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Research article

Some fractional integral inequalities involving extended Mittag-Leffler function with applications

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Abstract: Integral inequalities and the Mittag-Leffler function play a crucial role in many branches of mathematics and applications, including fractional calculus, mathematical physics, and engineering. In this paper, we introduced an extended generalized Mittag-Leffler function that involved several well-known Mittag-Leffler functions as a special case. We also introduced an associated generalized fractional integral to obtain some estimates for fractional integral inequalities of the Hermite-Hadamard and Hermite-Hadamard-Fejér types. This article offered several analytical tools that will be useful to anyone working in this field. To demonstrate the veracity of our findings, we offered a few numerical and graphical examples. A few applications of modified Bessel functions and unitarily invariant norm of matrices were also given.

Keywords: Mittag-Leffler functions and generalizations; Hermite-Hadamard type inequalities; fractional integrals; inequalities involving matrices; modified Bessel functions
Mathematics Subject Classification: 33E12, 26D15, 26A33, 15A45, 33C10

1. Introduction

The Mittag-Leffler function (MLF) is an important special function in mathematics with applications in many different fields. Complex differential equations are solved and the exponential function is expanded. It is illustrated by the analysis of stochastic processes and the resolution of some forms of Lévy and random walk processes. It can also be used to characterize the asymptotic behavior of solutions to particular types of differential equations. It is used to model a variety of physical processes, including viscoelastic materials and anomalous diffusion. It is particularly useful for elucidating memory-effect mechanisms and nonlocal interactions. The importance of the MLF is raised in fractional calculus because it actually occurs naturally in the solution of fractional differential equations and fractional integrals.

In the 19th century, M. G. Mittag-Leffler started to find out the answer to a classical question of complex analysis: How to explain the process of power series analytic continuation outside the disc of their convergence? The answer was given in the form of the MLF with one parameter

$$E_{m_1}(\omega) = \sum_{k=0}^{\infty} \frac{\omega^k}{\Gamma(km_1 + 1)}, \quad \omega \in \mathbb{C};\ \mathbb{R}(m_1) > 0,$$

where

$$\Gamma(\omega) = \int_0^{\infty} \varpi^{\omega-1} e^{-\varpi} d\varpi, \quad \mathbb{R}(\omega) > 0.$$

The MLF with two parameters was proposed by Wiman [1]

$$E_{m_1,m_2}(\omega) = \sum_{k=0}^{\infty} \frac{\omega^k}{\Gamma(km_1 + m_2)}, \quad \omega \in \mathbb{C};\ \mathbb{R}(m_1), \mathbb{R}(m_2) > 0.$$

The extension of the MLF with two parameters was introduced by Wright [2]

$$E_{m_1,m_2}(\omega) = \sum_{k=0}^{\infty} \frac{\omega^k}{\Gamma(km_1 + m_2)k!}, \quad \omega \in \mathbb{C};\ \mathbb{R}(m_1), \mathbb{R}(m_2) > 0.$$

The MLF with three parameters was introduced by Prabhakar [3]

$$E^{e}_{m_1,m_2}(\omega) = \sum_{k=0}^{\infty} \frac{(e)_k}{k!\Gamma(km_1 + m_2)} \omega^k, \quad \omega \in \mathbb{C};\ \mathbb{R}(m_1), \mathbb{R}(m_2), \mathbb{R}(e) > 0,$$

with Pochhammer symbol $(e)_k = \frac{\Gamma(e+k)}{\Gamma(e)}$. The next period in the development of the theory of the MLF is connected with increasing the number of parameters. Shukla and Prajapati [4] (see also Srivastava and Tomovski [5]) generalized the MLF as

$$E^{e,q}_{m_1,m_2}(\omega) = \sum_{k=0}^{\infty} \frac{(e)_{kq}}{k!\Gamma(km_1 + m_2)} \omega^k, \quad \omega \in \mathbb{C};\ \mathbb{R}(m_1), \mathbb{R}(m_2), \mathbb{R}(e) > 0;\ q \in (0, 1) \cup \mathbf{N}.$$

Salim and Faraj [6] introduced the MLF as

$$E^{e,\varkappa,q}_{m_1,m_2,v}(\omega) = \sum_{k=0}^{\infty} \frac{(e)_{qk}}{\Gamma(km_1 + m_2)} \frac{\omega^k}{(\varkappa)_{kv}}, \quad \omega \in \mathbb{C};\ min\{\mathbb{R}(m_1), \mathbb{R}(m_2), \mathbb{R}(e), \mathbb{R}(\varkappa)\} > 0;\ v, q > 0; q \leq v+\mathbb{R}(m_1).$$

Andric et al. [7] defined the MLF as

$$E^{b,v,q,e}_{m_1,m_2,\varkappa}(\omega; p) = \sum_{k=0}^{\infty} \frac{\mathbf{B}_p(b+qk, e-b)}{\mathbf{B}(b, e-b)} \frac{(e)_{kq}}{\Gamma(km_1 + m_2)} \frac{\omega^k}{(\varkappa)_{kv}},$$

where $\omega \in \mathbb{C}; min\{\mathbb{R}(m_1), \mathbb{R}(m_2), \mathbb{R}(\varkappa)\} > 0; \mathbb{R}(e) > \mathbb{R}(b) > 0; p \geq 0; v > 0; 0 < q \leq v + \mathbb{R}(m_1)$, with beta function $\mathbf{B}(\varsigma, \varphi) = \frac{\Gamma(\varsigma)\Gamma(\varphi)}{\Gamma(\varsigma+\varphi)} = \int_0^1 \varpi^{\varsigma-1}(1-\varpi)^{\varphi-1} d\varpi;\ \ \varsigma, \varphi > 0$, and its extension

$$\mathbf{B}_p(\varsigma, \varphi) = \int_0^1 \varpi^{\varsigma-1}(1-\varpi)^{\varphi-1} e^{-\frac{p}{\varpi(1-\varpi)}} d\varpi;\ \ \mathbb{R}(\varsigma), \mathbb{R}(\varphi) > 0;\ \ p \geq 0.$$

Bansal and Mehrez [8] defined the MLF as

$$E_{m_1,m_2}(\omega; \mu) = \sum_{k=0}^{\infty} \frac{\omega^k}{\Gamma(km_1 + m_2)\,(\mu k! + (1-\mu))}, \quad \omega \in \mathbb{C};\ \mathbb{R}(m_1), \mathbb{R}(m_2) > 0;\ 0 \leq \mu \leq 1.$$

Raina [9] generalized the MLF by involving the bounded sequence $\sigma(k)$ of real numbers as

$$E_{m_1}^{m_2,\sigma}(\omega) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(km_1 + m_2)} \omega^k, \quad \omega \in \mathbb{C};\ \mathbb{R}(m_1), \mathbb{R}(m_2) > 0.$$

Here, we did not give another multiparameter generalized Mittag-Leffler function or associated fractional integral operators; for details, the readers are suggested the references therein [10–12]. The multiparameter Mittag-Leffler function (MPMLF) was introduced for a number of reasons. It is used to describe fractional dynamics in multidimensional systems, which are common in advanced physics, such as quantum mechanics and multi-agent systems. In materials science, materials that behave fractionally in several dimensions or under different constraints can be modeled using the multi-parameter version. In order to create controllers for systems with numerous fractional orders, control theory uses the MPMLF, which enables more complex control techniques. It can be used in financial modeling to explain the asset returns and hazards in markets with non-Gaussian behaviors, improving comprehension of how prices change over time. In short, the MPMLF offers a more flexible framework for complex systems, especially those with interactions and dynamics that span multiple dimensions. For precise modeling and analysis, the MPMLF is helpful when different processes or dimensions display distinct fractional behaviors. The aforementioned MPMLFs have garnered significant attention in a number of recent studies, primarily because of their potential application to certain reaction-diffusion problems and the different generalizations that they exhibit in the solutions of fractional-order differential and integral equations [13–15].

On the other hand, fractional integral inequalities play a significant role in analyzing the solution of fractional differential equations, especially in the uniqueness of initial value problems. One effective method for establishing integral inequalities is to use a function's convexity property. The most renowned and glittering result for the convex function is the Hermite-Hadamard integral inequality. The classical Hermite-Hadamard inequality provides us an estimation of the mean value of a convex function $f : [a_1, a_2] \to \mathbb{R}$ and $a_1, a_2 \in \mathbb{R}$ with $a_1 < a_2$,

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x)dx \leq \frac{f(a_1) + f(a_2)}{2}.$$

Another well-known inequality for the integral mean of a convex function is Fejér, which is the weighted version of the Hermite-Hadamard inequality

$$f\left(\frac{a_1 + a_2}{2}\right) \int_{a_1}^{a_2} w(x)dx \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x)w(x)dx \leq \frac{f(a_1) + f(a_2)}{2} \int_{a_1}^{a_2} w(x)dx,$$

where $w : [a_1, a_2] \to \mathbb{R}$ is nonnegative, integrable, and symmetric to $\frac{a+b}{2}$.

A lot of integral inequalities have been produced by researchers utilizing different fractional integrals because numerical integration estimation is a fundamental component of applied science. Although there are several fractional integral operators, the Riemann-Liouville fractional integral is the most well-known because it offers a tangible way to extend integration to non-integer orders, which is crucial for modeling systems with memory and nonlocal interactions. It provides a strong framework for researching physical systems that display behaviors that are outside the scope of classical calculus and makes it possible to generate fractional derivatives.

Although there are numerous extended generalized forms of Riemann-Liouville fractional integrals that have been studied to establish various fractional integral inequalities, our focus here is on the generalizations of Riemann-Liouville fractional integrals that involve the MLF in their kernel. In this regard, Srivastava and Tomovski introduced the fractional integral, discussed its composition and bounding, and applied it to determine the fractional integral inequality's estimation [5]. Salim and Faraj also introduced a fractional integral and discussed its different properties [6]. Abbas and Farid used the Salim-Faraj fractional integral to obtain Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for m-convex functions [16]. Andric et al. further extended the integral and used it to obtain Opial-type inequalities [7]. Using the same integral, Andric derived Hermite-Hadamard type inequalities for the (h, g, m)-convex function [17]. Raina defined the fractional integral in a novel way using bound sequences and then employed it to generalize Wright's function [9]. For a generalized class of m-convex functions, Vivas-Cortez et al. derived Hermite-Hadamard-Fejér type inequalities using the Raina fractional integral [18]. Khan et al. [19] applied the Laplace transform to the generalized pathway fractional integral and discussed some interesting results. Recently, Du and Long used Riemann-Liouville fractional integrals to present Hermite-Hadamard-type inequalities for multiplicative convex functions [20]. For more details regarding fractional integrals and integral inequalities, the readers are suggested to refer to [21–23] and the references therein.

Inspired and motivated by the aforementioned works, as well as the success of the MPMLF and its associated integral's applications in many scientific and engineering domains, we present a new extended and generalized MPMLF and its corresponding fractional integral operator, which we use to derive some Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities. Furthermore, we illustrate the accuracy of our findings with graphical and numerical examples and provide applications in modified Bessel functions and matrix theory.

