Identification of malfunctioning quantum devices

Michalis Skotiniotis,[1,2,3] Santiago Llorens,[3] Ronja Hotz,[4] John Calsamiglia,[3] and Ramon Muñoz-Tapia[3]
[1]*Departamento de Electromagnetismo y Física de la Materia, Universidad de Granada, 18010 Granada, Spain*
[2]*Institute Carlos I for Theoretical and Computational Physics, Universidad de Granada, 18010 Granada, Spain*
[3]*Física Teòrica: Informació i Fenòmens Quàntics, Departament de Física, Universitat Autònoma de Barcelona, 08193 Bellatera (Barcelona), Spain*
[4]*Department of Physics, Freie Universität Berlin, 14195 Berlin, Germany*

(Received 10 August 2018; accepted 1 September 2024; published 23 September 2024)

We consider the problem of correctly identifying a malfunctioning quantum device that forms part of a network of N such devices, which can be considered as the quantum analog of classical anomaly detection. In the case where the devices in question are sources assumed to prepare identical quantum pure states, with the faulty source producing a different anomalous pure state, we show that the optimal probability of successful identification requires a global quantum measurement. We also put forth several local measurement strategies—both adaptive and nonadaptive—that achieve the same optimal probability of success in the limit where the number of devices to be checked is large. In the case where the faulty device performs a known unitary operation, we show that the use of entangled probes provides an improvement that even allows perfect identification for values of the unitary parameter that surpass a certain threshold. Finally, if the faulty device implements a known qubit channel, we find that the optimal probability for detecting the position of rank-one and rank-two Pauli channels can be achieved by product state inputs and separable measurements for any size of network, whereas for rank-three and general amplitude damping channels, optimal identification requires entanglement with N qubit ancillas.

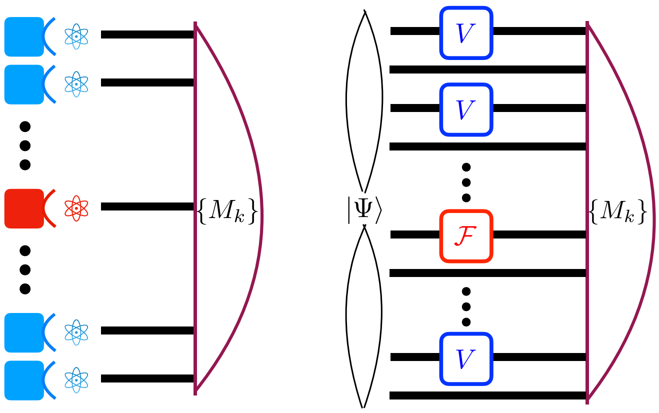
DOI: 10.1103/PhysRevResearch.6.033329

I. INTRODUCTION

Recent advancements in quantum technologies, such as quantum computing devices [1–6], quantum communication [7–9], and quantum sensors [10], lend credence to the notion that one day soon such devices will be readily available and, hopefully, part of an interconnected quantum network [11]. In turn, the existence of such quantum networks gives rise to new technical challenges, such as the correct identification of possible malfunctions. In a vast network of quantum devices—be they sources that produce quantum states, quantum channels that transmit information, or the vast array of gates in quantum computers—it is imperative that we are able to find efficient ways to identify faulty components.

The identification of rare events that differ significantly from the majority of all other observations is known as anomaly detection and is a fundamental primitive in classical data analysis and signal processing [12] with a wide range of applications, from the identification of denial of service attacks [13–15] to the identification of fraudulent financial activity [16] (for a survey of anomaly detection, see [17]). Two important algorithms for anomaly detection, namely, kernel principal component analysis and support vector machines, have been shown to be efficiently applicable in detecting anomalies in quantum data [18]. Here we consider a more direct application of anomaly detection where the anomaly is not restricted to classical data, but in the most general quantum device. Specifically, given N *identical* devices that are programmed to perform a particular task, with one of the devices developing a known malfunction, our goal is to optimally identify the malfunctioning device while only being allowed to query the network once (see Fig. 1). This task that we term *error position identification* (EPI) is a key ingredient in pulse-position modulated quantum communication [19,20], quantum illumination [21], quantum reading [22], and target detection [23].

We note that instances of EPI have appeared elsewhere in the literature under the name of channel position finding [24–27], where the task is to identify the position of a target bosonic channel, with a given reflectivity, among m background bosonic channels of a different reflectivity. The case of identifying the position of a singular unitary channel was also considered recently in [28]. Here, we provide a more comprehensive analysis of EPI by considering more general devices—including states, unitary gates, and quantum channels—using a variety of different strategies employing probes in separable, entangled, as well as ancilla-assisted strategies. Unlike Refs. [24–27], we consider channels acting on two-dimensional systems and consider both rank-one and rank-two Pauli noise channels, depolarizing, and amplitude damping channels. Moreover, our aim is to identify the position of the erroneous device by querying the network only once, as opposed to the more general multiquerying adaptive strategies considered in [24]. For the case of unitary channels,



(a) **(b)**

FIG. 1. Position error identification for sources and channels. (a) N quantum sources are programmed to produce a given state $|0\rangle$, except for a faulty device (depicted in red) which produces a known pure state $|\phi\rangle$. (b) N quantum gates are programmed to perform a given unitary V, except for a faulty device (depicted in red) which may perform a different unitary W or a noisy channel $\mathcal{F}$. While in (a) we only have to optimize over all possible measurement strategies, in (b) we can also optimize over all possible input state strategies, including those that make use of N additional ancillas.

we show that there exist instances for which the position of the unitary channel can be identified with certainty, a result that was missed in [28]. Note that EPI is fundamentally different from the anomaly detection scenario considered in [18], which focused on the classification of states using quantum machine learning techniques.

We first address the setting when the devices are quantum sources known to produce a specific pure state in $\mathcal{H}_d$, with the faulty source producing a different anomalous pure quantum state (the multi-anomaly case has recently been studied in [29]). We show that the maximum probability of success is achieved by a global measurement strategy given by the so-called *square-root measurement*, which yields a success probability that converges to a nonzero constant as N increases. We next show that when the devices are known to perform a specific unitary gate, with the anomalous device performing a different unitary, a probe state with entanglement among the N input states—that can be effectively taken to be qubits—achieves the maximal probability of success without the need to use ancillary systems. In fact, we show that for certain nontrivial errors, the optimal probability of success can even reach unity. While the use of entangled probe states does yield an improvement over strategies employing separable probe states, this improvement diminishes with the size of the network.

We then address the case when the devices are qubit channels and show that for rank-one and rank-two Pauli channels, the maximal probability of success is achievable by preparing probes and measuring their outputs in a common, suitably chosen local basis. For rank-three Pauli channels and amplitude damping, we show that one benefits most from appropriately entangling the N probes with additional N ancillas.

Finally, for the case where the quantum devices are sources, we provide several local strategies, both adaptive and nonadaptive, whose performance is nearly optimal, which is of particular interest for realistic applications.

II. BACKGROUND

The task of successfully identifying the position of a faulty device is equivalent to optimally discriminating among N quantum states, in the case of source EPI, or N quantum channels. For the remainder of this work, we will be concerned with devices that either prepare systems in a known pure state (sources) or perform a known quantum operation (channels). Without loss of generality, a faultless quantum source prepares systems in the state $|0\rangle$, whereas the faulty source—equally likely to be located at any of the N positions—prepares the kth system in some state $|\phi\rangle$. Similarly, a faultless quantum operation performs the unitary operation V, whereas the anomalous operation performs a completely positive, trace-preserving (CPTP) map, $\mathcal{F}^{(k)}$, on the kth quantum system [30]. The probability of successfully identifying the position of the faulty device is given by

$$P_S = \frac{1}{N}\sum_{k=1}^{N} q(k|k), \tag{1}$$

where $q(l|k)$ is the conditional probability given by $\mathrm{Tr}[M_l\,|\phi_k\rangle\langle\phi_k|]$ in the case of sources, and $\mathrm{Tr}[M_l\,\mathcal{E}^{(k)}[|\psi\rangle\langle\psi|]]$ in the case of channels, with $\{M_l \geqslant 0 \,|\, \sum_l M_l = \mathbb{1}\} := \mathcal{M}$ forming a general positive operator-valued measure (POVM). Observe that for source EPI, the optimization over all of the measurement strategies of Eq. (1) is a semidefinite program (SDP). For channel EPI, however, Eq. (1) needs to be optimized both over the input state as well as the measurement. Luckily enough, the formalism of quantum testers [31] allows one to linearize the problem and formulate it as an SDP—as long as the optimization is carried over strategies that make use of idler or ancillary systems, as shown in Fig. 1.

By and large, analytical solutions to either state or channel discrimination problems are very difficult and known only for two [32,33] or three [34] pure or mixed states or between states possessing a certain symmetry [35–43] (see, also, [44,45]). For channel discrimination, analytic results are known for distinguishing among a finite number of unitaries in a single or finite number of runs [46–48], or between two arbitrary CPTP maps [49–51]. We stress that here, unlike most channel discrimination instances to date, the identity of the channel (unitary or otherwise) is known, and what one is looking for is the position at which the channel is acting.

III. RESULTS

A. Source EPI

We first consider the case of successfully identifying the location of a faulty source as in Fig. 1(a). The problem simplifies to optimally discriminating among the set of N linearly independent states,

$$|\psi_k\rangle = |0\rangle^{\otimes(k-1)} \otimes |\phi\rangle \otimes |0\rangle^{\otimes(N-k)}, \quad k \in (1,\dots,N). \tag{2}$$

Notice that the set of states also enjoys cyclic symmetry, which allows one to explicitly work out the solution [20,29,36–38].

