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ORIGINAL RESEARCH ARTICLE

Competitive Equilibrium Always Exists for Combinatorial Auctions with Graphical Pricing Schemes

Marie-Charlotte Brandenburg[1] · **Christian Haase**[2] · **Ngoc Mai Tran**[3]

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Abstract

We show that a competitive equilibrium always exists in combinatorial auctions with anonymous graphical valuations and pricing, using discrete geometry. This is an intuitive and easy-to-construct class of valuations that can model both complementarity and substitutes, and to our knowledge, it is the first class besides gross substitutes that have guaranteed competitive equilibrium. We prove through counter-examples that our result is tight, and we give explicit algorithms for constructing competitive pricing vectors. We also give extensions to multi-unit combinatorial auctions (also known as product-mix auctions). Combined with theorems on graphical valuations and pricing equilibrium of Candogan, Ozdagar and Parrilo, our results indicate that quadratic pricing is a highly practical method to run combinatorial auctions.

Keywords Competitive equilibrium · Graphical pricing · Lattice polytopes · Combinatorial auctions · Regular subdivisions · Correlation polytope

Mathematics Subject Classification 91B26 · 14T90 · 52B20

✉ Marie-Charlotte Brandenburg
marie.brandenburg@mis.mpg.de

Christian Haase
haase@math.fu-berlin.de

Ngoc Mai Tran
ntran@math.utexas.edu

[1] Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, 04103 Leipzig, Germany

[2] Freie Universität Berlin, Arnimallee 3, 14195 Berlin, Germany

[3] The University of Texas at Austin, Speedway 2515, Austin, TX 78712, USA

1 Introduction

In a *multi-unit combinatorial auction*, of which product-mix auction is a special case [1, 2], multiple agents can make simultaneous bids on multiple subsets of indivisible goods of distinct types. The running example in this paper is the Cutlery Auction of three items, a fork, a knife and a spoon, among three agents, Fruit, Spaghetti and Steak (cf. Examples 1, 2 and 5). Each agent is willing to pay at most 1 dollar for their favourite combination and no other: for Fruit, it is (knife, spoon), for Spaghetti, it is (fork, spoon) and for Steak, it is (fork, knife).

In the auctioneer's disposal, there is the ability to *design* the auction. While this is a vast topic [3, 4], for this paper, we follow the setup of algorithmic auction design [5, 6], namely that the auctioneer can restrict the class of valuations that the agents can submit as well as the class of price functions that can be announced. In mathematical terms, an auction is a tuple of functions: there are valuations of the bidders, a pricing function and an allocation function. The problem faced by the auctioneer is the following: Given a bundle of items (i.e., a set of types of items and specified quantities for each type) and an equilibrium concept E, find a valuation class $\mathcal{V}$ and a pricing class $\mathcal{P}$ such that for any auction with valuations in $\mathcal{V}$, E is guaranteed to exist for some allocation μ and some pricing function $p \in \mathcal{P}$.

An ideal market would allocate the resources efficiently (maximize welfare) and set prices to achieve a competitive equilibrium where no participant wants to deviate. Existence of such equilibria is a function of the class of valuations $\mathcal{V}$, the pricing rules $\mathcal{P}$ and the definition of equilibrium.

In this paper, we discuss two commonly studied notions of equilibrium: competitive equilibrium (CE) and pricing equilibrium (PE). The formal definitions are given in Sect. 2.1. The difference lies in the involvement of the seller. For CE, the seller simply wants all other participants to be happy with their allocations at the announced price. Such allocations are called competitive allocations. For PE, the seller plays the same role as all the other agents, namely, they personally want to maximize their profit without caring for others. In this case, PE exists if the allocation and prices satisfy both the sellers' and all the other agents' agenda.

The simplest pricing scheme is linear pricing, where each type of item is assigned a price p_i, and the price of a set S of items is $\sum_{i \in S} p_i$. For linear pricing, both notions coincide and are known as *the* Walrasian equilibrium. This is a classical concept in economics, see [7] for a recent survey. In many combinatorial auctions, however, a Walrasian equilibrium may not exist. The Unimodularity Theorem [2, 8] gives a guarantee of Walrasian equilibrium for auctions whose valuation class $\mathcal{V}$ have unimodular edges. We state the Unimodularity Theorem in the language of regular subdivisions as pioneered by [2].

Theorem 1 (The Unimodularity Theorem) *Walrasian equilibrium (that is, competitive equilibrium under anonymous linear pricing) is guaranteed at* all *eligible bundles for* all *auctions with valuations in $\mathcal{V}$ if and only if the set of primitive edges in the regular subdivision of each valuation $v \in \mathcal{V}$ is a unimodular set.*

When demand edges are unimodular, the Unimodularity Theorem guarantees a Walrasian equilibrium in the strongest possible sense. However, the unimodularity of the demand edges is a strong condition that exclude many intuitive valuation profiles. For example, in combinatorial auctions with two items, the theorem would guarantee Walrasian equilibrium if and only if agents' valuations are either all submodular, or all supermodular. The Cutlery auction does not have unimodular demand edges. The most well-studied large class of valuations with unimodular edges are gross substitutes (GS). Unfortunately, GS valuations have several well-documented undesirable properties that hinder their applications [5, 6, 9]. First, the GS condition imposes exponentially many linear inequalities on the valuation v. It takes $O(2^m - \text{poly}(m))$ operations to check if an arbitrary function v is gross substitutes. Second, the GS valuations cannot be supermodular in *any pair* of items.