This paper is organized as follows: After this introduction, in Section 2, some preliminary topics are discussed; Sections 3 and 4 are related to the results; in Section 5, numerical and graphical analysis have been discussed; in Section 6, some applications to matrix theory and to some special functions are given; and in Section 7, the paper's conclusion is discussed.

2. Preliminaries and new extended operators

Definition 1. *[24] The function $\hbar : I \subset \mathbb{R} \to \mathbb{R}$, is said to be convex, if*

$$\hbar(\eta x + (1-\eta)y) \leq \eta\hbar(x) + (1-\eta)\hbar(y);\ \forall\ x, y \in I,\ \eta \in [0, 1].$$

Definition 2. *[25] Let $\hbar \in L^1([\xi, \varrho])$. The Riemann-Liouville integrals $J^{\Phi}_{\xi^+}\hbar$ and $J^{\Phi}_{\varrho^-}\hbar$ of order $\Phi > 0$*

with $\xi \geq 0$ are defined as:

$$J^{\Phi}_{\xi^+}\hbar(\vartheta) := \frac{1}{\Gamma(\Phi)}\int_{\xi}^{\vartheta}(\vartheta-\varpi)^{\Phi-1}\hbar(\varpi)d\varpi; \qquad \vartheta > \xi > 0,$$

and

$$J^{\Phi}_{\varrho^-}\hbar(\vartheta) := \frac{1}{\Gamma(\Phi)}\int_{\vartheta}^{\varrho}(\varpi-\vartheta)^{\Phi-1}\hbar(\varpi)d\varpi; \qquad 0 < \vartheta < \varrho,$$

respectively, provided that $\Gamma(\Phi) = \int_0^\infty e^{-u}u^{\Phi-1}du$.

Definition 3. *[26] Let $\alpha,\beta,\gamma,p,\varsigma,\varphi \in \mathbb{C}$ be such that $\mathbb{R}(\alpha),\mathbb{R}(\beta),\mathbb{R}(\gamma),\mathbb{R}(\varsigma),\mathbb{R}(\varphi) > 0$, and $\mathbb{R}(p) \geq 0$, then the extended beta function is defined as:*

$$\mathbf{B}^{\alpha,\beta,\gamma}_{p}(\varsigma,\varphi) := \int_0^1 \varpi^{\varsigma-1}(1-\varpi)^{\varphi-1}E^{\gamma}_{\alpha,\beta}\left(\frac{-p}{\varpi(1-\varpi)}\right)d\varpi. \tag{2.1}$$

Here, we provide the following definition of the generalized MLF in the form of Eq (2.1).

Definition 4. *Let $\omega,\alpha,\beta,\gamma,p,\rho,\delta,\tau,\varkappa,b,e \in \mathbb{C}$ such that*

$$min\{\mathbb{R}(\alpha),\mathbb{R}(\beta),\mathbb{R}(\gamma),\mathbb{R}(\rho),\mathbb{R}(\delta),\mathbb{R}(\tau),\mathbb{R}(\varkappa)\} > 0, \mathbb{R}(p) \geq 0, \mathbb{R}(e) > \mathbb{R}(b) > 0,$$

$0 \leq \mu \leq 1,\ r,s,v > 0$, *and* $0 < q \leq r+v+\mathbb{R}(\rho)$. *Then, the extended generalized MLF* $E^{\alpha,\beta,\gamma,\tau,\varkappa,s,\sigma}_{p,q,r,v,\mu,\rho,\delta}(b,e;\omega)$ *is defined as:*

$$E^{\alpha,\beta,\gamma,\tau,\varkappa,s,\sigma}_{p,q,r,v,\mu,\rho,\delta}(b,e;\omega) := \sum_{k=0}^{\infty}\sigma(k)\frac{\mathbf{B}^{\alpha,\beta,\gamma}_{p}(b+sk,e-b)(e)_{kq}}{\mathbf{B}(b,e-b)[\mu(\tau)_{kr}+(1-\mu)](\varkappa)_{kv}}\frac{\omega^k}{\Gamma(k\rho+\delta)}, \tag{2.2}$$

provided that $\sigma(k)$ ($k \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$) is a bounded sequence of positive real numbers.
Instead of employing a laborious manuscript form, we shall make use of a simpler notation:

$$\mathbf{E}(b,e;\omega) := E^{\alpha,\beta,\gamma,\tau,\varkappa,s,\sigma}_{p,q,r,v,\mu,\rho,\delta}(b,e;\omega).$$

Remark 1. *Several generalizations of the MLF can be obtained for different choices of parameters and form the function defined by Eq* (2.2)

1. *the Wright function for $\sigma(k) = \beta = \tau = \mu = r = 1$, $e = \varkappa$, $q = v$, and $p = s = 0$ [2],*
2. *the Shukla-Prajapati function for $\sigma(k) = \beta = \varkappa = v = 1$, and $p = s = r = 0$ [4],*
3. *the Salim-Faraj function for $\sigma(k) = \beta = 1$, and $p = s = r = 0$ [6],*
4. *Andric et al. function for $\sigma(k) = \alpha = \beta = \gamma = \mu = 1$, $s = q$, and $v = 0$ [7],*
5. *the Bansal-Mehrez function for $\sigma(k) = \beta = \tau = r = 1$, $e = \varkappa$, $q = v$, and $p = s = 0$ [8],*
6. *the Raina function for $\beta = 1$, $e = \varkappa$, $q = v$, and $p = s = r = 0$ [9],*
7. *the generalized Wright function for $\sigma(k) = \beta = q = \varkappa = \mu = r = v = 1$, and $p = s = 0$ [10],*
8. *the Pochhammer-Barnes confluent hypergeometric function for $\sigma(k) = \beta = q = v = \rho = \delta = 1$, and $p = s = r = 0$ [27].*

Here, we define the left and right-sided generalized fractional integral operators involving the generalized MLF defined by (2.2).

Definition 5. *Let $\chi, \alpha, \beta, \gamma, p, \rho, \delta, \tau, \varkappa, b, e \in \mathbb{C}$ such that*

$$min\{\mathbb{R}(\alpha), \mathbb{R}(\beta), \mathbb{R}(\gamma), \mathbb{R}(\rho), \mathbb{R}(\delta), \mathbb{R}(\tau), \mathbb{R}(\varkappa)\} > 0, \mathbb{R}(p) \geq 0, \mathbb{R}(e) > \mathbb{R}(b) > 0,$$

$0 \leq \mu \leq 1$, $r, s, v > 0$, *and* $0 < q \leq r + v + \mathbb{R}(\rho)$, *and let* $\hbar \in L^1([\xi, \varrho])$ *with* $\xi \geq 0$. *Then, the generalized fractional integral operators* $\varepsilon^{\alpha,\beta,\gamma,\tau,\varkappa,s,\sigma}_{p,q,r,v,\mu,\rho,\delta,\xi+;\chi}\hbar$ *and* $\varepsilon^{\alpha,\beta,\gamma,\tau,\varkappa,s,\sigma}_{p,q,r,v,\mu,\rho,\delta,\varrho-;\chi}\hbar$ *satisfying all the convergence conditions of the extended MLF are defined as:*

$$\left(\varepsilon^{\alpha,\beta,\gamma,\tau,\varkappa,s,\sigma}_{p,q,r,v,\mu,\rho,\delta,\xi+;\chi}\hbar\right)(b, e; \vartheta) = \int_{\xi}^{\vartheta} (\vartheta - \varpi)^{\delta-1} E(b, e; \chi(\vartheta - \varpi)^{\rho})\hbar(\varpi)d\varpi;\ \vartheta > \xi > 0, \tag{2.3}$$

$$\left(\varepsilon^{\alpha,\beta,\gamma,\tau,\varkappa,s,\sigma}_{p,q,r,v,\mu,\rho,\delta,\varrho-;\chi}\hbar\right)(b, e; \vartheta) = \int_{\vartheta}^{\varrho} (\varpi - \vartheta)^{\delta-1} E(b, e; \chi(\varpi - \vartheta)^{\rho})\hbar(\varpi)d\varpi;\ 0 < \vartheta < \varrho. \tag{2.4}$$

Instead of employing a laborious manuscript form, we shall make use of a simpler notation:

$$\left(\varepsilon^{\chi}_{\xi+}\hbar\right)(\vartheta) := \left(\varepsilon^{\alpha,\beta,\gamma,\tau,\varkappa,s,\sigma}_{p,q,r,v,\mu,\rho,\delta,\xi+;\chi}\hbar\right)(b, e; \vartheta),$$

$$\left(\varepsilon^{\chi}_{\varrho-}\hbar\right)(\vartheta) := \left(\varepsilon^{\alpha,\beta,\gamma,\tau,\varkappa,s,\sigma}_{p,q,r,v,\mu,\rho,\delta,\varrho-;\chi}\hbar\right)(b, e; \vartheta).$$

Remark 2. *For different choices of parameters, several known fractional integral operators can be deduced from the operators* (2.3) *and* (2.4). *Indeed, we have*

1. *the Prabhakar fractional integral operator for* $\sigma(k) = \beta = q = \varkappa = v = 1$, *and* $p = s = r = 0$ *[3],*
2. *the Srivastava-Tomovski fractional integral operator for* $\sigma(k) = \beta = \varkappa = v = 1$, *and* $p = s = r = 0$ *[5],*
3. *the Salim-Faraj fractional integral operator for* $\sigma(k) = \beta = 1$, *and* $p = s = r = 0$ *[6],*
4. *Andric et al. fractional integral operator for* $\sigma(k) = \alpha = \beta = \gamma = \mu = 1$, $s = q$, *and* $v = 0$ *[7],*
5. *the Raina fractional integral operator for* $\beta = 1$, $e = \varkappa$, $q = v$, *and* $p = s = r = 0$ *[9],*
6. *the Riemann-Liouville fractional integral for* $\sigma(0) = \beta = 1$ *and* $p = \chi = 0$, *that is, Definition* 2.

