The geometric fixed points of real topological cyclic homology revisited

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Abstract

We use the results of Quigley and Shah to give a formula for the geometric fixed points of real topological cyclic homology of a bounded below ring spectrum with anti-involution. The anti-involution on a ring spectrum A gives rise to a spectrum with canonical left and right A-module structures, whose tensor product over A can be equipped with an action by the cyclic group of order 2. Our formula is then given by the homotopy equalizer of two maps from the homotopy fixed points to the Tate construction. Furthermore, we show that this homotopy equalizer is equivalent to the one given in the computation by Dotto, Moi and Patchkoria, thereby proving their result with different methods.

As an application of our result we calculate the real topological cyclic homology of group ring spectra for abelian groups and certain classes of dihedral groups. We do this for arbitrary ground ring spectra, whose underlying spectra are bounded below. This is accomplished via a decomposition formula for the dihedral bar construction of a group.

Contents

Chapter 1

Introduction

1.1 Statement of results

Let A be a ring with anti-involution, that is a map of rings

$$\omega\colon A^{\mathrm{op}} \to A$$

such that $\omega^{\mathrm{op}} \circ \omega = \mathrm{id}_A$. Then one can form its real algebraic K-theory spectrum $\mathrm{KR}(A)$, which was originally introduced by Hesselholt and Madsen [HM15] (see [DO19] and [Cal+20a, Section 4.5] for modern references). This is the algebraic analogue to Atiyah's real topological K-theory spectrum $\mathrm{KR}^{\mathrm{top}}$ [Ati66], which classifies complex vector bundles carrying an anti-linear involution that is compatible with an involution on the base space. The spectrum $\mathrm{KR}(A)$ is a Σ_2-equivariant spectrum[1], where Σ_2 is the symmetric group on 2 elements. Its underlying non-equivariant spectrum is its connective K-theory spectrum $\mathrm{K}(A)$, just as $\mathrm{KR}^{\mathrm{top}}$ is a Σ_2-equivariant refinement of complex topological K-theory KU.

For any (left) A-module M let $DM := \mathrm{Hom}_A(M, A)$ be its dual. Then DM is a left A-module via the assignment

$$A \times DM \to DM, (a, f) \mapsto [m \mapsto f(m)\omega(a)].$$

If P is a finitely generated A-module, then D^2P is isomorphic to P[2], thus we obtain a duality functor

$$D\colon \mathrm{Proj}(A)^{\mathrm{op}} \to \mathrm{Proj}(A)$$

on the category of finitely generated projective right A-modules, which induces a Σ_2-action on $\mathrm{K}(A)$. By [Cal+20b, Theorem 1] the isotropy separation sequence[3] of $\mathrm{KR}(A)$ then takes the

[1] By this we mean what other authors often call a *genuine* equivariant spectrum.

[2] This isomorphism is given by $\varphi\colon P \to D^2P, \varphi(x)(f) = \omega(f(x))$.

[3] Recall that for any Σ_2-spectrum X there is a fiber sequence $X_{h\Sigma_2} \to X^{\Sigma_2} \to \Phi^{\Sigma_2}X$ connecting homotopy orbits, (categorical) fixed points and geometric fixed points.

form

$$\mathrm{K}(A)_{h\Sigma_2} \to \mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(A) \to \mathrm{L}^{\mathrm{short}}(A),$$

where the middle term is the Grothendieck-Witt-spectrum of unimodular forms and the right hand term is a connective spectrum whose homotopy groups are Ranicki's (non-periodic) symmetric L-theory groups defined in [Ran80].

One important method for computing algebraic K-theory is the use of trace maps into other theories. A particularly useful one is the cyclotomic trace map

$$\mathrm{tr}\colon \mathrm{K}(A) \to \mathrm{TC}(A)$$

into topological cyclic homology, which was first constructed by Bökstedt, Hsiang and Madsen in [BHM93]. Its main usefulness comes from the theorem of Dundas-Goodwillie-McCarthy [DGM13], which states that if $A \to B$ is a map of connective ($\mathbb{E}_1$-)ring spectra such that the induced map $\pi_0 A \to \pi_0 B$ is surjective with nilpotent kernel, then the square

$$\begin{array}{ccc} \mathrm{K}(A) & \xrightarrow{\mathrm{tr}} & \mathrm{TC}(A) \\ \downarrow & & \downarrow \\ \mathrm{K}(B) & \xrightarrow{\mathrm{tr}} & \mathrm{TC}(B) \end{array}$$

is a homotopy pullback. Just as for algebraic K-theory there is a real version of topological cyclic homology, which was originally introduced by Høgenhaven in [Høg16] in analogy to the classical definition of [BHM93] and later refined by Quigley-Shah in [QS21a]. It is expected that a real cyclotomic trace

$$\mathrm{tr}_{\mathbb{R}}\colon \mathrm{KR}(A) \to \mathrm{TCR}(A)$$

exists and that a real version of the Dundas-Goodwille-McCarthy theorem is true.

The first main result of this thesis computes the geometric fixed points of the p-typical version $\Phi^{\Sigma_2}\mathrm{TCR}(A;p)$ for any orthogonal ring spectrum A with anti-involution and any prime p under the assumption that A is bounded below as a spectrum. Before we state the result we need to give some preliminary explanations. Note that the anti-involution of A amounts to a Σ_2-action on the underlying orthogonal spectrum, thus it gives rise to a Σ_2-equivariant orthogonal spectrum. If A is discrete, then the resulting Σ_2-spectrum is the Eilenberg-MacLane spectrum of the Σ_2-Mackey functor with underlying abelian group A, fixed points A^{Σ_2} and the algebraic transfer

$$A \to A^{\Sigma_2}, a \mapsto a + \omega(a).$$

In both the discrete and non-discrete case we can form the geometric fixed point spectrum $\Phi^{\Sigma_2}A$. It has a natural structure as both a left and right A-module. Recall that the Hill-Hopkins-Ravenel norm $N_1^{\Sigma_2}A$ is the Σ_2-equivariant orthogonal ring spectrum with underlying spectrum $A \otimes A$, coordinatewise multiplication, and Σ_2-action given by switching the smash factors. Furthermore, there is a natural diagonal equivalence of ring spectra $\Delta\colon A \xrightarrow{\simeq} \Phi^{\Sigma_2} N_1^{\Sigma_2} A$. The left and right actions described in elements by

$$A \otimes N_1^{\Sigma_2} A \to A, x \otimes a \otimes b \mapsto \omega(a) \cdot x \cdot b,$$
$$N_1^{\Sigma_2} A \otimes A \to A, a \otimes b \otimes x \mapsto a \cdot x \cdot \omega(b),$$

are Σ_2-equivariant and since Φ^{Σ_2} is monoidal this yields the claimed A-module structures on $\Phi^{\Sigma_2} A$. The computation of $\Phi^{\Sigma_2}\mathrm{TCR}(A;p)$ involves the (derived) tensor product $\Phi^{\Sigma_2} A \otimes_A \Phi^{\Sigma_2} A$, which comes equipped with a canonical C_2-action given in the commutative case by swapping the factors, as well as a canonical map

$$\Phi^{\Sigma_2} A \otimes_A \Phi^{\Sigma_2} A \to (\Phi^{\Sigma_2} A \otimes_A \Phi^{\Sigma_2} A)^{tC_2}. \tag{1.1}$$

For more details we refer to Remark 3.5.11.

We can now state our main result.

Theorem A. *Let A be a (orthogonal) ring spectrum with anti-involution such that the underlying spectrum of A is bounded below.*

(i) For any odd prime p there is an equivalence

$$\Phi^{\Sigma_2}\mathrm{TCR}(A;p) \simeq \Phi^{\Sigma_2} A \otimes_A \Phi^{\Sigma_2} A.$$

(ii) The spectrum $\Phi^{\Sigma_2}\mathrm{TCR}(A;2)$ is equivalent to the homotopy fiber of

$$(\Phi^{\Sigma_2} A \otimes_A \Phi^{\Sigma_2} A)^{hC_2} \xrightarrow{\mathrm{can}-\phi_A} (\Phi^{\Sigma_2} A \otimes_A \Phi^{\Sigma_2} A)^{tC_2},$$

where can *is the canonical map of the homotopy fixed points to the Tate construction and ϕ_A is (1.1) composed with the inclusion of fixed points $(\Phi^{\Sigma_2} A \otimes_A \Phi^{\Sigma_2} A)^{hC_2} \to \Phi^{\Sigma_2} A \otimes_A \Phi^{\Sigma_2} A$.*

Suppose A is a discrete ring with $\frac{1}{2} \in A$. Then the Σ_2-spectrum obtained from A is a module over $\mathbb{Z}\left[\frac{1}{2}\right]$, where the latter has the trivial anti-involution. By the isotropy separation sequence $\Phi^{\Sigma_2}\left(\mathbb{Z}\left[\frac{1}{2}\right]\right)$ is trivial, thus we immediately obtain the following result from Theorem A.

Theorem B. *If A is a discrete ring with anti-involution such that $\frac{1}{2} \in A$, then $\Phi^{\Sigma_2}\mathrm{TCR}(A;p)$ vanishes for all primes p.*

An equivalent version of Theorem A has already appeared in [DMP21, Theorem A], which is proven by different methods. The proof given in loc. cit. relies on genuine real cyclotomic methods and a description of $\widetilde{E\mathcal{F}}$[4] as a pushout for a certain family of subgroups $\mathcal{F}$ of the dihedral group D_{2p^n} of order $2p^n$. For odd p we can completely rely on the theory of real cyclotomic spectra

[4] If $\mathcal{F}$ is a family of subgroups of a group G, then $E\mathcal{F}$ is the associated universal space and $\widetilde{E\mathcal{F}}$ is the (pointed) cofiber of the map $E\mathcal{F}_+ \to S^0$ that collapses $E\mathcal{F}$ to the non-basepoint.

developed by Quigley and Shah in [QS21a] and [QS21b]. In the case $p = 2$ we do need to use genuine real cyclotomic spectra to prove boundedness results on $\mathrm{TCR}(A;2)$. We refer to the next section for more details. The rest of the proof however, again only uses results from [QS21a] and [QS21b]. Additionally, Dotto, Moi and Patchkoria assume that $\Phi^{\Sigma_2}A$ is bounded below, but we will explain in Section 3.5.3 that [DMP21, Theorem A] without this assumption is still true and in this version equivalent to Theorem A. In contrast to [DMP21, Theorem A] the terms in the fiber sequence of Theorem A only involve the homotopy fixed points and the Tate construction of $\Phi^{\Sigma_2}A \otimes_A \Phi^{\Sigma_2}A$, whereas one term in the fiber sequence of [DMP21, Theorem A] is given by the *genuine* fixed points of $\Phi^{\Sigma_2}A \otimes_A \Phi^{\Sigma_2}A$[5]. In particular, one can immediately derive Theorem B in the case of $p = 2$, whereas it is not apparent how this follows from [DMP21, Theorem A].

In Chapter 4 we apply Theorem A to compute the geometric fixed points of TCR in the case of group rings. In the context of group rings we consider discrete (but possibly infinite) groups G equipped with the anti-involution given by inversion

$$\iota_G \colon G^{\mathrm{op}} \to G, g \mapsto g^{-1}.$$

If A is a ring spectrum with anti-involution we smash the anti-involutions, that is we equip the ring spectrum $A[G] = A \otimes G_+$ with the anti-involution

$$\omega_{A[G]} \colon (A[G])^{\mathrm{op}} = (A \otimes G_+)^{\mathrm{op}} = A^{\mathrm{op}} \otimes G_+^{\mathrm{op}} \xrightarrow{\omega_A \otimes \iota_{G+}} A \otimes G_+ = A[G].$$

In particular we do not consider the geometrically relevant situation where the involution is twisted by an orientation character. However, we emphasize that our results apply to arbitrary ground rings (provided their underlying spectra are bounded below). Our results on group rings are complementary to those in [DMP21], where only spherical coefficients are considered, but more general groups and involutions than in this thesis.

We shall consider two classes of groups. For any group G we denote by G_2 (the set of) its 2-torsion and let Σ_2 act on $G_2 \times G_2$ via $(g, h) \mapsto (g, gh)$. The first class of groups is that of groups G such that G_2 is contained in the center. We prove the following result in Theorem 4.2.

Theorem C. *Let G be a discrete group such that G_2, the elements of order 2, are contained in the center of G. For any ring spectrum A with anti-involution whose underlying spectrum is bounded below, there is a fiber sequence*

$$\bigoplus_{G_2} \Phi^{\Sigma_2}\mathrm{TCR}(A;2) \otimes BG_+ \to \Phi^{\Sigma_2}\mathrm{TCR}(A[G];2) \to \bigoplus_{(G_2 \setminus \{1\} \times G_2)/\Sigma_2} (\Phi^{\Sigma_2}A \otimes_A \Phi^{\Sigma_2}A) \otimes BG_+.$$

In particular, if G is 2-torsion-free the assembly map is an equivalence

$$\Phi^{\Sigma_2}\mathrm{TCR}(A;2) \otimes BG_+ \simeq \Phi^{\Sigma_2}\mathrm{TCR}(A[G];2).$$

[5]This is advantageous for the following reason. If $f \colon X \to Y$ is a map of C_2-spectra which is an equivalence on the underlying spectra, then it induces equivalences $X^{hC_2} \simeq Y^{hC_2}$ and $X^{tC_2} \simeq Y^{tC_2}$, but in general not on the genuine fixed points. The same remark applies if we replace C_2 with any finite group G.

The second class of groups is that of dihedral groups. We denote by D_{2n} the dihedral group with $2n$ elements, D_∞ the infinite dihedral group and $D_{2^\infty} = \mathrm{colim}_n D_{2^{n+1}}$, where the colimit is taken along the inclusions $D_{2^{n+1}} \to D_{2^{n+2}}$. Again all results hold for arbitrary ground rings. The following table indicates which result holds for which group.

Group	Result
D_{2n} with $n \equiv 2 \mod 4$ and $n \neq 2$	Theorem 4.3
D_{2n} with $n \equiv 0 \mod 4$	Theorem 4.5 (i)
D_{2^∞}	Theorem 4.5 (ii)
D_∞	Theorem 4.6

Some of the results are fairly technical. We describe those results (or special cases of them) which are easy to state to give an idea. We start with D_{2n} in the case that $n \equiv 2 \mod 4$ and $n \neq 2$. Theorem C applies in particular to abelian groups, hence also to $G = D_4$. The assumption on n implies that the inclusion $D_4 \to D_{2n}$ has a retraction, hence also

$$\mathrm{TCR}(A[D_4];2) \to \mathrm{TCR}(A[D_{2n}];2)$$

has a retraction. We describe the geometric fixed points of the cofiber. The cleanest statement is for coefficients in a discrete ring. We shall index over (subsets of) the orbit set $(C_n \setminus C_2)/\Sigma_2$ and we write $\Phi^{\Sigma_2} A^{\otimes_A 2}$ as a shorthand for $\Phi^{\Sigma_2} A \otimes_A \Phi^{\Sigma_2}$ for typographical reasons. Here Σ_2 acts as usual on C_n by sending an element to its inverse and $C_n \setminus C_2$ denotes the set theoretic difference. For discrete A the statement specializes as follows.

Theorem D. *Let A be a discrete ring with anti-involution and assume that $n \equiv 2 \mod 4$ and $n \neq 2$. Denote by $\mathrm{TCR}(A[D(D_{2n})/D(D_4)];2)$ the cofiber of the map*

$$\mathrm{TCR}(A[D_4];2) \to \mathrm{TCR}(A[D_{2n}];2)$$

induced by the inclusion $D_4 \to D_{2n}$. There is a pullback

$$\begin{array}{ccc}
\Phi^{\Sigma_2}\mathrm{TCR}(A[D(D_{2n})/D(D_4)];2) & \longrightarrow & \bigoplus_{\substack{[c^j]\in(C_n\setminus C_2)/\Sigma_2 \\ j \text{ even}}} \left(((\Phi^{\Sigma_2}A^{\otimes_A 2})\otimes BC_{2+})^{hC_2}\right)^{\oplus 2} \\
\downarrow & & \downarrow {\scriptstyle \mathrm{can}-\phi} \\
\bigoplus_{\substack{[c^j]\in(C_n\setminus C_2)/\Sigma_2 \\ j \text{ odd}}} (\Phi^{\Sigma_2}A^{\otimes_A 2})\otimes BC_{2+} & \xrightarrow{\phi} & \bigoplus_{\substack{[c^j]\in(C_n\setminus C_2)/\Sigma_2 \\ j \text{ even}}} \left(((\Phi^{\Sigma_2}A^{\otimes_A 2})\otimes BC_{2+})^{tC_2}\right)^{\oplus 2}.
\end{array}$$

where in the right hand terms BC_2 carries the trivial C_2-action.

In the case $n \equiv 0 \mod 4$ we can only describe $\Phi^{\Sigma_2}\mathrm{TCR}(A[D_{2n}];2)$ up to a filtration. The graded pieces have a particularly nice description for $D_{2^{n+1}}$ and D_{2^∞}.

Theorem E. *Let A be a ring spectrum with anti-involution whose underlying spectrum is bounded below.*

(i) There is a finite filtration of length n on $\Phi^{\Sigma_2}\mathrm{TCR}(A[D_{2^{n+1}}];2)$ such that the kth graded piece $\mathrm{gr}^k\Phi^{\Sigma_2}\mathrm{TCR}(A[D_{2^{n+1}}];2)$ is equivalent to

$$\begin{cases} (\Phi^{\Sigma_2}\mathrm{TCR}(A;2)\otimes BD_{2^{n+1}+}\oplus\Phi^{\Sigma_2}\mathrm{TCR}(A;2)\otimes BD_{4+})^{\oplus 2}, & \text{if } k=0,\\ (\Phi^{\Sigma_2}A\otimes_A\Phi^{\Sigma_2}A)\otimes BD_{2^{n+1}+}\oplus \\ \left((\alpha^*(\Phi^{\Sigma_2}A\otimes_A\Phi^{\Sigma_2}A))_{hD_8}\right)^{\oplus 2}\oplus((\Phi^{\Sigma_2}A\otimes_A\Phi^{\Sigma_2}A)\otimes BD_{4+})^{\oplus 4}, & \text{if } k=1,\\ \bigoplus_{(C_{2^k}\setminus C_{2^{k-1}})/\Sigma_2}\left((\Phi^{\Sigma_2}A\otimes_A\Phi^{\Sigma_2}A)_{hC_2}\otimes BC_{2+}\right)^{\oplus 2}, & \text{if } 2\le k<n,\\ \bigoplus_{(C_{2^n}\setminus C_{2^{n-1}})/\Sigma_2}(\Phi^{\Sigma_2}A\otimes_A\Phi^{\Sigma_2}A)\otimes BC_{2+}, & \text{if } k=n,\end{cases}$$

where $\alpha\colon D_8\to D_8/D_4\cong C_2$ is the projection.

(ii) There is a filtration of infinite length on $\Phi^{\Sigma_2}\mathrm{TCR}(A[D_{2^\infty}];2)$ with the kth graded piece $\mathrm{gr}^k\Phi^{\Sigma_2}\mathrm{TCR}(A[D_{2^\infty}];2)$ being equivalent to

$$\begin{cases} (\Phi^{\Sigma_2}\mathrm{TCR}(A;2)\otimes BD_{2^\infty+})^{\oplus 2}\oplus\Phi^{\Sigma_2}\mathrm{TCR}(A;2)\otimes BD_{4+}, & \text{if } k=0,\\ (\Phi^{\Sigma_2}A\otimes_A\Phi^{\Sigma_2}A)\otimes BD_{2^\infty+}\oplus(\alpha^*(\Phi^{\Sigma_2}A\otimes_A\Phi^{\Sigma_2}A))_{hD_8}\oplus \\ ((\Phi^{\Sigma_2}A\otimes_A\Phi^{\Sigma_2}A)\otimes BD_{4+})^{\oplus 2}, & \text{if } k=1,\\ \bigoplus_{(C_{2^k}\setminus C_{2^{k-1}})/\Sigma_2}(\Phi^{\Sigma_2}A\otimes_A\Phi^{\Sigma_2}A)_{hC_2}\otimes BC_{2+}, & \text{if } k\ge 2,\end{cases}$$

where $\alpha\colon D_8\to D_8/D_4\cong C_2$ is the projection.

This concludes our discussion on group rings and we turn our attention back to Theorem A. We will derive it as a special case of a statement for the real topological cyclic homology of a *real cyclotomic spectrum*. Recall from [NS18] that a p-cyclotomic spectrum is a spectrum X with C_{p^∞}-action[6] together with a C_{p^∞}-equivariant map

$$\varphi_X\colon X\to X^{tC_p}$$

into the Tate construction called the Frobenius. The most prominent examples of a (p-)cyclotomic spectrum are the topological Hochschild homology spectrum $\mathrm{THH}(A)$ of a ring (spectrum) A, which is constructed as the realization of a cyclic bar construction, and X^{triv}, which is obtained by equipping a spectrum X with the trivial C_{p^∞}-action. Then there is a natural C_{p^∞}-equivariant map $X\to X^{hC_p}$ and the Frobenius of X^{triv} is obtained by composing this map with the map from homotopy fixed points into the Tate construction.

[6] By this we mean an object of $\mathrm{Fun}(BC_{p^\infty},\mathbf{Sp})$.

The p-typical topological cyclic homology $\mathrm{TC}(X;p)$ of X is defined to be the fiber of the map

$$X^{hC_{p^\infty}} \xrightarrow{\mathrm{can}_p - \varphi_X^{hC_{p^\infty}}} (X^{tC_p})^{hC_{p^\infty}}.$$

and for a ring (spectrum) A one defines $\mathrm{TC}(A;p) = \mathrm{TC}(\mathrm{THH}(A);p)$. Here can_p is the composite

$$X^{hC_{p^\infty}} \simeq (X^{hC_p})^{h(C_{p^\infty}/C_p)} \simeq (X^{hC_p})^{hC_{p^\infty}} \to (X^{tC_p})^{hC_{p^\infty}},$$

where the arrow is the canonical map from homotopy fixed points into the Tate construction.

The real version of a p-cyclotomic spectrum has been introduced Quigley and Shah in [QS21a]. First, they replace a spectrum with C_{p^∞}-action by a Σ_2-spectrum X with *twisted* C_{p^∞} action (see Definition 3.4.1). Next, for such a Σ_2-spectrum X Quigley and Shah construct *parametrized* versions of homotopy orbits $X_{h_{\Sigma_2}C_{p^n}}$, homotopy fixed points $X^{h_{\Sigma_2}C_{p^n}}$ and the Tate construction $X^{t_{\Sigma_2}C_{p^n}}$, which are Σ_2-spectra by construction and whose underlying spectra are the usual homotopy orbits, fixed points and Tate construction.

A real p-cyclotomic spectrum X is then a Σ_2-spectrum with twisted C_{p^∞}-action together with a C_{p^∞}-equivariant map of Σ_2-spectra

$$\varphi_X \colon X \to X^{t_{\Sigma_2}C_p}$$

into the parametrized Tate construction, which we also call the Frobenius. The prime examples are the *real* version of topological Hochschild homology $\mathrm{THR}(A)$ of a ring (spectrum) with anti-involution A, which is now obtained as the geometric realization of a *dihedral* bar construction, and the real version of X^{triv}. If X is a Σ_2-spectrum, we equip it with the trivial twisted C_{p^∞}-action and then there is again a natural C_{p^∞}-equivariant map $X \to X^{h_{\Sigma_2}C_p}$ and we compose it with the natural map $X^{h_{\Sigma_2}C_p} \to X^{t_{\Sigma_2}C_p}$ to obtain the Frobenius.

For a real p-cyclotomic spectrum X one then defines $\mathrm{TCR}(X;p)$ in analogy to TC as the fiber of

$$X^{h_{\Sigma_2}C_{p^\infty}} \xrightarrow{\mathrm{can}_p - \varphi_X^{h_{\Sigma_2}C_{p^\infty}}} (X^{t_{\Sigma_2}C_p})^{h_{\Sigma_2}C_{p^\infty}},$$

and for a ring spectrum A with anti-involution one defines $\mathrm{TCR}(A;p) = \mathrm{TCR}(\mathrm{THR}(A);p)$. The main technical result of this thesis computes $\Phi^{\Sigma_2}\mathrm{TCR}(X;p)$. For any Σ_2-spectrum X with twisted C_{2^∞}-action (or even a C_2-action) the geometric fixed points of $X^{t_{\Sigma_2}C_2}$ carry a residual C_2-action and there is a projection map

$$\mathrm{pr}_t \colon \Phi^{\Sigma_2} X^{t_{\Sigma_2}C_2} \to (\Phi^{\Sigma_2}X)^{tC_2}.$$

If X is a real 2-cyclotomic spectrum its Frobenius thus gives rise to a map

$$\phi_X \colon (\Phi^{\Sigma_2}X)^{hC_2} \to \Phi^{\Sigma_2}X \xrightarrow{\Phi^{\Sigma_2}\varphi_X} \Phi^{\Sigma_2}X^{t_{\Sigma_2}C_2} \xrightarrow{\mathrm{pr}_t} (\Phi^{\Sigma_2}X)^{tC_2},$$

where the undecorated arrow is the inclusion of fixed points. In Section 3.5.3 we prove the following result.

Theorem F. *Let X a real p-cyclotomic spectrum such that its underlying spectrum is bounded below.*

(i) If p is odd, there is an equivalence

$$\Phi^{\Sigma_2}\mathrm{TCR}(X;p) \simeq \Phi^{\Sigma_2}X. \tag{1.2}$$

(ii) The spectrum $\Phi^{\Sigma_2}\mathrm{TCR}(X;2)$ is equivalent to

$$\mathrm{fib}((\Phi^{\Sigma_2}X)^{hC_2} \xrightarrow{\mathrm{can}-\phi_X} (\Phi^{\Sigma_2}X)^{tC_2}), \tag{1.3}$$

where can *is the canonical map from homotopy fixed points into the Tate construction.*

We obtain Theorem A from Theorem F by plugging in $X = \mathrm{THR}(A)$, applying [DMPR21, Theorem 2.26] and carefully examining the residual C_2-action as well as the map $\phi_{\mathrm{THR}(A)}$.

Finally, we point out that an integral version of Theorem F for real cyclotomic spectra has been announced in [QS21a] (but at the time of of writing this thesis a proof has not been published). Roughly, a real cyclotomic spectrum is a Σ_2-spectrum X with twisted $\mathbb{T}$-action, where $\mathbb{T}$ denotes the circle group, together with a real p-cyclotomic structure for all p and $\mathrm{TCR}(X)$ is defined as the fiber of

$$X^{h_{\Sigma_2}\mathbb{T}} \xrightarrow{(\varphi_p-\mathrm{can}_p)_{p\in\mathbb{P}}} \prod_{p\in\mathbb{P}}(X^{t_{\Sigma_2}C_p})^{h_{\Sigma_2}\mathbb{T}}.$$

Then [QS21a, Theorem A.2] states that $\Phi^{\Sigma_2}\mathrm{TCR}(X)$ is equivalent to (1.3), in other words there is an equivalence

$$\Phi^{\Sigma_2}\mathrm{TCR}(X) \simeq \Phi^{\Sigma_2}\mathrm{TCR}(X;2).$$

We do not know how to derive this result from Theorem F, nor the other way around, not even after p-completion. For odd p it is unclear whether there is an equivalence $\mathrm{TCR}(X) \simeq \mathrm{TCR}(X;p)^\wedge_p$ and in general it is not true that the p-completion of (1.3) is equivalent to $\Phi^{\Sigma_2}X^\wedge_p \simeq \Phi^{\Sigma_2}(X^\wedge_p)$. We believe that $\mathrm{TCR}(X)^\wedge_2 \simeq \mathrm{TCR}(X;2)^\wedge_2$, but in general 2-completion only commutes with Φ^{Σ_2} if the underlying spectrum is bounded below. We do not know how to prove that the underlying spectrum of $\mathrm{TCR}(X)$ is bounded below if the underlying spectrum of X is bounded below.

1.2 Outline and strategy of the proofs

We start by giving an overview of the proof of Theorem F. It is contained entirely in Chapter 3, which is independent from the other chapters. For $p = 2$ the main idea is to reduce the proof of Theorem F to the case where the underlying spectrum of X is contractible. The proof of this special case is relatively easy, but the reduction to this case is non-trivial. We first sketch the argument for this special case. By definition $X^{h_{\Sigma_2} C_{2^\infty}} = \lim_n X^{h_{\Sigma_2} C_{2^n}}$ and if the underlying spectrum of X is contractible, it turns out that we can directly calculate

$$\lim_n \Phi^{\Sigma_2} X^{h_{\Sigma_2} C_{2^n}} \quad \text{and} \quad \lim_n \Phi^{\Sigma_2} (X^{t_{\Sigma_2} C_2})^{h_{\Sigma_2} C_{2^n}}$$

to be equivalent to

$$(\Phi^{\Sigma_2} X)^{hC_2} \quad \text{and} \quad (\Phi^{\Sigma_2})^{tC_2}.$$

We perform this calculation in Section 3.3. To derive Theorem F we now apply the following result, which should be well known to experts and goes back to Adams. Its proof uses the isotropy separation sequence and the fact that $E\Sigma_2$ has a Σ_2-CW-structure with a finite $n-$skeleton for all n. Note that the assumptions of the lemma are fulfilled if the underlying spectrum of X is contractible, since then the underlying spectra of $X^{h_{\Sigma_2} C_{2^n}}$ and $(X^{t_{\Sigma_2} C_2})^{h_{\Sigma_2} C_{2^n}}$ are contractible as well.

Lemma G. *Consider a tower*

$$\cdots \to X_3 \to X_2 \to X_1$$

of Σ_2-spectra and suppose there is an integer k such that for all n the underlying spectrum of X_n is at least k-connected. Then the natural map of spectra

$$\Phi^{\Sigma_2} \lim_n X_n \to \lim_n \Phi^{\Sigma_2} X_n$$

is an equivalence.

To reduce the general case to the special case above, note that the cofiber sequence of pointed Σ_2-spaces

$$E\Sigma_{2+} \to S^0 \to \widetilde{E\Sigma_2}$$

gives rise to the cofiber sequence

$$\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes (E\Sigma_{2+})^{\mathrm{triv}}; 2) \to \Phi^{\Sigma_2}\mathrm{TCR}(X; 2) \to \Phi^{\Sigma_2}\mathrm{TCR}(X \otimes (\widetilde{E\Sigma_2})^{\mathrm{triv}}; 2).$$

Since the underlying spectrum of $X \otimes (\widetilde{E\Sigma_2})^{\mathrm{triv}}$ is contractible and the equivalence

$$\Phi^{\Sigma_2} X \simeq \Phi^{\Sigma_2}(X \otimes \widetilde{E\Sigma_2})$$

is C_2-equivariant if we equip $\widetilde{E\Sigma_2}$ with the trivial C_2-action, we need to show that $\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes (E\Sigma_{2+})^{\mathrm{triv}}; 2)$ vanishes.

Showing that $\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes (E\Sigma_{2+})^{\mathrm{triv}}; 2)$ vanishes is more involved. The key step here is to define $\mathrm{TCR}^n(X; p)$ as the fiber of

$$X^{h_{\Sigma_2} C_{p^n}} \xrightarrow{\varphi_{X,n} - \mathrm{can}_p^n} (X^{t_{\Sigma_2} C_p})^{h_{\Sigma_2} C_{p^{n-1}}},$$

where $\varphi_{X,n}$ is induced by the Frobenius of X and can_p^n is defined analogously to the map can_p used in the fiber sequence defining $\mathrm{TCR}(X;p)$. Then $\mathrm{TCR}(X;p) \simeq \lim_n \mathrm{TCR}^n(X;p)$ and we prove the following boundedness result using genuine cyclotomic methods in order to exchange the limit with the geometric fixed points.

Proposition H. *Let X be a real p-cyclotomic spectrum.*

(i) If the underlying spectrum of X is k-connected, then the underlying spectrum of $\mathrm{TCR}^n(X;p)$ *is $(k-1)$-connected and the underlying spectrum of* $\mathrm{TCR}(X;p)$ *is $(k-2)$-connected.*

(ii) If the underlying Σ_2-spectrum of X is k-connected, then $\mathrm{TCR}^n(X;p)$ *is $(k-1)$-connected and* $\mathrm{TCR}(X;p)$ *is $(k-2)$-connected.*

We stress that this is the only part of the argument which uses genuine real cyclotomic methods. Also, for odd p one can prove this result without resorting to genuine real cyclotomic spectra, see Remark 3.5.3.

Showing the vanishing of $\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes (E\Sigma_{2+})^{\mathrm{triv}};2)$ is now done in two steps. First, if we equip $E\Sigma_2$ with its standard Σ_2-CW-structure with one free Σ_2-cell in each dimension, then we use Proposition H to reduce the claim to showing that $\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes (\Sigma_2/1_+)^{\mathrm{triv}};2)$ vanishes. Using Proposition H a second time, it suffices to show that $\Phi^{\Sigma_2}\mathrm{TCR}^n(X \otimes (\Sigma_2/1_+)^{\mathrm{triv}};2)$ vanishes. This is easy. The underlying $D_{2^{n+1}}$-spectrum of $X \otimes (\Sigma_2/1_+)^{\mathrm{triv}}$ is induced up from C_{2^n} and we show in Section 3.2 that both $\Phi^{\Sigma_2}(-)^{h_{\Sigma_2}C_{2^n}}$ and $\Phi^{\Sigma_2}((-)^{t_{\Sigma_2}C_2})^{h_{\Sigma_2}C_{2^n}}$ vanish on $D_{2^{n+1}}$-spectra of this form.

Showing Theorem F for odd p is much easier than for $p = 2$. The results of [QS21b] and Section 3.3 imply that $\Phi^{\Sigma_2}(X^{t_{\Sigma_2}C_p})^{h_{\Sigma_2}C_{p^n}}$ vanishes for all n and if the underlying spectrum of X is bounded below it follows essentially from the dihedral Tate orbit lemma [QS21a, Lemma 3.20] that

$$\Phi^{\Sigma_2}\mathrm{TCR}^n(X;p) \simeq \Phi^{\Sigma_2}X_{h_{\Sigma_2}C_{p^n}}.$$

Finally, in Lemma 3.3.1 we prove that

$$\Phi^{\Sigma_2}X_{h_{\Sigma_2}C_{p^n}} \simeq \Phi^{\Sigma_2}X$$

and that under this equivalence the maps in the limit system are constant, hence Theorem F follows from Proposition H and Lemma G.

Our results on group rings rest on the observation that there is an equivalence of real cyclotomic spectra

$$\mathrm{THR}(A[G]) \simeq \mathrm{THR}(A) \otimes DN(G)_+,$$

where $DN(G)$ is the *dihedral nerve* (also called *dihedral bar construction* by other authors) of G 2.1.10. The geometric fixed points have the property that

$$\Phi^{\Sigma_2}(\mathrm{THR}(A) \otimes DN(G)_+) \simeq \Phi^{\Sigma_2}\mathrm{THR}(A) \otimes DN(G)_+^{\Sigma_2}$$

The main idea here is to prove a decomposition formula for $DN(G)^{\Sigma_2}$ in terms of classifying spaces (Proposition 2.3.2) and study the parts of this decomposition separately. This decomposition does not seem to be new, but to the best of our knowledge the residual C_2-action on the involved classifying spaces has not been studied before [7].

[7] Again, Theorem A allows for easier computations. If one wants to use [DMP21, Theorem A] then one

1.3 Notation and conventions

1.3.1 Group-theoretic notation

Throughout this thesis G denotes a discrete group unless otherwise stated. The only topological groups we consider are the circle group $\mathbb{T}$ and the orthogonal group $O(2)$. We denote by Σ_2 the symmetric group on 2 elements and by $\sigma \in \Sigma_2$ the non-trivial element. It is often useful to identify Σ_2 with the multiplicative group $\{\pm 1\}$. For $n = 1, 2, \ldots, \infty$ we write C_n for the cyclic group of order n and denote its generator by c. If p is a prime we let $C_{p^\infty} = \mathrm{colim}_n C_{p^n}$ denote the Prüfer group. Here, the colimit is taken along the inclusions $C_{p^n} \to C_{p^{n+1}}$. The group Σ_2 acts on C_n and C_{p^∞} by sending an element to its inverse. We denote the resulting semi-direct products by D_{2n} and D_{2p^∞}. We refer to D_{2n} as the dihedral group of order $2n$. In the case $p = 2$ we write D_{2^∞} to avoid awkward notation. Similarly, we write $D_{2^{k+1}}$ if $n = 2^k$ and D_∞ instead of $D_{2\cdot\infty}$ for the infinite dihedral group.

There are preferred embeddings of Σ_2 into D_{2n}, D_∞ and D_{2p^∞}. We also denote the image of these embeddings by Σ_2. We also specify a section of the determinant $\det\colon O(2) \to \Sigma_2$ by sending $\sigma \in \Sigma_2$ to the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in O(2).$$

Similarly, for finite n we specify a preferred embedding $C_n \to \mathbb{T}$ by sending c to $\exp(\frac{2\pi i}{n}) \in \mathbb{T}$. Combining the previous two embeddings then yields a preferred embedding $D_{2n} \to O(2)$ for finite n.

All actions of G (including $\mathbb{T}$ and $O(2)$) on a set, space or spectrum are left actions. Recall that any group G acts on itself via conjugation:

$$G \times G \to G, (g, h) \mapsto ghg^{-1}.$$

We denote the set of orbits by $\mathrm{conj}(G)$ and refer to its elements as the *conjugacy classes* of G. We use the notation $[g]$ for the conjugacy class of $g \in G$. For any $g \in G$ its isotropy group under the conjugation action is its centralizer, which we denote by $Z_G\langle g\rangle$.

Similarly, there is a $G \times \Sigma_2$-action on G:

$$G \times \Sigma_2 \times G \to G, (g, \tau, h) \mapsto gh^\tau g^{-1}. \tag{1.4}$$

We denote the set of orbits by $\mathrm{conj}_{\mathbb{R}}(G)$ and refer to its elements as the *real conjugacy classes* of G. To distinguish them from the conjugacy classes, we use the notation $[[g]]$ for the real conjugacy class of $g \in G$. We denote the isotropy group of g under this action by $SZ_G\langle g\rangle$ and refer to it as the semi-centralizer. We will be mostly interested in the (real) conjugacy classes of

has to determine the homotopy type of $DN(G)$ as a D_4-space, whereas in our situation it suffices to know the Σ_2-equivariant homotopy type and the residual C_2-action.

C_n, D_{2n} and D_{2p^∞}. In the dihedral groups the conjugacy classes coincide with the real conjugacy classes, since conjugation with σ sends every element to its inverse.

Finally, if H is a subgroup of G, we write $N_G H$ for its normalizer. The Weyl group of H in G is $W_G H := N_G H/H$.

1.3.2 Homotopy and category theory

We freely make use of the framework of ∞-categories except in Chapter 2. The standard reference is [Lur09]. This is primarily for convenience and is in no way essential. The so inclined reader can translate everything in the language of model categories. In that case, all functors in sight should be replaced by their derived functors and (co)limits have to be interpreted as homotopy (co)limits.

Any stable ∞-category, in particular the ∞-category $\mathbf{Sp}$ of spectra and the ∞-category $\mathbf{Sp}^G$ of G-spectra, comes equipped with a biproduct, which we denote by $\oplus$. We denote the tensor of a monoidal ∞-category by $\otimes$, in particular we use this notation for the smash product of(G-)spectra and (G-)spaces. By G-spectra we mean what are also called *genuine* equivariant spectra. The theory is well developed for a compact Lie group G. We put $\mathbf{Sp}^{D_{2p^\infty}} = \lim_n \mathbf{Sp}^{D_{2p^n}}$ and $\mathbf{Sp}^{C_{p^\infty}} = \lim_n \mathbf{Sp}^{C_{p^n}}$, where the limit is taken along the restrictions $\mathbf{Sp}^{D_{2p^{n+1}}} \to \mathbf{Sp}^{D_{2p^n}}$ induced by the inclusion $D_{2p^n} \subset D_{2p^{n+1}}$. We refer to the appendix for more details.

Next, we need to specify some conventions for classifying spaces. If G is a (well-pointed topological) group, then we use for EG the simplicial model $EG_\bullet = G^{\bullet+1}$ with G acting on the first coordinate from the left. The face and degeneracy maps are given by

$$d_i(g_0, \dots, g_n) = \begin{cases} (g_0, \dots, g_{i-1}, g_i g_{i+1}, g_{i+1}, \dots, g_n) & \text{if } i < n, \\ (g_0, \dots, g_{n-1}) & \text{if } i = n \end{cases}$$

and

$$s_i(g_0, \dots, g_n) = (g_0, \dots, g_i, 1, g_{i+1}, \dots, g_n).$$

For a category $\mathcal{C}$ we define its classifying space as the simplicial set

$$B_n\mathcal{C} = \mathrm{Fun}([n], \mathcal{C}).$$

We also need to fix some conventions on action groupoids. Let G be a discrete group and S is a G-set. Then let $G//S$ be its action groupoid, i.e. the groupoid with objects elements of S, morphisms given by the set

$$\mathrm{Hom}_{G//S}(s, t) = \{g \in G \colon gs = t\}$$

and composition given by the multiplication in G. We define $G \smallint S = (G//S)^{\mathrm{op}}$. This notation is non-standard, but it ensures that in $G \smallint S$ the composite of

$$s \xleftarrow{g} t \xleftarrow{h} u$$

is $gh\colon s \leftarrow u$ and hence that $B_\bullet G = B_\bullet(G \int *)$. Furthermore, if we let G act on $G \int G$ by left multiplication on the objects, there is a G-equivariant isomorphism

$$E_\bullet G \xrightarrow{\cong} B_\bullet(G \int G),$$
$$(g_0, g_1, \ldots, g_n) \mapsto (g_0 \xleftarrow{g_0(g_0g_1)^{-1}} g_0g_1 \xleftarrow{g_0g_1(g_0g_1g_2)^{-1}} \cdots \xleftarrow{g_0\cdots g_{n-1}(g_0\cdots g_n)^{-1}} g_0g_1\cdots g_n).$$

1.3.3 Rings and ring spectra with anti-involution

Let A be a ring or orthogonal ring spectrum. Recall that its opposite A^{op} has the same underlying abelian group or spectrum, but multiplication given by

$$A \otimes A \cong A \otimes A \xrightarrow{\mu} A,$$

where the isomorphism is given by swapping the factors and μ is the multiplication of A. A ring (spectrum) with anti-involution is a ring or orthogonal ring spectrum A together with a map of rings or orthogonal ring spectra $\omega\colon A^{\mathrm{op}} \to A$ such that $\omega^{\mathrm{op}} \circ \omega = \mathrm{id}_A$. The anti-involution amounts to a Σ_2-action on the underlying orthogonal spectrum or abelian group, hence gives rise to an orthogonal Σ_2-spectrum by [HHR16, Proposition A.19]. It thus makes sense to talk about the (geometric) Σ_2-fixed points of a ring spectrum with anti-involution. Note that if A is a ring with anti-involution, its Eilenberg-MacLane spectrum HA is a ring spectrum with anti-involution, since $HA^{\mathrm{op}} \cong H(A^{\mathrm{op}})$. The resulting orthogonal Σ_2-equivariant spectrum is equivalent to the Eilenberg-MacLane spectrum of the Σ_2-Mackey functor with underlying abelian group A, fixed points group A^{Σ_2}, and transfer the norm map

$$A \to A^{\Sigma_2}, a \mapsto a + \omega(a),$$

see [DMPR21, Example 2.4]. If A is an orthogonal ring spectrum and G a (topological) group, then we let $A[G] = A \otimes G_+$. This is again a ring spectrum and for a discrete ring A and discrete group G we have $H(A[G]) = HA[G]$, so this definition generalizes the discrete case. If A is commutative we will always equip $A[G]$ with the anti-involution that is the identity on A and sends elements of G to their inverses.

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Chapter 2

Real, cyclic and dihedral objects

In this chapter we review the theory of real, cyclic and dihedral objects. Just as topological Hochschild homology can be constructed as the geometric realization of a cyclic spectrum its real version is constructed as the realization of a dihedral object in spectra. Moreover, from the construction of real topological Hochschild homology it will be immediate that for any (topological) group G and ring spectrum A there is an equivalence of real p-cyclotomic spectra

$$\mathrm{THR}(A[G]) \simeq \mathrm{THR}(A) \otimes DN(G)_+,$$

where $DN(G)$ is the dihedral nerve of G (Example 2.1.10). We shall study the dihedral nerve and its Σ_2-fixed points with their residual C_2-action in Section 2.3 by means of subdivision functors described in Section 2.2.

