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The structure of exceptional sequences on toric varieties of Picard rank two

Klaus Altmann & Frederik Witt

ABSTRACT For a smooth projective toric variety of Picard rank two we classify all exceptional sequences of invertible sheaves which have maximal length. In particular, we prove that unlike non-maximal sequences, they (a) remain exceptional under lexicographical reordering, (b) satisfy strong spatial constraints in the Picard lattice, and (c) are full, that is, they generate the derived category of the variety.

1. INTRODUCTION

1.1. FULLNESS OF EXCEPTIONAL SEQUENCES. Let X be a smooth projective variety over an algebraically closed field K and $\mathcal{D}(X)$ the bounded category of coherent sheaves on X . In his ICM talk [19], Kuznetsov posed the following

Fullness Conjecture. If $\mathcal{D}(X)$ is generated by an exceptional sequence, then any exceptional sequence of the same length is full.

Though a counterexample to Kuznetsov's conjecture was recently given on a rational surface [18], this question is still of interest. For instance, the fullness property for X implies the absence of so-called phantom categories appearing in [12] and [7].

In this paper we shall address the question of fullness in the case of exceptional sequences of line bundles on a toric variety of Picard rank 2 defined over an algebraically closed field K . Let us first explain the general setting before we comment on our assumptions. A sequence of elements $\mathcal{E}_1, \dots, \mathcal{E}_N$ in $\mathcal{D}(X)$ is called *exceptional* if the derived homomorphisms satisfy

$$\mathbb{R}\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_i) = K \quad \text{and} \quad \mathbb{R}\mathrm{Hom}(\mathcal{E}_j, \mathcal{E}_i) = 0 \quad \text{for all } j > i.$$

For instance, if the \mathcal{E}_i are given by invertible sheaves and X satisfies $\mathbb{R}\Gamma(X, \mathcal{O}_X) = K$, the first condition holds automatically. Further, the second condition becomes $\mathrm{Ext}^\bullet(\mathcal{E}_j, \mathcal{E}_i) = 0$.

An exceptional sequence is *full* if it generates the derived category. This means that $\langle \mathcal{E}_1, \dots, \mathcal{E}_N \rangle$, the smallest triangulated full subcategory of $\mathcal{D}(X)$ containing the \mathcal{E}_i , is $\mathcal{D}(X)$ itself. The length N equals the rank of the K-group $K_0(X)$. Any other exceptional sequence has at most N elements; it is called *maximal* if it attains this bound. In particular, full sequences are maximal.

The simplest example is Beilinson's full exceptional sequence on \mathbb{P}^d [6], namely

$$\mathcal{D}(\mathbb{P}^d) = \langle \mathcal{O}_{\mathbb{P}^d}(0), \dots, \mathcal{O}_{\mathbb{P}^d}(d) \rangle.$$

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KEYWORDS. toric variety, derived category, exceptional sequence.

More generally, Kawamata proved existence of full exceptional sequences consisting of complexes of coherent sheaves on any smooth projective toric variety X (or even a smooth toric DM-stack for that matter) [14, 15, 16]. Note that, for $\dim X = d$, the rank of $K_0(X)$ equals the number of d -dimensional cones $\#\Sigma(d)$ of the underlying fan Σ . Equivalently, this coincides with the number of vertices of any polytope associated with an ample divisor on X .

In view of Beilinson’s example one could even hope of finding full exceptional sequences consisting of invertible sheaves \mathcal{L}_i instead of general complexes \mathcal{E}_i , but this fails even for toric Fano varieties [10]. However, existence of such sequences was established in [9] if Σ is a splitting fan. Geometrically, this means that the toric variety $X = \mathbb{T}\mathbb{V}(\Sigma)$ arises as the total space of a sequence of successive fibrations via $X_0 = \mathbb{P}^n, X_1, \dots, X_r = X$ with $X_i = \mathbb{P}(E_{i-1})$ for a sum of invertible sheaves E_{i-1} on X_{i-1} .

1.2. EXCEPTIONAL SEQUENCES OF LINE BUNDLES ON TORIC VARIETIES OF PICARD RANK TWO. From now on we solely consider exceptional sequences of line bundles on smooth projective toric varieties of Picard rank 2, the easiest examples after \mathbb{P}^d among toric varieties $X = \mathbb{T}\mathbb{V}(\Sigma)$ with splitting fan Σ . The basic invariant of X is

$$\text{the pair } (\ell_1, \ell_2) \text{ of integers } \ell_1, \ell_2 \geq 2$$

which indicates that X fibres over \mathbb{P}^{ℓ_1-1} with fibre \mathbb{P}^{ℓ_2-1} . Moreover,

$$d = \dim X = \ell_1 + \ell_2 - 2,$$

and the defining fan Σ contains exactly two rays more than d ; see Subsection 3.1 for further details. In particular, $\#\Sigma(d) = \ell_1\ell_2$. We refer to the trivial fibration $\mathbb{P}^{\ell_1-1} \times \mathbb{P}^{\ell_2-1}$ as the *product case* and to a nontrivial fibration as the *twisted case*. For the latter, the order of the two numbers $\ell_1, \ell_2 \geq 2$ really matters. In dimension two where $\ell_1 = \ell_2 = 2$, we just find the family of Hirzebruch surfaces but the complexity quickly increases with dimension. The fibration structure also implies that we have a canonical identification

$$\text{Pic}(X) \cong \mathbb{Z}^2$$

given by the primitive generators of the nef cone, see Subsection 3.1.3. Geometrically, these generators are given by the pullback of $\mathcal{O}_{\mathbb{P}^{\ell_1-1}}(1)$ and a relative hyperplane section of the fibration.

Since $\text{Ext}^*(\mathcal{L}_j, \mathcal{L}_i) = H^*(X, \mathcal{L}_j^{-1} \otimes \mathcal{L}_i)$ a sequence of line bundles $\mathcal{L}_0, \dots, \mathcal{L}_N$ is exceptional if and only if $\mathcal{L}_j^{-1} \otimes \mathcal{L}_i, i < j$, lies in the locus of cohomologically trivial line bundles inside the Picard group. This locus is explicitly known for toric varieties given by a splitting fan [2]. Second, we also understand the extensions provided by nontrivial cohomology, cf. [5] and [3].

1.3. MAXIMAL EXCEPTIONAL SEQUENCES ARE LEXICOGRAPHIC. Since properties of exceptional sequences such as fullness only depend on their underlying set, it is natural to look for a canonical order. We call an exceptional sequence *vertically* respectively *horizontally orderable* if it remains exceptional for the lexicographic order on $\text{Pic}(X) \cong \mathbb{Z}^2$ where priority is given either to the “vertical” or “horizontal” direction. In general, lexicographic reordering destroys exceptionality of the sequence, but remarkably, this does not happen for maximal exceptional sequences.

THEOREM A (see Theorems 6.4 and 8.2). *Let $s \subseteq \mathbb{Z}^2$ be a maximal exceptional sequence of invertible sheaves on a smooth projective toric variety X of Picard rank two. In the product case, s is either vertically or horizontally orderable. In the twisted case, s is vertically orderable.*

In contrast, it was shown in [1, Example 3.5] that there are maximal exceptional sequences on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ that are not orderable with respect to any of the six possible lexicographic orders on $\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z}^3$.

1.4. MAXIMAL EXCEPTIONAL SEQUENCES ARE DENSELY PACKED. It is well-known that the shape of exceptional sequences also impacts on the derived category, see for instance [20], [21] or [24]. Our next structure result concerns the spatial size of maximal exceptional sequences.

THEOREM B (see Theorems 6.1 and 9.8). *Let $s \subseteq \mathbb{Z}^2 = \text{Pic}(X)$ be a maximal exceptional sequence of invertible sheaves on a projective toric variety X of Picard rank two. In the twisted case, the height, which is the minimal number of rows of a horizontal strip containing s , is bounded by $2\ell_2$. In the product case, either the height or the width (the minimal number of columns of a vertical strip containing s) is bounded by $2\ell_2 - 1$ or $2\ell_1 - 1$, respectively.*

Again, it is easy to construct counterexamples for non maximal sequences. What is striking about this result is that it is false for higher Picard rank. In [1, Example 3.4] it was shown that maximal exceptional sequences on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ can spread arbitrarily far and simultaneously in all three directions.

1.5. THE CLASSIFICATION OF MAXIMAL EXCEPTIONAL SEQUENCES. Let

$$\mathcal{R}_{\ell_1, \ell_2} := \{(a, b) \in \mathbb{Z}^2 \mid 0 \leq a < \ell_1, 0 \leq b < \ell_2\}$$

be the *standard rectangle* associated with the pair (ℓ_1, ℓ_2) . The sequence given by

$$s_{a+b\ell_1} := (a, b) \in \mathcal{R}_{\ell_1, \ell_2}$$

is maximal exceptional with respect to the vertical lexicographical order, and so is any sequence obtained by a global shift, or by shifting each row of points (\bullet, b) independently by some $(a_b, 0)$. We refer to these maximal exceptional sequences as *vertically trivial*, see also Subsection 5.4. Similarly, there are also horizontally trivial sequences in the product case. Composing the well-known helix operator (e.g. [25]) with lexicographical reordering yields the heLex operator \tilde{h}_{lex} , see Subsection 7.1.

THEOREM C (see Theorems 7.2 and 10.1). *Any vertically orderable maximal exceptional sequence can be transformed into a vertically trivial maximal exceptional sequence by applying \tilde{h}_{lex} at most $\ell_1\ell_2$ times. Mutatis mutandis, the statement holds for horizontally maximal exceptional sequences in the product case.*

Theorem C can be recast into a more constructive version:

THEOREM D (see Theorems 7.7 and 10.2). *Any maximal exceptional sequence is determined and explicitly described by a so-called admissible set $X \subseteq (-\beta, \ell_2) + \mathcal{R}_{\ell_1, \ell_2}$ and a complementing partner Y which consist either of horizontal or vertical lines of consecutive points.*

The precise definition of admissible sets and complementing partners is given in Definition 7.3 and 10.2. In this way we can classify the totality of maximal exceptional sequences.

1.6. FULLNESS OF MAXIMAL EXCEPTIONAL SEQUENCES. Finally, we show that maximality is sufficient for fullness. Viewing exceptionality as an orthogonality condition in the derived category, this is comparable to the fact that in a finitely generated vector space any linearly independent set of maximal cardinality generates the space. This follows either directly from Theorem A (admitting that the helix (!) operator preserves fullness from the general theory), or from a combinatorial argument building on Theorems A, B and D.

THEOREM E (see Theorems 7.9 and 11.2). *An exceptional sequence of invertible sheaves on a smooth projective toric variety of Picard rank two is full if and only if it is maximal.*

For toric Fano varieties of Picard rank two and dimension less than five this was shown in [22] by direct calculations in a case-by-case analysis (special cases were already considered in [23]). In contrast, [1] required aid of a computer to prove the same result for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, the easiest example of Picard rank three. In [8], the claim that maximality implies fullness was proven (with completely different methods) for toric DM stacks X with Picard rank two under the additional assumptions that X is Fano and that the sequence is even strongly exceptional. See also [13] for a much more general approach.

In [4, Theorem 5.3] one can find a precise characterization of strongly exceptional sequences among all maximal exceptional sequences in terms of the building pairs (X, Y) consisting of an admissible set and a complementing partners in the sense of Subsection 7.2 and (10.2) of the present paper.

On the other hand, the structural Theorems A-E go beyond this fullness issue as they provide a completely general and conceptual treatment of maximal exceptional sequences for Picard rank two – including a complete classification of all maximal exceptional sequences. As a final comment we note that our arguments are purely combinatorial (at the expense of possible shortcuts, cf. for instance Remark 4.6). A more geometrical approach is given in the sequel [4].

1.7. THE EXAMPLE $\mathbb{P}^1 \times \mathbb{P}^1$. As an illustration of our theorems we consider the case of $\mathbb{P}^1 \times \mathbb{P}^1$ and show how we generate the whole lattice $\text{Pic}(X) = \mathbb{Z}^2$ out of the maximal exceptional sequence

$$s = (s_0, s_1, s_2, s_3, \dots) = (0, (-3, 1), (-2, 1), (1, 2)),$$

see also (a) of Figure 1. It is vertically ordered and of height 3 in accordance with Theorems A and B (for horizontally ordered examples of unbounded height see Example 4.2).

The main tool is the Beilinson sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow \mathcal{O}_{\mathbb{P}^1}(2) \rightarrow 0$$

on \mathbb{P}^1 . As we explain in Example 4.5 it allows us to generate or “fill” the whole (horizontal or vertical) line whenever it carries two consecutive points of $\langle s \rangle$.

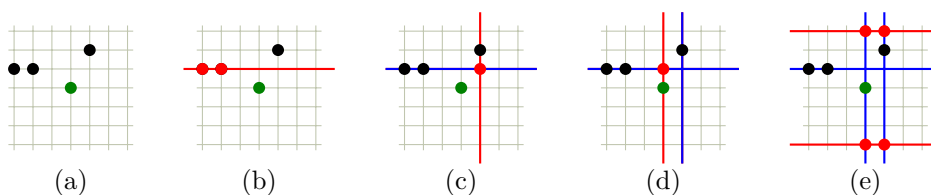


FIGURE 1. Filling $\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}^2$ from s . The green dot in (a) – (e) marks the origin $0 \in \mathbb{Z}^2$.

Here and in the sequel, let

$$[x = a] \text{ and } [y = b] \text{ denote the lines } \{(a, j) \mid j \in \mathbb{Z}\} \text{ and } \{(i, b) \mid i \in \mathbb{Z}\} \text{ in } \mathbb{Z}^2.$$

Right from the beginning in (a), we have a consecutive pair of elements in s on $[y = 1]$ so that we can generate or “fill” the entire line $[y = 1]$, cf. (b). Hence $[y = 1] \subseteq \langle s \rangle$.

Next, we fill the vertical line $[x = 1]$ using the consecutive pair $(1, 1), (1, 2)$ in (c) – we showed in (b) that $(1, 1) \in \langle s \rangle$. Similarly, we can fill the line $[x = 0]$ since $(0, 1) \in \langle s \rangle$, cf. (d).

It follows that we can fill any horizontal line $[y = b]$ for $(0, b), (1, b) \in \langle s \rangle$, $b \in \mathbb{Z}$, see (e). Hence $\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \subseteq \langle s \rangle$. As $\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ generates $\mathcal{D}(\mathbb{P}^1 \times \mathbb{P}^1)$, s is full.

The proof of Theorem E for $\mathbb{P}^{\ell_1-1} \times \mathbb{P}^{\ell_2-1}$ is a direct generalisation of this example. The key step consists in proving existence of sufficiently many “horizontal” and “vertical” consecutive points inside the sequence, see Section 6. The twisted case works differently. In particular, it requires a suitable “vertical” Beilinson sequence which reflects the fine structure of the twist, see Subsection 3.2.

1.8. PLAN OF THE PAPER. The first Sections 2 to 5 provide the necessary background and establish the main technical tools. Though large parts of the proofs in the product and the twisted case are similar and equally technical we found it more perspicuous to treat them separately with the product case serving as guideline. Consequently, Sections 6 and 7 prove Theorems A-E for the product case, while the remaining Sections 8 to 11 are devoted to proving the twisted versions.

2. SOME BACKGROUND ON TORIC GEOMETRY

We briefly review some features of toric geometry which we shall use in the paper. For a short introduction to toric geometry, see [11].

2.1. TORUS INVARIANT DIVISORS. Let $X = \mathbb{T}\mathbb{V}(\Sigma)$ be a smooth toric variety with underlying fan Σ . The r -dimensional cones in Σ form the subset $\Sigma(r)$. Similarly, for any $\sigma \in \Sigma$, the set $\sigma(r) \subseteq \Sigma(r)$ denotes the set of its r -dimensional faces. All these cones live in the real vector spaces $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, where N is the lattice of one-parameter groups of rank equal to $d = \dim X$. It is dual to the character lattice M of T , the torus of X .

These lattices link to the group of torus invariant Weil divisors $\text{Div}_T(X)$ and the class group $\text{Cl}(X)$ of X as follows. First, any ray, that is, an element $\rho \in \Sigma(1)$, corresponds to a unique torus orbit $\text{orb}(\rho)$ of codimension one. Let $D_\rho = \overline{\text{orb}(\rho)}$ be its closure. For $m \in M$ we define $\rho^*(m) = \text{div}(\chi^m) = \sum_{\rho} \langle m, \rho \rangle D_\rho$, where ρ denotes both the ray and its primitive generator in N .

Moreover, let ∇ be a lattice polytope in M which is *compatible* with Σ , i.e., Σ is a refinement of the normal fan $\mathcal{N}(\nabla)$ of ∇ . This comes with an associated Weil divisor

$$D_{\nabla} = - \sum_{\rho} \min \langle \nabla, \rho \rangle \cdot D_{\rho}.$$

The induced line bundle $\mathcal{O}(\nabla) := \mathcal{O}(D_{\nabla})$ is globally generated by the monomials χ^m with $m \in \nabla \cap M$. Further, for any $\sigma \in \Sigma(d)$ we have an associated vertex $v(\sigma) \in \nabla \cap M$ which is characterised by

$$\langle v(\sigma), \rho \rangle = \min \langle \nabla, \rho \rangle \quad \text{for } \rho \in \sigma(1).$$

It gives rise to a local generator of $\mathcal{O}(\nabla)$ on U_σ , the affine subset determined by σ , namely

$$\mathcal{O}(\nabla)|_{U_\sigma} = \chi^{v(\sigma)} \cdot G_X|_{U_\sigma}.$$

For non-maximal cones $\sigma \in \Sigma$ this works similarly. However, the vertices $v(\sigma)$ are only determined up to σ^\perp .

Finally, we have the exact sequence

$$(1) \quad 0 \longrightarrow M \xrightarrow{\rho^*} \left[\text{Div}_T(X) \cong \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}D_\rho \right] \xrightarrow{\pi} \text{Cl}(X) \longrightarrow 0$$

if the primitive generators span $N_{\mathbb{R}}$ (for instance if X is complete). In particular, $\text{rk Cl}(X) = \#\Sigma(1) - \dim X$.

