - \dim_{\mathbb{C}} \mathrm{Triv}^1(\Delta) = (3 - 1) - 2 = 0$$
$$\dim_{\mathbb{C}} T^1(\Delta, k \ge 2) = \dim_{\mathbb{C}} \mathrm{Def}^k(\Delta) - \dim_{\mathbb{C}} \mathrm{Triv}^k(\Delta) = (3 - 2) - 1 = 0.$$



(a) $r = [0, k]$ (b) $r = [1, 2]$

Figure 6.1: Calculating T^1 of the quadric cone for different degrees

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Zusammenfassung

Diese Arbeit befasst sich mit Deformationen normaler affiner Varietäten mit Toruswirkung. Solche Varietäten lassen sich durch polyedrische Divisoren beschreiben, wie von Altmann und Hausen gezeigt wurde. Wesentliche Objekte sind weiterhin gewisse Deformationen torischer Varietäten, die durch Altmann konstruiert wurden.

Der Text gliedert sich wie folgt. Zunächst werden relevante Aspekte der Theorie der p-Divisoren und T-Varietäten zusammengefasst. Danach wird das technische Mittel der sogenannten Upgrades von p-Divisoren entwickelt. Dabei geht es darum, die Erweiterung einer Toruswirkung in der Sprache der p-Divisoren nachzuvollziehen. Hierbei handelt es sich um ein Ergebnis, dessen Beweis gemeinsam mit Nathan Ilten entstand.

Daraufhin werden Begriffe der Deformationstheorie und die oben erwähnten torischen Deformationen Altmanns eingeführt. In diesem Zusammenhang wird auch gezeigt, dass der Vektorraum T^1 jeder T-Varietät eine Gradierung zulässt.

Im Anschluss werden invariante Deformationen rationaler T-Varietäten der Komplexität eins konstruiert, die durch Zerlegung der Koeffizienten in Minkowskisummanden entstehen. Diese kann man auch als Verschiebung von Punkten auf der projektiven Geraden auffassen, wobei sich die polyedrischen Koeffizienten beim Aufeinandertreffen von Punkten addieren. Es wird gezeigt, dass diese Deformationen Altmanns torische Deformationen umfassen. Danach wird ein Upgrade der entsprechenden p-Divisoren durchgeführt, das eine Beschreibung als T-Varietäten niedrigerer Komplexität zulässt.

Schließlich wird auf infinitesimale Deformationen erster Ordnung eingegangen. Neben einer expliziten Beschreibung der Vektorraumoperationen auf T^1 wird ein konvex-geometrischer Ansatz zur Beschreibung des torischen T^1 entwickelt, der Deformationen torischer Varietäten durch gewisse konkave Funktionen nach $\mathbb{C} \times \mathbb{Z}$ darstellt.

Summary

The topic of this thesis are deformations of normal affine varieties with torus action. Following Altmann and Hausen, such varieties can be described using polyhedral divisors. Another central ingredient is a certain class of deformations of toric varieties as constructed by Altmann.

The text is organized as follows. After summarizing relevant aspects of the theory of p-divisors and T-varieties, we develop the technical tool of so-called upgrades of p-divisors. This involves a translation of extensions of torus actions into the language of p-divisors. The proof of the main upgrade theorem is the result of joint work with Nathan Ilten.

This is followed by a summary of relevant parts of deformation theory. This part includes the definition of Altmann's toric deformations and a proof that the vector space T^1 of a T-variety admits a natural grading.

Then, we construct invariant deformations of rational T-varieties of complexity one that arise from Minkowski decompositions of coefficient polyhedra. These can be thought of as moving points on the projective line, where coefficients are summed up when such points meet. We show that this class of deformations encompasses Altmann's toric deformations. An upgrade of the resulting p-divisors yields a description of these deformations as T-varieties of lower complexity.

Finally we turn to first order deformations. Besides an explicit description of the vector space operations on the graded T^1, we develop an approach to toric T^1 that describes deformations of toric varieties with certain concave functions to $\mathbb{C} \times \mathbb{Z}$.