2.1 Crossed simplicial groups

In this section we review the definitions and main properties of real, cyclic, or dihedral objects. To treat all three simultaneously it is convenient to do this in the framework of crossed simplicial groups. We will see that real, cyclic and dihedral objects can all be interpreted as a kind of module over a certain crossed simplicial group. The main reference for the material in this section is [FL91].

Definition 2.1.1 ([FL91] Definition 1.1)**.** A crossed simplicial group is a sequence of groups $\{G_n\}_{n\in\mathbb{N}_0}$ together with the following data. There is a (small) category ΔG such that:

(i) The category ΔG contains the simplex category Δ as a wide subcategory.

(ii) $\mathrm{Aut}_{\Delta G}([n]) = G_n^{\mathrm{op}}$.

(iii) Any morphism $[m] \to [n]$ in ΔG can be uniquely written as a composite $\phi \circ g$, where $\phi \in \mathrm{Hom}_\Delta([m],[n])$ and $g \in G_m^{\mathrm{op}} = \mathrm{Aut}_{\Delta G}([m])$.

As one expects, $G_\bullet$ forms a simplicial set [FL91, Lemma 1.3] and simplicial groups are special instances of crossed simplicial groups [FL91, Proposition 1.4].

Example 2.1.2. The following are the main examples of crossed simplicial groups of interest to us.

(i) Let $G_n = \Sigma_2$ for all n. We give it the structure of a crossed simplicial group as follows. We let $\Delta\mathbb{R}$ be the category obtained from Δ by adding an automorphism $\omega_n : [n] \to [n]$ and impose the relations $\omega_n^2 = \mathrm{id}_{[n]}, \omega_n \circ \delta_i = \delta_{n-i} \circ \omega_{n-1}$, and $\omega_n \circ \sigma_i = \sigma_{n-i} \circ \omega_{n+1}$. We refer to $\Delta\mathbb{R}$ as the *real simplex category*. The imposed relations ensure that $G_\bullet$ is a constant simplicial set, hence $|G_\bullet| \cong \Sigma_2$.

(ii) The *cyclic category* ΔC first described by Connes [Con83] (but see also [Lod98, Chapter 6]) is an example of a crossed simplicial group. It is obtained by adding an extra automorphism $\tau_n : [n] \to [n]$ subject to the relations

$$\begin{aligned} \tau_n\delta_i &= \delta_{i-1}\tau_{n-1}, \quad \text{for } 1 \le i \le n, & \tau_n\delta_0 &= \delta_n, \\ \tau_n\sigma_i &= \sigma_{i-1}\tau_{n+1}, \quad \text{for } 1 \le i \le n, & \tau_n\sigma_0 &= \sigma_n\tau_{n+1}^2, \\ \tau_n^{n+1} &= \mathrm{id}_{[n]}. \end{aligned}$$

One checks this gives rise to a crossed simplicial group $C_\bullet$ and by [Lod98, p. 7.1.2] we have $|C_\bullet| = \mathbb{T}$.

(iii) Let $G_n = D_{2n+1} = C_{n+1} \rtimes \Sigma_2$. We construct the *dihedral category* ΔD by adding to Δ the automorphisms $\tau_n, \omega_n : [n] \to [n]$ satisfying the relations from the previous two examples and impose the additional relations $\omega_n^2 = \mathrm{id}_{[n]}$ and $\omega_n \circ \tau_n = \tau_n^n \circ w_n$. It follows from [FL91, Proposition 3.4] that this indeed yields the structure of a crossed simplicial group, which we denote by $D_\bullet$. We remark that there are natural inclusions $\Delta C \to \Delta D$ and $\Delta\mathbb{R} \to \Delta D$. Finally, $|D_\bullet| \cong O(2)$ by [Lod87, Proposition 3.10].

Definition 2.1.3. Let $\mathcal{C}$ be a category or an ∞-category.

(i) A *real simplicial object* in $\mathcal{C}$ is a functor $X : (\Delta\mathbb{R})^{\mathrm{op}} \to \mathcal{C}$.

(ii) A *cyclic object* in $\mathcal{C}$ is a functor $X : (\Delta C)^{\mathrm{op}} \to \mathcal{C}$.

(iii) A *dihedral object* in $\mathcal{C}$ is a functor $X : (\Delta D)^{\mathrm{op}} \to \mathcal{C}$.

These are special cases of the following definition.

Definition 2.1.4. Let $G_\bullet$ be a crossed simplicial group with corresponding category ΔG. A $G_\bullet$-object in a category of ∞-category $\mathcal{C}$ is a functor $X : (\Delta G)^{\mathrm{op}} \to \mathcal{C}$. We will often use the notation $X_\bullet$ or X_n just as in the case of simplicial objects.

Remark 2.1.5. If $\mathcal{C}$ is tensored over **Set**[1], then by [FL91, Lemma 4.2] a $G_\bullet$-object is equivalently given by a simplicial object $X_\bullet$ with the following structure:

(i) Left group actions $G_n \otimes X_n \to X_n$.

(ii) Face relations $d_i(gx) = d_i(g) \cdot d_{g^{-1}i}(x)$[2].

(iii) Degeneracy relations $s_i(gx) = s_i(g) \cdot s_{g^{-1}i}(x)$.

In fact, it suffices that the face and degeneracy conditions hold for generators of G_n. In particular, a real simplicial (resp. cyclic) object in $X_\bullet$ is the same as a cyclic object together with a map $w_n \colon X_n \to X_n$ (resp. $t_n \colon X_n \to X_n$) satisfying the dual relations of Example 2.1.2. Similarly, one sees that a dihedral object is an object which is simultaneously a real simplicial object and a cyclic object such that the two structures satisfy the dual compatibility relations from Example 2.1.2 (iii). Finally, a map $f \colon X_\bullet \to Y_\bullet$ is the same as a map of simplicial objects that is $G_\bullet$-equivariant.

We now list the examples which will be of interest to us throughout this thesis.

Example 2.1.6. Let G be a topological group. Then we let $B_{\mathbb{R},\bullet}G$ be the real simplicial space with underlying simplicial space $B_\bullet G$ and anti-involution given on n-simplices by

$$w_n(g_1, \ldots, g_n) = (g_n^{-1}, \ldots, g_1^{-1}).$$

We call its realization $B_{\mathbb{R}}G$ the *real classifying space* of G.

It is useful to generalize this construction to the situation of a category with anti-involution, i.e. a category $\mathcal{C}$ with a functor $\omega_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ such that $\omega_{\mathcal{C}}^{\mathrm{op}} \circ \omega_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}}$. The *real classifying space* $B_{\mathbb{R}}\mathcal{C}$ is the geometric realization of the real simplicial set $B_{\mathbb{R},\bullet}\mathcal{C}$ with underlying simplicial set $B_\bullet\mathcal{C}$ and anti-involution given on n-simplices by

$$w_n(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n) = \omega_{\mathcal{C}}(x_n) \xrightarrow{\omega_{\mathcal{C}}(f_n)} \omega_{\mathcal{C}}(x_{n-1}) \xrightarrow{\omega_{\mathcal{C}}(f_{n-1})} \cdots \xrightarrow{\omega_{\mathcal{C}}(f_1)} \omega_{\mathcal{C}}(x_0).$$

Note that if S is a $(G \times \Sigma_2)$-set, then $G \int S$[3] can be equipped with the anti-involution that sends an element $s \in S$ to σs and elements of G to their inverse. We point out that $B_{\mathbb{R}}G = B_{\mathbb{R}}(G \int *)$.

Example 2.1.7. Let $z \in G$ be a central element. The twisted classifying space $B_\bullet(G, z)$ is the cyclic set with underlying simplicial set $B_\bullet G$ and cyclic operator

$$t_n(g_1, \ldots, g_n) = (z(g_1 \cdots g_n)^{-1}, g_1, \ldots, g_{n-1}).$$

[1] In particular, this is the case for **Top**, $\mathbf{Top}_*$ and **OSp**.

[2] Recall that $G_\bullet$ is itself a simplicial set and that $\mathrm{Aut}_{\Delta G}([n]) = G_n^{\mathrm{op}}$ to make sense of this expression.

[3] For this we assume that G is discrete. In principle one can assume that G is a topological group, give $G \int S$ the structure of a topological groupoid and define $B_{\mathbb{R}}(G \int S)$ as the realization of a real simplicial space, but we have no need for this generality.

To generalize the previous example to the dihedral case, we recall from (1.4) that $G \times \Sigma_2$ acts on G via

$$G \times \Sigma_2 \times G \to G, (g, \tau, h) \mapsto gh^\tau g^{-1},$$

and that for $x \in G$ the isotropy group is denoted by $SZ_G\langle x \rangle$. We equip Σ_2 with an action of $SZ_G\langle x \rangle \times \Sigma_2$ by

$$SZ_G\langle x \rangle \times \Sigma_2 \times \Sigma_2 \to \Sigma_2, (g, \tau, \rho, \pi) \mapsto \tau\rho\pi,$$

so that we can form the real classifying space of $SZ_G\langle x \rangle \wr \Sigma_2$.

Example 2.1.8. Let G be a discrete group.

(i) Let $x \in G$ be a central element of order 2. Then we let $B_{\mathbb{R},\bullet}(G, x)$ be the dihedral set with underlying real simplicial set the real classifying space and underlying cyclic set the twisted classifying space. One checks that the real and cyclic structures are compatible. We denote its realization by $B_{\mathbb{R}}(G, x)$.

(ii) Let $x \in G$ be an element with $x^2 \neq 1$. We let $B_{\mathbb{R},\bullet}(SZ_G\langle x \rangle \wr \Sigma_2, x)$ be the dihedral set with underlying real simplicial set the real classifying space of $SZ_G\langle x \rangle \wr \Sigma_2$ and equip it with the cyclic structure given by

$$t_n(\sigma_0 \xleftarrow{g_1} \sigma_1 \xleftarrow{g_1} \cdots \xleftarrow{g_n} \sigma_n) = \sigma_n \xleftarrow{x^{\sigma_n}(g_1\cdots g_n)^{-1}} \sigma_0 \xleftarrow{g_1} \cdots \xleftarrow{g_{n-1}} \sigma_{n-1}.$$

We denote the realization by $B_{\mathbb{R}}(SZ_G\langle x \rangle \wr \Sigma_2, x)$.

The next two examples are the main objects of interest for studying (real) TC of group rings.

Example 2.1.9. Let G be a (topological) group. Then we let $CN_\bullet(G) = G^{\bullet+1}$ be the cyclic space with face maps

$$d_i(g_0, g_1, \dots, g_n) = \begin{cases} (g_0, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n) & \text{if } i < n, \\ (g_n g_0, g_1, \dots, g_{n-1}) & \text{if } i = n, \end{cases}$$

degeneracy maps

$$s_i(g_0, \dots, g_n) = (g_0, \dots, g_i, 1, g_{i+1}, \dots, g_n)$$

and cyclic operator

$$t_n(g_0, g_1, \dots, g_n) = (g_n, g_0, \dots, g_{n-1}).$$

We denote its realization by $CN(G)$ and refer to it as the *cyclic nerve* of G.

Example 2.1.10. Let G be a (topological) group. Then $DN_\bullet(G)$ is the dihedral space with underlying cyclic space $CN_\bullet(G)$ and anti-involution given by

$$w_n(g_0, g_1, \dots, g_n) = (g_0^{-1}, g_n^{-1}, \dots, g_1^{-1}).$$

We denote the realization of $DN_\bullet(G)$ by $DN(G)$ and refer to it as the *dihedral nerve* of G.

Our main reason for treating the theory of crossed simplicial groups is the following statement.

Theorem 2.1.11 ([FL91] Theorem 5.3). *(i) If $G_\bullet$ is a crossed simplicial group, then $|G_\bullet|$ is a topological group.*

(ii) If $X_\bullet$ is a $G_\bullet$-space or $G_\bullet$-spectrum, then there is a natural $|G_\bullet|$-action on $|X_\bullet|$.

Corollary 2.1.12. *Let $\mathcal{C} = \mathbf{Top}$ or $\mathcal{C} = \mathbf{OSp}$.*

(i) The realization of a real simplicial object in $\mathcal{C}$ has a natural Σ_2-action.

(ii) The realization of a cyclic object in $\mathcal{C}$ has a natural $\mathbb{T}$-action.

(iii) The realization of a dihedral object in $\mathcal{C}$ has a natural $O(2)$-action.

Remark 2.1.13. If $X_\bullet$ is a real simplicial or dihedral space, then we can explicitly describe the Σ_2-action on $|X_\bullet|$. Namely, if $x \in X_n$ and $(t_0, \dots, t_n) \in \Delta^n$, then it follows from [FL91, Lemma 5.6] that the action is given by $\sigma \cdot [(x, t_0, \dots, t_n)] = [(w_n(x), t_n, \dots, t_0)]$

Remark 2.1.14. By [MM02, Theorem V.1.5], the realization of a real simplicial orthogonal spectrum gives rise to an orthogonal Σ_2-spectrum. Similarly, from the realization of a cyclic (respectively dihedral) orthogonal spectrum we obtain a orthogonal $\mathbb{T}$-spectrum (respectively $O(2)$-spectrum).

2.2 Subdivision of dihedral objects.

Given a real simplicial object or a dihedral object, we would like to understand the fixed points of the realizations for (finite) cyclic and dihedral groups. The main tools for this purpose are the edgewise and Segal subdivision functors. The edgewise subdivision gives us access to C_a-fixed points and the Segal subdivision allows us to study the Σ_2-action. In the situation of cyclic objects the edgewise subdivision was first introduced by [BHM93] and later cast into a different light by [Dri04]. We largely follow [Spa00, Section 2], which generalizes the results in [BHM93] to the dihedral situation, and [Sai13], which expands on [Dri04], in this section. Note for the following definition that the simplex category has a monoidal structure given by concatenation, i.e. $[n] \amalg [m] = [n+m+1]$.

Definition 2.2.1. Let a be a natural number. Then the a-fold edgewise subdivision functor $\mathrm{sd}_a \colon \Delta \to \Delta$ is given by

$$\mathrm{sd}_a([n]) = \coprod_{i=1}^{a} [n] = [a(n+1)-1], \mathrm{sd}_a(\alpha) = \alpha \amalg \cdots \amalg \alpha.$$

If $X_\bullet$ is a simplicial object in a category or ∞-category $\mathcal{C}$, its a-fold subdivision $\mathrm{sd}_a X_\bullet$ is given by the composition

$$\Delta^{\mathrm{op}} \xrightarrow{\mathrm{sd}_a^{\mathrm{op}}} \Delta^{\mathrm{op}} \xrightarrow{X} \mathcal{C}.$$

If $X_\bullet$ is a cyclic or dihedral object, its a-fold edgewise subdivision is the a-fold edgewise subdivision of its underlying simplicial object.

To access the Σ_2-fixed points of a real object we need a second subdivision functor.

Definition 2.2.2. The Segal subdivision functor $\mathrm{sd}_S\colon \Delta \to \Delta$ is given by

$$\mathrm{sd}_S([n]) = [n] \amalg [n]^{\mathrm{op}} \cong [2n+1], \mathrm{sd}_S(\alpha) = \alpha \amalg \alpha^{\mathrm{op}}.$$

If $X_\bullet$ is a simplicial object in a category or ∞-category $\mathcal{C}$, its Segal subdivision $\mathrm{sd}_S X_\bullet$ is given by the composition

$$\Delta^{\mathrm{op}} \xrightarrow{\mathrm{sd}_S^{\mathrm{op}}} \Delta^{\mathrm{op}} \xrightarrow{X} \mathcal{C}.$$

If $X_\bullet$ is a real or dihedral object, its Segal subdivision is the Segal subdivision of its underlying simplicial object.

We can equip the Segal subdivision $\mathrm{sd}_S X$ with a simplicial Σ_2-action via

$$(\mathrm{sd}_S X)_k = X_{2k+1} \xrightarrow{w_{2k+1}} X_{2k+1} = (\mathrm{sd}_S X)_k. \tag{2.1}$$

If $X_\bullet$ is a dihedral object, then we can equip its a-fold edgewise subdivision with a simplicial C_a-action via

$$(\mathrm{sd}_a X)_k = X_{a(k+1)-1} \xrightarrow{t_{a(k+1)-1}^{(k+1)}} X_{a(k+1)-1} = (\mathrm{sd}_a X)_k, \tag{2.2}$$

and the subdivision $\mathrm{sd}_S \mathrm{sd}_a X_\bullet$ with a simplicial D_{2a}-action via

$$\begin{aligned} (\mathrm{sd}_S\mathrm{sd}_a X)_k &= X_{2a(k+1)-1} \xrightarrow{t_{2a(k+1)-1}^{2(k+1)}} X_{2a(k+1)-1} = (\mathrm{sd}_S\mathrm{sd}_a X)_k, \\ (\mathrm{sd}_S\mathrm{sd}_a X)_k &= X_{2a(k+1)-1} \xrightarrow{w_{2a(k+1)-1}} X_{2a(k+1)-1} = (\mathrm{sd}_S\mathrm{sd}_a X)_k. \end{aligned} \tag{2.3}$$

Proposition 2.2.3.

(i) Let $X_\bullet$ be a real simplicial space or spectrum. Then the map

$$d_S\colon \Delta^n_{\mathrm{top}} \to \Delta^{2n+1}_{\mathrm{top}}, (t_0, \cdots, t_n) \mapsto \frac{1}{2}(t_0, \ldots, t_n, t_n, \ldots, t_0)$$

induces a natural Σ_2-equivariant homeomorphism $|\mathrm{sd}_S X_\bullet| \xrightarrow{\cong} |X_\bullet|$.

(ii) Let $X_\bullet$ be a dihedral set. Then there is a canonical $O(2)$-action on $|\mathrm{sd}_a X_\bullet|$ extending the simplicial C_a-action (2.2) and the map

$$d_a\colon \Delta^n_{\mathrm{top}} \to \Delta^{a(n+1)-1}_{\mathrm{top}}, (t_0, \ldots, t_n) \mapsto \frac{1}{a}(t_0, \ldots, t_n, \ldots, t_0, \ldots, t_n)$$

induces a natural $O(2)$-equivariant homeomorphism $|\mathrm{sd}_a X| \xrightarrow{\cong} |X|$.

(iii) Let $X_\bullet$ be a dihedral set. Then there is a canonical $O(2)$-action on $|\mathrm{sd}_S \mathrm{sd}_a X_\bullet|$ extending the simplicial D_{2a}-action (2.3) and the map

$$d_{S,a}\colon \Delta^n_{\mathrm{top}} \to \Delta^{2a(n+1)-1}_{\mathrm{top}}, (t_0,\dots,t_n) \mapsto \frac{1}{2a}(t_0,\dots,t_n,t_n,\dots,t_0,\dots,t_0,\dots,t_n,t_n,\dots,t_0)$$

induces a natural $O(2)$-equivariant homeomorphism $|\mathrm{sd}_S \mathrm{sd}_a X| \xrightarrow{\cong} |X|$.

Proof. The map induced by d_S is Σ_2-equivariant by Remark 2.1.13, therefore the first statement follows from [Spa00, Lemma 2.4]. The second and third statement follow from [Sai13, Theorem 1.5], but we need to check that our maps agree with the homeomorphisms constructed there. We do this for $\mathrm{sd}_S \mathrm{sd}_a X_\bullet$. The argument for the a-fold edgewise subdivision is similar.

We denote by $\mathcal{F}$ (resp. $\mathcal{F}_{2a}$) the poset (with respect to inclusion) of finite subsets of $[0,1]$ (resp. $[0,2a]$) and by $\mathcal{G}$ (resp. $\mathcal{G}_{2a}$) the poset (also with respect to inclusion) of finite subsets of $\mathbb{R}/\mathbb{Z}$ (resp. $\mathbb{R}/2a\mathbb{Z}$). For $F \in \mathcal{F}$ we put

$$F_{2a} = \{2k + x, 2(k+1) - x \colon 0 \le k < a, x \in F \cup \{0,1\}\} \subset [0,2a].$$

Let $p\colon [0,1] \to \mathbb{R}/\mathbb{Z}$ and $p_{2a}\colon [0,2a] \to \mathbb{R}/2a\mathbb{Z}$ be the projections. If $F \in \mathcal{G}$, then we put

$$F_{2a} = \{2k + x + 2a\mathbb{Z}, 2(k+1) - x + 2a\mathbb{Z} \colon 0 \le k < a, x \in p^{-1}(F) \cup \{0,1\}\} \subset \mathbb{R}/2a\mathbb{Z}.$$

This yields a commutative square of posets

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{F \mapsto F_{2a}} & \mathcal{F}_{2a} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_{2a}} \\ \mathcal{G} & \xrightarrow{F \mapsto F_{2a}} & \mathcal{G}_{2a} \end{array} \tag{2.4}$$

such that the horizontal arrows are cofinal inclusions and $\mathcal{F}$ and $\mathcal{F}_{2a}$ contain cofinal subsets that are mapped isomorphically to $\mathcal{G}$ resp. $\mathcal{G}_{2a}$ by p resp. p_{2a} (namely all sets $F \in \mathcal{F}$ respectively $F \in \mathcal{F}_{2a}$ that contain both 0 and 1 respectively both 0 and $2a$ or neither). We need some additional notation. For $\mathbf{t} = (t_0,\dots,t_n) \in \Delta^n_{\mathrm{top}}$, let $F(\mathbf{t}) = \{t_0 + \dots + t_i \colon 0 \le i < n\}$ and define $f^{\mathbf{t}}\colon \pi_0([0,1] \setminus F(\mathbf{t})) \to [n]$ by

$$f^{\mathbf{t}}(x) = \begin{cases} 0 & \text{if } x < t_0, \\ i & \text{if } t_0 + \cdots t_{i-1} < x < t_0 + \cdots + t_i \text{ for some } i < n, \\ n & \text{if } t_0 + \cdots t_{n-1} < x. \end{cases}$$

For any simplicial set $X_\bullet$ there is a homeomorphism

$$|X_\bullet| \xrightarrow{\cong} \operatorname*{colim}_{F \in \mathcal{F}} X[\pi_0([0,1] \setminus F)] \tag{2.5}$$

by [Sai13, Theorem 1.1]. It is induced by the map determined by the requirement that for any n-simplex $x\colon \Delta^n \to X_\bullet$ the following diagram commutes:

$$\begin{array}{ccc} |\Delta^n| & \xrightarrow{\mathbf{t}\mapsto f^{\mathbf{t}}} & \operatorname*{colim}_{F\in\mathcal{F}} \Delta^n[\pi_0([0,1]\setminus F)] \\ \big\downarrow{\scriptstyle |x|} & & \big\downarrow{\scriptstyle x} \\ |X_\bullet| & \xrightarrow{(2.5)} & \operatorname*{colim}_{F\in\mathcal{F}} X[\pi_0([0,1]\setminus F)]. \end{array} \tag{2.6}$$

For a dihedral set $X_\bullet$ the isomorphism $|\mathrm{sd}_S\mathrm{sd}_a X| \cong |X|$ is then given by the lower row of the following diagram

$$\begin{array}{ccccccc} |\mathrm{sd}_S\mathrm{sd}_a X| & \xrightarrow{\cong} & \operatorname*{colim}_{F\in\mathcal{F}} X[\pi_0([0,2a]\setminus F_{2a})] & \xrightarrow{\cong} & \operatorname*{colim}_{F\in\mathcal{F}_{2a}} X[\pi_0([0,2a]\setminus F)] & \xleftarrow{\cong} & |X| \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle\cong} & & \big\downarrow{\scriptstyle\cong} & & \big\downarrow{\scriptstyle \mathrm{id}} \\ |\mathrm{sd}_S\mathrm{sd}_a X| & \xrightarrow{\cong} & \operatorname*{colim}_{F\in\mathcal{G}} X[\pi_0(\mathbb{R}/2a\mathbb{Z}\setminus F_{2a})] & \xrightarrow{\cong} & \operatorname*{colim}_{F\in\mathcal{G}_{2a}} X[\pi_0(\mathbb{R}/2a\mathbb{Z}\setminus F)] & \xleftarrow{\cong} & |X|. \end{array}$$

where the inner square is induced by (2.4), the upper right horizontal arrow is (2.5) together with the rescaling isomorphism $\mathcal{F} \cong \mathcal{F}_{2a}$, and the upper left vertical arrow is also defined by (2.5) and the following observation to identify the target. If $F \in \mathcal{F}$ and the cardinality of $\pi_0([0,1]\setminus F)$ is n, then $X[\pi_0([0,1]\setminus F)] = X_n$ and the cardinality of $\pi_0([0,2a]\setminus F_{2a})$ is $2a(n+1)$, therefore

$$\mathrm{sd}_S\mathrm{sd}_a X[\pi_0([0,1]\setminus F)] = (\mathrm{sd}_S\mathrm{sd}_a X)_n = X_{2a(n+1)-1} = X[\pi_0([0,2a]\setminus F_{2a})]$$

and similarly

$$\mathrm{sd}_S\mathrm{sd}_a X[\pi_0(\mathbb{R}/\mathbb{Z}\setminus F)] = (\mathrm{sd}_S\mathrm{sd}_a X)_n = X_{2a(n+1)-1} = X[\pi_0(\mathbb{R}/2a\mathbb{Z}\setminus F_{2a})]$$

for any $F \in \mathcal{G}$.

The fact that our maps agree with those constructed in [Sai13, Theorem 1.5] now follows by using the diagram (2.6) once for $X_\bullet$ and once for $\mathrm{sd}_S\mathrm{sd}_a X_\bullet$ and from the following two assertions, which are easily checked:

(a) $F(\mathbf{t})_{2a} = F(d_{S,a}(\mathbf{t}))$.

(b) There are isomorphisms $\pi_0([0,2a]\setminus F_{2a}) \cong \coprod_{i=1}^a \pi_0([0,1]\setminus F(\mathbf{t})) \amalg \pi_0([0,1]\setminus F(\mathbf{t}))^{\mathrm{op}}$ and $\coprod_{i=1}^a [n] \amalg [n]^{\mathrm{op}} \cong [2a(n+1)-1]$ and under these isomorphisms the map $f^{d_{S,a}(\mathbf{t})}$ is equal to

$$\coprod_{i=1}^a f^{\mathbf{t}} \amalg (f^{\mathbf{t}})^{\mathrm{op}} \colon \coprod_{i=1}^a \pi_0([0,1]\setminus F(\mathbf{t})) \amalg \pi_0([0,1]\setminus F(\mathbf{t}))^{\mathrm{op}} \to \coprod_{i=1}^a [n] \amalg [n]^{\mathrm{op}}.$$

□

2.3 The homotopy type of the dihedral nerve and its Σ_2-fixed points.

In this section we apply the subdivision functors to prove an additive decomposition formula for the dihedral nerve. We use this formula to determine the Σ_2-fixed points of $DN(G)$, as well as the residual C_2-action on the Σ_2-fixed points, which is necessary to calculate $\Phi^{\Sigma_2}\mathrm{TCR}(A[G];2)$.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories with anti-involution. A functor $F\colon \mathcal{C}\to\mathcal{D}$ is a functor of categories with anti-involution if it satisfies $\omega_{\mathcal{D}}F^{\mathrm{op}} = F\omega_{\mathcal{C}}$. Such a functor induces a map on the real classifying spaces. We will need the following lemma, which is well-known and states a condition under which F induces an equivalence $B_{\mathbb{R}}\mathcal{C}\simeq B_{\mathbb{R}}\mathcal{D}$ of Σ_2-spaces. We give the proof for completeness.

Lemma 2.3.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories with anti-involution and $F\colon \mathcal{C}\to\mathcal{D}$ a functor of categories with anti-involution which is an equivalence on the underlying categories. Then the map $B_{\mathbb{R}}F\colon B_{\mathbb{R}}\mathcal{C}\to B_{\mathbb{R}}\mathcal{D}$ is an equivalence of Σ_2-spaces.*

Proof. For any category with anti-involution $\mathcal{C}$ define the category $\mathcal{C}^{\omega}$ as follows. Its objects are maps $\alpha\colon x\to\omega_{\mathcal{C}}(x)$ such that $\omega_{\mathcal{C}}(\alpha)=\alpha$ and a morphism from $\alpha\colon x\to\omega_{\mathcal{C}}(x)$ to $\beta\colon y\to\omega_{\mathcal{C}}(y)$ is a morphism $f\colon x\to y$ in $\mathcal{C}$ such that the diagram

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ \omega_{\mathcal{C}}(x) & \xleftarrow{\omega_{\mathcal{C}}(f)} & \omega_{\mathcal{C}}(y) \end{array}$$

commutes[4]. We have an isomorphism of simplicial sets

$$B_\bullet\mathcal{C}^{\omega}\xrightarrow{\cong}(\mathrm{sd}_S B_{\mathbb{R},\bullet}\mathcal{C})^{\Sigma_2} \tag{2.7}$$

by mapping an n-simplex

$$\begin{array}{ccccccc} x_0 & \xrightarrow{f_1} & x_1 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_n} & x_n \\ \downarrow{\scriptstyle\alpha_0} & & \downarrow{\scriptstyle\alpha_1} & & & & \downarrow{\scriptstyle\alpha_n} \\ \omega_{\mathcal{C}}(x_0) & \xleftarrow{\omega_{\mathcal{C}}(f_1)} & \omega_{\mathcal{C}}(x_1) & \xleftarrow{\omega_{\mathcal{C}}(f_1)} & \cdots & \xleftarrow{\omega_{\mathcal{C}}(f_n)} & \omega_{\mathcal{C}}(x_n) \end{array}$$

of the source to the n-simplex

$$x_0\xrightarrow{f_1}x_1\xrightarrow{f_2}\cdots\xrightarrow{f_n}x_n\xrightarrow{\alpha_n}\omega_{\mathcal{C}}(x_n)\xrightarrow{\omega_{\mathcal{C}}(f_n)}\cdots\xrightarrow{\omega_{\mathcal{C}}(f_2)}\omega_{\mathcal{C}}(x_1)\xrightarrow{\omega_{\mathcal{C}}(f_1)}\omega_{\mathcal{C}}(x_0).$$

Now, any functor $F\colon\mathcal{C}\to\mathcal{D}$ of categories with anti-involution induces a functor $F^{\omega}\colon\mathcal{C}^{\omega}\to\mathcal{D}^{\omega}$ and one checks that F^{ω} is an equivalence of categories if F is an equivalence of the underlying

[4] Another way to describe this is by noting that $\omega_{\mathcal{C}}$ induces a Σ_2-action on (the opposite of) the twisted arrow category of $\mathcal{C}$ and $\mathcal{C}^{\omega}$ is the resulting fixed point category.

categories. By assumption $B_{\mathbb{R}}F$ is an equivalence on the underlying spaces and we see that $(B_{\mathbb{R}}F)^{\Sigma_2}$ is an equivalence by observing that the following diagram commutes

$$\begin{array}{ccc} B\mathcal{C}^{\omega} & \xrightarrow{BF^{\omega}} & B\mathcal{D}^{\omega} \\ \downarrow \cong & & \downarrow \cong \\ (B_{\mathbb{R}}\mathcal{C})^{\Sigma_2} & \xrightarrow{(B_{\mathbb{R}}F)^{\Sigma_2}} & (B_{\mathbb{R}}\mathcal{D})^{\Sigma_2}, \end{array}$$

where the vertical arrows are the geometric realizations of (2.7). □

We can now formulate the decomposition formula for $DN(G)$. It involves the twisted real classifying spaces from Example 2.1.8. We denote by $\mathcal{F}_{O(2)}[S^1]$ the family of subgroups $H \leqslant O(2)$ such that $H \cap S^1 = 1$. Recall from (1.4) that $G \times \Sigma_2$ acts on G via

$$G \times \Sigma_2 \times G \to G, (g, \tau, h) \mapsto gh^{\tau}g^{-1}.$$

For any $x \in G$ the real conjugacy class $[[x]]$ is an orbit of this action, hence itself is acted upon by $G \times \Sigma_2$, which means that we can form the real classifying space $B_{\mathbb{R}}(G \int [[x]])$. We shall use this fact in the proof.

Proposition 2.3.2. *Let G be a discrete group. There is a $\mathcal{F}_{O(2)}[S^1]$-equivalence*[5]

$$\coprod_{\substack{[[x]]\in \mathrm{conj}_{\mathbb{R}}(G),\\ x^2=1}} B_{\mathbb{R}}(Z_G\langle x\rangle, x) \amalg \coprod_{\substack{[[x]]\in \mathrm{conj}_{\mathbb{R}}(G),\\ x^2\neq 1}} B_{\mathbb{R}}(SZ_G\langle x\rangle \int \Sigma_2, x) \simeq DN(G), \tag{2.8}$$

which depends on a choice of representatives for the real conjugacy classes.

Proof. It suffices to show that there is an $O(2)$-equivariant map which induces an equivalence on the underlying Σ_2-spaces, since any subgroup in $\mathcal{F}_{O(2)}[S^1]$ is subconjugate to Σ_2. For any element $x \in G$ let $DN(G)_{[[x]]}$ denote the sub-dihedral set consisting of k-simplices $(g_0, \dots, g_k)$ such that $g_0 \cdots g_k \in [[x]]$. Now choose a set of representatives for the set of real conjugacy classes. We obtain a decomposition

$$DN(G) = \coprod_{\substack{[[x]]\in \mathrm{conj}_{\mathbb{R}}(G),\\ x^2=1}} DN(G)_{[[x]]} \amalg \coprod_{\substack{[[x]]\in \mathrm{conj}_{\mathbb{R}}(G),\\ x^2\neq 1}} DN(G)_{[[x]]}$$

of dihedral sets. For any representative x there is an isomorphism

$$B_{\mathbb{R}}(G \int [[x]]) \xrightarrow{\cong} DN(G)_{[[x]]}, (x_0 \xleftarrow{g_1} \cdots \xleftarrow{g_k} x_k) \mapsto (x_k(g_1 \cdots g_k)^{-1}, g_1, \dots, g_k).$$

If $x^2 = 1$, then the map

$$B_{\mathbb{R}}Z_G\langle x\rangle \to B_{\mathbb{R}}(G \int [[x]]), (g_1, \dots, g_k) \mapsto (x \xleftarrow{g_1} \cdots \xleftarrow{g_k} x)$$

[5] See Definition A.2.4 for this notion. Note that the definition given there also makes sense for spaces and the group $O(2)$.

is an equivalence of Σ_2-spaces by Lemma 2.3.1, since it is induced by the functor of categories with anti-involution

$$Z_G\langle x\rangle \wr * \to G \wr [[x]],$$

which sends the point to x and this functor is an equivalence of the underlying categories. One checks that the resulting map

$$B_{\mathbb{R}}(Z_G\langle x\rangle, x) \to DN(G)_{[[x]]}$$

is compatible with the dihedral structure. Similarly, if $x^2 \neq 1$, the functor of categories with anti-involution $SZ_G\langle x\rangle \wr \Sigma_2 \to G \wr [[x]]$ that sends the object τ to x^τ and is the restriction of the projection $SZ_G\langle x\rangle \to G$ on morphisms induces a map

$$B_{\mathbb{R}}(SZ_G\langle x\rangle \wr \Sigma_2) \to B_{\mathbb{R}}(G \wr [[x]]),$$

which is an equivalence of Σ_2-spaces by Lemma 2.3.1 and again one checks the resulting map

$$B_{\mathbb{R}}(SZ_G\langle x\rangle \wr \Sigma_2, x) \to DN(G)_{[[x]]}$$

is one of dihedral sets. The coproduct of all these maps is the desired equivalence. □

Using the subdivision functors from the previous section it is easy to determine the Σ_2-fixed points of the left hand side of (2.8).

Proposition 2.3.3. *Let S be a $G \times \Sigma_2$-set. Then the Σ_2-fixed points of the real classifying space of $G \wr S$ are given by*

$$B_{\mathbb{R}}(G \wr S)^{\Sigma_2} \cong \coprod_{\substack{[x]\in \mathrm{conj}(G)\\ x^2=1}} S^{\langle (x,-1)\rangle} \times_{Z_G\langle x\rangle} EG, \tag{2.9}$$

where the right hand side depends on a choice of representative x for each conjugacy class.

Proof. We use the subdivision functors[6] to determine the fixed points of the real classifying space. We will define mutually inverse maps between the simplicial sets $(\mathrm{sd}_S\mathrm{sd}_2 B_{\mathbb{R}}(G \wr S))^{\Sigma_2}$ and $\coprod_{[x]\in\mathrm{conj}(G),x^2=1} \mathrm{sd}_S(S^{\langle (x,-1)\rangle} \times_{Z_G\langle x\rangle} EG)$. The k-simplices of $(\mathrm{sd}_S\mathrm{sd}_2 B_{\mathbb{R}}(G \wr S))^{\Sigma_2}$ have the form

$$s_0 \xleftarrow{g_1} s_1 \xleftarrow{g_2} \cdots \xleftarrow{g_{2k+1}} s_{2k+1} \xleftarrow{y} \overline{s_{2k+1}} \xleftarrow{g_{2k+1}^{-1}} \cdots \xleftarrow{g_1^{-1}} \overline{s_0} \tag{2.10}$$

with $y^2 = 1$. Now if $x = g_0^{-1} y g_0$ is the chosen representative of $[y]$, then we map (2.10) to the k-simplex $[(g_0^{-1}s_{2k+1}, g_0^{-1}(g_1\cdots g_{2k+1})^{-1}, g_1, \ldots, g_{2k+1})] \in \mathrm{sd}_S(S^{\langle (x,-1)\rangle} \times_{Z_G\langle x\rangle} EG)_k$.

[6]Strictly speaking we only need to use the Segal subdivision of the real classifying space, but subdividing further is convenient to determine the residual C_2-action on our examples of interest.

The inverse map is defined as follows: If $[(s, g_0, \dots, g_{2k+1})] \in \mathrm{sd}_S(S^{\langle(x,-1)\rangle} \times_{Z_G\langle x\rangle} EG)_k$, then we map this k-simplex to the k-simplex

$$\begin{array}{ccccccc} g_0^{-1}s & \xleftarrow{g_1} & (g_0g_1)^{-1}s & \xleftarrow{g_2} & \cdots & \xleftarrow{g_{2k+1}} & (g_0\dots g_{2k+1})^{-1}s \\ & & & & & & \uparrow (g_0\cdots g_{2k+1})^{-1}xg_0\cdots g_{2k+1} \\ g_0^{-1}\overline{s} & \xrightarrow{g_1^{-1}} & (g_0g_1)^{-1}\overline{s} & \xrightarrow{g_2^{-1}} & \cdots & \xrightarrow{g_{2k+1}^{-1}} & (g_0\cdots g_{2k+1})^{-1}\overline{s} \end{array}$$

of $(\mathrm{sd}_S\mathrm{sd}_2 B_{\mathbb{R}}(G \wr S))^{\Sigma_2}$. $\square$

Remark 2.3.4. If $x \in G$ is central and of order 2, then the residual C_2-action on

$$B_{\mathbb{R}}(G, x)^{\Sigma_2} \cong \coprod_{\substack{[y]\in\mathrm{cong}(G),\\ y^2=1}} BZ_G\langle y\rangle$$

sends the component indexed by $[y]$ to the component indexed by $[xy]$. Similarly, if $x \in G$ with $x^2 \neq 1$, then the residual C_2-action on

$$B_{\mathbb{R}}(SZ_G\langle x\rangle \wr \Sigma_2, x)^{\Sigma_2} \cong \coprod_{\substack{[y]\in\mathrm{conj}(SZ_G\langle x\rangle),\\ y^2=1}} \Sigma_2^{\langle(y,-1)\rangle} \times_{Z_{SZ_G\langle x\rangle}} ESZ_G\langle x\rangle$$

also maps the component indexed by $[y]$ to the component indexed by $[xy] = [x^{-1}y]$. Both assertions are easily checked by tracing through the maps constructed in the proof of Proposition 2.3.3.

Remark 2.3.5. If one is not interested in the residual C_2-action, one can also define a homeomorphism

$$\coprod_{\substack{[x]\in\mathrm{conj}(G),\\ x^2=1}} S^{\langle(x,-1)\rangle} \times_{Z_G\langle x\rangle} EG \xrightarrow{\cong} (\mathrm{sd}_S B_{\mathbb{R}}(G \wr S))^{\Sigma_2},$$

via the simplicial map

$$[(s, g_0, g_1, \dots, g_n)] \mapsto \begin{array}{ccccccc} g_0^{-1}s & \xleftarrow{g_1} & (g_0g_1)^{-1}s & \xleftarrow{g_2} & \cdots & \xleftarrow{g_n} & (g_0\dots g_n)^{-1}s \\ & & & & & & \uparrow (g_0\cdots g_n)^{-1}xg_0\cdots g_n \\ g_0^{-1}\overline{s} & \xrightarrow{g_1^{-1}} & (g_0g_1)^{-1}\overline{s} & \xrightarrow{g_2^{-1}} & \cdots & \xrightarrow{g_n^{-1}} & (g_0\cdots g_n)^{-1}\overline{s}, \end{array} \tag{2.11}$$

where the left hand n-simplex is in the component indexed by $[x]$. One checks that under the homeomorphisms of Proposition 2.2.3 this map coincides with the map in the proof of Proposition 2.3.3

We also want to determine the residual C_2-action on the Σ_2-fixed points of the left hand side of (2.8) in certain cases of interest to us. Here we need to distinguish between the summands indexed by elements of order 2 and the remaining summands. We first treat the elements of order 2, where we again need to distinguish between two cases. The first case is that of the summand indexed by the trivial element.

Proposition 2.3.6. *As a Σ_2-space with C_2-action, $B_{\mathbb{R}}(G,1)$ is equivalent to $B_{\mathbb{R}}G$ with the trivial C_2-action.*

Proof. Consider the map of real simplicial sets

$$B_{\mathbb{R}}G \to \mathrm{sd}_2 B_{\mathbb{R}}(G,1), (g_1,\dots,g_k) \mapsto (g_1,\dots,g_k,(g_1\cdots g_k)^{-1},g_1,\dots,g_k), \tag{2.12}$$

which factors through the C_2-fixed points of the target, hence is C_2-equivariant if we equip the source with the trivial C_2-action. We need to show it is an equivalence of Σ_2-spaces. On the underlying spaces it is an equivalence, since it is induced from a G-equivariant map $EG \to \mathrm{sd}_2 EG$. Thus, we need to check that the induced map on Σ_2-fixed points is an equivalence. We check this by using the Segal subdivision. Let $x \in G$ be an element of order 2 and consider the diagram

$$\begin{array}{ccc} EG/Z_G\langle x\rangle & \dashrightarrow & \mathrm{sd}_S EG/Z_G\langle x\rangle \\ \downarrow{\scriptstyle (2.11)} & & \downarrow \\ (\mathrm{sd}_S B_{\mathbb{R}}G)^{\Sigma_2} & \xrightarrow{(2.12)} & (\mathrm{sd}_S \mathrm{sd}_2 B_{\mathbb{R}}(G,1))^{\Sigma_2}, \end{array}$$

where the right vertical arrow is the map constructed in the proof of Proposition 2.3.3. The dashed horizontal arrow is then induced by the simplicial map

$$[(g_0,g_1,\dots,g_k)] \mapsto [(g_0,g_1,\dots,g_k,(g_0\cdots g_k)^{-1}xg_0\cdots g_k,g_k^{-1},\dots,g_1^{-1})]. \tag{2.13}$$

Consider the maps

$$\begin{aligned} BZ_G\langle x\rangle \to EG/Z_G\langle x\rangle, &(g_1,\dots,g_k) \mapsto [(1,g_1,\dots,g_k)] \\ BZ_G\langle x\rangle \to \mathrm{sd}_S EG/Z_G\langle x\rangle, &(g_1,\dots,g_k) \mapsto [(1,g_1,\dots,g_k,x,g_k^{-1},\dots,g_1^{-1})], \end{aligned}$$

which are both equivalences and fit into the triangle

$$\begin{array}{ccc} & BZ_G\langle x\rangle & \\ \swarrow{\scriptstyle \simeq} & & \searrow{\scriptstyle \simeq} \\ EG/Z_G\langle x\rangle & \xrightarrow{(2.13)} & \mathrm{sd}_S EG/Z_G\langle x\rangle, \end{array}$$

therefore also (2.13) is an equivalence. □

In Chapter 4 we will want to show that

$$\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes B_{\mathbb{R}}(G,1)_+;2) \simeq \Phi^{\Sigma_2}\mathrm{TCR}(X \otimes (B_{\mathbb{R}}G_+)^{\mathrm{triv}};2)$$

(see Example 3.4.2, Example 3.4.5 and Lemma 3.4.8 for the definition of the 2-cyclotomic spectra involved), since we can explicitly describe the right hand side, and the map implementing this equivalence will be that of the previous proposition. The next lemma will ensure that it indeed induces a map after applying $\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes -;2)$.