Let $\{|m_j\rangle\}_{j=1}^N$ be an orthonormal basis for the N-dimensional space spanned by $\{|\psi_k\rangle\}_{k=1}^N$. Then the probability of success is given by [41]

$$P_S = \frac{1}{N}\sum_{k=1}^{N} |\langle\psi_k|m_k\rangle|^2 = \sum_{k=1}^{N} |B_{kk}|^2, \tag{3}$$

where B is but one of an infinitude of square roots of the *Gram* matrix—the matrix of overlaps $G_{kl} = \frac{1}{N}\langle\psi_k|\psi_l\rangle$, which contains all the relevant information to assess the discrimination properties of a set of quantum states. As $G > 0$, the optimization of Eq. (3) is achieved by maximizing over all polar decompositions of $B = VS$, i.e.,

$$P_S = \max_V \sum_{k=1}^{N} |(VS)_{kk}|^2, \tag{4}$$

where S is the unique, self-adjoint square root of G. The corresponding measurement is simply given by $\{|m_k\rangle = \frac{(VS)^{-1}}{\sqrt{N}}|\psi_k\rangle\}$ (see [41]).

For the set in Eq. (2), the Gram matrix can be easily shown to be

$$G = \frac{1-b}{N}\mathbb{1} + b|\mathbf{1}\rangle\langle\mathbf{1}|, \tag{5}$$

where $b = \langle\psi_k|\psi_l\rangle$ for $k \neq l$ and $|\mathbf{1}\rangle = \frac{1}{\sqrt{N}}(1, \dots, 1)^T$. The matrix G has two distinct eigenvalues, $\lambda_1 = \frac{1+(N-1)b}{N}$ and $\lambda_2 = \frac{1-b}{N}$, where the latter is $(N-1)$-fold degenerate. Note that G is circulant [52] as is any function of G, in particular $S = \sqrt{G}$, whose diagonal entries are all equal to

$$S_{kk} = \frac{\mathrm{Tr}S}{N} = \sum_{j=1}^{N} \frac{\sqrt{\lambda_j}}{N}; \quad \forall k = 1, 2, \dots, N. \tag{6}$$

The condition that $S_{kk} = S_{ll}$, $\forall k \neq l$, is necessary and sufficient to show that Eq. (3) is maximized by $B = S$ [20,41], i.e., $P_S^* = NS_{kk}^2 = (\mathrm{Tr}S)^2/N$, which reads

$$P_S^* = \left(\frac{\sqrt{1+(N-1)b} + (N-1)\sqrt{1-b}}{N}\right)^2. \tag{7}$$

To compare with the results of the next section, it is convenient to write the b in terms of the angle ϕ defined from the overlap between the default and mutated state as $b = |\langle 0|\phi\rangle|^2 = \cos^2 \phi/2$. We get

$$P_S^* = \left(\frac{\sqrt{1+(N-1)\cos^2 \phi/2} + (N-1)\sin \phi/2}{N}\right)^2. \tag{8}$$

The measurement that achieves this optimal value is the so-called square-root measurement [20,36–38]. Notice that $P_S^* = 1$ if and only if $b = 0$, i.e., $|\phi\rangle = |1\rangle$. For large strings of states, the success probability is finite and reads

$$\begin{aligned} P_S^* &= (1-b) + \frac{2\sqrt{b(1-b)}}{\sqrt{N}} + O\left(\frac{1}{N}\right) \\ &= \sin^2 \phi/2 + \frac{\sin\phi}{\sqrt{N}} + O\left(\frac{1}{N}\right). \end{aligned} \tag{9}$$

B. Unitary EPI

Let us now consider the case of successfully identifying the location of a faulty unitary gate W, i.e., $\mathcal{F}(\cdot) = W(\cdot)W^\dagger$ in Fig. 1(b). As both V and W are known—only the location of the latter is unknown—we can assume without loss of generality that the states to be discriminated have the form

$$\begin{aligned} |\psi_k\rangle &= (\mathbb{1}^{\otimes(k-1)} \otimes U \otimes \mathbb{1}^{\otimes(N-k)})_p \otimes \mathbb{1}_a|\psi\rangle \\ &:= U_p^{(k)} \otimes \mathbb{1}_a|\psi\rangle, \quad k \in (1, \dots, N), \end{aligned} \tag{10}$$

where $U = V^\dagger W$, and $|\psi\rangle \in \mathcal{H}_p \otimes \mathcal{H}_a$ is a probe state which, in principle, can contain ancillary systems. However, we note that the use of ancillas here is redundant since, for any probe-plus-ancilla state, there exists an N probe state that gives the same Gram matrix (see Appendices). Observe that if $|\psi\rangle = |\gamma\rangle^{\otimes N}$ with $U|\gamma\rangle = |\phi\rangle$, we recover the source scenario discussed above with success probability given by Eq. (7). The question is whether choosing a more suitable initial state, perhaps involving entanglement, improves the probability of success.

Using the symmetry of the problem, for any given input state $\rho \in \mathcal{B}(\mathcal{H}^{\otimes N})$ and optimal POVM $\{M_\sigma\}$, there exists a permutationally invariant state $\tilde{\rho} \in \mathcal{B}(\mathcal{H}^{\otimes N})$ and permutationally covariant POVM $\{\tilde{M}_\sigma\}$ that achieves the same probability of success (see Appendices). On the other hand, as the probability of success is a convex function, the optimal input state can always be chosen to be pure. A seemingly natural, though unproven, assumption is to restrict the optimization over states to those that are both pure and permutationally invariant. For $N > 2$, this assumption amounts to searching over pure states that belong to the totally symmetric subspace of $\mathcal{H}^{\otimes N}$ [see Eq. (11)]. We have numerically verified that this is indeed the case for small values of N.

As U is known, we may write any permutationally invariant probe state with respect to the eigenbasis of U, with all amplitudes taken to be real and positive without loss of generality. Furthermore, an optimal probe state only involves those eigenstates of U whose eigenvalues have the largest distance in the complex plane [53,54] as these yield the smallest overlap. We write these as $|0\rangle$ and $|1\rangle$, respectively. The action of the unitary can be taken to be $U|0\rangle = |0\rangle$ and $U|1\rangle = e^{i\phi}|1\rangle$ without loss of generality, where $|\phi|$ is the largest phase distance among the eigenvalues of the unitary. To keep track of the phase, we denote the unitary by $U(\phi)$.

An orthonormal basis of these permutationally symmetric states is given by the well-known Dicke states [55],

$$|N, m\rangle = \frac{1}{\sqrt{\binom{N}{m}}} \sum_{g\in\mathcal{S}_N} \pi_g[|1\rangle^{\otimes m}|0\rangle^{\otimes(N-m)}], \tag{11}$$

where $\pi : \mathcal{S}_N \to \mathbb{U}(2^N)$ is a permutation of the N qubits, and our input state can be taken to be

$$|\psi\rangle = \sum_{m=0}^{N} \sqrt{c_m}|N, m\rangle, \tag{12}$$

with $c_m \geqslant 0$, $\sum c_m = 1$. The overlaps between the states of Eq. (10) are given by

$$G_{kl}(\phi) = \frac{1}{N}\sum_{m=0}^{N} c_m\, b_m(\phi),\ \ k \neq l, \tag{13}$$

where

$$b_m(\phi) = 1 - \frac{4m(N-m)\sin^2 \phi/2}{N(N-1)}. \tag{14}$$

As $b_m(\phi)$ is independent of k and l, the Gram matrix is again circulant and we can immediately write the optimal probability of success. The latter is maximal whenever the off-diagonal terms of the Gram matrix are minimal. Whatever the value of ϕ, $b_{\lceil \frac{N}{2}\rceil} \leqslant b_m$, with $\lceil x \rceil$ denoting the minimum integer not smaller than x. The latter is given by

$$b_{\lceil \frac{N}{2}\rceil}(\phi) = 1 - \frac{2\lceil \frac{N}{2}\rceil \sin^2 \frac{\phi}{2}}{2\lceil \frac{N}{2}\rceil - 1}. \tag{15}$$

Observe that $b_{\frac{N}{2}}(\phi)$ is a positive, monotonically decreasing function for $\phi \in [0, \phi_{\min}(N)]$, where $\cos[\phi_{\min}(N)] = (-1 + \frac{1}{\lceil \frac{N}{2}\rceil})$ is the value at which $b_{\lceil \frac{N}{2}\rceil} = 0$. Therefore, if $\phi \in [0, \phi_{\min}(N)]$, the optimal strategy consists of preparing $|\psi\rangle = |N, \lceil \frac{N}{2}\rceil\rangle$ and performing the square-root measurement, with the corresponding probability of success being the same as in Eq. (7) substituting b by $b_{\lceil \frac{N}{2}\rceil}(\phi)$ given in Eq. (15),

$$P_S^{\mathrm{U}}[0 \leqslant \phi \leqslant \phi_{\min}(N)] = \frac{1}{N}\bigg[\sqrt{\cos^2 \phi/2 + \xi_N \sin^2 \phi/2} + \sqrt{(N-1)(1-\xi_N)}\sin \phi/2\bigg]^2, \tag{16}$$

where $\xi_N = 0$ for N even and $\xi_N = 1/N^2$ for N odd. Observe that the differences between the even and odd expressions are of the order of $O(1/N^2)$ and, as expected, disappear as N grows large.