3. Hermite-Hadamard type inequalities for generalized Mittag-Leffler function

Theorem 1. *Let* $\hbar \in L^1[\mathfrak{a}, \mathfrak{m}]$ *be a nonnegative convex function with* $0 \leq \mathfrak{a} < \mathfrak{m}$ *and* $\mathfrak{a} < \mathfrak{e}_1 + 1 < \mathfrak{e}_2 < \mathfrak{m}$, $\varphi_1 \in \mathbb{R}$, $p \geq 0$, $\delta \geq 1$, $e > b > 0$, $0 \leq \mu \leq 1$, $\alpha, \beta, \gamma, \rho, \Omega, \tau, \varkappa, r, s, v > 0$ *such that* $0 < q \leq r + v + \rho$, *then*

$$\begin{aligned} \hbar\left(\frac{(2\mathfrak{e}_2 - 2\mathfrak{e}_1 - 1)\mathfrak{a} + (2\mathfrak{e}_1 + 1)\mathfrak{m}}{2\mathfrak{e}_2}\right) &\leq \frac{\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}\hbar\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right) + \left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}\hbar\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)}{2\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}1\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)} \\ &\leq \frac{\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right) + \hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)}{2}. \end{aligned} \tag{3.1}$$

Proof. Let $\eta \in [0, 1]$. Since $\hbar$ is a convex function on $[\mathfrak{a}, \mathfrak{m}]$, for $x_1, y_1 \in [\mathfrak{a}, \mathfrak{m}]$,

$$\hbar\left(\frac{x_1 + y_1}{2}\right) \leq \frac{\hbar(x_1) + \hbar(y_1)}{2}, \tag{3.2}$$

setting

$$x_1 = \frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}, \tag{3.3}$$

and

$$y_1 = \frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}. \tag{3.4}$$

In this case (3.2) reduces to

$$\begin{aligned}2\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right) &\le \hbar\left(\frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)\\ &\quad+\hbar\left(\frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right).\end{aligned} \tag{3.5}$$

Multiplying both sides of (3.5) by $\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})$ and integrating over $\eta\in[0,1]$ yields

$$\begin{aligned}&2\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})d\eta\\ &\quad\le\int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})\hbar\left(\frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)d\eta\\ &\qquad+\int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})\hbar\left(\frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)d\eta.\end{aligned}$$

Equivalently,

$$I_1 \le I_2 + I_3, \tag{3.6}$$

provided that

$$\begin{aligned}I_1 &:= 2\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})d\eta,\\ I_2 &:= \int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})\hbar\left(\frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)d\eta,\\ I_3 &:= \int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})\hbar\left(\frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)d\eta.\end{aligned}$$

Now,

$$I_1 = 2\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\int_0^1\mathbf{E}(b,e;\chi\eta^{\rho})\frac{d\eta}{\eta^{1-\Omega}}.$$

From (3.3), we have $\eta=\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}-\mathfrak{e}_2x_1}{\mathfrak{m}-\mathfrak{a}}$, and letting $\varphi_1=\frac{(\mathfrak{e}_2)^{\rho}\chi}{(\mathfrak{m}-\mathfrak{a})^{\rho}}$,

$$I_1 = 2\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\int_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}}^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}}\left(\mathfrak{e}_2\frac{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-x_1}{\mathfrak{m}-\mathfrak{a}}\right)^{\Omega-1}$$

$$\times\mathbf{E}\left(b,e;\varphi_1\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-x_1\right)^{\rho}\right)\frac{\mathfrak{e}_2 dx_1}{\mathfrak{m}-\mathfrak{a}}$$
$$=\frac{2(\mathfrak{e}_2)^{\Omega}}{(\mathfrak{m}-\mathfrak{a})^{\Omega}}\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}1\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right), \quad (3.7)$$

$$I_2 = \int_0^1 \mathbf{E}(b,e;\chi\eta^{\rho})\hbar\left(\frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)\frac{d\eta}{\eta^{1-\Omega}}$$
$$=\left(\frac{\mathfrak{e}_2}{\mathfrak{m}-\mathfrak{a}}\right)^{\Omega}\int_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}}^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}}\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-x_1\right)^{\Omega-1}$$
$$\times\mathbf{E}\left(b,e;\varphi_1\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-x_1\right)^{\rho}\right)\hbar(x_1)dx_1$$
$$=\left(\frac{\mathfrak{e}_2}{\mathfrak{m}-\mathfrak{a}}\right)^{\Omega}\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}\hbar\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right), \quad (3.8)$$

$$I_3 = \int_0^1 \mathbf{E}(b,e;\chi\eta^{\rho})\hbar\left(\frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)\frac{d\eta}{\eta^{1-\Omega}}.$$

By (3.4), we have

$$\eta=\frac{\mathfrak{e}_2 y_1-(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}-\mathfrak{e}_1\mathfrak{m}}{\mathfrak{m}-\mathfrak{a}},$$

$$I_3=\left(\frac{\mathfrak{e}_2}{\mathfrak{m}-\mathfrak{a}}\right)^{\Omega}\int_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}}^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}}\left(y_1-\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)^{\Omega-1}\mathbf{E}\left(b,e;\varphi_1\left(y_1-\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)^{\rho}\right)\hbar(y_1)dy_1$$
$$=\left(\frac{\mathfrak{e}_2}{\mathfrak{m}-\mathfrak{a}}\right)^{\Omega}\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}\hbar\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right). \quad (3.9)$$

Putting (3.7)–(3.9) in (3.6),

$$2\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}1\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)$$
$$\le\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}\hbar\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)+\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}\hbar\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right). \quad (3.10)$$

Again, we note by the convexity of $\hbar$ that

$$\hbar\left(\frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)$$
$$\le\eta\,\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)+(1-\eta)\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right), \quad (3.11)$$

$$\hbar\left(\frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)$$

$$\leq (1-\eta)\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}}{\mathfrak{e}_2}\right)+\eta\,\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right). \tag{3.12}$$

By addition of (3.11) and (3.12),

$$\begin{aligned}
&\hbar\left(\frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)\\
&+\hbar\left(\frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)\\
&\leq \hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)+\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right).
\end{aligned} \tag{3.13}$$

Multiplying both sides of (3.13) by $\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})$ and integrating over $\eta\in[0,1]$ yields

$$\begin{aligned}
&\int_0^1 \eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})\hbar\left(\frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)d\eta\\
&+\int_0^1 \eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})\hbar\left(\frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)d\eta\\
&\leq\left[\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)+\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)\right]\int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})d\eta.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
&\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}\hbar\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)+\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}\hbar\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)\\
&\leq\left[\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)+\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)\right]\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}1\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right).
\end{aligned} \tag{3.14}$$

The desired inequality (3.1) is produced by combining (3.10) and (3.14). □

Corollary 1. *Under assumption of Theorem 1 for $\sigma(0)=\beta=1$ and $p=\varphi_1=0$, we have*

$$\begin{aligned}
&\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\leq\frac{(\mathfrak{e}_2)^{\Omega}\Gamma(\Omega+1)}{2(\mathfrak{m}-\mathfrak{a})^{\Omega}}\left[J^{\Omega}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)\right.\\
&\left.+J^{\Omega}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)\right]\leq\frac{\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)+\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)}{2}.
\end{aligned} \tag{3.15}$$

Remark 3. *Theorem 1 is the generalization of the Corollary 1, and Corollary 1 is the generalization of the classical Hermite-Hadamard inequality for $\Omega=\mathfrak{e}_2=1$ and $\mathfrak{e}_1=0$ in (3.15).*

Theorem 2. *Let $\hbar\in L^1[\mathfrak{a},\mathfrak{m}]$ be a nonnegative convex function with $0\leq\mathfrak{a}<\mathfrak{m}$ and $\mathfrak{a}<\mathfrak{e}_2<\mathfrak{m}$, $\varphi_2\in\mathbb{R}$, $p\geq 0$, $\delta\geq 1$, $e>b>0$, $0\leq\mu\leq 1$, $0\leq\mathfrak{k}_1<1$, $\alpha,\beta,\gamma,\rho,\Omega,\tau,\varkappa,r,s,v>0$ such that $0<q\leq r+v+\rho$, then*

$$\begin{aligned}
\hbar\left(\frac{2\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}{2}\right)&\leq\frac{(\varepsilon^{\varphi_2}_{\mathfrak{a}+}\hbar)\,(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1))+\left(\varepsilon^{\varphi_2}_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}\hbar\right)(\mathfrak{a})}{2\,(\varepsilon^{\varphi_2}_{\mathfrak{a}+}1)\,(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1))}\\
&\leq\frac{(1+\mathfrak{k}_1)\hbar(\mathfrak{a})+(1-\mathfrak{k}_1)\hbar(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)}{2}.
\end{aligned} \tag{3.16}$$

Proof. Let $\eta \in [0,1]$. Since $\hbar$ is a convex function on $[\mathfrak{a}, \mathfrak{m}]$, for $x_2, y_2 \in [\mathfrak{a}, \mathfrak{m}]$,

$$\hbar\left(\frac{x_2 + y_2}{2}\right) \leq \frac{\hbar(x_2) + \hbar(y_2)}{2}, \tag{3.17}$$

provided that

$$x_2 = \eta\, \mathfrak{a} + (1-\eta)(\mathfrak{k}_1 \mathfrak{a} + (1-\mathfrak{k}_1)(\mathfrak{a} + \mathfrak{m} - \mathfrak{e}_2)), \tag{3.18}$$

$$y_2 = (1-\eta)\mathfrak{a} + \eta\, (\mathfrak{k}_1 \mathfrak{a} + (1-\mathfrak{k}_1)(\mathfrak{a} + \mathfrak{m} - \mathfrak{e}_2)). \tag{3.19}$$

In this case, (3.17) reduces to

$$\begin{aligned} 2\hbar\left(\frac{2\mathfrak{a} + (\mathfrak{m} - \mathfrak{e}_2)(1-\mathfrak{k}_1)}{2}\right) &\leq \hbar(\eta\mathfrak{a} + (1-\eta)(\mathfrak{k}_1 \mathfrak{a} + (1-\mathfrak{k}_1)(\mathfrak{a} + \mathfrak{m} - \mathfrak{e}_2))) \\ &\quad + \hbar((1-\eta)\mathfrak{a} + \eta(\mathfrak{k}_1 \mathfrak{a} + (1-\mathfrak{k}_1)(\mathfrak{a} + \mathfrak{m} - \mathfrak{e}_2))). \end{aligned} \tag{3.20}$$

Multiplying both sides of (3.20) by $\eta^{\Omega-1}\mathbf{E}(b, e; \chi\eta^{\rho})$ and integrating over $\eta \in [0,1]$ yields

$$\begin{aligned} & 2\hbar\left(\frac{2\mathfrak{a} + (\mathfrak{m} - \mathfrak{e}_2)(1-\mathfrak{k}_1)}{2}\right)\int_0^1 \eta^{\Omega-1}\mathbf{E}(b, e; \chi\eta^{\rho})d\eta \\ \leq & \int_0^1 \eta^{\Omega-1}\mathbf{E}(b, e; \chi\eta^{\rho})\hbar(\eta\mathfrak{a} + (1-\eta)(\mathfrak{k}_1 \mathfrak{a} + (1-\mathfrak{k}_1)(\mathfrak{a} + \mathfrak{m} - \mathfrak{e}_2)))d\eta \\ + & \int_0^1 \eta^{\Omega-1}\mathbf{E}(b, e; \chi\eta^{\rho})\hbar((1-\eta)\mathfrak{a} + \eta(\mathfrak{k}_1 \mathfrak{a} + (1-\mathfrak{k}_1)(\mathfrak{a} + \mathfrak{m} - \mathfrak{e}_2)))d\eta. \end{aligned}$$