Lemma 2.3.7. *For any discrete group G and central element $z \in G$ the map on geometric realizations induced by*

$$BG_\bullet \to \mathrm{sd}_S BG_\bullet, (g_1, \cdots, g_n) \mapsto (g_1, \ldots, g_n, z, g_n^{-1}, \ldots, g_1^{-1}), \tag{2.14}$$

is homotopic to the identity after composing with the homeomorphism $|\mathrm{sd}_S BG_\bullet| \cong BG$ of Proposition 2.2.3.

Proof. It is straightforward to check that the induced map on π_1 is the identity. We spell out the details. The element $g \in G \cong \pi_1(BG)$ is represented by the loop

$$[0,1] \to BG, t \mapsto [(g, 1-t, t)] \tag{2.15}$$

and the composition of (2.14) with the homeomorphism $|\mathrm{sd}_S BG_\bullet| \cong BG$ maps this loop to the loop

$$[0,1] \to BG, t \mapsto [(g, z, g^{-1}, \frac{1}{2}(1-t), \frac{1}{2}t, \frac{1}{2}t, \frac{1}{2}(1-t))]. \tag{2.16}$$

We lift this to the path

$$[0,1] \to EG, t \mapsto [(1, g, z, g^{-1}, \frac{1}{2}(1-t), \frac{1}{2}t, \frac{1}{2}t, \frac{1}{2}(1-t))],$$

which starts at

$$[(1, g, z, g^{-1}, \frac{1}{2}, 0, 0, \frac{1}{2})] = [(1, z, \frac{1}{2}, \frac{1}{2})]$$

and ends at

$$[(1, g, z, g^{-1}, 0, \frac{1}{2}, \frac{1}{2}, 0)] = [(g, z, \frac{1}{2}, \frac{1}{2})] = g \cdot [(1, z, \frac{1}{2}, \frac{1}{2})],$$

so that (2.16) also represents $g \in G \cong \pi_1(BG)$. □

Next, we consider the components of $DN(G)^{\Sigma_2}$ indexed by the non-trivial elements of order 2.

Example 2.3.8. Let $z \in G$ be a central element of order 2. Then we obtain the isomorphism

$$B_{\mathbb{R}}(G, z)^{\Sigma_2} \cong \coprod_{\substack{[x] \in \mathrm{conj}(G) \\ x^2 = 1}} BZ_G\langle x \rangle,$$

where $BZ_G\langle x\rangle$ is modeled by $EG/Z_G\langle x\rangle$. Recall from Remark 2.3.4 that the residual C_2-action maps the component indexed by $[x]$ to the component indexed by $[zx]$.

(i) Suppose $G = \Sigma_2$ and $z = \sigma$. Then we see that $B_{\mathbb{R}}(\Sigma_2, \sigma)^{\Sigma_2} \cong B\Sigma_2 \amalg B\Sigma_2$ and the residual C_2-action switches the components, hence is induced after untwisting.

(ii) Suppose $G = D_4$ and $z \neq 1$. Since D_4 is abelian we obtain $B_{\mathbb{R}}(D_4, z)^{\Sigma_2} \cong \coprod_{d \in D_4} BD_4$, and again after untwisting the residual C_2-action this is induced.

(iii) Suppose $n \equiv 2 \mod 4$. We consider $G = D_{2n}$ and $z = c^{\frac{n}{2}}$. Up to conjugacy the elements of order 2 are $1, c^{\frac{n}{2}}, \sigma$, and $\sigma c^{\frac{n}{2}}$, since $\frac{n}{2}$ is odd. The first two elements are central and $Z_{D_{2n}}\langle \sigma c^i \rangle = \langle \sigma c^{\frac{in}{2}}, c^{\frac{n}{2}} \rangle = D_4$ for $i = 0, 1$. Thus,

$$B_{\mathbb{R}}(D_{2n}, c^{\frac{n}{2}})^{\Sigma_2} \cong BD_{2n} \amalg BD_{2n} \amalg BD_4 \amalg BD_4.$$

Furthermore, $[c^{\frac{n}{2}}\sigma c^i] = [\sigma c^{i+(-1)^i}]$ for $i = 0, 1$, since $\frac{n}{2}$ is odd, so that the residual C_2-action switches the components and is therefore induced after untwisting, that is

$$B_{\mathbb{R}}(D_{2n}, c^{\frac{n}{2}})^{\Sigma_2} \cong \mathrm{ind}_1^{C_2}(BD_{2n} \amalg BD_4).$$

(iv) Next, we assume $n \equiv 0 \mod 4$. We again consider $G = D_{2n}$ and $z = c^{\frac{n}{2}}$. The same considerations as before show that

$$B_{\mathbb{R}}(D_{2n}, c^{\frac{n}{2}})^{\Sigma_2} \cong BD_{2n} \amalg BD_{2n} \amalg BZ_{D_{2n}}\langle\sigma\rangle \amalg BZ_{D_{2n}}\langle\sigma c\rangle,$$

and the residual C_2-action switches both copies of BD_{2n}. The other two summands however, are invariant under the action, since $[c^{\frac{n}{2}}\sigma c^i] = [c^i]$ for $i = 0, 1$. We describe the C_2-action. Recall that for $i = 0, 1$ the space $BZ_{D_{2n}}\langle\sigma c^i\rangle$ is modeled by the Segal subdivision of $ED_{2n}/Z_{D_{2n}}\langle\sigma c^i\rangle$. Tracing through the maps in the proof of Proposition 2.3.3 we see that the C_2-action is given on k-simplices as

$$[(d_0, d_1, \ldots, d_{2k+1})] \mapsto [(c^{\frac{n}{4}} d_0 \cdots d_{2k+1}, d_{2k+1}^{-1}, d_{2k}^{-1}, \ldots, d_1^{-1})].$$

Note that $N_{D_{2n}}Z_{D_{2n}}\langle\sigma c^i\rangle = \langle\sigma c^i, c^{\frac{n}{4}}\rangle$. If we equip the left hand side with the residual action by $W_{D_{2n}}Z_{D_{2n}}\langle\sigma c^i\rangle \cong C_2$, the map of simplicial sets

$$\begin{aligned} EN_{D_{2n}}Z_{D_{2n}}\langle\sigma c^i\rangle/Z_{D_{2n}}\langle\sigma c^i\rangle &\to \mathrm{sd}_S ED_{2n}/Z_{D_{2n}}\langle\sigma c^i\rangle, \\ [(d_0, d_1, \ldots, d_k)] &\mapsto [(d_0, d_1, \ldots, d_k, 1, d_k^{-1}, \ldots, d_1^{-1})] \end{aligned}$$

is C_2-equivariant, and it is an equivalence, since it is induced by a $Z_{D_{2n}}\langle\sigma c^i\rangle$-equivariant map $EN_{D_{2n}}Z_{D_{2n}}\langle\sigma c^i\rangle \to \mathrm{sd}_S ED_{2n}$. Putting this all together we obtain an equivalence of spaces with C_2-action

$$B_{\mathbb{R}}(D_{2n}, c^{\frac{n}{2}})^{\Sigma_2} \simeq \mathrm{ind}_1^{C_2} BD_{2n} \amalg \coprod_{i=0,1} EN_{D_{2n}}Z_{D_{2n}}\langle\sigma c^i\rangle/Z_{D_{2n}}\langle\sigma c^i\rangle.$$

(v) Finally, we consider $B_{\mathbb{R}}(D_{2^\infty}, c_1)$, where c_1 denotes the generator of C_2. We have that

$$B_{\mathbb{R}}(D_{2^\infty}, c_1)^{\Sigma_2} \cong BD_{2^\infty} \amalg BD_{2^\infty} \amalg BZ_{D_{2^\infty}}\langle\sigma\rangle.$$

As in the previous example, the residual C_2-action switches both copies of BD_{2^∞}, hence is induced on these summands, and $BZ_{D_{2^\infty}}\langle\sigma\rangle = BD_4$ is invariant under the C_2-action. In fact, BD_4 is modeled by $\mathrm{sd}_S ED_{2^\infty}/D_4$ with the C_2-action given on k-simplices by

$$[(d_0, d_1, \ldots, d_{2k+1})] \mapsto [(c_2 d_0 \cdots d_{2k+1}, d_{2k+1}^{-1}, d_{2k}^{-1}, \ldots, d_1^{-1})],$$

where c_2 denotes a chosen generator of D_8. Just as before one then shows that there is a C_2-equivariant equivalence

$$ED_8/D_4 \simeq \mathrm{sd}_S ED_{2^\infty}/D_4,$$

where the left hand side carries the residual action by $D_8/D_4 \cong C_2$, so that all in all there is a C_2-equivariant equivalence

$$B_{\mathbb{R}}(D_{2^\infty}, c_1)^{\Sigma_2} \simeq \mathrm{ind}_1^{C_2} BD_{2^\infty} \amalg ED_8/D_4.$$

Example 2.3.9. We now describe the residual C_2-action on the Σ_2-fixed points of the components of $DN(D_{2n})$ corresponding to the elements not of order 2, where we include the cases $n = \infty$ and $n = 2^\infty$. Any element x with $x^2 \neq 1$ must be contained in C_n and we have an isomorphism $D_{2n} \cong SZ_{D_{2n}}\langle x\rangle$ by mapping $c \in C_n$ to $(c, 1) \in SZ_{D_{2n}}\langle x\rangle$ and σ to $(\sigma, -1)$. Under this isomorphism the action of D_{2n} on Σ_2 is given by restriction of the action of Σ_2 on itself via the projection $D_{2n} \to D_{2n}/C_n \cong \Sigma_2$. We see that $\Sigma_2^{\langle(y,-1)\rangle}$ is non-empty iff $y \notin C_n$ and in that case it is equal to Σ_2. Now we need to treat the different cases separately.

(i) Assume that $n < \infty$ is odd. Then

$$B_{\mathbb{R}}(D_{2n} \wr \Sigma_2, x)^{\Sigma_2} \cong \Sigma_2 \times_{\langle\sigma\rangle} ED_{2n} \simeq \Sigma_2/\langle\sigma\rangle = *$$

since in this case σc^j is conjugate to σ for all j, $Z_{D_{2n}}\langle\sigma\rangle = \langle\sigma\rangle$ and $\langle\sigma\rangle$ acts freely on Σ_2. Thus, the residual C_2-action is trivial.

(ii) Similarly, we have

$$B_{\mathbb{R}}(D_\infty \wr \Sigma_2, x) \simeq \coprod_{i=0,1} \Sigma_2 \times_{\langle\sigma c^i\rangle} ED_\infty \simeq * \amalg *,$$

since $Z_{D_{2n}}\langle\sigma c^i\rangle = \langle\sigma c^i\rangle$ acts freely on Σ_2. One checks that if $x = c^j$ with j odd, the the residual C_2-action switches the components indexed by $[\sigma]$ and $[\sigma c]$, that is the action is induced after untwisting. If $x = c^j$ with j even, then both components are invariant under the C_2-action, hence the C_2-action is trivial.

(iii) Suppose that $n < \infty$ is even. Then

$$B_{\mathbb{R}}(D_{2n} \wr \Sigma_2, x)^{\Sigma_2} \cong \coprod_{i=0,1} \Sigma_2 \times_{Z_{D_{2n}}\langle\sigma c^i\rangle} ED_{2n}.$$

If $x = c^j$ with j odd, then the residual C_2-action switches the components indexed by $[\sigma]$ and $[\sigma c]$, therefore the action is induced after untwisting. If on the other hand $x = c^j$ with j even, then both components are invariant under the C_2-action. A careful check shows that on k-simplices the C_2-action is given by

$$[(\tau, d_0, d_1, \dots, d_{2k+1})] \mapsto [(-\tau, (c^{\frac{j}{2}})^\tau d_0 d_1 \cdots d_{2k+1}, d_{2k+1}^{-1}, d_{2k}^{-1}, \dots, d_1^{-1})].$$

We consider for $i = 0, 1$ the map

$$\begin{aligned}\Sigma_2 \times_{Z_{D_{2n}}\langle \sigma c^i \rangle} ED_{2n} &\to \mathrm{sd}_S(\Sigma_2 \times_{D_4} ED_{2n}),\\ [(\tau, d_0, d_1, \ldots, d_k)] &\mapsto [(\tau, d_0, d_1, \ldots, d_k, 1, d_k^{-1}, \ldots, d_1^{-1})],\end{aligned}$$

which is an equivalence and C_2-equivariant if we give the source the action defined by

$$[(\tau, d_0, d_1, \ldots, d_k)] \mapsto [(-\tau, (c^{\frac{j}{2}})^{\tau} d_0, d_1, \ldots, d_k)].$$

We claim that there is a C_2-equivariant equivalence $\Sigma_2 \times_{Z_{D_{2n}}\langle \sigma c^i \rangle} ED_{2n} \simeq BW_{D_{2n}}\langle \sigma c^i \rangle$, where the right hand side carries the trivial C_2-action. The C_2-action on $\Sigma_2 \times_{Z_{D_{2n}}\langle \sigma c^i \rangle} ED_{2n}$ can clearly be lifted to a $(C_2 \times Z_{D_{2n}}\langle \sigma c^i \rangle)$-action on $\Sigma_2 \times ED_{2n}$. Denote by $\alpha\colon C_2 \times Z_{D_{2n}}\langle \sigma c^i \rangle \to W_{D_{2n}}\langle \sigma c^i \rangle$ the composite

$$C_2 \times Z_{D_{2n}}\langle \sigma c^i \rangle = C_2 \times N_{D_{2n}}\langle \sigma c^i \rangle \to N_{D_{2n}}\langle \sigma c^i \rangle \xrightarrow{\beta} W_{D_{2n}}\langle \sigma c^i \rangle,$$

where both arrows are the projection. Then the projection

$$(\Sigma_2 \times_{\langle \sigma c^i \rangle} ED_{2n}) \times_{W_{D_{2n}}\langle \sigma c^i \rangle} \beta^* EW_{D_{2n}}\langle \sigma c^i \rangle \to \Sigma_2 \times_{Z_{D_{2n}}\langle \sigma c^i \rangle} ED_{2n}$$

is a C_2-equivariant homotopy equivalence, since it is induced by the projection

$$\Sigma_2 \times ED_{2n} \times \alpha^* EW_{D_{2n}}\langle \sigma c^i \rangle \to \Sigma_2 \times ED_{2n}$$

and $Z_{D_{2n}}\langle \sigma c^i \rangle$ acts freely on both source and target. Similarly, the projection

$$(\Sigma_2 \times_{\langle \sigma c^i \rangle} ED_{2n}) \times_{W_{D_{2n}}\langle \sigma c^i \rangle} \beta^* EW_{D_{2n}}\langle \sigma c^i \rangle \to \beta^* EW_{D_{2n}}\langle \sigma c^i \rangle / W_{D_{2n}}\langle \sigma c^i \rangle$$

is also a C_2-equivariant homotopy equivalence, since it is induced by the projection

$$(\Sigma_2 \times_{\langle \sigma c^i \rangle} ED_{2n}) \times \beta^* EW_{D_{2n}}\langle \sigma c^i \rangle \to \beta^* EW_{D_{2n}}\langle \sigma c^i \rangle.$$

The latter is a homotopy equivalence since $\langle \sigma c^i \rangle$ acts freely on Σ_2 and $W_{D_{2n}}\langle \sigma c^i \rangle$ acts freely on both source and target, therefore the induced map on orbits is also a homotopy equivalence. Finally, note that the residual C_2-action on $(\beta^* EW_{D_{2n}}\langle \sigma c^i \rangle)/Z_{D_{2n}\langle \sigma c^i \rangle} = BW_{D_{2n}}\langle \sigma c^i \rangle$ is trivial. This yields the claim and all in all we obtain an equivalence of spaces with C_2-action

$$B_{\mathbb{R}}(D_{2n}, c^j)^{\Sigma_2} \simeq \coprod_{i=0,1} BW_{D_{2n}}\langle \sigma c^i \rangle.$$

(iv) Finally, we consider D_{2^∞} and fix generators $\{c_n\}_{n \in \mathbb{N}}$ of C_{2^n} such that $c_n^2 = c_{n-1}$. Note that σc_n^j is conjugate to σ for all n and j and $Z_{D_{2^\infty}}\langle \sigma \rangle = \langle \sigma, c_1 \rangle = D_4$. Thus,

$$B_{\mathbb{R}}(D_{2^\infty} \wr \Sigma_2, c_n^j) \simeq \Sigma_2 \times_{D_4} ED_{2^\infty},$$

and the residual C_2-action is given by

$$[(\tau, d_0, d_1, \ldots, d_{2k+1})] \mapsto [(-\tau, (c_{n+1}^j)^{\tau} d_0 d_1 \cdots d_{2k+1}, d_{2k+1}^{-1}, d_{2k}^{-1}, \ldots, d_1^{-1})].$$

The same argument as for D_{2n} with $n < \infty$ even now shows that $\Sigma_2 \times_{D_4} ED_{2^\infty}$ is equivalent to BC_2 with the trivial C_2-action.

Chapter 3

Real Topological Cyclic Homology

In this chapter we introduce real p-cyclotomic spectra and real topological cyclic homology following [QS21a]. Recall from [NS18] that a p-cyclotomic spectrum is a spectrum X with C_{p^∞}-action[1] and a C_{p^∞}-equivariant map $\varphi_X\colon X \to X^{tC_p}$ called the Frobenius. Its topological cyclic homology can then be computed via the fiber sequence

$$\mathrm{TC}(X;p) \to X^{hC_{p^\infty}} \xrightarrow{\varphi^{hC_{p^\infty}} - \mathrm{can}} (X^{tC_p})^{hC_{p^\infty}},$$

where

$$\mathrm{can}\colon X^{hC_{p^\infty}} \simeq (X^{hC_p})^{h(C_{p^\infty}/C_p)} \simeq (X^{hC_p})^{hC_{p^\infty}} \to (X^{tC_p})^{hC_{p^\infty}}.$$

To define the real versions we need to define so called parametrized versions of homotopy orbits, homotopy fixed points and the Tate construction. This is done in section 3.2. In Section 3.3 we specialize our discussion of parametrized homotopy orbits, homotopy fixed points and the Tate construction to the case of dihedral groups. Section 3.4 treats the theory of real cyclotomic spectra and additionally we give the definition of real topological cyclic homology there. Finally, in section 3.5 we will calculate $\Phi^{\Sigma_2}\mathrm{TCR}(X;p)$ for all primes p, reproving [DMP21, Corollary 2.5] and [DMP21, Theorem 2.13] by different methods.

3.1 Group-theoretic preliminaries

In this section we lay down some conventions and review some facts regarding group theory which we will need in the sequel. Recall from the introduction that except from $\mathbb{T}$ and $O(2)$ all groups are discrete, and although we will encounter infinite groups, an abstract group G will always be finite in this chapter. We denote the orbit category by $\mathcal{O}_G$. Up to isomorphism its objects consist of G-sets of the form G/K for K any subgroup. For $g \in G$ and K any subgroup

[1] By this we mean an object of $\mathrm{Fun}(BC_{p^\infty}, \mathbf{Sp})$.

put $K^g = gKg^{-1}$. For any $h \in G$ we have $hgK = gK$ iff $h \in K^g$, thus

$$\mathrm{Hom}_{\mathcal{O}_G}(G/H, G/K) \cong (G/K)^H = \{gK : H \subset K^g\} = \{gK : H^{g^{-1}} \subset K\}, \tag{3.1}$$

where the bijection is given by evaluating a map at the coset $1H$. It is easy to see that there is an isomorphism $f\colon G/H \xrightarrow{\cong} G/K$ iff H is conjugate to K. The only if direction is immediate from (3.1) and if $K = H^g$, then the maps $f\colon G/H \to G/K$ determined by $g^{-1}K$ and $f'\colon G/K \to G/H$ determined by gH are mutually inverse.

If we restrict the action on G/K to a subgroup H, then the above implies that the isotropy of the H-action at gK is $H \cap K^g$, thus we obtain (a special case of) the double coset formula:

$$\mathrm{res}^G_H G/K \cong \coprod_{HgK \in H\backslash G/K} H/(H \cap K^g). \tag{3.2}$$

We now state some facts about conjugacy classes of subgroups of order 2 in dihedral groups, and also about their normalizers, centralizers and Weyl groups. In general, if $x \in G$ is an element of order 2, then by injectivity of the conjugation map we have $g\langle x\rangle g^{-1} = \langle x\rangle$ iff $gxg^{-1} = x$ and this is the case iff $g \in Z_G\langle x\rangle$, hence $Z_G\langle x\rangle = N_G\langle x\rangle$. In the case of dihedral groups one can now check directly for finite n and $j = 1, \dots, n$ that we have

$$N_{D_{2n}}\langle \sigma c^j\rangle = \begin{cases} \langle \sigma c^j\rangle & \text{for odd}\, n, \\ \langle \sigma c^j, c^{\frac{n}{2}}\rangle & \text{for even}\, n, \end{cases} \tag{3.3}$$

but $N_{D_{2p^\infty}}\langle \sigma c^j\rangle = \langle \sigma c^j\rangle$ for any prime p and $N_{D_\infty}\langle \sigma c^j\rangle = \langle \sigma c^j\rangle$. Thus, $W_{D_{2n}}\langle \sigma c^j\rangle = C_2$ for even n and $W_{D_{2n}}\langle \sigma c^j\rangle$ is trivial for odd $n, n = \infty$, and $n = p^\infty$.

We will use the notion of a G-family $\mathcal{F}$ and particular examples of G-families. We refer to section A.2 for the definition of a G-family and the G-spaces $E\mathcal{F}$ and $\widetilde{E\mathcal{F}}$. We also recall from section A.2 the particular G-families of interest to us:

(1) If N is a normal subgroup, then we denote by $\mathcal{F}_G[N]$ the N-free family, that is the subgroups H such that $H \cap N = 1$.

(2) Let N again be a normal subgroup. Then $\mathcal{F}_{G,\not\supset N}$ denotes the family of subgroups H that do not contain N.

(3) If H is any subgroup, then we let $\mathcal{F}_{G,\leqslant H}$ be the family of all subgroups that are conjugate to a subgroup of H.

If the group is clear from context, then we often drop the subscript G in the notation.

Remark 3.1.1. We specialize to the case $G = D_{2p^n}$ and $N = C_{p^n}$, where we allow $n = \infty$. Then one easily checks that the following statements hold.

(i) For $1 \leq k \leq n$ we have $\mathcal{F}_{D_{2p^n}}[C_{p^k}] = \mathcal{F}_{D_{2p^n}}[C_{p^n}]$.

(ii) If p is odd then up to conjugacy $\mathcal{F}_{D_{2p^n}}[C_{p^n}]$ consists of the trivial group and $\langle\sigma\rangle$. The same is true for $p = 2$ if $n = 0, 1, \infty$.

(iii) If $1 < n < \infty$, then up to conjugacy $\mathcal{F}_{D_{2^{n+1}}}[C_{2^n}]$ consists of the trivial group, $\langle\sigma\rangle$ and $\langle\sigma c\rangle$. The subgroups $\langle\sigma c^{2j}\rangle$ are conjugate to $\langle\sigma\rangle$ and $\langle\sigma c^{2j+1}\rangle$ is conjugate to $\langle\sigma c\rangle$ for all j.

(iv) We always have $\mathcal{F}_{D_{2p^n}}[C_{p^n}] = \mathcal{F}_{D_{2p^n},\not\supset C_p}$.

Finally, we introduce some notation and state some facts for sections of group extensions. These will appear in the computation of the geometric fixed points of the parametrized homotopy orbits. We consider a group extension

$$1 \longrightarrow N \longrightarrow G \xrightarrow{p} \Sigma \longrightarrow 1.$$

We denote by Sec_p the set of (group theoretic) sections of p. The group N acts on Sec_p via conjugation, that is if $s \in \mathrm{Sec}_p$, then gsg^{-1} is also a section of p for any $g \in N$. Sending a section to its image yields an inclusion

$$\mathrm{Sec}_p \hookrightarrow \mathcal{F}_G[N].$$

The image of this inclusion consists exactly of all $H \in \mathcal{F}_G[N]$ such that the restriction of p to H induces an isomorphism $H \cong \Sigma$. For this reason we shall also refer to such subgroups as sections of p. Finally, for two orbits $[H], [K] \in \mathrm{Sec}_p/N$ we have that $[H] = [K]$ iff H and K are conjugate in G. To see why, we observe that the fact that H is the image of a section s implies $G = HN$, therefore for any $g \in G$ we obtain from $g = hn$ that

$$H^g = H^{hn} = H^n,$$

and hence H^g is the image of nsn^{-1}.

3.2 The parametrized Tate construction

In this section we introduce parametrized versions of the homotopy orbits, homotopy fixed points and the Tate construction. Our presentation differs from that of [QS21b], which uses parametrized ∞-categories. By [QS21b, Remark 4.31] the constructions given in loc. cit. are a special case of the constructions given in [GM95, Part IV], which is written in the language of equivariant stable homotopy theory and it is their language we adopt. We give the definitions for a general group G with normal subgroup N and specialize to the case of dihedral groups in the next section. At the end of this section we will compute $\Phi^{\Sigma} X_{h_{\Sigma}N}$, the geometric fixed points of the parametrized homotopy orbits (see Lemma 3.2.6).

Definition 3.2.1. Let X be a G-spectrum and N a normal subgroup.

(i) The parametrized homotopy orbits are $X_{h_{G/N}N} := (X \otimes E\mathcal{F}_G[N]_+)^N$.

(ii) The parametrized homotopy fixed points are $X^{h_{G/N}N} := F(E\mathcal{F}_G[N]_+, X)^N$.

(iii) The parametrized Tate construction $X^{t_{G/N}N}$ is defined to be the cofiber of

$$\mathrm{Nm}\colon (X \otimes E\mathcal{F}_G[N]_+)^N \simeq (F(E\mathcal{F}_G[N]_+, X) \otimes E\mathcal{F}_G[N]_+)^N \to F(E\mathcal{F}_G[N]_+, X)^N,$$

the map that collapses $E\mathcal{F}_G[N]$ to a point, which we also refer to as the *(parametrized) norm* map.

Remark 3.2.2. The parametrized homotopy orbits, fixed points and Tate construction have the following properties:

(i) By construction, the parametrized homotopy orbits, fixed points and Tate construction are G/N-spectra.

(ii) Since $X \otimes E\mathcal{F}_+$ is $\mathcal{F}$-torsion and $F(E\mathcal{F}_+, X)$ is $\mathcal{F}$-complete, Lemma A.2.5 implies that parametrized homotopy orbits, parametrized homotopy fixed points and the parametrized Tate construction send $\mathcal{F}_G[N]$-equivalences to equivalences of G/N-spectra.

(iii) If X is $\mathcal{F}_G[N]$-torsion, then

$$X^N \simeq X_{h_{G/N}N}$$

and similarly if X is $\mathcal{F}_G[N]$-complete,

$$X^N \simeq X^{h_{G/N}N}.$$

(iv) By using the cofiber sequence $E\mathcal{F}_+ \to S^0 \to \widetilde{E\mathcal{F}}$, we see that

$$X^{t_{G/N}N} = (F(E\mathcal{F}_G[N]_+, X) \otimes \widetilde{E\mathcal{F}_G[N]})^N.$$

In particular, $(-)^{t_{G/N}N}$ is lax monoidal, since fixed points are lax monoidal and $F(E\mathcal{F}_+, -)$ and $- \otimes \widetilde{E\mathcal{F}}$ are lax monoidal for any G-family $\mathcal{F}$.

(v) If $N = G$, this recovers the classical definition of homotopy orbits, fixed points and the Tate construction. More generally, the underlying non-equivariant spectra of $X_{h_{G/N}N}, X^{h_{G/N}N}$ and $X^{t_{G/N}N}$ are equivalent to X_{hN}, X^{hN} and X^{tN}.

(vi) If H is a subgroup containing N, then the underlying H/N-spectrum of $X_{h_{G/N}N}$ is equivalent to $X_{h_{H/N}N}$. Therefore, whenever we consider $X_{h_{H/N}N}$ as a G/N-spectrum, we will actually mean $X_{h_{G/N}N}$ and the analogous remark holds for $X^{h_{H/N}N}$. Note that for any normal subgroup N' containing N there are equivalences

$$(X_{h_{H/N}N})_{h_{(G/N)/(N'/N)}N'/N} \simeq X_{h_{G/N'}N'}, \quad (X^{h_{H/N}N})^{h_{(G/N)/(N'/N)}N'/N} \simeq X^{h_{G/N'}N'}.$$

The first equivalence follows from the projection formula (A.23) and (A.28). For the second equivalence one additionally uses Lemma A.1.9 and the equivalence of G-spectra

$$F(X, F(Y,Z)) \simeq F(X \otimes Y, Z),$$

which is a consequence of the ∞-categorical Yoneda lemma.

The main example to have in mind is that of a D_{2p^n}-spectrum X (where we allow $n = \infty$). We usually want to consider $X_{h_{\Sigma_2}C_{p^k}}$, $X^{h_{\Sigma_2}C_{p^k}}$ and $X^{t_{\Sigma_2}C_{p^k}}$ as $D_{2p^{n-k}}$-spectra. In this case the equivalences above become

$$(X_{h_{\Sigma_2}C_{p^k}})_{h_{\Sigma_2}C_{p^{n-k}}} \simeq X_{h_{\Sigma_2}C_{p^n}} \quad \text{and} \quad (X^{h_{\Sigma_2}C_{p^k}})^{h_{\Sigma_2}C_{p^{n-k}}} \simeq X^{h_{\Sigma_2}C_{p^n}}.$$

(vii) After choosing a G-CW-model for $E\mathcal{F}_G[N]$, its skeletal filtration gives rise to spectral sequences

$$E^2_{s,t} = H^G_s(E\mathcal{F}_G[N]; \pi^{(-)}_t X) \Rightarrow \pi^{G/N}_{s+t} X_{h_{G/N}N},$$
$$E^{s,t}_2 = H^s_G(E\mathcal{F}_G[N]; \pi^{(-)}_t X) \Rightarrow \pi^{G/N}_{t-s} X^{h_{G/N}N},$$

where $H^G_s(E\mathcal{F}_G[N]; \pi^{(-)}_t X)$ and $H^s_G(E\mathcal{F}_G[N]; \pi^{(-)}_t X)$ denote Bredon homology and cohomology (see [MNN19, Chapters 2 and 3]). As a consequence, if $\mathrm{res}^G_H X$ is n-connected for all $H \in \mathcal{F}_G[N]$, then $X_{h_{G/N}N}$ is n-connected. Similarly, if if $\mathrm{res}^G_H X$ is n-coconnected for all $H \in \mathcal{F}_G[N]$, then $X^{h_{G/N}N}$ is n-coconnected.

The previous definition only applies to finite G, but we also need a version of the parametrized homotopy fixed points in the case of $G = D_{2p^\infty}$ and $N = C_{p^\infty}$.

Definition 3.2.3. Let X be a D_{2p^∞}-spectrum. Then we define

$$X^{h_{\Sigma_2}C_{p^\infty}} = \lim_n X^{h_{\Sigma_2}C_{p^n}},$$

where the limit is taken along the inclusion of fixed points.

In the special case $G = D_{2^{n+1}}$ the following definition will play a part in the computation of $\Phi^{\Sigma_2} X^{t_{\Sigma_2}C_2}$ (see Lemma 3.3.3).

Definition 3.2.4. Let X be a G-spectrum and H a subgroup. The generalized Tate spectrum of X is defined to be the spectrum given by

$$X^{\tau H} = \Phi^H F(EG_+, X),$$

the geometric fixed points of its Borel completion.

We point out that the generalized Tate construction only depends on the underlying homotopy type of X. To be precise, if $f\colon X \to Y$ is a map of G-spectra which is an equivalence on the underlying spectra, then f induces for any subgroup H an equivalence $X^{\tau H} \simeq Y^{\tau H}$. This follows from Lemma A.2.5.

In Section 3.5 we will prove that $\Phi^{\Sigma_2}\mathrm{TCR}(X;2)$ vanishes in a special case. The key ingredient for this is the following vanishing result. It is probably known to experts and the proof is purely formal.

Proposition 3.2.5. *If N is a normal subgroup of G, then for any N-spectrum X the spectra $\Phi^{G/N}(\mathrm{ind}_N^G X)_{h_{G/N}N}$, $\Phi^{G/N}(\mathrm{ind}_N^G X)^{h_{G/N}N}$, and $\Phi^{G/N}(\mathrm{ind}_N^G X)^{t_{G/N}N}$ are contractible.*

Proof. It suffices to show that the spectra $\Phi^{G/N}(\mathrm{coind}_N^G X)_{h_{G/N}N}$, $\Phi^{G/N}(\mathrm{coind}_N^G X)^{h_{G/N}N}$, and $\Phi^{G/N}(\mathrm{coind}_N^G X)^{t_{G/N}N}$ are contractible by the Wirthmüller isomorphism, and it suffices to prove this for the first two by exactness of geometric fixed points. Consider the following diagram of groups

$$\begin{array}{ccc} N & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ G & \longrightarrow & G/N, \end{array}$$

where the vertical arrows are inclusions and the horizontal arrows projections. Since right adjoints compose, we obtain a natural equivalence $(\mathrm{coind}_N^G X)^N \simeq \mathrm{coind}_1^{G/N} X^N$ of G/N-spectra.

Combined with the projection formula (A.22) this equivalence implies

$$\begin{aligned}(\mathrm{coind}_N^G X)_{h_{G/N}N} &= (\mathrm{coind}_N^G X \otimes E\mathcal{F}_G[N]_+)^N \\ &\simeq (\mathrm{coind}_N^G (X \otimes EN_+))^N \\ &\simeq \mathrm{coind}_1^{G/N}(X \otimes EN_+)^N,\end{aligned}$$

where we used that $\mathrm{res}_N^G E\mathcal{F}_G[N] \simeq EN$. Using Lemma A.1.9 we similarly obtain

$$\begin{aligned}(\mathrm{coind}_N^G X)^{h_{G/N}N} &= F(E\mathcal{F}_G[N]_+, \mathrm{coind}_N^G X)^N \\ &\simeq (\mathrm{coind}_N^G F(EN_+, X))^N \\ &\simeq \mathrm{coind}_1^{G/N} F(EN_+, X)^N,\end{aligned}$$

and we conclude with the observation that geometric fixed points of coinduced spectra vanish. □

We end this section by computing the geometric fixed points of the parametrized homotopy orbits. This computation will involve equivalences between (geometric) fixed points of conjugate subgroups. Before we state the result we review the definition of these equivalences. First we fix notation. If G is a finite group and $g \in G$, then we denote by $c_g \colon G \to G$ the conjugation map. The restriction to any subgroup H induces an isomorphism $H \xrightarrow{\cong} H^g$, which we also denote by c_g. After passing to the quotients we obtain an isomorphism of the Weyl groups

$$\omega_H^g \colon W_G H \xrightarrow{\cong} W_G H^g$$

and for any G-spectrum X there is a natural equivalence of W_GH-spectra

$$(\omega_H^g)^* X^{H^g} \simeq ((c_g)^* X)^H$$

We refer to Section A.3 of the appendix for the construction of this equivalence and its general properties.

Left multiplication by g also induces an equivalence of G-spectra $l_g\colon X \xrightarrow{\simeq} (c_g)^*X$ and composing this with the previous equivalence we obtain an equivalence of W_GH-spectra

$$X^H \simeq (\omega_H^g)^* X^{H^g}.$$

If $g \in H$, then ω_H^g is the identity and so is the previous equivalence.

For the geometric fixed points we note that there is an equivalence of N_GH-spaces

$$\widetilde{E\mathcal{F}}_{N_GH,\not\supseteq H} \simeq (c_g)^* \widetilde{E\mathcal{F}}_{N_GH^g,\not\supseteq H^g},$$

therefore a natural equivalence of W_GH-spectra

$$\begin{aligned}\Phi^H X = (X \otimes \widetilde{E\mathcal{F}}_{\not\supseteq H})^H \simeq (X \otimes (c_g)^* \widetilde{E\mathcal{F}}_{\not\supseteq H^g})^H \xrightarrow{(l_g \otimes \mathrm{id})^H} ((c_g)^*(X \otimes \widetilde{E\mathcal{F}}_{\not\supseteq H^g}))^H \simeq \\ (\omega_H^g)^*(X \otimes \widetilde{E\mathcal{F}}_{\not\supseteq H^g})^{H^g} = (\omega_H^g)^* \Phi^{H^g} X, \quad (3.4)\end{aligned}$$

where we used that $(c_g)^*$ is monoidal. In a similar vein, if $\alpha\colon \Gamma \to G$ is an isomorphism, we have that $E\Gamma \simeq \alpha^* EG$ and since α^* is a monoidal equivalences,

$$(\alpha^* X)_{h\Gamma} \simeq X_{hG}.$$

Since α is an isomorphism, the unit map $\eta_\alpha^!\colon \mathrm{id} \to \alpha^*\alpha_!$ and counit map $\epsilon_\alpha^!\colon \alpha_!\alpha^* \to \mathrm{id}$ are equivalence, therefore the projection formula (A.21) also holds in this case and the (∞-categorical) Yoneda Lemma yields

$$F(E\Gamma_+, \alpha^* X) \simeq F(\alpha^* EG_+, \alpha^* X) \simeq \alpha^* F(EG_+, X).$$

Thus, (after combining this with the equivalence $\alpha^*\widetilde{EG} \simeq \widetilde{E\Gamma}$) we also obtain conjugation equivalences for homotopy fixed points and the Tate construction:

$$(\alpha^* X)^{h\Gamma} \simeq X^{hG}, \quad \text{and} \quad (\alpha^* X)^{t\Gamma} \simeq X^{tG}.$$

Combining this with the conjugation equivalences for geometric fixed points we obtain equivalences of spectra:

$$(\Phi^H X)^{W_GH} \simeq (\Phi^{H^g} X)^{W_GH^g} \quad (3.5)$$

$$(\Phi^H X)_{hW_GH} \simeq (\Phi^{H^g} X)_{hW_GH^g} \quad (3.6)$$

$$(\Phi^H X)^{hW_GH} \simeq (\Phi^{H^g} X)^{hW_GH^g} \quad (3.7)$$

$$(\Phi^H X)^{tW_GH} \simeq (\Phi^{H^g} X)^{tW_GH^g}. \quad (3.8)$$

It is easily checked that if $\Phi^H X$ is Borel torsion, then so is $\Phi^{H^g} X$ and (3.5) and (3.6) agree under the equivalences $(\Phi^H X)^{W_G H} \simeq (\Phi^H X)_{hW_G H}$ and $(\Phi^{H^g} X)^{W_G H^g} \simeq (\Phi^{H^g} X)_{hW_G H^g}$. Similarly, if $\Phi^H X$ is Borel complete, then so is $\Phi^{H^g} X$ and (3.5) and (3.7) agree under the equivalences $(\Phi^H X)^{W_G H} \simeq (\Phi^H X)^{hW_G H}$ and $(\Phi^{H^g} X)^{W_G H^g} \simeq (\Phi^{H^g} X)^{hW_G H^g}$.

Lemma 3.2.6. *Let*

$$1 \longrightarrow N \longrightarrow G \xrightarrow{p} \Sigma \longrightarrow 1$$

be a group extension and let X be a G-spectrum.

(i) There is a natural equivalence

$$\Phi^\Sigma X_{h_\Sigma N} \simeq \bigoplus_{[H] \in \mathrm{Sec}_p/N} (\Phi^H X)_{hW_G H}, \tag{3.9}$$

which only depends on a choice of representatives in the following sense. If $[H] = [K]$, then the following diagram commutes

$$\begin{array}{ccc} & \Phi^\Sigma X_{h_\Sigma N} & \\ \swarrow & & \searrow \\ (\Phi^H X)_{hW_G H} & \xrightarrow{\simeq} & (\Phi^K X)_{hW_G K}, \end{array}$$

where the diagonal arrows are (3.9) composed with the projection to the summand indexed by $[H] = [K]$ for the different choices of representatives and the horizontal arrow is the conjugation equivalence (3.6).

(ii) Consider the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \tilde{N} & \longrightarrow & \tilde{G} & \xrightarrow{\tilde{p}} & \Sigma & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \| & & \\ 1 & \longrightarrow & N & \longrightarrow & G & \xrightarrow{p} & \Sigma & \longrightarrow & 1, \end{array}$$

where the rows are group extensions and the vertical arrows are inclusions of subgroups. The inclusion $G \to \tilde{G}$ induces a map $f\colon \mathrm{Sec}_p/N \to \mathrm{Sec}_{\tilde{p}}/\tilde{N}$ and under the equivalence of (i) the restriction of the inclusion of fixed points $\Phi^\Sigma X_{h_\Sigma \tilde{N}} \to \Phi^\Sigma X_{h_\Sigma N}$ to the summand indexed by $[\tilde{H}] \in \mathrm{Sec}_{\tilde{p}}/\tilde{N}$ factors through $\bigoplus_{[H] \in f^{-1}([\tilde{H}])} (\Phi^H X)_{hW_G H}$ and the composite

$$(\Phi^{\tilde{H}} X)_{hW_{\tilde{G}} \tilde{H}} \to \bigoplus_{[H] \in f^{-1}([\tilde{H}])} (\Phi^H X)_{hW_G H} \to (\Phi^H X)_{hW_G H},$$

where the second arrow is the projection, is homotopic to the conjugation equivalence followed by inclusion of fixed points

$$(\Phi^{\tilde{H}} X)_{hW_{\tilde{G}} \tilde{H}} \xrightarrow{\simeq} (\Phi^H X)_{hW_{\tilde{G}} H} \to (\Phi^H X)_{hW_G H}.$$

Proof. We start by constructing a natural transformation

$$\Phi^{\Sigma} X_{h_\Sigma N} \to \bigoplus_{[H]\in \mathrm{Sec}_p/N} (\Phi^H X)_{hW_G H}. \tag{3.10}$$

Let $\mathcal{F} = \mathcal{F}_G[N]$. Note that if $H \in \mathrm{Sec}_p$, then $E\mathcal{F}^H \simeq EW_G H$ as a $W_G H$-space, therefore

$$(\Phi^H(X \otimes E\mathcal{F}_+))^{W_G H} \simeq (\Phi^H X)_{hW_G H}.$$

Indeed, $E\mathcal{F}^H$ is (non-equivariantly) contractible since $H \in \mathcal{F}$, and if K/H is a non-trivial subgroup of $W_G H$, then it must be the case that $K \cap N \neq 1$, therefore $(E\mathcal{F}^H)^{K/H} = E\mathcal{F}^K = \emptyset$. We shall construct a natural transformation

$$\psi\colon \Phi^{\Sigma} X^N \to \bigoplus_{[H]\in \mathrm{Sec}_p/N} (\Phi^H X)^{W_G H}$$

and then define (3.10) as the composite

$$\Phi^{\Sigma} X_{h_\Sigma N} = \Phi^{\Sigma}(X \otimes E\mathcal{F}_+)^N \xrightarrow{\psi} \bigoplus_{[H]\in \mathrm{Sec}_p/N} (\Phi^H(X \otimes E\mathcal{F}_+))^{W_G H} \simeq \bigoplus_{[H]\in \mathrm{Sec}_p/N} (\Phi^H X)_{hW_G H}.$$

For the construction of ψ let $H \in \mathrm{Sec}_p$. We use the projection formula (A.23) to rewrite the source

$$\Phi^{\Sigma} X^N = (X^N \otimes \widetilde{E\mathcal{P}_\Sigma})^{\Sigma} \simeq (X \otimes p^* \widetilde{E\mathcal{P}_\Sigma})^G,$$

and the target is by definition

$$(\Phi^H X)^{W_G H} = ((X \otimes \widetilde{E\mathcal{F}}_{\not\supseteq H})^H)^{W_G H} \simeq (X \otimes \widetilde{E\mathcal{F}}_{\not\supseteq H})^{N_G H}.$$

Our assumption on H implies $\mathrm{res}^G_{N_G H} p^* \mathcal{P}_\Sigma \subset \mathcal{F}_{\not\supseteq H}$, therefore there is a map of $N_G H$-spaces

$$p^* \widetilde{E\mathcal{P}_\Sigma} \to \widetilde{E\mathcal{F}}_{\not\supseteq H} \tag{3.11}$$

(see (A.30) for the construction of this map) and using the equivalences above we define

$$\psi_H\colon \Phi^{\Sigma} X^N \to (\Phi^H X)^{W_G H}$$

as the composite

$$(X \otimes p^* \widetilde{E\mathcal{P}_\Sigma})^G \to (X \otimes p^* \widetilde{E\mathcal{P}_\Sigma})^{N_G H} \to (X \otimes \widetilde{E\mathcal{F}}_{\not\supseteq H})^{N_G H},$$

where the first map is the inclusion of fixed points and the second map is obtained by smashing (3.11) with X. After a choice of representatives of the elements in Sec_p, the map ψ is then obtained from the various ψ_H.