Recall that Eq. (16) assumes the use of pure fully symmetric probe states. We have performed a numerical optimization without any assumption on the probe states using SDPs with seesaw techniques, where one first optimizes the measurement for a fixed state and subsequently optimizes the input state for the optimal measurement in the previous step. We attain the same values as those given by Eq. (16) for up to $N = 7$ systems.

Notice that something remarkable happens for $\phi_{\min}(N) < \phi \leqslant \pi$; for this range of values, $b_{\lceil \frac{N}{2}\rceil}(\phi) \leqslant 0$ and we can exploit superpositions between symmetric basis states in order to identify the malfunctioning device *with certainty*.

Specifically, as $b_0(\phi) = 1$ is independent of ϕ, initializing the N probes in the state $|\psi\rangle = \sqrt{c_0}|N, 0\rangle + \sqrt{c_{\lceil \frac{N}{2}\rceil}}|N, \lceil \frac{N}{2}\rceil\rangle$ with

$$c_{\lceil \frac{N}{2}\rceil} = \frac{2\lceil \frac{N}{2}\rceil - 1}{2\lceil \frac{N}{2}\rceil \sin^2 \frac{\phi}{2}} \tag{17}$$

and $c_0 = 1 - c_{\lceil \frac{N}{2}\rceil}$ guarantees that $P_S^{\mathrm{U}}[\phi_{\min}(N) < \phi \leqslant \pi] = 1$.

In the limit of large N, perfect discrimination is possible for angles $\pi - \frac{2}{\sqrt{N}} \leqslant \phi \leqslant \pi$. For a fixed angle ϕ, the probability of success attains the same asymptotic expression as for the source EPI, given by Eq. (9), up to subleading order (the gap closes as at a rate equal to $\frac{\sin^2 \phi/2}{N}$). Thus, in the limit where the number of devices is large, entangling the initial probes does not enhance the probability of success. Note, however, that for a small number of devices, this improvement can be quite sizable.

We would like to point out that part of our results here has been reproduced in a recent publication [28]. Notwithstanding, this work assumes symmetric probe states of qubits *ab initio* and fails to notice that this probability can be made equal to one for nontrivial values of the angle ϕ.

C. Channel EPI

We now consider the successful identification of faulty channels [Fig. 1(b)] described by CPTP maps acting on qubit systems. Ideally, well-functioning components implement the unitary operation V, with the faulty component implementing the CPTP map $\mathcal{F} : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_2)$ with Kraus operator decomposition $\{F_i\}_{i=1}^r$. As both V and $\mathcal{F}$ are known—only the position of the latter is unknown—our task reduces to discriminating among the states,

$$\rho_k = \mathcal{I}^{\otimes(k-1)} \otimes \mathcal{E} \otimes \mathcal{I}^{\otimes(N-k)} \otimes \mathcal{I}^{\otimes N}(|\psi\rangle\langle\psi|), \tag{18}$$

where $\mathcal{E} : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_2)$ is a channel with Kraus operator decomposition given by $\{K_i = V^\dagger F_i\}_{i=1}^r$. $|\psi\rangle \in \mathcal{H}_2^{\otimes N} \otimes \mathcal{H}_2^{\otimes N}$ is the initial state of N qubits and N ancillas.

For any Kraus decomposition of $\mathcal{E}$ and for any input state, the following holds:

$$P_S(\mathcal{E}, |\psi\rangle\langle\psi|) \leqslant \sum_{i=1}^{r} P_S(K_i, |\psi\rangle\langle\psi|), \tag{19}$$

where

$$P_S(K_i, |\psi\rangle\langle\psi|) = \frac{1}{N}\max_{\mathcal{M}}\sum_{k=1}^{N}\mathrm{Tr}\big(M_k K_i^{(k)}|\psi\rangle\langle\psi|K_i^{(k)\dagger}\big) \tag{20}$$

denotes the optimal probability of successfully identifying the position of the action of the Kraus operator K_i. Equality holds in Eq. (19) if and only if there exists a POVM $\mathcal{M}$ that simultaneously optimizes all $P_S(K_i, |\psi\rangle\langle\psi|)$.

Consider first the Pauli channels

$$\mathcal{E}_{\mathrm{P}}[\rho] = p_0\rho + p_1\sigma_x\rho\sigma_x + p_2\sigma_y\rho\sigma_y + p_3\sigma_z\rho\sigma_z, \tag{21}$$

where $p_1 \geqslant p_2 \geqslant p_3$, (x, y, z) form an orthonormal Cartesian frame, $\{\sigma_x, \sigma_y, \sigma_z\}$ the standard Pauli matrices, and $\sum_{k=0}^{3} p_k = 1$. The rank of a Pauli channel is the number of nonzero weights p_i with $i \in (1, 2, 3)$. For rank-one and rank-two Pauli channels, the bound of Eq. (19) can be saturated by a simple strategy involving just product states and projective measurements. Indeed, for $p_3 = 0$, the optimal strategy corresponds to preparing each probe in the state $|0\rangle$ and measuring in the eigenbasis of σ_z. Hence, for both rank-one and rank-two Pauli channels, the optimal probability of success reads

$$P_S(\mathcal{E}_{\mathrm{rank}-1(2)}) = 1 - p_0 + \frac{p_0}{N}. \tag{22}$$

For rank-three Pauli channels, any strategy involving N probes prepared in a product state and projective measurements is suboptimal. This is because one cannot perfectly discriminate between the Pauli matrices using a single qubit. Indeed, the best one can hope for using such a strategy is (see Appendices)

$$P_S^{\text{sep}}(\mathcal{E}_{\text{rank}-3}) = 1 - (p_0 + p^*) + \frac{p_0 + p^*}{N}, \tag{23}$$

where $p^* = \min\{p_1, p_2, p_3\}$ and corresponds to unambiguously discriminating the most likely rank-two Pauli noise. One can, however, optimally distinguish among the Pauli matrices, and thus saturate Eq. (19), by introducing N ancilla qubits and preparing each probe-plus-ancilla in the maximally entangled state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. As the corresponding set of states $\{\mathbb{1} \otimes \sigma_i|\Phi^+\rangle\}_{i=0}^3$ is orthogonal, it can be distinguished with certainty and one has to make a random guess only in the case the identity acts. We then recover Eq. (22) as the optimal probability of success. We note that this is in agreement with the result of [24], which proved that for the optimal probability of success for telecovariant channels—of which the depolarizing channel is a special case—the optimal strategy for detecting the position of the channel is a nonadaptive ancilla-assisted strategy. It remains to check whether the upper bound to $P_S(\mathcal{E}_{\text{rank}-3})$ is achievable without the use of ancillas. For small network sizes, we can numerically determine the optimal probability of success where we observe a clear gap. By way of example, for $N = 3$ and the completely depolarizing channel ($p_0 = p_1 = p_2 = p_3 = 1/4$), we numerically find $\max_{\rho\in\mathcal{B}(\mathcal{H}_2^{\otimes 3})} P_S(\mathcal{E}_{\text{rank}-3}) = 0.71$, which indeed is larger than the suboptimal $2/3$ value in Eq. (23), and smaller than the ancilla-assisted value $5/6$ [Eq. (22)]. Just as in the case of unitary EPI, we find that fully symmetric probe states attain the optimal probability of success.

We now consider an amplitude damping channel with Kraus operators

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \tag{24}$$

with $0 \leqslant \gamma \leqslant 1$ the damping parameter. The best strategy utilizing separable states can be obtained by noting that the action of amplitude damping on an arbitrary Bloch vector results in $\vec{r} \to [r_x\sqrt{1-\gamma}, r_y\sqrt{1-\gamma}, \gamma + r_z(1-\gamma)]$. It follows that the probability of detecting the action of amplitude damping is highest if one prepares the N product-state probes in the direction where amplitude damping is most pronounced (here, in the $\hat{z}$ direction), and measuring each qubit along that same direction which results in

$$P_S^{\text{sep}}(\mathcal{E}_{AD}) = \gamma + \frac{1-\gamma}{N}. \tag{25}$$

Clearly, Eq. (25) provides a lower bound to the probability of success for the most general, ancilla-assisted strategy. An upper bound can be obtained from the following chain of inequalities:

$$\begin{aligned} P_S(\mathcal{E}_{AD}, \rho) &\leqslant P_S(K_0, \rho) + P_S(K_1, \rho) \\ &\leqslant P_S(K_0, \rho) + p(K_1, \rho), \end{aligned} \tag{26}$$

where $p(K_1, \rho)$ is the probability that the K_1 Kraus operator of the amplitude damping channel has acted and is independent of the position of the channel. Moreover, $P_S(K_0, \rho)$ depends solely on the overlap of the conditional (unnormalized) pure states $\rho_k = (\mathbb{1} \otimes K_0^{(k)})\rho(\mathbb{1} \otimes K_0^{(k)\dagger})$.