Equivalently,

$$I_4 \leq I_5 + I_6, \tag{3.21}$$

$$\begin{aligned} I_4 &= 2\hbar\left(\frac{2\mathfrak{a} + (1-\mathfrak{k}_1)(\mathfrak{m} - \mathfrak{e}_2)}{2}\right)\int_0^1 \eta^{\Omega-1}\mathbf{E}(b, e; \chi\eta^{\rho})d\eta, \\ I_5 &= \int_0^1 \eta^{\Omega-1}\mathbf{E}(b, e; \chi\eta^{\rho})\hbar(\eta\mathfrak{a} + (1-\eta)(\mathfrak{k}_1 \mathfrak{a} + (1-\mathfrak{k}_1)(\mathfrak{a} + \mathfrak{m} - \mathfrak{e}_2)))d\eta, \\ I_6 &= \int_0^1 \eta^{\Omega-1}\mathbf{E}(b, e; \chi\eta^{\rho})\hbar((1-\eta)\mathfrak{a} + \eta(\mathfrak{k}_1 \mathfrak{a} + (1-\mathfrak{k}_1)(\mathfrak{a} + \mathfrak{m} - \mathfrak{e}_2)))d\eta. \end{aligned}$$

Now,

$$I_4 = 2\hbar\left(\frac{2\mathfrak{a} + (1-\mathfrak{k}_1)(\mathfrak{m} - \mathfrak{e}_2)}{2}\right)\int_0^1 \eta^{\Omega-1}\mathbf{E}(b, e; \chi\eta^{\rho})d\eta.$$

By (3.18), we have $\eta = \frac{\mathfrak{a} + (\mathfrak{m} - \mathfrak{e}_2)(1-\mathfrak{k}_1) - x_2}{(\mathfrak{m} - \mathfrak{e}_2)(1-\mathfrak{k}_1)}$, and letting $\varphi_2 = \frac{\chi}{((\mathfrak{m} - \mathfrak{e}_2)(1-\mathfrak{k}_1))^{\rho}}$,

$$\begin{aligned} I_4 &= 2\hbar\left(\frac{2\mathfrak{a} + (\mathfrak{m} - \mathfrak{e}_2)(1-\mathfrak{k}_1)}{2}\right)\int_0^1 \eta^{\Omega-1}\mathbf{E}(b, e; \chi\eta^{\rho})d\eta \\ &= \frac{2}{((\mathfrak{m} - \mathfrak{e}_2)(1-\mathfrak{k}_1))^{\Omega}}\hbar\left(\frac{2\mathfrak{a} + (\mathfrak{m} - \mathfrak{e}_2)(1-\mathfrak{k}_1)}{2}\right)\int_{\mathfrak{a}}^{\mathfrak{a} + (\mathfrak{m} - \mathfrak{e}_2)(1-\mathfrak{k}_1)} (\mathfrak{a} + (\mathfrak{m} - \mathfrak{e}_2)(1-\mathfrak{k}_1) - x_2)^{\Omega-1} \end{aligned}$$

$$\times\mathbf{E}(b,e;\varphi_2(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)-x_2)^{\rho})dx_2$$
$$= \frac{2}{((\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1))^{\Omega}}\hbar\left(\frac{2\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}{2}\right)(\varepsilon^{\varphi_2}_{\mathfrak{a}+}1)(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)), \tag{3.22}$$

$$\begin{aligned} I_5 &= \int_0^1 \eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})\hbar(\eta\mathfrak{a}+(1-\eta)(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)))d\eta \\ &= \frac{1}{((\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1))^{\Omega}}\int_{\mathfrak{a}}^{\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)-x_2)^{\Omega-1} \\ &\quad \times\mathbf{E}(b,e;\varphi_2(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)-x_2)^{\rho})\hbar(x_2)dx_2 \\ &= \frac{1}{((\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1))^{\Omega}}(\varepsilon^{\varphi_2}_{\mathfrak{a}+}\hbar)(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)), \end{aligned} \tag{3.23}$$

$$I_6 = \int_0^1 \eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})\hbar((1-\eta)\mathfrak{a}+\eta(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)))d\eta.$$

By (3.19), we have

$$\eta = \frac{y_2-\mathfrak{a}}{(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)},$$

$$\begin{aligned} I_6 &= \frac{1}{((\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1))^{\Omega}}\int_{\mathfrak{a}}^{\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}(y_2-\mathfrak{a})^{\Omega-1}\mathbf{E}(b,e;\varphi_2(y_2-\mathfrak{a})^{\rho})\hbar(y_2)dy_2 \\ &= \frac{1}{((\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1))^{\Omega}}\left(\varepsilon^{\varphi_2}_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}\hbar\right)(\mathfrak{a}). \end{aligned} \tag{3.24}$$

Putting (3.22)–(3.24) in (3.21),

$$\begin{aligned} &2\hbar\left(\frac{2\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}{2}\right)(\varepsilon^{\varphi_2}_{\mathfrak{a}+}1)(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)) \\ &\leq (\varepsilon^{\varphi_2}_{\mathfrak{a}+}\hbar)(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)) + \left(\varepsilon^{\varphi_2}_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}\hbar\right)(\mathfrak{a}). \end{aligned} \tag{3.25}$$

Again, we note by the convexity of $\hbar$ that

$$\hbar(\eta\mathfrak{a}+(1-\eta)(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2))) \leq \eta\hbar(\mathfrak{a})+\mathfrak{k}_1(1-\eta)\hbar(\mathfrak{a})+(1-\eta)(1-\mathfrak{k}_1)\hbar(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2), \tag{3.26}$$

$$\hbar((1-\eta)\mathfrak{a}+\eta(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2))) \leq (1-\eta)\hbar(\mathfrak{a})+\mathfrak{k}_1\eta\hbar(\mathfrak{a})+\eta(1-\mathfrak{k}_1)\hbar(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2). \tag{3.27}$$

By addition of (3.26) and (3.27), we have

$$\begin{aligned} &\hbar(\eta\mathfrak{a}+(1-\eta)(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)))+\hbar((1-\eta)\mathfrak{a}+\eta(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2))) \\ &\leq (1+\mathfrak{k}_1)\hbar(\mathfrak{a})+(1-\mathfrak{k}_1)\hbar(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2). \end{aligned} \tag{3.28}$$

Multiplying both sides of (3.28) by $\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})$ and integrating over $\eta\in[0,1]$ yields

$$\int_0^1 \eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})\hbar(\eta\mathfrak{a}+(1-\eta)(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)))d\eta$$

$$+\int_0^1 \eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})\hbar((1-\eta)\mathfrak{a}+\eta(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)))d\eta$$
$$\leq (1+\mathfrak{k}_1)\hbar(\mathfrak{a})\int_0^1 \eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})d\eta+(1-\mathfrak{k}_1)\hbar(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)\int_0^1 \eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})d\eta.$$

Equivalently,

$$\begin{aligned}&\left(\varepsilon^{\varphi_2}_{\mathfrak{a}+}\hbar\right)(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1))+\left(\varepsilon^{\varphi_2}_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}\hbar\right)(\mathfrak{a})\\ &\leq[(1+\mathfrak{k}_1)\hbar(\mathfrak{a})+(1-\mathfrak{k}_1)\hbar(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)]\left(\varepsilon^{\varphi_2}_{\mathfrak{a}+}1\right)(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)).\end{aligned} \tag{3.29}$$

The desired inequality (3.16) is produced by combining (3.25) and (3.29). □

Corollary 2. *Under assumption of Theorem 2 with $\sigma(0)=\beta=1$ and $p=\varphi_2=0$, we have*

$$\begin{aligned}\hbar\left(\frac{2\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}{2}\right)&\leq\frac{\Gamma(\Omega+1)[J^{\Omega}_{\mathfrak{a}+}\hbar(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1))+J^{\Omega}_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}\hbar(\mathfrak{a})]}{2\left((\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)\right)^{\Omega}}\\ &\leq\frac{(1+\mathfrak{k}_1)\hbar(\mathfrak{a})+(1-\mathfrak{k}_1)\hbar(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)}{2}.\end{aligned}$$

Remark 4. *On letting $\mathfrak{k}_1=0$, $\mathfrak{e}_2=\mathfrak{a}$, and $\Omega=1$, Corollary 2 coincides with the classical Hermite-Hadamard inequality.*

Theorem 3. *Let $\hbar\in L^1[\mathfrak{a},\mathfrak{m}]$ be a nonnegative convex function with $0\leq\mathfrak{a}<\mathfrak{m}$ and $\mathfrak{a}<\mathfrak{e}_1+1<\mathfrak{e}_2<\mathfrak{m}$, $\varphi_3\in\mathbb{R}$, $p\geq0$, $\delta\geq1$, $e>b>0$, $0\leq\mu\leq1$, $\alpha,\beta,\gamma,\rho,\Omega,\tau,\varkappa,r,s,v>0$ such that $0<q\leq r+v+\rho$, then*

$$\begin{aligned}&\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\\ &\leq\frac{\left(\varepsilon^{\varphi_3}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac12)\mathfrak{a}+\left(\mathfrak{e}_1+\frac12\right)\mathfrak{m}}{\mathfrak{e}_2}+}\hbar\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)+\left(\varepsilon^{\varphi_3}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac12)\mathfrak{a}+\left(\mathfrak{e}_1+\frac12\right)\mathfrak{m}}{\mathfrak{e}_2}-}\hbar\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)}{2\left(\varepsilon^{\varphi_3}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac12)\mathfrak{a}+\left(\mathfrak{e}_1+\frac12\right)\mathfrak{m}}{\mathfrak{e}_2}+}1\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)}\\ &\leq\frac{\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)+\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)}{2}.\end{aligned} \tag{3.30}$$

Proof. Let $\eta\in[0,1]$. Since $\hbar$ is a convex function on $[\mathfrak{a},\mathfrak{m}]$, for $x_3,y_3\in[\mathfrak{a},\mathfrak{m}]$,

$$\hbar\left(\frac{x_3+y_3}{2}\right)\leq\frac{\hbar(x_3)+\hbar(y_3)}{2}, \tag{3.31}$$

provided that

$$x_3=\frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(2-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{2\mathfrak{e}_2}, \tag{3.32}$$

$$y_3=\frac{(2-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{2\mathfrak{e}_2}. \tag{3.33}$$

In this case, (3.31) reduces to

$$2\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\leq\hbar\left(\frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(2-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{2\mathfrak{e}_2}\right)$$
$$+\hbar\left(\frac{(2-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{2\mathfrak{e}_2}\right). \quad (3.34)$$

Multiplying both sides of (3.34) by $\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})$ and integrating over $\eta\in[0,1]$ yields

$$2\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})d\eta$$
$$\leq\int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})\hbar\left(\frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(2-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{2\mathfrak{e}_2}\right)d\eta$$
$$+\int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})\hbar\left(\frac{(2-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{2\mathfrak{e}_2}\right)d\eta.$$