Before we show that ψ is an equivalence, we want to show that for any $g \in G$ and any $H \in \mathrm{Sec}_p$ the maps ψ_H and ψ_{H^g} agree up to the conjugation equivalence. First note that left multiplication by g induces an equivalence of G-spaces

$$l_g\colon p^* \widetilde{E\mathcal{P}_\Sigma} \xrightarrow{\simeq} (c_g)^* p^* \widetilde{E\mathcal{P}_\Sigma}, \tag{3.12}$$

and as explained before there is an equivalence of N_GH-spaces

$$(c_g)^*\widetilde{E\mathcal{F}}_{\not\supseteq H^g} \simeq \widetilde{E\mathcal{F}}_{\not\supseteq H}. \tag{3.13}$$

It follows from the explanation surrounding (A.30) that the following diagram of N_GH-spaces commutes

$$\begin{array}{ccc} p^*\widetilde{E\mathcal{P}}_\Sigma & \longrightarrow & \widetilde{E\mathcal{F}}_{\not\supseteq H} \\ \Big\downarrow{\scriptstyle l_g} & & \Big\downarrow{\scriptstyle \simeq} \\ (c_g)^*p^*\widetilde{E\mathcal{P}}_\Sigma & \longrightarrow & (c_g)^*\widetilde{E\mathcal{F}}_{\not\supseteq H^g}, \end{array} \tag{3.14}$$

where the right vertical arrow is (3.13) and the horizontal arrows are (3.11) and $(c_g)^*$ applied to (3.11).

Now we consider the following diagram

$$\begin{array}{ccccc} (X\otimes p^*\widetilde{E\mathcal{P}}_\Sigma)^G & \longrightarrow & (X\otimes p^*\widetilde{E\mathcal{P}}_\Sigma)^{N_GH} & \xrightarrow{(3.11)} & (X\otimes \widetilde{E\mathcal{F}}_{\not\supseteq H})^{N_GH} \\ \Big\downarrow{\scriptstyle (l_g\otimes p^*\widetilde{E\mathcal{P}}_\Sigma)^G} & & \Big\downarrow{\scriptstyle (l_g\otimes p^*\widetilde{E\mathcal{P}}_\Sigma)^{N_GH}} & & \Big\downarrow{\scriptstyle (l_g\otimes \widetilde{E\mathcal{F}}_{\not\supseteq H})^{N_GH}} \\ ((c_g)^*X\otimes p^*\widetilde{E\mathcal{P}}_\Sigma)^G & \longrightarrow & ((c_g)^*X\otimes p^*\widetilde{E\mathcal{P}}_\Sigma)^{N_GH} & \xrightarrow{(3.11)} & ((c_g)^*X\otimes \widetilde{E\mathcal{F}}_{\not\supseteq H})^{N_GH} \\ {\scriptstyle\simeq}\Big\downarrow{\scriptstyle ((c_g)^*X\otimes l_g)^G} & & {\scriptstyle\simeq}\Big\downarrow{\scriptstyle ((c_g)^*X\otimes l_g)^{N_GH}} & & {\scriptstyle (3.13)}\Big\downarrow{\scriptstyle\simeq} \\ ((c_g)^*(X\otimes p^*\widetilde{E\mathcal{P}}_\Sigma))^G & \longrightarrow & ((c_g)^*(X\otimes p^*\widetilde{E\mathcal{P}}_\Sigma))^{N_GH} & \xrightarrow{(3.13)} & ((c_g)^*(X\otimes \widetilde{E\mathcal{F}}_{\not\supseteq H^g}))^{N_GH} \\ \Big\downarrow{\scriptstyle\simeq} & & \Big\downarrow{\scriptstyle\simeq} & & \Big\downarrow{\scriptstyle\simeq} \\ (X\otimes p^*\widetilde{E\mathcal{P}}_\Sigma)^G & \longrightarrow & (X\otimes p^*\widetilde{E\mathcal{P}}_\Sigma)^{N_GH^g} & \xrightarrow{(3.13)} & (X\otimes \widetilde{E\mathcal{F}}_{\not\supseteq H^g})^{N_GH^g}, \end{array}$$

where the horizontal arrows on the left are the inclusion of fixed points and the lower vertical arrows are the natural equivalences of spectra $((c_g)^*(-))^G \simeq (-)^G$ and $((c_g)^*(-))^{N_GH} \simeq (-)^{N_GH^g}$. We also used the fact that $(c_g)^*$ is a monoidal functor. The upper two squares on the left commute by naturality of the inclusion of fixed points and the lower left square is (A.45). The upper square on the right is obtained by applying $(-)^{N_GH}$ to a commutative square of N_GH-spectra, hence is itself commutative, the middle square on the right is obtained from (3.14), and the lower right square commutes by naturality of the equivalence $((c_g)^*(-))^{N_GH} \simeq (-)^{N_GH^g}$. We conclude by observing that the upper row is ψ_H, the lower row is ψ_{H^g}, the right column is the conjugation equivalence (3.5) and the left column is the identity by our discussion of the conjugation equivalences and the fact that G operates diagonally on $X\otimes p^*\widetilde{E\mathcal{P}}_\Sigma$.

To show that (3.10) is an equivalence, it suffices to show that ψ_X is an equivalence for any $\mathcal{F}$-torsion spectrum X, since $X\otimes E\mathcal{F}_+$ is $\mathcal{F}$-torsion and by a localizing subcategory argument (see Remark A.1.2) it suffices to do so for $X = G/K_+$ with $K\in\mathcal{F}$. We first consider the case where K is not the image of a section $s\colon \Sigma\to G$ and claim that in this case both the source and target of ψ are contractible. For the source we use the projection formula (A.21):

$$G/K_+\otimes p^*\widetilde{E\mathcal{P}}_\Sigma \simeq \mathrm{ind}_K^G\mathrm{res}_K^G p^*\widetilde{E\mathcal{P}}_\Sigma \simeq 0,$$

since $K \in p^*\mathcal{P}_\Sigma$ by our assumption on K. For the target we have for any $H \in \mathrm{Sec}_p$

$$\Phi^H(G/K_+) \simeq \Sigma^\infty (G/K)_+^H \simeq 0,$$

by (3.1), since our assumption on K implies that H cannot be subconjugate to K.

Next, we consider the case where K is the image of a section of p. We have seen in the previous section that $[K] = [H] \in \mathrm{Sec}_p/N$ iff K and H are conjugate in G, thus $\Phi^H(G/K_+) \simeq 0$ as $W_G H$-spectrum if $[K] \neq [H]$. Furthermore, we can assume that K is our chosen representative of $[K]$, since $G/K_+ \simeq G/H_+$ as a G-spectrum if $[K] = [H]$ by (3.1) and the surrounding remarks. Putting this together, we need to show that

$$\psi_K \colon \Phi^\Sigma (G/K_+)^N \to (\Phi^K G/K_+)^{W_G K}.$$

is an equivalence. By (3.2) there is an isomorphism of $N_G K$-sets

$$\mathrm{res}^G_{N_G K} G/K \cong \coprod_{N_G K g K \in N_G K \backslash G/K} N_G K/(N_G K \cap K^g)$$

and we let

$$\mathrm{pr} \colon \mathrm{res}^G_{N_G K} G/K_+ \to N_G K/K_+$$

be the map of pointed $N_G K$-sets which under the above isomorphism is the identity on the summand indexed by $N_G K 1 K$ and sends all other orbits to the basepoint. We consider the diagram of spectra

$$\begin{array}{ccc}
(G/K_+ \otimes p^*\widetilde{E\mathcal{P}_\Sigma})^G & & \\
\downarrow & & \\
(G/K_+ \otimes p^*\widetilde{E\mathcal{P}_\Sigma})^{N_G K} & \longrightarrow & (G/K_+ \otimes \widetilde{E\mathcal{F}}_{\not\supseteq K})^{N_G K} \\
\downarrow{\scriptstyle (\mathrm{pr} \otimes p^*\widetilde{E\mathcal{P}_\Sigma})^{N_G K}} & & \downarrow{\scriptstyle (\mathrm{pr} \otimes \widetilde{E\mathcal{F}}_{\not\supseteq K})^{N_G K}} \\
(N_G K/K_+ \otimes p^*\widetilde{E\mathcal{P}_\Sigma})^{N_G K} & \longrightarrow & (N_G K/K_+ \otimes \widetilde{E\mathcal{F}}_{\not\supseteq K})^{N_G K},
\end{array}$$

where the top left vertical arrow is given by the inclusion of fixed points and the horizontal arrows are obtained by smashing G/K_+ respectively $N_G K/K_+$ with (3.11) and taking fixed points. One checks on fixed points that (3.11) becomes an equivalence of $N_G K$-spaces after smashing with $N_G K/K_+$, therefore the lower horizontal map is an equivalence. Similarly $\mathrm{pr} \otimes \widetilde{E\mathcal{F}}_{\not\supseteq K}$ is an equivalence of $N_G K$-spaces, therefore the right vertical arrow is an equivalence as well. The composition of the two left vertical arrows is an equivalence by [Sch18, Theorem 3.2.15] (see Equation 3.2.6 in loc. cit. for an explanation that the map constructed there agrees with pr). Since ψ_K is the composition of the upper left vertical arrow and upper right horizontal arrow, it is an equivalence as well, as desired, concluding the proof of (i).

We now turn to (ii) and first prove that the restriction of the inclusion of fixed points $\Phi^\Sigma X_{h_\Sigma \tilde{N}} \to \Phi^\Sigma X_{h_\Sigma N}$ to the summand indexed by $[\tilde{H}] \in \mathrm{Sec}_{\tilde{p}}/\tilde{N}$ factors as claimed. For any representative $\tilde{H}$, recall that $\mathcal{F}_{\tilde{G},\leqslant\tilde{H}}$ is the $\tilde{G}$-family of subgroups that are subconjugate to $\tilde{H}$. Then for $K \in \mathrm{Sec}_p$ we have that $\Phi^K(X \otimes E\mathcal{F}_{\tilde{G},\leqslant\tilde{H}+})$ is non-trivial iff K is conjugate to $\tilde{H}$ in $\tilde{G}$, that is iff $[K] \in f^{-1}([\tilde{H}])$, and in that case it is equivalent to $\Phi^K X$. Consider the following diagram

$$\begin{array}{ccc} \Phi^\Sigma(X \otimes E\mathcal{F}_{\tilde{G},\leqslant\tilde{H}+})_{h_\Sigma \tilde{N}} & \longrightarrow & \Phi^\Sigma X_{h_\Sigma \tilde{N}} \\ \downarrow & & \downarrow \\ \Phi^\Sigma(X \otimes E\mathcal{F}_{\tilde{G},\leqslant\tilde{H}+})_{h_\Sigma N} & \longrightarrow & \Phi^\Sigma X_{h_\Sigma N}, \end{array}$$

where the vertical arrows are the inclusion of fixed points and the horizontal arrows are given by collapsing $E\mathcal{F}_{\tilde{G},\leqslant\tilde{H}}$ to a point. Under the equivalence of (i) this diagram is equivalent to the diagram

$$\begin{array}{ccc} (\Phi^{\tilde{H}} X)_{hW_{\tilde{H}}\tilde{H}} & \longrightarrow & \bigoplus_{[\tilde{K}]\in \mathrm{Sec}_{\tilde{p}}/\tilde{N}} (\Phi^{\tilde{K}} X)_{hW_{\tilde{G}}\tilde{K}} \\ \downarrow & & \downarrow \\ \bigoplus_{[H]\in f^{-1}([\tilde{H}])} (\Phi^H X)_{hW_G H} & \longrightarrow & \bigoplus_{[K]\in \mathrm{Sec}_p/N} (\Phi^K X)_{hW_G K}, \end{array}$$

with the horizontal arrows being the inclusion of summands. This shows the claimed factorization.

Finally, we identify

$$(\Phi^{\tilde{H}} X)_{hW_{\tilde{G}}\tilde{H}} \to \bigoplus_{[H]\in f^{-1}([\tilde{H}])} (\Phi^H X)_{hW_G H} \to (\Phi^H X)_{hW_G H}.$$

It suffices to show that the following diagram commutes

$$\begin{array}{ccccc} & & \Phi^\Sigma X^{\tilde{N}} & \longrightarrow & \Phi^\Sigma X^N \\ & \swarrow^{\tilde{\psi}_{\tilde{H}}} & \downarrow{\scriptstyle \tilde{\psi}_H} & & \downarrow{\scriptstyle \psi_H} \\ (\Phi^{\tilde{H}} X)^{W_{\tilde{G}}\tilde{H}} & \xrightarrow{\simeq} & (\Phi^H X)^{W_{\tilde{G}} H} & \longrightarrow & (\Phi^H X)^{W_G H}, \end{array}$$

where the lower left horizontal arrow is the conjugation equivalence (3.5) and the horizontal arrows in the square are the inclusions of fixed points. The triangle commutes by (the proof of) (i) and the square consists of the outer arrows of the following diagram

$$\begin{array}{ccc} (X \otimes \tilde{p}^*\widetilde{E\mathcal{P}_\Sigma})^{\tilde{G}} & \longrightarrow & (X \otimes p^*\widetilde{E\mathcal{P}_\Sigma})^G \\ \downarrow & & \downarrow \\ (X \otimes \tilde{p}^*\widetilde{E\mathcal{P}_\Sigma})^{N_{\tilde{G}}H} & \longrightarrow & (X \otimes p^*\widetilde{E\mathcal{P}_\Sigma})^{N_G H} \\ \downarrow{\scriptstyle (3.11)} & & \downarrow{\scriptstyle (3.11)} \\ (X \otimes \widetilde{E\mathcal{F}}_{N_{\tilde{G}}H,\not\supseteq H})^{N_{\tilde{G}}H} & \longrightarrow & (X \otimes \widetilde{E\mathcal{F}}_{N_G H,\not\supseteq H})^{N_G H}, \end{array}$$

where all undecorated arrows are inclusions of fixed points. The upper square obviously commutes and the lower square commutes by naturality since $\mathrm{res}^{\tilde{G}}_{G}\tilde{p}^* \simeq p^*$ and

$$\mathrm{res}^{N_{\tilde{G}}H}_{N_G H}\mathcal{F}_{N_{\tilde{G}}H,\not\supset H} = \mathcal{F}_{N_G H,\not\supset H}.$$

□

3.3 Parametrized homotopy orbits and Tate construction in the special case of dihedral groups

We specialize the results of the previous section to the case of $G = D_{2p^n}$ and $N = C_{p^n}$. Additionally, we shall compute $\Phi^{\Sigma_2}X^{h_{\Sigma_2}C_{2^n}}$ and $\Phi^{\Sigma_2}(X^{t_{\Sigma_2}C_2})^{h_{\Sigma_2}C_{2^{n-1}}}$ in an important special case. We will use these results in section 3.5 to compute $\Phi^{\Sigma_2}\mathrm{TCR}^n(X;p)$ (see Definition 3.4.10). We start with the computation of $\Phi^{\Sigma_2}X_{h_{\Sigma_2}C_{p^k}}$ for odd p. It follows directly from Lemma 3.2.6.

Lemma 3.3.1. *Let p be odd and X be a D_{2p^n}-spectrum.*

(i) There is a natural equivalence $\Phi^{\Sigma_2}X_{h_{\Sigma_2}C_{p^n}} \simeq \Phi^{\Sigma_2}X$.

(ii) The inclusion of fixed points induces a map $X_{h_{\Sigma_2}C_{p^n}} \to X_{h_{\Sigma_2}C_{p^{n-1}}}$ and after passing to geometric fixed points this is the identity $\mathrm{id}\colon \Phi^{\Sigma_2}X \to \Phi^{\Sigma_2}X$ under the equivalence of (i).

To state the analogous result to Lemma 3.3.1 for $p = 2$ we fix a generator c_n of C_{2^n} for each $n \in \mathbb{N}$ such that under the inclusion $D_{2^n} \subset D_{2^{n+1}}$ we have $c_{n-1} = c_n^2$. Again the next result is a special case of Lemma 3.2.6.

Lemma 3.3.2. *Let X be a $D_{2^{n+1}}$-spectrum.*

(i) There is a natural equivalence $\Phi^{\Sigma_2}X_{h_{\Sigma_2}C_{2^n}} \simeq (\Phi^{\langle\sigma\rangle}X)_{hW_{D_{2^{n+1}}}\langle\sigma\rangle} \oplus (\Phi^{\langle\sigma c_n\rangle}X)_{hW_{D_{2^{n+1}}}\langle\sigma c_n\rangle}$.

(ii) The inclusion of fixed points induces a map $X_{h_{\Sigma_2}C_{2^n}} \to X_{h_{\Sigma_2}C_{2^{n-1}}}$, which on geometric fixed points corresponds to

$$\begin{pmatrix} 1 & 0 \\ c_n^{-1} & 0 \end{pmatrix} : \bigoplus_{i=0,1} (\Phi^{\langle\sigma c_n^i\rangle}X)_{hW_{D_{2^{n+1}}}\langle\sigma c_n^i\rangle} \to \bigoplus_{i=0,1} (\Phi^{\langle\sigma c_{n-1}^i\rangle}X)_{hW_{D_{2^n}}\langle\sigma c_{n-1}\rangle},$$

where

$$c_n^{-1}\colon (\Phi^{\langle\sigma\rangle}X)_{hW_{D_{2^{n+1}}}\langle\sigma\rangle} \xrightarrow{\sim} (\Phi^{\langle\sigma c_{n-1}\rangle}X)_{hW_{D_{2^n}}\langle\sigma c_{n-1}\rangle}$$

is the conjugation equivalence (3.6)[2].

[2] Note that $(\Phi^{\langle\sigma c_{n-1}\rangle}X)_{hW_{D_{2^{n+1}}}\langle\sigma c_{n-1}\rangle} \simeq (\Phi^{\langle\sigma c_{n-1}\rangle}X)_{hW_{D_{2^n}}\langle\sigma c_{n-1}\rangle}$ via the inclusion of fixed points, since $W_{D_{2^{n+1}}}\langle\sigma c_{n-1}\rangle \cong W_{D_{2^n}}\langle\sigma c_{n-1}\rangle$.

The following result computes $\Phi^{\Sigma_2} X^{t_{\Sigma_2} C_p}$ and follows basically from [QS21b, Example 2.53]. It plays a key role in the proof of the dihedral Tate orbit lemma [QS21a, Lemma 3.20]. We will use it in section 3.5 in our computation of $\Phi^{\Sigma_2}\mathrm{TCR}(X;p)$ and later in this section to compute $\Phi^{\Sigma_2} X^{h_{\Sigma_2} C_{2^n}}$ and $\Phi^{\Sigma_2}(X^{t_{\Sigma_2} C_2})^{h_{\Sigma_2} C_{2^{n-1}}}$. For the statement we make some preliminary remarks. Recall from the previous section that for any $D_{2^{n+1}}$-spectrum X we defined $X^{\tau D_4} = \Phi^{D_4} F(ED_{2^{n+1}+}, X)$. Now let $c_1 \in C_2$ be the generator. Then the Borel completion map induces canonical maps

$$X^{\tau D_4} \simeq \Phi^{W\langle\sigma c_1^i\rangle}\Phi^{\langle\sigma c_1^i\rangle} F(ED_{2^{n+1}}, X) \to (\Phi^{\langle\sigma c_1^i\rangle} F(ED_{2^{n+1}} X))^{tW\langle\sigma c_1^i\rangle} = (X^{t\langle\sigma c_1^i\rangle})^{tW\langle\sigma c_1^i\rangle}$$

for $i = 0, 1$. Similarly, the Borel completion map induces for $i = 0, 1$ canonical maps

$$(\Phi^{\langle\sigma c_1^i\rangle} X)^{tW\langle\sigma c_1^i\rangle} \to (X^{t\langle\sigma c_1^i\rangle})^{tW\langle\sigma c_1^i\rangle}.$$

Lemma 3.3.3. *Let X be a D_{2p^n}-spectrum.*

(i) If p is odd, then $\Phi^{\Sigma_2} X^{t_{\Sigma_2} C_p} \simeq 0$.

(ii) For $p = 2$ there is a natural equivalence of spectra

$$\Phi^{\Sigma_2} X^{t_{\Sigma_2} C_2} \simeq \mathrm{fib}\left(X^{\tau D_4} \oplus \bigoplus_{i=0,1} (\Phi^{\langle\sigma c_1^i\rangle} X)^{tW\langle\sigma c_1^i\rangle} \to \bigoplus_{i=0,1} (X^{t\langle\sigma c_1^i\rangle})^{tW\langle\sigma c_1^i\rangle} \right).$$

Proof. This follows from [QS21b, Example 2.51 and Example 2.53] after observing that the $\mathcal{F}_{D_{2p^n}}[C_{p^n}]$-completion map induces an equivalence

$$X^{t_{\Sigma_2} C_p} \simeq F(E\mathcal{F}_{D_{2p^n}}[C_{p^n}]_+, X)^{t_{\Sigma_2} C_p}$$

of Σ_2-spectra by Remark 3.2.2 (ii) and that for any $\mathcal{F}_{D_{2p^n}}[C_{p^n}]$-complete spectrum X we have $X^{t_{\Sigma_2} C_p} \simeq \Phi^{C_p} X$, since $\mathcal{F}_{D_{2p^n}}[C_{p^n}] = \mathcal{F}_{D_{2p^n}, \not\supseteq C_p}$ (see Remark 3.1.1). □

For future reference we explain the definition of the map

$$\Phi^{\Sigma_2} X^{t_{\Sigma_2} C_2} \to (\Phi^{\langle\sigma c_1^i\rangle} X)^{tW\langle\sigma c_1^i\rangle} \tag{3.15}$$

for $i = 0, 1$. Let $\mathcal{F} = \mathcal{F}_{D_{2^{n+1}}}[C_{2^n}]$. Then we have

$$X^{t_{\Sigma_2} C_2} = \Phi^{C_2} F(E\mathcal{F}_+, X)$$

and (3.15) is defined as the composite

$$\begin{aligned}\Phi^{\Sigma_2}\Phi^{C_2} F(E\mathcal{F}_+, X) &\simeq \Phi^{W\langle\sigma c_1^i\rangle}\Phi^{\langle\sigma c_1^i\rangle} F(E\mathcal{F}_+, X) \\ &\to (\Phi^{\langle\sigma c_1^i\rangle} F(E\mathcal{F}_+, X))^{tW\langle\sigma c_1^i\rangle} \simeq (\Phi^{\langle\sigma c_1^i\rangle} X)^{tW\langle\sigma c_1^i\rangle},\end{aligned}$$

where the arrow is the Borel completion map and the final equivalence follows from the fact that $\langle \sigma c_1^i \rangle \in \mathcal{F}$. This map will play a role in the computation of $\Phi^{\Sigma_2}\mathrm{TCR}(X;2)$.

The rest of this section will deal with the case $p = 2$. In general it is difficult to compute $\Phi^{\Sigma_2} X^{h_{\Sigma_2} C_{2n}}$, but in the special case that the underlying spectrum of X is contractible it is possible to give an answer that is similar to Lemma 3.3.2. This special case will play a key role in the computation of $\Phi^{\Sigma_2}\mathrm{TCR}(X;2)$. Our computation of $\Phi^{\Sigma_2} X^{h_{\Sigma_2} C_{2n}}$ will proceed by induction on n and we will need the following statement in the induction step.

Lemma 3.3.4. *Let X be a D_8-spectrum whose underlying spectrum is trivial. Then*

$$\Phi^{\Sigma_2} X^{t_{\Sigma_2} C_2} \simeq \bigoplus_{i=0,1} (\Phi^{\langle \sigma c_1^i \rangle} X)^{tW\langle \sigma c_1^i \rangle}$$

and the residual C_2-action is given by

$$\begin{pmatrix} 0 & c_2^{-1} \\ c_2^{-1} & 0 \end{pmatrix} : \bigoplus_{i=0,1} (\Phi^{\langle \sigma c_1^i \rangle} X)^{tW\langle \sigma c_1^i \rangle} \to \bigoplus_{i=0,1} (\Phi^{\langle \sigma c_1^i \rangle} X)^{tW\langle \sigma c_1^i \rangle},$$

where

$$c_2^{-1} : (\Phi^{\langle \sigma c_1^i \rangle} X)^{tW\langle \sigma c_1^i \rangle} \xrightarrow{\sim} (\Phi^{\langle \sigma c_1^{i-(-1)^i} \rangle} X)^{tW\langle \sigma c_1^{i-(-1)^i} \rangle}$$

is the conjugation equivalence (3.8).

Proof. We assume for simplicity that X is $\mathcal{F}_{D_8}[C_4]$-complete, so that

$$\Phi^{\Sigma_2} X^{t_{\Sigma_2} C_2} \simeq \Phi^{\Sigma_2} \Phi^{C_2} X \simeq \Phi^{D_4} X$$

by Remark 3.1.1 (iv) and Remark 3.2.2 (iv). The stated equivalence follows from Lemma 3.3.3 and our assumptions on X, therefore we only need to identify the residual C_2-action. Recall from (3.15) that the equivalence from the statement is induced by the map

$$\Phi^{D_4} X \simeq \Phi^{W\langle \sigma c_1^i \rangle} \Phi^{\langle \sigma c_1^i \rangle} X \to (\Phi^{\langle \sigma c_1^i \rangle} X)^{tW\langle \sigma c_1^i \rangle},$$

where the arrow is given by the Borel completion. Note that the residual C_2-action on the source is given by the conjugation equivalence

$$\Phi^{D_4} X \xrightarrow{\Phi^{D_4} l_{c_2^{-1}}} \Phi^{D_4} (c_{c_2^{-1}})^* X \simeq \Phi^{D_4} X$$

for both $i = 0$ and $i = 1$. The claim now follows from the commutativity of the following diagram by observing that the right column is c_2^{-1} and the composition of the upper and lower horizontal arrows is the projection onto one of the summands:

$$
\begin{array}{ccccc}
\Phi^{D_4}X & \xrightarrow{\simeq} & \Phi^{W\langle\sigma c_1^i\rangle}\Phi^{\langle\sigma c_1^i\rangle}X & \longrightarrow & (\Phi^{\langle\sigma c_1^i\rangle}X)^{tW\langle\sigma c_1^i\rangle} \\
\downarrow{\scriptstyle \Phi^{D_4}l_{c_2^{-1}}} & & \downarrow{\scriptstyle \Phi^{W\langle\sigma c_1^i\rangle}\Phi^{\langle\sigma c_1^i\rangle}l_{c_2^{-1}}} & & \downarrow{\scriptstyle (\Phi^{\langle\sigma c_1^i\rangle}l_{c_2^{-1}})^{tW\langle\sigma c_1^i\rangle}} \\
\Phi^{D_4}(c_{c_2^{-1}})^*X & \xrightarrow{\simeq} & \Phi^{W\langle\sigma c_1^i\rangle}\Phi^{\langle\sigma c_1^i\rangle}c^*_{c_2^{-1}}X & \longrightarrow & (\Phi^{\langle\sigma c_1^i\rangle}c^*_{c_2^{-1}}X)^{tW\langle\sigma c_1^i\rangle} \\
\Big\downarrow & & \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} \\
{\scriptstyle\simeq} & & \Phi^{W\langle\sigma c_1^i\rangle}\left(\omega^{c_2^{-1}}_{\langle\sigma c_1^i\rangle}\right)^*\Phi^{\langle\sigma c_1^{i+(-1)^i}\rangle}X & \longrightarrow & \left(\left(\omega^{c_2^{-1}}_{\langle\sigma c_1^i\rangle}\right)^*\Phi^{\langle\sigma c_1^{i+(-1)^i}\rangle}X\right)^{tW\langle\sigma c_1^i\rangle} \\
\Big\downarrow & & \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} \\
\Phi^{D_4}X & \xrightarrow{\simeq} & \Phi^{W\langle\sigma c_1^{i+(-1)^i}\rangle}\Phi^{\langle\sigma c_1^{i+(-1)^i}\rangle}X & \longrightarrow & (\Phi^{\langle\sigma c_1^{i+(-1)^i}\rangle}X)^{tW\langle\sigma c_1^{i+(-1)^i}\rangle}.
\end{array}
$$

The upper squares commute by naturality of Borel completion and the commutativity of the right middle square follows from our discussion of the conjugation equivalences. The lower left square is (A.47). To see the commutativity of the lower right square we observe that for any group G if $B_G\colon X \to F(EG_+, X)$ is the Borel completion map and $\alpha\colon \Gamma \to G$ is an isomorphism, then $\alpha^*F(EG_+, X) \simeq F(E\Gamma_+, \alpha^*X)$ and the composite $\alpha^*X \xrightarrow{\alpha^*B_G} \alpha^*F(EG_+, X) \simeq F(E\Gamma_+, \alpha^*X)$ is the Borel completion of α^*X. □

Lemma 3.3.5. *Let X be a $D_{2^{n+1}}$-spectrum whose underlying spectrum is trivial. Then there is a natural equivalence*

$$\Phi^{\Sigma_2}X^{h_{\Sigma_2}C_{2^n}} \simeq \bigoplus_{i=0,1}(\Phi^{\langle\sigma c_n^i\rangle}X)^{hW\langle\sigma c_n^i\rangle}.$$

Under this equivalence the inclusion of fixed points $\Phi^{\Sigma_2}X^{h_{\Sigma_2}C_{2^n}} \to \Phi^{\Sigma_2}X^{h_{\Sigma_2}C_{2^{n-1}}}$ is given by

$$\begin{pmatrix} 1 & 0 \\ c_n^{-1} & 0 \end{pmatrix}\colon \bigoplus_{i=0,1}(\Phi^{\langle\sigma c_n^i\rangle}X)^{hW_{D_{2^{n+1}}}\langle\sigma c_n^i\rangle} \to \bigoplus_{i=0,1}(\Phi^{\langle\sigma c_{n-1}^i\rangle}X)^{hW_{D_{2^n}}\langle\sigma c_{n-1}\rangle},$$

where

$$c_n^{-1}\colon (\Phi^{\langle\sigma\rangle}X)^{hW\langle\sigma\rangle} \xrightarrow{\sim} (\Phi^{\langle\sigma c_{n-1}\rangle}X)^{hW\langle\sigma c_{n-1}\rangle}$$

is the conjugation equivalence (3.7). Consequently, there are natural equivalences

$$\Phi^{\Sigma_2}X^{h_{\Sigma_2}C_{2^\infty}} \simeq (\Phi^{\Sigma_2}X)^{hC_2},$$
$$\Phi^{\Sigma_2}(X^{t_{\Sigma_2}C_2})^{h_{\Sigma_2}C_{2^\infty}} \simeq (\Phi^{\Sigma_2}X)^{tC_2}.$$

Proof. For notational simplicity we assume that X is $\mathcal{F}_{D_{2^{n+1}}}[C_{2^n}]$-complete, so that $X^{h_{\Sigma_2}C_{2^n}} \simeq X^{C_{2^n}}$. We also drop the subscripts for the normalizer and Weyl group, since they do not depend on n. We start by constructing the map

$$\Phi^{\Sigma_2}X^{h_{\Sigma_2}C_{2^n}} \to \bigoplus_{i=0,1}(\Phi^{\langle\sigma c_n^i\rangle}X)^{hW\langle\sigma c_n^i\rangle}. \tag{3.16}$$

Let $\alpha\colon D_{2^{n+1}} \to D_{2^{n+1}}/C_{2^n} \cong \Sigma_2$ denote the projection and note that $\alpha^*\widetilde{E\Sigma_2} \simeq \widetilde{E\mathcal{F}}_{\subset C_{2^n}}$. By the projection formula (A.23) we have

$$\Phi^{\Sigma_2} X^{C_{2^n}} = (X^{C_{2^n}} \otimes \widetilde{E\Sigma_2})^{\Sigma_2} \simeq (X \otimes \widetilde{E\mathcal{F}_{\subset C_{2^n}}})^{D_{2^{n+1}}}.$$

We define (3.16) as the composition of the inclusion of fixed points

$$(X \otimes \widetilde{E\mathcal{F}_{\subset C_{2^n}}})^{D_{2^{n+1}}} \to (X \otimes \widetilde{E\mathcal{F}_{\subset C_{2^n}}})^{N\langle \sigma c_n^i\rangle}$$

with the map

$$(X \otimes \widetilde{E\mathcal{F}_{\subset C_{2^n}}})^{N\langle \sigma c_n^i\rangle} \to (X \otimes \widetilde{E\mathcal{F}_{\not\supset\langle \sigma c_n^i\rangle}})^{N\langle \sigma c_n^i\rangle} \simeq (\Phi^{\langle \sigma c_n^i\rangle} X)^{W\langle \sigma c_n^i\rangle} \to (\Phi^{\langle \sigma c_n^i\rangle} X)^{hW\langle \sigma c_n^i\rangle},$$

where the first arrow is induced by (A.30) (note that $\mathrm{res}^{D_{2^{n+1}}}_{N\langle \sigma c_n^i\rangle} \mathcal{F}_{\subset C_{2^n}} \subset \mathcal{F}_{\not\supset\langle \sigma c_n^i\rangle}$) and the final arrow is induced by the Borel completion of $\Phi^{\langle \sigma c_n^i\rangle} X$.

We show that (3.16) is an equivalence by induction on n. For $n = 1$ this follows from Lemma 3.3.2 and Lemma 3.3.3. For the induction step we consider the diagram of cofiber sequences

$$\begin{array}{ccccc}
\Phi^{\Sigma_2}(X_{h_{\Sigma_2} C_2})^{h_{\Sigma_2} C_{2^{n-1}}} & \xrightarrow{\Phi^{\Sigma_2}\mathrm{Nm}} & \Phi^{\Sigma_2} X^{h_{\Sigma_2} C_{2^n}} & \longrightarrow & \Phi^{\Sigma_2}(X^{t_{\Sigma_2} C_2})^{h_{\Sigma_2} C_{2^{n-1}}} \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{i=0,1} (\Phi^{\langle \sigma c_n^i\rangle} X)_{hW\langle \sigma c_n^i\rangle} & \xrightarrow{\mathrm{Nm}\oplus\mathrm{Nm}} & \bigoplus_{i=0,1} (\Phi^{\langle \sigma c_n^i\rangle} X)^{hW\langle \sigma c_n^i\rangle} & \longrightarrow & \bigoplus_{i=0,1} (\Phi^{\langle \sigma c_n^i\rangle} X)^{tW\langle \sigma c_n^i\rangle}.
\end{array}$$

The left vertical arrow is an equivalence by Lemma 3.3.2 and the dihedral Tate orbit lemma [QS21a, Lemma 3.20] and the right vertical arrow is an equivalence by induction. By hypothesis we have

$$\Phi^{\Sigma_2}(X^{t_{\Sigma_2} C_2})^{h_{\Sigma_2} C_{2^{n-1}}} \simeq \bigoplus_{i=0,1} (\Phi^{\langle \sigma c_{n-1}^i\rangle} X^{t_{\Sigma_2} C_2})^{hW\langle \sigma c_{n-1}^i\rangle},$$

and Lemma 3.3.4 implies that

$$(\Phi^{\langle \sigma c_{n-1}^i\rangle} X^{t_{\Sigma_2} C_2})^{hW\langle \sigma c_{n-1}^i\rangle} \simeq \left(\bigoplus_{j=0,1} (\Phi^{\langle \sigma c_n^i c_1^j\rangle} X)^{tW\langle \sigma c_n^i c_1^j\rangle}\right)^{hW\langle \sigma c_{n-1}^i\rangle} \simeq (\Phi^{\langle \sigma c_n^i\rangle} X)^{tW\langle \sigma c_n^i\rangle}.$$

Therefore the right vertical arrow in the diagram is an equivalence, thus also the middle vertical arrow, as desired. The identification of the map

$$\bigoplus_{i=0,1} (\Phi^{\langle \sigma c_n^i\rangle} X)^{hW_{D_{2^{n+1}}}\langle \sigma c_n^i\rangle} \to \bigoplus_{i=0,1} (\Phi^{\langle \sigma c_{n-1}^i\rangle} X)^{hW_{D_{2^n}}\langle \sigma c_{n-1}\rangle}$$

is done analogously to the proof of Lemma 3.2.6.

The equivalence $\Phi^{\Sigma_2} X^{h_{\Sigma_2} C_{2^\infty}} \simeq (\Phi^{\Sigma_2} X)^{hC_2}$ follows from Proposition 3.5.1 and the observation that the tower

$$\cdots \xrightarrow{\begin{pmatrix} 1 & 0 \\ c_3^{-1} & 0 \end{pmatrix}} \bigoplus_{i=0,1} (\Phi^{\langle \sigma c_2^i\rangle} X)^{hW\langle \sigma c_2^i\rangle} \xrightarrow{\begin{pmatrix} 1 & 0 \\ c_2^{-1} & 0 \end{pmatrix}} \bigoplus_{i=0,1} (\Phi^{\langle \sigma c_1^i\rangle} X)^{hW\langle \sigma c_1^i\rangle}$$

is pro-equivalent to the constant tower with value $(\Phi^{\Sigma_2} X)^{hC_2}$. Finally, we have

$$\Phi^{\Sigma_2}(X^{t_{\Sigma_2} C_2})^{h_{\Sigma_2} C_{2^\infty}} \simeq (\Phi^{\Sigma_2} X^{t_{\Sigma_2} C_2})^{hC_2} \simeq (\Phi^{\Sigma_2} X)^{tC_2},$$

where the first equivalence follows from what we have shown before and the second equivalence from Lemma 3.3.4. ☐

3.4 Real p-cyclotomic spectra.

In this section we define real p-cyclotomic and genuine p-cyclotomic spectra following [QS21a], discuss their main properties we need and treat the examples which are relevant to us. We give a slightly different definition from the one given in loc cit. and we will explain afterwards why both are equivalent.

Definition 3.4.1. A real p-cyclotomic spectrum is a $\mathcal{F}_{D_{2p^\infty}}[C_{p^\infty}]$-complete D_{2p^∞}-spectrum together with a map $\varphi_X\colon X \to X^{t_{\Sigma_2} C_p}$ of D_{2p^∞}-spectra. We refer to the map φ_X as the *Frobenius*.

The reason to insist that X is $\mathcal{F}_{D_{2p^\infty}}[C_{p^\infty}]$-complete is the following. In the classical setting a p-cyclotomic spectrum has an underlying spectrum with C_{p^∞}-action, by which we mean an object of $\mathrm{Fun}(BC_{p^\infty}, \mathbf{Sp})$. It is well known that $\mathrm{Fun}(BC_{p^\infty}, \mathbf{Sp})$ embeds fully faithfully into $\mathbf{Sp}^{C_{p^\infty}}$ with essential image given by the Borel complete spectra (see for example [MNN17, Proposition 6.17]). For the real analogon one wants to replace C_{p^∞} with D_{2p^∞}, but the Σ_2-part should be genuine and the C_{p^∞}-part should be Borel. Quigley and Shah construct an ∞-category $\mathbf{Sp}^{D_{2p^\infty}}_{C_{p^\infty}-\mathrm{Borel}}$ in [QS21b] using parametrized ∞-categories and they show this category embeds fully faithfully into $\mathbf{Sp}^{D_{2p^\infty}}$ with the $\mathcal{F}[C_{p^\infty}]$-complete spectra as essential image [QS21b, Theorem A], therefore our definition is equivalent to [QS21a, Definition 2.5]. To avoid the use of parametrized ∞-categories, we have opted to give the above definition in the language of equivariant stable homotopy theory.

From the definition and the discussion above one immediately sees that any real p-cyclotomic spectrum has an underlying p-cyclotomic spectrum. Just as in the non-real case, one can define the (stable) ∞-category of real p-cyclotomic spectra $\mathbb{R}\mathbf{CycSp}_p$ as a lax equalizer. We refer to [QS21a, Section 2.1] for more details. The crucial part for us is that [NS18, Proposition II.1.5 and Construction IV.2.1] imply the following:

(i) There is a canonical forgetful functor $\mathbb{R}\mathbf{CycSp}_p \to \mathbf{Sp}^{D_{2p^\infty}}$, which preserves colimits. We refer to the image of a real p-cyclotomic spectrum under this functor as its underlying D_{2p^∞}-spectrum. Similarly, any real p-cyclotomic spectrum has an underlying (Σ_2-)spectrum.

(ii) A map of real cyclotomic spectra $f\colon (X, \varphi_X) \to (Y, \varphi_Y)$ is given by a map $f\colon X \to Y$ of

D_{2p^∞}-spectra such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\varphi_X} & & \downarrow{\varphi_Y} \\ X^{t_{\Sigma_2}C_p} & \xrightarrow{f^{t_{\Sigma_2}C_p}} & Y^{t_{\Sigma_2}C_p} \end{array}$$

commutes. Furthermore, f is an equivalence of real p-cyclotomic spectra iff it is an equivalence on the underlying Σ_2-spectra.

(iii) There is a symmetric monoidal structure on $\mathbb{R}\mathbf{CycSp}_p$ with the property that the underlying D_{2p^∞}-spectrum of $X \otimes Y$ is the smash product of D_{2p^∞}-spectra and the Frobenius is given as the composition

$$X \otimes Y \xrightarrow{\varphi_X \otimes \varphi_Y} X^{t_{\Sigma_2}C_p} \otimes Y^{t_{\Sigma_2}C_p} \to (X \otimes Y)^{t_{\Sigma_2}C_p},$$

where the second arrow is the lax monoidal structure of the parametrized Tate construction.

Example 3.4.2. Let X be a Σ_2-spectrum, denote by

$$\pi_p \colon D_{2p^\infty}/C_p \xrightarrow{\cong} D_{2p^\infty} \quad \text{and} \quad \rho_p \colon D_{2p^\infty} \xrightarrow{\cong} D_{2p^\infty}/C_p$$

the pth power and root map respectively, and let $\alpha \colon D_{2p^\infty} \to \Sigma_2$ and $\beta \colon D_{2p^\infty} \to D_{2p^\infty}/C_p$ be the projections. Note that π_p^* and ρ_p^* are inverse equivalences, therefore $(\pi_p)_* \simeq \rho_p^*$. We consider the counit map

$$\alpha^* X \to (\pi_p\beta)_*(\pi_p\beta)^*\alpha^* X \simeq \rho_p^*\beta_*\alpha^* X = \rho_p^*(\alpha^* X)^{C_p}, \tag{3.17}$$

where we used that $\alpha\pi_p\beta = \alpha$. If we compose this with the $\mathcal{F}_{D_{2p^\infty}}[C_{p^\infty}]$-completion map we obtain a map of D_{2p^∞}-spectra $\alpha^* X \to \rho_p^*(\alpha^* X)^{h_{\Sigma_2}C_p}$. We can further compose this map with the canonical map $(-)^{h_{\Sigma_2}C_p} \to (-)^{t_{\Sigma_2}C_p}$ to obtain a real p-cyclotomic structure on $\alpha^* X$. We denote the resulting real p-cyclotomic spectrum by X^{triv}.

We will also need the notion of genuine real p-cyclotomic spectra. The reason is twofold. Firstly, the construction of real topological Hochschild homology we present is an instance of a genuine real p-cyclotomic spectrum. Secondly, we will use the genuine version of real topological cyclic homology to prove boundedness results, see Remark 3.5.3.

Definition 3.4.3. A genuine real p-cyclotomic spectrum is a D_{2p^∞}-spectrum together with an equivalence

$$X \xrightarrow{\simeq} \Phi^{C_p} X$$

of D_{2p^∞}-spectra, where we use the identification $D_{2p^\infty}/C_p \cong D_{2p^\infty}$ to view the right hand side as a D_{2p^∞}-spectrum.

Remark 3.4.4. One can also define the (stable) ∞-category of genuine real p-cyclotomic spectra $\mathbb{R}\mathbf{CycSp}_p^{\mathrm{gen}}$ as a lax equalizer. Again by [NS18, Construction IV.2.1] we see that $\mathbb{R}\mathbf{CycSp}_p^{\mathrm{gen}}$ has a symmetric monoidal structure such that the underlying D_{2p^∞}-spectrum of $X \otimes Y$ is the smash product of the underlying D_{2p^∞}-spectra. For any D_{2p^∞}-spectrum X the $\mathcal{F}_{D_{2p^\infty}}[C_{p^\infty}]$-completion induces a map $\Phi^{C_p} X \to X^{t_{\Sigma_2} C_p}$ of D_{2p^∞}-spectra. Thus, if X is a genuine real p-cyclotomic spectrum we can give it the structure of a real p-cyclotomic spectrum by defining its Frobenius as the composite

$$X \simeq \Phi^{C_p} X \to X^{t_{\Sigma_2} C_p}.$$

This extends to a monoidal functor $\mathbb{R}\mathbf{CycSp}_p^{\mathrm{gen}} \to \mathbb{R}\mathbf{CycSp}_p$, which by [QS21a, Theorem 3.3] is an equivalence on the full subcategories of objects whose underlying spectra are bounded below.