It is straightforward to check that the upper bound in Eq. (26) can be attained by preparing the $2N$ probe-ancilla systems in the state

$$|N, m\rangle_{pa} \equiv \binom{N}{m}^{-1/2} \sum_{g\in S_N} \Pi_g |11\rangle_{pa}^{\otimes m} |00\rangle_{pa}^{\otimes N-m}, \tag{27}$$

where each ancilla system acts as a flag for its respective probe system and $\Pi : \mathcal{S}_N \to \mathcal{H}_p^{\otimes N} \otimes \mathcal{H}_a^{\otimes N}$ is a representation of the permutation group S_N acting on the total probe-plus-ancilla space, i.e., $\Pi_g = \pi_g \otimes \pi_g$, $\forall\, g \in \mathcal{S}_N$, $\pi : \mathcal{S}_N \to \mathcal{H}_{p(a)}$. Observe that the total number of excitations of $|N, m\rangle_{pa}$ is even for all $m \in (0, \dots, N)$ and that K_0 preserves this number, whereas K_1 removes one excitation from the probe systems but not the ancilla, resulting in an odd number of total excitations. This implies that (i) the conditional set of states $\{|\Psi_1^{(k)}\rangle\}$, which arises from the action of $K_1^{(k)}$ on an input state

$$|\Psi\rangle = \sum_{m=0}^{N} \sqrt{c_m} |N, m\rangle_{pa}, \tag{28}$$

is orthogonal to the conditional states $\{|\Psi_0^{(k)}\rangle\}$ and (ii) the conditional states $\{|\Psi_1^{(k)}\rangle\}$ form an orthonormal set of states of odd number of excitations. Properties (i) and (ii) ensure that the first and second inequalities in Eq. (26) are achievable with $p(K_1, \rho) = 1 - p(K_0, \rho) = \gamma \frac{\langle\hat{n}\rangle}{N}$, where $\hat{n}$ is the total excitations number operator. By computing the Gram matrix for the branch corresponding to the action of K_0, the optimal probability of success can be explicitly determined to be

$$P_S(\mathcal{E}_{AD}) \leqslant \gamma + \frac{\sqrt{1-\gamma}(\sqrt{1-\gamma}+1)}{2N} + O(N^{-2}), \tag{29}$$

and is achievable by the ancilla-assisted state with coefficients $c_N = p$, $c_{N-1} = 1 - p$, where

$$p = \frac{\sqrt{1-\gamma}(3N-2)+2}{\sqrt{1-\gamma}(4N-2)+2}. \tag{30}$$

We note that upper and lower bounds for the probability of success can be derived by suitably modifying the approach of [24] to the single-shot scenario, but that these bounds are not tight and are also not demonstrably known to be achievable.

Numerically, we find that for $N = 3$ and $\gamma = 1/2$, the optimal success probability without ancillas is $\max_{\rho\in\mathcal{B}(\mathcal{H}_2^{\otimes 3})} P_S(\mathcal{E}_{AD}) = 0.68$—achievable by a pure fully symmetric probe state—while the ultimate limit is $P_S(\mathcal{E}_{AD}) = 0.69$. The difference is small, but sufficient to show that there is a gap in performance. Notice, however, that Eq. (29) shows that the improvement over the optimal product state strategy is subleading in N. Indeed, in the limit where the number of devices to be checked is large, the gap between Eqs. (25) and (29) closes, and it suffices to deploy our probes in the optimal product state.

IV. LOCAL MEASUREMENT STRATEGIES FOR SOURCE EPI

In this section, we investigate four local measurement strategies and gauge their performance in determining the position of a malfunctioning quantum source, i.e., detecting faulty prepared states. For ease of exposition, we will describe each of the local measurement strategies and state their performance only. For a more detailed treatment of each strategy, we direct the reader to the Appendices.

The simplest conceivable local strategy is to unambiguously determine the malfunctioning source. For this, each party measures in the $\{|0\rangle, |1\rangle\}$ basis. We term such measurement strategy as the *basic* local strategy. The probability of obtaining outcome one is $\sin^2 \phi/2$, whereas if all measurement outcomes are zero, which occurs with probability $\cos^2 \phi/2$, we have to guess the position of the malfunctioning source at random. Hence, the optimal probability of success using the basic local strategy is

$$P_S^{\mathrm{BL}}(\phi) = \sin^2\left(\frac{\phi}{2}\right) + \frac{\cos^2\left(\frac{\phi}{2}\right)}{N} = 1 - \frac{(N-1)}{N}\cos^2\left(\frac{\phi}{2}\right). \quad (31)$$

Notice that in the limit of $N \to \infty$, this strategy asymptotically approaches the constant value of the optimal square-root measurement, given by Eq. (7), but at a slower rate since the subleading term here is $\sim 1/N$ instead of $1/\sqrt{N}$ of the optimal performance.

Next, we consider the most general local strategy involving N independent measurements. Here each party measures in a different basis, and for each of the 2^N possible measurement outcomes, $\mathbf{m} \in \{0, 1\}^{N-1}$, our guessing rule is the maximum likelihood, i.e., we choose the hypothesis, $k \in (1, \dots, N)$, that maximizes the corresponding conditional probability distribution $q(\mathbf{m}|k)$,

$$P_S^{\mathrm{GL}} = \frac{1}{N}\sum_{\mathbf{m}=0}^{2^N-1} \max_k\{q(\mathbf{m}|k)\}. \quad (32)$$

The best such strategy corresponds to maximizing Eq. (32) over the N local measurement basis.

Our third local strategy is a *greedy* strategy, a sequential strategy that uses forward communication in order to optimally distinguish among the most likely hypothesis at every step. In this strategy, each party $n \in \{1, \dots, N\}$, starting with $n = 1$, performs a measurement on the nth subsystem of $|\psi_k\rangle \in \mathcal{H}^{\otimes N}$ corresponding to their location and attempts to identify the location of the error. As we show in the Appendices, the optimal strategy for each party n corresponds to a binary hypothesis testing scenario where the hypotheses are that the error is at location n or at the most likely position among the remaining hypotheses given the prior information at party n's disposal, by means of the prior distributions $p^{(n)}(k)$, $k \in (1, \dots N)$. These probabilities are updated sequentially, using Bayes' rule, depending on the outcomes of all parties up to and including party $n-1$. Party N then measures their subsystem using the optimal measurement derived from the prior distribution $p^{(N)}(k)$. The corresponding probability of success, which depends on the measurement record $\mathbf{m} \in \{0, 1\}^{N-1}$, is $P_S^{(N)}(\mathbf{m})$ [see Eq. (D4)] and the average probability of success is given by

$$P_S^{\mathrm{Gr}} = \sum_{\mathbf{m}} q(\mathbf{m})\, P_S^{(N)}(\mathbf{m}), \quad (33)$$

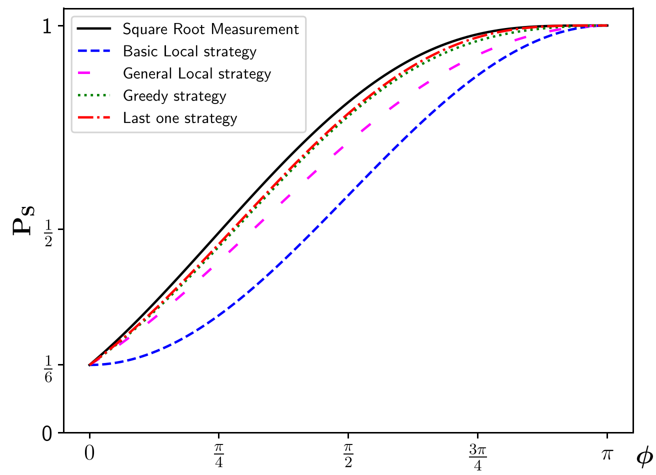


FIG. 2. The probability of success for EPI of sources using the square-root measurement (solid black line), the basic local strategy (blue dashed line), the greedy strategy (green dotted line), the general local strategy (magenta dashed line), and the last-one strategy (red dot-dashed line). All strategies are evaluated for the case of $N = 6$.

where $q(\mathbf{m})$ denotes the probability of obtaining the measurement record $\mathbf{m} \in \{0, 1\}^{N-1}$.

Our last local measurement strategy is again a sequential adaptive strategy where we fix our guess for the most likely hypothesis to be the position of the last positive outcome in any given measurement record (the *last-one* strategy). By way of example, suppose that $N = 3$. Then, for the measurement records $\{001, 011, 101, 111\}$, we hypothesize that the error occurred at position $k = 3$; for the measurement records $\{010, 110\}$, that the error occurred at $k = 2$; and for $\{100, 000\}$, that the error occurred at $k = 1$. Notice that here we need to optimize over $2^N - 1$ parameters since the optimal measurement basis at each position n depends on all previous $k - 1$ measurement outcomes. The corresponding probability of success is given by

$$P_S^{\mathrm{L1}} = \frac{1}{N}\sum_{k=1}^{N}\sum_{\boldsymbol{m}\in S_k} q(\mathbf{m}|k), \quad (34)$$

where

$$q(\mathbf{m}|k) = \prod_{i=1}^{N} |\langle m_i|\psi_k\rangle|^2, \quad (35)$$

and $S_k \equiv \{\mathbf{m}|m_{k+1}\dots m_N = 0\}$.

The performance of all four strategies for the case where $N = 6$ is shown in Fig. 2. The worst performing strategy is the basic local strategy, with the general local, greedy, and last-one strategies performing much better. This is hardly surprising since adaptive strategies utilizing information from previous measurement outcomes are expected to perform better. Moreover, the last-one strategy performs best as it utilizes the entire past measurement record, contrary to the greedy strategy which only utilizes the information from the previous measurement. While the improvement of the last-one strategy over the greedy one is only slight—the corresponding probabilities of success differ only in the third digit—we have checked that this improvement does persist also for moderate network sizes and that there is a clear nonzero gap with the optimal measurement strategy employing the square-root measurement.