Equivalently,

$$I_7\leq I_8+I_9, \quad (3.35)$$

provided that

$$I_7 = 2\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})d\eta,$$
$$I_8 = \int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})\hbar\left(\frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(2-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{2\mathfrak{e}_2}\right)d\eta,$$
$$I_9 = \int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})\hbar\left(\frac{(2-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{2\mathfrak{e}_2}\right)d\eta.$$

Now,

$$I_7=2\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\int_0^1\mathbf{E}(b,e;\chi\eta^{\rho})\frac{d\eta}{\eta^{1-\Omega}}.$$

By (3.32), we have $\eta=2\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}-\mathfrak{e}_2x_3}{\mathfrak{m}-\mathfrak{a}}$, and letting $\varphi_3=\frac{(2\mathfrak{e}_2)^{\rho}\chi}{(\mathfrak{m}-\mathfrak{a})^{\rho}}$,

$$2\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})d\eta$$
$$=\frac{2^{\Omega+1}(\mathfrak{e}_2)^{\Omega}}{(\mathfrak{m}-\mathfrak{a})^{\Omega}}\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\int_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac{1}{2})\mathfrak{a}+\left(\mathfrak{e}_1+\frac{1}{2}\right)\mathfrak{m}}{\mathfrak{e}_2}}^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}}\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-x_3\right)^{\Omega-1}$$
$$\times\mathbf{E}\left(b,e;\varphi_3\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-x_3\right)^{\rho}\right)dx_3$$
$$=\frac{2^{\Omega+1}(\mathfrak{e}_2)^{\Omega}}{(\mathfrak{m}-\mathfrak{a})^{\Omega}}\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\left(\varepsilon^{\varphi_3}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac{1}{2})\mathfrak{a}+\left(\mathfrak{e}_1+\frac{1}{2}\right)\mathfrak{m}}{\mathfrak{e}_2}+}1\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right), \quad (3.36)$$

$$\begin{aligned}
I_8 &= \int_0^1 \mathbf{E}(b,e;\chi\eta^\rho)\hbar\left(\frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(2-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{2\mathfrak{e}_2}\right)\frac{d\eta}{\eta^{1-\Omega}} \\
&= \left(\frac{2\mathfrak{e}_2}{\mathfrak{m}-\mathfrak{a}}\right)^\Omega \int_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac{1}{2})\mathfrak{a}+\left(\mathfrak{e}_1+\frac{1}{2}\right)\mathfrak{m}}{\mathfrak{e}_2}}^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}} \left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-x_3\right)^{\Omega-1} \\
&\quad\times\mathbf{E}\left(b,e;\varphi_3\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-x_3\right)^\rho\right)\hbar(x_3)dx_3 \\
&= \left(\frac{2\mathfrak{e}_2}{\mathfrak{m}-\mathfrak{a}}\right)^\Omega \left(\varepsilon^{\varphi_3}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac{1}{2})\mathfrak{a}+\left(\mathfrak{e}_1+\frac{1}{2}\right)\mathfrak{m}}{\mathfrak{e}_2}+}\hbar\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right), \qquad (3.37)
\end{aligned}$$

$$I_9 = \int_0^1 \mathbf{E}(b,e;\chi\eta^\rho)\hbar\left(\frac{(2-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{2\mathfrak{e}_2}\right)\frac{d\eta}{\eta^{1-\Omega}}.$$

By (3.33), we have

$$\eta = 2\frac{\mathfrak{e}_2y_3-(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}-\mathfrak{e}_1\mathfrak{m}}{\mathfrak{m}-\mathfrak{a}},$$

$$\begin{aligned}
I_9 &= \left(\frac{2\mathfrak{e}_2}{\mathfrak{m}-\mathfrak{a}}\right)^\Omega \int_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}}^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac{1}{2})\mathfrak{a}+\left(\mathfrak{e}_1+\frac{1}{2}\right)\mathfrak{m}}{\mathfrak{e}_2}} \left(y_3-\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)^{\Omega-1}\mathbf{E}\left(b,e;\varphi_3\left(y_3-\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)^\rho\right)\hbar(y_3)dy_3 \\
&= \left(\frac{2\mathfrak{e}_2}{\mathfrak{m}-\mathfrak{a}}\right)^\Omega \left(\varepsilon^{\varphi_3}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac{1}{2})\mathfrak{a}+\left(\mathfrak{e}_1+\frac{1}{2}\right)\mathfrak{m}}{\mathfrak{e}_2}-}\hbar\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right). \qquad (3.38)
\end{aligned}$$

Putting (3.36)–(3.38) in (3.35),

$$\begin{aligned}
&2\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\left(\varepsilon^{\varphi_3}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac{1}{2})\mathfrak{a}+\left(\mathfrak{e}_1+\frac{1}{2}\right)\mathfrak{m}}{\mathfrak{e}_2}+}1\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right) \\
&\leq \left(\varepsilon^{\varphi_3}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac{1}{2})\mathfrak{a}+\left(\mathfrak{e}_1+\frac{1}{2}\right)\mathfrak{m}}{\mathfrak{e}_2}+}\hbar\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)+\left(\varepsilon^{\varphi_3}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac{1}{2})\mathfrak{a}+\left(\mathfrak{e}_1+\frac{1}{2}\right)\mathfrak{m}}{\mathfrak{e}_2}-}\hbar\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right). \qquad (3.39)
\end{aligned}$$

Again, we note by the convexity of $\hbar$ that

$$\begin{aligned}
&\hbar\left(\frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(2-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{2\mathfrak{e}_2}\right) \\
&\leq \frac{\eta}{2}\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)+\frac{2-\eta}{2}\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right), \qquad (3.40)
\end{aligned}$$

$$\begin{aligned}
&\hbar\left(\frac{(2-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{2\mathfrak{e}_2}\right) \\
&\leq \frac{2-\eta}{2}\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)+\frac{\eta}{2}\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right). \qquad (3.41)
\end{aligned}$$

By addition of (3.40) and (3.41),

$$\hbar\left(\frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(2-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{2\mathfrak{e}_2}\right)$$
$$+\hbar\left(\frac{(2-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{2\mathfrak{e}_2}\right)$$
$$\leq \hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)+\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right). \tag{3.42}$$

Multiplying both sides of (3.42) by $\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})$ and integrating over $\eta\in[0,1]$ yields

$$\int_0^1 \eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})\hbar\left(\frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(2-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{2\mathfrak{e}_2}\right)d\eta$$
$$+\int_0^1 \eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})\hbar\left(\frac{(2-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{2\mathfrak{e}_2}\right)d\eta$$
$$\leq\left[\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)+\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)\right]\int_0^1 \eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})d\eta.$$

Equivalently,

$$\left(\varepsilon^{\varphi_3}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac{1}{2})\mathfrak{a}+\left(\mathfrak{e}_1+\frac{1}{2}\right)\mathfrak{m}}{\mathfrak{e}_2}+}\hbar\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)+\left(\varepsilon^{\varphi_3}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac{1}{2})\mathfrak{a}+\left(\mathfrak{e}_1+\frac{1}{2}\right)\mathfrak{m}}{\mathfrak{e}_2}-}\hbar\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)$$
$$\leq\left(\varepsilon^{\varphi_3}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac{1}{2})\mathfrak{a}+\left(\mathfrak{e}_1+\frac{1}{2}\right)\mathfrak{m}}{\mathfrak{e}_2}+}1\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac{1}{2})\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)\left[\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)+\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)\right]. \tag{3.43}$$

The required inequality (3.30) is produced when (3.39) and (3.43) are combined. □

Corollary 3. *Under assumption of Theorem* 3 *with* $\sigma(0)=\beta=1$ *and* $p=\varphi_3=0$, *we have*

$$\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\leq\frac{(2\mathfrak{e}_2)^{\Omega}\Gamma(\Omega+1)}{(\mathfrak{m}-\mathfrak{a})^{\Omega}}\left[J^{\Omega}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac{1}{2})\mathfrak{a}+\left(\mathfrak{e}_1+\frac{1}{2}\right)\mathfrak{m}}{\mathfrak{e}_2}+}\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)\right.$$
$$\left.+J^{\Omega}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-\frac{1}{2})\mathfrak{a}+\left(\mathfrak{e}_1+\frac{1}{2}\right)\mathfrak{m}}{\mathfrak{e}_2}-}\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)\right]\leq\frac{\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)+\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)}{2}.$$

4. Hermite-Hadamard-Fejér type inequalities for generalized Mittag-Leffler function

Theorem 4. *Let* $\hbar\in L^1[\mathfrak{a},\mathfrak{m}]$ *be a nonnegative convex function with* $0\leq\mathfrak{a}<\mathfrak{m}$ *and* $\mathfrak{a}<\mathfrak{e}_1+1<\mathfrak{e}_2<\mathfrak{m}$, $\varphi_1\in\mathbb{R}$, $p\geq 0$, $\delta\geq 1$, $e>b>0$, $0\leq\mu\leq 1$, $\alpha,\beta,\gamma,\rho,\Omega,\tau,\varkappa,r,s,v>0$ *such that* $0<q\leq r+v+\rho$. *Further, let* $w\in L^1([\mathfrak{a},\mathfrak{m}])$ *be a nonnegative and symmetric with respect to* $\frac{x_1+y_1}{2}$, *that is,* $w(x_1+y_1-x)=w(x)$ *with* $\mathfrak{a}\leq x_1\leq x\leq y_1\leq\mathfrak{m}$, *where* x_1 *and* y_1 *are defined by* (3.3) *and* (3.4), *respectively. Then,*

$$\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\left[\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}w\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)\right.$$

$$
+\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}w\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)\Bigg]
$$
$$
\leq\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}(\hbar w)\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)+\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}(\hbar w)\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)
$$
$$
\leq\left[\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}w\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)+\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}w\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)\right]
$$
$$
\times\frac{\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)+\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)}{2}. \tag{4.1}
$$

Proof. The proof is followed by inequality (3.5) by multiplying both sides by

$$
\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})w\left(\frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right),
$$

and integrating over $\eta\in[0,1]$, we obtain

$$
2\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})
$$
$$
\times w\left(\frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)d\eta
$$
$$
\leq\int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})w\left(\frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)
$$
$$
\times\hbar\left(\frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)d\eta
$$
$$
+\int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})w\left(\frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)
$$
$$
\times\hbar\left(\frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)d\eta
$$

$$
\Rightarrow\ 2\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\int_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}}^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}}\left(y_1-\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)^{\Omega-1}
$$
$$
\times\mathbf{E}\left(b,e;\varphi_1\left(y_1-\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)^{\rho}\right)w(y_1)dy_1
$$
$$
\leq\int_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}}^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}}\left(y_1-\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)^{\Omega-1}\mathbf{E}\left(b,e;\varphi_1\left(y_1-\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)^{\rho}\right)
$$
$$
\times\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-y_1\right)w(y_1)dy_1+\int_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}}^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}}\left(y_1-\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)^{\Omega-1}
$$
$$
\times\mathbf{E}\left(b,e;\varphi_1\left(y_1-\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)^{\rho}\right)\hbar(y_1)w(y_1)dy_1,
$$

or

$$2\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\times\int_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}}^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}}\left(y_1-\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)^{\Omega-1}$$
$$\times\mathbf{E}\left(b,e;\varphi_1\left(y_1-\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)^{\rho}\right)w(y_1)dy_1$$
$$\leq\int_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}}^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}}\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-s_1\right)^{\Omega-1}\mathbf{E}\left(b,e;\varphi_1\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-s_1\right)^{\rho}\right)\hbar(s_1)$$
$$\times w\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-s_1\right)ds_1+\int_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}}^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}}\left(y_1-\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)^{\Omega-1}$$
$$\times\mathbf{E}\left(b,e;\varphi_1\left(y_1-\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)^{\rho}\right)\hbar(y_1)w(y_1)dy_1.$$