Example 3.4.5. Let A be a ring spectrum with anti-involution. Let $DN_\bullet^{\otimes}(A) = A^{\otimes \bullet+1}$. The dihedral structure is defined similarly to the dihedral structure of $DN_\bullet(G)$. Its realization gives rise to an $O(2)$-spectrum, which we call the *real topological Hochschild homology* spectrum of A and denote by $\mathrm{THR}(A)$.

This is a genuine real p-cyclotomic spectrum for all p. We describe the equivalence $\mathrm{THR}(A) \simeq \Phi^{C_p}\mathrm{THR}(A)$. If we equip $A^{\otimes p}$ with the C_p-action given by cyclic permutation, then there is an equivalence

$$A \xrightarrow{\sim} \Phi^{C_p} A^{\otimes p}, \tag{3.18}$$

(see [HHR16, Proposition B.209] after noting that the right hand side are the geometric fixed points of the Hill-Hopkins-Ravenel norm construction), which is induced by the diagonal map. Therefore, if we take the $(n+1)$-fold smash product of (3.18) and use that geometric fixed points are monoidal we get an equivalence of dihedral objects

$$DN_\bullet^{\otimes}(A) \xrightarrow{\sim} \Phi^{C_p}(\mathrm{sd}_p DN_\bullet^{\otimes}(A)),$$

and since geometric fixed points also commute with colimits we obtain the equivalence

$$\mathrm{THR}(A) \simeq \Phi^{C_p}\mathrm{THR}(A). \tag{3.19}$$

We refer to [DPM22, Section 3.3] for details on the fact that this is an equivalence of D_{2p^∞}-spectra (in fact it is one of $O(2)$-spectra).

An important special case is the following. Let G be a discrete group with anti-involution given by sending an element to its inverse. Then the spherical group ring $\mathbb{S}[G] = \Sigma^\infty G_+$ is a ring spectrum with anti-involution and we have an isomorphism of dihedral objects

$$\Sigma^\infty DN_\bullet(G)_+ \cong DN_\bullet^{\otimes}(\mathbb{S}[G]).$$

The genuine real p-cyclotomic structure can be described as follows. By Proposition 2.2.3 the map of dihedral sets

$$DN_\bullet(G) \xrightarrow{\cong} (\mathrm{sd}_p DN_\bullet(G))^{C_p}, (g_0, \dots, g_n) \mapsto (g_0, \dots, g_n, g_0 \dots, g_n, \dots, g_0, \dots, g_n). \tag{3.20}$$

induces an $O(2)$-equivariant homeomorphism $DN(G) \cong DN(G)^{C_p}$ and we observe that by general properties of the Hill-Hopkins-Ravenel norm the following triangle commutes

$$\begin{array}{ccc} \Sigma^\infty G_+ & \xrightarrow{\simeq} & \Phi^{C_p}(\Sigma^\infty G_+)^{\otimes p} \\ & {}_{\Delta_p}\searrow \quad \swarrow_{\simeq} & \\ & \Sigma^\infty((G^{\times p})_+)^{C_p}, & \end{array}$$

where the horizontal arrow is (3.18) and Δ_p is the diagonal map of spaces. From this we see that in the case at hand (3.19) is equal to (3.20).

Remark 3.4.6. For two ring spectra A and B with anti-involution, the shuffle isomorphism induces an isomorphism of dihedral orthogonal spectra $DN^{\otimes}_{\bullet}(A \otimes B) \cong DN^{\otimes}_{\bullet}(A) \otimes DN^{\otimes}_{\bullet}(B)$, and therefore after realization an equivalence

$$\mathrm{THR}(A \otimes B) \simeq \mathrm{THR}(A) \otimes \mathrm{THR}(B),$$

which clearly is compatible with the genuine (p-)cyclotomic structure. In particular we obtain an equivalence of genuine (p-)cyclotomic spectra

$$\mathrm{THR}(A[G]) \simeq \mathrm{THR}(A) \otimes DN(G)_+.$$

The following example will also be useful in the context of group rings.

Example 3.4.7. Let X be a Σ_2-spectrum. We can also realize X^{triv} as a genuine real p-cyclotomic spectrum. Denote by $\rho_p\colon D_{2p^\infty} \to D_{2p^\infty}/C_p$ the pth root map and by $\alpha\colon D_{2p^\infty} \to \Sigma_2$ the projection. Then the composite

$$\alpha^* X \to \rho_p^*(\alpha^* X)^{C_p} \to \rho_p^* \Phi^{C_p} \alpha^* X,$$

where the first map is (3.17) and the second map is the canonical map from fixed points to geometric fixed points, is an equivalence of D_{2p^∞}-spectra[3] and we denote the resulting genuine real p-cyclotomic spectrum by $X^{\mathrm{triv,gen}}$. Consider the diagram

$$\begin{array}{ccccc} \alpha^* X & \longrightarrow & \rho_p^*(\alpha^* X)^{C_p} & \longrightarrow & \rho_p^* \Phi^{C_p} \alpha^* X \\ & \searrow & \downarrow & & \downarrow \\ & & \rho_p^*(\alpha^* X)^{h_{\Sigma_2} C_p} & \longrightarrow & \rho_p^*(\alpha^* X)^{t_{\Sigma_2} C_p}, \end{array}$$

where the undecorated vertical arrows are induced by $\mathcal{F}_{D_{2p^\infty}}[C_{p^\infty}]$-completion. We see that the real p-cyclotomic spectrum obtained from $X^{\mathrm{triv,gen}}$ is X^{triv}.

In chapter 4 we will construct a finite filtration $F_k(X \otimes DN(D_{2n})_+)$ on $X \otimes DN(D_{2n})_+$ such that the Frobenius maps $F_k(X \otimes DN(D_{2n})_+)$ into $F_{k-1}(X \otimes DN(D_{2n})_+)$. The existence and desired properties of this filtration will follow from the next lemma.

[3] Use that both sides commute with colimits and Lemma A.2.7 in conjunction with the localizing subcategory argument from Remark A.1.2.

Lemma 3.4.8. *Under the equivalence*

$$\Sigma^\infty DN(G)_+ \simeq \bigoplus_{\substack{[[x]]\in \mathrm{conj}_{\mathbb{R}}(G)\\ x^2=1}} \Sigma^\infty B_{\mathbb{R}} Z_G\langle x\rangle_+ \oplus \bigoplus_{\substack{[[x]]\in \mathrm{conj}_{\mathbb{R}}(G)\\ x^2\neq 1}} \Sigma^\infty B_{\mathbb{R}}(SZ_G\langle x\rangle \wr \Sigma_2)_+$$

of Proposition 2.3.2, the Frobenius

$$\Sigma^\infty DN(G)_+ \to (\Sigma^\infty DN(G)_+)^{t_{\Sigma_2}C_p}$$

sends the summand indexed by $[[x]]$ *to the summand indexed by* $[[x^p]]$*.*

Proof. Recall from the proof of Proposition 2.3.2 that there is a decomposition of dihedral sets

$$DN_\bullet(G) \cong \coprod_{[[x]]\in \mathrm{conj}_{\mathbb{R}}(G)} DN_{\bullet,[[x]]}(G),$$

where $DN_{\bullet,[[x]]}(G)$ consists of the n-simplices $(g_0,\dots,g_n)$ such that $g_0g_1\cdots g_n \in [[x]]$. We denote its geometric realization by $DN_{[[x]]}(G)$. Consider the commutative diagram

$$\begin{array}{ccc}
\bigoplus_{[[x]]\in \mathrm{conj}_{\mathbb{R}}(G)} \Sigma^\infty DN_{[[x]]}(G)_+ & \xrightarrow{\simeq} & \Sigma^\infty DN(G)_+ \\
\downarrow \simeq & & \downarrow \simeq \\
\bigoplus_{[[x]]\in \mathrm{conj}_{\mathbb{R}}(G)} \Phi^{C_p}\Sigma^\infty DN_{[[x]]}(G)_+ & \xrightarrow{\simeq} & \Phi^{C_p}\Sigma^\infty DN(G)_+ \\
\downarrow & & \downarrow \\
\bigoplus_{[[x]]\in \mathrm{conj}_{\mathbb{R}}(G)} (\Sigma^\infty DN_{[[x]]}(G)_+)^{t_{\Sigma_2}C_p} & \xrightarrow{\simeq} & (\Sigma^\infty DN(G)_+)^{t_{\Sigma_2}C_p}
\end{array}$$

and note that the equivalence from Proposition 2.3.2 maps the summand indexed by $[[x]]$ into $DN_{[[x]]}(G)$. The claim then follows by observing that the lower left vertical arrow in the diagram preserves the summands and that the map induced by (3.20) maps $DN_{[[x]]}(G)$ into $DN_{[[x^p]]}(G)$. □

Now that we have treated the relevant material on real cyclotomic spectra we can define real topological cyclic homology. The definition we give is not the original definition given in [QS21a], but it is equivalent by [QS21a, Proposition 2.23].

Definition 3.4.9. Let (X,φ_X) be a real p-cyclotomic spectrum. Then its (p-typical) real topological cyclic homology is the Σ_2-spectrum defined via the fiber sequence

$$\mathrm{TCR}(X;p) \to X^{h_{\Sigma_2}C_{p^\infty}} \xrightarrow{\varphi_X^{h_{\Sigma_2}C_{p^\infty}} - \mathrm{can}_p} (X^{t_{\Sigma_2}C_p})^{h_{\Sigma_2}C_{p^\infty}}, \tag{3.21}$$

where can_p denotes the composition

$$X^{h_{\Sigma_2}C_{p^\infty}} \simeq (X^{h_{\Sigma_2}C_p})^{h_{\Sigma_2}(C_{p^\infty}/C_p)} \simeq (X^{h_{\Sigma_2}C_p})^{h_{\Sigma_2}C_{p^\infty}} \to (X^{t_{\Sigma_2}C_p})^{h_{\Sigma_2}C_{p^\infty}}$$

with the arrow induced by the canonical map into the cofiber and the second equivalence by the isomorphism of groups $C_{p^\infty}/C_p \cong C_{p^\infty}$. If A is a ring spectrum with anti-involution we define $\mathrm{TCR}(A;p) = \mathrm{TCR}(\mathrm{THR}(A);p)$.

It is also convenient to define a finitary version of TCR.

Definition 3.4.10. Let (X, φ_X) be a real p-cyclotomic spectrum. Denote by $\varphi_{X,n} \colon X^{h\Sigma_2 C_{p^n}} \to (X^{t\Sigma_2 C_p})^{h\Sigma_2 C_{p^{n-1}}}$ the composite

$$X^{h\Sigma_2 C_{p^n}} \to X^{h\Sigma_2 C_{p^{n-1}}} \xrightarrow{\varphi_X^{h\Sigma_2 C_{p^{n-1}}}} (X^{t\Sigma_2 C_p})^{h\Sigma_2 C_{p^{n-1}}},$$

where the first arrow is the inclusion of fixed points. We define $\mathrm{TCR}^n(X;p)$ as the fiber of

$$X^{h\Sigma_2 C_{p^n}} \xrightarrow{\varphi_{X,n} - \mathrm{can}_p} (X^{t\Sigma_2 C_p})^{h\Sigma_2 C_{p^{n-1}}},$$

where can_p is defined analogously as above. Again, if A is a ring spectrum with anti-involution we write $\mathrm{TCR}^n(A;p)$ for $\mathrm{TCR}^n(\mathrm{THR}(A);p)$.

3.5 Geometric fixed points of $\mathrm{TCR}(X;p)$.

In this section we calculate the geometric fixed points of $\mathrm{TCR}(X;p)$. We distinguish between odd primes p and the case $p = 2$. For odd primes it is easy to show that $\Phi^{\Sigma_2}(X^{t\Sigma_2 C_p})^{h\Sigma_2 C_{p^n}}$ vanishes and the identification of $\Phi^{\Sigma_2}\mathrm{TCR}(X;p)$ follows almost immediately. At the prime $p = 2$ a similar vanishing result does not hold and consequently the calculation is more involved. We start with some preliminaries.

3.5.1 Preliminaries

The goal of this subsection is twofold. First, we want to show that under certain boundedness conditions one can exchange the limit of Σ_2-spectra with geometric fixed points. Second, in order to apply this result we want to show that if X satisfies certain boundedness conditions then also $\Phi^{\Sigma_2}\mathrm{TCR}(X;p)$ and $\Phi^{\Sigma_2}\mathrm{TCR}^n(X;p)$ do. In the next result we say that a family $\{X_i\}_{i\in I}$ of spectra is uniformly bounded below if there is an integer k such that all the X_i are at least k-connected.

Proposition 3.5.1. *Let* $\cdots \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0$ *be a tower of* Σ_2*-spectra such that the underlying tower of spectra is uniformly bounded below. Then* $\Phi^{\Sigma_2} \lim_n X_n \simeq \lim_n \Phi^{\Sigma_2} X_n$.

Proof. We let $X = \lim_n X_n$. By the isotropy separation sequence the proposition is equivalent to the statement that $X_{h\Sigma_2} \simeq \lim_n (X_n)_{h\Sigma_2}$. Let k be an integer such that all the X_n are at least k-connected. By the Milnor exact sequence the underlying spectrum of X is at least $(k-1)$-connected. We use the geometric realization of the bar construction $\Sigma_2^{\times \bullet + 1}$ as a model for $E\Sigma_2$

and denote the m-skeleton of $E\Sigma_2$ by $\mathrm{sk}_m E\Sigma_2$. Then $E\Sigma_2/\mathrm{sk}_m E\Sigma_2$ is an m-connected pointed Σ_2-CW complex with free Σ_2-action away from the basepoint. This implies that

$$E\Sigma_2/\mathrm{sk}_m E\Sigma_2 \simeq (E\Sigma_2/\mathrm{sk}_m E\Sigma_2) \wedge E\Sigma_{2+},$$

so that

$$(X \otimes E\Sigma_2/\mathrm{sk}_m E\Sigma_2)^{\Sigma_2} \simeq (X \otimes E\Sigma_2/\mathrm{sk}_m E\Sigma_2)_{h\Sigma_2}.$$

Since homotopy orbits preserve connectivity of the underlying spectra we obtain that $(X \otimes E\Sigma_2/\mathrm{sk}_m E\Sigma_2)^{\Sigma_2}$ is at least $(k-1+m)$-connected and similarly $(X_n \otimes E\Sigma_2/\mathrm{sk}_m E\Sigma_2)^{\Sigma_2}$ is $(k+m)$-connected for all n. Again by the Milnor exact sequence $\lim_n (X_n \otimes E\Sigma_2/\mathrm{sk}_m E\Sigma_2)^{\Sigma_2}$ is then also $(k-1+m)$-connected. We consider the diagram

$$\begin{array}{ccc}
(X \otimes \mathrm{sk}_m E\Sigma_{2+})^{\Sigma_2} & \xrightarrow{\simeq} & \lim_n (X_n \otimes \mathrm{sk}_m E\Sigma_{2+})^{\Sigma_2} \\
\downarrow & & \downarrow \\
(X \otimes E\Sigma_{2+})^{\Sigma_2} & \longrightarrow & \lim_n (X_n \otimes E\Sigma_{2+})^{\Sigma_2} \\
\downarrow & & \downarrow \\
(X \otimes E\Sigma_2/\mathrm{sk}_m E\Sigma_2)^{\Sigma_2} & \longrightarrow & \lim_n (X_n \otimes E\Sigma_2/\mathrm{sk}_m E\Sigma_2)^{\Sigma_2},
\end{array}$$

where the columns are cofiber sequences. The upper horizontal arrow is an equivalence, since $\mathrm{sk}_m E\Sigma_2$ is a finite Σ_2-CW complex. After passing to colimits the lower left and lower right corners vanish, because their connectivity goes to infinity with m, thus we see that also the middle horizontal arrow is an equivalence, which is the statement. □

To show boundedness conditions on $\mathrm{TCR}(X;p)$ we need to use the genuine variant of real topological cyclic homology. We recall the necessary details. If X is a real genuine p-cyclotomic spectrum it comes equipped with the *restriction* map $R\colon X^{C_{p^n}} \to X^{C_{p^{n-1}}}$ for all n given by the composite

$$X^{C_{p^n}} \simeq (X^{C_p})^{C_{p^{n-1}}} \to (\Phi^{C_p} X)^{C_{p^{n-1}}} \simeq X^{C_{p^{n-1}}},$$

where the arrow is the canonical map from fixed points to geometric fixed points and the final equivalence comes from the genuine cyclotomic structure. There is also a map $F\colon X^{C_{p^n}} \to X^{C_{p^{n-1}}}$ called the *Frobenius* given by inclusion of fixed points. These maps have the property that the following diagram commutes:

$$\begin{array}{ccc}
X^{C_{p^n}} & \xrightarrow{R} & X^{C_{p^{n-1}}} \\
\downarrow F & & \downarrow F \\
X^{C_{p^{n-1}}} & \xrightarrow{R} & X^{C_{p^{n-2}}}.
\end{array}$$

Therefore, if we define

$$\mathrm{TCR}^{n,\mathrm{gen}}(X;p) = \mathrm{fib}\left(X^{C_{p^n}} \xrightarrow{R-F} X^{C_{p^{n-1}}} \right)$$

we get induced maps $\mathrm{TCR}^{n,\mathrm{gen}}(X;p) \to \mathrm{TCR}^{n-1,\mathrm{gen}}(X;p)$ and we define

$$\mathrm{TCR}^{\mathrm{gen}}(X;p) = \lim_n \mathrm{TCR}^{n,\mathrm{gen}}(X;p).$$

By [QS21a, Corollary 3.30] we have $\mathrm{TCR}^{\mathrm{gen}}(X;p) \simeq \mathrm{TCR}(X;p)$. In fact, the proof there shows that $\mathrm{TCR}^{n,\mathrm{gen}}(X;p) \simeq \mathrm{TCR}^n(X;p)$ and then passes to the limit. We now leverage this to prove the following connectivity estimate, which will form a key part of our calculations.

Lemma 3.5.2. *Let X be a real p-cyclotomic spectrum.*

(i) If the underlying spectrum of X is k-connected, then the underlying spectrum of $\mathrm{TCR}^n(X;p)$ is $(k-1)$-connected and the underlying spectrum of $\mathrm{TCR}(X;p)$ is $(k-2)$-connected.

(ii) If the underlying Σ_2-spectrum of X is k-connected, then $\mathrm{TCR}^n(X;p)$ is $(k-1)$-connected and $\mathrm{TCR}(X;p)$ is $(k-2)$-connected.

Proof. The claim for $\mathrm{TCR}(X;p)$ follows from the claim for $\mathrm{TCR}^n(X;p)$ by application of the Milnor exact sequence. By Remark 3.4.4 we can assume that X is a genuine real p-cyclotomic spectrum and by the discussion above we can show the claim for $\mathrm{TCR}^{n,\mathrm{gen}}(X;p)$ instead. Since the fiber of a map between k-connected spectra is $(k-1)$-connected, it suffices to show that $X^{C_{p^n}}$ is k-connected for all n. This follows by induction on n and the fiber sequence

$$X_{h_{\Sigma_2}C_{p^n}} \to X^{C_{p^n}} \to (\Phi^{C_p}X)^{C_{p^{n-1}}} \simeq X^{C_{p^{n-1}}},$$

where the existence of this fiber sequence follows from the fact that $\mathcal{F}_{D_{2p^n}}[C_{p^n}] = \mathcal{F}_{\not\supseteq C_p}$. □

Remark 3.5.3. For odd p one can prove this without resorting to genuine real p-cyclotomic spectra. In fact, by Lemma 3.5.5 it suffices to show that the underlying spectrum of $\mathrm{TCR}^n(X;p)$ (i.e. $\mathrm{TC}^n(X;p)$) is $(k-1)$-connected. For this it is relatively straightforward to adapt the argument given in [CMM21, Remark 2.14]. However, in the case $p=2$ one has to additionally show that $\Phi^{\Sigma_2}\mathrm{TCR}^n(X;2)$ is $(k-1)$-connected, but here the problem is that $\Phi^{\Sigma_2}X^{h_{\Sigma_2}C_{p^n}}$ and $\Phi^{\Sigma_2}(X^{t_{\Sigma_2}C_p})^{h_{\Sigma_2}C_{p^{n-1}}}$ do not have to be bounded below.

We will also repeatedly use the following easy corollary in our calculations.

Corollary 3.5.4. *If X is a real p-cyclotomic spectrum whose underlying spectrum is bounded below, then*

$$\Phi^{\Sigma_2}\mathrm{TCR}(X;p) \simeq \lim_n \Phi^{\Sigma_2}\mathrm{TCR}^n(X;p).$$

Proof. This follows immediately from the previous lemma and Proposition 3.5.1. □

3.5.2 Calculation for odd primes

We now have all the necessary ingredients to calculate $\Phi^{\Sigma_2}\mathrm{TCR}(X;p)$ for odd primes p. This recovers the result [DMP21, Corollary 2.5], which was proven in loc. cit. by using genuine cyclotomic methods, whereas we only need the methods of [QS21a] (see also Remark 3.5.3). We start by proving the vanishing result we mentioned at the start of this section.

Lemma 3.5.5. *Let p be an odd prime and X a D_{2p^n}-spectrum. Then $\Phi^{\Sigma_2}(X^{t_{\Sigma_2}C_p})^{h_{\Sigma_2}C_{p^{n-1}}} \simeq 0$.*

Proof. We prove the claim by induction on n. The case $n = 1$ follows from Lemma 3.3.3. Now assume that $\Phi^{\Sigma_2}(X^{t_{\Sigma_2}C_p})^{h_{\Sigma_2}C_{p^{n-1}}} \simeq 0$. Then we have the fiber sequence

$$((X^{t_{\Sigma_2}C_p})^{h_{\Sigma_2}C_{p^{n-1}}})_{h_{\Sigma_2}C_p} \to (X^{t_{\Sigma_2}C_p})^{h_{\Sigma_2}C_{p^n}} \to ((X^{t_{\Sigma_2}C_p})^{h_{\Sigma_2}C_{p^{n-1}}})^{t_{\Sigma_2}C_p},$$

where we want to show that the geometric fixed points of the middle term vanish. Again by Lemma 3.3.3 we see that $\Phi^{\Sigma_2}((X^{t_{\Sigma_2}C_p})^{h_{\Sigma_2}C_{p^{n-1}}})^{t_{\Sigma_2}C_p} \simeq 0$ and for the left hand term we apply Lemma 3.3.1 and the induction hypothesis to obtain

$$\Phi^{\Sigma_2}((X^{t_{\Sigma_2}C_p})^{h_{\Sigma_2}C_{p^{n-1}}})_{h_{\Sigma_2}C_p} \simeq \Phi^{\Sigma_2}(X^{t_{\Sigma_2}C_p})^{h_{\Sigma_2}C_{p^{n-1}}} \simeq 0.$$

□

It is now easy to compute $\Phi^{\Sigma_2}\mathrm{TCR}(X;p)$.

Theorem 3.5.6. *Let p be an odd prime and X a real p-cyclotomic spectrum whose underlying spectrum is bounded below. Then $\Phi^{\Sigma_2}\mathrm{TCR}(X;p) \simeq \Phi^{\Sigma_2}X$.*

Proof. By Corollary 3.5.4 we have $\Phi^{\Sigma_2}\mathrm{TCR}(X;p) \simeq \lim_n \Phi^{\Sigma_2}\mathrm{TCR}^n(X;p)$. By Lemma 3.5.5, Lemma 3.3.1 and [QS21a, Lemma 3.25] we have

$$\Phi^{\Sigma_2}\mathrm{TCR}^n(X;p) \simeq \Phi^{\Sigma_2}X_{h_{\Sigma_2}C_{p^n}} \simeq \Phi^{\Sigma_2}X,$$

and the maps in the limit system are constant, yielding the claim. □

Remark 3.5.7. In the special case $X = \mathrm{THR}(A)$ we have the additional identification

$$\Phi^{\Sigma_2}\mathrm{THR}(A) \simeq \Phi^{\Sigma_2}A \otimes_A \Phi^{\Sigma_2}A$$

by [DMPR21, Theorem 2.6]. We recall the left and right A-module structure on $\Phi^{\Sigma_2}A$ in Remark 3.5.11. Note that for any discrete group G we have $\Phi^{\Sigma_2}A[G] \simeq \bigoplus_{g^2=1}\Phi^{\Sigma_2}A$.

On the other hand Proposition 2.3.2 and Proposition 2.3.3 imply that

$$\begin{aligned}\Phi^{\Sigma_2}\mathrm{THR}(A[G]) \simeq & \bigoplus_{\substack{[[x]]\in \mathrm{conj}_{\mathbb{R}}(G),\\ x^2=1}} \bigoplus_{\substack{[y]\in\mathrm{conj}(Z_G\langle x\rangle),\\ y^2=1}} \Phi^{\Sigma_2}\mathrm{THR}(A)\otimes BZ_{Z_G\langle x\rangle}\langle y\rangle_+ \oplus \\ & \bigoplus_{\substack{[[x]]\in \mathrm{conj}_{\mathbb{R}}(G),\\ x^2\neq 1}} \bigoplus_{\substack{[(y,\sigma)]\in\mathrm{conj}(SZ_G\langle x\rangle),\\ y^2=1}} \Phi^{\Sigma_2}\mathrm{THR}(A)\otimes(\Sigma_2\times_{Z_{SZ_G\langle x\rangle}\langle (y,\sigma)\rangle} ESZ_G\langle x\rangle)_+,\end{aligned}$$

so that we can identify $\Phi^{\Sigma_2}\mathrm{TCR}(A[G];p)$ in two different ways.

3.5.3 The case $p = 2$

Now that we have dealt with the case of odd p we will treat the case $p = 2$. In this case the formula for $\Phi^{\Sigma_2}\mathrm{TCR}(X;2)$ becomes more interesting and depends also on the residual C_2-action on $\Phi^{\Sigma_2}X$ as well as the Frobenius. We have remarked before that in general we cannot directly compute $\Phi^{\Sigma_2}X^{h_{\Sigma_2}C_{2^n}}$ and $\Phi^{\Sigma_2}(X^{t_{\Sigma_2}C_2})^{h_{\Sigma_2}C_{2^{n-1}}}$, consequently we cannot directly compute $\Phi^{\Sigma_2}\mathrm{TCR}^n(X;2)$. Instead, we will use an isotropy separation argument to reduce the computation to the case where the underlying spectrum of X is contractible. For this case we have already proven all the necessary ingredients in Section 3.2. For the reduction itself we need the connectivity results from the previous section. We stress that these connectivity results are the only part of our computation for which we use genuine real cyclotomic methods.

We will now state the main result of this thesis. Recall that for a D_4-spectrum X there is a projection map

$$\Phi^{\Sigma_2}X^{t_{\Sigma_2}C_2} \to (\Phi^{\Sigma_2}X)^{tC_2}. \tag{3.22}$$

Its definition was explained in (3.15). If X is a real 2-cyclotomic spectrum with Frobenius φ_X, denote by $\phi_X\colon (\Phi^{\Sigma_2}X)^{hC_2} \to (\Phi^{\Sigma_2}X)^{tC_2}$ the composite

$$(\Phi^{\Sigma_2}X)^{hC_2} \to \Phi^{\Sigma_2}X \xrightarrow{\Phi^{\Sigma_2}\varphi_X} \Phi^{\Sigma_2}X^{t_{\Sigma_2}C_2} \to (\Phi^{\Sigma_2}X)^{tC_2},$$

where the first arrow is the inclusion of fixed points and the final map is (3.22).

Theorem 3.5.8. *Let X be a real 2-cyclotomic spectrum whose underlying spectrum is bounded below. Then there is an equivalence*

$$\Phi^{\Sigma_2}\mathrm{TCR}(X;2) \simeq \mathrm{fib}\left((\Phi^{\Sigma_2}X)^{hC_2} \xrightarrow{\phi_X - \mathrm{can}} (\Phi^{\Sigma_2}X)^{tC_2}\right),$$

where $\mathrm{can}\colon (\Phi^{\Sigma_2}X)^{hC_2} \to (\Phi^{\Sigma_2}X)^{tC_2}$ *is the canonical map from the homotopy fixed points to the Tate construction.*

As stated above, we start with the following special case and will reduce the statement to this case.

Lemma 3.5.9. *Theorem 3.5.8 is true if the underlying spectrum of X is trivial.*

Proof. Consider the commutative square

$$\begin{array}{ccc}
\Phi^{\Sigma_2}X^{h_{\Sigma_2}C_{2^\infty}} & \xrightarrow{\varphi_X^{h_{\Sigma_2}C_{2^\infty}} - \mathrm{can}} & \Phi^{\Sigma_2}(X^{t_{\Sigma_2}C_2})^{h_{\Sigma_2}C_{2^\infty}} \\
\downarrow & & \downarrow \\
(\Phi^{\Sigma_2}X)^{hC_2} & \xrightarrow{\phi_X - \mathrm{can}} & (\Phi^{\Sigma_2}X)^{tC_2}.
\end{array}$$

By Lemma 3.3.5 the vertical arrows are equivalences, therefore the square is a pullback and the claim follows by observing that the fiber of the upper horizontal arrow is $\Phi^{\Sigma_2}\mathrm{TCR}(X;2)$. $\square$

Proof of Theorem 3.5.8. We consider the cofiber sequence

$$X \otimes (E\Sigma_{2+})^{\mathrm{triv}} \to X \to X \otimes (\widetilde{E\Sigma_2})^{\mathrm{triv}}$$

of real 2-cyclotomic spectra. Recall that the underlying D_{2^∞}-spectrum of $(\widetilde{E\Sigma_2})^{\mathrm{triv}}$ is $\alpha^*\widetilde{E\Sigma_2} \simeq \widetilde{E\mathcal{F}}_{\subset C_{2^\infty}}$, where $\alpha\colon D_{2^\infty} \to D_{2^\infty}/C_{2^\infty} \cong \Sigma_2$ is the projection. By the previous lemma the statement is true for the right hand term and since we have a C_2-equivariant equivalence $\Phi^{\Sigma_2}X \simeq \Phi^{\Sigma_2}(X \otimes \widetilde{E\mathcal{F}}_{\subset C_{2^\infty}})$, we need to show that $\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes (E\Sigma_{2+})^{\mathrm{triv}}; 2)$ is trivial.

The argument is similar to the proof of Proposition 3.5.1 and we use the same Σ_2-CW structure on $E\Sigma_2$ we used there. Denote by $\mathrm{sk}_k E\Sigma_2$ the k-skeleton of $E\Sigma_2$. The Σ_2-action on $E\Sigma_2/\mathrm{sk}_k E\Sigma_2$ is free away from the basepoint, therefore the connectivity of the underlying Σ_2-spectra of $X \otimes (E\Sigma_2/\mathrm{sk}_k E\Sigma_2)^{\mathrm{triv}}$ tends to infinity with k, thus we obtain

$$\mathrm{colim}_k \mathrm{TCR}(X \otimes (\mathrm{sk}_k E\Sigma_{2+})^{\mathrm{triv}}; 2) \simeq \mathrm{TCR}(X \otimes E\Sigma_{2+}^{\mathrm{triv}}; 2)$$

from Lemma 3.5.2. Since geometric fixed points commute with colimits and $\mathrm{sk}_k E\Sigma_{2+}$ is a finite Σ_2-CW complex we can reduce the claim to showing that $\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes (\Sigma_2/1_+)^{\mathrm{triv}}; 2)$ is trivial and by Corollary 3.5.4 it is enough to show that $\Phi^{\Sigma_2}\mathrm{TCR}^n(X \otimes (\Sigma_2/1_+)^{\mathrm{triv}}; 2)$ is trivial for all n. We observe that the underlying $D_{2^{n+1}}$-spectrum of $X \otimes (\Sigma_2/1_+)^{\mathrm{triv}}$ is $X \otimes D_{2^{n+1}}/C_{2^n +}$, which is equivalent to $\mathrm{ind}_{C_{2^n}}^{D_{2^{n+1}}} \mathrm{res}_{C_{2^n}}^{D_{2^{n+1}}} X$ by the projection formula (A.21). As a consequence of Proposition 3.2.5 and [QS21a, Lemma 3.25] both

$$\Phi^{\Sigma_2}(X \otimes D_{2^{n+1}}/C_{2^n +})^{h_{\Sigma_2}C_{2^n}} \quad \text{and} \quad \Phi^{\Sigma_2}((X \otimes D_{2^{n+1}}/C_{2^n +})^{t_{\Sigma_2}C_2})^{h_{\Sigma_2}C_{2^{n-1}}}$$

vanish, therefore $\Phi^{\Sigma_2}\mathrm{TCR}^n(X \otimes (\Sigma_2/1_+)^{\mathrm{triv}}; 2)$ is the fiber of a map between two trivial spectra, hence itself trivial. □

We immediately obtain the following corollary from Theorem 3.5.8 and [DMPR21, Corollary 2.28].

Corollary 3.5.10. *Let A be a discrete ring with anti-involution such that $\frac{1}{2} \in A$. Then $\Phi^{\Sigma_2}\mathrm{TCR}(A; 2) \simeq 0$.*

Remark 3.5.11. Suppose X is a genuine real 2-cyclotomic spectrum. We want to describe ϕ_X in this case and give a more detailed description in the case $X = \mathrm{THR}(A)$.

(a) Recall that (on the underlying D_4-spectra) the Frobenius of X is given by the composite

$$X \simeq \Phi^{C_2}X \to \Phi^{C_2}F(E\mathcal{F}_{D_4}[C_2]_+, X) = X^{t_{\Sigma_2}C_2},$$

where the equivalence comes from the genuine real cyclotomic structure and the arrow is the $\mathcal{F}_{D_4}[C_2]$-completion map. Let $f'\colon (\Phi^{\Sigma_2}X)^{hC_2} \to \Phi^{\Sigma_2}X$ be the inclusion of fixed points and $\phi'_X\colon \Phi^{\Sigma_2}X \to (\Phi^{\Sigma_2}X)^{hC_2}$ the composite

$$\Phi^{\Sigma_2}X \simeq \Phi^{\Sigma_2}\Phi^{C_2}X \simeq \Phi^{C_2}\Phi^{\Sigma_2}X \to (\Phi^{\Sigma_2}X)^{tC_2},$$

with the first equivalence being induced by the genuine real 2-cyclotomic structure of X and the arrow the Borel completion map. We consider the following diagram

$$\begin{array}{ccc}
\Phi^{\Sigma_2}X & & \\
\downarrow{\scriptstyle\simeq} & & \\
\Phi^{\Sigma_2}\Phi^{C_2}X & \longrightarrow & \Phi^{\Sigma_2}\Phi^{C_2}F(E\mathcal{F}_{D_4}[C_2]_+, X) \\
\downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} \\
\Phi^{C_2}\Phi^{\Sigma_2}X & \longrightarrow & \Phi^{C_2}\Phi^{\Sigma_2}F(E\mathcal{F}_{D_4}[C_2]_+, X) \\
\downarrow & & \downarrow \\
(\Phi^{\Sigma_2}X)^{tC_2} & \xrightarrow{\simeq} & (\Phi^{\Sigma_2}F(E\mathcal{F}_{D_4}[C_2]_+, X))^{tC_2},
\end{array}$$

where the lower vertical arrows are the Borel completion maps and the horizontal maps are the $\mathcal{F}_{D_4}[C_2]$-completion maps. By definition the left column is ϕ'_X and $\phi_X\colon (\Phi^{\Sigma_2}X)^{hC_2} \to (\Phi^{\Sigma_2}X)^{tC_2}$ is given by going from $\Phi^{\Sigma_2}X$ to the lower left corner in the diagram through the right column and precomposing with f', hence ϕ_X is homotopic to $\phi'_X \circ f'$.

(b) In the case $X = \mathrm{THR}(A)$ we shall give an additional description in terms of the Hill-Hopkins-Ravenel norm [HHR16, Section A.4]. In the case of finite groups G and H and an orthogonal G-spectrum Y it is easily checked that the Hill-Hopkins-Ravenel norm $N_G^{G\times H}Y$ is obtained by equipping $Y^{\otimes H}$ with the $(G \times H)$-action given by the diagonal G-action and H-action by permuting the smash factors. Here we view $G \cong G \times 1$ as a subgroup of $G \times H$. By [BDS16, Theorem 5.3.5] there is a natural diagonal equivalence of G-spectra

$$\Delta_G^{G\times H}\colon Y \xrightarrow{\simeq} \Phi^H N_G^{G\times H}Y.$$

The map $\Delta_G^{G\times H}$ has the property that the composite

$$\Phi^G X \xrightarrow{\Phi^G\Delta_G^{G\times H}} \Phi^G\Phi^H N_G^{G\times H}X \simeq \Phi^{G\times H}N_G^{G\times H}X$$

is homotopic to the diagonal equivalence of [HHR16, Proposition B.209]. We shall apply this to the case at hand. Recall that $\mathrm{THR}(A)$ is the geometric realization of the dihedral nerve. As a Σ_2-spectrum A is both a left and right module over the Hill-Hopkins-Ravenel norm of (the underlying spectrum of) A, with the module structures given by

$$N_1^{\Sigma_2}A \otimes A \to A, a \otimes b \otimes x \mapsto ax\omega(b),$$
$$A \otimes N_1^{\Sigma_2}A \to A, a \otimes b \otimes x \mapsto \omega(a)xb.$$

Since geometric fixed points are monoidal, these descend to left and right A-module structures on $\Phi^{\Sigma_2}A$. There is a Σ_2-equivariant simplicial isomorphism

$$\mathrm{sd}_S DN_\bullet^{\otimes}(A) \xrightarrow{\cong} B_\bullet(A; N_1^{\Sigma_2}A; A),$$
$$a_0 \otimes \cdots \otimes a_{2n+1} \mapsto a_0 \otimes a_1 \otimes \omega(a_{2n+1}) \otimes \cdots a_n \otimes \omega(a_{n+2}) \otimes a_{n+1},$$

between the Segal subdivision of the dihedral nerve and the two-sided bar construction and similarly a D_4-equivariant simplicial isomorphism

$$\begin{aligned}
&\mathrm{sd}_S \mathrm{sd}_2 D N^{\otimes}_{\bullet}(A) \xrightarrow{\cong} \mathrm{sd}_S B_{\bullet}(A; N_1^{\Sigma_2} A; A),\\
&a_0 \otimes \cdots a_{4n+3} \mapsto a_0 \otimes a_1 \otimes \omega(a_{4n+3}) \otimes \cdots a_{2n+1} \otimes \omega(a_{2n+3}) \otimes a_{2(n+1)},
\end{aligned}$$

where we equip the target with the C_2-action given by

$$x \otimes (a_1 \otimes a_2) \cdots \otimes (a_{4n+1} \otimes a_{4n+2}) \otimes y \mapsto y \otimes (\omega(a_{4n+2}) \otimes \omega(a_{4n+1})) \otimes \cdots \otimes (\omega(a_2) \otimes \omega(a_1)) \otimes x.$$

Finally, there is a D_4-equivariant simplicial isomorphism

$$\mathrm{sd}_S B_{\bullet}(A; N_1^{\Sigma_2} A; A) \xrightarrow{\cong} B_{\bullet}(N_{\Sigma_2}^{D_4} A; N_{\Sigma_2}^{D_4} N_1^{\Sigma_2} A; \alpha^* N_{\Sigma_2}^{D_4} A) \tag{3.23}$$

given by sending

$$x \otimes a_1 \otimes b_1 \otimes \cdots \otimes a_{2n+1} \otimes b_{2n+1} \otimes y$$

to

$$x \otimes y \otimes a_1 \otimes b_1 \otimes \omega(a_{2n+1}) \otimes \omega(b_{2n+1}) \otimes \cdots \otimes a_n \otimes b_n \otimes \omega(a_{n+2}) \otimes \omega(b_{n+2}) \otimes a_{n+1} \otimes \omega(b_{n+1}),$$

where $\alpha\colon D_4 \xrightarrow{\cong} D_4$ is defined by $\alpha(c) = c, \alpha(\sigma) = \sigma c$. Under these isomorphisms $\varphi_{\mathrm{THR}(A)}$ is given by the composite

$$\begin{aligned}
B_{\bullet}(A; N_1^{\Sigma_2} A; A) \xrightarrow{B_{\bullet}(\Delta_{\Sigma_2}^{D_4};\Delta_{\Sigma_2}^{D_4};\Delta_{\Sigma_2}^{D_4})} &B_{\bullet}(\Phi^{C_2} N_{\Sigma_2}^{D_4} A; \Phi^{C_2} N_{\Sigma_2}^{D_4} N_1^{\Sigma_2} A; \Phi^{C_2} N_{\Sigma_2}^{D_4} A)\\
&\simeq B_{\bullet}(\Phi^{C_2} N_{\Sigma_2}^{D_4} A; \Phi^{C_2} N_{\Sigma_2}^{D_4} N_1^{\Sigma_2} A; \Phi^{C_2} \alpha^* N_{\Sigma_2}^{D_4} A)\\
&\simeq \Phi^{C_2} B_{\bullet}(N_{\Sigma_2}^{D_4} A; N_{\Sigma_2}^{D_4} N_1^{\Sigma_2} A; \alpha^* N_{\Sigma_2}^{D_4} A)\\
&\cong \Phi^{C_2} \mathrm{sd}_S B_{\bullet}(A; N_1^{\Sigma_2} A; A),
\end{aligned}$$

where the first equivalence is the conjugation equivalence $\Phi^{C_2} \simeq \Phi^{C_2}\alpha^*$ and the second equivalence is the monoidal structure of geometric fixed points. After applying Φ^{Σ_2} and postcomposing with the natural transformation

$$\Phi^{\Sigma_2}\Phi^{C_2}(-) \simeq \Phi^{C_2}\Phi^{\Sigma_2}(-) \to (\Phi^{\Sigma_2}(-))^{tC_2}$$

we see that $\phi_{\mathrm{THR}(A)}$ is given by applying geometric realization to the composite

$$\begin{aligned}
B_{\bullet}(\Phi^{\Sigma_2} A; A; \Phi^{\Sigma_2} A) \xrightarrow{B(\Delta_2;\Delta_2;\mathrm{can})} &B_{\bullet}((\Phi^{\Sigma_2} A \otimes \Phi^{\Sigma_2} A)^{tC_2}; (A \otimes A)^{tC_2}; A^{t\Sigma_2})\\
&\simeq B_{\bullet}((\Phi^{\Sigma_2} A \otimes \Phi^{\Sigma_2} A)^{tC_2}; (A \otimes A)^{tC_2}; (\beta^* A)^{tC_2})\\
&\to B_{\bullet}(\Phi^{\Sigma_2} A \otimes \Phi^{\Sigma_2} A; A \otimes A; \beta^* A)^{tC_2}\\
&\cong \mathrm{sd}_S B_{\bullet}(\Phi^{\Sigma_2} A; A; \Phi^{\Sigma_2} A)^{tC_2}
\end{aligned}$$

and commuting geometric realization with the Tate construction. Here the undecorated arrow is the monoidal structure of the Tate construction, the isomorphism is Φ^{C_2} applied to (3.23) and $\beta\colon C_2 \cong \Sigma_2$. We used the following facts:

(1) The natural transformation $\Phi^{C_2} \to (-)^{tC_2}$ is lax monoidal.

(2) The composite $X \xrightarrow{\Delta_1^{C_2}} \Phi^{C_2} N_1^{C_2} X \to (X \otimes X)^{tC_2}$ is the Tate diagonal by [NS18, Remark III.1.5].

(3) For any G-spectrum X we have that

$$N_1^H \Phi^G X \simeq \Phi^G N_G^{G\times H} X$$

and the following diagram commutes:

$$\begin{array}{ccc} \Phi^G X & \xrightarrow{\Phi^G \Delta_G^{G\times H}} & \Phi^G \Phi^H N_G^{G\times H} X \\ \downarrow{\scriptstyle \Delta_1^H} & & \downarrow{\scriptstyle \simeq} \\ \Phi^H N_1^H \Phi^G X & \xrightarrow{\simeq} & \Phi^H \Phi^G N_G^{G\times H} X. \end{array}$$

(4) The composite

$$\Phi^{\Sigma_2} A \xrightarrow{\Phi^{\Sigma_2}\Delta_{\Sigma_2}^{D_4}} \Phi^{\Sigma_2}\Phi^{C_2} N_{\Sigma_2}^{D_4} A \simeq \Phi^{\Sigma_2}\Phi^{C_2}\alpha^* N_{\Sigma_2}^{D_4} A \simeq \Phi^{C_2}\Phi^{\Sigma_2}\alpha^* N_{\Sigma_2}^{D_4} A$$
$$\simeq \Phi^{C_2}\beta^*\Phi^{\langle\sigma c\rangle} N_{\Sigma_2}^{D_4} A \xleftarrow{\Phi^{C_2}\beta^*\Delta_{\Sigma_2}^{\Sigma_2\times\langle\sigma c\rangle}} \Phi^{C_2}\beta^* A$$

is homotopic to the conjugation equivalence $\Phi^{\Sigma_2} A \simeq \Phi^{C_2}\beta^* A$. Here we used that $D_4 = \Sigma_2 \times \langle\sigma c\rangle$ and the fact that the diagonal map commutes with the conjugation equivalence in a suitable way (see [Wim19, Proposition 2.23] for a precise statement).