How can one go beyond the Unimodularity Theorem? One option is to allow for anonymous (each agent is offered the same price for the same package of items), but *non-linear* pricing. The hope is that by having a richer pricing profile $\mathcal{P}$, one can achieve equilibrium with a simpler, more practical valuation profile $\mathcal{V}$. Our main theorem (cf. Theorem 2) precisely establishes such a result for the class of *graphical valuations* with *anonymous graphical pricing* under the concept of *competitive equilibrium*.

Definition 1 Let m be the number of distinct types of items. A valuation is an assignment $\text{val} : 2^{[m]} \to \mathbb{R}$. It is called *graphical* if there exists a simple, undirected graph G with vertices $V(G) = [m]$ and edges $E(G)$, and a vector $w \in \mathbb{R}^{V(G)\sqcup E(G)}$ such that

$$\text{val}(S) = \sum_{i \in S} w_i + \sum_{\substack{ij \in E(G)\\ i,j \in S}} w_{ij}.$$

for every $S \subseteq [m]$, where S represents a set of items of distinct types. Here, $V(G) \sqcup E(G)$ denotes the disjoint union of the sets $V(G)$ and $E(G)$. The graph G is called the *underlying value graph* of the valuation. Similarly, a pricing function is called graphical if there exists a pricing vector p such that the price of a set S of items of distinct types is given by $\sum_{i \in S} p_i + \sum_{\substack{ij \in E(G)\\ i,j \in S}} p_{ij}$.

A graphical valuation val can be interpreted as follows. If $w_{ij} > 0$ for some edge $ij \in E(G)$, then the agent views items of types i and j as *complementary*, that is, they are rather interested in buying both items together than only a single one of them. The higher the value of w_{ij}, the higher is the agent's preference of buying both i and j at the same time. Similarly, if $w_{ij} < 0$, then items of types i and j are viewed as *substitutable*, that is, the agent asks for a discount when they have items i and j together. An analogous interpretation holds for a graphical pricing function. In this case, when the pair of items i and j is bought together, they require a premium of p_{ij} if $p_{ij} > 0$. If $p_{ij} < 0$, then this is a discount.

Theorem 2 *If $\mathcal{V}$ is the set of graphical valuations with the complete graph K_m as the underlying graph, then a competitive equilibrium with anonymous graphical pricing is guaranteed to exist for* any *quantities $a^* \in \{0, 1\}^m$ for* any *auction with valuations in $\mathcal{V}$.*

Our theorem is tight, in that one cannot replace CE in the statement with PE, or replace graphical pricing with linear pricing. In particular, we stress that because our pricing function is non-linear, our result is *not* a statement about Walrasian equilibria and is, thus, not related to the setup of [10–13]. The Cutlery Auction is an instance of an auction with graphical valuations on the complete graph that does *not* have a Walrasian equilibrium nor a pricing equilibrium with graphical pricing, but it does have a CE with graphical pricing (cf. Example 2). In other words, for graphical valuations on the complete graph, graphical pricing is *truly necessary* to achieve competitive equilibrium, while a pricing equilibrium might still be impossible.

We present two generalizations of this statement in Sect. 3. In Theorem 8, we allow the agents' graphs to be complete subgraphs K_S where $S \subsetneq [m]$. Theorem 7 allows the auctioneer to sell more than one item per type.

Our proof is constructive, and not only so, we give two *different* constructions of pricing functions that support the competitive equilibrium. This yields a practical algorithm for computing a quadratic CE price (cf. Algorithm 1). Theorem 2 and the algorithm, thus, make a strong case for implementing graphical valuations for combinatorial auctions.

We stress that Theorem 2 is not obtained directly from the Unimodularity Theorem. There are three different proofs of the Unimodularity Theorem, whose key ideas are: the commutativity of Minkowski sum and convex hull operation for generalized permutohedra [8], the unimodularity of demand edges [2] and the total unimodularity of a certain integer program [1]. In contrast, our proof relies on the geometry of the *correlation polytope*, which does not have a unimodular edge set and does not enjoy the integer decomposition property of generalized permutohedra.

Paper Organization Our paper is organized as follows. In Sects. 2.1 and 2.2 we collect necessary definitions from multi-unit combinatorial auctions and discrete convex geometry. We formulate the problems in Sect. 2.3 using the language of regular subdivisions and integer polytopes. The main results are presented and discussed in Sect. 3 and proved in Sect. 4.

Notation Throughout this paper, except for examples, we fix an integer m and denote $[m] = \{1, \ldots, m\}$. Let G be a graph on $[m]$ vertices. For a subgraph $H \subseteq G$, let $V(H) \subseteq [m]$ denote its vertex set and $E(H) \subseteq \binom{[m]}{2}$ its edge set. Let $d := m + |E(G)|$. The characteristic vector of H is $\chi_H \in \{0, 1\}^d$, where the first m coordinates $((\chi_H)_i, i \in [m])$ are indicators for the vertices of H, while the next $|E(G)|$ coordinates $((\chi_H)_{ij}, ij \in E(G))$ are indicators for the edges of H. In general, a vector $a \in \mathbb{R}^d$ inherits the same indexing system, that is, the first m coordinates are indexed by $i \in [m]$, while the next $|E(G)|$ coordinates are indexed by $ij \in E(G)$.

2 Discrete Convex Geometry and Economic Equilibria

The product-mix auction with linear pricing forms a fundamental connection between discrete convex geometry and economic equilibria. We recommend [2] as an introduction for economists and [1] for geometers. In this section, we extend this bridge to the case of combinatorial auctions with non-linear pricings.