Equivalently,

$$2\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}w\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)$$
$$\leq\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}(\hbar w)\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)+\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}(\hbar w)\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right).\quad(4.2)$$

However, by symmetry of w with respect to $\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}$, we have

$$\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}w\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)=\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}w\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)$$
$$=\frac{1}{2}\left\{\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}w\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)+\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}w\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)\right\}.\quad(4.3)$$

By combining (4.2) and (4.3), we have

$$\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\times\left[\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}w\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)\right.$$
$$\left.+\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}w\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)\right]$$
$$\leq\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}\hbar w\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)+\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}\hbar w\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right).\quad(4.4)$$

Again, multiplying inequality (3.13) to $\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})$
$\times\, w\left(\frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)$ and integrating over $\eta\in[0,1]$, we obtain

$$\int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})w\left(\frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)$$
$$\times\hbar\left(\frac{\eta\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)d\eta$$

$$+\int_0^1 \eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})w\left(\frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)$$
$$\times\hbar\left(\frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)d\eta$$
$$\leq\left[\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)+\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)\right]$$
$$\times\int_0^1 \eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})w\left(\frac{(1-\eta)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+\eta\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}\right)d\eta.$$

Equivalently,

$$\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}(\hbar w)\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)+\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}(\hbar w)\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)$$
$$\leq\left[\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}w\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)+\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}w\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)\right]$$
$$\times\frac{\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)+\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)}{2}. \tag{4.5}$$

The desired inequality (4.1) is obtained by combining (4.4) and (4.5). □

Corollary 4. *Under assumption of Theorem* 4 *with* $\sigma(0)=\beta=1$ *and* $p=\varphi_1=0$, *we have*

$$\hbar\left(\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}\right)\times\left[J^{\Omega}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}w\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)\right.$$
$$\left.+J^{\Omega}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}w\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)\right]$$
$$\leq J^{\Omega}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}(\hbar w)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)+J^{\Omega}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}(\hbar w)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)$$
$$\leq\left[J^{\Omega}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}w\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)+J^{\Omega}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}-}w\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)\right]$$
$$\times\frac{\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}\right)+\hbar\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)}{2}.$$

Remark 5. *On letting* $\Omega=\mathfrak{e}_2=1$ *and* $\mathfrak{e}_1=0$, *Corollary 4 coincides with the classical Hermite-Hadamard-Fejér inequality.*

Theorem 5. *Let* $\hbar\in L^1[\mathfrak{a},\mathfrak{m}]$ *be a nonnegative convex function with* $0\leq\mathfrak{a}<\mathfrak{m}$ *and* $\mathfrak{a}<\mathfrak{e}_2<\mathfrak{m}$, $\varphi_2\in\mathbb{R}$, $p\geq 0$, $\delta\geq 1$, $e>b>0$, $0\leq\mu\leq 1$, $0\leq\mathfrak{k}_1<1$, $\alpha,\beta,\gamma,\rho,\Omega,\tau,\varkappa,r,s,v>0$ *such that* $0<q\leq r+v+\rho$. *Further, let* $w\in L^1([,])$ *be nonnegative and symmetric with respect to* $\frac{x_2+y_2}{2}$, *that is,* $w(x_2+y_2-x)=w(x)$ *with* $\mathfrak{a}\leq x_2\leq x\leq y_2\leq\mathfrak{m}$, *where* x_2 *and* y_2 *are defined by* (3.17) *and* (3.18), *respectively. Then,*

$$\hbar\left(\frac{2\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}{2}\right)\left[\left(\varepsilon^{\varphi_2}_{\mathfrak{a}+}w\right)(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1))+\left(\varepsilon^{\varphi_2}_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}w\right)(\mathfrak{a})\right]$$

$$\leq \left(\varepsilon_{\mathfrak{a}+}^{\varphi_2}(\hbar w)\right)(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1))+\left(\varepsilon_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}^{\varphi_2}(\hbar w)\right)(\mathfrak{a})$$
$$\leq \frac{(1+\mathfrak{k}_1)\hbar(\mathfrak{a})+(1-\mathfrak{k}_1)\hbar(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)}{2}\left[\left(\varepsilon_{\mathfrak{a}+}^{\varphi_2}w\right)(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1))+\left(\varepsilon_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}^{\varphi_2}w\right)(\mathfrak{a})\right]. \quad (4.6)$$

Proof. The proof is followed by inequality (3.20) by multiplying both sides by

$$\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})w((1-\eta)\mathfrak{a}+\eta(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2))),$$

and integrating over $\eta\in[0,1]$, we obtain

$$2\hbar\left(\frac{2\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}{2}\right)\int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})w((1-\eta)\mathfrak{a}+\eta(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)))d\eta$$
$$\leq\int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})w((1-\eta)\mathfrak{a}+\eta(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)))\hbar(\eta\mathfrak{a}+(1-\eta)(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)))d\eta$$
$$+\int_0^1\eta^{\Omega-1}\mathbf{E}(b,e;\chi\eta^{\rho})w((1-\eta)\mathfrak{a}+\eta(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)))\hbar((1-\eta)\mathfrak{a}+\eta(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)))d\eta$$

$$\Rightarrow 2\hbar\left(\frac{2\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}{2}\right)\int_{\mathfrak{a}}^{\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}(y_2-\mathfrak{a})^{\Omega-1}\mathbf{E}(b,e;\varphi_2(y_2-\mathfrak{a})^{\rho})w(y_2)dy_2$$
$$\leq\int_{\mathfrak{a}}^{\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{m}-\mathfrak{e}_2)}(y_2-\mathfrak{a})^{\Omega-1}\mathbf{E}(b,e;\varphi_2(y_2-\mathfrak{a})^{\rho})\hbar(2\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)-y_2)w(y_2)dy_2$$
$$+\int_{\mathfrak{a}}^{\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}(y_2-\mathfrak{a})^{\Omega-1}\mathbf{E}(b,e;\varphi_2(y_2-\mathfrak{a})^{\rho})\hbar(y_2)w(y_2)dy_2,$$

or

$$2\hbar\left(\frac{2\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}{2}\right)\int_{\mathfrak{a}}^{\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}(y_2-\mathfrak{a})^{\Omega-1}\mathbf{E}(b,e;\varphi_2(y_2-\mathfrak{a})^{\rho})w(y_2)dy_2$$
$$\leq\int_{\mathfrak{a}}^{\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)-s_2)^{\Omega-1}\times$$
$$\mathbf{E}(b,e;\varphi_2(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)-s_2)^{\rho})\hbar(s_2)w(2\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)-s_2)ds_2$$
$$+\int_{\mathfrak{a}}^{\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}(y_2-\mathfrak{a})^{\Omega-1}\mathbf{E}(b,e;\varphi_2(y_2-\mathfrak{a})^{\rho})\hbar(y_2)w(y_2)dy_2.$$

Equivalently,

$$2\hbar\left(\frac{2\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}{2}\right)\left(\varepsilon_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}^{\varphi_2}w\right)(\mathfrak{a})$$
$$\leq\left(\varepsilon_{\mathfrak{a}+}^{\varphi_2}(\hbar w)\right)(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1))+\left(\varepsilon_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}^{\varphi_2}(\hbar w)\right)(\mathfrak{a}). \quad (4.7)$$

Provided that w is symmetric with regard to $\frac{2\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}{2}$, we have

$$\left(\varepsilon_{\mathfrak{a}+}^{\varphi_2}w\right)(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1))=\left(\varepsilon_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}^{\varphi_2}w\right)(\mathfrak{a})$$

$$= \frac{\left(\varepsilon_{\mathfrak{a}+}^{\varphi_2} w\right)(\mathfrak{a} + (\mathfrak{m} - \mathfrak{e}_2)(1 - \mathfrak{k}_1)) + \left(\varepsilon_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}^{\varphi_2} w\right)(\mathfrak{a})}{2}. \tag{4.8}$$

By combining (4.7) and (4.8), we have

$$\begin{aligned} &\hbar\left(\frac{2\mathfrak{a} + (\mathfrak{m} - \mathfrak{e}_2)(1 - \mathfrak{k}_1)}{2}\right)\left[\left(\varepsilon_{\mathfrak{a}+}^{\varphi_2} w\right)(\mathfrak{a} + (\mathfrak{m} - \mathfrak{e}_2)(1 - \mathfrak{k}_1)) + \left(\varepsilon_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}^{\varphi_2} w\right)(\mathfrak{a})\right] \\ &\leq \left(\varepsilon_{\mathfrak{a}+}^{\varphi_2}(\hbar w)\right)(\mathfrak{a} + (\mathfrak{m} - \mathfrak{e}_2)(1 - \mathfrak{k}_1)) + \left(\varepsilon_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}^{\varphi_2}(\hbar w)\right)(\mathfrak{a}). \end{aligned} \tag{4.9}$$

Again, multiplying both sides of (3.28) by $\eta^{\Omega-1}\mathbf{E}(b, e; \chi\eta^\rho) \times$
$w((1 - \eta)\mathfrak{a} + \eta(\mathfrak{k}_1\mathfrak{a} + (\mathfrak{a} + \mathfrak{m} - \mathfrak{e}_2)(1 - \mathfrak{k}_1)))$ and integrating over $\eta \in [0, 1]$, we obtain

$$\begin{aligned} &\int_0^1 \eta^{\Omega-1}\mathbf{E}(b, e; \chi\eta^\rho) w((1-\eta)\mathfrak{a}+\eta(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)))\hbar(\eta\mathfrak{a}+(1-\eta)(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a} + \mathfrak{m} - \mathfrak{e}_2)))d\eta \\ &+ \int_0^1 \eta^{\Omega-1}\mathbf{E}(b, e; \chi\eta^\rho) w((1-\eta)\mathfrak{a}+\eta(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)))\hbar((1-\eta)\mathfrak{a}+\eta(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)))d\eta \\ &\leq [(1+\mathfrak{k}_1)\hbar(\mathfrak{a})+(1-\mathfrak{k}_1)\hbar(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)] \int_0^1 \eta^{\Omega-1}\mathbf{E}(b, e; \chi\eta^\rho) w((1-\eta)\mathfrak{a}+\eta(\mathfrak{k}_1\mathfrak{a}+(1-\mathfrak{k}_1)(\mathfrak{a} + \mathfrak{m} - \mathfrak{e}_2)))d\eta. \end{aligned}$$

