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Existence, regularity, and a strong Itô formula for the isochronal phase of SPDE*

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Abstract

We prove the existence and regularity of the isochron map for stable invariant manifolds of a large class of evolution equations. Our results apply in particular to the isochron map of reaction-diffusion equations and neural field equations. Using the regularity properties proven here, we prove a strong Itô formula for the isochronal phase of stochastically perturbed travelling waves and other patterns appearing in SPDEs driven by white noise, even for SPDEs that only admit mild solutions.

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1 Introduction

This note presents simple arguments on the existence and regularity of the isochron map for a class of dynamical systems $(\phi_t)_{t\geq 0}$ on a separable Banach space in the vicinity of a stable, normally hyperbolic invariant manifold Γ. The isochron map was introduced for stable limit cycles of dynamical systems in $\mathbb{R}^d$ by [15] and [35]. Our framework encompasses infinite dimensional dynamical systems, in particular reaction-diffusion systems and neural field equations. From the regularity properties of the isochron map proven here, we obtain a strong Itô formula for the isochronal phase of stochastic perturbations of reaction-diffusion systems, even when these systems only admit a mild solution theory.

The proofs in this note are short and general, and recover results on the isochron map proven elsewhere in the literature using different techniques [1]. As will be seen, while [1] requires the invariant manifold to be smooth, our arguments only require Γ to be C^1. Additionally, we make no assumptions on the spectral properties of the generator of the linearization of ϕ_t about Γ. This is of particular interest in the case where Γ is the orbit of a non-commutative Lie group (such as $SE(n)$ when Γ consists of spiral wave solutions to a parabolic PDE), where trying to track the linearized dynamics along Γ can lead to significant complications, as noted in both [24, 32].

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As a consequence, the approach taken in this note may lead to significant generalizations of results obtained for travelling pulses and other transient patterns appearing in stochastic evolution equations using other notions of “phase”. For instance, those studied in neural field equations by [20, 25, 26], and in reaction-diffusion equations by [13, 16, 17, 18, 22, 23, 31]. The approach of this note also allows us to prove a strong Itô formula for the isochronal phase of SPDEs with only mild solutions, which is stronger than the mild Itô formula proven using more complicated techniques in [11, 19].

2 Setup

We consider an evolution equation of the form

$$\partial_t x \;=\; Lx + N(x) \;=:\; V(x), \tag{2.1}$$

and its stochastic perturbation

$$dX \;=\; (LX + N(X))\; dt + \sigma B\, dW. \tag{2.2}$$

Here, L is a linear operator generating a C_0-semigroup on a Hilbert space H, N is a nonlinearity defined on a subspace of H, B is a linear operator on H, $(W_t)_{t\geq 0}$ is a cylindrical Wiener process, as in [12, Chapter 4.1.2], and $\sigma \geq 0$. We make the following assumptions on the deterministic dynamics of (2.1).

Assumption 2.1. *The system* (2.1) *satisfies the following.*

(a) *N is a nonlinearity defined on a dense subset E of H such that E is a Banach space with norm $\|\cdot\|_E$, and the embedding $E \hookrightarrow H$ is dense and continuous. N is four times continuously Fréchet differentiable in the topology of E, with first and second Fréchet derivatives in this topology denoted DN and D^2N.*

(b) *L is a linear operator with a dense domain of definition $D(L)$ in H, and generates a C_0-semigroup $(\Lambda_t)_{t\geq 0}$ on H that restricts to E. Moreover, letting $\|\cdot\|$ denote the norm of H, there exists $\omega > 0$ such that $\sup\{\|\Lambda_t\|,\, \|\Lambda_t\|_E\} \;\leq\; e^{-\omega t}$ for all $t \geq 0$.*

Let the flow map of (2.1) be $(t, x) \;\mapsto\; \phi_t(x)$, and suppose that the flow can be expressed by the variation of constants formula as

$$\phi_t(x) \;=\; \Lambda_t x + \int_0^t \Lambda_{t-s} N(\phi_s(x))\, ds.$$

Since the nonlinearity N is C^4 in the topology of E, the flow map is C^3 in this topology. The first and second derivatives of ϕ_t at x_0 in directions $x, y \in E$ are denoted

$$x \;\mapsto\; D\phi_t(x_0)[x], \qquad (x, y) \;\mapsto\; D^2\phi_t(x_0)[x, y]. \tag{2.3}$$

We now state our assumptions on the flow of (2.1). Specifically, we assume the existence of a stable invariant manifold Γ of (2.1) that models spatiotemporal patterns. In particular, the following guarantees that the dynamics of (2.1) is not chaotic on Γ.

Assumption 2.2. *The deterministic system* (2.1) *has a stable, finite dimensional, normally hyperbolic invariant manifold Γ, as defined in [4, Condition (H3)]. Γ is parameterized by a manifold $\mathcal{S} \subset \mathbb{R}^m$, where $m \in \mathbb{N}$. We write $\Gamma = \{\gamma_\alpha\}_{\alpha\in\mathcal{S}}$, and let $B(\Gamma)$ denote the basin of attraction of Γ in H. Moreover, the following hold.*

(a) *ϕ_t and ϕ_t^{-1} are Lipschitz on Γ with uniform in time Lipschitz constants.*

(b) *$D\phi_t(\gamma_\alpha)$ is invertible with uniformly bounded inverse for all $\alpha \in \mathcal{S}$.*

(c) *The map $\alpha \mapsto \gamma_\alpha$ is continuously differentiable with derivative $D\gamma_\alpha$, and $D\gamma_\alpha$ is invertible with a uniformly bounded inverse.*

Definition 2.3. *Assumption 2.2 implies that for each $x \in B(\Gamma)$ there is a unique $\pi(x) \in \mathcal{S}$ such that*

$$\left\|\phi_t(x) - \phi_t(\gamma_{\pi(x)})\right\| \xrightarrow[t\to\infty]{} 0, \tag{2.4}$$

providing a well-defined map $\pi : B(\Gamma) \to \mathcal{S}$. The existence and uniqueness of this map is proven in Theorem 3.1. We refer to π as the isochron map *of Γ.*

Throughout this note, for $\delta > 0$ we define $\Gamma_\delta := \left\{x \in E : \left\|x - \gamma_{\pi(x)}\right\|_E \leq \delta\right\} \subset E$. When considering the SDE (2.2), we require the following additional assumptions.