2.2. EXACT SEQUENCES REFLECTING POLYHEDRAL COVERS. We recall a method from [3] to transform polyhedral inclusion / exclusion sequences as in Example 2.2 below into exact sequences of sheaves on toric varieties $X = \mathbb{T}\mathbb{V}(\Sigma)$.

We start with a so-called Σ -family of lattice polytopes $S = \{\nabla_i \mid i = 1, \dots, n\}$ in $M_{\mathbb{R}}$. This means that for all $I \subseteq [n] := \{1, \dots, n\}$ the intersections

$$\nabla_I := \bigcap_{i \in I} \nabla_i \text{ for } I \neq \emptyset$$

as well as

$$\nabla_{\emptyset} := \bigcup_{i \in [n]} \nabla_i$$

are compatible with Σ , i.e., $\Sigma \leq \mathcal{N}(\nabla_I)$.

The second ingredient is the standard Koszul complex $(\Lambda^{\bullet} K^n, d)$ where

$$d : \Lambda^{p+1} K^n \rightarrow \Lambda^p K^n, \quad e_I \mapsto \sum_{i \in I} (-1)^i e_{I \setminus \{i\}}$$

for any $I \subseteq [n]$ with $\#I = p + 1$, and $e_I := \wedge_{i \in I} e_i$ for the standard basis vectors $e_1, \dots, e_n \in K^n$. Tensoring with $K[M]$ yields the exact complex

$$0 \rightarrow K[M] \cdot e_{[n]} \rightarrow \bigoplus_{\substack{I \subseteq [n] \\ \#I=n-1}} K[M] \cdot e_I \rightarrow \dots \rightarrow \bigoplus_{i \in [n]} K[M] \cdot e_i \rightarrow K[M] \cdot e_{\emptyset} \rightarrow 0.$$

For each $I \subseteq [n]$ the vector space $K[M] \cdot e_I$ appears as a direct summand inside this complex and contains the finite-dimensional subspace

$$S(I) := K[\nabla_I \cap M] \cdot e_I := \bigoplus_{m \in \nabla_I \cap M} K \cdot \chi^m \cdot e_I.$$

It follows from [3, Section 3] that these subvector spaces define an exact subcomplex $S^{\bullet} \subseteq K[M] \otimes \Lambda^{\bullet} K^n$. Moreover, we have the

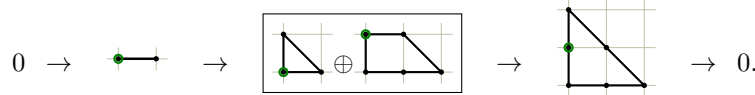
THEOREM 2.1. [3, Theorem 12] *S^{\bullet} is the complex of global sections of the equivariant, exact complex \mathcal{S}^{\bullet} given by the globally generated sheaves*

$$\mathcal{S}^k = \bigoplus_{I \subseteq [n], \#I=k} \mathcal{O}_X(\nabla_I)$$

on $X = \mathbb{T}\mathbb{V}(\Sigma)$, namely

$$0 \rightarrow \mathcal{O}_X(\nabla_{[n]}) \rightarrow \bigoplus_{i=1}^n \mathcal{O}_X(\nabla_{[n] \setminus \{i\}}) \rightarrow \dots \rightarrow \bigoplus_{i=1}^n \mathcal{O}_X(\nabla_i) \rightarrow \mathcal{O}_X(\nabla_{\emptyset}) \rightarrow 0.$$

EXAMPLE 2.2. Let $X = \mathcal{H}_1$ be the first Hirzebruch surface, see also the picture on the left hand side of Figure 2. We consider the Σ -family $S = \{\nabla_1, \nabla_2\}$ provided by the triangle and the quadrangle in the middle box of the polyhedral exact sequence



Here, the green dots indicate the position of the origin in each of the polyhedra. Using the notation from Subsection 3.1.2 below, the leftmost polyhedron $\nabla_1 \cap \nabla_2$ equals U , and the triangle ∇_1 itself is just V . The sequence may be therefore rephrased by

$$(2) \quad 0 \rightarrow \mathcal{O}_{\mathcal{H}_1}(U) \rightarrow \mathcal{O}_{\mathcal{H}_1}(V) \oplus \mathcal{O}_{\mathcal{H}_1}(U + V) \rightarrow \mathcal{O}_{\mathcal{H}_1}(2V) \rightarrow 0.$$

Let us translate this sequence into classical language. The right hand side of Figure 2 displays the fan of the blow-up $b : \mathcal{H}_1 \rightarrow \mathbb{P}^1$.

The labeling of the rays in $\Sigma(\mathcal{H}_1)$ is concordant with the notation of Subsection 3.1.2. In particular, the closed orbit $\overline{\text{orb}}(v^2)$ equals the exceptional divisor $E \subseteq$

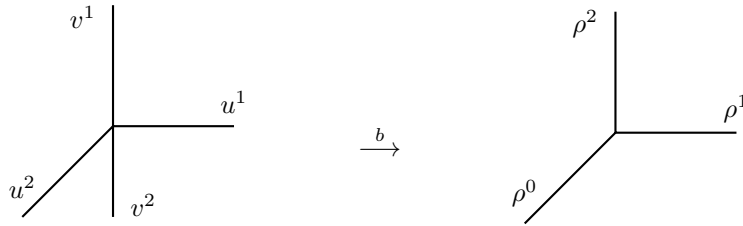


FIGURE 2.

\mathcal{H}_1 . Since the blow-up b contracts E to the point $\overline{\text{orb}(\rho^0, \rho^1)} = 0 \in \mathbb{P}^2$, the remaining ray ρ^2 encodes the line $L_\infty \subset \mathbb{P}^2$ at infinity. Moreover, the restriction $b : \overline{\text{orb}(v^1)} \rightarrow \overline{\text{orb}(\rho^2)} = L_\infty$ is an isomorphism.

Therefore, the exact sequence (2) is obtained from

$$0 \rightarrow \mathcal{O}_{\mathcal{H}_1} \rightarrow \mathcal{O}_{\mathcal{H}_1}(E) \oplus \mathcal{O}_{\mathcal{H}_1}(L_\infty) \rightarrow \mathcal{O}_{\mathcal{H}_1}(E + L_\infty) \rightarrow 0.$$

after replacing the polyhedra with toric Weil divisors and twisting by $\mathcal{O}_{\mathcal{H}_1}(U) = \mathcal{O}_{\mathcal{H}_1}(\overline{\text{orb}(u^2)})$. This is the Koszul complex of the exceptional line and the line at infinity.

2.3. DEALING WITH NON-FACES. Next we apply the formalism of Subsection 2.2 and fix an arbitrary subset $S \subseteq \Sigma(1)$. Let $n = \#S$ and choose an order on S . We think of S as a sequence ρ_1, \dots, ρ_n and identify $\rho_i \in S$ with $i \in [n]$.

For subsets $I \subseteq S$ we define integral tuples $k_I \in \mathbb{Z}^{\Sigma(1)}$ by

$$(k_I)_\rho := \begin{cases} 1 & \text{if } \rho \in S \setminus I \\ 0 & \text{if otherwise, i.e., } \rho \in I \cup (\Sigma(1) \setminus S). \end{cases}$$

Interpreting $k_I \in \mathbb{Z}^{\Sigma(1)}$ as a T -invariant, effective and reduced Weil divisor on $X = \text{TV}(\Sigma)$ we denote by $\mathcal{O}_X(k_I)$ the associated sheaf. Since X is smooth, these divisors are automatically Cartier. In particular, the sheaves $\mathcal{O}_X(k_I)$ are invertible albeit not nef in general.

PROPOSITION 2.3. Assume that $S \subseteq \Sigma(1)$ is a non-face, that is, S does not define a cone in Σ . Then, with I running through the subsets of S , the following complex \mathcal{C}_S^\bullet of invertible sheaves with the usual Koszul-like differentials is exact:

$$0 \rightarrow \mathcal{O}_X(k_S) \rightarrow \bigoplus_{\#I=k-1} \mathcal{O}_X(k_I) \rightarrow \dots \rightarrow \bigoplus_{\#I=1} \mathcal{O}_X(k_I) \rightarrow \mathcal{O}_X(k_\emptyset) \rightarrow 0.$$

Here, $\mathcal{O}_X(k_S) = \mathcal{O}_X$ and $\mathcal{O}_X(k_\emptyset) = \mathcal{O}_X(\underline{1}_S)$ which is the sheaf associated with the effective and reduced divisor $\sum_{\rho \in S} D_\rho$.

Proof. We choose a sufficiently ample polytope Δ such that all polytopes

$$\nabla_I := \Delta(k_I) := \{m \in M_{\mathbb{R}} \mid \langle m, \rho \rangle \geq \min\langle \Delta, \rho \rangle - (k_I)_\rho\}$$

are at least nef. It follows immediately that

$$\nabla_I \cap \nabla_J = \Delta(k_I) \cap \Delta(k_J) = \Delta(k_{I \cup J}) = \nabla_{I \cup J}$$

for all subsets $I, J \subseteq [n]$. In particular, $\nabla_I = \bigcap_{i \in I} \nabla_i$ if $I \neq \emptyset$. We assert that $\bigcup_{i \in [n]} \nabla_i = \nabla_\emptyset$ whence $\{\nabla_i\}$ is a Σ -family. This immediately implies the claim of the proposition by tensoring the sequence in Theorem 2.1 with $\mathcal{O}(\Delta)^{-1}$.

We claim that for sufficiently ample Δ , $\bigcup_{i \in [n]} \Delta(k_i) = \Delta(k_\emptyset)$ if and only if S is a non-face.

Let $\bigcup_{i \in [n]} \Delta(k_i) = \Delta(k_\emptyset)$. Further, assume that $S \subseteq \sigma(1)$ for some (smooth) and full-dimensional cone $\sigma \in \Sigma$ with set of rays $\sigma(1)$. The vertex $v_\emptyset(\sigma)$ of $\nabla_\emptyset = \Delta(k_\emptyset)$

associated with σ satisfies

$$\langle v_\emptyset(\sigma), \rho \rangle = \begin{cases} \min\langle \Delta, \rho \rangle - 1 & \text{if } \rho \in S \subseteq \sigma(1) \\ \min\langle \Delta, \rho \rangle & \text{if } \rho \in \sigma(1) \setminus S. \end{cases}$$

However, this contradicts the inequality $\langle v_\emptyset(\sigma), \rho_i \rangle \geq \min\langle \Delta, \rho_i \rangle$ of $\Delta(k_i)$ for $\rho_i \in S$.

Conversely, assume that $S \notin \Sigma$. For each $\rho \in \Sigma(1)$ we consider the associated facet

$$\text{face}(\Delta, \rho) := \{r \in \Delta \mid \langle r, \rho \rangle = \min\langle \Delta, \rho \rangle\}$$

and define the “thickened ρ -facet” by

$$F(\Delta, \rho) := \{r \in \Delta \mid \langle r, \rho \rangle \leq \min\langle \Delta, \rho \rangle + 1\}.$$

More generally, these definitions work for all cones $\sigma \in \Sigma \setminus \{0\}$: The face associated to σ is

$$\text{face}(\Delta, \sigma) := \bigcap_{\rho \in \sigma(1)} \text{face}(\Delta, \rho)$$

and the corresponding “thickened σ -face” is

$$F(\Delta, \sigma) := \bigcap_{\rho \in \sigma(1)} F(\Delta, \rho).$$

The usual one-to-one correspondence between faces of Δ and cones of $\Sigma = \mathcal{N}(\Delta)$ implies that $\sigma, \sigma' \in \Sigma \setminus \{0\}$ are not contained in some common cone $\tilde{\sigma} \in \Sigma$ if and only if the $\text{face}(\Delta, \sigma)$ and $\text{face}(\Delta, \sigma')$ are disjoint. Therefore, the thickened faces $F(\Delta, \sigma)$ and $F(\Delta, \sigma')$ are also disjoint for sufficiently ample Δ .

Applying this to the polytope $\nabla_\emptyset = \Delta(k_\emptyset)$ shows that for a non-face $S \subseteq \Sigma(1)$ the corresponding thickened facets are disjoint, that is, $\bigcap_{\rho \in S} F(\nabla_\emptyset, \rho) = \emptyset$. On the other hand,

$$\nabla_\emptyset \setminus F(\nabla_\emptyset, \rho) \subseteq \nabla_\rho$$

implies

$$\bigcup_{\rho \in S} \nabla_\rho \supseteq \bigcup_{\rho \in S} [\nabla_\emptyset \setminus F(\nabla_\emptyset, \rho)] = \nabla_\emptyset \setminus \bigcap_{\rho \in S} F(\nabla_\emptyset, \rho) = \nabla_\emptyset. \quad \square$$

REMARK 2.4. (i) Though we will not use this observation in our later arguments, we note in passing that for a primitive collection S the exact sequence of Proposition 2.3 represents the unique extension arising from $\text{Ext}^{n-1}(\mathcal{O}_X(k_\emptyset), \mathcal{O}_X) = K$.

(ii) Proposition 2.3 is the homological counterpart to the multiplicative Stanley–Reisner presentation of the equivariant K-theory ring of a smooth toric variety, see [26].

3. TORIC VARIETIES OF PICARD RANK TWO

3.1. KLEINSCHMIDT’S CLASSIFICATION. Let $X = \mathbb{T}\mathbb{V}(\Sigma)$ be a complete and smooth toric variety of dimension d and Picard rank two. These varieties are characterised by the following data [17]:

(i) Natural numbers $\ell_1, \ell_2 \geq 2$ with $\ell_1 + \ell_2 = d + 2$.

(ii) An integer vector $c \in \mathbb{Z}^{\ell_2}$ with nonpositive components

$$0 = c^1 \geq c^2 \geq \dots \geq c^{\ell_2}.$$

We write the corresponding variety

$$X = (\ell_1, \ell_2; c).$$

Here are some key properties.

3.1.1. *The class map.* The varieties $(\ell_1, \ell_2; c)$ arise as fibre bundles over \mathbb{P}^{ℓ_1-1} with typical fibre \mathbb{P}^{ℓ_2-1} . Actually, we have

$$(\ell_1, \ell_2; c) = \mathbb{P}\left(\bigoplus_{j=1}^{\ell_2} \mathcal{O}_{\mathbb{P}^{\ell_1-1}}(-c^j)\right) \rightarrow \mathbb{P}^{\ell_1-1}.$$

The best known instance is the Hirzebruch surface $\mathcal{H}_\alpha = (2, 2; (0, -\alpha))$, cf. Example 3.4 or Subsection 8.1. The fibration is trivial (*product case*) if and only if $c = 0$. Identifying $\text{Div}_T(\ell_1, \ell_2; c)$ with $\mathbb{Z}^{\ell_1+\ell_2}$ we can rearrange this data in terms of the $2 \times (\ell_1 + \ell_2)$ -matrix

$$\pi_c := \begin{pmatrix} 1 & \dots & 1 & 0 & c^2 & \dots & c^{\ell_2} \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 \end{pmatrix} : \mathbb{Z}^{\ell_1+\ell_2} \twoheadrightarrow \mathbb{Z}^2.$$

For $X = (\ell_1, \ell_2; c)$ this provides the map $\pi : \text{Div}_T(X) \rightarrow \text{Cl}(X)$ in the exact sequence (1) on page 1043. In particular, $M \cong \ker \pi_c$ and $N \cong \text{coker } \pi_c^*$ where $\pi_c^* : \mathbb{Z}^2 \hookrightarrow \mathbb{Z}^{\ell_1+\ell_2}$ is the transpose.

3.1.2. *The fan.* Let $\{e_1, \dots, e_{\ell_1}, f_1, \dots, f_{\ell_2}\}$ and $\{e^1, \dots, e^{\ell_1}, f^1, \dots, f^{\ell_2}\}$ be the mutually dual bases of $\text{Div}_T(\ell_1, \ell_2; c)$ and $\text{Div}_T(\ell_1, \ell_2; c)^*$, respectively. Under ρ_c from the exact sequence

$$0 \longrightarrow \text{Cl}(X)^* \xrightarrow{\pi_c^*} [\text{Div}_T(X)^* \cong \mathbb{Z}^{\Sigma(1)}] \xrightarrow{\rho_c} N \longrightarrow 0$$

dual to (1) the latter vectors are mapped to $\{u^1, \dots, u^{\ell_1}, v^1, \dots, v^{\ell_2}\}$. These images define the rays of the fan $\Sigma = \Sigma(\ell_1, \ell_2; c)$ generated by the d -dimensional smooth cones

$$\sigma_{ij} := \langle \Sigma(1) \setminus \{u^i, v^j\} \rangle, \quad i = 1, \dots, \ell_1 \text{ and } j = 1, \dots, \ell_2.$$

In particular, $\#\Sigma(d) = \ell_1 \ell_2$ as stated before. Note that $\sum_{j=1}^{\ell_2} v^j = 0$, but $\sum_{i=1}^{\ell_1} u^i = \sum_{j=1}^{\ell_2} (-c^j) \cdot v^j$.

3.1.3. *The nef divisors.* Regarded as a map $\text{Div}_T(X) \rightarrow \text{Cl}(X)$, π_c sends the equivariant prime divisors $\overline{\text{orb}}(u^i)$ and $\overline{\text{orb}}(v^j)$ to their classes. The effective cone $\text{Eff} \subseteq \text{Cl}(X)$ is generated by $[\overline{\text{orb}}(u^1)]$ and $[\overline{\text{orb}}(v^{\ell_2})]$. The nef cone $\text{Nef} \subseteq \text{Eff}$ is generated by $[\overline{\text{orb}}(u^1)]$ and $[\overline{\text{orb}}(v^1)]$. This provides a natural identification $\text{Cl}(X) \cong \mathbb{Z}^2$ by sending $[\overline{\text{orb}}(u^1)]$ to $(1, 0)$ and $[\overline{\text{orb}}(v^1)]$ to $(0, 1)$. In particular, $[\overline{\text{orb}}(v^{\ell_2})]$ is sent to $(c^{\ell_2}, 1)$. These classes are also represented by the lattice polytopes

$$U := \Delta(u^1) = \text{conv}\{e_i - e_1 \mid i = 1, \dots, \ell_1\}$$

and

$$V := \Delta(v^1) = \text{conv}\{f_j - f_1 - c^j e_i \mid i = 1, \dots, \ell_1; j = 1, \dots, \ell_2\}$$

in $M_{\mathbb{R}} \subseteq \mathbb{R}^{\ell_1+\ell_2}$. Note that U equals the $(\ell_1 - 1)$ -dimensional standard simplex Δ^{ℓ_1-1} while V can be understood as the Caley product

$$V = (-c^1 \cdot \Delta^{\ell_1-1}) * \dots * (-c^{\ell_2} \cdot \Delta^{\ell_1-1}).$$

3.1.4. *The anti-canonical divisor.* The divisor class of $-K_X$ is

$$[-K_X] = \pi_c(1, \dots, 1) = (\ell_1 - \beta, \ell_2)$$

with

$$\beta := - \sum_j c^j \geq 0.$$

Hence, $(\ell_1, \ell_2; c)$ is Fano if and only if $\beta < \ell_1$.