Notice that to zeroth order in N, all four strategies achieve the maximum probability of success in the large-N limit, since $\lim_{N\to\infty} P_S^{\mathrm{BL}}(\phi) = \lim_{N\to\infty} P_S^{*}(\phi) = \sin^2 \phi/2$ and $P_S^{\mathrm{BL}} < P_S^{\mathrm{GL}} < P_S^{\mathrm{Gr}} < P_S^{\mathrm{L1}} < P_S^{*}$. The differences between all strategies is the speed with which one approaches this constant. As shown in Eq. (31), the basic local strategy approaches the asymptotic value from above with a scaling of N^{-1}, i.e., the success probability decreases at a faster rate than the optimal strategy, which shows a $N^{-1/2}$ scaling [Eq. (9)]. We can only assess the performance of the other local strategies by numerical optimization, which becomes exceedingly hard for large values of N. The most tractable case is that of the greedy strategy for which we have computed $P_S^{\mathrm{Gr}}(N)$ up to $N = 20$, exhibiting a scaling that is compatible with $N^{-2/3}$.

It is worth stressing that aside from being easily implementable, there are other advantages that one needs to consider when choosing among local strategies. For example, the basic local strategy may identify the position of an error with high confidence in an online fashion, even in settings where the number of samples, N, is not fixed in advance. The use of quantum sequential methodologies in settings without a finite horizon has been recently studied in [56–59].

V. CONCLUSIONS

We have addressed how to optimally identify the position of a malfunctioning quantum device that forms part of an interconnected quantum network in the simplified case where the latter consists of N identical devices that can be addressed in a parallel fashion. For unitary EPI, we discover that entanglement enhances the probability of correct identification, and even allows for perfect identification of the device if the unitary rotation angle is greater than some threshold. For rank-one and rank-two Pauli channels the optimal strategy involves separable states and measurements, whereas for rank-three and amplitude damping channels, we find that the optimal identification strategy requires entanglement with N additional ancillas. However, the use of entanglement among the input probes only pays dividends if the number of devices needed to be checked is small; as the network size grows large, strategies employing separable states do just as well.

We have numerically verified that for unitary EPI, ancilla-free totally symmetric pure states attain the optimal performance. While we have given a formal proof that ancillas are not necessary to achieve optimality, the sufficiency of pure permutationally symmetric states remains unproven. In channel identification using probes without ancillas, we find again that symmetric pure states stand as the optimal choice. Although the numerical evidence in both cases is very compelling, an analytical proof presents an intriguing open challenge of theoretical interest.

There are several interesting extensions of the current work that are of both theoretical and practical significance. For instance, in this work, we have assumed throughout that devices act on *separate* systems. In a quantum computer or a general quantum network, the same quantum system may be subject to a *sequence* of quantum operations. It is natural then to ask how to optimally identify malfunctioning devices that act in sequence on the same quantum system. Furthermore, it is interesting to know what the optimal performance is in cases where both the position and the identity of the anomaly are required.

ACKNOWLEDGMENTS

This work has been supported by Grant PID2022-141283NB-I00 funded by MCIN/AEI/10.13039/501100011033 and funding from European Union NextGenerationEU (PRTR-C17.I1) and Generalitat de Catalunya. We also acknowledge financial support from the Ministry for Digital Transformation and of Civil Service of the Spanish Government through the QUANTUM ENIA project call Quantum Spain project, and by the European Union through the Recovery, Transformation and Resilience Plan NextGeneration EU within the framework of the Digital Spain 2026 agenda. M.S. also acknowledges funding by Grant PID2021-128970OA-I00 funded by MCIN/AEI/10.13039/501100011033 and by “ERDF A way of making Europe”, the Ministerio de Ciencia e Innovación of the Spanish Government under the Ramon y Cajal funding scheme (RYC2021-032032-I), and FEDER funds C.EXP.256.UGR23 from the Junta Andalucia. J.C. also acknowledges support from the ICREA Academia award, and form the QuantERA project C’MON-QSENS!, by Spanish MICINN PCI2019-111869-2.

APPENDIX A: OPTIMAL STATES FOR UNITARY EPI REQUIRE NO ANCILLAS

In this Appendix, we provide the proof that for the case of unitary EPI, the optimal probability of success can be achieved without ancillas. To that end, let us write the malfunctioning unitary of dimension d as

$$U = \sum_{a=0}^{d-1} e^{\mathrm{i}\phi_a} |a\rangle\langle a|, \tag{A1}$$

where $\{|a\rangle\}_{a=0}^{d-1}$ are the d eigenvectors of the unitary U. Using the same orthonormal basis, we may write any probe-plus-ancilla state as

$$|\psi\rangle = \sum_{\mathbf{m},\mathbf{n}=\mathbf{0}}^{d^N-1} c_{\mathbf{nm}} |\mathbf{n}\rangle \otimes |\mathbf{m}\rangle, \tag{A2}$$

where $\mathbf{n}$, $\mathbf{m}$ denote N d-nary strings. The elements of the Gram matrix are then given by

$$\begin{aligned} G_{kl} &= \sum_{\mathbf{n},\mathbf{n}',\mathbf{m}=\mathbf{0}}^{d^N-1} c^*_{\mathbf{nm}} c_{\mathbf{n}'\mathbf{m}} \langle \mathbf{n}| U_k^\dagger U_l |\mathbf{n}'\rangle \\ &= \sum_{\mathbf{n},\mathbf{m}=\mathbf{0}}^{d^N-1} |c_{\mathbf{nm}}|^2 e^{-\mathrm{i}(\phi_{n_k}-\phi_{n_l})} \\ &= \sum_{\mathbf{n}=\mathbf{0}}^{d^N-1} |\psi_{\mathbf{n}}|^2 e^{-\mathrm{i}(\phi_{n_k}-\phi_{n_l})}, \end{aligned} \tag{A3}$$

where $|\psi_{\mathbf{n}}|^2 = \sum_{\mathbf{m}=\mathbf{0}}^{d^N-1} |c_{\mathbf{nm}}|^2$. But this is precisely the same Gram matrix element one would obtain using the no-ancilla state,

$$|\psi\rangle = \sum_{\mathbf{n}=\mathbf{0}}^{d^N-1} \psi_{\mathbf{n}}|\mathbf{n}\rangle, \tag{A4}$$

proving the claim.

APPENDIX B: OPTIMALITY OF PERMUTATIONAL INVARIANT STATES AND COVARIANT MEASUREMENTS

In this Appendix, we prove that given any state $\rho = |\psi\rangle\langle\psi|$ and POVM $\{M_k\}$, there exists a permutationally invariant state τ and covariant POVM that achieves the same probability of success.

Recall that the probability of successfully identifying the position of a malfunctioning device is given by

$$P_S(\mathcal{E}, |\psi\rangle\langle\psi|) = \max_{\{E_k\geqslant 0\}} \frac{1}{N}\sum_{k=1}^{N} \mathrm{Tr}[\mathcal{E}_k(|\psi\rangle\langle\psi|)E_k], \tag{B1}$$

where $|\psi\rangle \in \mathcal{H}_2^{\otimes N}$ and $\mathcal{E}_k : \mathcal{B}(\mathcal{H}_2^{(k)}) \to \mathcal{B}(\mathcal{H}_2^{(k)})$. Alternatively, we may think of the position of the channel as fixed, acting only on the last qubit, with a pre- and postprocessing of the input and output states by the shift superoperator $\mathcal{T}^{N-k}(\cdot) = T^{N-k}(\cdot)T^{-(N-k)}$, where $T : \mathbb{Z}_N \to \mathbb{U}(2^N)$, $T^N = \mathbb{1}$ is a unitary representation of $\mathbb{Z}_N$, i.e,

$$\mathcal{E}_k = \mathcal{T}^{-(N-k)} \circ \mathcal{E}_N \circ \mathcal{T}^{N-k}, \tag{B2}$$

with $\mathcal{T} \circ \mathcal{E}(A) = \mathcal{T}[\mathcal{E}(A)]$. Moreover, as we are promised that only a single error occurs, we can extend the translation symmetry of the channel to the full permutation group S_N by noting that

$$\mathcal{T}^{N-k}(\cdot) = \frac{1}{N!}\sum_{\sigma\in S_N|N\xrightarrow{\sigma}k} \pi_\sigma(\cdot)\pi_\sigma^\dagger, \tag{B3}$$

where $\pi : S_N \to \mathbb{U}(2^N)$ is a unitary representation of S_N.

Equation (B3) simply states that the translation of the channel from position N to k is equivalent to the $(N-1)!$ permutations that map position N to k. We can also modify our search over the optimal POVM accordingly by associating all $(N-1)!$ measurement outcomes E_σ that map N to k so that Eq. (B1) reads

$$P_S(\mathcal{E}, |\psi\rangle\langle\psi|) = \max_{\{E_\sigma\geqslant 0\}} \frac{1}{N}\sum_{\sigma\in S_N} \mathrm{Tr}[\mathcal{V}_\sigma^\dagger \circ \mathcal{E}_N \circ \mathcal{V}_\sigma(|\psi\rangle\langle\psi|)\,E_\sigma], \tag{B4}$$

where $\mathcal{V}_\sigma(\cdot) = \pi_\sigma(\cdot)\pi_\sigma^\dagger$ and $\mathcal{V}_\sigma^\dagger = \mathcal{V}_\sigma^{-1} = \mathcal{V}_{\sigma^{-1}}$.