Assumption 2.4. *Assumption 2.1 holds, and the SDE (2.2) satisfies the following.*

(a) *There exists a unique E-valued mild solution $(X_t)_{t\geq 0}$ to (2.2), satisfying*

$$X_t \;=\; \Lambda_t X_0 + \int_0^t \Lambda_{t-s} N(X_s)\, ds + \sigma \int_0^t \Lambda_{t-s} B\, dW_s, \qquad X_0 \in H, \tag{2.5}$$

for $X_0 \in E$ and all $t < \tau_{blowup}$, where τ_{blowup} is in $(0, \infty]$ almost surely.

(b) *For $x \in H$, $(\phi_t(x))_{t>0}$ is continuous in $D(L)$, equipped with the graph norm of L.*

(c) *There exists an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ of H consisting of eigenfunctions of L, and $e_k \in E$ for $k \in \mathbb{N}$. Letting λ_k denote the eigenvalue of $-L$ corresponding to e_k,*

$$K_{s,r} \;\coloneqq\; \sup_{t\in[s,r]} \sum_{k\in\mathbb{N}} \|\Lambda_t B e_k\|_E \;<\; \infty \qquad \textit{and} \qquad \int_0^t K_{s,r}\, ds \;<\; \infty \tag{2.6}$$

for $t \in (0, r)$. Moreover, for $y \in E \subset H$, written $y = \sum_{k\in\mathbb{N}} y^k e_k$ in H,

$$L\Lambda_t y \;=\; \Lambda_t L y \;\coloneqq\; \sum_{k\in\mathbb{N}} y^k \Lambda_t L e_k, \qquad \textit{with convergence in } E. \tag{2.7}$$

Assumption 2.4 is needed for our strong Itô formula. We remark that the assumption is weaker than requiring $X_t \in D(L)$ or that $E = H$, so that the Itô formulas of [2, 9, 19] do not apply to (2.2). See Remark 2.5 below. The condition (2.6) is only needed when the noise in (2.2) is white in space, but not when the noise is trace class.

Note that with additive noise, solutions of (2.2) may be pushed out of $B(\Gamma)$ in finite time. We therefore define the exit time of the solution from $B(\Gamma)$ as

$$\tau \;\coloneqq\; \min\left\{\tau_{blowup}, \inf\left\{t > 0 \,:\, X_t \in E/B(\Gamma)\right\}\right\}. \tag{2.8}$$

Since the geometry of $B(\Gamma)$ is generally unknown and may be very complicated, we will sometimes restrict our attention to Γ_δ for $\delta > 0$. We define the exit time from Γ_δ as

$$\tau_\delta \;\coloneqq\; \min\left\{\tau_{blowup}, \inf\left\{t > 0 \,:\, \left\|X_t - \gamma_{\pi(X_t)}\right\|_E = \delta\right\}\right\}. \tag{2.9}$$

When Γ consists of fixed points of ϕ_t, estimates on the distribution of τ_δ can be found in [27], and under more general assumptions in [32]. See also [5]. Further work is needed to estimate the distribution of τ for more general stable invariant manifolds. We let $\pi_t = \pi(X_t)$ be the isochronal phase of X_t for $t < \tau$.

Remark 2.5. The present study is motivated by two classes of examples of (2.1). The first class consists of reaction-diffusion systems

$$\partial_t x \;=\; (\Delta - a)x + N(x), \tag{2.10}$$

where $a > 0$ and N is a vector of polynomials. With $L = \Delta - a$, $B = I$, and O bounded, we can show that (2.10) satisfies Assumptions 2.1 & 2.4. For instance if $O = [0, \ell]$ with periodic boundaries, take $H = L^2(O)$ and $E = C(O)$, the space of continuous functions on O with the supremum norm. In this case, it is more convenient to index the eigenpairs by $\mathbb{Z}$ rather than $\mathbb{N}$. We then have $e_k(x) = \sin\left(\frac{2k\pi}{\ell}x\right)$ and $\lambda_k = \frac{4\pi^2k^2}{\ell^2}$ for $k < 0$, and $e_k(x) = \cos\left(\frac{1k\pi}{\ell}x\right)$ and $\lambda_k = \frac{4\pi^2k^2}{\ell^2}$ for $k \geq 0$. Hence $\|e_k\|_E \leq 1$, and

$$\|\Lambda_t e_k\|_E \leq \exp\left(-\frac{4\pi^2k^2t}{\ell^2} - at\right) \quad \text{for } k \in \mathbb{Z} \text{ and } t > 0.$$

For fixed $r > s > 0$ we then have $K_{s,r} = \sup_{t\in[s,r]} \sum_{k\in\mathbb{N}} \|\Lambda_t e_k\|_E < \infty$. Indeed, in this case $K_{s,r}$ takes the form of a Jacobi ϑ-function (as in [34, Section 21.11]), which is integrable over $[0, 1]$, composed with a continuous function taking values in $[0, 1]$, verifying (2.6). Assumption 2.4(b) holds by the regularity of solutions to the heat equation, which can be proven as in [14, Theorem 2.3.1]. The regularity of the heat equation also immediately implies (2.7), since $L\Lambda_t y$ is clearly equal to $\Lambda_t L y$, as defined in (2.7) for $y \in E$, and in this case the graph norm of L is just $\|\cdot\|_{C^2}$. Finally, when $B = I$, Assumption 2.4(a) is verified using the local inversion theorem as in [12], since in this case the stochastic convolution $\int_0^t \Lambda_{t-s}\, dW_s$ is contained in E.