REMARK 3.1. Rather than the full vector c the non-negative parameters

$$\alpha := -c^{\ell_2} \quad \text{and} \quad \beta = -\sum_{j=1}^{\ell_2} c^j$$

are the most important ones for our purposes. Almost by definition we get the inequalities

$$(3) \quad 0 \leq \alpha \leq \beta \leq \alpha(\ell_2 - 1).$$

The latter will be referred to as the *basic inequality*. For further simplification we also introduce

$$\gamma := \alpha\ell_2 - \beta$$

which satisfies $\gamma \geq \alpha \geq 0$.

3.2. APPLICATION OF SUBSECTION 2.3 TO THE SITUATION OF PICARD RANK TWO. Next we apply Proposition 2.3 to the toric varieties $X = (\ell_1, \ell_2; c)$. From the description of their fan in Subsection 3.1.2 we derive that there are exactly two primitive collections, namely

$$p_u = \{u^1, \dots, u^{\ell_1}\} \quad \text{and} \quad p_v = \{v^1, \dots, v^{\ell_2}\}.$$

COROLLARY 3.2. *The primitive collections p_u and p_v give rise to the “U-sequence”*

$$0 \rightarrow \mathcal{O}_X \rightarrow \dots \rightarrow \mathcal{O}_X(U)^{\otimes \ell_1} \rightarrow 0$$

and the “V-sequence”

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{F}_1 \rightarrow \dots \rightarrow \mathcal{F}_{\ell_2-1} \rightarrow \mathcal{O}_X(U)^{\otimes (-\beta)} \otimes \mathcal{O}_X(V)^{\otimes \ell_2} \rightarrow 0$$

where the sheaves \mathcal{F}_k are direct sums indexed by subsets $I \subseteq \{1, \dots, \ell_2\}$, namely

$$\mathcal{F}_k = \bigoplus_{\#I=k} \mathcal{O}_X(c^I U + kV) \quad \text{with} \quad c^I := \sum_{i \in I} c^i \leq 0.$$

REMARK 3.3. Setting $\mathcal{F}_0 := \mathcal{O}_X$ and $\mathcal{F}_{\ell_2} := \mathcal{O}_X(-\beta U + \ell_2 V)$ we can extend this notation to $k \in \{0, \dots, \ell_2\}$. Moreover, using the identification $\text{Pic } X \cong \mathbb{Z}^2$ with $U = (1, 0)$ and $V = (0, 1)$ from Subsection 3.1.3 we usually write $\mathcal{F}_k = \bigoplus_{\#I=k} \mathcal{O}_X(c^I, k)$.

Proof. We deal with the primitive collection $S = \{v^1, \dots, v^{\ell_2}\}$ giving rise to the V-sequence; the case of the U-sequence works similarly.

By Proposition 2.3 we obtain an exact sequence for $\mathcal{F}_k = \bigoplus_{\#J=\ell_2-k} \mathcal{O}_X(k_J)$ with $k_J = \underline{1}_{S \setminus J}$. Renaming $I := S \setminus J$ this becomes $\mathcal{F}_k = \bigoplus_{\#I=k} \mathcal{O}_X(\underline{1}_I)$. On the other hand, assigning the Weil divisor $\underline{1}_I = \sum_{\rho \in I} D_\rho$ to its class means applying the map π_c from Subsection 3.1.1. Therefore, $\underline{1}_I$ becomes (c^I, k) since $\pi_c(f_i) = (c^i, 1)$. \square

EXAMPLE 3.4. One of the very first examples one comes across when computing the cohomology of invertible sheaves on toric varieties is $\text{Ext}^1(2V, U) = K$ on the Hirzebruch surface $X = \mathcal{H}_1 = (2, 2; (0, 1))$. This is represented by the exact sequence from Example 2.2. After twisting with $\mathcal{O}(-U)$ this becomes the V-sequence of Corollary 3.2.

3.3. THE CO-IMMACULATE LOCUS. From now on we will work with toric varieties of the form $X = (\ell_1, \ell_2; c)$ and identify $\text{Pic}(X)$ with \mathbb{Z}^2 via the map π_c from Subsection 3.1.1.

As pointed out in the introduction the locus of invertible sheaves with vanishing cohomology plays a crucial rôle in this paper. We call

$$\mathcal{I}(X) := \{\mathcal{L} \in \text{Pic}(X) \mid H^j(X, \mathcal{L}^{-1}) = 0, j \in \mathbb{Z}_{\geq 0}\}.$$

the *co-immaculate locus* of $\text{Pic}(X)$. This distinguished subset was referred to as the negative immaculate locus in [2] and [1]. In passing we remark that here and in the sequel we shall not distinguish between invertible sheaves and their isomorphism classes in the Picard group.

Rewriting [2, Theorem VI.2] in terms of the co-immaculate locus, we can describe $\mathcal{I}(\ell_1, \ell_2; \alpha, \beta) = \mathcal{H} \cup \mathcal{P}$ as the union of the *horizontal strip*

$$\mathcal{H} := \{(a, b) \in \mathbb{Z}^2 \mid 0 < b < \ell_2\}$$

and the *parallelogram*

$$\mathcal{P} = \{(a, b) \in \mathbb{Z}^2 \mid -\beta < a < \ell_1 \text{ and } 0 < \langle (a, b), (1, \alpha) \rangle < \ell_1 + \gamma\}.$$

The co-immaculate locus is point symmetric with respect to $(\ell_1 - \beta, \ell_2)/2$. In particular, the origin is point symmetric to the anti-canonical class $[-K] = (\ell_1 - \beta, \ell_2)$. Figure 3 illustrates the typical shape of the co-immaculate locus for $\ell_1 = 7$ and $\ell_2 = 4$. The parallelogram is indicated by the shaded area; lattice points in \mathcal{H} are in grey while lattice points in \mathcal{P} but not in \mathcal{H} are blue.

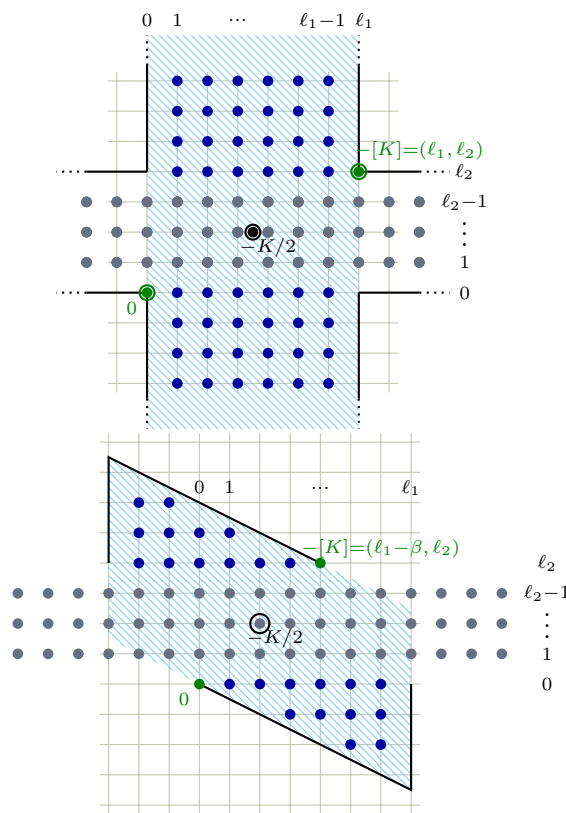


FIGURE 3. The co-immaculate locus of $(7, 4; 0)$ (left hand side) and $(7, 4; (0, 0, -1, -2))$ (right hand side).

REMARK 3.5.

- (i) The sequel of this article entirely depends on the combinatorics of the co-immaculate locus, not the underlying fan. The former depends solely on ℓ_1 , ℓ_2 , $\alpha = -c^{\ell_2}$ and $\beta = -\sum_j c^j$. The co-immaculate locus will be therefore written as $\mathcal{I}(\ell_1, \ell_2; \alpha, \beta)$ (or simply \mathcal{I} depending on the context).
- (ii) The co-immaculate locus is *horizontally integral convex*, that is, if $s, t \in \mathcal{I} \cap \ell$ for any horizontal line $\ell \subseteq \mathbb{Z}^2$, then the segment $[s, t] \subseteq \ell$ is contained in \mathcal{I} , too.

- (iii) Since $[-K] = (\ell_1 - \beta, \ell_2)$ the anti-canonical class sits always to the left of the line $[x = \ell_1 + 1]$. By Subsection 3.1.4 it sits to the right of $[x = 0]$ if and only if $(\ell_1, \ell_2; c)$ is Fano.

3.4. THE ASSOCIATED LATTICE. The outer “boundary points” $(\ell_1, 0)$ and $(-\beta, \ell_2)$ of the co-immaculate locus $\mathcal{I}(\ell_1, \ell_2; \alpha, \beta)$ define the *associated lattice*

$$L := \mathbb{Z}(\ell_1, 0) \oplus \mathbb{Z}(-\beta, \ell_2).$$

We denote $\mathcal{T} = \mathbb{Z}^2/L$ the induced quotient and $\Phi : \mathbb{Z}^2 \rightarrow \mathcal{T}$ the projection map which sends (a, b) to its class $[a, b]$ in \mathcal{T} , cf. also [1, Section 4].

LEMMA 3.6. *We have $L \cap \mathcal{I}(\ell_1, \ell_2; \alpha, \beta) = \emptyset$.*

Proof. This is obvious for $c = 0$ so assume that $c \neq 0$.

Suppose we could pick $(a, b) \in L \cap \mathcal{I}$ so that $(a, b) = (n\ell_1 - m\beta, m\ell_2) \in \mathcal{I}$ for some $n, m \in \mathbb{Z}$. In particular, we necessarily have $(a, b) \in \mathcal{P}$. If $m \leq 0$, then we exploit the inequalities $\langle (a, b), (1, 0) \rangle < \ell_1$ and $0 < \langle (a, b), (1, \alpha) \rangle$; they imply

$$0 \leq -m\alpha\ell_2 < n\ell_1 - m\beta < \ell_1$$

and therefore, by adding $m\beta$,

$$m\beta \leq [-m\gamma = m(\beta - \alpha\ell_2)] < n\ell_1 < \ell_1 + m\beta \leq \ell_1.$$

By the basic inequality (3) on page 1048 we have even $0 \leq -m\gamma$ whence a contradiction for both cases $n \geq 1$ and $n \leq 0$. On the other hand, for $m \geq 1$ the remaining two inequalities of \mathcal{P} , namely $-\beta < \langle (a, b), (1, 0) \rangle$ and $\langle (a, b), (1, \alpha) \rangle < \ell_1 + \gamma$ imply

$$0 \leq (m - 1)\beta < n\ell_1 < \ell_1 - (m - 1)\gamma \leq \ell_1.$$

Again, this leads to a contradiction. □

4. EXCEPTIONAL SEQUENCES OF INVERTIBLE SHEAVES

4.1. THE EXCEPTIONALITY CONDITION. Recall from the introduction that a sequence $\mathcal{L}_0, \dots, \mathcal{L}_N$ of invertible sheaves on a variety X is said to be *exceptional* if all backward Ext-groups vanish. Equivalently,

$$H^k(X, \mathcal{L}_i \otimes \mathcal{L}_j^{-1}) = 0 \text{ if } i < j.$$

Consequently, we can rephrase the exceptionality condition as follows. Denoting the isomorphism classes in $\text{Pic}(X)$ by $s_i := \mathcal{L}_i$, the sequence $s_0, s_1, \dots, s_N \in \text{Pic}(X)$ is exceptional on X if and only if

$$\overrightarrow{s_i s_j} = s_j - s_i \in \mathcal{I}(X)$$

for all $i < j$, or equivalently,

$$s_j \in \bigcap_{i < j} (s_i + \mathcal{I}(X)) \quad \text{for all } j \geq 1.$$

This condition persists under a simultaneous shift so that we may replace the original sequence by $\mathcal{L}'_i := \mathcal{L}_i \otimes \mathcal{L}_0^{-1}$. Therefore, we may assume that $\mathcal{L}_0 = \mathcal{O}_X$ is trivial whenever this is convenient. In particular, $s_0 = 0$ which implies $s_i \in \mathcal{I}(X)$ for all $i \geq 1$.

EXAMPLE 4.1. For $X = \mathbb{P}^{\ell-1}$, Serre duality and the well-known vanishing theorems for invertible sheaves on projective space yield

$$\mathcal{I}(\mathbb{P}^{\ell-1}) = \{\mathcal{O}_{\mathbb{P}^{\ell-1}}(1), \dots, \mathcal{O}_{\mathbb{P}^{\ell-1}}(\ell - 1)\} \cong \{1, 2, \dots, \ell - 1\} \subseteq \mathbb{Z}.$$

EXAMPLE 4.2. The previous example behaves well under products. Figure 4 visualises the case of $X = \mathbb{P}^1 \times \mathbb{P}^1$. Let us determine on $\mathbb{P}^1 \times \mathbb{P}^1$ all possible types of exceptional sequences of maximal length which is 4. The shape of the co-immaculate locus immediately implies that we have at most two elements on the lines $[x = 1]$ and $[y = 1]$, and any two elements on the same line must be consecutive. Computing the sets $\mathcal{I} \cap ((a, 1) + \mathcal{I})$ and $\mathcal{I} \cap ((1, b) + \mathcal{I})$ for $a, b \in \mathbb{Z}$ shows that we have four families s^1, \dots, s^4 of maximal exceptional sequences given by

$$s^1 = ((0, 0), (1, 0), (a, 1), (a + 1, 1)), s^2 = ((0, 0), (0, 1), (1, b), (1, b + 1))$$

$$s^3 = ((0, 0), (a, 1), (a + 1, 1), (1, 2)), s^4 = ((0, 0), (1, b), (1, b + 1), (2, 1)).$$

Examples are displayed in Figure 5.

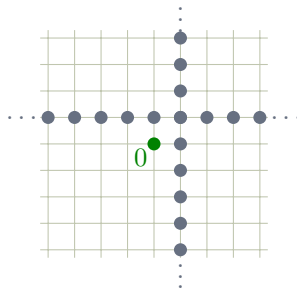


FIGURE 4. The co-immaculate locus of $\mathbb{P}^1 \times \mathbb{P}^1$ is given by the grey points. Note that the origin does not belong to $\mathcal{I}(\mathbb{P}^1 \times \mathbb{P}^1)$.

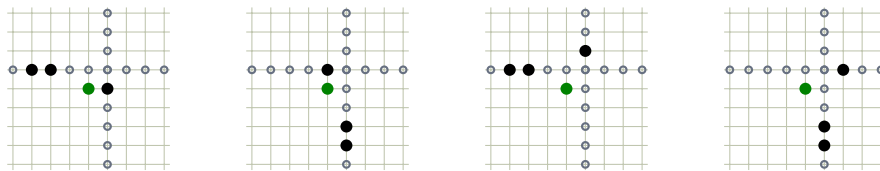


FIGURE 5. s^1, s^2, s^3, s^4 for $a = b = -3$.

4.2. THE MAXIMALITY CONDITION. Using the identification $\text{Pic}(\text{TV}(\Sigma)) \cong \mathbb{Z}^2$ we think of exceptional sequences as subsets of \mathbb{Z}^2 together with a certain ordering.

DEFINITION 4.3. Consider the smooth toric variety of Picard rank two $(\ell_1, \ell_2; c)$ which is of dimension $d := \dim X = \ell_1 + \ell_2 - 2$, cf. Subsection 3.1. The subset inside \mathbb{Z}^2 underlying an exceptional sequence $s = (s_0, s_1, \dots, s_N)$ is called

- (i) non-extendable if it is not strictly contained in some other subset of \mathbb{Z}^2 underlying an exceptional sequence.
- (ii) maximal if $N + 1$ equals $\#\Sigma(d) = \ell_1 \ell_2$.

By abuse of language, we refer to the sequence s itself as non-extendable or maximal if the underlying set is non-extendable or maximal.

Note that “maximal” implies “non-extendable”, the converse being false, see Example 8.1. A crucial property of maximal exceptional sequences is this.

LEMMA 4.4. If s is a maximal exceptional sequence, then the restriction of $\Phi : \mathbb{Z}^2 \rightarrow \mathcal{T}$ from Subsection 3.4 defines a bijection.

Proof. If $\Phi(s_i) = \Phi(s_j)$ for some pair $s_i < s_j$ in s , then $s_j - s_i \in \mathcal{I}(\ell_1, \ell_2; \alpha, \beta) \cap L$ which is empty by Lemma 3.6. Hence $\Phi|_s$ is injective.

Since s and \mathcal{T} have the same cardinality, $\Phi|_s$ is also surjective. □

4.3. THE FULLNESS CONDITION. Recall from the introduction that an exceptional sequence s is said to be *full* if the underlying set $\{s_0, \dots, s_N\}$ generates $\mathcal{D}(X)$. Since X is smooth, $\mathcal{D}(X)$ is generated by $\text{Pic}(X)$. Regarding s as a subset of $\text{Pic}(X)$ it is therefore sufficient to show that we can generate any invertible sheaf.