For a fixed, equiprobable set of quantum states $\{\rho_k\}_{k=1}^N$, the optimal probability of successful discrimination can be written as the following semidefinite program (SDP) in dual form [32,33,35,60]:

$$\min_{\Gamma\geqslant 0} \mathrm{Tr}\,\Gamma \text{subject to} \quad \Gamma \geqslant \frac{1}{N}\rho_k \quad \forall k \in (1,\ldots,N). \tag{B5}$$

Using Eq. (B5) and the properties of the trace, we have

$$\begin{aligned} \mathrm{Tr}\,\Gamma &= \mathrm{Tr}(\pi_\sigma\,\Gamma\,\pi_\sigma^\dagger) \quad \forall\sigma\in S_N \\ &= \frac{1}{N!}\mathrm{Tr}\left(\sum_{\sigma\in S_N}\pi_\sigma\Gamma\,\pi_\sigma^\dagger\right) \\ &\equiv \mathrm{Tr}\,\tilde{\Gamma}, \end{aligned} \tag{B6}$$

where $\tilde{\Gamma}$ is a permutationally invariant operator. Furthermore, the optimality conditions of Eq. (B5) imply

$$\begin{aligned} \Gamma &\geqslant \frac{1}{N!}\pi_\sigma^\dagger\,\mathcal{E}_N(\pi_\sigma|\psi\rangle\langle\psi|\pi_\sigma^\dagger)\pi_\sigma, \quad \forall\sigma\in S_N, \\ \pi_\sigma\,\Gamma\,\pi_\sigma^\dagger &\geqslant \frac{1}{N!}\mathcal{E}_N(\pi_\sigma|\psi\rangle\langle\psi|\pi_\sigma^\dagger), \quad \forall\sigma\in S_N, \\ \tilde{\Gamma} &\geqslant \frac{1}{N!}\mathcal{E}_N(\mathcal{G}[|\psi\rangle\langle\psi|]), \end{aligned} \tag{B7}$$

where $\mathcal{G}[|\psi\rangle\langle\psi|] = \frac{1}{N!}\sum_{\sigma\in S_N}\pi_\sigma|\psi\rangle\langle\psi|\pi_\sigma^\dagger$, and has the property of symmetrizing $|\psi\rangle\langle\psi|$.

As it is sufficient to restrict to permutationally invariant states, Eq. (B4) reduces to

$$\begin{aligned} P_S(\mathcal{E},\tau) &= \max_{\{E_\sigma\geqslant 0\}}\max_{\{\tau\}}\frac{1}{N}\sum_{\sigma\in S_N}\mathrm{Tr}[\mathcal{E}_N(\tau)\mathcal{V}_\sigma^\dagger(E_\sigma)] \\ &= (N-1)!\max_{\{E\geqslant 0\}}\max_{\{\tau\}}\mathrm{Tr}[\mathcal{E}_N(\tau)E], \end{aligned} \tag{B8}$$

where the second maximization is over all τ that satisfy $[\pi_\sigma,\tau]=0, \forall\sigma\in S_N$, and we have made use of the fact that $\{E_\sigma = \pi_\sigma E\pi_\sigma^\dagger\}$ constitutes a covariant measurement whose fiducial element is E, and used the identity $\mathcal{G}[E] = \frac{1}{N!}\sum_{\sigma\in S_N}\mathcal{V}_\sigma(E) = \frac{1}{N!}\mathbb{1}$.

Equation (B8) tells us that for any given input state $|\psi\rangle$ and optimal POVM $\{M_k\}$, there exists a permutationally symmetric state τ and permutationally covariant POVM $\{E_\sigma = \pi_\sigma E\pi_\sigma^\dagger\}$ that achieves the same probability of success. Moreover, the optimization over the covariant measurement requires us to optimize over a single *fiducial* POVM element E such that

$$N!\,\mathcal{G}(E) = \mathbb{1}. \tag{B9}$$

APPENDIX C: OPTIMAL PRODUCT STATE STRATEGY FOR RANK-THREE PAULI CHANNELS

In this Appendix, we show that the optimal, ancilla-free, product state strategy for detecting rank-three Pauli noise using projective measurements has a probability of success given by Eq. (23). To do so, we will first show that the optimal probability of success can be achieved by unambiguously determining the position of the error channel and then show that the optimal unambiguous strategy is indeed given by Eq. (23).

Consider then the case where each party measures its corresponding system locally using a fixed, nonadaptive, but otherwise completely arbitrary measurement. Without loss of generality, let use denote the measurement outcomes of each party's measurement by 0 and 1. In a completely arbitrary fashion, let us also make the assignment that outcome zero means that the channel did not act, whereas outcome one means the channel has acted. After all parties have performed their measurement, we will obtain one of the 2^N possible measurement outcomes $\{\mathbf{m} \equiv m_1\ldots,m_N\,|\,m_i\in(0,1), \forall i\in(1,\ldots,N)\}$.

For the outcome $\mathbf{0}$, we simply guess at random the position of the channel, whereas for the N measurement outcomes with Hamming weight one, our guess for the position of the channel is that it acted at the position for which the measurement outcome $m_i = 1$. For example, if $m_1 = 1,\ m_i = 0,\ \forall i \in$

$(2, \ldots, N)$, we guess that the channel acted on the first qubit. Our probability of success is then given by

$$P_S = p(g=1|k=1)p(k=1)\Pi_{i=2}^{N} q(i=0|i=0) \equiv \frac{1}{N} p(g=1|k=1) q^{N-1}. \tag{C1}$$

Here, $p(g=i|k=i) = \mathrm{Tr}[|1\rangle_i\langle 1|\mathcal{E}(|\psi\rangle_i\langle\psi|)]$ denotes the probability that $m_i = 1$ given that the channel $\mathcal{E}$ acted at position i (i.e., correctly identifying the channel); $1 - p(g=i|k=i)$ is the conditional probability that $m_i = 0$ given the channel acted at position i; and $q(i=0|i=0)$ is the conditional probability that $m_i = 0$ given the state at position i is $|\psi\rangle$ (i.e., correctly identifying that the channel did not happen at position i), with $1 - q(i=0|i=0)$ denoting the conditional probability that $m_i = 1$ given the state at position i is $|\psi\rangle$. As all the latter probabilities are the same, we simply drop the label i in what follows and write q for simplicity. It follows that if the measurement record contains a single element different from zero at position i, the probability of successfully identifying the channel is given by

$$P_S = \frac{1}{N} p(g=i|k=i) q^{N-1}. \tag{C2}$$

Now consider a measurement record with Hamming weight $1 \leqslant r \leqslant N$ and suppose that for such measurement outcome, we randomly choose one of the r positions $\{i \in (1, \ldots, r)\,|m_i = 1\}$ as our guess. The probability of success now reads

$$\begin{aligned} P_S &= \frac{1}{N}\sum_{i=1}^{N} p(g=1|k=1) \sum_{r=1}^{N} \binom{N-1}{r-1} \frac{q^{N-r}(1-q)^{r-1}}{r} \\ &= p(g=1|k=1) \sum_{r=1}^{N} \binom{N-1}{r-1} \frac{q^{N-r}(1-q)^{r-1}}{r} \\ &= \frac{p(g=1|k=1)}{N} \sum_{r=1}^{N} \binom{N}{r} q^{N-r}(1-q)^{r-1} \\ &= \frac{p(g=1|k=1)}{N} \frac{1}{1-q} \sum_{r=1}^{N} \binom{N}{r} q^{N-r}(1-q)^{r} \\ &= \frac{p(g=1|k=1)}{N} \frac{1}{1-q} \left[\sum_{r=0}^{N} \binom{N}{r} q^{N-r}(1-q)^{r} - q^N\right] \\ &= \frac{p(g=1|k=1)}{N} \frac{1-q^N}{1-q}, \end{aligned} \tag{C3}$$

where we have used the fact that $p(g=i|k=i)$ is the same for all $i \in (1, \ldots, N)$ in going from the first to the second line in Eq. (C3). For $q < 1$, the success probability scales as $\frac{1}{N}$, whereas for $q = 1$, i.e., for unambiguously discriminating when the channel acted, the success probability is given by

$$P_S = p(g=1|k=1). \tag{C4}$$

It follows that the optimal minimum error strategy for product states with projective measurements is the one that unambiguously detects the action of the channel.

All that is left now is for us to determine the initial state $|\psi\rangle$ and corresponding unambiguous measurement strategy that maximizes $p(g=1|k=1)$. Looking at the definition of rank-three Pauli channels [Eq. (21)], it follows that the maximum of $p(g=1|k=1)$ occurs by tailoring the initial state $|\psi\rangle$ such that we can distinguish with certainty the action of two out of the three Pauli operators with the largest probability of occurrence, and randomly guessing the position of the channel for the remaining Pauli operator and the identity. Denoting by $p^* = \min\{p_1, p_2, p_3\}$, it follows that

$$P_S = 1 - (p_0 + p^*) + \frac{p_0 + p^*}{N}. \tag{C5}$$

Interestingly enough, one may arrive at the conclusion that the optimal probability of success is achievable by unambiguous discrimination by employing the following strategy. For any measurement record other than **0**—for which we simply guess at random—we nominate as our guess for the position of the channel to be that corresponding to the last one in the measurement record. For example, for $N = 3$, we nominate the position of the channel to be $k = 2$ for both measurement records 010 and 110. The probability that we successfully identify the position of the channel is given by

$$P_S = p(g=2|k=2)q^2 + p(g=2|k=2)q(1-q) = p(g=2|k=2)q, \tag{C6}$$

which is clearly maximal if $q = 1$. More generally, this strategy yields, for the optimal probability of success,

$$P_S = \frac{1}{N}\sum_{i=1}^{N} p(g=i|k=i) q^{N-i}, \tag{C7}$$

which is clearly optimized by an unambiguous strategy, namely, $q = 1$. Notice the remarkable simplicity of the argument by cleverly choosing our guessing strategy for each measurement record.

APPENDIX D: LOCAL MEASUREMENT STRATEGIES

In this Appendix, we provide the details for all local strategies described in Sec. IV.