The second class consists of integro-differential equations on a spatial domain O,

$$\partial_t x_t = -x_t + \omega * f(x_t), \qquad \partial_t y_t = -\varepsilon^{-1} y_t + x_t, \tag{2.11}$$

where ω and f are usually bounded and Lipschitz, and the convolution is taken over O. These equations appear in neuroscience, where they are referred to as *neural field equations* [8], and ecology [30]. Mild solutions to stochastic perturbations of (2.11) only exist when the perturbing noise is trace class [23], and in this case Assumption 2.4 is satisfied. With $L = [-1, -\epsilon^{-1}]$, Assumptions 2.1(a), (b), are satisfied, though (c) is not. However, Assumption 2.1 (c) is only needed to prove the Itô formula for the isochronal phase when W is not trace class, as seen in the arguments of Section 3 below.

For reaction-diffusion or neural field equations, the main examples of Γ that we have in mind are stationary or travelling waves [3, 21, 28], and spiral waves in excitable media, as studied in [6, 7] on unbounded domains and in [36] on bounded spatial domains – see also [24]. We are also interested in the case of stationary patterns that remain stationary under translation or rotation [10, 28, 33]. So long as (2.10) or (2.11) possesses a sufficiently regular invariant manifold Γ, and the parameterization $\alpha \mapsto \gamma_\alpha$ is non-degenerate, these examples satisfy Assumption 2.2. For instance, if Γ consists of travelling wave solutions on a periodic domain, this is the case.

3 Regularity of the isochron map & a strong Itô formula

Theorem 3.1. *Under Assumptions 2.1 & 2.2, there exists a unique function* $\pi : B(\Gamma) \to \mathcal{S} \subset \mathbb{R}^m$ *satisfying* (2.4)*. This function is twice continuously Fréchet differentiable at each* $x_0 \in B(\Gamma) \subset E$ *in the topology of* E.

Proof. We begin by proving that $\pi : B(\Gamma) \to \mathcal{S}$ is well-defined. Fix $x \in B(\Gamma)$. Since Γ is a stable manifold of (2.1), taking any sequence of positive numbers $\{\epsilon_n\}_{n\in\mathbb{N}}$ decreasing to zero, there exists a sequence of times $\{t_n\}_{n\in\mathbb{N}}$ increasing to infinity and a sequence of closed sets $U_n \subset \Gamma$ such that for all $n \in \mathbb{N}$ and $\gamma \in U_n$, we have

$$\|\phi_{t_n}(x) - \gamma\|_E \leq \epsilon_n / 2M_1 \qquad \text{for a constant } M_1. \tag{3.1}$$

The collection $\{\phi_{t_n}^{-1} U_n\}_{n\in\mathbb{N}}$ is a sequence of non-empty, closed, nested subsets of the complete metric space E. Using the bi-Lipschitz condition of ϕ_t on Γ and (3.1), we see that $\operatorname{diam}\left(\phi_{t_n}^{-1}U_n\right) \le \epsilon_n$. In particular, the diameter of $\phi_{t_n}^{-1}U_n$ tends to zero as $n \to \infty$. By Cantor's Intersection Theorem, there is a unique γ_* such that $\bigcap_{n\in\mathbb{N}} \phi_{t_n}^{-1}U_n = \{\gamma_*\}$. Since $\alpha \mapsto \gamma_\alpha$ is invertible, there is a unique $\pi(x) \in \mathcal{S}$ such that $\gamma_* = \gamma_{\pi(x)}$. To prove that π is C^2, note that our assumptions on N implies that $x \mapsto \phi_t(x)$ is C^3 on $B(\Gamma) \cap E$ in the topology of E. Since C^3 is a Banach space, the map $x \mapsto \gamma_{\pi(x)} = \lim_{t\to\infty} \phi_t(x)$ is C^3 on $B(\Gamma) \cap E$. By assumption, $(r \mapsto \gamma_r) : \mathcal{S} \to \Gamma$ is invertible with C^2 inverse. Denoting this inverse by γ^{-1}, the map $x \mapsto \pi(x) = \gamma^{-1}(\gamma_{\pi(x)})$ is C^2 on $B(\Gamma) \cap E$. □

We denote the first and second derivatives of the isochron map at $x_0 \in B(\Gamma) \cap E$ in the directions $x, y \in E$ by $D\pi(x_0)x$ and $D^2\pi(x_0)[x, y]$. The following theorem yields additional regularity of the isochron map when Assumption 2.1 holds, showing in particular that its second derivative is, in a sense, trace class.

Theorem 3.2. *Under Assumptions 2.1 & 2.2, there exists $M_\pi \in (0, \infty)$ such that*

$$\begin{aligned} \sum_{k\in\mathbb{N}} \left\| D^2\pi(x_0)[Be_k, Be_k] \right\|_{\mathbb{R}^m} < M_\pi, \qquad & \sum_{k\in\mathbb{N}} \| D\pi(x_0)Be_k \|_{\mathbb{R}^m} < M_\pi, \\ \textit{and} \qquad & \sum_{k\in\mathbb{N}} \| D\pi(x_0)Be_k \|_{\mathbb{R}^m}^2 < M_\pi, \end{aligned} \tag{3.2}$$

uniformly in $x_0 \in \Gamma_\delta$. Moreover, these quantities are Lipschitz continuous in $x_0 \in \Gamma_\delta$.