EXAMPLE 4.5. The celebrated Beilinson exact sequence [6] yields

$$(4) \quad 0 \rightarrow \Lambda^\ell K^\ell \otimes \mathcal{O}_{\mathbb{P}^{\ell-1}}(0) \rightarrow \Lambda^{\ell-1} K^\ell \otimes \mathcal{O}_{\mathbb{P}^{\ell-1}}(1) \rightarrow \dots \rightarrow \Lambda^0 K^\ell \otimes \mathcal{O}_{\mathbb{P}^{\ell-1}}(\ell) \rightarrow 0.$$

In our language, this is just the exact sequence from Theorem 2.1 for the standard $(\ell - 1)$ -simplex in \mathbb{R}^ℓ .

At any rate, given the sequence $\mathcal{O}_{\mathbb{P}^{\ell-1}}(1), \dots, \mathcal{O}_{\mathbb{P}^{\ell-1}}(\ell - 1)$ the bundle $\mathcal{O}_{\mathbb{P}^{\ell-1}}(0)$ generates $\mathcal{O}_{\mathbb{P}^{\ell-1}}(\ell)$ in $\mathcal{D}(\mathbb{P}^{\ell-1})$ and vice versa. Therefore, any sequence of ℓ consecutive classes of invertible sheaves generates $\text{Pic}(\mathbb{P}^{\ell-1}) \cong \mathbb{Z}$ and thus the derived category $\mathcal{D}(\mathbb{P}^{\ell-1})$. Note that any such sequence actually defines a maximal exceptional sequence.

4.4. THE HELIX OPERATOR. Let $s = (s_0, s_1, \dots, s_N)$ be a maximal exceptional sequence on $(\ell_1, \ell_2; c)$. We define

$$\hbar(s) := (s_1, s_2, \dots, s_N, -[K] + s_0) = (s_1, s_2, \dots, s_N, (\ell_1 - \beta, \ell_2) + s_0)$$

and call \hbar the *helix operator*. It preserves exceptionality as follows directly from the point symmetry of \mathcal{I} with respect to $-[K]/2$, cf. Subsection 3.3.

REMARK 4.6. Helixing is a standard operation in the theory of exceptional sequences, cf. for instance [25], and it is well-known that the helix operator also preserves fullness. From our combinatorial point of view, this follows as a corollary to Theorem E; assuming the general theory, Theorem E is a simple corollary of Theorem C.

5. LEXICOGRAPHICAL ORDER AND CHAINS

5.1. LEXICOGRAPHICALLY ORDERABLE EXCEPTIONAL SEQUENCES. We consider an exceptional sequence $s = (s_0, \dots, s_N)$ on $X = (\ell_1, \ell_2; c)$, that is, $s_j - s_i \in \mathcal{I} = \mathcal{I}(\ell_1, \ell_2; \alpha, \beta)$ for all $0 \leq i < j \leq N$. The underlying set $\{s_0, \dots, s_N\}$ can give rise to exceptional sequences for various orders. Still, we necessarily have the

LEMMA 5.1. *Let $s = (s_0, \dots, s_N)$ be an exceptional sequence on $(\ell_1, \ell_2; c)$. Then for any $s_i = (a_i, b)$ and $s_j = (a_j, b)$ in $s \cap [y = b]$ we have $a_i < a_j$ if and only if $i < j$. Moreover, $|a_j - a_i| \leq \ell_1 - 1$.*

Proof. By definition of exceptionality, $i < j$ implies

$$s_j - s_i = (a_j - a_i, 0) \in \mathcal{I}$$

which holds if and only if $0 < a_j - a_i < \ell_1$. □

DEFINITION 5.2. *The subset underlying an exceptional sequence s is called vertically orderable if it also defines an exceptional sequence with respect to the vertical (lexicographical) order*

$$(a_1, b_1) < (a_2, b_2) \Leftrightarrow b_1 < b_2 \text{ or } (b_1 = b_2 \text{ and } a_1 < a_2).$$

Similarly, it is called horizontally orderable if it defines an exceptional sequence with respect to the horizontal (lexicographical) order

$$(a_1, b_1) < (a_2, b_2) \Leftrightarrow a_1 < a_2 \text{ or } (a_1 = a_2 \text{ and } b_1 < b_2).$$

In general, exceptionality cannot be expected to be preserved under lexicographic reordering. For instance, the sequence $s = (s_0, s_1, s_2)$ on $(\ell_1, \ell_2; c)$ with $c \neq 0$ and $\alpha\ell_2 + 1 < \ell_1$ which is given by

$$s_0 = (0, 0), \quad s_1 = (\alpha\ell_2 + 1, -\ell_2), \quad s_2 = (\alpha\ell_2 + 2 - \beta, 0) = (\gamma + 2, 0)$$

is certainly exceptional, but neither the horizontally ordered sequence (s_0, s_2, s_1) nor the vertically ordered sequence (s_1, s_0, s_2) are as $\beta < \alpha\ell_2$ in virtue of the basic inequality (3). However, these big “downward jumps” are the only obstruction against vertical order.

PROPOSITION 5.3. *An exceptional sequence $s = (s_0, \dots, s_N)$ on $(\ell_1, \ell_2; c)$ with $s_i = (a_i, b_i)$ is vertically orderable if and only if*

$$(5) \quad b_j - b_i > -\ell_2$$

for all $0 \leq i < j \leq N$.

Proof. Assume that s is vertically orderable and that we have a pair $s_i < s_j$ with

$$-\ell_2 \geq b := b_j - b_i.$$

In particular, $s_j - s_i$ must lie in the parallelogram \mathcal{P} , hence

$$-\alpha b < a := a_j - a_i.$$

On the other hand, we also have $s_i - s_j \in \mathcal{I}$. Since $-b \geq \ell_2$, the difference $s_i - s_j = (-a, -b)$ must be in the parallelogram \mathcal{P} as well whence $-\beta < -a$. By the basic inequality (3), $a > -\alpha b > \beta$ and thus $-a < -\beta$, contradiction.

Conversely, assume that the bound (5) holds. We show that a permutation $\sigma: \{0, \dots, N\} \rightarrow \{0, \dots, N\}$ exists such that $s_{\sigma(0)}, \dots, s_{\sigma(N)}$ is lexicographically ordered and still exceptional.

From Lemma 5.1, $i < j$ implies $a_i < a_j$ if $s_i = (a_i, b)$ and $s_j = (a_j, b)$. In addition, we show that we can interchange the order of two adjacent sequence elements $s_i = (a_i, b_i)$ and $s_{i+1} = (a_{i+1}, b_{i+1})$ with $b_i > b_{i+1}$ while keeping the sequence exceptional.

Indeed, the difference $\overrightarrow{s_i s_{i+1}} = (a_{i+1} - a_i, b_{i+1} - b_i)$ lies in \mathcal{I} . The assumptions imply

$$0 < b_i - b_{i+1} \leq \ell_2 - 1.$$

Hence $\overrightarrow{s_{i+1} s_i} = -\overrightarrow{s_i s_{i+1}}$ belongs to the horizontal strip and therefore lies in \mathcal{I} , too.

Replacing in the sequence the pair s_i, s_{i+1} by s_{i+1}, s_i does not affect the remaining difference vectors of $\{s_i, s_{i+1}\}$ with elements from $\{s_0, \dots, s_{i-1}\}$ and $\{s_{i+2}, \dots, s_N\}$ so that $s_0, \dots, s_{i-1}, s_{i+1}, s_i, s_{i+2}, \dots, s_N$ is still exceptional. Furthermore, it still satisfies (5) since $b_{i+1} - b_i \leq 0$. After a finite number of pairwise permutations which preserve (5) and exceptionality, we obtain the desired permutation. \square

COROLLARY 5.4 (“no horizontal gaps”). *Let $s = (s_0, s_1, \dots, s_N)$ define an exceptional sequence which is not extendable. If s is vertically orderable, then the restriction $s \cap [y = b]$ to any horizontal line has no gaps. Namely, if (a, b) and $(a', b) \in s$ with $a < a'$, then $(a'', b) \in s \cap [y = b]$ for all $a \leq a'' \leq a'$.*

Proof. Reordering if necessary we may assume that s is vertically ordered. Suppose we have $s_i = (a_i, b)$, $s_{i+1} = (a_{i+1}, b)$ with $a_i + 2 \leq a_{i+1}$. Then we can choose a point $(a, b) \in \mathbb{Z}^2$ and $a_i < a < a_j$. We define a new sequence s' by

$$s'_k := \begin{cases} s_k & \text{if } k \leq i \\ (a, b) & \text{if } k = i + 1 \\ s_{k-1} & \text{if } k \geq i + 2 \end{cases}.$$

We show that s' is exceptional, contradicting the nonextendability of s . Consider the vectors $\overrightarrow{s'_k s'_{k+r}} = s'_{k+r} - s'_k$ for $k \geq 1$. If none of the indices k or $k+r$ equals $i+1$, then s'_k and s'_{k+r} belong to the original sequence s , whence $\overrightarrow{s'_k s'_{k+r}} \in \mathcal{I}$.

If $k \leq i$, consider $s'_{i+1} - s'_k = (a, b) - s_k$ which sits between $s_i - s_k$ and $s_{i+1} - s_k$ on the same horizontal line. Hence $(a, b) - s_k \in \mathcal{I}$ by horizontal integral convexity, cf. Subsection 3.3.

Finally, if $k \geq i+1$, we can reason as before and conclude that the difference $s'_k - s'_{i+1}$ belongs to \mathcal{I} . \square

Since by Proposition 5.3 any pair of adjacent elements with $b_i - b_{i+1} < \ell_2$ can be switched we obtain the following

LEMMA 5.5. *Let $s = (s_0, \dots, s_N)$ be an exceptional sequence on $(\ell_1, \ell_2; c)$ which is not vertically orderable. Then we can reorder s in such a way that the new sequence is still exceptional and there is a consecutive pair $s_i < s_{i+1}$ with $b_i - b_{i+1} \geq \ell_2$.*

REMARK 5.6. For $c = 0$ the \mathbb{Z}^2 -involution $(a, b) \mapsto (b, a)$ maps any $(\ell_1, \ell_2; 0)$ -exceptional sequence to an $(\ell_2, \ell_1; 0)$ -exceptional sequence. In particular, it follows that s is horizontally orderable if and only if

$$(6) \quad a_j - a_i > -\ell_1$$

for all $0 \leq i < j \leq N$. Similarly, Lemma 5.1, 5.5 and Corollary 5.4 hold mutatis mutandis.

5.2. ORDERABLE VARIETIES. For $c \neq 0$ we define the *integral depth* of \mathcal{I} as the smallest integer d_{int} such that the line $[y = d_{\text{int}}]$ meets \mathcal{I} , namely

$$(7) \quad d_{\text{int}} = -\left\lfloor \frac{\ell_1 - 2}{\alpha} \right\rfloor.$$

Proposition 5.3 immediately implies the

COROLLARY 5.7. *If $d_{\text{int}} \geq 1 - \ell_2$, then any exceptional sequence on $(\ell_1, \ell_2; c)$ with $c \neq 0$ can be vertically ordered.*

We call the toric variety $(\ell_1, \ell_2; c)$ itself *vertically orderable* if every exceptional sequence is vertically orderable. Subsection 3.1.4 and the basic inequality (3) immediately imply the

PROPOSITION 5.8. *A toric variety $(\ell_1, \ell_2; c)$ with $c \neq 0$ and $\beta \geq \ell_1 - 2$ is vertically orderable.*

This holds, for instance, for varieties of dimension less than or equal to three as well as for all non-Fano varieties.

5.3. EXISTENCE. To generalise the generation strategy from Subsection 1.7 we make the following

DEFINITION 5.9. *A horizontal chain in \mathbb{Z}^2 is a subset of the form*

$$(a, b) + \{(0, 0), \dots, (\ell_1 - 1, 0)\}.$$

Similarly, a vertical chain is a subset of the form $(a, b) + \{(0, 0), \dots, (0, \ell_2 - 1)\}$.

REMARK 5.10. Thinking of \mathbb{Z}^2 as the Picard group $\text{Pic}(\ell_1, \ell_2; c)$ we can fill, that is, generate in the derived category any line $[y = b]$ which contains a horizontal chain via the Beilinson sequence (4) in Subsection 4.3. The product case also requires vertical chains.

Given an exceptional sequence s we quantify its “spatial size” as follows. We let

$$\underline{a} := \underline{a}(s) := \min\{a \mid (a, b) \in s\}, \quad \underline{b} := \underline{b}(s) := \min\{b \mid (a, b) \in s\}$$

and

$$\bar{a} := \bar{a}(s) := \max\{a \mid (a, b) \in s\}, \quad \bar{b} := \bar{b}(s) := \max\{b \mid (a, b) \in s\},$$

The *height* and the *width* of s are then defined by

$$H(s) := \bar{b} - \underline{b} + 1 \quad \text{and} \quad W(s) := \bar{a} - \underline{a} + 1.$$

PROPOSITION 5.11. *Let s be a maximal exceptional sequence on $(\ell_1, \ell_2; c)$ and $H(s) \leq 2\ell_2$. For all integers b with $0 \leq b - \underline{b} < \ell_2$, the sets*

$$Y_{b+\ell_2} := (-\beta, \ell_2) + ([y = b] \cap s) \quad \text{and} \quad X_{b+\ell_2} := [y = b + \ell_2] \cap s$$

define the horizontal chain

$$S_{b+\ell_2} := Y_{b+\ell_2} \cup X_{b+\ell_2} \subseteq \mathbb{Z}^2.$$

In particular, $Y_{b+\ell_2} \cap X_{b+\ell_2} = \emptyset$ and $\#S_{b+\ell_2} = \ell_1$.

Proof. Consider the lattice L with associated map $\Phi : \mathbb{Z}^2 \rightarrow \mathcal{T}$ from Subsection 3.4. By Lemma 4.4 its restriction to s is bijective. We denote by $[a, b] = \Phi(a, b)$ the equivalence class of $(a, b) \in s$ in \mathcal{T} .

To ease notation we assume that $\underline{b} = 0$. Since $H(s) \leq 2\ell_2$, $0 \leq b \leq 2\ell_2 - 1$ whenever $(a, b) \in s$. Hence, the ℓ_1 classes $[a_1, b], \dots, [a_{\ell_1}, b]$ come either from $[y = b] \cap s$ or $[y = b + \ell_2] \cap s$. In particular, the union $Y_{b+\ell_2} \cup X_{b+\ell_2}$ is disjoint and has precisely ℓ_1 elements.

Let $Y'_b := \{y \in \mathbb{Z} \mid (y, b + \ell_2) \in Y_{b+\ell_2}\}$ and $X'_b := \{x \in \mathbb{Z} \mid (x, b + \ell_2) \in X_{b+\ell_2}\}$. Since both sets are disjoint, we can define

$$s(z) := \begin{cases} (y + \beta, b) & \text{if } z = y \in Y'_b \\ (x, b + \ell_2) & \text{if } z = x \in X'_b \end{cases}$$

which is just the element of s giving rise to z . By Lemma 5.1, $x, x' \in X'_b$ satisfy $s(x) < s(x')$ if and only if $x < x'$, and similarly for Y'_b . A more involved characterisation holds for mixed pairs $(x, y) \in X'_b \times Y'_b$, namely

$$s(y) < s(x) \iff 0 < x - y < \ell_1 \quad \text{and} \quad s(x) < s(y) \iff \gamma < y - x < \ell_1 - \beta.$$

Indeed, $s(y) < s(x)$ if and only if $(x - y - \beta, \ell_2) \in \mathcal{I}(\ell_1, \ell_2; \alpha, \beta)$. Further, $s(x) < s(y)$ if and only if $(y + \beta - x, -\ell_2) \in \mathcal{I}(\ell_1, \ell_2; \alpha, \beta)$. The claim follows from the inequalities defining the co-immaculate locus as well as $\gamma \geq 0$, cf. Remark 3.1. In particular, $Y'_b \cup X'_b$ forms a sequence of ℓ_1 consecutive integers. \square

REMARK 5.12. If in addition $c \neq 0$, the proof of Proposition 5.11 also yields a horizontal no-gap lemma for $Y_{b+\ell_2} \cup X_{b+\ell_2}$ without assuming that the maximal exceptional sequence s is orderable, compare Corollary 5.4. Moreover, it implies for this case that $Y_{b+\ell_2}$ lies to the left of $X_{b+\ell_2}$. Indeed, if there exists $x \in X'_b$ with $y = x + 1 \in Y'_b$, then $1 = y - x > \gamma \geq 1$, contradiction.

In the sequel we say that an element s_i of s is at *level* h , if $s_i = (a, \underline{b} + h)$.

COROLLARY 5.13. *Let s be a maximal exceptional sequence with $l := 2\ell_2 - H(s) > 0$. Then there exist horizontal chains in s at level $\ell_2 - l, \dots, \ell_2 - 1$.*

Proof. The assertion is invariant under shifts so we may assume that $\underline{b} = \underline{b}(s) = 0$. If $(a, b) \in s$, then $\underline{b} = 0 \leq b \leq \bar{b} = 2\ell_2 - l - 1$. Therefore, $[y = 2\ell_2 - l] \cap s, \dots, [y = 2\ell_2 - 1] \cap s = \emptyset$. By Proposition 5.11 we must have horizontal chains in $[y = \ell_2 - l] \cap s, \dots, [y = \ell_2 - 1] \cap s$. \square

The converse statement also holds. For this, let

$$\Delta^{\text{up}} := \{(a, b) \in \mathcal{P} \mid b \geq \ell_2\} = \{(a, b) \in \mathbb{Z}^2 \mid b \geq \ell_2, -\beta < a \text{ and } a + b\alpha < \ell_1 + \gamma\}.$$

Note that the inequalities $b \geq \ell_2$ and $-\beta < a$ alone already imply that $a + b\alpha > \gamma$. The subsequent lemma states, roughly speaking, that a point of s at sufficiently high level prevents horizontal chains in s .