Finally, we point out that under suitable point set conditions the realization of the two sided bar construction $B_\bullet(\Phi^{\Sigma_2} A; A; \Phi^{\Sigma_2} A)$ models the (derived) tensor product $\Phi^{\Sigma_2} A \otimes_A \Phi^{\Sigma_2} A$ (see for example [DMPR21, Lemma 2.13]).

We now deduce [DMP21, Theorem 2.13] from Theorem 3.5.8, which in loc. cit. was proven using genuine real cyclotomic methods [4]. The proof will show that both statements are essentially equivalent and is similar to that of [NS18, Theorem II.4.10]. In the statement and the proof we will use the fact that for $n \geq 1$ and any $D_{2^{n+1}}$-spectrum X there are natural equivalences $\Phi^{C_2}\Phi^{\Sigma_2} X \simeq \Phi^{D_4} X \simeq \Phi^{\Sigma_2}\Phi^{C_2} X$.

Theorem 3.5.12. *Let X be a genuine real 2-cyclotomic spectrum whose underlying spectrum is bounded below. Denote by $f\colon (\Phi^{\Sigma_2} X)^{C_2} \to \Phi^{\Sigma_2} X$ the inclusion of fixed points and define $r\colon (\Phi^{\Sigma_2} X)^{C_2} \to \Phi^{\Sigma_2} X$ to be the composite*

$$(\Phi^{\Sigma_2} X)^{C_2} \to \Phi^{C_2}\Phi^{\Sigma_2} X \simeq \Phi^{\Sigma_2}\Phi^{C_2} X \simeq \Phi^{\Sigma_2} X,$$

[4] In fact our statement is slightly more general. In [DMP21, Theorem 2.13] it is assumed that the underlying Σ_2-spectrum of X is bounded below, whereas we only assume boundedness for the underlying non-equivariant spectrum. This can however also be shown using the methods in loc. cit. if one uses Proposition 3.5.1 in the proof given there instead of [DMP21, Lemma 2.4].

where the arrow is the canonical map from fixed points to geometric fixed points and the third equivalence is induced by the genuine real 2-cyclotomic structure of X. Then there is an equivalence

$$\Phi^{\Sigma_2}\mathrm{TCR}(X;2) \simeq \mathrm{fib}\left((\Phi^{\Sigma_2}X)^{C_2} \xrightarrow{r-f} \Phi^{\Sigma_2}X\right).$$

Proof. Recall that the Frobenius $\varphi_X \colon X \to X^{t_{\Sigma_2}C_2}$ is given by the composite

$$X \simeq \Phi^{C_2}X \to X^{t_{\Sigma_2}C_2},$$

where the equivalence comes from the genuine real cyclotomic structure and the arrow is the $\mathcal{F}_{D_{2^\infty}}[C_{2^\infty}]$-completion map. Let $f' \colon (\Phi^{\Sigma_2}X)^{hC_2} \to \Phi^{\Sigma_2}X$ be the inclusion of fixed points and $\phi'_X \colon \Phi^{\Sigma_2}X \to (\Phi^{\Sigma_2}X)^{hC_2}$ the composite

$$\Phi^{\Sigma_2}X \simeq \Phi^{\Sigma_2}\Phi^{C_2}X \simeq \Phi^{C_2}\Phi^{\Sigma_2}X \to (\Phi^{\Sigma_2}X)^{tC_2},$$

with the first equivalence being induced by the genuine real 2-cyclotomic structure of X and the arrow being the Borel completion map. By Remark 3.5.11 the map $\phi_X \colon (\Phi^{\Sigma_2}X)^{hC_2} \to (\Phi^{\Sigma_2}X)^{tC_2}$ of Theorem 3.5.8 is given by the composition of f' and ϕ'_X.

The isotropy separation sequence and the equivalence $\Phi^{\Sigma_2}X \simeq \Phi^{C_2}\Phi^{\Sigma_2}X$ imply that the following diagram is a pullback:

$$\begin{array}{ccc} (\Phi^{\Sigma_2}X)^{C_2} & \xrightarrow{r} & \Phi^{\Sigma_2}X \\ \downarrow & & \downarrow{\scriptstyle \phi'_X} \\ (\Phi^{\Sigma_2}X)^{hC_2} & \xrightarrow{\mathrm{can}} & (\Phi^{\Sigma_2}X)^{tC_2}. \end{array}$$

To compute the fiber of $r - f \colon (\Phi^{\Sigma_2}X)^{C_2} \to \Phi^{\Sigma_2}X$ we consider the following square

$$\begin{array}{ccc} \Phi^{\Sigma_2}X \oplus (\Phi^{\Sigma_2}X)^{hC_2} & \xrightarrow{r'-f'} & \Phi^{\Sigma_2}X \\ \downarrow{\scriptstyle \phi'_X-\mathrm{can}} & & \downarrow \\ (\Phi^{\Sigma_2}X)^{tC_2} & \longrightarrow & 0, \end{array}$$

where r' is the projection onto the first factor. The vertical fibers are $(\Phi^{\Sigma_2}X)^{C_2}$ and $\Phi^{\Sigma_2}X$ with the induced map being $r - f$. The upper horizontal fiber is $(\Phi^{\Sigma_2}X)^{hC_2}$ via the map $(f', \mathrm{id}) \colon (\Phi^{\Sigma_2}X)^{hC_2} \to \Phi^{\Sigma_2}X \oplus (\Phi^{\Sigma_2}X)^{hC_2}$ and the lower horizontal fiber is $(\Phi^{\Sigma_2}X)^{tC_2}$. The induced map between the horizontal fibers is then $\phi_X - \mathrm{can}$ so the claim follows from Theorem 3.5.8 after taking the total fiber of the square. ☐

The next corollary follows immediately from Theorem 3.5.12, [QS21a, Proposition], and the fact that fixed points and geometric fixed points preserve colimits (Lemma A.1.4 and Lemma A.2.7)[5].

[5] Here we also use that by definition the restriction functor $\mathbf{Sp}^{D_{2^\infty}} \to \mathbf{Sp}^{D_4}$ preserves colimits, see Section A.1 for a discussion of this fact.

Corollary 3.5.13. *Let k be an integer and denote by $\mathbb{R}\mathbf{CycSp}_{2,\geq k} \subset \mathbb{R}\mathbf{CycSp}_2$ the full subcategory of real 2-cyclotomic spectra whose underlying spectra are k-connective. Then the functor*

$$\Phi^{\Sigma_2}\mathrm{TCR}(-;2)\colon \mathbb{R}\mathrm{CycSp}_{2,\geq k} \to \mathbf{Sp}$$

preserves all colimits.

We use this corollary to prove the following statement. We will use it in the next chapter to identify parts of $\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes DN(G)_+;2)$.

Proposition 3.5.14. *Let A be a Σ_2-space and let X be a real 2-cyclotomic spectrum whose underlying spectrum is bounded below. Then the assembly map*[6]

$$\Phi^{\Sigma_2}\mathrm{TCR}(X;2) \otimes A_+^{\Sigma_2} \to \Phi^{\Sigma_2}\mathrm{TCR}(X \otimes (A_+)^{\mathrm{triv}};2) \tag{3.24}$$

is an equivalence.

Proof. By Corollary 3.5.13 the functor

$$\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes (-)^{\mathrm{triv}};2)\colon \mathbf{Spc}_*^{\Sigma_2} \to \mathbf{Sp}$$

commutes with colimits, therefore it suffices to prove the claim for $A = \Sigma_2/1$ and $A = *$[7]. For the former the source of (3.24) is contractible and we have seen in the proof of Theorem 3.5.8 that the target of (3.24) is contractible as well. For the latter the statement is obvious. □

In chapter 4 we will define for various n a filtration on $X \otimes DN(D_{2n})_+$ such that the Frobenius of $F_k(X \otimes DN(D_{2n})_+)$ factors through $F_{k-1}(X \otimes DN(D_{2n})_+)$. The next proposition describes the geometric fixed points of real topological cyclic homology at the prime 2 in this situation.

Proposition 3.5.15. *Let X be a real 2-cyclotomic spectrum whose underlying spectrum is bounded below, and suppose there is a splitting of the underlying D_{2^∞}-spectrum*

$$X \simeq_{\mathcal{F}_{D_{2^\infty}}[C_{2^\infty}]} Y \oplus Z^{8}$$

such that the Frobenius $X \to X^{t_{\Sigma_2}C_2}$ factors through $Y^{t_{\Sigma_2}C_2}$. Then Y inherits the structure of a real 2-cyclotomic spectrum and there is a cofiber sequence

$$\Phi^{\Sigma_2}\mathrm{TCR}(Y;2) \to \Phi^{\Sigma_2}\mathrm{TCR}(X;2) \to (\Phi^{\Sigma_2}Z)_{hC_2}. \tag{3.25}$$

[6] Let $*$ be a final object of $\mathbf{Spc}^{\Sigma_2}$. If $T\colon \mathbf{Spc}^{\Sigma_2} \to \mathbf{Sp}$ is a functor, then $\mathrm{colim}_{*\to A}T(*) \simeq T(*) \otimes A_+^{\Sigma_2}$, where the colimit is taken over all maps $* \to A$ in $\mathbf{Spc}^{\Sigma_2}$, and the universal property of the colimit yields the assembly map $\mathrm{colim}_{*\to A}T(*) \to T(A)$.

[7] This is a version of the localizing subcategory argument (Remark A.1.2) in the unstable case: The smallest subcategory of $\mathbf{Spc}^{\Sigma_2}$ that is closed under colimits and contains $\Sigma_2/1$ and $*$ is $\mathbf{Spc}^{\Sigma_2}$ itself.

[8] See Definition A.2.4 for this notation. Note that the definition given there also makes sense for D_{2^∞}.

Proof. It is clear that Y inherits a real 2-cyclotomic structure such that the inclusion $Y \to X$ is a map of real 2-cyclotomic spectra. The assumption implies that the following diagram, where the vertical sequences are the standard split cofiber sequences, is a diagram of cofiber sequences

$$\begin{array}{ccc} (\Phi^{\Sigma_2} Y)^{hC_2} & \xrightarrow{\text{can}-\phi_Y} & (\Phi^{\Sigma_2} Y)^{tC_2} \\ \downarrow & & \downarrow \\ (\Phi^{\Sigma_2} Y)^{hC_2} \oplus (\Phi^{\Sigma_2} Z)^{hC_2} & \xrightarrow{\text{can}-\phi_X} & (\Phi^{\Sigma_2} Y)^{tC_2} \oplus (\Phi^{\Sigma_2} Z)^{tC_2} \\ \downarrow & & \downarrow \\ (\Phi^{\Sigma_2} Z)^{hC_2} & \xrightarrow{\text{can}} & (\Phi^{\Sigma_2} Z)^{tC_2}. \end{array}$$

By Theorem 3.5.8 the fibers of the upper and middle horizontal arrow are $\Phi^{\Sigma_2}\mathrm{TCR}(Y;2)$ and $\Phi^{\Sigma_2}\mathrm{TCR}(X;2)$ respectively, and the fiber of the lower horizontal arrow is $(\Phi^{\Sigma_2} Z)_{hC_2}$. □

Chapter 4

Real Topological Cyclic Homology of group rings for cyclic and dihedral groups

In this chapter we use Theorem 3.5.8 and our results on the dihedral nerve from Section 2.3 to calculate $\Phi^{\Sigma_2}\mathrm{TCR}(A[G];2)$ for cyclic and dihedral groups. Here and throughout this chapter A is a bounded below ring spectrum with anti-involution and $A[G]$ the group ring spectrum, where the involution on G is given by sending an element to its inverse. Recall that there is an equivalence of real 2-cyclotomic spectra $\mathrm{THR}(A[G]) \simeq \mathrm{THR}(A) \otimes DN(G)_+$ and that if A is bounded below then the underlying spectrum of $\mathrm{THR}(A)$ is as well. We will actually consider $X \otimes DN(G)_+$ for an arbitrary real 2-cyclotomic spectrum X whose underlying spectrum is bounded below instead of $\mathrm{THR}(A)$.

Instead of G being cyclic we can in fact be somewhat more general and give a calculation for the situation where the 2-torsion of G is contained in the center, which also includes the case that G is abelian. This result is given in Theorem 4.2. For dihedral groups we give results in the following cases: D_{2n} with $n \equiv 2 \mod 4$ (Theorem 4.3), D_{2n} with $n \equiv 0 \mod 4$ (Theorem 4.5), D_∞ and D_{2^∞} (Theorem 4.6). All our computations will use the additive decomposition of $DN(G)$ given in Proposition 2.3.6. We will analyze the summands directly and start by computing the summand indexed by the neutral element.

Proposition 4.1. *Let X be a real 2-cyclotomic spectrum whose underlying spectrum is bounded below and G a discrete (possibly infinite) group. Then there is an equivalence*

$$\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes B_{\mathbb{R}}(G,1)_+;2) \simeq \bigoplus_{\substack{[x]\in \mathrm{conj}(G)\\ x^2=1}} \Phi^{\Sigma_2}\mathrm{TCR}(X;2) \otimes BZ_G\langle x\rangle_+,$$

which depends on a choice of representatives for each conjugacy class.

Proof. By Proposition 2.3.6 there is a C_2-equivariant equivalence of Σ_2-spaces

$$f\colon B_{\mathbb{R}}G \simeq B_{\mathbb{R}}(G,1)$$

induced by (2.12), where the left hand side has the trivial C_2-action and the right hand side the C_2-action by restricting the $\mathbb{T}$-action coming from the cyclic structure. We claim that this induces an equivalence

$$\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes (B_{\mathbb{R}}G_+)^{\mathrm{triv}};2) \simeq \Phi^{\Sigma_2}\mathrm{TCR}(X \otimes B_{\mathbb{R}}(G,1)_+;2).$$

The proposition will then follow from Proposition 3.5.14 and Proposition 2.3.3.

Let $\alpha\colon D_4 \to \Sigma_2$ be the projection. Consider the following diagram of spectra

$$\begin{array}{ccc}
\Phi^{\Sigma_2}\Sigma^\infty B_{\mathbb{R}}G_+ & \xrightarrow{\Phi^{\Sigma_2}f} & \Phi^{\Sigma_2}\Sigma^\infty B_{\mathbb{R}}(G,1)_+ \\
\downarrow & & \downarrow \\
\Phi^{D_4}\Sigma^\infty \alpha^* B_{\mathbb{R}}G_+ & \xrightarrow{\Phi^{D_4}f} & \Phi^{D_4}\Sigma^\infty B_{\mathbb{R}}(G,1)_+ \\
\downarrow & & \downarrow \\
(\Phi^{\Sigma_2}\Sigma^\infty \alpha^* B_{\mathbb{R}}G_+)^{tC_2} & \xrightarrow{(\Phi^{\Sigma_2}f)^{tC_2}} & (\Phi^{\Sigma_2}\Sigma^\infty B_{\mathbb{R}}(G,1)_+)^{tC_2},
\end{array}$$

where the lower vertical arrows are the Borel completion maps, the upper right vertical arrow is induced by (3.20) and the upper left vertical arrow is obtained by applying Φ^{Σ_2} to the composite

$$\Sigma^\infty B_{\mathbb{R}}G_+ \to (\Sigma^\infty \alpha^* B_{\mathbb{R}}G_+)^{C_2} \to \Phi^{C_2}\Sigma^\infty \alpha^* B_{\mathbb{R}}G_+ \tag{4.1}$$

with the first map being the unit and the second map the natural map from fixed points to geometric fixed points. By Example 3.4.7 the composition of the left vertical arrows is $\phi_{(\Sigma^\infty B_{\mathbb{R}}G_+)^{\mathrm{triv}}}$ and by the proof of Proposition 4.1 the composition of the right vertical arrows is $\phi_{\Sigma^\infty B_{\mathbb{R}}(G,1)_+}$, therefore by Theorem 3.5.8 we must show that the outer square in the diagram commutes. By naturality of the Borel completion it suffices to show that the upper square commutes.

Note that the the composition of (4.1) with the equivalences

$$\Phi^{C_2}\Sigma^\infty \alpha^* B_{\mathbb{R}}G_+ \simeq \Sigma^\infty(\alpha^* B_{\mathbb{R}}G_+)^{C_2} \simeq \Sigma^\infty B_{\mathbb{R}}G_+$$

is the identity. After taking Σ_2-fixed points f fits into the square of spaces

$$\begin{array}{ccc}
B_{\mathbb{R}}G^{\Sigma_2} & \xrightarrow{f^{\Sigma_2}} & B_{\mathbb{R}}(G,1)^{\Sigma_2} \\
\downarrow & & \downarrow \\
B_{\mathbb{R}}G^{\Sigma_2} & \xrightarrow{f^{D_4}} & B_{\mathbb{R}}(G,1)^{D_4},
\end{array}$$

where the right vertical arrow is induced by (3.20) and the left vertical arrow is on each component of $B_{\mathbb{R}}G^{\Sigma_2}$ given by (2.14), which is homotopic to the identity. □

We will now compute $\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes DN(G)_+; 2)$ for the case that all 2-torsion is contained in the center of G. In the statement we denote by G_2 the set of 2-torsion elements and we let Σ_2 act on $G_2 \times G_2$ via the map

$$G_2 \times G_2 \to G_2 \times G_2, (g, h) \mapsto (g, gh).$$

We shall use this in our indexing notation. Note that the above Σ_2 action restricts to a free action on $G_2 \setminus \{1\} \times G_2$.

Theorem 4.2. *Let G be a discrete group such that G_2, the elements of order 2, are contained in the center of G. For any real 2-cyclotomic spectrum X whose underlying spectrum is bounded below there is a natural pullback*

$$\begin{array}{ccc} \Phi^{\Sigma_2}\mathrm{TCR}(X \otimes DN(G)_+; 2) & \longrightarrow & \bigoplus_{(G_2\setminus\{1\}\times G_2)/\Sigma_2} \Phi^{\Sigma_2} X \otimes BG_+ \\ \downarrow & & \downarrow \phi \\ (\Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}} G_+^{\mathrm{triv}}))^{hC_2} & \xrightarrow{\mathrm{can}-\phi} & (\Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}} G_+^{\mathrm{triv}}))^{tC_2}, \end{array}$$

and the horizontal fiber is $\bigoplus_{G_2} \Phi^{\Sigma_2}\mathrm{TCR}(X; 2) \otimes BG_+$. Consequently, if G is 2-torsion-free there is a natural equivalence

$$\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes DN(G)_+; 2) \simeq \Phi^{\Sigma_2}\mathrm{TCR}(X; 2) \otimes BG_+.$$

Proof. By Proposition 2.3.2 we have a decomposition

$$X \otimes DN(G)_+ \simeq \bigoplus_{x \in G_2} X \otimes B_{\mathbb{R}}(G, x)_+ \oplus \bigoplus_{\substack{[x] \in \mathrm{conj}_{\mathbb{R}}(G), \\ x^2 \neq 1}} X \otimes B_{\mathbb{R}}(SZ_G\langle x\rangle \wr \Sigma_2, x)_+,$$

and by Proposition 2.3.3 and the assumption on G the inclusion induces a C_2-equivariant equivalence

$$\Phi^{\Sigma_2}\left(\bigoplus_{x \in G_2} X \otimes B_{\mathbb{R}}(G, x)_+\right) \simeq \Phi^{\Sigma_2}(X \otimes DN(G)_+). \tag{4.2}$$

Note that $\bigoplus_{x \in G_2} X \otimes B_{\mathbb{R}}(G, x)_+$ is a real 2-cyclotomic spectrum by Lemma 3.4.8 such that the inclusion into $X \otimes DN(G)_+$ is a map of real 2-cyclotomic spectra. The left hand side of (4.2) is equivalent to

$$\Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}}(G, 1)_+) \oplus \bigoplus_{G_2\setminus\{1\}\times G_2} \Phi^{\Sigma_2} X \otimes BG_+$$

and by Example 2.3.8 the residual C_2-action sends the summand indexed by $(x, y) \in G_2 \times G_2$ to the summand indexed by (x, xy), therefore the C_2-action on $\bigoplus_{G_2\setminus\{1\}\times G_2} \Phi^{\Sigma_2} X \otimes BG_+$ is induced after untwisting. Combining the above with Theorem 3.5.8 and [NS18, Lemma I.3.8] we obtain that $\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes DN(G)_+; 2)$ is equivalent to the fiber of

$$\Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}}(G, 1)_+)^{hC_2} \oplus \bigoplus_{(G_2\setminus\{1\}\times G_2)/\Sigma_2} \Phi^{\Sigma_2} X \otimes BG_+ \xrightarrow{\mathrm{can}-\phi} \Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}}(G, 1)_+)^{tC_2}.$$

Since the restriction of can to $\bigoplus_{(G_2\setminus\{1\}\times G_2)/\Sigma_2} \Phi^{\Sigma_2}X \otimes BG_+$ is zero, this is equivalent to the fact that the following diagram is a pullback

$$\begin{array}{ccc} \Phi^{\Sigma_2}\mathrm{TCR}(X \otimes DN(G)_+;2) & \longrightarrow & \bigoplus_{(G_2\setminus\{1\}\times G_2)/\Sigma_2} \Phi^{\Sigma_2}X \otimes BG_+ \\ \downarrow & & \downarrow \phi \\ (\Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}}(G,1)_+)^{hC_2} & \xrightarrow{\text{can}-\phi} & (\Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}}(G,1)_+))^{tC_2}, \end{array}$$

and by (the proof of) Proposition 4.1 the lower row is equivalent to

$$(\Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}}G_+^{\text{triv}}))^{hC_2} \xrightarrow{\text{can}-\phi} (\Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}}G_+^{\text{triv}}))^{tC_2}$$

with fiber $\bigoplus_{G_2} \Phi^{\Sigma_2}\mathrm{TCR}(X;2) \otimes BG_+$. □

We now turn to dihedral groups. For D_{2n} with n finite we restrict ourselves to even n and we need to distinguish between the cases $n \equiv 2 \mod 4$ and $n \equiv 0 \mod 4$. We start with the former. Note that the previous result also applies to D_4, since it is abelian. Therefore, we assume that $n \equiv 2 \mod 4$ but $n \neq 2$. The condition on n implies that the inclusion $D_4 \to D_{2n}$ is a split injection of groups, hence it induces a split injection of spectra

$$\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes DN(D_4)_+;2) \to \Phi^{\Sigma_2}\mathrm{TCR}(X \otimes DN(D_{2n})_+;2). \tag{4.3}$$

We shall describe the cofiber of (4.3), which is equivalent to $\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes D(D_{2n})/D(D_4);2)$, where

$$DN(D_{2n})/DN(D_4) = \mathrm{cofib}(DN(D_4) \to DN(D_{2n})).$$

The inclusion $C_2 \to C_n$ induces an isomorphism in group homology $H_*(C_2;\mathbb{F}_2) \cong H_*(C_n;\mathbb{F}_2)$, hence an application of the Lyndon-Hochschild-Serre spectral sequence shows that also the inclusion $D_4 \to D_{2n}$ induces an isomorphism on group homology $H_*(D_4;\mathbb{F}_2) \cong H_*(D_{2n};\mathbb{F}_2)$. In particular, the map of suspension spectra $BD_{4+} \to BD_{2n+}$ is a 2-adic equivalence, that is the suspension spectrum of BD_{2n}/BD_4 vanishes after 2-completion. We will exploit this fact in the proof of the next result. In it we will use sums indexed over (certain subsets of) the orbit set $(C_n \setminus C_2)/\Sigma_2$. Here, as usual Σ_2 acts on C_n by sending an element to its inverse. This Σ_2-action clearly restricts to $C_n \setminus C_2$[1].

Theorem 4.3. *Let X be a real 2-cyclotomic spectrum whose underlying spectrum is bounded below.*

[1]Here by $C_n \setminus C_2$ we mean the set theoretic difference, not the right orbits.

(i) After 2-completion there is a pullback

$$\begin{array}{ccc}
\Phi^{\Sigma_2}\mathrm{TCR}(X\otimes D(D_{2n})/D(D_4);2) & \longrightarrow & \bigoplus_{\substack{[c^j]\in(C_n\backslash C_2)/\Sigma_2\\ j \text{ even}}} \left((\Phi^{\Sigma_2}X\otimes BC_{2+})^{hC_2}\right)^{\oplus 2} \\
\downarrow & & \downarrow{\scriptstyle \mathrm{can}-\phi} \\
\bigoplus_{\substack{[c^j]\in(C_n\backslash C_2)/\Sigma_2\\ j \text{ odd}}} \Phi^{\Sigma_2}X\otimes BC_{2+} & \xrightarrow{\phi} & \bigoplus_{\substack{[c^j]\in(C_n\backslash C_2)/\Sigma_2\\ j \text{ even}}} \left((\Phi^{\Sigma_2}X\otimes BC_{2+})^{tC_2}\right)^{\oplus 2},
\end{array}$$

where in the right hand terms BC_2 carries the trivial C_2-action.

(ii) If the underlying Σ_2-spectrum of X is bounded below, then there is a pullback

$$\begin{array}{ccc}
\Phi^{\Sigma_2}\mathrm{TCR}(X\otimes D(D_{2n})/D(D_4);2) & \longrightarrow & \begin{array}{c}\bigoplus_{\substack{[c^j]\in(C_n\backslash C_2)/\Sigma_2\\ j \text{ even}}} \left((\Phi^{\Sigma_2}X\otimes BC_{2+})^{hC_2}\right)^{\oplus 2} \\ \oplus \bigoplus_{\substack{[c^j]\in(C_n\backslash C_2)/\Sigma_2\\ j \text{ odd}}} \Phi^{\Sigma_2}X\otimes BC_{2+}\end{array} \\
\downarrow & & \downarrow{\scriptstyle \mathrm{can}-\phi} \\
\left(\left((\Phi^{\Sigma_2}X)_{hC_2}\right)^{\oplus_2}\oplus\Phi^{\Sigma_2}X\right)\otimes BD_{2n}/BD_4 & \xrightarrow{0} & \bigoplus_{\substack{[c^j]\in(C_n\backslash C_2)/\Sigma_2\\ j \text{ even}}} \left((\Phi^{\Sigma_2}X\otimes BC_{2+})^{tC_2}\right)^{\oplus 2},
\end{array}$$

where in the right hand terms BC_2 carries the trivial C_2-action.

Remark 4.4. Before we give the proof we point out the following two remarks on the pullback in (ii).

(a) Denote by F the vertical fiber of the pullback square in (ii). Then one can see from the form of the given pullback square that

$$\Phi^{\Sigma_2}\mathrm{TCR}(X\otimes D(D_{2n})/D(D_4);2)\simeq F\oplus\left(\left((\Phi^{\Sigma_2}X)_{hC_2}\right)^{\oplus_2}\oplus\Phi^{\Sigma_2}X\right)\otimes BD_{2n}/BD_4,$$

since the lower left corner is a direct summand of the horizontal fiber. Since the restriction of can to $\bigoplus_{\substack{[c^j]\in(C_n\backslash C_2)/\Sigma_2\\ j \text{ odd}}} \Phi^{\Sigma_2}X\otimes BC_{2+}$ is zero, F also fits into a pullback

$$\begin{array}{ccc}
F & \longrightarrow & \bigoplus_{\substack{[c^j]\in(C_n\backslash C_2)/\Sigma_2\\ j \text{ even}}} \left((\Phi^{\Sigma_2}X\otimes BC_{2+})^{hC_2}\right)^{\oplus 2} \\
\downarrow & & \downarrow{\scriptstyle \mathrm{can}-\phi} \\
\bigoplus_{\substack{[c^j]\in(C_n\backslash C_2)/\Sigma_2\\ j \text{ odd}}} \Phi^{\Sigma_2}X\otimes BC_{2+} & \xrightarrow{\phi} & \bigoplus_{\substack{[c^j]\in(C_n\backslash C_2)/\Sigma_2\\ j \text{ even}}} \left((\Phi^{\Sigma_2}X\otimes BC_{2+})^{tC_2}\right)^{\oplus 2}.
\end{array}$$

(b) As remarked before, for an orthogonal ring spectrum A with anti-involution the underlying spectrum of $\mathrm{THR}(A)$ is bounded below if A is bounded below as a spectrum so that $X = \mathrm{THR}(A)$ fulfills the condition in (i). If the (geometric) fixed points of the orthogonal Σ_2-spectrum associated to A are additionally bounded below, then the underlying Σ_2 spectrum of $\mathrm{THR}(A)$ is bounded below by [DMPR21, Theorem 2.26] and the condition in (ii) applies to $X = \mathrm{THR}(A)$. This is in particular the case if A is a discrete ring with anti-involution by [DMPR21, Example 2.4]. Furthermore, in this case $\Phi^{\Sigma_2}\mathrm{THR}(A)$ is a module over $\Phi^{\Sigma_2}\mathrm{THR}(\mathbb{Z})$ by [DMPR21, Section 4.1]. Now $\Phi^{\Sigma_2}\mathrm{THR}(\mathbb{Z})$ is itself an $H\mathbb{F}_2$-algebra by [DMPR21, Theorem 5.23], therefore the lower left corner in the pullback of (ii) vanishes for $X = \mathrm{THR}(A)$ in this case and $\Phi^{\Sigma_2}\mathrm{TCR}(\mathrm{THR}(A) \otimes D(D_{2n})/D(D_4); 2)$ is equivalent to the vertical fiber, hence by (a) fits into the pullback

$$\begin{array}{ccc}
\Phi^{\Sigma_2}\mathrm{TCR}(\mathrm{THR}(A) \otimes D(D_{2n})/D(D_4); 2) & \to & \bigoplus_{\substack{[c^j]\in(C_n\setminus C_2)/\Sigma_2 \\ j \text{ even}}} \left((\Phi^{\Sigma_2}\mathrm{THR}(A) \otimes BC_{2+})^{hC_2}\right)^{\oplus 2} \\
\downarrow & & \downarrow{\scriptstyle \mathrm{can}-\phi} \\
\bigoplus_{\substack{[c^j]\in(C_n\setminus C_2)/\Sigma_2 \\ j \text{ odd}}} \Phi^{\Sigma_2}\mathrm{THR}(A) \otimes BC_{2+} & \xrightarrow{\phi} & \bigoplus_{\substack{[c^j]\in(C_n\setminus C_2)/\Sigma_2 \\ j \text{ even}}} \left((\Phi^{\Sigma_2}\mathrm{THR}(A) \otimes BC_{2+})^{tC_2}\right)^{\oplus 2}.
\end{array}$$

Proof of Theorem 4.3. By Proposition 2.3.2 there is an equivalence

$$DN(D_4) \simeq \coprod_{x\in D_4} B_{\mathbb{R}}(D_4, x),$$

and $DN(D_{2n})$ is equivalent to

$$B_{\mathbb{R}}(D_{2n}, 1) \amalg B_{\mathbb{R}}(D_{2n}, c^{\frac{n}{2}}) \amalg \coprod_{i=0,1} B_{\mathbb{R}}(Z_{D_{2n}}\langle \sigma c^{\frac{in}{2}}\rangle, \sigma c^{\frac{in}{2}}) \amalg \coprod_{[c^j]\in(C_n\setminus C_2)/\Sigma_2} B_{\mathbb{R}}(D_{2n} \wr \Sigma_2, c^j).$$

Note that $Z_{D_{2n}}\langle \sigma c^{\frac{in}{2}}\rangle = \langle \sigma c^{\frac{in}{2}}, c^{\frac{n}{2}}\rangle = D_4$. One checks that under the above equivalences the map $DN(D_4) \to DN(D_{2n})$ maps the summands indexed by $[[1]], [[\sigma c^i]]$ and $[[c]]$ respectively to the summands indexed by $[[1]], [[\sigma c^{\frac{in}{2}}]]$ and $[[c^{\frac{n}{2}}]]$ respectively and on these summands the map is given by either the identity $B_{\mathbb{R}}(D_4) \to B_{\mathbb{R}}(D_4)$ or the map $B_{\mathbb{R}}(D_4) \to B_{\mathbb{R}}(D_{2n})$. We obtain that $DN(D_{2n})/DN(D_4)$ is equivalent to

$$B_{\mathbb{R}}(D_{2n}, 1)/B_{\mathbb{R}}(D_4, 1) \vee B_{\mathbb{R}}(D_{2n}, c^{\frac{n}{2}})/B_{\mathbb{R}}(D_4, c) \vee \bigvee_{[c^j]\in(C_n\setminus C_2)/\Sigma_2} B_{\mathbb{R}}(D_{2n} \wr \Sigma_2, c^j)_+.$$

We take suspension spectra and smash with X. We want to apply Theorem 3.5.8 and start by analyzing $(\Phi^{\Sigma_2}(-))^{tC_2}$ and $(\Phi^{\Sigma_2}(-))^{hC_2}$ of each summand separately.

(1) By Proposition 2.3.3 we have

$$\left(\Phi^{\Sigma_2}\left(X \otimes B_{\mathbb{R}}(D_{2n}, 1)/B_{\mathbb{R}}(D_4, 1)\right)\right)^{tC_2} \simeq \left((\Phi^{\Sigma_2}X \otimes BD_{2n}/BD_4)^{tC_2}\right)^{\oplus 2},$$

where C_2 acts trivially on BD_{2n}/BD_4. As the mod 2 Moore spectrum is a finite spectrum, smashing with it commutes with the Tate construction and the discussion above shows that $\Phi^{\Sigma_2}X \otimes BD_{2n}/BD_4$ vanishes after smashing with the mod 2 Moore spectrum. Therefore, $(\Phi^{\Sigma_2}X \otimes BD_{2n}/BD_4)^{tC_2}$ vanishes after 2-completion and it vanishes unconditionally if $\Phi^{\Sigma_2}X$ is bounded below by [NS18, Lemma I.2.9]. The same argument shows that $(\Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}}(D_{2n},1)/B_{\mathbb{R}}(D_4,1)))^{hC_2}$ vanishes after 2-completion. If $\Phi^{\Sigma_2}X$ is bounded below, the vanishing of the Tate construction together with Proposition 2.3.6 implies that there is an equivalence

$$\begin{aligned}(\Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}}(D_{2n},1)/B_{\mathbb{R}}(D_4,1)))^{hC_2} &\simeq ((\Phi^{\Sigma_2}X \otimes BD_{2n}/BD_4)_{hC_2})^{\oplus 2}\\ &\simeq ((\Phi^{\Sigma_2}X)_{hC_2} \otimes BD_{2n}/BD_4)^{\oplus 2}.\end{aligned}$$

(2) By Proposition 2.3.3 and Example 2.3.8

$$\Phi^{\Sigma_2}\left(X \otimes B_{\mathbb{R}}(D_{2n}, c^{\frac{n}{2}})/B_{\mathbb{R}}(D_4,c)\right) \simeq \mathrm{ind}_1^{C_2}\left(\Phi^{\Sigma_2}X \otimes BD_{2n}/BD_4\right),$$

which vanishes after applying the Tate construction and we obtain

$$\left(\Phi^{\Sigma_2}\left(X \otimes B_{\mathbb{R}}(D_{2n}, c^{\frac{n}{2}})/B_{\mathbb{R}}(D_4,c)\right)\right)^{hC_2} \simeq \Phi^{\Sigma_2}X \otimes BD_{2n}/BD_4.$$

Note that also this spectrum vanishes after 2-completion.

(3) If j is odd, then Proposition 2.3.3 and Example 2.3.9 imply that

$$\Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}}(D_{2n} \wr \Sigma_2, c^j)_+) \simeq \mathrm{ind}_1^{C_2}\left(\Phi^{\Sigma_2}X \otimes BC_{2+}\right),$$

which again vanishes after applying the Tate construction and

$$\left(\Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}}(D_{2n} \wr \Sigma_2, c^j)_+)\right)^{hC_2} \simeq \Phi^{\Sigma_2}X \otimes BC_{2+}.$$

(4) If j is even, then we obtain from Proposition 2.3.3 and Example 2.3.9 that

$$\Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}}(D_{2n} \wr \Sigma_2, c^j)_+) \simeq \left(\Phi^{\Sigma_2}X \otimes BC_{2+}\right)^{\oplus 2},$$

where BC_2 carries the trivial C_2-action.

Combining the above with Theorem 3.5.8 yields that after 2-completion there is a fiber sequence

$$\begin{gathered}\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes DN(D_{2n})/DN(D_4);2)\\ \downarrow\\ \bigoplus_{\substack{[c^j]\in(C_n\setminus C_2)/\Sigma_2\\ j\text{ even}}}\left((\Phi^{\Sigma_2}X \otimes BC_{2+})^{hC_2}\right)^{\oplus 2} \oplus \bigoplus_{\substack{[c^j]\in(C_n\setminus C_2)/\Sigma_2\\ j\text{ odd}}}\Phi^{\Sigma_2}X \otimes BC_{2+}\\ \downarrow{\scriptstyle \mathrm{can}-\phi}\\ \bigoplus_{\substack{[c^j]\in(C_n\setminus C_2)/\Sigma_2\\ j\text{ even}}}\left((\Phi^{\Sigma_2}X \otimes BC_{2+})^{tC_2}\right)^{\oplus 2},\end{gathered}$$

and by the vanishing of the Tate construction the restriction of can to the summand indexed by $[c^j]$ with j odd is zero, thus we obtain the pullback in (i) by rewriting the above fiber sequence as a pullback. Similarly, if $\Phi^{\Sigma_2}X$ is bounded below, then $\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes DN(D_{2n})/DN(D_4); 2)$ is equivalent to the fiber of

$$\begin{aligned}
&(((\Phi^{\Sigma_2}X)_{hC_2})^{\oplus 2} \oplus \Phi^{\Sigma_2}X) \otimes BD_{2n}/BD_4 \oplus \\
&\bigoplus_{\substack{[c^j]\in(C_n\setminus C_2)/\Sigma_2\\ j \text{ even}}} \left((\Phi^{\Sigma_2}X \otimes BC_{2+})^{hC_2}\right)^{\oplus 2} \oplus \\
&\bigoplus_{\substack{[c^j]\in(C_n\setminus C_2)/\Sigma_2\\ j \text{ odd}}} \Phi^{\Sigma_2}X \otimes BC_{2+} \xrightarrow{\mathrm{can}-\phi} \bigoplus_{\substack{[c^j]\in(C_n\setminus C_2)/\Sigma_2\\ j \text{ even}}} \left((\Phi^{\Sigma_2}X \otimes BC_{2+})^{tC_2}\right)^{\oplus 2}.
\end{aligned}$$

If we apply Lemma 3.4.8 to D_4 and D_{2n} and subsequently pass to the cofiber, then by the vanishing of the Tate construction we see that the map $\mathrm{can} - \phi$ is zero when restricted to $(((\Phi^{\Sigma_2}X)_{hC_2})^{\oplus 2} \oplus \Phi^{\Sigma_2}X) \otimes BD_{2n}/BD_4$, thus we obtain the claimed pullback of (ii) by rewriting the above fiber sequence as a pullback. □

Next, we want to treat the case of finite n with $n \equiv 0 \mod 4$. We denote by ν_2 the 2-adic valuation, i.e. $\nu_2(n) = k$ if $n = 2^k m$ with m being odd. In this case for $k = 0, 1, \dots \nu_2(n)$ there is an inclusion $C_{2^k m} \to C_n$. As before we let Σ_2 act on C_n by sending an element to its inverse. This restricts to an action on $C_{2^k m} \setminus C_{2^{k-1}m}$. We will again use this fact in our indexing notation.

Theorem 4.5. *Let X be a real 2-cyclotomic spectrum whose underlying spectrum is bounded below.*

(i) Suppose $n \equiv 0 \mod 4$. There is a finite filtration of length $\nu_2(n)$ on $\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes DN(D_{2n})_+; 2)$ such that for $k \geq 1$ the kth graded piece $\mathrm{gr}^k\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes DN(D_{2n})_+; 2)$ is equivalent to

$$\begin{cases}
\Phi^{\Sigma_2}X \otimes BD_{2n+} \oplus \bigoplus_{i=0,1} \left((\alpha_i^*\Phi^{\Sigma_2}X)_{h\langle \sigma c^i, c^{\frac{n}{4}}\rangle} \oplus (\Phi^{\Sigma_2}X \otimes BZ_{D_{2n}}\langle\sigma c^i\rangle_+)^{\oplus 2}\right) & \\
\quad \oplus \bigoplus_{(C_{2m}\setminus C_2)/\Sigma_2} \left((\Phi^{\Sigma_2}X)_{hC_2} \otimes BC_{2+}\right)^{\oplus 2} & \text{if } k=1, \\
\bigoplus_{(C_{2^k m}\setminus C_{2^{k-1}m})/\Sigma_2} \left((\Phi^{\Sigma_2}X)_{hC_2} \otimes BC_{2+}\right)^{\oplus 2} & \text{if } 2 \leq k < \nu_2(n), \\
\bigoplus_{(C_n\setminus C_{\frac{n}{2}})/\Sigma_2} \Phi^{\Sigma_2}X \otimes BC_{2+} & \text{if } k = \nu_2(n),
\end{cases}$$

where $\alpha_i\colon \langle\sigma c^i, c^{\frac{n}{4}}\rangle = N_{D_{2n}}Z_{D_{2n}}\langle\sigma c^i\rangle \to W_{D_{2n}}Z_{D_{2n}}\langle\sigma c^i\rangle \cong C_2$ is the projection. Finally, there is a split injection

$$(\Phi^{\Sigma_2}\mathrm{TCR}(X;2) \otimes BD_{2n+})^{\oplus 2} \oplus \bigoplus_{i=0,1} \Phi^{\Sigma_2}\mathrm{TCR}(X;2) \otimes BZ_{D_{2n}}\langle\sigma c^i\rangle_+ \\ \to \mathrm{gr}^0\Phi^{\Sigma_2}(\mathrm{TCR}(X \otimes DN(D_{2n})_+; 2)),$$

which is an equivalence of n is a power of 2.

(ii) There is a filtration of infinite length on $\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes DN(D_{2^\infty})_+;2)$ with the kth graded piece $\mathrm{gr}^k\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes DN(D_{2n})_+;2)$ being equivalent to

$$\begin{cases} (\Phi^{\Sigma_2}\mathrm{TCR}(X;2) \otimes BD_{2^\infty +})^{\oplus 2} \oplus \Phi^{\Sigma_2}\mathrm{TCR}(X;2) \otimes BD_{4+} & \text{if } k=0, \\ \Phi^{\Sigma_2}X \otimes BD_{2^\infty +} \oplus (\alpha^*\Phi^{\Sigma_2}X)_{hD_8} \oplus (\Phi^{\Sigma_2}X \otimes BD_{4+})^{\oplus 2} & \text{if } k=1, \\ \bigoplus_{(C_{2^k}\setminus C_{2^{k-1}})/\Sigma_2} (\Phi^{\Sigma_2}X)_{hC_2} \otimes BC_{2+} & \text{if } k \geq 2, \end{cases}$$

where $\alpha\colon D_8 \to D_8/D_4 \cong C_2$ is the projection.

Proof. We put a filtration on $X \otimes DN(D_{2n})_+$ itself, which induces a filtration on $\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes DN(D_{2n})_+;2)$, since $\Phi^{\Sigma_2}\mathrm{TCR}(-;2)$ is exact. Just as before, Proposition 2.3.2 implies that $X \otimes DN(D_{2n})_+$ is equivalent to

$$\bigoplus_{x\in\{1,c^{\frac{n}{2}},\sigma,\sigma c\}} X \otimes B_{\mathbb{R}}(Z_{D_{2n}}\langle x\rangle, x)_+ \oplus \bigoplus_{[c^j]\in(C_n\setminus C_2)/\Sigma_2} X \otimes B_{\mathbb{R}}(SZ_{D_{2n}}\langle c^j\rangle \wr \Sigma_2, c^j)_+,$$

and we let $F_k(X \otimes DN(D_{2n})_+)$ consist of the summands indexed by elements x such that $\nu_2(\mathrm{ord}(x)) \leq k$. By Lemma 3.4.8 $F_k(X \otimes DN(D_{2n})_+)$ is a real 2-cyclotomic spectrum such that for $k \geq 1$ its Frobenius factors through $F_{k-1}(X \otimes DN(D_{2n})_+)$, therefore Proposition 3.5.15 yields the identification

$$\Phi^{\Sigma_2}\mathrm{gr}^k\mathrm{TCR}(X \otimes DN(D_{2n})_+;2) \simeq (\Phi^{\Sigma_2}\mathrm{gr}^k(X \otimes DN(D_{2n})_+))_{hC_2}$$

for $k \geq 1$. We treat the different cases $k = 1, \dots, \nu_2(n)$.