To prove Theorem 3.2, define $\tilde{\pi} : \Gamma_\delta \to \Gamma$ as $\tilde{\pi}(x) := \gamma_{\pi(x)}$, and note that

$$\phi_t(\tilde{\pi}(x)) = \tilde{\pi}(\phi_t(x)) \qquad \forall x \in \Gamma_\delta, \;\; t > 0. \tag{3.3}$$

We show that (3.2) is satisfied by π replaced with $\tilde{\pi}$. Then, since the map $\alpha \mapsto \gamma_\alpha$ is C^2 with uniformly bounded first derivative, (3.2) follows. From (3.3) and Assumption 2.2,

$$D\tilde{\pi}(x)y = [D\phi_t(\tilde{\pi}(x))]^{-1} D\tilde{\pi}(\phi_t(x)) D\phi_t(x)y, \qquad \text{and} \tag{3.4}$$

$$\begin{aligned} D^2\tilde{\pi}(x)[y, z] = [D\phi_t(\tilde{\pi}(x))]^{-1} \Big(& D^2\tilde{\pi}(\phi_t(x))[D\phi_t(x)y, D\phi_t(x)z] \\ & + D\tilde{\pi}(\phi_t(x)) D^2\phi_t(x)[y, z] - D^2\phi_t(\tilde{\pi}(x))[D\tilde{\pi}(x)y, D\tilde{\pi}(x)z] \Big). \end{aligned} \tag{3.5}$$

We now let $\|\cdot\|_1 := \|\cdot\|_{E\to E}$ and $\|\cdot\|_2 := \|\cdot\|_{E\times E\to E}$. Since $D\phi_t(x)^{-1}$ and $D\tilde{\pi}(\phi_t(x))$ are bounded operators, it follows that for some $C > 0$ independent of $x \in B(\Gamma)$ and $t > 0$,

$$\begin{aligned} \sum_{k\in\mathbb{N}} \| D\tilde{\pi}(x)Be_k \|_E &\le \left\| D\phi_t(\tilde{\pi}(x))^{-1} D\tilde{\pi}(\phi_t(x)) \right\|_1 \sum_{k\in\mathbb{N}} \| D\phi_t(x)Be_k \|_E \\ &\le C \sum_{k\in\mathbb{N}} \| D\phi_t(x)Be_k \|_E. \end{aligned} \tag{3.6}$$

Similarly, since $D\phi_t(\tilde{\pi}(x))^{-1}$, $D^2\tilde{\pi}(\phi_t(x))$, $D\tilde{\pi}(\phi_t(x))$, and $D^2\phi_t(x)$ are bounded uniformly in $x \in \Gamma_\delta$, for some $C_1, C_2, C_3 > 0$ independent of $x \in B(\Gamma)$ and $t > 0$,

$$\begin{aligned} \sum_{k\in\mathbb{N}} \left\| D^2\tilde{\pi}(x)[Be_k, Be_k] \right\|_1 \le{}& \left\| D\phi_t(\tilde{\pi}(x))^{-1} \right\|_E \Bigg(\left\| D^2\tilde{\pi}(\phi_t(x)) \right\|_2 \sum_{k\in\mathbb{N}} \| D\phi_t(x)Be_k \|_E^2 \\ & + \| D\tilde{\pi}_E(\phi_t(x)) \|_1 \sum_{k\in\mathbb{N}} \left\| D^2\phi_t(x)[Be_k, Be_k] \right\|_E \\ & + \left\| D^2\phi_t(\tilde{\pi}(x)) \right\|_2 \sum_{k\in\mathbb{N}} \| D\tilde{\pi}(x)Be_k \|_E^2 \Bigg) \\ \le{}& C_1 \sum_{k\in\mathbb{N}} \| D\phi_t(x)Be_k \|_E^2 + C_2 \sum_{k\in\mathbb{N}} \left\| D^2\phi_t(x)[Be_k, Be_k] \right\|_E + C_3 \sum_{k\in\mathbb{N}} \| D\tilde{\pi}(x)Be_k \|_E^2. \end{aligned}$$

Hence, Theorem 3.2 immediately follows as a corollary to the following result.

Lemma 3.3. *Under Assumptions 2.1 & 2.2, there exists $M_\phi \in (0,\infty)$ such that for any $t > 0$ and small $\delta > 0$, the following hold uniformly in $x_0 \in \Gamma_\delta$ and $s \in (0,t)$*

$$\begin{aligned} &\sum_{k\in\mathbb{N}} \|D\phi_s(x_0)[Be_k]\|_E \,<\, M_\phi, \\ &\sum_{k\in\mathbb{N}} \|D\phi_s(x_0)[Be_k]\|_E^2 \,<\, M_\phi, \quad \text{and} \quad \sum_{k\in\mathbb{N}} \left\|D^2\phi_s(x_0)[Be_k, Be_k]\right\|_E \,<\, M_\phi. \end{aligned} \tag{3.7}$$

Moreover, these quantities are Lipschitz continuous in $x_0 \in \Gamma_\delta$.

Proof. We consider the evolution equations of the operators defined in (2.3), written as

$$\begin{aligned} D\phi_t(x_0)[x] \,&=\Lambda_t D\phi_0(x_0)[x] + \int_0^t \Lambda_{t-s} DN(\phi_s(x_0)) D\phi_s(x_0)[x]\,ds, \\ D^2\phi_t(x_0)[x,y] \,&=\, \Lambda_t D^2\phi_0(x_0)[x,y] + \int_0^t \Lambda_{t-s} DN(\phi_s(x_0)) D^2\phi_s(x_0)[x,y]\,ds \\ &\qquad + \int_0^t \Lambda_{t-s} D^2N(\phi_s(x_0))\left[D\phi_s(x_0)[x], D\phi_s(x_0)[y]\right] ds, \end{aligned} \tag{3.8}$$

where DN and D^2N are the first and second Fréchet derivatives of N. Observing that $D\phi_0(x_0)[x] = x$ and applying Tonelli's theorem,