LEMMA 5.14. *Let s be an exceptional sequence on $(\ell_1, \ell_2; c)$ which is vertically orderable. If $(a'', b'') \in s$, then for all pairs $(a, b) < (a', b)$ with $b'' - b \geq \ell_2$ we have $a' - a < \ell_1 - 1$ (instead of $\leq \ell_1 - 1$ as asserted in Lemma 5.1).*

Proof. From the definition of the co-immaculate locus it follows that

$$\Delta^{\text{up}} \subseteq \{(a, b) \in \mathbb{Z}^2 \mid -\beta < a < \ell_1 - \beta\}.$$

Reordering vertically if necessary we have $(a, b) < (a'', b'')$ for any $(a, b) \in s$ hence $[y = b] \cap s$ is contained in

$$(a'', b'') - \Delta^{\text{up}} \subseteq \{(a, b) \in \mathcal{P} \mid \beta + a'' - \ell_1 < a < \beta + a''\}.$$

Consequently, $a' - a < \ell_1 - 1$. □

If an exceptional sequence s starts at $s_0 = 0$, any horizontal chain in s must be necessarily located at level h with $0 \leq h \leq \ell_2 - 1$. We therefore immediately deduce the

COROLLARY 5.15. *Let s be an exceptional sequence on $(\ell_1, \ell_2; c)$ which is vertically orderable. If there exists a horizontal chain in s at level h , then $H(s) \leq \ell_2 + h \leq 2\ell_2 - 1$.*

COROLLARY 5.16. *Let s be a maximal exceptional sequence on $(\ell_1, \ell_2; c)$ which is vertically orderable. If $H(s) \leq 2\ell_2$, then $[y = b] \cap s \neq \emptyset$ for all $\underline{b} \leq b \leq \bar{b}$. In particular, $H(s) = \#$ of rows occupied by s .*

Proof. Since the assertion concerns only the underlying set we may suppose that s is vertically ordered and $s_0 = 0$. Assume that $[y = b] \cap s = \emptyset$. If $0 \leq b < \ell_2$, then $[y = b + \ell_2] \cap s$ must have ℓ_1 elements by Proposition 5.11. But $\ell_2 \leq b + \ell_2$ whence $([y = b + \ell_2] \cap s) \subseteq \Delta^{\text{up}}$ which is impossible.

On the other hand, $b \geq \ell_2$ implies that $[y = b - \ell_2] \cap s$ has ℓ_1 elements. By Corollary 5.15, $H(s) \leq b$ which contradicts $b \leq H(s) - 1$. □

REMARK 5.17. For $c = 0$ Proposition 5.11 and Corollaries 5.4 5.13, 5.15 and 5.16 hold mutatis mutandis for the horizontal case.

5.4. THE TRIVIAL MAXIMAL EXCEPTIONAL SEQUENCES. In Subsection 1.5 we introduced the standard rectangle

$$\mathcal{R}_{\ell_1, \ell_2} = \{(a, b) \in \mathbb{Z}^2 \mid 0 \leq a < \ell_1, 0 \leq b < \ell_2\}.$$

If the pair (ℓ_1, ℓ_2) is clear from the context we simply write \mathcal{R} . With respect to the vertical lexicographical order there is the maximal exceptional sequence given by

$$s_{a+b\ell_1} := (a, b) \in \mathcal{R}.$$

Indeed, $\#\mathcal{R} = \ell_1\ell_2$, and the difference of s_i and s_{i+k} sitting in a common row is $(k, 0) \in \mathcal{I}$. In all other cases, the difference is of the form (\cdot, b) with $b \in \{1, 2, \dots, \ell_2 - 1\}$. It is thus contained in $\mathcal{I}(\ell_1, \ell_2; \alpha, \beta)$, too.

Furthermore, we obtain maximal exceptional sequences by

- (i) applying an overall shift (a, b) to the entire sequence.
- (ii) shifting any of the rows at level $b = 1, 2, \dots, \ell_2 - 1$ by some $(a_b, 0) \in \mathbb{Z}^2$ depending on the level b .

The resulting maximal exceptional sequences will be referred to as *vertically trivial*. They are always vertically orderable and have the maximal number $z = \ell_2$ of horizontal chains. In particular, a maximal exceptional sequence s is vertically trivial if and only if $H(s) = \ell_2$.

Mutatis mutandis we also have horizontally trivial maximal exceptional sequences in the product case $c = 0$.

6. THE DICHOTOMY OF THE PRODUCT CASE

Choose two integers $\ell_1, \ell_2 \geq 2$. We set out to tackle the Theorems A-E from the introduction for the product case $(\ell_1, \ell_2; 0) = \mathbb{P}^{\ell_1-1} \times \mathbb{P}^{\ell_2-1}$. The reader which is solely interested in the twisted case can continue with Section 8. For the remainder of this section let $\mathcal{I} := \mathcal{I}(\ell_1, \ell_2; 0, 0)$.

6.1. EXCEPTIONAL SEQUENCES ARE SEMI-BOUNDED.

THEOREM 6.1 (Theorem B, product version). *An exceptional sequence s on $(\ell_1, \ell_2; 0)$ is semi-bounded, that is, we have either $H(s) \leq 2\ell_2 - 1$ or $W(s) \leq 2\ell_1 - 1$.*

Proof. Suppose to the contrary that both $H(s) \geq 2\ell_2$ and $W(s) \geq 2\ell_1$.

We consider (a priori not necessarily distinct points) $A = (a, b_A), B = (a_B, \underline{b}), C = (\bar{a}, b_C), D = (a_D, \bar{b}) \in s$, see Figure 6. We decompose

$$R := \{(a, b) \in \mathbb{Z}^2 \mid \underline{a} \leq a \leq \bar{a}, \underline{b} \leq b \leq \bar{b}\}$$

into pairs of horizontal and vertical strips, namely

$$H^{\text{down}} := \{(a, b) \in R \mid b < \underline{b} + \ell_2\}, \quad H^{\text{up}} := \{(a, b) \in R \mid b \geq \bar{b} - \ell_2\}$$

and

$$V^{\text{left}} := \{(a, b) \in R \mid a < \underline{a} + \ell_1\}, \quad V^{\text{right}} := \{(a, b) \in R \mid a \geq \bar{a} - \ell_1\}.$$

see Figure 6. In particular, $A \in V^{\text{left}}$ and $B \in H^{\text{down}}$. We distinguish two cases.

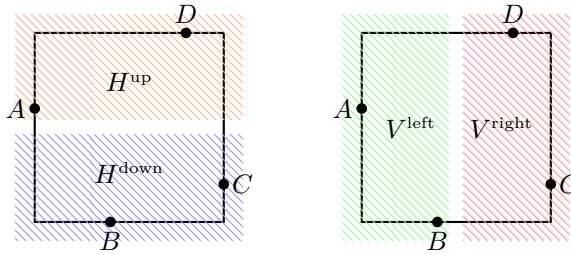


FIGURE 6. The horizontal and the vertical subdivision of R .

Case 1: $A \in H^{\text{up}}$. Then $A \neq B$ and the exceptionality condition imply that either $A - B$ or $B - A$ lies in \mathcal{I} . However, $A - B = (\underline{a} - a_B, b_A - \underline{b}) \in \mathcal{I}$ is impossible for $b_A - \underline{b} \geq \ell_2$ and $\underline{a} - a_B \leq 0$. Hence $A < B$ and in particular $B \in V^{\text{left}}$ as $\underline{b} - b_A < 0$, so $a_B - \underline{a} < \ell_1$. We conclude in a similar way that necessarily $B < C$ and $C \in H^{\text{down}}$. It follows that $A < C$, but this is impossible since $\bar{a} - \underline{a} \geq W(s) - 1 \geq \ell_1$ while $b_C - b_A < 0$.

Case 2: $A \in H^{\text{down}}$. Since $\underline{a} - a_D \leq 0$ and $b_A - \bar{b} < 0$ we necessarily have $A < D$ and $D \in V^{\text{left}}$. We conclude in a similar way that $C < D$ and $C \in H^{\text{up}}$, thus $B < C$ and $B \in V^{\text{right}}$. But then $B < D$ which is impossible for $a_D - a_B < 0$ and $\bar{b} - \underline{b} \geq H(s) - 1 \geq \ell_2$. \square

6.2. AN INDUCTIVE ARGUMENT. Our mainstream development for proving fullness will pursue a rather algorithmic approach based on the lexicographical order from Subsection 6.3 and the classification of maximal exceptional sequences in Subsection 7.2. As an aside, we briefly sketch an inductive approach to Theorem E which is based on the following *collapsing procedure*.

LEMMA 6.2. For $\ell_2 \geq 3$ let s be an exceptional sequence on $(\ell_1, \ell_2; 0)$ with $s_0 = 0$ and $H(s) \leq 2\ell_2 - 1$. Then we obtain a sequence s' on

$$(\ell'_1, \ell'_2; 0) = (\ell_1, \ell_2 - 1; 0)$$

via the following procedure:

- (i) Remove the horizontal line at level $\ell_2 - 1$ from s .
- (ii) For every $s_i = (a_i, b_i)$ with $b_i \geq \ell_2$ put $s'_i := s_i - (0, 1)$.
- (iii) For all remaining s_i put $s'_i := s_i$.

If we endow s' with the order induced by s , then s' defines an exceptional sequence, too.

Proof. Consider $s_i = (a_i, b_i) < s_j = (a_j, b_j)$ and assume that $b_i, b_j \neq \ell_2 - 1$. We denote the co-immaculate locus of $(\ell'_1, \ell'_2; 0)$ by \mathcal{I}' .

Case 1. If $b_i \geq b_j$ or $b_i + 2 \leq b_j \leq b_i + \ell_2 - 2$, then $s'_1 < s'_2$ is immediate.

Case 2. If $b_j = b_i + 1$, then both s_i and s_j belong to the same side either above or below the removed line. Hence, $s'_j - s'_i = s_j - s_i = (a_j - a_i, 1) \in \mathcal{I}'$.

Case 3. If $b_j \geq b_i + \ell_2 - 1$, then s_i sits below and s_j sits above the line at level $\ell_2 - 1$. Thus, $s'_j - s'_i = s_j - s_i - (0, 1) \in \mathcal{I}'$. □

We say that s' is obtained from s by *collapsing along* ℓ_2 . Similarly, we can collapse along ℓ_1 if $W(s) \leq 2\ell_1 - 1$, cf. also Theorem 6.7 and Lemma 9.7.

As an immediate consequence of the collapsing procedure we obtain that for a maximal exceptional sequence where $\#s = \ell_1\ell_2$, the inequalities $\#s' \leq \ell_1(\ell_2 - 1)$ and $\#[y = \ell_2 - 1] \cap s \leq \ell_1$ imply that

$$\#s' = \ell_1(\ell_2 - 1) \quad \text{and} \quad \#(s \cap [y = \ell_2 - 1]) = \ell_1.$$

Therefore, s' is maximal, too, and we actually removed a horizontal chain in the sense of Definition 5.9.

This allows to prove fullness in a rather implicit way:

COROLLARY 6.3 (Theorem E, product version). On $(\ell_1, \ell_2; 0)$ every maximal exceptional sequence s with $H(s) \leq 2\ell_2 - 1$ is full.

Of course, the same result holds for $W(s) \leq 2\ell_1 - 1$.

Sketch of the proof. Proceeding by induction we may assume that the collapsed sequence s' generates \mathbb{Z}^2 by filling horizontal and vertical lines whenever there are ℓ_1 or ℓ_2 consecutive points, respectively.

Given s we begin by using the horizontal chain at level $\ell_2 - 1$ to fill the entire horizontal line on which it lies. Afterwards, we may lift all line fillings from s' to s since any vertical chain of s' must reach height $\ell_2 - 1$. □

6.3. MAXIMAL EXCEPTIONAL SEQUENCES ARE ORDERABLE.

THEOREM 6.4 (Theorem A, product version). Let $s = (s_0, \dots, s_N)$ be a maximal exceptional sequence on $(\ell_1, \ell_2; 0)$. Then s is vertically or horizontally orderable.

Proof. Assume that s is a maximal exceptional sequence which is not vertically orderable.

Reordering if necessary, Lemma 5.5 implies that there exists a pair $s_i < s_{i+1}$ with $b_i - b_{i+1} \geq \ell_2$. Upon applying i -times the helix operator from Subsection 4.4 we may replace s by a sequence where $i = 0$. Choosing suitable coordinates we may therefore suppose without loss of generality that

$$s_0 = (0, 0), \quad s_1 = (\kappa, \lambda)$$

for $\lambda \leq -\ell_2$ and $0 < \kappa < \ell_1$.

Next, there exists precisely one $i_0 > 0$ such that $\Phi(s_{i_0}) = [1, 0]$ by Lemma 4.4. Hence, $s_{i_0} \in ((1, 0) + L) \cap \mathcal{I}$, where we recall that

$$L = \mathbb{Z}(\ell_1, 0) \oplus \mathbb{Z}(0, \ell_2).$$

Consequently,

$$s_{i_0} = (1, m\ell_2) \in \mathcal{P}$$

for some $m \in \mathbb{Z}$, where \mathcal{P} is the parallelogram of the co-immaculate locus, cf. Subsection 3.3.

On the other hand, s_{i_0} is equal to or a successor of $s_1 = (\kappa, \lambda)$, hence

$$s_{i_0} \in ((\kappa, \lambda) + \mathcal{I}) \cup \{(\kappa, \lambda)\}.$$

But $\lambda \leq -\ell_2$ implies $\mathcal{P} \cap ((\kappa, \lambda) + \mathcal{P}) = \emptyset$, so that $s_{i_0} \in \mathcal{I}$ yields the contradiction $0 < \kappa < 1$ unless $i_0 = 1$, that is, $\kappa = 1$ and $m \leq -1$.

Now any point $s_j = (a_j, b_j)$, $j \geq 2$ must be in

$$\mathcal{I} \cap ((1, m\ell_2) + \mathcal{I}) = \{(a, b) \in \mathbb{Z}^2 \mid 1 < a < \ell_1\}.$$

By Remark 5.6 it follows that s must be horizontally orderable. □

REMARK 6.5. As s^1 and s^2 in Example 4.2 show for suitable a and b , a maximal exceptional sequence is not necessarily both vertically and horizontally orderable. In particular, there are maximal exceptional sequences examples of height equal to or less than $2\ell_2 - 1$ which are either vertically or horizontally orderable, but not both. Similarly for constrained width.

6.4. THE DICHOTOMY OF MAXIMAL EXCEPTIONAL SEQUENCES. As a first step towards the classification of maximal exceptional sequence we want to combine a lexicographical order with a spatial constraint.

LEMMA 6.6. *Let s be a maximal exceptional sequence with $H(s) \geq 2\ell_2$. Then s is horizontally orderable.*

Proof. By Theorem 6.4, it is enough to show that vertical orderability implies horizontal orderability. So if $<$ denotes the vertical order on s , we will show that $a_j - a_i > -\ell_1$ whenever $s_i < s_j$, and appeal again to Remark 5.6.

By definition, $s_N - s_i = (a_N - a_i, b_N - b_i)$ is in the co-immaculate locus for all $i < N$. If $b_N - b_i \geq \ell_2$, then

$$0 < a_N - a_i < \ell_1$$

by the equations defining \mathcal{I} . If $b_N - b_i < \ell_2$, then $H(s) \geq 2\ell_2$ implies that b_i is at least at level ℓ_2 whence

$$0 < a_i - a_0, a_N - a_0 < \ell_1.$$

At any rate, $-\ell_1 < a_N - a_i < \ell_1$.

Next consider $s_i = (a_i, b_i) < s_j = (a_j, b_j)$ for $j < N$. By the above, $a_j - a_N > -\ell_1$. Since $b_i \leq b_j$, $b_N - b_i < \ell_2$ implies $b_N - b_j < \ell_2$ and therefore $0 < a_i - a_0, a_j - a_0 < \ell_1$. In particular, $a_j - a_i > -\ell_1$. On the other hand, $b_N - b_i \geq \ell_2$ yields also

$$a_j - a_i = a_j - a_N + a_N - a_i > -\ell_1.$$

□

This gives rise to the following dichotomy. If s satisfies $H(s) \geq 2\ell_2$, then Lemma 6.6 and Theorem 6.1 imply horizontal orderability and $W(s) \leq 2\ell_1 - 1$. Similarly, $W(s) \geq 2\ell_1$ implies vertical orderability and $H(s) \leq 2\ell_2 - 1$. Moreover, independently on $H(s)$ or $W(s)$, s is either vertically or horizontally orderable by Theorem 6.4. This implies the

THEOREM 6.7 (Dichotomy of maximal exceptional sequences). *To any maximal exceptional sequence s on $(\ell_1, \ell_2; 0)$ at least one of the following two items applies:*

- (i) s is vertically orderable with $H(s) \leq 2\ell_2 - 1$
- (ii) s is horizontally orderable with $W(s) \leq 2\ell_1 - 1$.

For sake of concreteness we will concentrate on the first case for the remainder of this paper since this fits into the twisted case; mutatis mutandis everything which follows also applies to the second case.

7. THE CLASSIFICATION FOR THE PRODUCT CASE

7.1. HELEXING. Recall from Subsection 4.4 the definition of the helix operator \hbar which sends a maximal exceptional sequence s on $(\ell_1, \ell_2; 0)$ to

$$\hbar(s) = (s_1, \dots, s_N, s_0 + (\ell_1, \ell_2)).$$

LEMMA 7.1. *Let s be a vertically ordered maximal exceptional sequence on $(\ell_1, \ell_2; 0)$ with $s_0 = 0$ and $\ell_2 < H(s) \leq 2\ell_2 - 1$. Then $\hbar(s)$ is vertically orderable with $H(\hbar(s)) \leq 2\ell_2 - 1$.*

Proof. Let

$$s' = (s'_0, \dots, s'_N) := \hbar(s).$$

Since $H(s) > \ell_2$ there is a point $s_i = (a_i, b_i)$ with $b_i \geq \ell_2$ for some $i = 1, \dots, N$. Therefore, $H(\hbar(s)) \leq H(s) \leq 2\ell_2 - 1$.