2.1 The Economic Setup

Consider an auction with m types of indivisible goods on sale to n agents, where $a_i^* \in \mathbb{N}$ is the number of goods of type $i \in [m]$. Throughout this paper, we assume that each agent wants to buy at most one item of each type, even though the auctioneer might sell more than one item of a this type to the group of agents. The vector $a^* \in \mathbb{N}^m$ can be thought of as a vector which keeps track of the quantities which are for sale in the auction. We note that, in contrast to notational conventions in optimization, where a^* would typically denote an optimal solution, in our context, it is merely the number of goods of each type which are given as input of the auction.

A graphical pricing function $p : 2^{[m]} \to \mathbb{R}$ specifies the price to be paid for each bundle. Each agent $b \in [n]$ has a graphical valuation function $\text{val}^b : 2^{[m]} \to \mathbb{R}$, where $\text{val}^b(S) \in \mathbb{R}$ measures how much the agent values the bundle S.

Following [6], we model the relations between the goods via a graph G on m vertices. Each vertex represents a type of goods, while edges between vertices model the existence of a relationship between types, such as complementarity (supermodularity) and substitutability (submodularity). Given a subset of items $S \subseteq [m]$, we construct a vector $a_S \in \{0, 1\}^{[m]\sqcup E(G)}$, where the first m coordinates are indexed by the vertices of G and the following coordinates are indexed by the edges of G. Explicitly, we define a_S by

$$a_{S,i} = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}, \qquad a_{S,ij} = \begin{cases} 1 & \text{if } i, j \in S \text{ and } ij \in E(G) \\ 0 & \text{otherwise} \end{cases}.$$

Thus, any such vector a_S can be seen as an extended incidence vector of the induced subgraph of G with vertex set S.

We can then rewrite the graphical valuation from Definition 1 as $\text{val} : \{a_S \mid S \subseteq [m]\} \to \mathbb{R}$, $\text{val}(a) = \langle w, a \rangle$ and a graphical price function as $\langle p, a \rangle$.

Definition 2 The *polytope of the graph G* is

$$P(G) = \text{conv}\left(\left\{a_S \in \{0, 1\}^{[m]\sqcup E(G)} \,\middle|\, S \subseteq [m]\right\}\right),$$

i.e. the convex hull of characteristic vectors (or incidence vectors) of induced subgraphs of G. Let $\pi = \pi_G$ denote the coordinate projection

$$\pi : nP(G) \cap \mathbb{Z}^{[m]\sqcup E(G)} \to \mathbb{Z}^{[m]}$$

that forgets the coordinates which correspond to edges in G. Here, n denotes the number of agents in the auction.

Let $a^* \in \mathbb{N}^m$ be the vector of quantities for sale, and $a_{S_1}, \ldots, a_{S_n} \in \{0, 1\}^{[m]\sqcup E(G)}$ such that $S_1, \ldots, S_n \subseteq [m]$ and $a_{S_1} + \cdots + a_{S_n} = a \in \pi^{-1}(a^*)$. The vector a can be thought of as packaging of the items in a^*, consisting of packages a_{S_b}, which is offered for sale to agent $b \in [n]$ during the auction. The coordinate a_{ij} indicates the number of packages which contain the pair of items $i, j \in [m]$ together. The set $\pi^{-1}(a^*)$ can, thus, be thought of as the set of possible packagings of the set of items in the auction, whose quantities are indicated by a^*.

The auction works as follows. First, the agents submit their valuations to the auctioneer, then the auctioneer announces the price p, and an allocation $a^1, \ldots, a^n \in \{0,1\}^{[m]\sqcup E(G)}$, where agent b is assigned the package a^b and $a = \sum_{b=1}^n a^b \in \pi^{-1}(a^*)$. If all of the agents and the seller agree, then agent b gets the package a^b and pays $\langle p, a^b \rangle$ to the seller. If one of them does not agree, then in theory, 'equilibrium failed' (and perhaps the auctioneer should be fired). Note that a choice of $a \in \pi^{-1}(a^*)$ amounts to choosing values a_{ij} for each pair ij of types of goods. As mentioned above, this specifies the number of times that an item of type i and an item of type j are sold together as a pair to an agent.

The bare-minimum goal for the auctioneer is to compute an allocation-pricing pair such that all of the market participants can agree to, that is, a non-trivial economic equilibrium is reached. In other words, the auctioneer proposes an allocation of packages and a price under which the packages are offered to the individual bidders. It is assumed that agent b is satisfied with the allocation-price pair $((a^1, \ldots, a^n), p)$ when they are allocated a bundle that maximizes their own utility at this price. At a given price p, the set of such bundles for agent b is their *demand set*

$$D(\mathrm{val}^b, p) = \arg\max_{S\subseteq[m]} \left\{ \mathrm{val}^b(a_S) - \langle p, a_S \rangle \right\},$$

where $a_S \in \{0,1\}^{[m]\sqcup E(G)}$ is the vector corresponding to the set S. Thus, agent b is satisfied with the allocation-price pair when $a^b \in D(\mathrm{val}^b, p)$. When all agents are satisfied, we have a *competitive equilibrium.*

Definition 3 (Competitive equilibrium) Let $(\mathrm{val}^b : b \in [n])$ be graphical valuations with a fixed value graph G. We say that the corresponding auction has a *competitive equilibrium allocating packages* $(a^1, \ldots, a^n)$ with price $p \in \mathbb{R}^{[m]\sqcup E(G)}$ if $a^b \in D(\mathrm{val}^b, p)$ for all $b \in [n]$. The auction has a *competitive equilibrium* allocating the packaging $a \in \mathbb{Z}^{[m]\sqcup E(G)}$ with price $p \in \mathbb{R}^{[m]\sqcup E(G)}$ if there exists an allocation $(a^1, \ldots, a^n)$ of packages such that $a = \sum a^b$ and $a^b \in D(\mathrm{val}^b, p)$ for all $b \in [n]$.