$$\begin{aligned} \sum_{k\in\mathbb{N}} \|D\phi_t(x_0)[Be_k]\|_E \,&\le\, \sum_{k\in\mathbb{N}} \|\Lambda_t Be_k\|_E + \sum_{k\in\mathbb{N}} \int_0^t \|\Lambda_{t-s} DN(\phi_s(x_0))\|_1 \|D\phi_s(x_0)[Be_k]\|_E\,ds \\ &\le\, \sum_{k\in\mathbb{N}} \|\Lambda_t Be_k\|_E + C_0 \int_0^t \sum_{k\in\mathbb{N}} \|D\phi_s(x_0)[Be_k]\|_E\,ds \end{aligned}$$

for some $C_0 > 0$ which is independent of x_0 (using Assumption 2.1, and the fact that N is uniformly Lipschitz on Γ_δ in the topology of E). Taking $s \in (0,t)$ and letting $r \mapsto K_{s,r}$ be the non-decreasing function defined in (2.6), we apply Grönwall's inequality to obtain

$$\sum_{k\in\mathbb{N}} \|D\phi_t(x_0)[Be_k]\|_E \,\le\, K_{s,r} e^{C_0 t} \,<\, \infty \qquad \text{for arbitrary } r > t. \tag{3.9}$$

Furthermore, (2.6) & (3.9) imply local integrability of the sum:

$$\sum_{k\in\mathbb{N}} \int_0^t \|D\phi_s(x_0)[Be_k]\|_E\,ds \,=\, \int_0^t \sum_{k\in\mathbb{N}} \|D\phi_s(x_0)[Be_k]\|_E\,ds \,<\, e^{C_0 t} \int_0^t K_{s,r}\,ds.$$

We remark that the constants appearing in this bound are independent of $x_0 \in \Gamma_\delta$, implying the first uniform bound (3.7). Similarly, observe that

$$\begin{aligned} \sum_{k\in\mathbb{N}} \|D\phi_t(x_0)[Be_k]\|_E^2 \,&\le\, \sum_{k\in\mathbb{N}} \|\Lambda_t Be_k\|_E^2 + 2C_0 \sum_{k\in\mathbb{N}} \left(\|\Lambda_t Be_k\|_E \int_0^t \|D\phi_s(x_0)[Be_k]\|_E\,ds \right) \\ &\qquad + C_0^2 \sum_{k\in\mathbb{N}} \left(\int_0^t \|D\phi_s(x_0)[Be_k]\|_E\,ds \right)^2. \end{aligned}$$

The first and second sums are bounded, respectively, by (2.6) & (3.9). Hence,

$$\begin{aligned} \sum_{k\in\mathbb{N}} \|D\phi_t(x_0)[Be_k]\|_E^2 \,&\le\, K_{s,r}^2 + 2K_{s,r}^2 e^{C_0 t} + C_0^2 \sum_{k\in\mathbb{N}} \left(\int_0^t \|D\phi_s(x_0)[Be_k]\|_E\,ds \right)^2 \\ &\le\, K_{s,r}^2 \left(1 + 2e^{C_0 t}\right) + C_0^2 \int_0^t \sum_{k\in\mathbb{N}} \|D\phi_s(x_0)[Be_k]\|_E^2\,ds, \end{aligned}$$

for arbitrary $s<t<r$. Applying Grönwall's inequality, we find that $\sum_{k\in\mathbb{N}}\|D\phi_t(x_0)[Be_k]\|_E^2$ is finite and locally integrable in $t>0$. We remark that the constants appearing in the second bound of (3.7) obtained by these arguments are independent of x_0.

Having obtained the first two bounds in (3.7), we can similarly obtain the third:

$$\begin{aligned}\sum_{k\in\mathbb{N}}\left\|D^2\phi_t(x_0)[Be_k,Be_k]\right\|_E \;\le\; &\sum_{k\in\mathbb{N}}\|\Lambda_t Be_k\|_E + C_1\int_0^t\sum_{k\in\mathbb{N}}\left\|D^2\phi_s(x_0)[Be_k,Be_k]\right\|_E ds\\ &+ C_1\int_0^t\sum_{k\in\mathbb{N}}\|D\phi_s(x_0)[Be_k]\|_E^2\,ds,\end{aligned}$$

since $D^2\phi_0(x_0)[x,x]=x$. By our previous results, the first and third sum in the above are finite. Therefore for arbitrary $s<t<r$, for another constant $L_{s,r}>0$ we have

$$\sum_{k\in\mathbb{N}}\left\|D^2\phi_t(x_0)[Be_k,Be_k]\right\|_E \;\le\; L_{s,r}+C\int_0^t\sum_{k\in\mathbb{N}}\left\|D^2\phi_s(x_0)[Be_k,Be_k]\right\|_E ds$$

Grönwall's inequality then yields the third bound in (3.7) with x_0-independent constants.

To see Lipschitz continuity of the maps defined by taking x_0 to the values in (3.7), we remark that the first part of this proposition implies that the finite sums

$$\begin{aligned}&\sum_{k=1}^K\|D\phi_t(x_0)[Be_k]\|_E \;<\; M_\phi,\\ &\sum_{k=1}^K\|D\phi_t(x_0)[Be_k]\|_E^2 \;<\; M_\phi, \quad \text{and} \quad \sum_{k=1}^K\left\|D^2\phi_t(x_0)[Be_k,Be_k]\right\|_E \;<\; M_\phi,\end{aligned}$$

converge uniformly in $x_0\in\Gamma_\delta$ to the series in (3.7) as $K\to\infty$. Hence we need only prove that the finite sums are Lipschitz. But, from the fact that $x_0\mapsto\phi_t(x_0)$ is C^3 in the topology of E (since N is C^4), we know that each of the summands is Lipschitz in Γ_δ. □

We now prove a strong Itô formula for $\pi(X_t)$ that holds for $t<\tau$. We need the following two lemmas.