(i) Up to real conjugacy, the elements $x \in D_{2n}$ with $\nu_2(\mathrm{ord}(x)) = 1$ are $\sigma, \sigma c, c^{\frac{n}{2}}$ and c^j with $[c^j] \in (C_{2m}\setminus C_2)/\Sigma_2$. For the latter we must have that j is even. We determine $(\Phi^{\Sigma_2}(-))_{hC_2}$ of these summands separately. Note that for $i = 0, 1$ we have $Z_{D_{2n}}\langle\sigma c^i\rangle = N_{D_{2n}}\langle\sigma c^i\rangle = \langle\sigma c^i, c^{\frac{n}{2}}\rangle \cong D_4$ and $N_{D_{2n}}Z_{D_{2n}}\langle\sigma c^i\rangle = \langle\sigma c^i, c^{\frac{n}{4}}\rangle \cong D_8$. Furthermore, $c^{\frac{n}{2}}$ is central in D_{2n}. By Example 2.3.8 we have

$$(\Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}}(Z_{D_{2n}}\langle\sigma c^i\rangle, \sigma c^i)_+))_{hC_2} \simeq (\Phi^{\Sigma_2}X \otimes BZ_{D_{2n}}\langle\sigma c^i\rangle_+)^{\oplus 2},$$

and

$$\left(\Phi^{\Sigma_2}\left(X \otimes B_{\mathbb{R}}(D_{2n}, c^{\frac{n}{2}})_+\right)\right)_{hC_2} \simeq \Phi^{\Sigma_2}X \otimes BD_{2n+} \oplus \bigoplus_{i=0,1} (\Phi^{\Sigma_2}X \otimes BZ_{D_{2n}}\langle\sigma c^i\rangle_+)_{hC_2},$$

where $BZ_{D_{2n}}\langle\sigma c^i\rangle = EN_{D_{2n}}Z_{D_{2n}}\langle\sigma c^i\rangle/Z_{D_{2n}}\langle\sigma c^i\rangle \simeq (*)_{hZ_{D_{2n}}\langle\sigma c^i\rangle}$ with the residual action by $W_{D_{2n}}Z_{D_{2n}}\langle\sigma c^i\rangle \cong C_2$. Now we observe that

$$\begin{aligned} &(\Phi^{\Sigma_2}X \otimes BZ_{D_{2n}}\langle\sigma c^i\rangle_+)_{hC_2} \simeq (\Phi^{\Sigma_2}X \otimes (S^0)_{hZ_{D_{2n}}\langle\sigma c^i\rangle})_{hC_2} \\ \simeq &((\alpha_i^*\Phi^{\Sigma_2}X)_{hZ_{D_{2n}}\langle\sigma c^i\rangle})_{hW_{D_{2n}}Z_{D_{2n}}\langle\sigma c^i\rangle} \simeq (\alpha_i^*\Phi^{\Sigma_2}X)_{hN_{D_{2n}}Z_{D_{2n}}\langle\sigma c^i\rangle}. \end{aligned}$$

For the final summand, it follows again from Example 2.3.9 that

$$(\Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}}(SZ_{D_{2n}}\langle c^j\rangle \wr \Sigma_2, c^j)_+))_{hC_2} \simeq \left((\Phi^{\Sigma_2}X \otimes BC_{2+})^{\oplus 2}\right)_{hC_2},$$

and since the C_2-action is on BC_{2+} is trivial we have

$$(\Phi^{\Sigma_2}X \otimes BC_{2+})_{hC_2} \simeq (\Phi^{\Sigma_2}X)_{hC_2} \otimes BC_{2+}.$$

(ii) For $2 \leq k < \nu_2(n)$ we have

$$\mathrm{gr}^k(X \otimes DN(D_{2n})_+) \simeq \bigoplus_{[c^j]\in(C_{2^k m}\setminus C_{2^{k-1}m})/\Sigma_2} X \otimes B_{\mathbb{R}}(SZ_{D_{2n}}\langle c^j\rangle \wr \Sigma_2, c^j)_+.$$

The corresponding j are all even. Therefore, the same argument as above shows that

$$\mathrm{gr}^k\Phi^{\Sigma_2}\mathrm{TCR}(X \otimes DN(D_{2n})_+;2) \simeq \bigoplus_{(C_{2^k m}\setminus C_{2^{k-1}m})/\Sigma_2} \left((\Phi^{\Sigma_2}X)_{hC_2} \otimes BC_{2+}\right)^{\oplus 2}.$$

(iii) The remaining graded piece is $\mathrm{gr}^{\nu_2(n)}\mathrm{TCR}(X \otimes DN(D_{2n})_+;2)$. One checks that

$$\mathrm{gr}^{\nu_2(n)}(X \otimes DN(D_{2n})_+) = \bigoplus_{[c^j]\in(C_n\setminus C_{\frac{n}{2}})/\Sigma_2} X \otimes B_{\mathbb{R}}(SZ_{D_{2n}}\langle c^j\rangle \wr \Sigma_2, c^j)_+,$$

and the corresponding j are all odd. Thus, the same argument as in the proof of Theorem 4.3 shows that

$$(\Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}}(SZ_{D_{2n}}\langle c^j\rangle \wr \Sigma_2, c^j)_+))_{hC_2} \simeq \Phi^{\Sigma_2}X \otimes BC_{2+}.$$

Finally, we treat the zeroth graded piece. We have that

$$\mathrm{gr}^0(X \otimes DN(D_{2n})_+) \simeq X \otimes B_{\mathbb{R}}(D_{2n},1)_+ \oplus \bigoplus_{[c^j]\in(C_m\setminus\{1\})/\Sigma_2} X \otimes B_{\mathbb{R}}(SZ_{D_{2n}}\langle c^j\rangle \wr \Sigma_2, c^j)_+$$

and by Lemma 3.4.8 the inclusion

$$X \otimes B_{\mathbb{R}}(D_{2n},1)_+ \to X \otimes B_{\mathbb{R}}(D_{2n},1)_+ \oplus \bigoplus_{[c^j]\in(C_m\setminus\{1\})/\Sigma_2} X \otimes B_{\mathbb{R}}(SZ_{D_{2n}}\langle c^j\rangle \wr \Sigma_2, c^j)_+$$

is a split injection of real 2-cyclotomic spectra. We conclude by applying $\Phi^{\Sigma_2}\mathrm{TCR}(-;2)$ and using Proposition 4.1 to identify the source.

For D_{2^∞} the proof is similar to the proof of Theorem 4.5. By Proposition 2.3.2 there is an equivalence

$$X \otimes DN(D_{2^\infty})_+ \simeq \bigoplus_{x\in\{1,c_1,\sigma\}} X \otimes B_{\mathbb{R}}(Z_{D_{2^\infty}}\langle x\rangle, x)_+ \oplus \bigoplus_{[c_n^j]\in(C_{2^\infty}\setminus C_2)/\Sigma_2} X \otimes B_{\mathbb{R}}(SZ_{D_{2^\infty}}\langle c_n^j\rangle \wr \Sigma_2, c^j)_+.$$

Note that $SZ_{D_{2^\infty}}\langle c_n^j\rangle = D_{2^\infty}$, that c_1 is central and that $Z_{D_{2^\infty}}\langle\sigma\rangle = D_4$. We put

$$\begin{aligned}
&F_0(X\otimes DN(D_{2^\infty})_+) = X\otimes B_{\mathbb{R}}(D_{2^\infty},1)_+,\\
&F_1(X\otimes DN(D_{2^\infty})_+) = X\otimes B_{\mathbb{R}}(D_{2^\infty},1)_+\oplus X\otimes B_{\mathbb{R}}(D_{2^\infty},c_1)\oplus X\otimes B_{\mathbb{R}}(D_4,\sigma)_+
\end{aligned}$$

and

$$F_k(X\otimes DN(D_{2^\infty})_+) = \bigoplus_{x\in\{1,c_1,\sigma\}} X\otimes B_{\mathbb{R}}(Z_{D_{2^\infty}}\langle x\rangle,x)_+\oplus \bigoplus_{[c_n^j]\in(C_{2^k}\setminus C_2)/\Sigma_2} X\otimes B_{\mathbb{R}}(D_{2^\infty}\wr\Sigma_2,c_n^j)_+$$

for $k\geq 2$. By Lemma 3.4.8 $F_k(X\otimes DN(D_{2^\infty})_+)$ is a real 2-cyclotomic spectrum such that for $k\geq 1$ its Frobenius factors through $F_{k-1}(X\otimes DN(D_{2^\infty})_+)$, and by exactness this filtration induces a filtration on $\Phi^{\Sigma_2}\mathrm{TCR}(X\otimes DN(D_{2^\infty})_+;2)$. Furthermore,

$$\mathrm{gr}^1(X\otimes DN(D_{2^\infty})_+)\simeq X\otimes B_{\mathbb{R}}(D_{2^\infty},c_1)_+\oplus X\otimes B_{\mathbb{R}}(D_4,\sigma)_+,$$

and

$$\mathrm{gr}^k(X\otimes DN(D_{2^\infty})_+)\simeq \bigoplus_{[c_k^j]\in(C_{2^k}\setminus C_{2^{k-1}})/\Sigma_2} X\otimes B_{\mathbb{R}}(D_{2^\infty}\wr\Sigma_2,c_n^j)_+$$

for $k\geq 2$. By Proposition 3.5.15

$$\mathrm{gr}^k\Phi^{\Sigma_2}\mathrm{TCR}(X\otimes DN(D_{2^\infty})_+;2)\simeq \Phi^{\Sigma_2}(\mathrm{gr}^k(X\otimes DN(D_{2^\infty})_+))_{hC_2}$$

for $k\geq 1$, so the identification of these graded pieces is now analogous to the proof of Theorem 4.5 using Example 2.3.8 and Example 2.3.9. The identification of the zeroth graded piece was done in Proposition 4.1. □

Theorem 4.6. *Let X be a real 2-cyclotomic spectrum whose underlying spectrum is bounded below.*

(i) There is a splitting

$$\begin{aligned}
&\Phi^{\Sigma_2}\mathrm{TCR}\left(X\otimes B_{\mathbb{R}}(D_\infty;1)_+\oplus\bigoplus_{i=0,1}X\otimes B_{\mathbb{R}}(\langle\sigma c^i\rangle;\sigma c^i)_+;2\right)\oplus\\
&\Phi^{\Sigma_2}\mathrm{TCR}\left(\bigoplus_{[x]\in(C_\infty\setminus\{1\})/\Sigma_2}^{\infty} X\otimes B_{\mathbb{R}}(D_\infty\wr\Sigma_2,x)_+;2\right)\simeq\Phi^{\Sigma_2}\mathrm{TCR}(X\otimes DN(D_\infty)_+;2).
\end{aligned}$$

The first summand fits into a pullback

$$\begin{array}{ccc}
\Phi^{\Sigma_2}\mathrm{TCR}\left(\bigoplus_{x\in\{1,\sigma,\sigma c\}}X\otimes B_{\mathbb{R}}(Z_{D_{2n}}\langle x\rangle,x)_+;2\right) & \longrightarrow & \bigoplus_{i=0,1}\Phi^{\Sigma_2}X\otimes B\langle\sigma c^i\rangle_+\\
\downarrow & & \downarrow{\scriptstyle\phi}\\
(\Phi^{\Sigma_2}(X\otimes(B_{\mathbb{R}}D_{\infty+})^{\mathrm{triv}}))^{hC_2} & \xrightarrow{\mathrm{can}-\phi} & (\Phi^{\Sigma_2}(X\otimes(B_{\mathbb{R}}D_{\infty+})^{\mathrm{triv}}))^{tC_2},
\end{array}$$

and the horizontal fiber is equivalent to

$$\Phi^{\Sigma_2}\mathrm{TCR}(X;2)\otimes BD_{\infty+}\oplus\bigoplus_{i=0,1}\Phi^{\Sigma_2}\mathrm{TCR}(X;2)\otimes B\langle\sigma c^i\rangle_+.$$

For the second summand there is a filtration indexed on $\mathbb{Z}^{\leq 0}$ *such that*

$$\mathrm{gr}^k\Phi^{\Sigma_2}\mathrm{TCR}\left(\bigoplus_{[x]\in(C_\infty\setminus\{1\})/\Sigma_2}X\otimes B_{\mathbb{R}}(D_\infty\wr\Sigma_2,x)_+;2\right)\simeq\bigoplus_{\mathbb{N}_0}((\Phi^{\Sigma_2}X)_{hC_2})^{\oplus 2}$$

for $k<0$ *and*

$$\mathrm{gr}^0\Phi^{\Sigma_2}\mathrm{TCR}\left(\bigoplus_{[x]\in(C_\infty\setminus\{1\})/\Sigma_2}X\otimes B_{\mathbb{R}}(D_\infty\wr\Sigma_2,x)_+;2\right)\simeq\bigoplus_{\mathbb{N}_0}\Phi^{\Sigma_2}X.$$

Proof. For D_∞ we also use Proposition 2.3.2 to see that $X\otimes DN(D_{2^\infty})_+$ is equivalent to

$$\bigoplus_{x\in\{1,\sigma,\sigma c\}}X\otimes B_{\mathbb{R}}(Z_{D_\infty}\langle x\rangle,x)_+\oplus\bigoplus_{[c^j]\in(C_\infty\setminus\{1\})/\Sigma_2}X\otimes B_{\mathbb{R}}(SZ_{D_\infty}\langle c^j\rangle\wr\Sigma_2,c^j)_+.$$

In this case the centralizer of $Z_{D_\infty}\langle\sigma c^i\rangle=\langle\sigma c^i\rangle$ for $i=0,1$ and $SZ_{D_\infty}\langle c^j\rangle=D_\infty$ for all j and by Lemma 3.4.8 the inclusions of $\bigoplus_{x\in\{1,\sigma,\sigma c\}}X\otimes B_{\mathbb{R}}(Z_{D_\infty}\langle x\rangle,x)_+$ $\bigoplus_{[c^j]\in(C_\infty\setminus\{1\})/\Sigma_2}X\otimes B_{\mathbb{R}}(SZ_{D_\infty}\langle c^j\rangle\wr\Sigma_2,c^j)_+$ into $X\otimes DN(D_\infty)_+$ are both split injections of real 2-cyclotomic spectra, therefore we immediately obtain the desired splitting of $\Phi^{\Sigma_2}\mathrm{TCR}(X\otimes DN(D_\infty)_+;2)$. The pullback

$$\begin{array}{ccc}\Phi^{\Sigma_2}\mathrm{TCR}\left(X\otimes B_{\mathbb{R}}(D_\infty;1)_+\oplus\bigoplus_{i=0,1}X\otimes B_{\mathbb{R}}(\langle\sigma c^i\rangle;\sigma c^i)_+;2\right) & \longrightarrow & \bigoplus_{i=0,1}\Phi^{\Sigma_2}X\otimes B\langle\sigma c^i\rangle_+\\ \downarrow & & \downarrow\phi\\ (\Phi^{\Sigma_2}(X\otimes(B_{\mathbb{R}}D_{\infty+})^{\mathrm{triv}}))^{hC_2} & \xrightarrow{\mathrm{can}-\phi} & (\Phi^{\Sigma_2}(X\otimes(B_{\mathbb{R}}D_{\infty+})^{\mathrm{triv}}))^{tC_2},\end{array}$$

as well as the identification of the horizontal fiber, is obtained as in the proof of Theorem 4.2 using Example 2.3.8.

To define the filtration on $\mathrm{TCR}\left(\bigoplus_{[x]\in(C_\infty\setminus\{1\})/\Sigma_2}X\otimes B_{\mathbb{R}}(D_\infty\wr\Sigma_2,x)_+;2\right)$ we first rewrite the index set as

$$\bigoplus_{[c^j]\in(C_\infty\setminus\{1\})/\Sigma_2}X\otimes B_{\mathbb{R}}(D_\infty\wr\Sigma_2,c^j)_+=\bigoplus_{s=0}^{\infty}\bigoplus_{i=0}^{\infty}X\otimes B_{\mathbb{R}}(D_\infty\wr\Sigma_2,c^{2^s(2i+1)})_+$$

by choosing the representative with $j>0$ for $[c^j]$ and then sorting the summands according to the 2-adic valuation of the index. Then, for $k\leq 0$ we put

$$F_k\left(\bigoplus_{s=0}^{\infty}\bigoplus_{i=0}^{\infty}X\otimes B_{\mathbb{R}}(D_\infty\wr\Sigma_2,c^{2^s(2i+1)})_+\right)=\bigoplus_{s=-k}^{\infty}\bigoplus_{i=0}^{\infty}X\otimes B_{\mathbb{R}}(D_\infty\wr\Sigma_2,c^{2^s(2i+1)})_+,$$

and again using Lemma 3.4.8 we see that

$$F_k\left(\bigoplus_{s=0}^{\infty}\bigoplus_{i=0}^{\infty} X \otimes B_{\mathbb{R}}(D_\infty \wr \Sigma_2, c^{2^s(2i+1)})_+\right)$$

is a real 2-cyclotomic spectrum such that the Frobenius factors through

$$F_{k-1}\left(\bigoplus_{s=0}^{\infty}\bigoplus_{i=0}^{\infty} X \otimes B_{\mathbb{R}}(D_\infty \wr \Sigma_2, c^{2^s(2i+1)})_+\right).$$

By Proposition 3.5.15 we have

$$\mathrm{gr}^k \Phi^{\Sigma_2}\mathrm{TCR}(X \otimes DN(D_\infty)_+; 2) \simeq \bigoplus_{i=0}^{\infty}(\Phi^{\Sigma_2}(X \otimes B_{\mathbb{R}}(D_\infty \wr \Sigma_2, c^{2^k(2i+1)})_+))_{hC_2}.$$

Finally, by Example 2.3.9 the right hand side is equivalent to $\bigoplus_{\mathbb{N}_0}((\Phi^{\Sigma_2}X)_{hC_2})^{\oplus 2}$ if $k < 0$ and $\bigoplus_{\mathbb{N}_0}\Phi^{\Sigma_2}X$ if $k = 0$. □

Appendix A

Equivariant homotopy theory

A.1 G-spectra and change of groups.

In this appendix we fix notation and conventions and collect some facts from equivariant homotopy theory. We will mostly deal with stable equivariant homotopy theory and use only basic concepts concerning G-spaces. For former the classic references are [LMS86], [MM02] and [HHR16, Appendix A and B] (but see also the somewhat more accessible lecture notes [Sch20]), for the latter [tom87] is an excellent reference. We denote the ∞-category of G-spaces by $\mathbf{Spc}^G$ and pointed G-spaces by $\mathbf{Spc}^G_*$. As a model for G-equivariant spectra we use orthogonal G-spectra as described in [MM02] with the stable model structure. Here we allow G to be a compact Lie group. We denote the category of orthogonal G-spectra by $\mathbf{OSp}^G$ and the resulting closed symmetric monoidal stable ∞-category by $\mathbf{Sp}^G$. Just as in the non-equivariant setting there is a suspension spectrum functor

$$\Sigma^\infty_G : \mathbf{Spc}^G_* \to \mathbf{Sp}^G$$

and a smash product, which we denote by $\otimes$. The internal hom is given by the function spectrum $F(-,-)$. For suspension spectra we often drop the subscript G if the group is clear from context. For convenience we will almost always write G/H_+ instead of $\Sigma^\infty G/H_+$.

If $\alpha\colon H \to G$ is a homomorphism (of Lie groups), then there exists an associated restriction functor

$$\alpha^*\colon \mathbf{Sp}^G \to \mathbf{Sp}^H. \tag{A.1}$$

Restriction functors are symmetric monoidal and have the property that if $\beta\colon G \to K$ is another group homomorphism, then

$$\alpha^*\beta^* \simeq (\beta\alpha)^{*1}. \tag{A.2}$$

[1] This follows from the fact that the restriction functors can be modeled by Quillen functors and on the level of Quillen functors this is an equality.

Furthermore, we have that the following diagrams commute:

$$\begin{array}{ccc} \mathbf{Spc}_*^G & \xrightarrow{\alpha^*} & \mathbf{Spc}_*^H \\ \downarrow{\scriptstyle \Sigma^\infty} & & \downarrow{\scriptstyle \Sigma^\infty} \\ \mathbf{Sp}^G & \xrightarrow{\alpha^*} & \mathbf{Sp}^H \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{Sp}_*^G & \xrightarrow{\alpha^*} & \mathbf{Sp}_*^H \\ \downarrow{\scriptstyle \Omega^\infty} & & \downarrow{\scriptstyle \Omega^\infty} \\ \mathbf{Spc}^G & \xrightarrow{\alpha^*} & \mathbf{Spc}^H. \end{array} \tag{A.3}$$

If H is a closed subgroup of G and $\alpha = i_H^G$ is the inclusion we write res_H^G instead of α^*. We use (A.1) to define $\mathbf{Sp}^{D_{2p^\infty}} = \lim_n \mathbf{Sp}^{D_{2p^n}}$, where the limit is taken in $\mathbf{CAlg}(\mathbf{Pr}^{L,\mathrm{st}})$, the ∞-category of symmetric monoidal presentable stable ∞-categories, and the structure maps are the restriction functors $\mathbf{Sp}^{D_{2p^{n+1}}} \to \mathbf{Sp}^{D_{2p^n}}$ along the inclusion $D_{2p^n} \subset D_{2p^{n+1}}$. Note that since in $\mathbf{Pr}^{L,\mathrm{st}}$ the morphisms are colimit preserving functors, the restriction functors $\mathbf{Sp}^{D_{2p^\infty}} \to \mathbf{Sp}^{D_{2p^n}}$ preserve colimits for all n.

From now on we will assume that G is finite throughout this appendix. In this case the restriction functor (A.1) preserves all limits and colimits and has a left adjoint

$$\alpha_! \colon \mathbf{Sp}^H \to \mathbf{Sp}^G$$

called *induction* with unit and counit

$$\eta_\alpha^! \colon \mathrm{id} \to \alpha^*\alpha_! \quad \text{and} \quad \epsilon_\alpha^! \colon \alpha_!\alpha^* \to \mathrm{id}$$

as well as a right adjoint

$$\alpha_* \colon \mathbf{Sp}^H \to \mathbf{Sp}^G$$

called *coinduction* with unit and counit

$$\eta_\alpha \colon \mathrm{id} \to \alpha_*\alpha^* \quad \text{and} \quad \epsilon_\alpha \colon \alpha^*\alpha_* \to \mathrm{id}.$$

Note that as the right adjoint to a monoidal functor α_* is lax monoidal. If $\alpha = i_H^G$ is the inclusion of a subgroup, then we write ind_H^G instead of $\alpha_!$ and coind_H^G instead of α_*. We can also define fixed points using this formalism. For any subgroup H of G we denote by $p_H^G \colon N_G H \to W_G H$ the projection. We shall also write $p_G \colon G \to 1$ instead of p_G^G to avoid cumbersome notation. Now if N is a normal subgroup we let

$$(-)^N := (p_N^G)_* \colon \mathbf{Sp}^G \to \mathbf{Sp}^{G/N}$$

and call it the N-fixed point functor[2]. In the case of subgroup H which is not normal, we let $(-)^H \colon \mathbf{Sp}^G \to \mathbf{Sp}^{W_G H}$ denote the composite

$$\mathbf{Sp}^G \xrightarrow{\mathrm{res}_{N_G H}^G} \mathbf{Sp}^{N_G H} \xrightarrow{(p_H^G)_*} \mathbf{Sp}^{W_G H}.$$

[2]In the literature these are sometimes called *categorical* fixed points.

The fixed point functor has the property that for a subgroup K containing N

$$\mathrm{res}^{G/N}_{K/N} X^N \simeq (p^K_N)_*(\mathrm{res}^G_K X). \tag{A.4}$$

We will prove this in Lemma A.1.3. For a non-normal subgroup H we then obtain

$$\mathrm{res}^{W_G H}_{K/H} X^H = \mathrm{res}^{W_G H}_{K/H} (p^G_H)_*(\mathrm{res}^G_{N_G H} X) \simeq (p^K_H)_* \mathrm{res}^{N_G H}_H \mathrm{res}^G_{N_G H} X \simeq (p^K_H)_*(\mathrm{res}^G_K X). \tag{A.5}$$

For this reason we use the notation X^H for both the $W_G H$-spectrum and the non-equivariant spectrum and it will be clear from the context which of both is meant.

The following lemma is useful for the proof of Lemma A.1.3 as well as other statements.

Lemma A.1.1. *Consider a square of groups*

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & H \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\ K & \xrightarrow{\delta} & L. \end{array}$$

(i) Let

$$\delta^* \beta_* \to \gamma_* \alpha^* \tag{A.6}$$

be the right adjunct of

$$\gamma^* \delta^* \beta_* \simeq \alpha^* \beta^* \beta_* \xrightarrow{\alpha^* \epsilon_\beta} \alpha^*$$

and

$$\alpha_! \gamma^* \to \beta^* \delta_! \tag{A.7}$$

the left adjunct of

$$\gamma^* \xrightarrow{\gamma^* \eta^!_\delta} \gamma^* \delta^* \delta_! \simeq \alpha^* \beta^* \delta_!.$$

Then (A.6) and (A.7) are dual to each other and hence that one is an equivalence iff the other is an equivalence. Similarly,

$$\beta^* \delta_* \to \alpha_* \gamma^* \tag{A.8}$$

is dual to

$$\gamma_! \alpha^* \to \delta^* \beta_!. \tag{A.9}$$

(ii) The map (A.6) is homotopic to

$$\delta^* \beta_* \xrightarrow{\delta^* \beta_* \eta_\alpha} \delta^* \beta_* \alpha_* \alpha^* \simeq \delta^* \delta_* \gamma_* \alpha^* \xrightarrow{\epsilon_\delta} \gamma_* \alpha^*. \tag{A.10}$$

Proof. We start with (i). Going through the left column of the next diagram is by definition the dual of (A.6):

$$\begin{tikzcd}
\alpha_!\gamma^* \arrow[r, "\alpha_!\gamma^*\eta^!_\delta"] & \alpha_!\gamma^*\delta^*\delta_! \arrow[r, "\alpha_!\gamma^*\delta^*\eta_\beta"] & \alpha_!\gamma^*\delta^*\beta_*\beta^*\delta_! \arrow[d, "\alpha_!\gamma^*\eta_\gamma"] \arrow[dr, "\mathrm{id}"] & & \\
& & \alpha_!\gamma^*\gamma_*\gamma^*\delta^*\beta_*\beta^*\delta_! \arrow[r, "\alpha_!\epsilon_\gamma"] \arrow[d, "\simeq"] & \alpha_!\gamma^*\delta^*\beta_*\beta^*\delta_! \arrow[d, "\simeq"] & \\
& & \alpha_!\gamma^*\gamma_*\alpha^*\beta^*\beta_*\beta^*\delta_! \arrow[r, "\alpha_!\epsilon_\gamma"] \arrow[d, "\alpha_!\gamma^*\gamma_*\alpha^*\epsilon_\beta"] & \alpha_!\alpha^*\beta^*\beta_*\beta^*\delta_! \arrow[d, "\alpha_!\alpha^*\epsilon_\beta"] & \\
& & \alpha_!\gamma^*\gamma_*\alpha^*\beta^*\delta_! \arrow[r, "\alpha_!\epsilon_\gamma"] & \alpha_!\alpha^*\beta^*\delta_! \arrow[r, "\epsilon^!_\alpha"] & \beta^*\delta_!.
\end{tikzcd}$$

Going through the right column of the previous diagram fits in the following diagram

$$\begin{tikzcd}
\alpha_!\gamma^* \arrow[d, "\alpha_!\gamma^*\eta^!_\delta"] & & & \\
\alpha_!\gamma^*\delta^*\delta_! \arrow[r, "\simeq"] \arrow[d, "\alpha_!\gamma^*\delta^*\eta_\beta"] & \alpha_!\alpha^*\beta^*\delta_! \arrow[d, "\alpha_!\alpha^*\beta^*\eta_\beta"] \arrow[dr, "\mathrm{id}"] & & \\
\alpha_!\gamma^*\delta^*\beta_*\beta^*\delta_! \arrow[r, "\simeq"] & \alpha_!\alpha^*\beta^*\beta_*\beta^*\delta_! \arrow[r, "\alpha_!\alpha^*\epsilon_\beta"] & \alpha_!\alpha^*\beta^*\delta_! \arrow[r, "\epsilon^!_\alpha"] & \beta^*\delta_!,
\end{tikzcd}$$

and going through the upper right corner of the square is equal to (A.7). This shows the first claim in (i) and the claim about (A.8) being dual to (A.9) follows by flipping the square of groups.

Now we show (ii). First we show that

$$\delta^*\beta_*\alpha_*\alpha^* \simeq \delta^*\delta_*\gamma_*\alpha^* \xrightarrow{\epsilon_\delta} \gamma_*\alpha^* \tag{A.11}$$

is homotopic to

$$\delta^*\beta_*\alpha_*\alpha^* \xrightarrow{\eta_\gamma} \gamma_*\gamma^*\delta^*\beta_*\alpha_*\alpha^* \simeq \gamma_*\alpha^*\beta^*\beta_*\alpha_*\alpha^* \xrightarrow{\gamma_*\alpha^*\epsilon_\beta} \gamma_*\alpha^*\alpha_*\alpha^* \xrightarrow{\gamma_*\epsilon_\alpha} \gamma_*\alpha^* \tag{A.12}$$

Note that the equivalence $\beta_*\alpha_* \simeq \delta_*\gamma_*$ is given by the composite

$$\beta_*\alpha_* \xrightarrow{\eta_\delta} \delta_*\delta^*\beta_*\alpha_* \xrightarrow{\delta_*\eta_\gamma} \delta_*\gamma_*\gamma^*\delta^*\beta_*\alpha_* \simeq \delta_*\gamma_*\alpha^*\beta^*\beta_*\alpha_* \xrightarrow{\delta_*\gamma_*\alpha^*\epsilon_\beta} \delta_*\gamma_*\alpha^*\alpha_* \xrightarrow{\delta_*\gamma_*\epsilon_\alpha} \delta_*\gamma_*.$$

We consider the following diagram

$$\begin{tikzcd}
\delta^*\beta_*\alpha_*\alpha^* \arrow[d, "\delta^*\eta_\delta"] & & & \\
\delta^*\delta_*\delta^*\beta_*\alpha_*\alpha^* \arrow[r, "\delta^*\delta_*\eta_\gamma"] \arrow[d, "\epsilon_\delta"] & \delta^*\delta_*\gamma_*\gamma^*\delta^*\beta_*\alpha_*\alpha^* \arrow[r, "\simeq"] \arrow[d, "\epsilon_\delta"] & \delta^*\delta_*\gamma_*\alpha^*\beta^*\beta_*\alpha_*\alpha^* \arrow[r, "\delta_*\delta^*\gamma_*\epsilon_{\beta\alpha}"] \arrow[d, "\epsilon_\delta"] & \delta^*\delta_*\gamma_*\alpha^* \arrow[d, "\epsilon_\delta"] \\
\delta^*\beta_*\alpha_*\alpha^* \arrow[r, "\eta_\gamma"] & \gamma_*\gamma^*\delta^*\beta_*\alpha_*\alpha^* \arrow[r, "\simeq"] & \gamma_*\alpha^*\beta^*\beta_*\alpha_*\alpha^* \arrow[r, "\gamma_*\epsilon_{\beta\alpha}"] & \gamma_*\alpha^*.
\end{tikzcd}$$

The lower row is (A.12) and going through the upper row is (A.11), hence the claim follows from the fact that by the rules for (co)unit maps the composition of the left two vertical arrows is the identity.

Now (ii) follows from what we have shown before and the following diagram

$$\begin{array}{ccccccccc}
\delta^*\beta_* & \xrightarrow{\eta_\gamma} & \gamma_*\gamma^*\delta^*\beta_* & \xrightarrow{\simeq} & \gamma_*\alpha^*\beta^*\beta_* & \xrightarrow{\gamma_*\alpha^*\epsilon_\beta} & \gamma_*\alpha^* & & \\
\downarrow{\scriptstyle \delta^*\beta_*\eta_\alpha} & & \downarrow{\scriptstyle \gamma_*\gamma^*\delta^*\beta_*\eta_\alpha} & & \downarrow{\scriptstyle \gamma_*\alpha^*\beta^*\beta_*\eta_\alpha} & & \downarrow{\scriptstyle \gamma_*\alpha^*\eta_\alpha} & \searrow^{\mathrm{id}} & \\
\delta^*\beta_*\alpha_*\alpha^* & \xrightarrow{\eta_\gamma} & \gamma_*\gamma^*\delta^*\beta_*\alpha_*\alpha^* & \xrightarrow{\simeq} & \gamma_*\alpha^*\beta^*\beta_*\alpha_*\alpha^* & \xrightarrow{\gamma_*\alpha^*\epsilon_\beta} & \gamma_*\alpha^*\alpha_*\alpha^* & \xrightarrow{\gamma_*\epsilon_\alpha} & \gamma_*\alpha^*,
\end{array}$$

since the upper row is by definition (A.6) and the lower row is (A.12). □

The following argument involving localizing subcategories is standard. We will record it in the following remark, since we will often use it.

Remark A.1.2 (The localizing subcategory argument)**.** Recall that if $\mathcal{C}$ is a presentable stable ∞-category, a full subcategory is called localizing if it is a stable subcategory which is closed under colimits. A set $\{X_\alpha\}_{\alpha\in A}$ of objects is called a generating set if the smallest localizing subcategory containing $\{X_\alpha\}_{\alpha\in A}$ is $\mathcal{C}$ itself. This is useful for the following reason. Suppose that $\mathcal{D}$ is an ∞-category, $F, F'\colon \mathcal{C}\to\mathcal{D}$ are two colimit preserving functors and $\varphi\colon F\Rightarrow F'$ is a natural transformation. Then the full subcategory spanned by objects X such that φ_X is an equivalence is localizing, therefore φ is an equivalence iff φ_{X_α} is an equivalence for all $\alpha\in A$.

There is the following useful criterion to check if a set is generating: By [HPS97, Theorem 2.3.3] the set $\{X_\alpha\}_{\alpha\in A}$ is generating iff $\mathrm{map}(X_\alpha, Y)\simeq *$ for all $\alpha\in A$ implies $Y\simeq 0$ for all objects Y[3]. In the case of Sp^G we see that $\{G/H_+\}$, where H ranges over all subgroups, is generating[4]. In fact, the generators are all compact objects.

Now we have all the necessary ingredients to show (A.4). We shall also show a compatibility between restriction along a surjective homomorphism and coinduction from a subgroup.

Lemma A.1.3. *Let N be a normal subgroup of G and K a subgroup containing N. There are natural equivalences*

$$\mathrm{res}^{G/N}_{K/N}(p^G_N)_*\simeq (p^K_N)_*\mathrm{res}^G_K, \tag{A.13}$$

$$(p^G_N)^*\mathrm{coind}^{G/N}_{K/N}\simeq \mathrm{coind}^G_K(p^K_N)^*, \tag{A.14}$$

which are both induced by (A.6).

[3]Strictly speaking the cited reference only shows one implication, but the converse is easy. For any object Y the functor $\mathrm{map}(-,Y)$ preserves colimits, therefore the subcategory of objects X such that $\mathrm{map}(X,Y)\simeq *$ is localizing, hence by the assumption is $\mathcal{C}$ itself. It follows that $\mathrm{map}(Y,Y)\simeq *$.

[4]This is simply another way of saying that a G-spectrum X is contractible iff $\pi^H_* X = 0$ for all subgroups H. Note that $\pi^H_q(X)\cong \pi_0\mathrm{map}_{\mathbf{Sp}^G}(S^q\otimes G/H_+, X)$ (in fact some authors take this as a definition of the homotopy groups).

Proof. By Lemma A.1.1 (i) it suffices to show that the following maps are equivalences:

$$\mathrm{ind}_K^G (p_N^K)^* \xrightarrow{(A.7)} (p_N^G)^* \mathrm{ind}_{K/N}^{G/N}, \tag{A.15}$$

$$(p_N^K)_! \mathrm{res}_K^G \xrightarrow{(A.7)} \mathrm{res}_{K/N}^{G/N} (p_N^G)_!. \tag{A.16}$$

By Remark A.1.2 and (A.3) it suffices to show that

$$\mathrm{ind}_K^G (p_N^K)^* ((K/N)/(M/N))_+ \to (p_N^G)^* \mathrm{ind}_{K/N}^{G/N} ((K/N)/(M/N))_+$$

is an equivalence of G-spaces and

$$(p_N^K)_! \mathrm{res}_K^G G/M_+ \to \mathrm{res}_{K/N}^{G/N} (p_N^G)_! G/M_+$$

is an equivalence of K/N-spaces, which is easily checked. □

We will make frequent use of the following lemma. It is well-known, but we include a proof for completeness, since we did not find a proof in the literature.

Lemma A.1.4. *The fixed point functor* $(-)^H \colon \mathbf{Sp}^G \to \mathbf{Sp}^{W_G H}$ *commutes with colimits.*

Proof. Since the restriction functor commutes with colimits, we can prove the statement for a normal subgroup N. Suppose $X = \mathrm{colim}_I X_i$. We use Remark A.1.2 and the ∞-categorical Yoneda Lemma [Lur09, Proposition 5.1.3.1] to reduce the claim to showing that

$$\mathrm{map}_{\mathbf{Sp}^{G/N}}((G/N)/(K/N)_+, \mathrm{colim}_I X_i^N) \simeq \mathrm{map}_{\mathbf{Sp}^{G/N}}((G/N)/(K/N)_+, X^N).$$

Let $\alpha \colon G \to G/N$ be the projection. Using that $(G/N)/(K/N)_+$ is compact we compute

$$\begin{aligned}
\mathrm{map}_{\mathbf{Sp}^{G/N}}((G/N)/(K/N)_+, \mathrm{colim}_I X_i^N) &\simeq \mathrm{colim}_I \mathrm{map}_{\mathbf{Sp}^{G/N}}((G/N)/(K/N)_+, X_i^N) \\
&\simeq \mathrm{colim}_I \mathrm{map}_{\mathbf{Sp}^{G}}(\alpha^*(G/N)/(K/N)_+, X_i) \\
&\simeq \mathrm{map}_{\mathbf{Sp}^{G}}(\alpha^*(G/N)/(K/N)_+, X) \\
&\simeq \mathrm{map}_{\mathbf{Sp}^{G/N}}((G/N)/(K/N)_+, X^N),
\end{aligned}$$

where we used for the third equivalence that $\alpha^*(G/N)/(K/N)_+ \simeq G/K_+$ is compact as well. □

The fixed point functors also have the property that they detect equivalences of G-spectra.

Lemma A.1.5. *A map* $f \colon X \to Y$ *of* G*-spectra is an equivalence iff* f^H *is an equivalence of spectra for all subgroups* H.

Proof. By passing to the cofiber it suffices to show that X is a contractible G-spectrum iff X^H is a contractible spectrum for all subgroups H. The collection of spectra Y such that $\mathrm{map}_{\mathbf{Sp}^G}(Y, X)$ is contractible is a localizing subcategory by the ∞-categorical Yoneda lemma

[Lur09, Proposition 5.1.3.1], since the smash product commutes with colimits. Thus, by Remark A.1.2 it suffices to show that $\mathrm{map}_{\mathbf{Sp}^G}(G/H_+, X)$ is contractible for any subgroup H. We have

$$\begin{aligned}\mathrm{map}_{\mathbf{Sp}^G}(G/H_+, X) &\simeq \mathrm{map}_{\mathbf{Sp}^G}(\mathrm{ind}_H^G S^0, X)\\ &\simeq \mathrm{map}_{\mathbf{Sp}^H}(S^0, \mathrm{res}_H^G X)\\ &\simeq \mathrm{map}_{\mathbf{Sp}^H}((p_H)^* S^0, \mathrm{res}_H^G X)\\ &\simeq \mathrm{map}_{\mathbf{Sp}}(S^0, (\mathrm{res}_H^G X)^H).\end{aligned}$$

By Lemma A.1.3 $(\mathrm{res}_H^G X)^H$ is equivalent to the underlying spectrum of X^H, hence contractible, yielding the claim. □

Lemma A.1.6 (Wirthmüller isomorphism)**.** *For any subgroup H there is a natural equivalence $\mathrm{ind}_H^G \simeq \mathrm{coind}_H^G$.*

Proof. See [Sch20, Theorem 4.9]. □

Corollary A.1.7. *For any subgroup H and any H-spectrum X there are natural equivalences $(\mathrm{ind}_H^G X)^G \simeq X^H$ and $(\mathrm{coind}_H^G X)^G \simeq X^H$.*

Proof. For coinduction this follows directly from (A.2) and the fact that adjoint functors compose. The statement for induction is implied by the statement for coinduction via the Wirthmüller isomorphism. □

For any group homomorphism α the adjunction between restriction and (co)induction gives rise to so called *projection morphisms*

$$\alpha_!(X \otimes \alpha^* Y) \to \alpha_! X \otimes Y, \tag{A.17}$$

$$\alpha_* X \otimes Y \to \alpha_*(X \otimes \alpha^* Y), \tag{A.18}$$

which are the left and right adjuncts of the map $X \otimes \alpha^* Y \to \alpha^* \alpha_! X \otimes \alpha^* Y$ obtained by smashing $\eta_\alpha^!$ with $\alpha^* Y$ respectively of the map $\alpha^* \alpha_* X \otimes \alpha^* Y \to X \otimes \alpha^* Y$ obtained by smashing ϵ_α with $\alpha^* Y$. We shall frequently use the following result in the main text.

Lemma A.1.8 (Projection formula)**.** *The projection morphisms have the following properties.*

(i) Consider a diagram of groups

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & H \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\ K & \xrightarrow{\delta} & L, \end{array}$$

let X be an H-spectrum and Y a K-spectrum. Consider the map

$$\begin{aligned}\beta_* X \otimes \delta_* Y \to \beta_*(X \otimes \beta^* \delta_* Y) \to \beta_*(X \otimes \alpha_* \gamma^* Y)\\ \to \beta_* \alpha_*(\alpha^* X \otimes \gamma^* Y) \simeq \delta_* \gamma_*(\alpha^* X \otimes \gamma^* Y),\end{aligned} \tag{A.19}$$

where the second arrow is obtained by smashing X with (A.8) *and the first and third arrow are (β_* applied to) the projection morphisms* (A.18). *Consider also the map*

$$\beta_* X \otimes \delta_* Y \to \delta_*(\delta^* \beta_* X \otimes Y) \to \delta_*(\gamma_* \alpha^* X \otimes Y) \to \delta_* \gamma_*(\alpha^* X \otimes \gamma^* Y), \tag{A.20}$$

where the second arrow is obtained by smashing Y with (A.6) *and the first and third arrow are (δ_* applied to) the projection morphisms* (A.18). *Then* (A.19) *and* (A.20) *are homotopic.*

(ii) If α is the inclusion of a subgroup, both projection morphisms are equivalences, that is

$$\operatorname{ind}_H^G(X \otimes \operatorname{res}_H^G Y) \simeq \operatorname{ind}_H^G X \otimes Y, \tag{A.21}$$

$$\operatorname{coind}_H^G(X \otimes \operatorname{res}_H^G Y) \simeq \operatorname{coind}_H^G X \otimes Y. \tag{A.22}$$

The projection morphism (A.18) *is an equivalence if $\alpha = p_N^G$ is the projection, that is*

$$X^N \otimes Y \simeq (X \otimes (p_N^G)^* Y)^N. \tag{A.23}$$

Proof. We start with (i). This follows from the fact that both (A.19) and (A.20) are the right adjuncts of the composite

$$\gamma^* \delta^* (\beta_* X \otimes \delta_* Y) \simeq \gamma^* \delta^* \beta_* X \otimes \gamma^* \delta^* \delta_* Y \simeq \alpha^* \beta^* \beta_* X \otimes \gamma^* \delta^* \delta_* Y \xrightarrow{\alpha^* \epsilon_\beta \otimes \gamma^* \epsilon_\delta} \alpha^* X \otimes \gamma^* Y. \tag{A.24}$$

Here we used the following facts:

(1) The equivalence $\alpha^* \beta^* \simeq \gamma^* \delta^*$ is monoidal.

(2) The smash product is functorial in both variables.

(3) If $\hat{\varphi} \colon L_1 F X \to X$ is the left adjunct of $\varphi \colon FX \to R_1 X$ and $\hat{\psi} \colon L_2 X \to X$ is the left adjunct of $\psi \colon X \to R_2 X$, then

$$L_2 L_1 F X \xrightarrow{L_2 \hat{\varphi}} L_2 X \xrightarrow{\hat{\psi}} X$$

is the left adjunct of

$$FX \xrightarrow{\varphi} R_1 X \xrightarrow{R_1 \psi} R_1 R_2 X.$$

One immediately verifies this using naturality of (co)unit maps and the fact that

$$\mathrm{id} \xrightarrow{\eta_2} R_2 L_2 \xrightarrow{R_2 \eta_1} R_2 R_1 L_1 L_2$$

is the unit of the adjunction $L_1 L_2 \dashv R_2 R_1$. Here η_i denotes the unit of the adjunction $L_i \dashv R_i$.