We say that the auction has a *competitive equilibrium (CE) allocating quantities* $a^* \in \mathbb{N}^m$ *of goods* if there exists some packaging $a \in \pi^{-1}(a^*)$ and some price $p \in \mathbb{R}^{[m]\sqcup E(G)}$ at which the auction has a competitive equilibrium.

Example 1 (Cutlery Auction) We consider the Cutlery Auction due to [14, Example 3.2]. Let $G = K_3$ be the complete graph on three vertices $V(K_3) = \{A, B, C\}$, let $n = 3$ be the number of agents, and $a^* = (1,1,1)^t$. With indexing $w = (w_A, w_B, w_B, w_{AB}, w_{AC}, w_{BC})^t$, the agents' valuations are given by the weight vectors

$$w^1 = (0,0,0,1,0,0)^t, \quad w^2 = (0,0,0,0,1,0)^t, \quad w^3 = (0,0,0,0,0,1)^t,$$

i.e. the first agent has weight one for edge AB, the second agent has weight one for edge AC, the third agent has weight one for edge BC and all remaining weights are zero. If we pick the graphical price vector $p = (0, 0, 0, 1, 1, 1)^t$, then for each $b \in \{1, 2, 3\}$, we have

$$D(\mathrm{val}^b, p) \supseteq \left\{ \begin{pmatrix} 0\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0\\0\\0 \end{pmatrix} \right\}.$$

Thus, we can decompose $a = (1, 1, 1, 0, 0, 0)^t \in \pi^{-1}(a^*)$ by assigning one item to each agent, e.g.

$$a^1 = (1, 0, 0, 0, 0, 0)^t, \quad a^2 = (0, 1, 0, 0, 0, 0)^t, \quad a^3 = (0, 0, 1, 0, 0, 0)^t,$$

(where each agent is charged the price 0) in order to achieve a competitive equilibrium allocating the packages (a^1, a^2, a^3) with price p in the sense of Definition 3. Hence, the auction has a competitive equilibrium at $a = a^1 + a^2 + a^3 = (1, 1, 1, 0, 0, 0)^t$. This implies that the auction has a competitive equilibrium allocating quantities $a^* = (1, 1, 1)^t$ of goods in the sense of Definition 3. Furthermore, for a fixed price the allocation at which a competitive equilibrium is achieved is not unique. Considering instead the price $p = (0, 1, 1, 0, 1, -1)$, we note that the auction has a competitive equilibrium allocating the packages

$$a_2^1 = (1, 0, 0, 0, 0, 0)^t, \quad a_2^2 = (0, 0, 0, 0, 0, 0)^t, \quad a_2^3 = (0, 1, 1, 0, 0, 1)^t,$$

with price p. At the same time, allocating the packages

$$a_3^1 = (1, 1, 1, 1, 1, 1)^t, \quad a_3^2 = (0, 0, 0, 0, 0, 0)^t, \quad a_3^3 = (0, 0, 0, 0, 0, 0)^t,$$

also constitutes a competitive equilibrium at the same price p. We continue with this in Example 2.

Remark 1 In Theorem 6, we show that if there is at most one quantity per item for sale in the auction (such as in Examples 1), then for any packaging $a \in \pi^{-1}(a^*)$ of the quantities $a^* \in \{0, 1\}^m$, there exists a price p such that the auction achieves a competitive equilibrium allocating this packaging with price p. Theorem 7 applies to general quantities $a^* \in \mathbb{Z}^m_{\geq 0}$, and we construct one explicit packaging $a \in \pi^{-1}(a^*)$ for which a competitive equilibrium exists under some price p. The packaging $a_3 = a_3^1 + a_3^2 + a_3^3$ in Examples 1 is the kind of packaging that we construct in the proof of Theorem 7, while both the packaging $a_2 = a_2^1 + a_2^2 + a_2^3$ and a_3 fit into the framework of Theorem 6. A discussion about the possible packagings which fit into the respective frameworks of these results can be found in Remark 2.

In this setup, we only consider allocations that are *complete*, that is, the seller must sell all items in the bundle. Note that the seller's revenue only depends on the sum of the bundles in the allocation, as $\sum_{b=1}^n \langle p, a^b \rangle = \langle p, \sum_{b=1}^n a^b \rangle = \langle p, a \rangle$. In other words, the seller's revenue only depends on the entire packaging and is independent of the allocation of the packages. Informally, a competitive equilibrium says that the agents are always satisfied. A stronger concept is pricing equilibrium, where the seller is another market participant who also needs to be satisfied. A revenue-maximization seller wants to maximize revenue at a given price p, that is,

$$D(\text{seller}, p) = \arg\max_{a\in\pi^{-1}(a^*)} \{\langle p, a \rangle\}.$$

This leads to the definition of a pricing equilibrium [6, Def. 2.3].