Lemma 3.4. *Let Assumptions 2.2 & 2.4 hold. For each $t>0$, the following Itô integral is well-defined as an $\mathbb{R}^m$-valued random variable:*

$$\int_0^{t\wedge\tau} D\pi(X_s)B\,dW_s. \tag{3.10}$$

Proof. For a collection of independent identically distributed Brownian motions $\{\beta^k\}_{k\in\mathbb{N}}$, write $W_t=\sum_{k\in\mathbb{N}}e_k\beta^k_t$. Of course, this sum does not converge in any sense as an H-valued random variable, but as an H_0-valued random variable for some Hilbert space $H_0\supset H$ (see [12, Chapter 4] for details). However, we need not specify H_0 here. Indeed, the integral (3.10) can be made sense of as the limit in expectation of the finite sums

$$\sum_{k=1}^K\int_0^{t\wedge\tau}D\pi(X_s)Be_k\,d\beta^k_s \;=:\; \sum_{k=1}^K\tilde\beta^k_t,$$

which converge in mean square. Using Theorem 3.2 and Itô's isometry, we obtain

$$\begin{aligned}\mathbb{E}\left[\left\|\sum_{k\in\mathbb{N}}\tilde\beta^k_t\right\|_{\mathbb{R}^m}\right]^2 \;&\le\; \sum_{k,\ell\in\mathbb{N}}\mathbb{E}\left[\left\langle\tilde\beta^k_t,\tilde\beta^\ell_t\right\rangle_{\mathbb{R}^m}\right]\\ &\le\; \sum_{k\in\mathbb{N}}\mathbb{E}\left[\int_0^{t\wedge\tau}\|D\pi(X_s)Be_k\|_{\mathbb{R}^m}^2\,ds\right]\\ &\le\; t\sup_{x\in\Gamma_\delta}\sum_{k\in\mathbb{N}}\|D\pi(x)Be_k\|_{\mathbb{R}^m}^2 \;<\; tM_\pi,\end{aligned}\tag{3.11}$$

where $\mathbb{E}\left[\left\langle \tilde{\beta}_t^k l, \tilde{\beta}_t^\ell \right\rangle\right] = 0$ for $k = \ell$ by the covariation of $\tilde{\beta}_t^k$ and $\tilde{\beta}_t^\ell$. □

Lemma 3.5. *Let Assumptions 2.2 & 2.4 hold. Then,* $D\pi(x)Ly := \sum_{k\in\mathbb{N}} y^k D\pi(x)Le_k$ *exists in* $\mathbb{R}^m$ *for all* $x, y \in E$*, where* $\{y^k\}_{k\in\mathbb{N}}$ *are the* $\{e_k\}_{k\in\mathbb{N}}$ *basis coefficients of* y *in* H*. Moreover,* $\partial_t \pi(\phi_t(x)) = D\pi(\phi_t(x))V(\phi_t(x))$.

Proof. Write $y = \sum_{k\in\mathbb{N}} y^k e_k$, converging in H. Note that the sum $Ly = \sum_{k\in\mathbb{N}} y^k \lambda_k e_k$ does not necessarily converge in H, but in a larger Hilbert space H_L. Recall that for $t > 0$ and $x \in E$, $D\pi(x) = M(x)D\phi_t(x)$ for a bounded linear operator $M(x)$ on E. Hence, $\|D\pi(x)Ly\|_E \le c\|D\phi_t(x)Ly\|_E$ for some $c > 0$, and

$$\|D\phi_t(x)Ly\|_E \;\le\; \|\Lambda_t Ly\|_E + \int_0^t \|\Lambda_{t-s} DN(\phi_s(x))\|_1 \|D\phi_s(x)Ly\|_E \, ds.$$

Assumption 2.4(c) and Grönwell's inequality complete the proof of the convergence of $D\pi(x)Ly$. The statement $\partial_t \pi(\phi_t(x)) = D\pi(\phi_t(x))V(\phi_t(x))$ follows from the chain rule. □

Theorem 3.6. *Let Assumptions 2.2 & 2.4 hold. Then,*

$$\begin{aligned} \pi(X_{t\wedge\tau}) \;=\; \pi(X_0) + \int_0^{t\wedge\tau} D\pi(X_s)V(X_s)\,ds + \frac{\sigma^2}{2}\int_0^{t\wedge\tau} \sum_{k\in\mathbb{N}} D^2\pi(X_s)[Be_k, Be_k]\,ds \\ + \sigma\int_0^{t\wedge\tau} D\pi(X_s)B\,dW_s, \end{aligned} \tag{3.12}$$

and all of the terms in the above expression are well-defined in $\mathbb{R}^m$*.*

Proof. The proof now follows as in [1]. Taking $t > 0$, partition $[0, t]$ as $\{t_k\}_{k=1}^M$. Let $t^* := t \wedge \tau$, and set $t_i^* := t_i \wedge \tau$ for $i \in \{1, \dots, M\}$. By Theorem 3.1, we write $\pi(X_{t^*})$ as

$$\pi(X_{t^*}) \;=\; \pi(X_0) + \sum_{i=1}^M \Big(D\pi(X_{t_i^*})[X_{t_{i+1}^*} - X_{t_i^*}] + D^2\pi(w_i)[X_{t_{i+1}^*} - X_{t_i^*}, X_{t_{i+1}^*} - X_{t_i^*}] \Big), \tag{3.13}$$

where $w_i = a_i X_{t_i^*} + (1 - a_i)X_{t_{i+1}^*}$ for $a_i \in [0, 1]$. As $(X_t)_{t\ge 0}$ is a mild solution of (2.2),

$$\begin{aligned} X_{t_{i+1}^*} - X_{t_i^*} \;&=\; \Big[\Lambda_{t_{i+1}^* - t_i^*} - I\Big] X_0 + \int_{t_i^*}^{t_{i+1}^*} \Lambda_{t_{i+1}^* - s} N(X_s)\,ds + \sigma\int_{t_i^*}^{t_{i+1}^*} \Lambda_{t_{i+1}^* - s} B(X_s)\,dW_s \\ &=:\; U_i^1 + U_i^2 + \sigma U_i^3. \end{aligned}$$