Now we prove (ii). For the statement about induction and coinduction see [MNN17, Proposition 5.14]. Alternatively, one uses that all functors in question can be modeled on the point-set level by Quillen functors and that for these Quillen functors the projection morphisms are isomorphisms.

To show the equivalence $X^N \otimes Y \simeq (X \otimes (p_N^G)^* Y)^N$, note that both sides commute with colimits in both variables. By Remark A.1.2 it suffices to show that (A.18) induces an equivalence

$$X^N \otimes (G/N)/(K/N)_+ \simeq (X \otimes (p_N^G)^*(G/N)/(K/N)_+)^N.$$

By (A.3) and Lemma A.1.6 we have

$$(G/N)/(K/N)_+ \simeq \mathrm{ind}_{K/N}^{G/N} S^0 \simeq \mathrm{coind}_{K/N}^{G/N} S^0.$$

We consider the diagram

$$\begin{array}{ccc}
(p_N^G)_* X \otimes \mathrm{coind}_{K/N}^{G/N} S^0 & \longrightarrow & (p_N^G)_*(X \otimes (p_N^G)^* \mathrm{coind}_{K/N}^{G/N} S^0) \\
{\scriptstyle (A.22)}\downarrow \simeq & & {\scriptstyle (A.8)}\downarrow \simeq \\
\mathrm{coind}_{K/N}^{G/N}(\mathrm{res}_{K/N}^{G/N}(p_N^G)_* X \otimes S^0) & & (p_N^G)_*(X \otimes \mathrm{coind}_K^G (p_N^K)^* S^0) \\
{\scriptstyle (A.6)}\downarrow \simeq & & \\
\mathrm{coind}_{K/N}^{G/N}((p_N^K)_*(\mathrm{res}_K^G X) \otimes S^0) & & {\scriptstyle (A.22)}\downarrow \simeq \\
\downarrow & & \\
\mathrm{coind}_{K/N}^{G/N}(p_N^K)_*(\mathrm{res}_K^G X \otimes (p_N^K)^* S^0) & \xrightarrow{\simeq} & (p_N^G)_* \mathrm{coind}_K^G(\mathrm{res}_K^G X \otimes (p_N^K)^* S^0),
\end{array}$$

where we want to show that the upper horizontal arrow is an equivalence, the lower horizontal arrow is the equivalence $\mathrm{coind}_{K/N}^{G/N}(p_N^K)_* \simeq (p_N^G)_* \mathrm{coind}_K^G$ and the lower left vertical arrow is $\mathrm{coind}_{K/N}^{G/N}$ applied to the projection morphism

$$(p_N^K)_*(\mathrm{res}_K^G X) \otimes S^0 \to (p_N^K)_*(\mathrm{res}_K^G X \otimes (p_N^K)^* S^0). \tag{A.25}$$

The diagram above commutes by (i) applied to the square

$$\begin{array}{ccc}
K & \xrightarrow{i_K^G} & G \\
\downarrow{\scriptstyle p_N^K} & & \downarrow{\scriptstyle p_N^G} \\
K/N & \xrightarrow{i_{K/N}^{G/N}} & G/N.
\end{array}$$

Additionally we used Lemma A.1.3 and the projection formula for coinduction (A.22) to see that the four vertical arrows are equivalences. Thus, it suffices to show that (A.25) is an equivalence. But by monoidality and the rules for (co)unit maps (A.25) is homotopic to the equivalence

$$(p_N^K)_*(\mathrm{res}_K^G X) \otimes S^0 \simeq (p_N^K)_*(\mathrm{res}_K^G X) \simeq (p_N^K)_*(\mathrm{res}_K^G X \otimes S^0) \simeq (p_N^K)_*(\mathrm{res}_K^G X \otimes (p_N^K)^* S^0),$$

where the first two equivalences are the fact that S^0 is the unit and the third equivalence is monoidality of $(p_N^K)^*$. $\square$

Lemma A.1.9. *Let $\alpha\colon H \to G$ be a group homomorphism. For any H-spectrum Y and any G-spectrum X there is a natural equivalence of G-spectra*

$$F(X, \alpha_* Y) \simeq \alpha_* F(\alpha^* X, Y).$$

If H is a subgroup of G then there is a natural equivalence of G-spectra

$$F(\mathrm{ind}_H^G Y, X) \simeq \mathrm{coind}_H^G F(Y, \mathrm{res}_H^G X).$$

Proof. Let Z be any G-spectrum. Since α^* is monoidal and $F(-,-)$ denotes the internal hom of G- and H-spectra, we obtain natural equivalences

$$\begin{aligned}
\mathrm{map}_{\mathbf{Sp}^G}(Z, F(X, \alpha_* Y)) &\simeq \mathrm{map}_{\mathbf{Sp}^G}(Z \otimes X, \alpha_* Y) \\
&\simeq \mathrm{map}_{\mathbf{Sp}^H}(\alpha^*(Z \otimes X), Y) \\
&\simeq \mathrm{map}_{\mathbf{Sp}^H}(\alpha^* Z \otimes \alpha^* X, Y) \\
&\simeq \mathrm{map}_{\mathbf{Sp}^H}(\alpha^* Z, F(\alpha^* X, Y)) \\
&\simeq \mathrm{map}_{\mathbf{Sp}^G}(Z, \alpha_* F(\alpha^* X, Y)).
\end{aligned}$$

The first equivalence of the lemma is thus a consequence of the ∞-categorical Yoneda Lemma [Lur09, Proposition 5.1.3.1]. Similarly, using the various adjunctions in conjunction with the projection formula (A.21), we obtain natural equivalences

$$\begin{aligned}
\mathrm{map}_{\mathbf{Sp}^G}(Z, F(\mathrm{ind}_H^G Y, X)) &\simeq \mathrm{map}_{\mathbf{Sp}^G}(Z \otimes \mathrm{ind}_H^G Y, X) \\
&\simeq \mathrm{map}_{\mathbf{Sp}^G}(\mathrm{ind}_H^G(\mathrm{res}_H^G Z \otimes Y), X) \\
&\simeq \mathrm{map}_{\mathbf{Sp}^H}(\mathrm{res}_H^G Z \otimes Y, \mathrm{res}_H^G X) \\
&\simeq \mathrm{map}_{\mathbf{Sp}^H}(\mathrm{res}_H^G Z, F(Y, \mathrm{res}_H^G X)) \\
&\simeq \mathrm{map}_{\mathbf{Sp}^G}(Z, \mathrm{coind}_H^G F(Y, \mathrm{res}_H^G X)),
\end{aligned}$$

and the second equivalence of the lemma follows again from the ∞-categorical Yoneda Lemma. □

A.2 Families of subgroups

Recall that we assume G to be finite. A family of subgroups $\mathcal{F}$ of G is a non-empty collection of subgroups such that if H is contained in $\mathcal{F}$, so is any subgroup K that is subconjugate to H. The most relevant examples for us are the following.

Example A.2.1. (i) Let N be a normal subgroup. Then we denote by $\mathcal{F}_G[N]$ the N-free family, i.e. all subgroups H such that $N \cap H = 1$. We sometimes drop the subscript if the ambient group G is clear from context.

(ii) If N is again a normal subgroup, we let $\mathcal{F}_{\not\supset N}$ be the family of subgroups H such that N is not contained in H. For $N = G$ this coincides with the family of proper subgroups $\mathcal{P}$.

(iii) Let H be any subgroup. We denote by $\mathcal{F}_{\leqslant H}$ the family of subgroups that are conjugate to a subgroup of H.

(iv) Let $\alpha\colon H \to G$ be a homomorphism and $\mathcal{F}$ a family of subgroups of G. Then we let $\alpha^*\mathcal{F}$ be the H-family of subgroups K such that $\alpha(K) \in \mathcal{F}$. As usual, if α is the inclusion of a subgroup we write $\mathrm{res}^G_H\mathcal{F}$.

(v) The intersection of two families of subgroups is again a family of subgroups.

To any family $\mathcal{F}$ we can associate a universal G-space $E\mathcal{F}$[5], which is uniquely determined up to equivalence by the property that

$$E\mathcal{F}^H \simeq \begin{cases} \emptyset & \text{if } H \notin \mathcal{F}, \\ * & \text{if } H \in \mathcal{F}. \end{cases}$$

The uniqueness follows from the fact that by [Lüc05, Section 1.2] $E\mathcal{F}$ can equivalently be characterized by the following universal property: For any G-space X such that $X^H = \emptyset$ if $H \notin \mathcal{F}$ there is a unique (up to homotopy) map of G-spaces $X \to E\mathcal{F}$. In particular if $\mathcal{F} \subset \mathcal{G}$ there is a unique map

$$E\mathcal{F} \to E\mathcal{G}. \tag{A.26}$$

There are several ways to realize $E\mathcal{F}$ as a G-CW-complex, see for example [MNN19, Proposition 2.17] for a concrete model. In the case that $\mathcal{F} = \{1\}$ is the trivial family we write EG instead of $E\mathcal{F}$. Note that if $\alpha\colon H \to G$ is a homomorphism, then by uniqueness

$$\alpha^* E\mathcal{F} \simeq E\alpha^*\mathcal{F}, \tag{A.27}$$

since $(\alpha^* E\mathcal{F})^K = (E\mathcal{F})^{\alpha(K)}$. Finally, again by uniqueness we have

$$E\mathcal{F} \times E\mathcal{G} \simeq E(\mathcal{F} \cap \mathcal{G}), \tag{A.28}$$

which is easily checked on fixed points.

Next, we define a pointed G-space $\widetilde{E\mathcal{F}}$ via the cofiber sequence

$$E\mathcal{F}_+ \to S^0 \to \widetilde{E\mathcal{F}}$$

of pointed G-spaces, where the first map collapses $E\mathcal{F}$ to the non-basepoint of S^0. We then have that

$$\widetilde{E\mathcal{F}}^H \simeq \begin{cases} S^0 & \text{if } H \notin \mathcal{F}, \\ * & \text{if } H \in \mathcal{F}. \end{cases}$$

[5] Actually one usually requires that $E\mathcal{F}$ is a G-CW complex, but since we normally work directly in the ∞-category of G-spaces we just say G-space.

Since α^* preserves cofiber sequences, we also have

$$\alpha^* \widetilde{E\mathcal{F}} \simeq \widetilde{E\alpha^*\mathcal{F}}, \tag{A.29}$$

and if $\mathcal{F} \subset \mathcal{G}$, the map (A.26) induces a canonical map

$$\widetilde{E\mathcal{F}} \to \widetilde{E\mathcal{G}} \tag{A.30}$$

by passing to cofibers. We will frequently use this map in our study of the parametrized homotopy orbits and fixed points.

Finally, we point out that

$$\widetilde{E\mathcal{F}} \otimes \widetilde{E\mathcal{G}} \simeq \widetilde{E(\mathcal{F} \cup \mathcal{G})}. \tag{A.31}$$

To see why, note that the total cofiber of the square

$$\begin{array}{ccc} E\mathcal{F}_+ \otimes E\mathcal{G}_+ & \longrightarrow & E\mathcal{G}_+ \\ \downarrow & & \downarrow \\ E\mathcal{F}_+ & \longrightarrow & S^0, \end{array}$$

where all arrows are the collapse maps, is $\widetilde{E\mathcal{F}} \otimes \widetilde{E\mathcal{G}}$. It is easy to check on fixed points that the square of G-spaces

$$\begin{array}{ccc} E\mathcal{F}_+ \otimes E\mathcal{G}_+ & \longrightarrow & E\mathcal{G}_+ \\ \downarrow & & \downarrow \\ E\mathcal{F}_+ & \longrightarrow & E(\mathcal{F} \cup \mathcal{G})_+ \end{array}$$

is a pushout, therefore we obtain a cofiber sequence $E(\mathcal{F} \cup \mathcal{G})_+ \to S^0 \to \widetilde{E\mathcal{F}} \otimes \widetilde{E\mathcal{G}}$, thus the claimed equivalence. In particular $\widetilde{E\mathcal{F}}$ is an idempotent pointed G-space.

The following terminology is standard in the literature. We shall frequently refer to it in the context of parametrized homotopy orbits and fixed points.

Definition A.2.2. Let X be a G-spectrum and $\mathcal{F}$ a family of subgroups.

(i) X is $\mathcal{F}$-torsion if $X \otimes E\mathcal{F}_+ \xrightarrow{\sim} X$ via the map that collapses $E\mathcal{F}$ to a point. If $\mathcal{F}$ is the trivial family, then we use the terminology Borel torsion instead.

(ii) X is $\mathcal{F}$-complete if $X \xrightarrow{\sim} F(E\mathcal{F}_+, X)$ via the map that collapses $E\mathcal{F}$ to a point. We use the term Borel complete for G-spectra which are complete with respect to the trivial family.

Remark A.2.3. It follows from $E\mathcal{F} \times E\mathcal{F} \simeq E\mathcal{F}$ that $X \otimes E\mathcal{F}_+$ is $\mathcal{F}$-torsion. Similarly, $F(E\mathcal{F}_+, X)$ is $\mathcal{F}$-complete and we refer to it as the $\mathcal{F}$-completion of X.

The homotopy type of $\mathcal{F}$-torsion and $\mathcal{F}$-complete G-spectra depends only on the groups contained in $\mathcal{F}$. The following definition and Lemma A.2.5 make this precise.

Definition A.2.4. Let $f\colon X \to Y$ be a map of G-spectra and $\mathcal{F}$ a family of subgroups of G. We say f is an $\mathcal{F}$-equivalence if $\mathrm{res}^G_H f$ is an equivalence of H-spectra for all $H \in \mathcal{F}$. We write $X \simeq_{\mathcal{F}} Y$ if there is an $\mathcal{F}$-equivalence between X and Y.

Note that for any G-spectrum X the maps $X \otimes E\mathcal{F}_+ \to X$ and $X \to F(E\mathcal{F}_+, X)$ are $\mathcal{F}$-equivalences. Thus, if $f\colon X \to Y$ is an $\mathcal{F}$-equivalence, so are the induced maps $X \otimes E\mathcal{F}_+ \to Y \otimes E\mathcal{F}_+$ and $F(E\mathcal{F}_+, X) \to F(E\mathcal{F}_+, Y)$. The following lemma shows why this is useful.

Lemma A.2.5. *Let $\mathcal{F}$ be a family of subgroups.*

(i) Suppose X and Y are $\mathcal{F}$-torsion spectra and $f\colon X \to Y$ is an $\mathcal{F}$-equivalence. Then f is an equivalence.

(ii) Suppose X and Y are $\mathcal{F}$-complete spectra and $f\colon X \to Y$ is an $\mathcal{F}$-equivalence. Then f is an equivalence.

(iii) For any G-spectrum X we have an equivalence $F(E\mathcal{F}_+, X) \otimes E\mathcal{F}_+ \simeq X \otimes E\mathcal{F}_+$.

Proof. (i) Clearly the cofiber of a map between $\mathcal{F}$-torsion spectra is itself $\mathcal{F}$-torsion. Therefore, it suffices to show that if X is $\mathcal{F}$-torsion and $\mathcal{F}$-equivalent to the trivial G-spectrum, it is contractible as a G-spectrum. We choose a G-CW-structure on $E\mathcal{F}$ with n-skeleton $\mathrm{sk}_n E\mathcal{F}$. Then

$$X \simeq X \otimes E\mathcal{F}_+ \simeq \mathrm{colim}_n (X \otimes \mathrm{sk}_n E\mathcal{F}_+).$$

Note that the n-cells of $E\mathcal{F}$ have the form $D^n \times G/H$ with $H \in \mathcal{F}$, therefore it suffices to show that

$$X \otimes G/H_+ \simeq X \otimes \mathrm{ind}^G_H S^0 \simeq \mathrm{ind}^G_H \mathrm{res}^G_H X$$

is a contractible G-spectrum for all $H \in \mathcal{F}$, where we used the projection formula (A.21) and (A.3). But by assumption $\mathrm{res}^G_H X$ is a contractible H-spectrum, so we are done.

(ii) We show that for any $\mathcal{F}$-equivalence $f\colon X \to Y$ between G-spectra, the map

$$f_*\colon F(E\mathcal{F}_+, X) \to F(E\mathcal{F}_+, Y)$$

is an equivalence on fixed points. We first do this for the G-fixed points. Choose a G-CW-structure on $E\mathcal{F}$ with n-skeleton $\mathrm{sk}_n E\mathcal{F}$. Then

$$F(E\mathcal{F}_+, X)^G \simeq \lim_n F(\mathrm{sk}_n E\mathcal{F}_+, X)^G \quad \text{and} \quad F(E\mathcal{F}_+, Y)^G \simeq \lim_n F(\mathrm{sk}_n E\mathcal{F}_+, Y)^G.$$

Here we used that $(-)^G$ preserves limits as a right adjoint and by the ∞-categorical Yoneda Lemma [Lur09, Proposition 5.1.3.1] $F(-,-)$ sends colimits in the first variable to limits, since the smash product preserves colimits.

Similarly to (i), using the above and exactness of function spectra in both variables it suffices to show $f_*\colon F(G/H_+, X)^G \to F(G/H_+, Y)^G$ is an equivalence. But

$$F(G/H_+, X)^G \simeq X^H \quad \text{and} \quad F(G/H_+, Y)^G \simeq Y^H$$

by Lemma A.1.9 and Corollary A.1.7, hence the claim follows. Now, if H is a proper subgroup, note that $\mathrm{res}^G_H f$ is a $\mathrm{res}^G_H \mathcal{F}$-equivalence and

$$\mathrm{res}^G_H F(E\mathcal{F}_+, X) \simeq F(\mathrm{res}^G_H E\mathcal{F}_+, \mathrm{res}^G_H X) \simeq F(E\mathrm{res}^G_H \mathcal{F}_+, \mathrm{res}^G_H X),$$

and similarly for Y, so the same argument as above shows that f induces an equivalence $F(E\mathcal{F}_+ X)^H \simeq F(E\mathcal{F}_+, Y)^H$.

(iii) By (i) it suffices to show that $X \to F(E\mathcal{F}_+, X)$ is an $\mathcal{F}$-equivalence, which is clear, since $\mathrm{res}^G_H F(E\mathcal{F}_+, X) \simeq F(\mathrm{res}^G_H E\mathcal{F}_+, \mathrm{res}^G_H X)$ and $\mathrm{res}^G_H E\mathcal{F} \simeq *$ if $H \in \mathcal{F}$.

□

We end this subsection with a discussion of the geometric fixed point functor.

Definition A.2.6. Let N be a normal subgroup of G. The geometric fixed point functor

$$\Phi^N\colon \mathbf{Sp}^G \to \mathbf{Sp}^{G/N}$$

is defined by $\Phi^N X = (X \otimes \widetilde{E\mathcal{F}}_{\not\supset N})^N$. If H is not normal we define Φ^H as the composite

$$\mathbf{Sp}^G \xrightarrow{\mathrm{res}^G_{N_G H}} \mathbf{Sp}^{N_G H} \xrightarrow{\Phi^H} \mathbf{Sp}^{W_G H}.$$

Note that for K a subgroup of $N_G H$ containing H it follows from (A.5) and monoidality of the restriction functor that

$$\begin{aligned}\mathrm{res}^{W_G H}_{K/H} \Phi^H X &= \mathrm{res}^{W_G H}_{K/H} ((\mathrm{res}^G_{N_G H} X) \otimes \widetilde{E\mathcal{F}}_{N_G H, \not\supset H})^H \\ &\simeq ((\mathrm{res}^G_K X) \otimes \mathrm{res}^{N_G H}_K \widetilde{E\mathcal{F}}_{N_G H, \not\supset H})^H \simeq ((\mathrm{res}^G_K X) \otimes \widetilde{E\mathcal{F}}_{K, \not\supset H})^H = \Phi^H \mathrm{res}^G_K X. \qquad \text{(A.32)}\end{aligned}$$

Therefore, just as for fixed points we will use the notation $\Phi^H X$ for both the $W_G H$-spectrum and the non-equivariant spectrum and it will be clear from the context which of both is meant.

We record the following standard properties of geometric fixed points.

Lemma A.2.7. *For any subgroup H, the geometric fixed point functor Φ^H has the following properties:*

(i) It preserves colimits.

(ii) It is symmetric monoidal.

(iii) If A is a pointed G-space, then $\Phi^H \Sigma^\infty_G A \simeq \Sigma^\infty_{W_G H} A^H$.

(iv) Suppose N is a normal subgroup of $N_G H$ contained in H. Then there is a natural equivalence of $W_G H$-spectra

$$\Phi^H X \simeq \Phi^{H/N}\Phi^N X.$$

(v) A map of G-spectra $f\colon X \to Y$ is an equivalence iff $\Phi^H f$ is an equivalence on the underlying spectra for all subgroups H.

Proof. The first claim follows from Lemma A.1.4 and the fact that smashing with a G-space and restricting to a subgroup both preserve colimits. We obtain (ii) and (iii) from [MM02, Corollary V.4.6 and Proposition V.4.7] and the fact that restricting to a subgroup is monoidal and commutes with taking suspension spectra. Note that our definition of geometric fixed points agrees with [MM02, Definition V.4.3] by [MM02, Proposition V.4.17].

Next, for (iv) we can assume that H is normal in G by replacing G with $N_G H$ if necessary. Then the projection formula (A.23) yields

$$\begin{aligned}\Phi^{H/N}\Phi^N X = ((X \otimes \widetilde{E\mathcal{F}}_{G,\not\supseteq N})^N \otimes \widetilde{E\mathcal{F}}_{G/N,\not\supseteq H/N})^{H/N} \simeq \\ ((X \otimes \widetilde{E\mathcal{F}}_{G,\not\supseteq N} \otimes (p_N^G)^*\widetilde{E\mathcal{F}}_{G/H,\not\supseteq H/N})^N)^{H/N} \simeq (X \otimes \widetilde{E\mathcal{F}}_{G,\not\supseteq H})^H = \Phi^H X.\end{aligned}$$

For the second equivalence we used (A.31) to see that

$$\widetilde{E\mathcal{F}}_{G,\not\supseteq N} \otimes (p_N^G)^*\widetilde{E\mathcal{F}}_{G/H,\not\supseteq H/N} \simeq \widetilde{E\mathcal{F}}_{G,\not\supseteq H},$$

since

$$\mathcal{F}_{G,\not\supseteq N} \cup (p_N^G)^*\mathcal{F}_{G/N,\not\supseteq H/N} = \mathcal{F}_{G,\not\supseteq H},$$

as well as the natural equivalence $((-)^N)^{H/N} \simeq (-)^H$.

Finally, for (v) it suffices to show that X is a trivial G-spectrum iff the underlying spectrum of $\Phi^H X$ is contractible for all subgroups H by passing to the cofiber. The only if direction is clear. We show the if direction by induction on the order of G. For $G = 1$ there is nothing to show, therefore we can proceed with the induction step and assume that $\mathrm{res}^G_H X$ is a contractible H-spectrum for any proper subgroup H of G. We consider the cofiber sequence

$$(X \otimes E\mathcal{P}_{G+})^G \to X^G \to (X \otimes \widetilde{E\mathcal{P}}_G)^G = \Phi^G X.$$

By assumption the right hand term is contractible and it follows from Lemma A.2.5 (ii) and Remark A.2.3 that the left hand term is contractible. Hence, also X^G is contractible and we obtain that X is a contractible G-spectrum from Lemma A.1.5. □

A.3 Conjugation equivalences between fixed points

If G is a finite group, H is a subgroup, and $g \in G$ we let $H^g = gHg^{-1}$ and denote by $c_g\colon G \to G$ the conjugation map. Note that the restriction to H induces an isomorphism

$$\kappa_H^g\colon H \xrightarrow{\cong} H^g.$$

Furthermore, the conjugation map also induces an isomorphism of the Weyl groups

$$\omega_H^g \colon W_G H \xrightarrow{\cong} W_G H^g.$$

We want to construct conjugation equivalences of $W_G H$-spectra.

$$X^H \simeq (\omega_H^g)^* X^{H^g}, \tag{A.33}$$

$$\Phi^H X \simeq (\omega_H^g)^* \Phi^{H^g} X. \tag{A.34}$$

We will explain below that there is a natural equivalence of $W_G H$-spectra

$$((c_g)^* X)^H \simeq (\omega_H^g)^* X^{H^g}, \tag{A.35}$$

$$\Phi^H (c_g)^* X \simeq (\omega_H^g)^* \Phi^{H^g} X. \tag{A.36}$$

Left multiplication with g induces an equivalence of G-spectra $l_g \colon X \xrightarrow{\simeq} (c_g)^* X$. Then we take H-fixed points and apply (A.35):

$$X^H \xrightarrow{(l_g)^H} ((c_g)^* X)^H \simeq (\omega_H^g)^* X^{H^g}. \tag{A.37}$$

If $g \in H$, then $H = H^g, \omega_H^g = \mathrm{id}_H$ and (A.37) is homotopic to the identity of X^H. This can be checked directly on orthogonal G-spectra. Similarly, we can define (A.34) as the composite

$$\Phi^H X \xrightarrow{\Phi^H l_g} \Phi^H (c_g)^* X \xleftarrow{(A.36)} (\omega_H^g)^* \Phi^{H^g} X$$

Again one checks on orthogonal G-spectra that if $g \in H$, then (A.34) is homotopic to the identity of $\Phi^H X$.

Now we construct (A.35). Recall that $p_H^G \colon N_G H \to W_G H$ and $p_H \colon H \to 1$ denote the projections and $i_H^G \colon H \to G$ the inclusion. Also, if α is any group homomorphism we denote by

$$\eta_\alpha \colon \mathrm{id} \to \alpha_* \alpha^* \quad \text{and} \quad \epsilon_\alpha \colon \alpha^* \alpha_* \to \mathrm{id}$$

the unit and counit maps. For any G-spectrum X the (co)unit maps together with the equalities $p_{H^g}^G \circ \kappa_{N_G H}^g = \omega_H^g \circ p_H$ then induce a natural equivalence of $W_G H$-spectra

$$\xi \colon (\omega_H^g)^* (p_{H^g}^G)_* \xrightarrow{\eta_{p_H^G}} (p_H^G)_* (p_H^G)^* (\omega_H^g)^* (p_{H^g}^G)_* \xrightarrow{(p_H^G)_* \hat{\xi}} (p_H^G)_* (\kappa_{N_G H}^g)^*, \tag{A.38}$$

which by definition is the right adjunct of the composite

$$\hat{\xi} \colon (p_H^G)^* (\omega_H^g)^* (p_{H^g}^G)_* \simeq (\kappa_{N_G H}^g)^* (p_{H^g}^G)_* (p_{H^g}^G)^* \xrightarrow{(\kappa_{N_G H}^g)^* \epsilon_{p_{H^g}^G}} (\kappa_{N_G H}^g)^*.$$

If we apply this to $(i_{N_G H^g}^G)^* X$ and use that $i_{N_G H^g}^G \circ \kappa_{N_G H}^g = c_g \circ i_{N_G H}^G$ we obtain the equivalence of $W_G H$-spectra

$$\begin{aligned}(\omega_H^g)^* X^{H^g} = (\omega_H^g)^* (p_{H^g}^G)_* (i_{N_G H^g}^G)^* X &\xrightarrow{\xi} (p_H^G)_* (\kappa_{N_G H}^g)^* (i_{N_G H^g}^G)^* X \\ &\simeq (p_H^G)_* (i_{N_G H}^G)^* (c_g)^* X = ((c_g)^* X)^H. \end{aligned} \tag{A.39}$$

We still need to show that (A.38) is indeed an equivalence. This is the content of the next lemma. We also show several other properties of (A.38).

Lemma A.3.1. *Let $\alpha\colon \tilde{G} \to G$ be an isomorphism of groups, $\tilde{N}$ a normal subgroup of $\tilde{G}$ and $N = \alpha(\tilde{N})$. Denote by $\omega\colon \tilde{G}/\tilde{N} \to G/N$ the isomorphism induced by α.*

(i) The composite of natural transformations

$$\omega^*(p_N^G)_* \xrightarrow{\eta_{p_{\tilde{N}}^{\tilde{G}}}} (p_{\tilde{N}}^{\tilde{G}})_*(p_{\tilde{N}}^{\tilde{G}})^*\omega^*(p_N^G)_* \simeq (p_{\tilde{N}}^{\tilde{G}})_*\alpha^*(p_N^G)^*(p_N^G)_* \xrightarrow{(p_{\tilde{N}}^{\tilde{G}})_*\alpha^*\epsilon_{p_N^G}} (p_{\tilde{N}}^{\tilde{G}})_*\alpha^* \tag{A.40}$$

is homotopic to the composite

$$\omega^*(p_N^G)_* \xrightarrow{\omega^*(p_N^G)_*\eta_\alpha} \omega^*(p_N^G)_*\alpha_*\alpha^* \simeq \omega^*\omega_*(p_{\tilde{N}}^{\tilde{G}})_*\alpha^* \xrightarrow{\epsilon_\omega} (p_{\tilde{N}}^{\tilde{G}})_*\alpha^*. \tag{A.41}$$

In particular it is an equivalence.

(ii) The equivalence above is compatible with the projection formula in the following sense. Let X be a G-spectrum and Y a G/N-spectrum. Consider the map of $\tilde{G}/\tilde{N}$-spectra

$$\omega^*(X^N \otimes Y) \to \omega^*(X \otimes (p_N^G)^*Y)^N \to (\alpha^*X \otimes \alpha^*(p_N^G)^*Y)^{\tilde{N}} \simeq (\alpha^*X \otimes (p_{\tilde{N}}^{\tilde{G}})^*\omega^*Y)^{\tilde{N}}, \tag{A.42}$$

where the first arrow is ω^* applied to the projection formula (A.23) and the second arrow is (A.40) combined with monoidality of α^*. Consider also the map

$$\omega^*(X^N \otimes Y) \simeq \omega^*X^N \otimes \omega^*Y \to (\alpha^*X)^{\tilde{N}} \otimes \omega^*Y \to (\alpha^*X \otimes (p_{\tilde{N}}^{\tilde{G}})^*\omega^*Y)^{\tilde{N}}, \tag{A.43}$$

where the first arrow is (A.40) smashed with ω^*Y and the second arrow is the projection formula (A.23). Then (A.42) and (A.43) are homotopic (as natural transformations).

(iii) The following diagram of spectra commutes:

$$\begin{array}{ccccc} X^G & & \xrightarrow{\simeq} & & (\alpha^*X)^{\tilde{G}} \\ \downarrow\simeq & & & & \downarrow\simeq \\ (X^N)^{G/N} & \xrightarrow{\simeq} & (\omega^*X^N)^{\tilde{G}/\tilde{N}} & \xrightarrow{\simeq} & ((\alpha^*X)^{\tilde{N}})^{\tilde{G}/\tilde{N}}. \end{array} \tag{A.44}$$

Here the vertical arrows are the equivalences

$$(p_G)_* \simeq (p_{G/N})_*(p_N^G)_* \quad \text{and} \quad (p_{\tilde{G}})_* \simeq (p_{\tilde{G}/\tilde{N}})_*(p_{\tilde{N}}^{\tilde{G}})_*,$$

and the horizontal arrows are (A.40) for various (sub)groups.

(iv) The following diagram of spectra commutes

$$\begin{array}{ccc} X^G & \longrightarrow & X^{H^g} \\ {\scriptstyle (A.39)}\downarrow\simeq & & {\scriptstyle (A.39)}\downarrow\simeq \\ ((c_g)^*X)^G & \longrightarrow & ((c_g)^*X)^H, \end{array} \tag{A.45}$$

where the horizontal arrows are inclusions of fixed points.

Proof. For (i) we first note that since ω and α are isomorphisms the unit η_α and the counit ϵ_ω are equivalences, hence also (A.41) is an equivalence. That (A.40) and (A.41) are homotopic is a special case of Lemma A.1.1 (ii) applied to the square

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\alpha} & G \\ \Big\downarrow{\scriptstyle p^{\tilde{G}}_{\tilde{N}}} & & \Big\downarrow{\scriptstyle p^{G}_{N}} \\ \tilde{G}/\tilde{N} & \xrightarrow{\omega} & G/N. \end{array}$$

This shows (i). Applying Lemma A.1.8 (i) to the same square shows (ii).

We turn to (iii). For typographic reasons we denote the map $G/N \to 1$ by q, the map $\tilde{G}/\tilde{N} \to 1$ by $\tilde{q}$, and put $p = p^G_N, \tilde{p} = p^{\tilde{G}}_{\tilde{N}}$. Then the vertical arrows of (A.44) are given by $(-)^G \simeq q_* p_*$ and $(-)^{\tilde{G}} \simeq \tilde{q}_* \tilde{p}_*$. Under this equivalence the upper horizontal arrow of (A.44) is given by tracing through the upper right corners of the upper square and the right square in the following diagram

$$\begin{array}{ccccccc} q_* p_* & & & & & & \\ \Big\downarrow{\scriptstyle \eta_{\tilde{q}}} & & & & & & \\ \tilde{q}_* \tilde{q}^* q_* p_* & \xrightarrow{\tilde{q}_* \eta_{\tilde{p}}} & \tilde{q}_* \tilde{p}_* \tilde{p}^* \tilde{q}^* q_* p_* & & & & \\ \Big\downarrow{\scriptstyle \simeq} & & \Big\downarrow{\scriptstyle \simeq} & & & & \\ \tilde{q}_* \omega^* q^* q_* p_* & \xrightarrow{\tilde{q}_* \eta_{\tilde{p}}} & \tilde{q}_* \tilde{p}_* \tilde{p}^* \omega^* q^* q_* p_* & \xrightarrow{\simeq} & \tilde{q}_* \tilde{p}_* \alpha^* p^* q^* q_* p_* & & \\ \Big\downarrow{\scriptstyle \tilde{q}_* \omega^* \epsilon_q} & & \Big\downarrow{\scriptstyle \tilde{q}_* \tilde{p}_* \tilde{p}^* \omega^* \epsilon_q} & & \Big\downarrow{\scriptstyle \tilde{q}_* \tilde{p}_* \alpha^* p^* \epsilon_q} & & \\ \tilde{q}_* \omega^* p_* & \xrightarrow{\tilde{q}_* \eta_{\tilde{p}}} & \tilde{q}_* \tilde{p}_* \tilde{p}^* \omega^* p_* & \xrightarrow{\simeq} & \tilde{q}_* \tilde{p}_* \alpha^* p^* p_* & \xrightarrow{\tilde{q}_* \tilde{p}_* \alpha^* \epsilon_p} & \tilde{q}_* \tilde{p}_* \alpha^*, \end{array}$$

and the lower horizontal arrows of (A.44) are given by passing through the lower left corner of this diagram.

Finally, we show (iv). We consider the following diagram

$$\begin{array}{ccc} \mathrm{id} & \xrightarrow{\eta_{i^G_{H^g}}} & (i^G_{H^g})_* (i^G_{H^g})^* \\ & & \Big\downarrow{\scriptstyle \eta_{\kappa^g_H}}\ {\scriptstyle \simeq} \\ \Big\downarrow{\scriptstyle \eta_{c_g}}\ {\scriptstyle \simeq} & & (i^G_{H^g})_* (\kappa^g_H)_* (\kappa^g_H)^* (i^G_{H^g})^* \\ & & \Big\downarrow{\scriptstyle \simeq} \\ (c_g)_* (c_g)^* & \xrightarrow{\eta_{i^G_H}} & (c_g)_* (i^G_H)_* (i^G_H)^* (c_g)^*, \end{array}$$

where the undecorated equivalence is induced by the equality $c_g \circ i^G_H = i^G_{H^g} \circ \kappa^g_H$ and all other arrows are unit maps of adjunctions. After applying $(-)^G$ the horizontal arrows become inclusions of fixed points and the columns are homotopic to (A.39) by (i). This concludes the proof. $\square$

The equivalence $\Phi^H (c_g)^* X \simeq (\omega^g_H)^* \Phi^{H^g} X$ can be defined similarly to (A.35). We do this in the slightly more general setting of Lemma A.3.1, that is we consider a group isomorphism $\alpha \colon \tilde{G} \to G$, a normal subgroup $\tilde{N}$ of $\tilde{G}$ with image $N = \alpha(\tilde{N})$, and the induced isomorphism

$\omega\colon \tilde{G}/\tilde{N} \to G/N$. Note that we have an equality of $\tilde{G}$-families $\alpha^*\mathcal{F}_{G,\not\supseteq N} = \mathcal{F}_{\tilde{G},\not\supseteq\tilde{N}}$, therefore by (A.29) an equivalence of pointed $\tilde{G}$-spaces $\alpha^*\widetilde{E\mathcal{F}}_{G,\not\supseteq N} \simeq \widetilde{E\mathcal{F}}_{\tilde{G},\not\supseteq\tilde{N}}$. Using (A.42) we can then define the equivalence $\omega^*\Phi^N X \simeq \Phi^{\tilde{N}}\alpha^* X$ as the composite

$$\omega^*\Phi^N X = \omega^*(X \otimes \widetilde{E\mathcal{F}}_{G,\not\supseteq N})^N \xrightarrow{(A.42)} (\alpha^*(X \otimes \widetilde{E\mathcal{F}}_{G,\not\supseteq N}))^{\tilde{N}} \simeq (\alpha^* X \otimes \widetilde{E\mathcal{F}}_{\tilde{G},\not\supseteq\tilde{N}}))^{\tilde{N}} = \Phi^N\alpha^* X. \tag{A.46}$$

This is indeed an equivalence by Lemma A.3.1 and we have additionally used that α^* is monoidal for the undecorated equivalence. We will need the following lemma in the main text.

Lemma A.3.2. *Let $\alpha\colon \tilde{G} \to G$ be an isomorphism of groups, $\tilde{N}$ a normal subgroup of $\tilde{G}$ and $N = \alpha(\tilde{N})$. Denote by $\omega\colon \tilde{G}/\tilde{N} \to G/N$ the isomorphism induced by α. Then the following diagram of spectra commutes:*

$$\begin{array}{ccc} \Phi^G X & \xrightarrow{\simeq} & \Phi^{G/N}\Phi^N X \\ & & \downarrow{\scriptstyle (A.46)\ \simeq} \\ \downarrow{\scriptstyle (A.46)\ \simeq} & & \Phi^{\tilde{G}/\tilde{N}}\omega^*\Phi^N X \\ & & \downarrow{\scriptstyle (A.46)\ \simeq} \\ \Phi^{\tilde{G}}\alpha^* X & \xrightarrow{\simeq} & \Phi^{\tilde{G}/\tilde{N}}\Phi^{\tilde{N}}\alpha^* X. \end{array} \tag{A.47}$$

Proof. We obtain (A.47) as the outer square of the following diagram:

$$\begin{array}{ccccc} \Phi^G X & & \dashrightarrow{\simeq} & & \Phi^{G/N}\Phi^N X \\ \| & & & & \| \\ (X \otimes \widetilde{E\mathcal{P}}_G)^G & \xrightarrow{\simeq} & ((X \otimes \widetilde{E\mathcal{P}}_G)^N)^{G/N} & \xleftarrow{\simeq} & ((X \otimes \widetilde{E\mathcal{F}}_{G,\not\supseteq N})^N \otimes \widetilde{E\mathcal{P}}_{G/N})^{G/N} \\ & & \downarrow{\simeq} & & \downarrow{\simeq} \\ \downarrow{\simeq} & & (\omega^*(X \otimes \widetilde{E\mathcal{P}}_G)^N)^{\tilde{G}/\tilde{N}} & \xleftarrow{\simeq} & (\omega^*((X \otimes \widetilde{E\mathcal{F}}_{G,\not\supseteq N})^N \otimes \widetilde{E\mathcal{P}}_{G/N}))^{\tilde{G}/\tilde{N}} \\ & & \downarrow{\simeq} & & \downarrow{\simeq} \\ (\alpha^*(X \otimes \widetilde{E\mathcal{P}}_G))^{\tilde{G}} & \xrightarrow{\simeq} & ((\alpha^*(X \otimes \widetilde{E\mathcal{P}}_G))^{\tilde{N}})^{\tilde{G}/\tilde{N}} & \xleftarrow{\simeq} & ((\alpha^*(X \otimes \widetilde{E\mathcal{F}}_{G,\not\supseteq N}))^{\tilde{N}} \otimes \omega^*\widetilde{E\mathcal{P}}_{G/N})^{\tilde{G}/\tilde{N}}, \\ \downarrow{\scriptstyle (A.29)\ \simeq} & & & & \downarrow{\scriptstyle (A.29)\ \simeq} \\ (\alpha^* X \otimes \widetilde{E\mathcal{P}}_{\tilde{G}})^{\tilde{G}} & & \dashrightarrow{\simeq} & & ((\alpha^* X \otimes \widetilde{E\mathcal{F}}_{\tilde{G},\not\supseteq\tilde{N}})^{\tilde{N}} \otimes \widetilde{E\mathcal{P}}_{\tilde{G}/\tilde{N}})^{\tilde{G}/\tilde{N}} \\ \| & & & & \| \\ \Phi^{\tilde{G}}\alpha^* X & & \dashrightarrow{\simeq} & & \Phi^{\tilde{G}/\tilde{N}}\Phi^{\tilde{N}}\alpha^* X. \end{array}$$

Note that by definition

$$(\omega^*((X \otimes \widetilde{E\mathcal{P}}_G)^N \otimes \widetilde{E\mathcal{P}}_{G/N}))^{\tilde{G}/\tilde{N}} \simeq (\omega^*(X \otimes \widetilde{E\mathcal{P}}_G)^N \otimes \widetilde{E\mathcal{P}}_{\tilde{G}/\tilde{N}})^{\tilde{G}/\tilde{N}} = \Phi^{\tilde{G}/\tilde{N}}\omega^*\Phi^N X.$$

Here the dashed arrows are the unique maps making the squares commute, the undecorated vertical arrows are given by (A.42) and the horizontal arrows pointing to the left by (A.23).

Of the squares involving solid arrows the upper right square commutes by naturality and the left square and lower right square commute by Lemma A.3.1 (iv) and (ii) respectively. We additionally used (A.31) to see that

$$\widetilde{E\mathcal{F}}_{G,\not\supset N} \otimes p^*\widetilde{E\mathcal{P}}_{G/N} \simeq \widetilde{E\mathcal{P}}_G,$$

since $\mathcal{F}_{G,\not\supset N} \cup p^*\mathcal{P}_{G/N} = \mathcal{P}_G$. □

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Zusammenfassung

Wir benutzen die Ergebnisse von Quigley und Shah um eine Formel für die geometrischen Fixpunkte von reeller topologischer zyklischer Homologie eines nach unten beschränkten Ringspektrums mit Antiinvolution herzuleiten. Die Antiinvolution auf einem Ringspektrum A induziert ein Spektrum mit sowohl der kanonischen Struktur eines A-Linksmoduls als der kanonischen Struktur eines A-Rechtsmoduls, deren Tensorprodukt über A mit einer Wirkung der zyklischen Gruppe der Ordnung 2 ausgestattet werden kann. Unsere Formel ist gegeben durch den homotopietheoretischen Differenzkern zweier Abbildungen von den Homotopiefixpunkten dieses Tensorprodukts in die Tate-Konstruktion dieses Tensorprodukts. Wir zeigen außerdem, dass dieser Differenzkern äquivalent ist zu dem, der von Dotto, Moi und Patchkoria in ihrer Berechnung der geometrischen Fixpunkte von reeller topologischer zyklischer Homologie hergeleitet wurde, sodass wir ihr Ergebnis mit anderen Methoden beweisen können.

Als Anwendung berechnen wir die reelle topologische zyklische Homologie von Gruppenringspektren für abelsche Gruppen und gewisse Klassen von Diedergruppen. Wir machen dies für beliebige Grundringspektren, unter der Voraussetzung, dass das zugrundeliegende Spektrum nach unten beschränkt ist. Für die Berechnung machen wir Gebrauch von einer Zerlegung der „dihedral-bar“-Konstruktion.

Lebenslauf

Der Lebenslauf wird aus Gründen des Datenschutzes in der elektronischen Fassung meiner Arbeit nicht veröffentlicht.

Selbstständigkeitserklärung

Ich erkläre gegenüber der Freien Universität Berlin, dass ich die vorliegende Dissertation selbstständig und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel angefertigt habe. Die vorliegende Arbeit ist frei von Plagiaten. Alle Ausführungen, die wörtlich oder inhaltlich aus anderen Schriften entnommen sind, habe ich als solche kenntlich gemacht. Diese Dissertation wurde in gleicher oder ähnlicher Form noch in keinem früheren Promotionsverfahren eingereicht.

Mit einer Prüfung meiner Arbeit durch ein Plagiatsprüfungsprogramm erkläre ich mich einverstanden.

Berlin, 04.09.2023 Thomas Vincent Boelens