Definition 4 (Pricing equilibrium) Let $(\text{val}^b : b \in [n])$ be graphical valuations. We say that the corresponding auction has a *pricing equilibrium (PE) allocating packages* $(a^1, \dots, a^n)$ with price $p \in \mathbb{R}^{[m]\sqcup E(G)}$ if $a^b \in D(\text{val}^b, p)$ for all $b \in [n]$ and in addition $a = \sum_{b=1}^n a^b \in D(\text{seller}, p)$.

In between, these two notions lies the *optimal competitive equilibrium*: an allocation-price pair such that the seller's revenue is maximized among all $a \in \pi^{-1}(a^*)$ at which a competitive equilibrium exists. The Cutlery Auction is a simple auction that illustrates the difference between these concepts.

Example 2 (An auction with CE but no PE for graphical pricing) Consider the auction from Examples 1. Note that the seller's revenue at the price $p = (0, 0, 0, 1, 1, 1)$ is $\langle p, a^1 + a^2 + a^3\rangle = 0$. If we instead decide to sell items A and B to agent 1, item C to agent 2, and nothing to agent 3, then this constitutes a competitive equilibrium in which the sellers revenue is $\langle p, (1, 1, 0, 1, 0, 0)^t + (0, 0, 1, 0, 0, 0)^t + (0, 0, 0, 0, 0, 0)^t\rangle = 1$. In fact, this is the maximum revenue that the seller can achieve under all allocations that constitute a competitive equilibrium and, thus, induces an optimal CE. This shows that the competitive equilibrium given in Examples 1 is not an optimal competitive equilibrium. On the other hand, the optimal competitive equilibrium from above also constitutes a competitive equilibrium in the sense of Definition 3. However, this does not constitute a pricing equilibrium, since assigning all items to one agent would increase the seller's profit to 3 (but none of the agents would be happy with this outcome). It was shown in [6, Example 3.13], that neither a pricing equilibrium nor a Walrasian equilibrium for this example exists. We will continue with this in Examples 5.

2.2 Convex Geometry Setup

We now describe a strong connection between CE and discrete convex geometry for graphical valuations and pricing. For a more detailed description of regular subdivisions, mixed subdivisions and the Cayley trick, we refer the reader to [15, Sect. 9.2].

Recall that the *polytope of the graph G* is

$$P(G) = \text{conv}\left(\left\{a_S \in \mathbb{R}^{[m]\sqcup E(G)} \;\middle|\; S \subseteq [m]\right\}\right),$$

where a_S is the characteristic vector of the set S as described in Sect. 2.1. Let $d = |[m]\sqcup E(G)|$ denote the dimension of $P(G)$. By construction, $P(G)$ is a 0/1–polytope, and

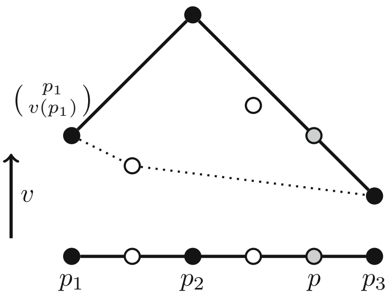


Fig. 1 The black points p_1, p_2 and p_3 are vertices of the regular subdivision of the line segment, induced by the lifting function v. The white points are not lifted points. The point p is a lifted point, but not a vertex of the regular subdivision: p lies in the interior of the face with vertices p_2 and p_3

therefore,

$$P(G) \cap \mathbb{Z}^d = \mathrm{vert}(P(G)) = \left\{ a_S \in \mathbb{R}^d \;\middle|\; S \subseteq [m] \right\},$$

where $\mathrm{vert}(P(G))$ denotes the set of vertices of $P(G)$. It is worth noting that $P(G) \subseteq \mathbb{R}^d$ is indeed of dimension $d = |[m] \sqcup E(G)|$. This can be seen from the fact, that the subset of vertices $\{a_S \mid S \subseteq [m], |S| \le 2\}$ form a set of size $d+1$ of affinely independent vectors.

Let val be a (not necessarily graphical) valuation on a polytope P, i.e. a function $\mathrm{val} : P \cap \mathbb{Z}^d \to \mathbb{R}$. For a face F of P we denote $\mathrm{lift}_{\mathrm{val}}(F) = \left\{ \binom{a}{\mathrm{val}(a)} \mid a \in F \cap \mathbb{Z}^d \right\} \subseteq \mathbb{R}^{d+1}$. We should think of val as giving a height function to each lattice point in $P \cap \mathbb{Z}^d$. If we take the convex hull of these points in $\mathbb{R}^{d+1}$, the collection of the facets visible when we look down at the convex hull from above is called the *upper convex hull*. This convex hull is a polyhedral complex in dimension $\mathbb{R}^{d+1}$. Its projection onto the first d coordinates is called the *regular subdivision of P induced by* val.

If val is a graphical valuation as defined in Definition 1, then the induced regular subdivision on $P(G)$ is trivial, since val can be seen as a linear functional on the polytope. However, the regular subdivision of a single valuation val only captures information about the valuation of a single agent. As soon as the number n of agents is bigger than 1 and not all agents have identical valuations, then competitive equilibrium concerns a regular subdivision of the dilated polytope $nP(G)$, which is guaranteed to be non-trivial. We now explain this in more details.