Combining this with (3.13), we obtain

$$\begin{aligned} \pi(X_{t^*}) - \pi(X_0) \;=\; &\sum_{i=1}^M D\pi(X_{t_i^*})\left[U_i^1\right] + \sum_{i=1}^M D\pi(X_{t_i^*})\left[U_i^2\right] + \sigma\sum_{i=1}^M D\pi(X_{t_i^*})\left[U_i^3\right] \\ &+ \sum_{i=1}^M D^2\pi(w_i)\left[U_i^1 + U_i^2, U_i^1 + U_i^2\right] + 2\sigma\sum_{i=1}^M D^2\pi(w_i)\left[U_i^1 + U_i^2, U_i^3\right] \\ &+ \sigma^2\sum_{i=1}^M D^2\pi(w_i)\left[U_i^3, U_i^3\right] \\ =:\; & I + II + III + IV + V + VI. \end{aligned} \tag{3.14}$$

Let the mesh size of $\{t_i\}_{i=1}^M$ be $h > 0$, and for arbitrary $i \in \{1, \dots, M\}$ let $(\hat{X}_s)_{s\in[t_i, t_{i+1}]}$ be defined by the solution to (2.1) with initial condition X_{t_i},

$$\hat{X}_{t_i} \;=\; X_{t_i}, \quad \hat{X}_s \;=\; \Lambda_{s-t_i} X_{t_i} + \int_{t_i}^s \Lambda_{s-r} N(\hat{X}_r)\,dr \quad \text{for} \quad s \in [t_i, t_{i+1}].$$

By Assumption 2.4(b), note that $(\hat{X}_s)_{s\in[t_i^*,t_{i+1}^*]}$ exists as a continuous path in $D(L)$ (in the topology of the graph norm of L), and hence

$$\sup_{s\in[t_i^*,t_{i+1}^*]} \lim_{h\to 0} h^{-1}\left\|(\Lambda_h - I)\,\hat{X}_s\right\|_E \,<\, \infty.$$

Consequently, we have the following fact, which will prove to be useful below:

$$\sup_{s\in[t_i^*,t_{i+1}^*]} \lim_{h\to 0} h^{-1/2}\left\|(\Lambda_h - I)\,\hat{X}_s\right\|_E \,=\, 0. \tag{3.15}$$

We now show that $(I+II) \to \int_0^{t^*} D\pi(X_s)V(X_s)\,ds$ as $h\to 0$. To do so, we apply the second order Taylor's theorem to $\pi(\hat{X}_{t_{i+1}^*})$ centered at $\hat{X}_{t_i}$ to obtain

$$D\pi(\hat{X}_{t_i^*})\left[\hat{X}_{t_{i+1}^*} - \hat{X}_{t_i^*}\right] + D^2\pi(U_i^5)\left[\hat{X}_{t_{i+1}^*} - \hat{X}_{t_i^*}, \hat{X}_{t_{i+1}^*} - \hat{X}_{t_i^*}\right] \,=\, \pi(\hat{X}_{t_{i+1}^*}) - \pi(\hat{X}_{t_i^*}),$$

where U_i^5 is defined for some $a_i^5 \in [0,1]$ as

$$U_i^5 \,\coloneqq\, a_i^5 \hat{X}_{t_i^*} + (1-a_i^5)U_i^4.$$

By Grönwall's inequality and the Lipschitz property of N (on Γ_δ), it can be seen that

$$\left\|X_s - \hat{X}_s\right\|_E \,\sim\, O(h^2) \quad \text{for} \quad s\in[t_i,t_{i+1}]. \tag{3.16}$$

It then follows that

$$\begin{aligned}\left\|\hat{X}_{t_{i+1}^*} - \hat{X}_{t_i^*} - \left[(\Lambda_{t_{i+1}^*-t_i^*} - I)X_{t_i^*} + \int_{t_i}^{t_{i+1}^*} \Lambda_{t_{i+1}^*-s}N(X_s)\,ds\right]\right\|_E \\ \le\, K_0 h \sup_{s\in[t_i^*,t_{i+1}^*]}\left\|\hat{X}_s - X_s\right\|_E \,\le\, K_1 h^2\end{aligned}$$

for some constants $K_0, K_1 > 0$. Using (3.15), for all $t\in(t_i,t_{i+1})$ we have,

$$\left\|\hat{X}_{t^*} - \hat{X}_{t_i^*}\right\|_E \,\sim\, o(h^{1/2}). \tag{3.17}$$

Hence, using Lemma 3.5 each summand in $(I+II)$ is estimated as

$$\begin{aligned}&D\pi(X_{t_i^*})\left[(\Lambda_{t_{i+1}^*-t_i^*} - I)X_{t_i^*} + \int_{t_i}^{t_{i+1}} \Lambda_{t_{i+1}-s}N(X_s)\,ds\right] \\ &\quad=\, -D^2\pi(U_i^5)\left[\hat{X}_{t_{i+1}^*} - \hat{X}_{t_i^*}, \hat{X}_{t_{i+1}^*} - \hat{X}_{t_i^*}\right] + O(h^2) + \pi(\hat{X}_{t_{i+1}^*}) - \pi(\hat{X}_{t_i^*}) \\ &\quad=\, o(h) + \pi(\hat{X}_{t_{i+1}^*}) - \pi(\hat{X}_{t_i^*}) \,=\, o(h) + hD\pi(\hat{X}_{t_i^*})V(\hat{X}_{t_i^*}).\end{aligned}$$

Noting that $\hat{X}_{t_i^*} = X_{t_i^*}$, summing over $\{t_i^*\}_{i=1}^M$, and taking $h\to 0$ yields the result.