Next assume that $s' = \hbar(s)$ is not vertically orderable. By Proposition 5.3, there exists a pair $s'_i < s'_j$ with $b'_i - b'_j \geq \ell_2$. Since $s'_i = s_{i+1}$ and $s'_j = s_{j+1}$ if $j < N$, vertical orderability of s implies that $j = N$. However, $b_{i+1} - \ell_2 \geq \ell_2$ implies $b_{i+1} \geq 2\ell_2$, contradicting $H(s) \leq 2\ell_2 - 1$. □

It follows that we can vertically reorder $\hbar(s)$. Shifting yields an exceptional sequence $\hbar_{\text{lex}}(s)$ starting at the origin. We call \hbar_{lex} the *heLex operator*.

THEOREM 7.2 (Theorem C, product version). *On $(\ell_1, \ell_2; 0)$ every vertically orderable maximal exceptional sequence s of height $H(s) \leq 2\ell_2 - 1$ can be transformed into a vertically trivial sequence by vertically reordering and successively applying \hbar_{lex} at most $\ell_1\ell_2$ times.*

Proof. If s is vertically ordered with $s_0 = 0$, the helix operator \hbar sends s_0 to (ℓ_1, ℓ_2) . Unless $H(s) \leq \ell_2$, that is, s is vertically trivial, either

- (i) the lowest row becomes empty, whence $H(s)$ decreases at least by one, or
- (ii) the height $H(s)$ remains unchanged, so that we are reducing the number of elements in $[y = 0] \cap s$ by one.

By induction we eventually arrive at a maximal exceptional sequence of height $H(s) \leq \ell_2$, i.e., a vertically trivial one. □

7.2. THE CLASSIFICATION OF MAXIMAL EXCEPTIONAL SEQUENCES. Next we discuss the classification of the sets underlying a maximal exceptional sequence on $(\ell_1, \ell_2; 0)$ by giving an explicit algorithm for their construction. We recall that we tacitly assume to work with vertically orderable sequences s of height $H(s) \leq 2\ell_2 - 1$.

For any subset $X \subseteq \Delta^{\text{up}} = \{(a, b) \in \mathbb{Z}^2 \mid b \geq \ell_2, 0 < a < \ell_1\}$ or, more generally, for any $X \subseteq \mathbb{Z}^2$ we let

$$X_k := [y = k] \cap X = \{(a, b) \in X \mid b = k\}.$$

DEFINITION 7.3. We call a non-empty set $X \subseteq \Delta^{\text{up}}$ admissible if

- (Ai) $X_k = \emptyset$ for $k \geq 2\ell_2 - 1$.
- (Aii) the layers $X_k \neq \emptyset$ consist of successive points $(x, k), (x+1, k), \dots, (x+q_k, k)$.

(Aiii) for each $k \geq \ell_2$ we have

$$(0, -1) + X_{k+1} \subseteq X_k.$$

(Aiv) the bottom layer X_{ℓ_2} is right-aligned, i.e., $(\ell_1 - 1, \ell_2) \in X_{\ell_2}$.

By convention, the empty set will be admissible, too.

In addition to an admissible set $X \subseteq \Delta^{\text{up}}$ we need a further set to completely classify maximal exceptional sequences.

DEFINITION 7.4. Let $\emptyset \neq X \subseteq \Delta^{\text{up}}$ be admissible. Then $Y \subseteq \mathbb{Z}^2$ is called a complementing partner of X if

- (Cv) $Y_k = \emptyset$ for $k \geq 2\ell_2$.
- (Cvi) $Y_k < X_k$, meaning that $(y, k) \in Y_k$ and $(x, k) \in X_k$ imply $y < x$.
- (Cvii) for each $k \in \{\ell_2, \dots, 2\ell_2 - 1\}$, the union $Y_k \cup X_k$ forms a horizontal chain.

For $X = \emptyset$ a complementing partner will be any set Y consisting of ℓ_2 horizontal chains $Y_{\ell_2}, \dots, Y_{2\ell_2-1}$ such that Y_{ℓ_2} starts at $(0, \ell_2)$.

REMARK 7.5. Whenever $X_k \neq \emptyset$, the complementary set Y_k is uniquely determined. In contrast, $X_k = \emptyset$ for $k \in \{\ell_2, \dots, 2\ell_2 - 1\}$ implies that Y_k defines a horizontal chain whose horizontal position is unrestricted unless $k = \ell_2$.

EXAMPLE 7.6. Figure 7 displays a typical maximal exceptional sequence together with its admissible set and complementing partner.

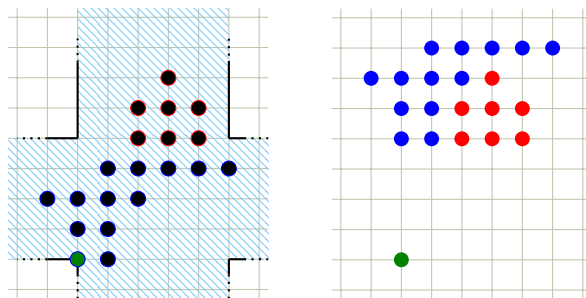


FIGURE 7. A maximal exceptional sequence on $(5, 4; 0)$ with starting point at the origin in green. The right hand side displays its corresponding admissible set X in red with complementing partner Y in blue.

THEOREM 7.7 (Theorem D, product version). *Let $X \subseteq \Delta^{\text{up}}$ be admissible and $Y \subseteq \mathbb{Z}^2$ a complementing partner. Then the union of*

$$s_{\downarrow} := Y + (0, -\ell_2) \subseteq \mathcal{H} \cup [y = 0] \quad \text{and} \quad s^{\uparrow} := X \subseteq \Delta^{\text{up}}$$

together with vertical order yields a maximal exceptional sequence s with $H(s) \leq 2\ell_2 - 1$ and $s_0 = 0$. Moreover, any vertically ordered maximal exceptional sequence starting at the origin arises this way.

Proof. If $X = \emptyset$, then Y consists of ℓ_2 consecutive horizontal chains. Shifting down by $(0, -\ell_2)$ yields a vertically trivial sequence starting at the origin.

We therefore assume that $X \neq \emptyset$. We order the set $s_{\downarrow} \cup s^{\uparrow}$ vertically to obtain the sequence s . From Definition 5.9, Definition 7.3 (iv), and Definition 7.4 (Cvii), it is clear that $Y_{\ell_2} \cup X_{\ell_2}$ forms a horizontal chain which ends at $(\ell_1 - 1, \ell_2)$. Consequently, Y_{ℓ_2} starts at $(0, \ell_2)$ and $s_0 = 0$.

For $s_i < s_j$ we have to show that $s_j - s_i \in \mathcal{I}$. If $s_i, s_j \in [y = b]$, then this follows from $s_i, s_j \in Y_{b+\ell_2} + (0, -\ell_2)$ or $s_i, s_j \in X_b$, (Cvi) and (Cvii). If they are at different levels $0 \leq b_i < b_j \leq 2\ell_2 - 1$, the only critical case arises from $b_j - b_i \geq \ell_2$ which implies $0 \leq b_i \leq \ell_2 - 1$ and $\ell_2 \leq b_j \leq 2\ell_2 - 1$. This means

$$s_i \in [y = b_i] \cap s_{\downarrow} = Y_{b_i+\ell_2} + (0, -\ell_2) \quad \text{and} \quad s_j \in [y = b_j] \cap s^{\uparrow} = X_{b_j}.$$

We proceed via induction over $m := b_j - b_i - \ell_2 \geq 0$.

If $m = 0$, then $s_i \in Y_{b_j} + (0, -\ell_2)$ whence

$$s_j - (s_i + (0, \ell_2)) \in X_{b_j} - Y_{b_j} \subseteq \{(1, 0), \dots, (\ell_1 - 1, 0)\}.$$

In particular, $s_j - s_i \in (0, \ell_2) + \{(1, 0), \dots, (\ell_1 - 1, 0)\} \subseteq \Delta^{\text{up}} \subseteq \mathcal{I}$.

Next let $m \geq 1$. By (Aiii) we know that $s_j - (0, 1) \in s^{\uparrow}$. On the other hand, the induction hypothesis implies $s_j - (0, 1) - s_i \in \mathcal{I}$, and since $b_j - 1 - b_i - \ell_2 \geq 0$ we even have $s_j - (0, 1) - s_i \in \Delta^{\text{up}}$. Further, $s_j - s_i \in \Delta^{\text{up}}$ by definition of Δ^{up} whence $s_j - s_i \in \mathcal{I}$.

Finally, we want to show that any vertically ordered maximal exceptional sequence s with $s_0 = 0$ and $H(s) \leq 2\ell_2 - 1$ arises this way. For this, we let

$$Y := (0, \ell_2) + (s \cap (\mathcal{H} \cup [y = 0])) \quad \text{and} \quad X := s \cap \Delta^{\text{up}}$$

where we identify the sequence s with its underlying set.

If s is vertically trivial, then $Y = (0, \ell_2) + s$ and $X = \emptyset$. We therefore assume that s is not trivial and check that X is admissible with complementing partner Y . Properties (Ai) and (Cv) follow from $H(s) \leq 2\ell_2 - 1$ and the definition of Y , respectively. Furthermore, (Aii) follows from Corollary 5.4, while (Cvii) is a consequence of Proposition 5.11.

By Corollary 5.16, $[y = b] \cap s \neq \emptyset$ for $0 \leq b \leq H(s) - 1$. Furthermore, we have $l = 2\ell_2 - H(s)$ horizontal lines by Corollary 5.13. For $b = 0, \dots, \ell_2 - l - 1$, we put here and in the sequel

$$(8) \quad 0 \leq r_b := \#[y = b] \cap s - 1 < \ell_1 - 1$$

so that $[y = b] \cap s = \{(a_b, b), \dots, (a_b + r_b, b)\}$.

In particular, we have $\{(r_0 + 1, \ell_2), \dots, (\ell_1 - 1, \ell_2)\} = [y = \ell_2] \cap s$ which implies (Aiv). For all other $0 < b < \ell_2 - l - 1$, $Y_{b+\ell_2} = (0, \ell_2) + [y = b] \cap s$ is either to the left or to the right of $X_{b+\ell_2}$. If $Y_{b+\ell_2}$ is to the right, then $(a_b - 1, b + \ell_2) \in X_{b+\ell_2}$ whence

$$(a_b - 1, b + \ell_2) - (a_b + r_b, b) = (-1 - r_b, \ell_2) \in \mathcal{I},$$

contradiction. This implies (Cvi).

Finally, Proposition 5.11 implies $(a_b + r_b + 1, b + \ell_2) \in X = s \cap \Delta^{\text{up}}$. It follows that $(a_b + r_b + 1, b + \ell_2) - (a_{b-1} + r_{b-1}, b - 1) \in \mathcal{I}$ whence $a_b + r_b \geq a_{b-1} + r_{b-1}$. Similarly, we

have $a_b \leq a_{b-1}$. In particular, $(0, 1) + Y_{b+\ell_2} \subseteq Y_{b+\ell_2+1}$ and thus $(0, -1) + X_{b+\ell_2+1} \subseteq X_{b+\ell_2}$ which yields (Aiii). \square

7.3. GENERATING THE DERIVED CATEGORY. We can use Theorem 7.7 to prove fullness of any maximal exceptional sequence on $(\ell_1, \ell_2; 0)$. By the dichotomy principle, it suffices to consider the case of a vertically orderable sequence s of height $H(s) \leq 2\ell_2 - 1$. The case of horizontally orderable sequences of width $W(s) \leq 2\ell_1 - 1$ follows analogously.

Let s be a maximal exceptional sequence. Since the standard rectangle $\mathcal{R}_{\ell_1, \ell_2}$ generates the Picard group via the horizontal and vertical Beilinson sequence it suffices to show that $\mathcal{R}_{\ell_1, \ell_2} \subseteq \langle s \rangle$.

EXAMPLE 7.8. We consider again the sequence s from Example 7.6. Figure 8 displays our strategy to fill all of \mathbb{Z}^2 starting from s . After filling horizontal and vertical lines in Steps (a)-(e) we see that $\mathcal{R}_{5,4} \subseteq \langle s \rangle$.

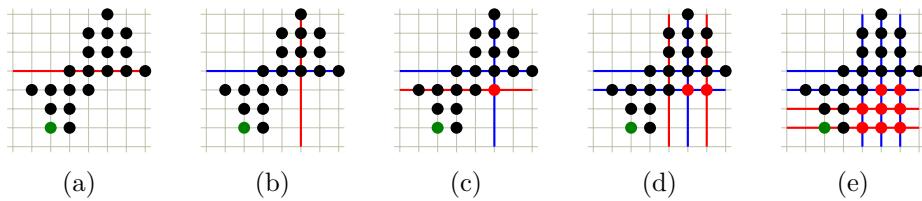


FIGURE 8. Filling $\text{Pic}(5, 4; 0, 0)$ from s . The green dot marks the origin. The red dots are generated by s and are used to fill further lines.

THEOREM 7.9 (Theorem E, product version). *On $(\ell_1, \ell_2; 0)$ every maximal exceptional sequence s is full.*

Proof. We are indebted to the referee for pointing out to us the following much more elegant version of the proof.

We may assume that s is of the form given in Theorem 7.7 and proceed by induction on $\#X$. If the admissible set $X \subseteq \Delta^{\text{up}}$ is empty, we have already observed in the proof of Theorem 7.7 that s is vertically trivial.

Next consider $\#X > 0$. First, we fill the lines $[y = b]$ for $b \in \{0, \dots, \ell_2 - 1\}$ with $X_{b+\ell_2} = \emptyset$, that is, $(s_\downarrow)_b = Y_{b+\ell_2} + (0, -\ell_2)$ is a horizontal chain. Second, we pick the left-most element (A, B) of the top row of X . Because of (Aiii), the vertical line $[x = A]$ contains a vertical chain built from the elements of X and the horizontal lines just filled. In particular, filling this vertical line yields $(A, B - \ell_2) \in \langle s \rangle$.

Now we may consider s' built from $X' := X \setminus \{(A, B)\}$. This is still an admissible set with $\#X' = \#X - 1$. Moreover, $s' \subseteq s \cup \{(A, B - \ell_2)\} \subseteq \langle s \rangle$, hence $\mathbb{Z}^2 = \langle s' \rangle \subseteq \langle s \rangle$. \square

8. MAXIMAL EXCEPTIONAL SEQUENCES IN THE TWISTED CASE

For the rest of this article we assume $c \neq 0$ and set out to prove the main Theorems A-E in the subsequent sections. We start with some examples.

8.1. MAXIMAL EXCEPTIONAL SEQUENCES WITH $\ell_1 = 2$. The integral depth was defined as

$$d_{\text{int}} = - \left\lfloor \frac{\ell_1 - 2}{\alpha} \right\rfloor,$$

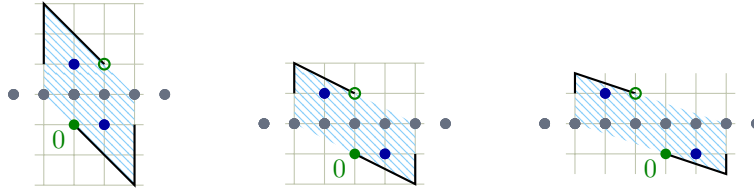


FIGURE 9. The Hirzebruch surfaces \mathcal{H}_1 , \mathcal{H}_2 and \mathcal{H}_3 .

cf. Subsection 5.2. If $\ell_1 = 2$, then $d_{\text{int}} = 0$. Therefore, the co-immaculate locus of $(2, \ell_2; c)$ is given by

$$\mathcal{I}(2, \ell_2; \alpha, \beta) = \mathcal{H} \cup \{0, (1, 0), (-\beta + 1, \ell_2)\}$$

and the variety is orderable. Hence, the only nontrivial family of maximal exceptional sequences up to shifts is given by

$$s = \{0, (a_1, 1), (a_1 + 1, 1), \dots, (a_{\ell_2-1}, \ell_2 - 1), (a_{\ell_2-1} + 1, \ell_2 - 1), (-\beta + 1, \ell_2)\}$$

with $a_i \in \mathbb{Z}$. We have $H(s) = \ell_2 + 1$. In the language of admissible sets and complementing partners which will be developed for the twisted case in Subsection 10.2, we have

$$X = \{(-\beta + 1, \ell_2)\} \text{ and } Y = \{(-\beta, \ell_2), (a_1 - \beta, 1 + \ell_2), \dots, (a_{\ell_2-1} + 1 - \beta, 2\ell_2 - 1)\}.$$

For instance, we obtain the family of Hirzebruch surfaces $\mathcal{H}_\alpha = (2, 2; (0, -\alpha))$ by setting $\ell_2 = 2$. Among these, $\mathcal{H}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ is the only Fano variety. Figure 9 represents $\mathcal{I}(\mathcal{H}_\alpha) = \mathcal{I}(2, 2; \alpha, \alpha)$ for $\alpha = 1, 2$ and 3 .

8.2. MAXIMAL EXCEPTIONAL SEQUENCES WITH $\ell_1 = \ell_2 = 3$. Since $d_{\text{int}} \geq -1$ any variety $(3, 3; c)$ is necessarily vertically orderable. In order to determine the nontrivial maximal exceptional sequences s up to shift we may therefore assume that s is vertically lexicographically ordered. We distinguish two cases:

Case 1: $\alpha \geq 2$. Then $d_{\text{int}} = 0$. As in Subsection 8.1 we find the maximal exceptional sequences

$$\begin{aligned} s^1 &= (0, (1, 0), (a_1, 1), (a_1 + 1, 1), (a_1 + 2, 1), \\ &\quad (a_2, 2), (a_2 + 1, 2), (a_2 + 2, 2), (-\beta + 2, 3)) \\ s^2 &= (0, (a_1, 1), (a_1 + 1, 1), (a_1 + 2, 1), \\ &\quad (a_2, 2), (a_2 + 1, 2), (a_2 + 2, 2), (-\beta + 1, 3), (-\beta + 2, 3)) \end{aligned}$$

for $a_1, a_2 \in \mathbb{Z}$. We have $H(s^1) = H(s^2) = 4$.

Case 2: $\alpha = 1$. By the basic inequality (3), $\beta = 1$ or 2 . In addition to the maximal exceptional sequences from the previous case we find

$$\begin{aligned} s^3 &= (0, (-1, 1), (0, 1), (a_2, 2), (a_2 + 1, 2), (a_2 + 2, 2), \\ &\quad (-\beta + 1, 3), (-\beta + 2, 3), (-\beta + 1, 4)) \end{aligned}$$

for $a_2 \in \mathbb{Z}$. We have $H(s^3) = 5$.