Let $\widetilde{\mathrm{val}}$ be the smallest concave function such that $\widetilde{\mathrm{val}}(a) \ge \mathrm{val}(a)$ for all $a \in P \cap \mathbb{Z}^d$. We call $a \in \mathbb{Z}^d$ a *lifted point* if $\widetilde{\mathrm{val}}(a) = \mathrm{val}(a)$, i.e. if $\binom{a}{\mathrm{val}(a)}$ lies in the upper convex hull of $\mathrm{conv}(\mathrm{lift}_{\mathrm{val}}(P))$, as depicted in Fig. 1. Note that, in particular, a vertex of a face of the regular subdivision is always a lifted point. Further,

$$\begin{aligned} D(\mathrm{val}, p) &= \arg\max_{a \in P(G) \cap \mathbb{Z}^d} \{\mathrm{val}(a) - \langle a, p \rangle\} \\ &= \arg\max_{a \in P(G) \cap \mathbb{Z}^d} \left\{ \langle \binom{a}{\mathrm{val}(a)}, \binom{-p}{1} \rangle \right\}. \end{aligned}$$

The set $D(\mathrm{val}, p)$ thus consists of all lifted points of the face of the upper convex hull of $\mathrm{lift}_{\mathrm{val}}(P(G))$ with normal vector $\binom{-p}{1}$. Since a vertex of such a face is always a lifted point, the set of faces of the regular subdivision is given by $\left\{ \mathrm{conv}(D(\mathrm{val}, p)) \;\middle|\; p \in \mathbb{R}^d \right\}$.

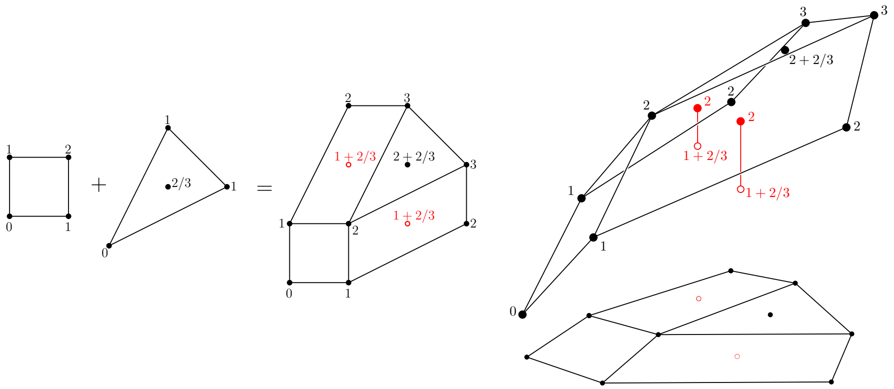


Fig. 2 The induced mixed regular subdivision on $P_1 + P_2$ from Examples 3

In the case of graphical valuations, this allows us to interpret the condition $a^b \in D(\mathrm{val}^b, p)$, $b \in [n]$ for competitive equilibrium in the following way: Let $\mathrm{val}^1, \ldots, \mathrm{val}^n$ be graphical valuations and consider the *aggregate valuation function* $\mathrm{Val} : nP(G) \cap \mathbb{Z}^d \to \mathbb{R}$ given by

$$\begin{aligned} \mathrm{Val}(a) &= \max\left\{ \sum_{b=1}^{n} \mathrm{val}^b(a^b) \,\Big|\, \sum_{b=1}^{n} a^b = a, a^b \in P(G) \cap \mathbb{Z}^d \right\} \\ &= \max\left\{ \sum_{b=1}^{n} \langle w^b, a^b \rangle \,\Big|\, \sum_{b=1}^{n} a^b = a, a^b \in P(G) \cap \mathbb{Z}^d \right\}. \end{aligned} \tag{1}$$

This is a valuation function that cannot be interpreted as a linear functional on $nP(G)$, and hence, the regular subdivision on $nP(G)$ is non-trivial. Indeed, this subdivision is the *mixed regular subdivision* induced by $\mathrm{val}^1, \ldots, \mathrm{val}^n$ and a face F of this mixed regular subdivision is of the form $F = \sum_{b=1}^{n} F^b$, where F^b is a face of $P(G)$. In other words, for the demand sets of the aggregate valuation holds

$$D(\mathrm{Val}, p) = \sum_{b=1}^{n} D(\mathrm{val}^b, p),$$

where $D(\mathrm{Val}, p)$ is the set of lifted points in the face of the mixed regular subdivision of $nP(G)$ which is maximized by the normal vector $\binom{-p}{1}$ and $D(\mathrm{val}^b, p)$ the set of vertices of the face of $\mathrm{lift}_{\mathrm{val}^b}(P(G))$ that is maximized by $\binom{-p}{1}$. In particular, given a point $a \in D(\mathrm{Val}, p)$ we can always write the point as a sum $a = \sum_{b=1}^{n} a^b$, where $a^b \in D(\mathrm{val}^b, p)$.