Equivalently,

$$\begin{aligned} &\left(\varepsilon_{\mathfrak{a}+}^{\varphi_2}(\hbar w)\right)(\mathfrak{a} + (\mathfrak{m} - \mathfrak{e}_2)(1 - \mathfrak{k}_1)) + \left(\varepsilon_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}^{\varphi_2}(\hbar w)\right)(\mathfrak{a}) \\ &\leq \frac{(1+\mathfrak{k}_1)\hbar(\mathfrak{a})+(1-\mathfrak{k}_1)\hbar(\mathfrak{a} + \mathfrak{m} - \mathfrak{e}_2)}{2}\left[\left(\varepsilon_{\mathfrak{a}+}^{\varphi_2} w\right)(\mathfrak{a} + (\mathfrak{m} - \mathfrak{e}_2)(1 - \mathfrak{k}_1)) + \left(\varepsilon_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}^{\varphi_2} w\right)(\mathfrak{a})\right]. \end{aligned} \tag{4.10}$$

The desired inequality (4.6) is demonstrated by combining (4.9) and (4.10). □

Corollary 5. *Under assumption of Theorem* 5 *with* $\sigma(0) = \beta = 1$ *and* $p = \varphi_2 = 0$, *we have*

$$\begin{aligned} &\hbar\left(\frac{2\mathfrak{a} + (\mathfrak{m} - \mathfrak{e}_2)(1 - \mathfrak{k}_1)}{2}\right)\left[J_{\mathfrak{a}+}^{\Omega} w(\mathfrak{a} + (\mathfrak{m} - \mathfrak{e}_2)(1 - \mathfrak{k}_1)) + J_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}^{\Omega} w(\mathfrak{a})\right] \\ &\leq J_{\mathfrak{a}+}^{\Omega}(\hbar w)(\mathfrak{a} + (\mathfrak{m} - \mathfrak{e}_2)(1 - \mathfrak{k}_1)) + J_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}^{\Omega}(\hbar w)(\mathfrak{a}) \\ &\leq \frac{[(1 + \mathfrak{k}_1)\hbar(\mathfrak{a}) + (1 - \mathfrak{k}_1)\hbar(\mathfrak{a} + \mathfrak{m} - \mathfrak{e}_2)]}{2}\left[J_{\mathfrak{a}+}^{\Omega} w(\mathfrak{a} + (\mathfrak{m} - \mathfrak{e}_2)(1 - \mathfrak{k}_1)) + J_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)]-}^{\Omega} w(\mathfrak{a})\right]. \end{aligned}$$

Remark 6. *On letting* $\mathfrak{k}_1 = 0$, $\mathfrak{e}_2 = \mathfrak{a}$, *and* $\Omega = 1$, *Corollary* 5 *coincides with the classical Hermite-Hadamard-Fejér inequality.*

5. Numerical and graphical analysis

This section covers the numerical and graphical analysis of our main results to help understand the theoretical results. In every example, there is no correlation between the tables and figures. Random selections were made for both sets of statistics. In our calculation, we find out values separately for the left, middle, and right sides of each inequality of the relevant theorem.

Example 1. *Let* $\hbar(x) = \exp(x)$ *such that* $x \in [0, \infty)$, *and* $\sigma(k) = \frac{1}{7k+1}$, *with* $\beta = 1$, $e = \varkappa$, $q = \nu$, $p = s = r = 0$, $\rho = 3$, *in the result* (3.16) *of Theorem* 2 *(Table 1). Additionally, Figure 1 displays a graphic representation of the result* (3.16) *in Theorem* 2 *by taking the above functions with* $\Omega = \mathfrak{a} = 1$, $\beta = 11$, $p = \varphi_2 = 0$, $\mathfrak{e}_2 = 7$, $\mathfrak{m} = 10$, *and* $0 \leq \mathfrak{t}_1 \leq 0.9$.

Table 1. Comparison of values in result of Theorem 2.

$\mathfrak{a}$	$\mathfrak{e}_2$	$\mathfrak{m}$	$\mathfrak{t}_1$	Ω	χ	LHS of (3.16)	Mid of (3.16)	RHS of (3.16)
1	4	9	0	0.7	2	33.1155	89.8721	203.0735
4	49	90	0.2	0.5	0	7.2378e+08	8.1666e+14	1.3974e+19
3	4	40	0.5	0.6	0.125	1.6275e+05	1.2652e+08	2.1648e+16
0.1	0.4	1	0.7	19	1	0.7047	0.7640	0.9476
0.9	101	400	0.9	11	0.27	7.6483e+06	3.2571e+12	8.7879e+128
21	50	100	0.999	34	10	1.3522e+09	1.3526e+09	3.4188e+27

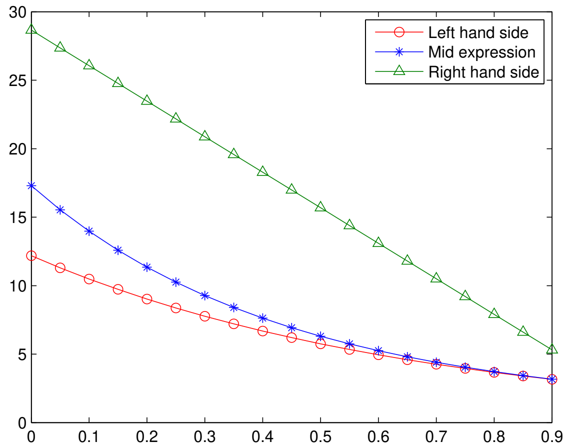

Figure 1. Validity of inequality (3.16) in Theorem 2.

Example 2. *Let* $\hbar(x) = x^3$ *such that* $x \in [0, \infty)$, *with* $\sigma(0) = 1$, $\beta = 3$, $p = \varphi_3 = 0$ *in the result* (3.30) *of Theorem* 3 *(Table 2). Additionally, Figure 2 displays a graphic representation of the result* (3.30) *in Theorem* 3 *by taking the above function with assumptions:* $\sigma(0) = 1$, $\beta = 3.5$, $p = \varphi_3 = 0$, $\mathfrak{a} = 0.2$, $\mathfrak{e}_1 = 0.5$, $\mathfrak{e}_2 = 3$, $\Omega = 2$, *and* $10 \leq \mathfrak{m} \leq 20$.

Table 2. Comparison of values in result of Theorem 3.

$\mathfrak{a}$	$\mathfrak{e}_1$	$\mathfrak{e}_2$	$\mathfrak{m}$	Ω	LHS of (3.30)	Mid of (3.30)	RHS of (3.30)
3	7	11	21	5	3.5625e+03	3.5639e+03	3.5931e+03
11	100	124	129	10	1.2126e+06	1.2126e+06	1.2127e+06
0.1	0.3	4	10	21	8.9989	9.0367	18.5549
0.9	1	3	612	11	2.8779e+07	2.8901e+07	3.8316e+07
7	8	19	20	7	2.1049e+03	2.1050e+03	2.1094e+03
1	10	40	47	8	2.2352e+03	2.2355e+03	2.2482e+03

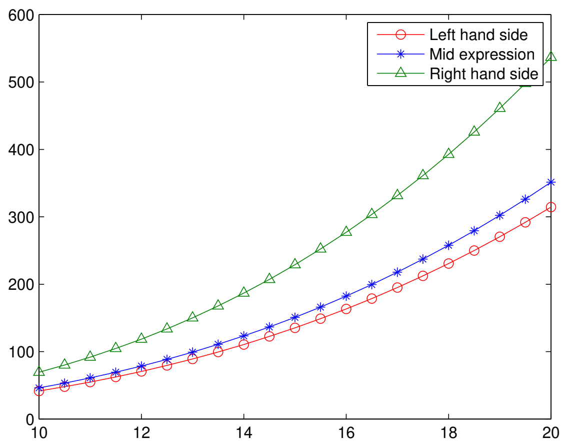

Figure 2. Validity of inequality (3.30) in Theorem 3.

Example 3. *Let* $\hbar(x) = x^4$ *such that* $x \in [0, \infty)$, *and* $w(x) = 2$, *with* $\sigma(0) = 1$, $\beta = 7.1$, $p = \varphi_1 = 0$ *in the result* (4.1) *of Theorem* 4 *(Table 3). Additionally, Figure 3 displays a graphic representation of the result* (4.1) *in Theorem* 4 *by taking the above functions with assumptions:* $\sigma(0) = 1$, $\beta = 1.2$, $p = \varphi_1 = 0$, $\mathfrak{a} = 0.3$, $\mathfrak{e}_1 = 1$, $\mathfrak{e}_2 = 5$, $\mathfrak{m} = 21$, *and* $0 \leq \Omega \leq 10$.

Table 3. Comparison of values in result of Theorem 4.

$\mathfrak{a}$	$\mathfrak{e}_1$	$\mathfrak{e}_2$	$\mathfrak{m}$	Ω	LHS of (4.1)	Mid of (4.1)	RHS of (4.1)
2	5	10	11	13	3.8095e-07	3.8817e-07	3.9054e-07
0.3	0.9	4	20	20	0.3106	0.4941	0.5331
10	950	980	990	5	2.8371e+10	2.8371e+10	2.8371e+10
11	20	50	71	2	4.6259e+06	4.6285e+06	4.6337e+06
22	30	100	130	9	200.7545	200.8328	200.8709
5	6	120	125	4	2.9150e+03	2.9304e+03	2.9481e+03

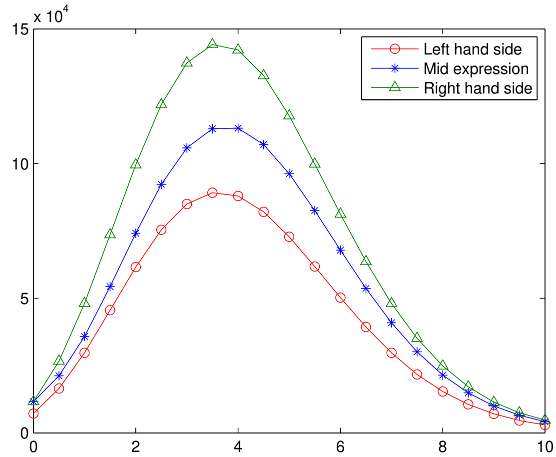

Figure 3. Validity of inequality (4.1) in Theorem 4.