To show $III \to \int_0^{t^*} D\pi(X_s)B\,dW_s$, we Taylor expand $\Lambda_{t_{i+1}-s}$ about $s = t_{i+1}$,

$$\Lambda_{t_{i+1}-s} \,=\, I + L\Lambda_u(t_{i+1}-s) \tag{3.18}$$

for some $u\in(s,t_{i+1})$. Hence, for arbitrary $x\in\Gamma_\delta$

$$\begin{aligned}D\pi(x)\int_{t_i^*}^{t_{i+1}^*} \Lambda_{t_{i+1}^*-s}B\,dW_s \,&=\, \int_{t_i^*}^{t_{i+1}^*} D\pi(x)\left(I + (t_{i+1}-s)L\Lambda_u\right)B\,dW_s \\ &=\, D\pi(x)B[W_{t_{i+1}^*} - W_{t_i^*}] + \int_{t_i^*}^{t_{i+1}^*}(t_{i+1}-s)D\pi(x)L\Lambda_u B\,dW_s.\end{aligned}$$

By Lemma 3.5, $D\pi(x)L\Lambda_u B : H \to \mathbb{R}^m$ is a bounded linear operator, which we may identify with an element of H. Hence, applying the Burkholder-Davis-Gundy from [29],

$$\begin{aligned}\mathbb{E}_x\left[\left\|\int_{t_i^*}^{t_{i+1}^*}(t_{i+1}-s)D\pi(x)L\Lambda_u B\,dW_s\right\|_{\mathbb{R}^m}^2\right] &\le h\,\mathbb{E}_x\left[\int_{t_i^*}^{t_{i+1}^*}\|D\pi(x)L\Lambda_u B\|_{H^m}^2\,ds\right]\\ &\le h^2\|D\pi(x)L\|^2\|B\|^2 e^{-2\omega t_i^*}.\end{aligned}$$

Therefore for small $h>0$ and any $i\in\mathbb{N}$,

$$D\pi(X_{t_i^*})\int_{t_i^*}^{t_{i+1}^*}\Lambda_{t_{i+1}^*-s}B\,dW_s = D\pi(X_{t_i^*})B[W_{t_{i+1}^*}-W_{t_i^*}]+o\left(h^2\right).$$

Observing that $K=\lfloor t/h\rfloor$, we have the following in probability

$$\begin{aligned}III &= \sum_{i=0}^K D\pi(X_{t_i^*})\left[\int_{t_i^*}^{t_{i+1}^*}\Lambda_{t_{i+1}^*-s}B\,dW_s\right]\\ &= \sum_{i=0}^K D\pi(X_{t_i^*})B[W_{t_{i+1}^*}-W_{t_i^*}]+o\left(h^{1/2}\right)\xrightarrow[h\to 0]{}\int_0^{t^*}D\pi(X_s)B\,dW_s.\end{aligned}$$

We now show that $VI\to\sigma^2\int_0^{t^*}\sum_{k\in\mathbb{N}}D^2\pi(X_s)\left[Be_k,Be_k\right]ds$ as $h\to 0$. By (3.18),

$$\begin{aligned}&\int_{t_i^*}^{t_{i+1}^*}\sum_{k\in\mathbb{N}}D^2\pi(w_i)\left[\Lambda_{t_{i+1}^*-s}Be_k,\,\Lambda_{t_{i+1}^*-s}Be_k\right]ds\\ &\quad= \int_{t_i^*}^{t_{i+1}^*}\sum_{k\in\mathbb{N}}\Big(D^2\pi(w_i)\left[Be_k,Be_k\right]+2D^2\pi(w_i)\left[L\Lambda_u Be_k,Be_k\right](t_{i+1}^*-s)\\ &\qquad+D^2\pi(w_i)\left[L\Lambda_u Be_k,L\Lambda_u Be_k\right](t_{i+1}^*-s)^2\Big)ds\\ &\quad= (t_{i+1}^*-t_i^*)\sum_{k\in\mathbb{N}}D^2\pi(w_i)\left[Be_k,Be_k\right]+O(h^2).\end{aligned}\tag{3.19}$$

Itô's isometry and (3.19) then imply that

$$\begin{aligned}&\Bigg|\mathbb{E}\Bigg[D^2\pi(w_i)\left[\int_{t_i^*}^{t_{i+1}^*}\Lambda_{t_{i+1}^*-s}B\sum_{k\in\mathbb{N}}e_k\,d\beta_s^k,\ \int_{t_i^*}^{t_{i+1}^*}\Lambda_{t_{i+1}^*-s}B\sum_{\ell\in\mathbb{N}}e_\ell\,d\beta_s^\ell,\right]\\ &\qquad\qquad-\int_{t_i^*}^{t_{i+1}^*}\sum_{k\in\mathbb{N}}D^2\pi(w_i)\left[\Lambda_{t_{i+1}^*-s}Be_k,\,\Lambda_{t_{i+1}^*-s}Be_k\right]ds\Bigg]\Bigg|\\ &\le \|\pi\|_{C^2}\left|\sum_{k\in\mathbb{N}}\mathbb{E}\left[\left\|\int_{t_i^*}^{t_{i+1}^*}\Lambda_{t_{i+1}^*-s}Be_k\,d\beta_s^k\right\|^2-\int_{t_i^*}^{t_{i+1}^*}\left\|\Lambda_{t_{i+1}^*-s}Be_k\right\|^2ds\right]\right| = 0,\end{aligned}\tag{3.20}$$

where we use (3.19) to exchange the sum and expectation in (3.20). Taking (3.19) & (3.20) together, summing over $k\in\{1,\dots K\}$, and letting $h\to 0$, we have convergence in probability of VI. Finally, IV and V tend to zero as $h\to 0$ by (3.15) and (3.16). □

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