Example 3 (Non-trivial mixed subdivisions where not all points are lifted)
We give an example of two polytopes P_1, P_2 with trivial regular subdivisions, where the induced mixed regular subdivision on $P_1 + P_2$ is non-trivial. Let $P_1 = [0, 1]^2$ be the square with lifting function $v_1(a) = \langle a, \binom{1}{1} \rangle$ defined for $a \in P_1 \cap \mathbb{Z}^2$ and

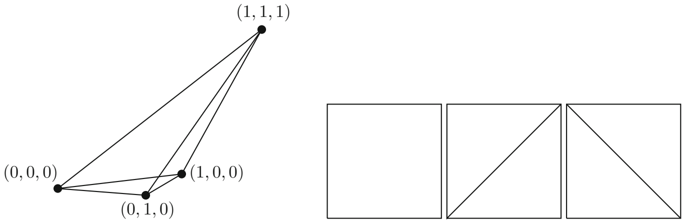


Fig. 3 The polytope $P(K_2)$ (left) and three possible subdivisions of $[0,1]^2$ (right)

$P_2 = \mathrm{conv}(\binom{0}{0}, \binom{2}{1}, \binom{1}{2})$ with lifting function $v_2(a) = \frac{1}{3}\langle a, \binom{1}{1} \rangle$ for $a \in P_2 \cap \mathbb{Z}^2$. The induced mixed regular subdivision on $P_1 + P_2$ is given by the lifting function

$$V(a) = \max\left\{ v^1(a^1) + v^2(a^2) \,\middle|\, a^1 + a^2 = a, a^1 \in P_1 \cap \mathbb{Z}^2, a^2 \in P_2 \cap \mathbb{Z}^2 \right\}.$$

The values of $V(a)$ on the integer points of $P_1 + P_2$ and the induced mixed regular subdivision can be seen in Fig. 2. Note that the points $\binom{2}{1}$ and $\binom{1}{2}$ are not lifted: indeed, $V(\binom{2}{1}) = 1 + \frac{2}{3}$, while $\widetilde{V}(\binom{2}{1}) = 2$.

Example 4 (Geometry of graphical pricing versus linear pricing) In an auction with two types of items, any valuation $v : \{0,1\}^2 \to \mathbb{R}$ is linearly equivalent to a graphical valuation on K_2. Indeed, by the linear transformation $v \longmapsto v - v((0,0))$, we can assume that $v((0,0)) = 0$. Then, we can set $w_1 = v((1,0))$, $w_2 = v((0,1))$ and $w_{12} = v((1,1)) - (w_1 + w_2)$, so this valuation v is indeed a graphical valuation on K_2. In this case, the correlation polytope $P(K_2)$ is a simplex in $\mathbb{R}^3$ (cf. Fig. 3).

Consider an auction with two types of items and two agents. When one works with linear pricing, competitive equilibrium concerns the regular subdivision Δ_v of the square $[0,1]^2$. There are three possible subdivisions of $[0,1]^2$ induced by v (cf. Fig. 3). In economics, valuations that induce these subdivisions are called linear, complements and substitutes, respectively. Thus, if v^1 is substitutes and v^2 is complements, then the set of edges of Δ_{v^1} and Δ_{v^2} together do not form a unimodular set: in particular, the vectors $(1,-1)$ and $(1,1)$ are not unimodular, as the determinant of the corresponding 2×2 matrix is bigger than 1. Therefore, by the Unimodularity Theorem [1, 2, 8], competitive equilibrium *can* fail for this auction.

In contrast, when we work with graphical pricing, competitive equilibrium concerns the regular subdivision Δ_v of the correlation polytope $P(K_2)$. In this case, Theorem 2 implies that competitive equilibrium *always exists* for graphical pricing, regardless of whether the agents' valuations are complements or substitutes.

Lemma 3 *Let $w \in \mathbb{R}^d$. The mixed subdivision induced by* $\mathrm{Val}(a)$ *is equal to the subdivision induced by*

$$\mathrm{Val}_w(a) = \max\left\{ \sum_{b=1}^{n} \langle w^b + w, a^b \rangle \,\middle|\, \sum_{b=1}^{n} a^b = a, a^b \in \mathrm{vert}(P(G)) \right\}.$$

That is, adding a constant vector w to all weight vectors w^b does not change the mixed subdivision of $nP(G)$.

The proof of this lemma can be found in Sect. 4. This lemma implies that, without loss of generality, we can assume that the weights w^b are nonnegative. Note that despite the fact that this does not change the existence of a competitive equilibrium, it does affect the prices at which a competitive equilibrium can be achieved.

Example 5 If we add the vector $(1, 1, 1, 1, 1, 1)^t$ to all weights in Example 2, then a pricing equilibrium can be achieved at the price $p = (3, 3, 1, 0, 0, 0)^t$ by selling all items to the first agent, in which case the seller's revenue is 7. In accordance to [6, Th. 3.7] this price vector even constitutes a Walrasian equilibrium, as the underlying graph $G = K_3$ is series-parallel.

2.3 Competitive Equilibrium and Convex Geometry

We now establish the connection between the existence of a competitive equilibrium and properties of the polytope $P(G)$. A key result is Corollary 5, which gives a necessary and sufficient condition for a CE allocating given quantities a^* to be guaranteed to exist when the agents' valuations and the pricing rules are both graphical. Compared to the linear pricing case, each prescribed vector of quantities a^* allows for a *set* $\pi^{-1}(a^*)$ of packagings and a CE exists *if and only if one* of these $a \in \pi^{-1}(a^*)$ is lifted in a certain regular mixed subdivision.