6. Application

In this section, we give some examples of our established results related to modified Bessel functions and matrices. Here, $\mathbb{C}^n$ is expressed as the set of $n \times n$ complex matrices, $\mathbb{M}_n$ as the algebra of $n \times n$ complex matrices, and $\mathbb{M}_n^+$ as the strictly positive matrices in $\mathbb{M}_n$. That is, $A \in \mathbb{M}_n^+$ if $\langle Au, u\rangle > 0$ for all nonzero $u \in \mathbb{C}^n$. Sababheh [28, Theorem 4] incorporated the concept of matrices and convexity together, i.e., $\hbar(z) = \|A^z XB^{1-z} + A^{1-z}XB^z\|$, $A, B \in \mathbb{M}_n^+$, $X \in \mathbb{M}_n$ is convex on $\mathbb{R}$ for all $z \in [0, 1]$, provided that $\|.\|$ is a unitarily invariant norm. Then, by using Theorem 1, we have

$$
\begin{aligned}
&\left\|A^{\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}} XB^{\frac{2\mathfrak{e}_2(1-\mathfrak{a})+(2\mathfrak{e}_1+1)(\mathfrak{a}-\mathfrak{m})}{2\mathfrak{e}_2}} + A^{\frac{2\mathfrak{e}_2(1-\mathfrak{a})+(2\mathfrak{e}_1+1)(\mathfrak{a}-\mathfrak{m})}{2\mathfrak{e}_2}} XB^{\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}}\right\| \\
&\leq \frac{1}{2\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}1\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right)} \times \left(\frac{\mathfrak{m}-\mathfrak{a}}{\mathfrak{e}_2}\right)^{\Omega} \left[\int_0^1 z^{\Omega-1}\mathbf{E}(b, e; \chi z^{\rho})\right. \\
&\times \left\|A^{\frac{z\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-z)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}} XB^{1-\frac{z\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-z)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}}\right. \\
&\left. + A^{1-\frac{z\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-z)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}} XB^{\frac{z\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-z)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}}\right\| dz \\
&+ \int_0^1 z^{\Omega-1}\mathbf{E}(b, e; \chi z^{\rho}) \left\|A^{\frac{(1-z)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+z\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}} XB^{1-\frac{(1-z)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+z\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}}\right. \\
&\left.\left. + A^{1-\frac{(1-z)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+z\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}} XB^{\frac{(1-z)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+z\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}}\right\| dz\right] \\
&\leq \frac{1}{2}\left[\left\|A^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}} XB^{\frac{\mathfrak{e}_2(1-\mathfrak{a})+\mathfrak{e}_1(\mathfrak{a}-\mathfrak{m})}{\mathfrak{e}_2}} + A^{\frac{\mathfrak{e}_2(1-\mathfrak{a})+\mathfrak{e}_1(\mathfrak{a}-\mathfrak{m})}{\mathfrak{e}_2}} XB^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}}\right\|\right. \\
&\left. + \left\|A^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}} XB^{\frac{\mathfrak{e}_2(1-\mathfrak{a})+(\mathfrak{e}_1+1)(\mathfrak{a}-\mathfrak{m})}{\mathfrak{e}_2}} + A^{\frac{\mathfrak{e}_2(1-\mathfrak{a})+(\mathfrak{e}_1+1)(\mathfrak{a}-\mathfrak{m})}{\mathfrak{e}_2}} XB^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}}\right\|\right].
\end{aligned} \tag{6.1}
$$

For more understanding of (6.1), let $\sigma(0) = \mathfrak{e}_1 = \Omega = 1$, $\mathfrak{e}_2 = \beta = 3$, $\mathfrak{m} = 4$, $\mathfrak{a} = p = \varphi_1 = 0$, $X = I$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, and $B = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. In unitarily invariant norm, we are taking Ky Fan k-norm,

$$
\begin{aligned}
&\left\|A^{\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}} XB^{\frac{2\mathfrak{e}_2(1-\mathfrak{a})+(2\mathfrak{e}_1+1)(\mathfrak{a}-\mathfrak{m})}{2\mathfrak{e}_2}} + A^{\frac{2\mathfrak{e}_2(1-\mathfrak{a})+(2\mathfrak{e}_1+1)(\mathfrak{a}-\mathfrak{m})}{2\mathfrak{e}_2}} XB^{\frac{(2\mathfrak{e}_2-2\mathfrak{e}_1-1)\mathfrak{a}+(2\mathfrak{e}_1+1)\mathfrak{m}}{2\mathfrak{e}_2}}\right\| \\
&= \left\|A^2B^{-1} + A^{-1}B^2\right\| = 18.6667,
\end{aligned} \tag{6.2}
$$

$$
\left\|A^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}} XB^{\frac{\mathfrak{e}_2(1-\mathfrak{a})+\mathfrak{e}_1(\mathfrak{a}-\mathfrak{m})}{\mathfrak{e}_2}} + A^{\frac{\mathfrak{e}_2(1-\mathfrak{a})+\mathfrak{e}_1(\mathfrak{a}-\mathfrak{m})}{\mathfrak{e}_2}} XB^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}}\right\| = \left\|A^{\frac{4}{3}}B^{-\frac{1}{3}} + A^{-\frac{1}{3}}B^{\frac{4}{3}}\right\| = 10.0402, \tag{6.3}
$$

$$
\left\|A^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}} XB^{\frac{\mathfrak{e}_2(1-\mathfrak{a})+(\mathfrak{e}_1+1)(\mathfrak{a}-\mathfrak{m})}{\mathfrak{e}_2}} + A^{\frac{\mathfrak{e}_2(1-\mathfrak{a})+(\mathfrak{e}_1+1)(\mathfrak{a}-\mathfrak{m})}{\mathfrak{e}_2}} XB^{\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}}\right\| = \left\|A^{\frac{8}{3}}B^{-\frac{5}{3}} + A^{-\frac{5}{3}}B^{-\frac{8}{3}}\right\| = 37.7620, \tag{6.4}
$$

$$
\left(\varepsilon^{\varphi_1}_{\frac{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}}{\mathfrak{e}_2}+}1\right)\left(\frac{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}}{\mathfrak{e}_2}\right) = \left(\varepsilon^{0}_{\frac{4}{3}+}1\right)\left(\frac{8}{3}\right) = 0.6667, \tag{6.5}
$$

$$
\int_0^1 z^{\Omega-1}\mathbf{E}(b,e;\chi z^\rho)\left\|A^{\frac{z\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-z)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}}XB^{1-\frac{z\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-z)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}}\right.
$$
$$
\left.+A^{1-\frac{z\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-z)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}}XB^{\frac{z\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+(1-z)\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}}\right\|dz
$$
$$
=\int_0^1 \mathbf{E}(b,e;0)\left\|A^{\frac{8-4z}{3}}B^{\frac{-5+4z}{3}}+A^{\frac{-5+4z}{3}}B^{\frac{8-4z}{3}}\right\|dz = 14.17225, \tag{6.6}
$$

$$
\int_0^1 z^{\Omega-1}\mathbf{E}(b,e;\chi z^\rho)\left\|A^{\frac{(1-z)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+z\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}}XB^{1-\frac{(1-z)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+z\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}}\right.
$$
$$
\left.+A^{1-\frac{(1-z)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+z\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}}XB^{\frac{(1-z)\{(\mathfrak{e}_2-\mathfrak{e}_1)\mathfrak{a}+\mathfrak{e}_1\mathfrak{m}\}+z\{(\mathfrak{e}_2-\mathfrak{e}_1-1)\mathfrak{a}+(\mathfrak{e}_1+1)\mathfrak{m}\}}{\mathfrak{e}_2}}\right\|dz
$$
$$
=\int_0^1 \mathbf{E}(b,e;0)\left\|A^{\frac{4+4z}{3}}B^{-\frac{1+4z}{3}}+A^{-\frac{1+4z}{3}}B^{\frac{4+4z}{3}}\right\|dz = 5.09522. \tag{6.7}
$$

Substituting the values from (6.2)–(6.7) in (6.1), we have $18.6667 < 19.26747 < 23.9011$.

Watson [29] defined the function $M_\theta : \mathbf{R} \to [1,\infty)$ as

$$
M_\theta(\nu) = 2^\theta\Gamma(\theta+1)\nu^{-\theta}I_\theta(\nu)\ \ \forall\, \nu\in\mathbf{R},\ \theta > -1,
$$

provided that the modified Bessel function of first kind is: $I_\theta(\nu) = \sum_{z=0}^{\infty}\frac{(\frac{\nu}{2})^{\theta+2z}}{z!\Gamma(\theta+z+1)}$. Employing the above two functions, one can have $M'_\theta(\nu) = \frac{\nu M_{\theta+1}(\nu)}{2(\theta+1)}$. If we use $\hbar(\nu) = M'_\theta(\nu)$ and the above identity in Theorem 2, then we have

$$
\frac{2\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}{4(\theta+1)}M_{\theta+1}\left(\frac{2\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}{2}\right)
$$
$$
\leq \frac{1}{2(\varepsilon_{\mathfrak{a}+}^{\varphi_2}1)(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1))}\left[\varepsilon_{\mathfrak{a}+}^{\varphi_2}\left(\frac{\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1)}{2(\theta+1)}M_{\theta+1}(\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1))\right)\right.
$$
$$
\left.+\varepsilon_{[\mathfrak{a}+(\mathfrak{m}-\mathfrak{e}_2)(1-\mathfrak{k}_1]-}^{\varphi_2}\left(\frac{\mathfrak{a}}{2(\theta+1)}M_{\theta+1}(\mathfrak{a})\right)\right]
$$
$$
\leq \frac{1}{2}\left[(1+\mathfrak{k}_1)\left(\frac{\mathfrak{a}}{2(\theta+1)}M_{\theta+1}(\mathfrak{a})\right)+(1-\mathfrak{k}_1)\left(\frac{\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2}{2(\theta+1)}M_{\theta+1}(\mathfrak{a}+\mathfrak{m}-\mathfrak{e}_2)\right)\right].
$$

7. Conclusions

A generalized MPMLF has been introduced as a generalization of the Wright function [2], the Shukla-Prajapati function [4], the Salim-Faraj function [6], Andric et al. function [7], the Bansal-Mehrez function [8], the Raina function [9], the generalized Wright function [10], and the Pochhammer-Barnes confluent hypergeometric function [27]. Moreover, the defined fractional integral operator is a generalization of the Prabhakar fractional integral [3], the Srivastava-Tomovski fractional integral [5], the Salim-Faraj fractional integral [6], Andric et al. fractional integral [7], the Raina fractional integral [9], and the Riemann-Liouville fractional integral [25]. The article is an elegant unification of several known Mittag-Leffler-type functions and related integral operators. Based on defined generalized fractional integrals, some new extended and generalized estimates for Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities are produced. These findings enable us to generalize existing functional inequality discoveries involving convex functions to a new class of inequalities. The validity of the derived results is ensured through matrix theory, special functions, and numerically and graphically. The results of the paper are expected to be of interest to readers. In the future, the defined generalized fractional integral can be helpful for the researchers to find more interesting results.

Author contributions

Sabir Hussain: Conceptualization, formal analysis, writing-review and editing, methodology, validation; Sobia Rafeeq: Conceptualization, formal analysis, writing-review and editing, writing-original draft preparation, validation, visualization, Software, investigation; Jongsuk Ro: Conceptualization, formal analysis, writing-review and editing, visualization, resources; Rida Khaliq: Methodology, writing-original draft preparation; Azhar Ali: Methodology, writing-original draft preparation. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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