Figure 10 displays the admissible sets and complementing partners for $\beta = 2$.

8.3. MAXIMAL EXCEPTIONAL SEQUENCES ARE ALWAYS VERTICALLY ORDERABLE. General non-extendable sequences might not be vertically orderable.

EXAMPLE 8.1. On $(4, 2; (0, 1))$ consider the exceptional sequence

$$s = (0, (3, -2), (2, -1), (3, -1), (3, 0), (3, 1), (4, 1)),$$

cf. Figure 11. Since $s_1 - s_0 = (3, -2)$ the order of these two elements cannot be

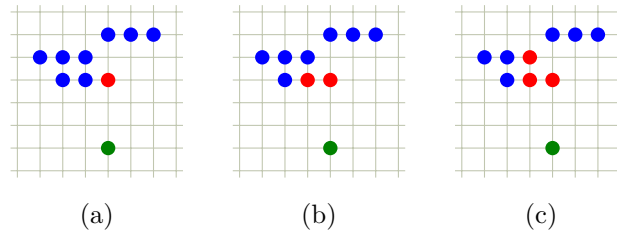


FIGURE 10. The admissible sets (red) and complementing partners (blue) for (a) s^1 (b) s^2 and (c) s^3 with $a_1 = -1$, $a_2 = 2$ and $\beta = 2$. The origin is marked in green.

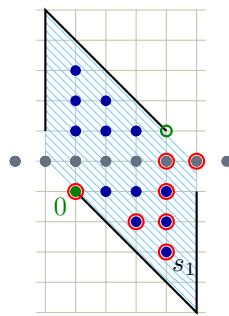


FIGURE 11. The points encircled in red define the sequence s in $\mathcal{I}(4, 2; 1, 1)$.

switched, cf. Proposition 5.3.

On the other hand, the sequence s is not maximal for $\#s = 7 < 8 = \ell_1\ell_2$ and eventually not extendable. An easy computation shows that $\mathcal{I} \cap ((3, -2) + \mathcal{I})$, the set of possible common successors of s_0 and s_1 , is

$$\{(2, -1), (3, -1), (3, 0), (3, 1), (4, 1)\}.$$

Alternatively, one may apply Theorem 8.2 below to show that s is the largest choice for extending $(0, (3, -2))$ to an exceptional sequence.

THEOREM 8.2 (Theorem A, twisted version). *Let $s = (s_0, \dots, s_{\ell_1\ell_2-1})$ be a maximal exceptional sequence on $(\ell_1, \ell_2; c)$, $c \neq 0$. Then s is vertically orderable.*

Proof. Assume to the contrary that s is a maximal exceptional sequence which is not vertically orderable.

By Lemma 5.5 we may assume that there exists a pair $s_i < s_{i+1}$ with $b_i - b_{i+1} \geq \ell_2$. Furthermore, upon applying i -times the helix operator from Subsection 4.4 we may replace s by a sequence where $i = 0$. Choosing suitable coordinates we therefore suppose without loss of generality that

$$s_0 = (0, 0), \quad s_1 = (\kappa, \lambda)$$

for $\lambda \leq -\ell_2$ and $0 < -\lambda\alpha < \kappa < \ell_1$.

By Lemma 4.4, there exists precisely one $i_0 > 0$ such that $\Phi(s_{i_0}) = [1, 0]$. Hence, $s_{i_0} \in ((1, 0) + L) \cap \mathcal{I}$. Reasoning as in the proof of Lemma 3.6 shows that

$$s_{i_0} = (n\ell_1 - m\beta + 1, m\ell_2) \in \mathcal{P}.$$

Comparing with the proof of Lemma 3.6 we have added 1 in the first argument. Instead of a contradiction, we now obtain a unique solution: $m = n = 0$ providing $s_{i_0} = (1, 0)$ for $m \leq 0$ and $m = 1, n = 0$ providing $s_{i_0} = (-\beta + 1, \ell_2)$ for $m \geq 1$.

On the other hand, s_{i_0} is equal to or a successor of $s_1 = (\kappa, \lambda)$, hence

$$(9) \quad s_{i_0} \in ((\kappa, \lambda) + \mathcal{I}) \cup \{(\kappa, \lambda)\}.$$

However, $(1, 0), (-\beta + 1, \ell_2) \notin (\kappa, \lambda) + \mathcal{I}$, contradicting (9). Indeed, we have $\kappa > -\lambda\alpha \geq \alpha\ell_2 > \beta$. Therefore, a point $(a, b) \in \mathbb{Z}^2$ with $a \leq 1$ and $b \geq 0$ cannot lie in $(\kappa, \lambda) + \mathcal{I}$ as

$$a - \kappa \leq 1 - \kappa \leq -\alpha\ell_2 < -\beta,$$

while $b - \lambda \geq \ell_2$ implies $-\beta < a - \kappa$. □

9. VERTICAL ENSEMBLES

We continue with the twisted case $c \neq 0$. In particular, $\alpha, \beta, \gamma \geq 1$.

9.1. REPLACING VERTICAL CHAINS BY ENSEMBLES. Rather than to vertical chains, the V -sequence from Corollary 3.2 gives rise to a complicated shape for the involved locus in $\text{Pic } X = \mathbb{Z}^2$. The sheaves \mathcal{F}_k occurring in Subsection 3.2 suggest the following

DEFINITION 9.1. We denote by

$$V = V(\ell_2; \alpha, \beta) \subseteq \mathbb{Z}^2$$

the set of lattice points $\sum_{j \in J} (f^j, 1) = (\sum_{j \in J} f^j, \#J)$ where $J \subseteq \{1, \dots, \ell_2\}$ is an arbitrary subset of cardinality $1 \leq \#J \leq \ell_2 - 1$ and $f = (f^1, f^2, \dots, f^{\ell_2})$ runs through all integral ℓ_2 -tuples satisfying

$$0 = f^1 \geq f^2 \geq \dots \geq f^{\ell_2-1} \geq f^{\ell_2} = -\alpha \quad \text{and} \quad \sum_{j=1}^{\ell_2} f^j = -\beta.$$

We call the subset $V + (a, b)$ the V -ensemble based at (a, b) .

Note that V fits into the rectangular box bounded by $-\beta \leq x \leq 0$ and $0 \leq y \leq \ell_2$ and whose diagonal of negative slope ends in $(0, 0)$ and $(-\beta, \ell_2)$. In particular, $V \subseteq \mathcal{H}$. Further, in the extreme case $\alpha = \beta$ which necessarily implies $c^1 = c^2 = \dots = c^{\ell_2-1} = 0$, the V -ensemble degenerates to

$$V(\ell_2; \beta, \beta) = \{(-\beta, b), (0, b) \mid b = 1, \dots, \ell_2 - 1\}.$$

See also Figure 12 for an illustration.

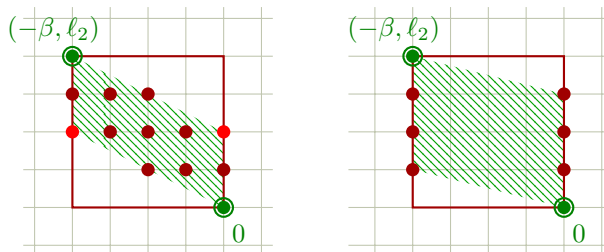


FIGURE 12. The sets $V(4; 2, 4)$ and $V(4; 4, 4)$. On the left hand side the dark red points come from the vector $c = (0, -1 - 1, -2)$. Additional points from $c = (0, 0, -2, -2)$ which are not dark red yet are marked in light red.

We turn to existence of V -ensembles inside exceptional sequences next. To simplify the shape we introduce the larger W -ensemble of $(\ell_1, \ell_2; c)$ by

$$W = W(\ell_2; \alpha) := \{(a, b) \in \mathbb{Z}^2 \mid 0 < b < \ell_2, -b\alpha \leq a \leq 0\}.$$

LEMMA 9.2. $V(\ell_2; \alpha, \beta)$ is symmetric under the transformation $(x, y) \mapsto (-\beta, \ell_2) - (x, y)$. Moreover, $V(\ell_2; \alpha, \beta) \subseteq W(\ell_2; \alpha)$.

Proof. For the symmetry note that $\sum_{j=1}^{\ell_2} (f^j, 1) = (-\beta, \ell_2)$. Therefore,

$$(-\beta, \ell_2) - \sum_{j \in J} (f^j, 1) = \sum_{i \in I} (f^i, 1)$$

with $I = \{1, \dots, \ell_2\} \setminus J$.

Further, $(a, b) \in V$ implies $(a, b) = (\sum_{j \in J_b} f^j, b)$ for some suitable $0 \geq f^j \geq -\alpha$ and $\#J_b = b$. Hence $a \geq -b\alpha$. \square

Let us define

$$\overline{\mathcal{P}}_V = \{(a, b) \in \mathbb{Z}^2 \mid -\beta \leq a \leq 0 \text{ and } 0 \leq \langle (a, b), (1, \alpha) \rangle \leq \gamma\}$$

as a smaller (and closed) version of the co-immaculate parallelogram \mathcal{P} from Subsection 3.3, cf. Figure 13. The $(-\beta, \ell_2)$ -symmetry of V implies the following observation.

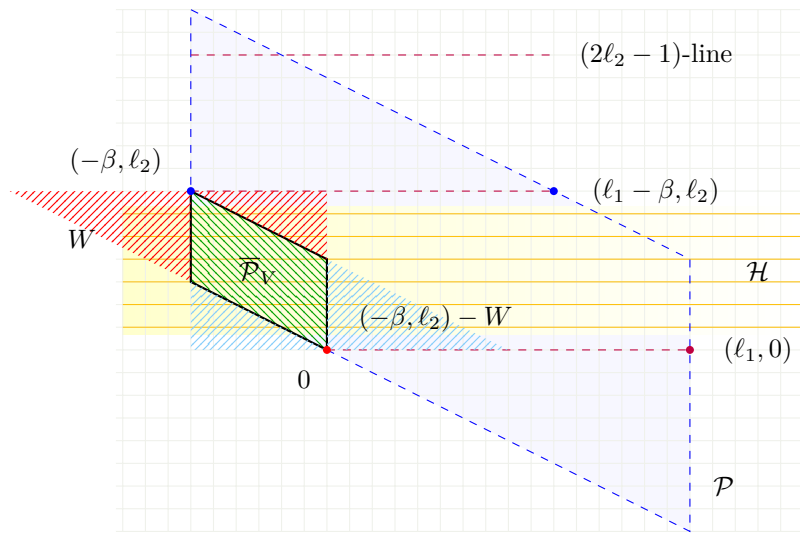


FIGURE 13. $\overline{\mathcal{P}}_V$ and \mathcal{P} and \mathcal{H} for $\mathcal{I}(\ell_1, \ell_2, \alpha, \beta) = \mathcal{I}(16, 7; 2, 6)$

LEMMA 9.3. We have

- (i) $W(\ell_2; \alpha) \cap ((-\beta, \ell_2) - W(\ell_2; \alpha)) = \overline{\mathcal{P}}_V \setminus \{0, (-\beta, \ell_2)\}$.
- (ii) $V(\ell_2; \alpha, \beta) \subseteq \overline{\mathcal{P}}_V \setminus \{0, (-\beta, \ell_2)\}$.

REMARK 9.4. The fact that $f^1 = 0$ and $f^{\ell_2} = -\alpha$ for the sequences defining V allows actually an even more refined description. It turns out that $V \cup \{0, (-\beta, \ell_2)\}$ equals the union of four smaller $\overline{\mathcal{P}}_V$ -like parallelograms located at the four corners of the ambient $\overline{\mathcal{P}}_V$. However, the rather coarse relationship $V \subseteq W$ from Lemma 9.2 will be sufficient for our purposes.

9.2. FINDING V - AND W -ENSEMBLES INSIDE MAXIMAL EXCEPTIONAL SEQUENCES. Our goal is to locate sufficiently many V -ensembles inside any given maximal exceptional sequence s .

LEMMA 9.5. *Let s be a vertically ordered exceptional sequence on $(\ell_1, \ell_2; c)$ with $s_0 = 0$ and $H(s) \geq 2\ell_2$. Then $s \cup W(\ell_2; \alpha)$ is also exceptional with respect to the vertical order.*

Proof. As suggested by Figure 14, $(a, b) - W \subseteq \mathcal{I}$ for all $(a, b) \in \Delta^{\text{up}}$. Indeed, let $(a', b') \in W$. By Subsection 3.3,

$$\ell_2 \leq b, \quad -\beta < a < \ell_1 + \gamma - b\alpha.$$

Since $b' < \ell_2$ it follows that $0 < b - b'$. If $b - b' < \ell_2$, then $(a - a', b - b') \in \mathcal{H}$, and we are done. If $\ell_2 \leq b - b'$, then $0 \leq -a' \leq b'\alpha$ implies

$$-\beta < a - a' < \ell_1 + \gamma - (b - b')\alpha = \ell_1 - \beta - (b - b' - \ell_2)\alpha < \ell_1$$

whence $(a - a', b - b') \in \mathcal{I}$.

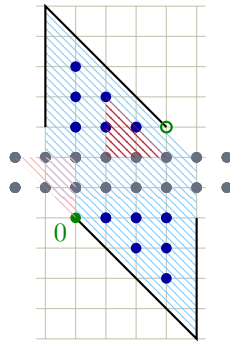


FIGURE 14. $\mathcal{I}(4, 3; 1, 1)$ with the W -ensemble in light red and the set $(1, 4) - W$ in red.

Furthermore, $H(s) \geq 2\ell_2$ implies existence of a point $(a, b) \in s$ with $b \geq 2\ell_2 - 1$. Therefore, $(a, b) - W$ and $(a, b) - (s \cap \mathcal{H})$ are contained in Δ^{up} . In virtue of Lemma 5.14 it follows that $|a' - a''| \leq \ell_1 - 2$ for any $(a', b') \in W$ and $(a'', b') \in (s \cap \mathcal{H}) \cup W$. Hence $s \cup W$ is exceptional. \square

COROLLARY 9.6. *Let s be a vertically ordered maximal exceptional sequence with $H(s) \geq 2\ell_2$. Then $s_0 + W \subseteq s$. In particular, $s_0 + (V \cup \{(0, b) \mid 0 \leq b < \ell_2\})$ is contained in s .*

Proof. Since $s \cup (s_0 + W)$ is an exceptional extension of s , maximality of s implies $s \cup (s_0 + W) \subseteq s$. The second claim follows from

$$s_0 + \{(0, b) \mid 0 \leq b < \ell_2\} \subseteq (s_0 + W) \cup \{s_0\}$$

and Lemma 9.2 which asserts that $V \subseteq W$. \square

9.3. BOUNDING THE HEIGHT. Our final goal in this section is to establish Theorem B in the twisted case. We first define

$$\mathcal{H}^- := \{(a, b) \in \mathbb{Z}^2 \mid a < 0, 0 < b < \ell_2\} \subseteq \mathcal{H}.$$

LEMMA 9.7. *Let s be a maximal exceptional sequence on $(\ell_1, \ell_2; c)$ with $c \neq 0$, $\ell_1 \geq 3$ and $H(s) \geq 2\ell_2$. If s is vertically ordered with $s_0 = 0$, then we obtain a sequence s' on*

$$(\ell'_1, \ell'_2; c') = (\ell_1 - 1, \ell_2; c)$$

via the following procedure:

- (i) Remove the set $\{(0, b) \mid 0 \leq b < \ell_2\}$ from s .
- (ii) For every $s_i = (a_i, b_i) \in \mathcal{H}^-$ put $s'_i := s_i + (1, 0)$.
- (iii) For all remaining s_i put $s'_i := s_i$.

If we endow s' with the order induced by s , then s' defines a maximal exceptional sequence. We say that s' is obtained from s by collapsing along ℓ_1 .

Proof. We denote the co-immaculate locus of $(\ell'_1, \ell'_2; c')$ by \mathcal{I}' . Let $s'_i < s'_j$ be a pair of elements in s' coming from $s_i = (a_i, b_i) < s_j = (a_j, b_j)$ in s . We need to show that $s'_j - s'_i \in \mathcal{I}'$. Figure 15 sketches how the co-immaculate locus adjusts \mathcal{I}' under passing from $(\ell_1, \ell_2; c)$ to $(\ell_1 - 1, \ell_2; c)$.

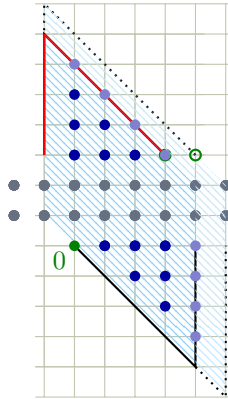


FIGURE 15. $\mathcal{I}(5, 3; 1, 1)$ and $\mathcal{I}' = \mathcal{I}(4, 3; 1, 1)$. The additional points in $\mathcal{I}(5, 3; 1, 1)$ are marked in light blue. The upper left boundary and the upper right boundary are marked in red.

Case 1: $s_i, s_j \notin \mathcal{H}^-$. Then $s'_j - s'_i = s_j - s_i$. If $\ell_2 > b_j - b_i$, then $s'_j - s'_i \in \mathcal{H} = \mathcal{H}'$ or $b_j = b_i$. In the latter case we have $a_j - a_i < \ell_1 - 1$ for otherwise, s would have a horizontal chain and thus $H(s) \leq 2\ell_2 - 1$ by Corollary 5.15.

If, on the other hand, $b_j - b_i \geq \ell_2$, then $s'_j - s'_i \in \mathcal{I}'$ unless

$$a_j - a_i + (b_j - b_i)\alpha = \ell_1 - 1 - \beta + \ell_2\alpha = \ell'_1 - \beta + \ell_2\alpha,$$

that is, $s_j - s_i$ sits in the upper right boundary of \mathcal{P}' , cf. Figure 15. Since $\ell_2 \leq b_j$ we have $a_j + b_j\alpha \leq \ell_1 - 1 - \beta + \ell_2\alpha$. But $0 < a_i + b_i\alpha$ - this is clear for $s_i \in \mathcal{H}$ and follows from the defining inequalities of the parallelogram if $s_i \in \mathcal{P}$. Hence

$$a_j - a_i + (b_j - b_i)\alpha < a_j + b_j\alpha \leq \ell_1 - 1 - \beta + \ell_2\alpha,$$

contradiction.