1. General (fixed) local strategy

We begin with the most general strategy involving N independent fixed measurements. Here, qubit i is measured in the basis

$$|m_i\rangle = \cos\left(\frac{\theta_i + m_i\pi}{2}\right)|0\rangle + \sin\left(\frac{\theta_i + m_i\pi}{2}\right)|1\rangle, \tag{D1}$$

where $m_i \in (0, 1)$. The conditional probability of obtaining any of the 2^N measurement records $\{\mathbf{m}\}$, given the state is $|\psi_k\rangle = |0\rangle^{\otimes k-1} \otimes |\phi\rangle \otimes |0\rangle^{\otimes N-k}$, is

$$q(\mathbf{m}|k) = \prod_{i=1}^{k-1} \cos^2\left(\frac{\theta_i + m_i\pi}{2}\right) \cos^2\left(\frac{\theta_k - \phi + m_k\pi}{2}\right) \times \prod_{i=k+1}^{N} \cos^2\left(\frac{\theta_i + m_i\pi}{2}\right). \tag{D2}$$

Upon obtaining a given measurement record, we need to assign a guess as to which one of the N hypotheses $k \in (1, \ldots, N)$ is the most likely one. This is achieved by choosing the hypothesis that maximizes the corresponding posterior probability distribution, $\{p(k)q(\mathbf{m}|k)\}_{k=1}^{N}$. Hence, the resulting probability of success is given by

$$P_S^{\mathrm{GL}} = \sum_{\mathbf{m}=0}^{2^N-1} \max_k \{p(k)q(\mathbf{m}|k)\}. \tag{D3}$$

The maximum probability of success under the general local strategy corresponds to optimizing over the N independent measurement angles $\{\theta_i\}$, which can be done efficiently using numerical techniques.

2. Greedy strategy

Next we consider a greedy local strategy that uses forward communication from each measurement to the next. Specifically, we imagine a total of N parties, each of which has access to a corresponding subsystem of the state $|\psi_k\rangle \in \mathcal{H}^{\otimes N}$, $k \in \{1, \dots, N\}$. The strategy proceeds sequentially, with the first party performing a measurement on their part of the state $|\psi_k\rangle$ communicating the outcome to the second party, and so on. Each party aims to maximize the probability of successfully identifying the position of the error based on the information received from the previous party and the information obtained from their measurement. The prior information available to party n is encapsulated in the prior probabilities for each hypothesis $\{p^{(n)}(k|\mathbf{m})\}_{k=1}^{N}$, where, to ease the notation, $\mathbf{m}$ is understood to run over the first $n-1$ outcomes, i.e., the measurement record $\{m_1, \dots, m_{n-1}\}$.

Party n now attempts to locate the position of the error by assigning a POVM element E_m to each of the $m = 1, \dots, N$ possible hypotheses. The corresponding probability of success is given by

$$\begin{aligned} P_S^{(n)}(\mathbf{m}) &= \mathrm{Tr}[E_n\rho_\phi]p^{(n)}(n|\mathbf{m}) + \sum_{k\neq n}\mathrm{Tr}[E_k\rho_0]p^{(n)}(k|\mathbf{m}) \\ &\leqslant \mathrm{Tr}[E_n\rho_\phi]p^{(n)}(n|\mathbf{m}) + p^{(n)}(k^*|\mathbf{m})\sum_{k\neq n}\mathrm{Tr}[E_k\rho_0] \\ &= \mathrm{Tr}[E_n\rho_\phi]p^{(n)}(n|\mathbf{m}) + p^{(n)}(k^*|\mathbf{m})\mathrm{Tr}[(\mathbb{1}-E_n)\rho_0] \\ &= p^{(n)}(k^*|\mathbf{m}) + \mathrm{Tr}\{E_n[p^{(n)}(n|\mathbf{m})\rho_\phi - p^{(n)}(k^*|\mathbf{m})\rho_0]\} \\ &\leqslant \frac{p^{(n)}(n|\mathbf{m}) + p^{(n)}(k^*|\mathbf{m})}{2} \\ &\quad + \frac{1}{2}\|p^{(n)}(n|\mathbf{m})\rho_\phi - p^{(n)}(k^*|\mathbf{m})\rho_0\|_1, \end{aligned} \tag{D4}$$

where $k^* = \mathrm{argmax}_k\{p^{(n)}(k|\mathbf{m})\}_{k\neq n}$, with the first inequality attained by picking $E_m = 0$ for $m \neq \{n, k^*\}$, and $E_{k^*} = \mathbb{1} - E_n := E_0$, and the second inequality attained by choosing E_n to be the projection onto the positive eigenspace of $p^{(n)}(n|\mathbf{m})\rho_\phi - p^{(n)}(k^*|\mathbf{m})\rho_0$, where $\rho_0 = |0\rangle\langle 0|$ and $\rho_\phi = |\phi\rangle\langle\phi|$, respectively.

That is, party n will always guess in favor of either the error being at location n (corresponding to the measurement outcome that we henceforth label by $m_n = 1$) or at the most likely alternative location k^* (corresponding to the measurement now labeled $m_n = 0$). As this is true for all parties $n \in \{1, \dots, N\}$, the corresponding measurement record after party n measures is one out of 2^n possible binary strings, i.e., $\mathbf{m} \in \{0, 1\}^n$.

The probability that party n obtains the measurement outcome $m \in \{0, 1\}$ is given by

$$q(m_n) = \mathrm{Tr}(E_{m_n}\{p^{(n)}(k|\mathbf{m})\rho_\phi + [1 - p^{(n)}(k|\mathbf{m})]\rho_0\}). \tag{D5}$$

Upon obtaining outcome m_n, we use Bayes' rule to update the priors to

$$p^{(n+1)}(k|\mathbf{m}) = \begin{cases} \frac{\mathrm{Tr}(E_{m_n}\rho_\phi)p^{(n)}(k|\mathbf{m})}{q(m_n)} & \text{for } k = n \\ \frac{\mathrm{Tr}(E_{m_n}\rho_0)p^{(n)}(k|\mathbf{m})}{q(m_n)} & \text{otherwise,} \end{cases} \tag{D6}$$

which are then used by party $n+1$ to pick the optimal measurement accordingly. This process is iterated until the last party N is reached. The success probability after all N samples have been measured is given by $P_S^{(N)}(\mathbf{m})$. Hence, the average probability of success is

$$P_S^{\mathrm{Gr}} = \sum_{\mathbf{m}=0}^{2^{N-1}-1} q(\mathbf{m})P_S^{(N)}(\mathbf{m}), \tag{D7}$$

where

$$q(\mathbf{m}) = \prod_{n=1}^{N-1} q(m_n). \tag{D8}$$

It is important to note that the greedy strategy described above is capable of providing a guess as to the location of the error even if, for some reason, only the first $n < N$ parties are able to perform measurements.

3. Last-one strategy

Finally, let us analyze a local measurement strategy that uses a guessing rule other than maximum likelihood. Specifically, we choose as our guess the hypothesis corresponding to the last one in the measurement record, and optimize over the measurement angles for each of the N measurements. For $\mathbf{m} = \mathbf{0}$, we nominate the first hypothesis as our guess.

The first von Neumann measurement is parametrized by a single angle $\theta_1(0)$ such that

$$|m_1 = a\rangle = \cos\left(\frac{\theta_1(a)}{2}\right)|0\rangle + \sin\left(\frac{\theta_1(a)}{2}\right)|1\rangle, \tag{D9}$$

where $\theta_1(1) = \theta_1(0) + \pi$ are the corresponding projection operators. Depending on the measurement outcome m_1, the second measurement is parametrized by two angles $\theta_2(0|m_1)$, $m_1 \in (0, 1)$; $\theta_2(1|m_1) = \theta_2(0|m_1) + \pi$, the third by four angles, and so on. For the N hypothesis, the total number of measurement angles that we need to optimize is $2^N - 1$, and the corresponding probability of success is given by

$$P_S^{\mathrm{L1}} = \sum_{n=1}^{N} p(n) \sum_{\mathbf{m}\in S_n} q(\mathbf{m}|n), \tag{D10}$$

where

$$q(\mathbf{m}|n) = \prod_{i=1}^{N} |\langle m_i|\psi_n\rangle|^2, \tag{D11}$$

and $S_n \equiv \{\mathbf{m}|m_{n+1}\dots m_N = 0\}$.

[1] D. Nigg, M. Mueller, E. A. Martinez, P. Schindler, M. Hennrich, T. Monz, M. A. Martin-Delgado, and R. Blatt, Quantum computations on a topologically encoded qubit, Science **345**, 302 (2014).

[2] T. H. Taminiau, J. Cramer, T. van der Sar, V. V. Dobrovitski, and R. Hanson, Universal control and error correction in multi-qubit spin registers in diamond, Nat. Nanotechnol. **9**, 171 (2014).

[3] A. D. Córcoles, E. Magesan, S. J. Srinivasan, A. W. Cross, M. Steffen, J. M. Gambetta, and J. M. Chow, Demonstration of a quantum error detection code using a square lattice of four superconducting qubits, Nat. Commun. **6**, 6979 (2015).

[4] J. Kelly, R. Barends, A. G. Fowler, A. Megrant, E. Jeffrey, T. C. White, D. Sank, J. Y. Mutus, B. Campbell, Y. Chen *et al.*, State preservation by repetitive error detection in a superconducting quantum circuit, Nature (London) **519**, 66 (2015).