Case 2: $s_i, s_j \in \mathcal{H}^-$. Again $s'_j - s'_i = s_j - s_i$, and $s_j - s_i \in \mathcal{H}$ or $b_j = b_i$. As for Case 1 we find that $s'_j - s'_i \in \mathcal{H}' \subseteq \mathcal{I}'$.

Case 3: $s_i \in \mathcal{H}^-, s_j \notin \mathcal{H}^-$. Then $s'_j - s'_i = s_j - s_i - (1, 0)$. If $\ell_2 > b_j - b_i$, we conclude as in Case 1.

If, on the other hand, $b_j - b_i \geq \ell_2$, then $s'_j - s'_i \in \mathcal{I}'$ unless

$$s_j - s_i = (a_j - a_i, b_j - b_i) = (-\beta + 1, b_j - b_i),$$

that is, $s_j - s_i$ sits in the upper left boundary of \mathcal{P}' , cf. Figure 15. Now $\ell_2 < b_j$ implies $a_j - a_i = -\beta + 1 \leq a_j$, but $a_i < 0$.

Case 4: $s_i \notin \mathcal{H}^-, s_j \in \mathcal{H}^-$. Then $s'_j - s'_i = s_j + (1, 0) - s_i$ and $\ell_2 > b_j > b_i \geq 0$, where the middle inequality follows from $a_j < a_i$. We conclude again as in Case 1. \square

COROLLARY 9.8 (Theorem B, twisted version). *Let s be a maximal exceptional sequence. Then $H(s) \leq 2\ell_2$.*

Proof. We assume that s is vertically ordered and starts at $s_0 = 0$. We proceed by induction on $\ell_1 \geq 2$.

If $\ell_1 = 2$, then $H(s) \leq \ell_2 + 1 < 2\ell_2$, cf. Subsection 8.1.

Next assume that $\ell_1 \geq 3$. Let s be a maximal exceptional sequence with $H(s) > 2\ell_2$. By Lemma 9.7 we can collapse s along ℓ_1 and obtain the maximal exceptional sequence s' on $(\ell_1 - 1, \ell_2; c)$ with $2\ell_2 \leq H(s) - 1 \leq H(s')$. Here, the latter inequality is a consequence of $W \subseteq s$ from Corollary 9.6. Further, $H(s') \leq 2\ell_2$ by induction hypothesis so that $H(s') = 2\ell_2 < H(s)$. In particular, the collapsed sequence s' starts at $s'_0 = (a, 1)$ with $a \leq 0$.

By Corollary 9.6, $s'_0 + W = (a, 1) + W \subseteq s'$. Therefore

$$(a, 1) + (-(\ell_2 - 1)\alpha, \ell_2 - 1) = (-(\ell_2 - 1)\alpha + a, \ell_2) \in s'.$$

By design of the collapsing procedure, $(-(\ell_2 - 1)\alpha + a, \ell_2)$ is also in s whence

$$-\beta < -(\ell_2 - 1)\alpha + a \leq -(\ell_2 - 1)\alpha.$$

But this contradicts the basic inequality (3). □

10. THE CLASSIFICATION FOR THE TWISTED CASE

We now discuss the twisted analogues of heLexing (cf. Subsection 7.1) and the structure of maximal exceptional sequences (cf. Subsection 7.2).

10.1. HELEXING. Let s be a maximal exceptional sequence starting at the origin which by Theorem 8.2 we may take to be vertically ordered. The helix operator \hbar sends s_0 , the leftmost element of the lowest row, to the point $(\ell_1 - \beta, \ell_2)$ at level ℓ_2 , using the terminology of Subsection 5.3.

In Subsection 7.1 we considered \hbar_{lex} which was the helix operator \hbar followed by vertically lexicographic reordering and a shift sending the resulting s_0 back to the origin. The proof of Proposition 7.2 applies verbatim and yields the

THEOREM 10.1 (Theorem C, twisted version). *Every maximal exceptional sequence on $(\ell_1, \ell_2; c)$, $c \neq 0$, can be transformed into a trivial sequence by successively applying \hbar_{lex} at most $\ell_1\ell_2$ times.*

10.2. THE CLASSIFICATION. Again we can establish an algorithmic recipe for the construction of maximal exceptional sequences.

The definition of admissible sets and complementing partners carries over from Subsection 7.2 except for the modified shape of Δ^{up} , namely

$$\Delta^{\text{up}} = \{(a, b) \in \mathcal{P} \mid b \geq \ell_2\} = \{(a, b) \in \mathbb{Z}^2 \mid b \geq \ell_2, -\beta < a \text{ and } a + b\alpha < \ell_1 + \gamma\}$$

and (Aiii) which gets replaced by

(Aiii') for each $k \geq \ell_2$ and $(x, k+1) \in X_{k+1}$ the points $(x, k), (x+1, k), \dots, (x+\alpha, k)$ belong to X_k .

See Subsections 8.1 and 8.2 for examples. Note that X_{ℓ_2} being right-aligned means now that $(\ell_1 - 1 - \beta, \ell_2) \in X_{\ell_2}$. Then we obtain the

THEOREM 10.2 (Theorem D, twisted version). *If $X \subseteq \Delta^{\text{up}}$ is admissible and $Y \subseteq \mathbb{Z}^2$ a complementing partner, then the union of*

$$s_{\downarrow} := Y + (\beta, -\ell_2) \subseteq \mathcal{H} \cup [y = 0] \quad \text{and} \quad s^{\uparrow} := X \subseteq \Delta^{\text{up}}$$

together with vertical order yields a maximal exceptional sequence s with $s_0 = 0$. Moreover, any vertically ordered maximal exceptional sequence starting at the origin arises this way.

Proof. The proof goes along the lines of the proof of Theorem 7.7.

If $X = \emptyset$, then Y consists of ℓ_2 consecutive horizontal chains. Shifting down by $(\beta, -\ell_2)$ yields a vertically trivial sequence starting at the origin.

We therefore assume that $X \neq \emptyset$. We order the set $s_\downarrow \cup s^\uparrow$ vertically to obtain the sequence s . From Definition 5.9, Definition 7.3 (iv), and Definition 7.4 (Cvii), it is clear that $Y_{\ell_2} \cup X_{\ell_2}$ forms a horizontal chain which ends at $(\ell_1 - 1 - \beta, \ell_2)$. Consequently, Y_{ℓ_2} starts at $(-\beta, \ell_2)$ and $s_0 = 0$.

For $s_i < s_j$ we have to show that $s_j - s_i \in \mathcal{I}$. If $s_i, s_j \in [y = b]$, this follows from $s_i, s_j \in Y_{b+\ell_2} + (\beta, -\ell_2)$ or $s_i, s_j \in X_b$, (Cvi) and (Cvii). If they are at different levels $0 \leq b_i < b_j \leq 2\ell_2 - 1$, the only critical case arises from $b_j - b_i \geq \ell_2$ which implies $0 \leq b_i \leq \ell_2 - 1$ and $\ell_2 \leq b_j \leq 2\ell_2 - 1$. This means that

$$s_i \in [y = b_i] \cap s_\downarrow = Y_{b_i+\ell_2} + (\beta, -\ell_2) \quad \text{and} \quad s_j \in [y = b_j] \cap s^\uparrow = X_{b_j}.$$

We proceed via induction over $m := b_j - b_i - \ell_2 \geq 0$.

If $m = 0$, then $s_i \in Y_{b_j} + (\beta, -\ell_2)$ whence

$$s_j - (s_i + (-\beta, \ell_2)) \in X_{b_j} - Y_{b_j} \subseteq \{(1, 0), \dots, (\ell_1 - 1, 0)\}.$$

In particular, $s_j - s_i \in (-\beta, \ell_2) + \{(1, 0), \dots, (\ell_1 - 1, 0)\} \subseteq \Delta^{\text{up}}$.

Next let $m \geq 1$. Writing $B := \{(0, -1), (1, -1), \dots, (\alpha, -1)\}$, (Aiii') implies that $s_j + B \in s^\uparrow$. On the other hand, the induction hypothesis implies $s_j - s_i + B \in \mathcal{I}$, and since $b_j - 1 - b_i - \ell_2 \geq 0$ we even have $s_j - s_i + B \in \Delta^{\text{up}}$. By definition of Δ^{up} we also have $s_j - s_i \in \Delta^{\text{up}}$, whence $s_j - s_i \in \mathcal{I}$.

Finally, we want to show that any vertically ordered maximal exceptional sequence s with $s_0 = 0$ arises this way. For this, we let

$$Y := (-\beta, \ell_2) + (s \cap (\mathcal{H} \cup [y = 0])) \quad \text{and} \quad X := s \cap \Delta^{\text{up}},$$

where we identify the sequence s with its underlying set.

If s is vertically trivial, then $Y = (-\beta, \ell_2) + s$ and $X = \emptyset$. We therefore assume that s is not trivial and check that X is admissible with complementing partner Y . By Lemma 5.14, $H(s) \leq 2\ell_2$. From this and the definition of Y , Properties (Ai) and (Cv) follow. Furthermore, (Aii) follows from Corollary 5.4, while (Cvii) is a consequence of Proposition 5.11.

By Corollary 5.16, $[y = b] \cap s \neq \emptyset$ for $0 \leq b \leq H(s) - 1$. Furthermore, we have $l = 2\ell_2 - H(s)$ horizontal lines by Corollary 5.13.

From Subsection 7.2 and in particular Inequality (8) on Page 1062 we recall the following notation: For $b = 0, \dots, \ell_2 - l - 1$, we let (a_b, b) and $(a_b + r_b, b) \in s$ be the minimal and maximal element of $[y = b] \cap s$, that is, $[y = b] \cap s = \{(a_b, b), \dots, (a_b + r_b, b)\}$ for some $0 \leq r_b < \ell_1 - 1$. It follows that $(r_0 + 1 - \beta, \ell_2), \dots, (\ell_1 - 1 - \beta, \ell_2) = [y = \ell_2] \cap s$ which implies (Aiv).

For all other $0 < b < \ell_2 - l - 1$, $Y_{b+\ell_2} = (-\beta, \ell_2) + [y = b] \cap s$ is to the left, cf. Remark 5.12. This implies (Cvi).

It remains to check (Aiii'). Let $(x, b + \ell_2 + 1) \in X_{b+\ell_2+1}$ for some $0 \leq b$. We need to show that $(x, b + \ell_2)$ and $(x + \alpha, b + \ell_2)$ belong to s . We first note that $X_{b+\ell_2} \neq \emptyset$ for otherwise, $Y_{b+\ell_2}$ and thus $[y = b] \cap s$ would consist of ℓ_1 consecutive points. Hence $X_{b+\ell_2+1}$ would be empty by Corollary 5.15, which is absurd.

Next, we show that $(x + \alpha, b + \ell_2) \in X_{b+\ell_2}$. Assume otherwise. Since $X_{b+\ell_2} \neq \emptyset$ and $\#(Y_{b+\ell_2} \cup X_{b+\ell_2}) = \ell_1$ this would imply $(x + \alpha - \ell_1, b + \ell_2) \in Y_{b+\ell_2}$, or equivalently, $(x + \alpha - \ell_1 + \beta, b) \in s$. This implies that

$$(x, b + \ell_2 + 1) - (x + \alpha - \ell_1 + \beta, b) = (\ell_1 - \alpha - \beta, \ell_2 + 1) \in \Delta^{\text{up}}.$$

However, the rightmost element of Δ^{up} is $(\ell_1 - \alpha - 1 - \beta, \ell_2 + 1)$ whence again a contradiction.

Finally, we show that $(x, b + \ell_2) \in X_{b+\ell_2}$. Again, assume otherwise. Since $(x + \alpha, b + \ell_2) \in X_{b+\ell_2}$ and $\#(Y_{b+\ell_2} \cup X_{b+\ell_2}) = \ell_1$, this would imply $(x, b + \ell_2) \in Y_{b+\ell_2}$ whence $(x + \beta, b) \in s$. However, this means that

$$(x, b + \ell_2 + 1) - (x + \beta, b) = (-\beta, \ell_2 + 1) \in \Delta^{\text{up}}.$$

But $[x = -\beta] \cap \Delta^{\text{up}}$ lies in the boundary of Δ^{up} , which gives a contradiction. \square

As $(x, k + 1) \in X_{k+1}$ implies $(x, k) \in X_k$ by (Aiii'), Theorem 10.2 immediately yields the following

COROLLARY 10.3. *On $(\ell_1, \ell_2; c)$ let s be a vertically ordered maximal exceptional sequence with $s_0 = 0$, and let $l = 2\ell_2 - H(s)$. Then*

$$a_{i-1} + r_{i-1} \leq a_i + r_i \quad \text{and} \quad a_i + \alpha \leq a_{i-1}$$

for $i = 1, \dots, \ell_2 - 1 - l$.

Proof. By (Cvi) and (Cvii) it follows for $i = 0, \dots, \ell_2 - 1 - l$ that

$$X_{i+\ell_2} = \{(a_i + r_i + 1 - \beta, i + \ell_2), \dots, (a_i + \ell_1 - 1 - \beta, i + \ell_2)\}.$$

Now if $i > 0$, then

$$\{(a_i + r_i + 1 - \beta, i + \ell_2 - 1), \dots, (a_i + \ell_1 - 1 - \beta + \alpha, i + \ell_2 - 1)\} \subseteq X_{i+\ell_2-1}$$

by (Aiii'). In particular, $a_{i-1} + r_{i-1} \leq a_i + r_i$ and $a_i + \alpha \leq a_{i-1}$. \square

11. GENERATING THE DERIVED CATEGORY

Finally, we set out to prove fullness of any nontrivial maximal exceptional sequence s on $(\ell_1, \ell_2; c)$ with $c \neq 0$.

Since $V \subseteq W$, a W -ensemble based at $(a, b) \in s$ generates the point $(a - \beta, b + \ell_2)$ in $\mathcal{D}(\ell_1, \ell_2; c)$ by Corollary 3.2. In particular, the standard rectangle $\mathcal{R}_{\ell_1, \ell_2}$ generates the Picard group in the twisted case, too.

Furthermore, Corollary 10.3 implies that s contains the set

$$(10) \quad \bigcup_{i=1}^l C_i \cup \{(a, b) \in \mathbb{Z}^2 \mid 0 < b < \ell_2 - l, -b\alpha \leq a \leq a_b + r_b\}$$

where again $l = 2\ell_2 - H(s)$, C_i is a horizontal chain in $[y = \ell_2 - 1 - i] \cap s$ if $l \geq 1$, and r_b was defined in Inequality (8) on Page 1062. Filling these lines via the C_i shows that s contains the W -ensembles centered at $[y = 0] \cap s$.

EXAMPLE 11.1. We illustrate our generation procedure on $(\ell_1, \ell_2; \alpha, \beta) = (3, 3; 1, 1)$ for the maximal exceptional sequence

$$s = (0, (-1, 1), (0, 1), (1, 2), (2, 2), (3, 2), (0, 3), (1, 3), (0, 4)).$$

First,

$$W = W(3; 1) = \{(-1, 1), (0, 1), (-2, 2), (-1, 2), (0, 2)\}.$$

Filling the line $[y = 2]$ in (a) shows that $\langle s \rangle$ contains W based at the origin. Hence we can generate in (b) the point $(-1, 3)$ which we use to fill the line $[y = 3]$ in (c). Therefore, the W -boxes based at $[y = 1] \cap s$ are contained in $\langle s \rangle$. They generate the points $(-2, 4)$ and $(-1, 4)$ in (d) so that together with $(0, 4) \in s$ we fill the line $[y = 4]$ in (e), too. It follows that $(-2, 2) + \mathcal{R}_{3,3} \subseteq \langle s \rangle$ whence s is full.

THEOREM 11.2 (Theorem E, twisted version). *On $(\ell_1, \ell_2; c)$, $c \neq 0$, any maximal exceptional sequence is full.*

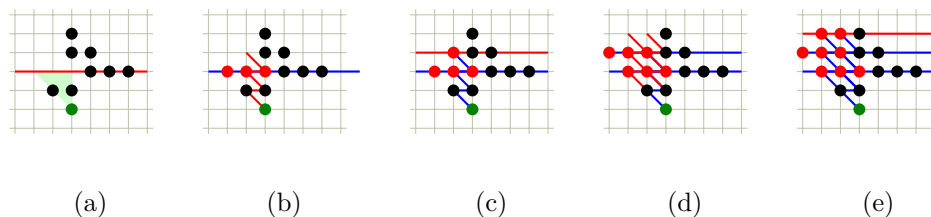


FIGURE 16. Filling $\text{Pic}(3, 3; 1, 1)$ from s . The green point marks $s_0 = 0$. The shaded area in (a) is the W -ensemble centered at $[y = 0] \cap s = \{0\}$.

Proof. We may assume that s is vertically ordered and starts at the origin. We continue to use the notation from Corollary 10.3.

First, we fill all lines containing a horizontal chain in s (if any). Consequently, the lines $[y = \ell_2 - 1], \dots, [y = \ell_2 - l]$ belong to $\langle s \rangle$. From Equation (10) on Page 1072 we conclude that $\langle s \rangle$ contains all the W -boxes centered at the points in $[y = 0] \cap s$. Hence we can generate the points $(-\beta, \ell_2) + ([y = 0] \cap s)$ and fill the line $[y = \ell_2]$ which therefore also belongs to $\langle s \rangle$. We can again appeal to Equation (10) to infer that the W -boxes based at $[y = 1] \cap s$ are contained in $\langle s \rangle$. Thus we can generate the points $(-\beta, \ell_2) + ([y = 1] \cap s)$ and fill the line $[y = \ell_2 + 1]$. After at most $\ell_2 - l$ repetitions we conclude that $(0, \ell_2 - l) + \mathcal{R}_{\ell_1, \ell_2} \subseteq \langle s \rangle$. Hence s is full. \square

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