[5] B. Náfrádi, M. Choucair, K. P. Dinse, and L. Forró, Room temperature manipulation of long lifetime spins in metallic-like carbon nanospheres, Nat. Commun. **7**, 12232 (2016).

[6] J. Cramer, N. Kalb, M. A. Rol, B. Hensen, M. S. Blok, M. Markham, D. J. Twitchen, R. Hanson, and T. H. Taminiau, Repeated quantum error correction on a continuously encoded qubit by real-time feedback, Nat. Commun. **7**, 11526 (2016).

[7] J. Yin, Y. Cao, Y.-H. Li, S.-K. Liao, L. Zhang, J.-G. Ren, W.-Q. Cai, W.-Y. Liu, B. Li, H. Dai *et al.*, Satellite-based entanglement distribution over 1200 kilometers, Science **356**, 1140 (2017).

[8] J.-G. Ren, P. Xu, H.-L. Yong, L. Zhang, S.-K. Liao, J. Yin, W.-Y. Liu, W.-Q. Cai, M. Yang, L. Li *et al.*, Ground-to-satellite quantum teleportation, Nature (London) **549**, 70 (2017).

[9] S.-K. Liao, W.-Q. Cai, W.-Y. Liu, L. Zhang, Y. Li, J.-G. Ren, J. Yin, Q. Shen, Y. Cao, Z.-P. Li *et al.*, Satellite-to-ground quantum key distribution, Nature (London) **549**, 43 (2017).

[10] F. Reiter, A. S. Sørensen, P. Zoller, and C. A. Muschik, Dissipative quantum error correction and application to quantum sensing with trapped ions, Nat. Commun. **8**, 1822 (2017).

[11] H. J. Kimble, The quantum internet, Nature (London) **453**, 1023 (2008).

[12] A. G. Tartakovsky, I. V. Nikiforov, and M. Basseville, *Sequential Analysis: Hypothesis Testing and Changepoint Detection* (CRC Press, Taylor and Francis, Boca Raton, FL, 2015).

[13] M. Thottan and C. Ji, Anomaly detection in IP networks, IEEE Trans. Signal Proc. **51**, 2191 (2003).

[14] V. A. Siris and F. Papagalou, Application of anomaly detection algorithms for detecting SYN flooding attacks, in *IEEE Global Telecommunications Conference* (IEEE, 2004), Vol. 4, pp. 2050–2054.

[15] A. G. Tartakovsky, A. S. Polunchenko, and G. Sokolov, Efficient computer network anomaly detection by changepoint detection methods, IEEE J. Sel. Top. Signa. **7**, 4 (2013).

[16] M. Ahmed, A. N. Mahmood, and M. R. Islam, A survey of anomaly detection techniques in financial domain, Future Gener. Comput. Syst. **55**, 278 (2016).

[17] V. Chandola, A. Banerjee, and V. Kumar, Anomaly detection: A survey, ACM Comput. Surv. **41**, 1 (2009).

[18] N. Liu and P. Rebentrost, Quantum machine learning for quantum anomaly detection, Phys. Rev. A **97**, 042315 (2018).

[19] S. J. Dolinar, J. Hamkins, B. E. Moision, and V. A. Vilnrotter, *Deep Space Optical Communications*, edited by H. Hammati (Wiley, New York, 2006), Vol. 11, chap. 4, pp. 219–289.

[20] N. Dalla Pozza and G. Pierobon, Optimality of square-root measurements in quantum state discrimination, Phys. Rev. A **91**, 042334 (2015).

[21] S. Lloyd, Enhanced sensitivity of photodetection via quantum illumination, Science **321**, 1463 (2008).

[22] S. Pirandola, Quantum reading of a classical digital memory, Phys. Rev. Lett. **106**, 090504 (2011).

[23] Q. Zhuang, Z. Zhang, and J. H. Shapiro, Entanglement-enhanced lidars for simultaneous range and velocity measurements, Phys. Rev. A **96**, 040304(R) (2017).

[24] Q. Zhuang and S. Pirandola, Ultimate limits for multiple quantum channel discrimination, Phys. Rev. Lett. **125**, 080505 (2020).

[25] Q. Zhuang and S. Pirandola, Entanglement-enhanced testing of multiple quantum hypotheses, Commun. Phys. **3**, 103 (2020).

[26] C. Harney and S. Pirandola, Idler-free multichannel discrimination via multipartite probe states, npj Quantum Inf. **7**, 153 (2021).

[27] J. L. Pereira, L. Banchi, Q. Zhuang, and S. Pirandola, Idler-free channel position finding, Phys. Rev. A **103**, 042614 (2021).

[28] M. Hillery, H. Gupta, and C. Zhan, Discrete outcome quantum sensor networks, Phys. Rev. A **107**, 012435 (2023).

[29] S. Llorens, G. Sentís, and R. Muñoz-Tapia, Quantum multi-anomaly detection, Quantum **8**, 1452 (2024).

[30] In the language of [18], EPI is an instance of *point* anomaly and, moreover, can be cast in the framework of supervised learning as we assume that the description of both normal and anomalous devices is known.

[31] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Quantum circuit architecture, Phys. Rev. Lett. **101**, 060401 (2008).

[32] A. S. Holevo, Statistical decision theory for quantum systems, J. Multivariate Anal. **3**, 337 (1973).

[33] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).

[34] D. Ha and Y. Kwon, Complete analysis for three-qubit mixed-state discrimination, Phys. Rev. A **87**, 062302 (2013).

[35] H. Yuen, R. Kennedy, and M. Lax, Optimum testing of multiple hypotheses in quantum detection theory, IEEE Trans. Inf. Theory **21**, 125 (1975).

[36] M. Ban, K. Kurokawa, R. Momose, and O. Hirota, Optimum measurements for discrimination among symmetric quantum states and parameter estimation, Intl. J. Theor. Phys. **36**, 1269 (1997).

[37] Y. C. Eldar and G. D. Forney, On quantum detection and the square-root measurement, IEEE Trans. Inf. Theory **47**, 858 (2001).

[38] Y. C. Eldar, A. Megretski, and G. C. Verghese, Optimal detection of symmetric mixed quantum states, IEEE Trans. Inf. Theory **50**, 1198 (2004).

[39] S. M. Barnett, Minimum-error discrimination between multiply symmetric states, Phys. Rev. A **64**, 030303(R) (2001).

[40] E. Andersson, S. M. Barnett, C. R. Gilson, and K. Hunter, Minimum-error discrimination between three mirror-symmetric states, Phys. Rev. A **65**, 052308 (2002).

[41] G. Sentís, E. Bagan, J. Calsamiglia, G. Chiribella, and R. Muñoz Tapia, Quantum change point, Phys. Rev. Lett. **117**, 150502 (2016).

[42] G. Sentís, J. Calsamiglia, and R. Muñoz Tapia, Exact identification of a quantum change point, Phys. Rev. Lett. **119**, 140506 (2017).

[43] G. Sentís, E. Martínez-Vargas, and R. Muñoz Tapia, Online strategies for exactly identifying a quantum change point, Phys. Rev. A **98**, 052305 (2018).

[44] S. M. Barnett and S. Croke, Quantum state discrimination, Adv. Opt. Photon. **1**, 238 (2009).

[45] J. Bae and L.-C. Kwek, Quantum state discrimination and its applications, J. Phys. A: Math. Theor. **48**, 083001 (2015).

[46] A. Acín, Statistical distinguishability between unitary operations, Phys. Rev. Lett. **87**, 177901 (2001).

[47] R. Duan, Y. Feng, and M. Ying, Entanglement is not necessary for perfect discrimination between unitary operations, Phys. Rev. Lett. **98**, 100503 (2007).

[48] R. Duan, Y. Feng, and M. Ying, Perfect distinguishability of quantum operations, Phys. Rev. Lett. **103**, 210501 (2009).

[49] M. F. Sacchi, Optimal discrimination of quantum operations, Phys. Rev. A **71**, 062340 (2005).

[50] M. F. Sacchi, Entanglement can enhance the distinguishability of entanglement-breaking channels, Phys. Rev. A **72**, 014305 (2005).

[51] M. Piani and J. Watrous, All entangled states are useful for channel discrimination, Phys. Rev. Lett. **102**, 250501 (2009).

[52] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, 2012).

[53] G. M. D'Ariano, P. Lo Presti, and M. G. A. Paris, Using entanglement improves the precision of quantum measurements, Phys. Rev. Lett. **87**, 270404 (2001).

[54] K. Nakahira, Identification of quantum change points for Hamiltonians, Phys. Rev. Lett. **131**, 210804 (2023).

[55] R. H. Dicke, Coherence in spontaneous radiation processes, Phys. Rev. **93**, 99 (1954).

[56] E. Martínez-Vargas, C. Hirche, G. Sentis, M. Skotiniotis, M. Carrizo, R. Munoz-Tapia, and J. Calsamiglia, Quantum sequential hypothesis testing, Phys. Rev. Lett. **126**, 180502 (2021).

[57] Y. Li, V. Y. F. Tan, and M. Tomamichel, Optimal adaptive strategies for sequential quantum hypothesis testing, Commun. Math. Phys. **392**, 993 (2022).

[58] M. Fanizza, C. Hirche, and J. Calsamiglia, Ultimate limits for quickest quantum change-point detection,Phys. Rev. Lett. **131**, 020602 (2023).

[59] G. Gasbarri, M. Bilkis, E. Roda-Salichs, and J. Calsamiglia, Sequential hypothesis testing for continuously-monitored quantum systems, Quantum **8**, 1289 (2024).

[60] J. Watrous, *The Theory of Quantum Information* (Cambridge University Press, Cambridge, 2018).