Proposition 4 (CE at an allocation for graphical valuations and pricings) *Consider an auction with n agents, anonymous graphical pricing and graphical valuations with a fixed underlying value graph G on m vertices and $d = m+|E(G|$. Let $a \in nP(G) \cap \mathbb{Z}^d$ be a fixed packaging. Then the following are equivalent:*

- *(i) For each set of valuations $\{\mathrm{val}^b \mid b \in [n]\}$ there exists an allocation of packages $(a^1, \ldots, a^n)$ forming the packaging $a = \sum_{b=1}^n a^b$ and there exists a price $p \in \mathbb{R}^d$ constituting a competitive equilibrium allocating packages $(a^1, \ldots, a^n)$ with price p.*
- *(ii) For each set of valuations $\{\mathrm{val}^b \mid b \in [n]\}$ there exists a price $p \in \mathbb{R}^d$ such that $a \in \sum_{b=1}^n D(\mathrm{val}^b, p)$.*
- *(iii) For any faces $F^1, \ldots, F^n$ of $P(G)$ such that $a \in \sum_{b=1}^n F^b$ holds: $a \in \sum_{b=1}^n \mathrm{vert}(F^b)$.*

The proof of this proposition can be found in Sect. 4. Informally, if a competitive equilibrium exists for any set of valuations, then this means that a CE is guaranteed to exist: Regardless of the agents' preferences, the auctioneer can always guarantee to make everyone happy with the outcome. Applying the definition of a CE allocating quantities a^* of goods immediately implies the following:

Corollary 5 (CE at a bundle for graphical valuations and pricings) *Consider an auction with n agents, anonymous graphical pricing and graphical valuations with a fixed underlying value graph G on m vertices and $d = m + |E(G)|$. Let $a^* \in \mathbb{N}^m$ be fixed quantities of goods. Then the following are equivalent:*

(i) There exists a packaging $a \in \pi^{-1}(a^*)$ such that for each set of valuations $\{\mathrm{val}^b \mid b \in [n]\}$ there exists an allocation of packages $(a^1, \ldots, a^n)$ forming the packaging $a = \sum_{b=1}^n a^b$ and there exists a price $p \in \mathbb{R}^d$ constituting a competitive equilibrium allocating packages $(a^1, \ldots, a^n)$ with price p.
(ii) There exists an $a \in \pi^{-1}(a^*)$ such that for any set of valuations $\{\mathrm{val}^b \mid b \in [n]\}$ there exists a price $p \in \mathbb{R}^d$ such that $a \in \sum_{b=1}^n D(\mathrm{val}^b, p)$.
(iii) There exists an $a \in \pi^{-1}(a^*)$ such that for any faces $F^1, \ldots, F^n$ of P such that $a \in \sum_{b=1}^n F^b$ holds: $a \in \sum_{b=1}^n \mathrm{vert}(F^b)$.

In particular, if conditions (i)–(iii) hold, then for any set of valuations, a competitive equilibrium allocating quantities a^* of goods exists.

Example 6 (Some packagings cannot be allocated) We illustrate that condition (iii) in Corollary 5 is not necessarily satisfied for all $a \in \pi^{-1}(a^*)$. Consider an auction with 4 agents and quantities $a^* = (2, 2, 2, 2)$, i.e. an auction with 4 types of items A, B, C, D, where each item has two identical copies. The packaging a with

$$(a_A, a_B, a_C, a_D, a_{AB}, a_{AC}, a_{AD}, a_{BC}, a_{BD}, a_{CD}) = (2, 2, 2, 2, 1, 1, 1, 1, 1, 1) \quad (2)$$

is a lattice point in $4P(K_4)$ and thus we have $a \in \pi^{-1}(a^*)$. This packaging indicates that each pair of distinct types should be allocated together in the same package exactly once. However, by a simple counting argument it can be checked that this is impossible. This translates to a violation of Corollary 5 (iii) in the following way. We note that a is the sum of the midpoints of the four edges

$$F_1 = \mathrm{conv}\left((0,0,0,0,0,0,0,0,0,0), (0,0,0,1,0,0,0,0,0,0)\right),$$
$$F_2 = \mathrm{conv}\left((0,1,1,0,0,0,0,1,0,0), (1,0,1,0,0,1,0,0,0,0)\right),$$
$$F_3 = \mathrm{conv}\left((0,1,1,1,0,0,0,1,1,1), (1,0,1,1,0,1,1,0,0,1)\right),$$
$$F_4 = \mathrm{conv}\left((1,1,0,0,1,0,0,0,0,0), (1,1,0,1,1,0,1,0,1,0)\right),$$

of the polytope $P(K_4)$, i.e. these four edges are faces of $P(K_4)$ such that $a \in F_1 + F_2 + F_3 + F_4$. But one can check that (2) cannot be written as the sum of any 4 lattice points of $P(K_4)$. By Proposition 4, this means that for *any* auction with graphical valuations on the complete graph K_4, it is impossible to allocate the fixed packaging a. However, we will see in Theorem 7 that for each set of graphical valuations, there exists a different packaging a' of $(2, 2, 2, 2)$ such that competitive equilibrium is achieved for a' with graphical pricing. More specifically, the construction in Theorem 7 guarantees for a competitive equilibrium allocating the packaging $a' = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$, and Theorem 6 guarantees for a CE allocating any packaging $a' \in \{0, 2\}^{10}$.

Moreover, this example can be seen in a more general framework, as it is known that the poltytope $P(K_m)$ does not have the so-called integer decomposition property [16, p. 337] for $m \geq 4$. This implies that this example can be extended to a series of examples: For any $m \geq 4$ and $n \geq 4$, there exists a choice of quantities a^* and packaging $a \in \pi^{-1}(a^*)$ such that a cannot be allocated under a competitive equilibrium.

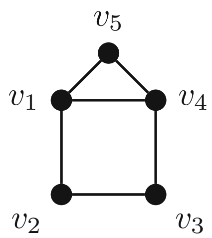


Fig. 4 A value graph at which CE may fail at $a^* = (1, 1, 1, 1, 1)$

3 Main Results

In this section we present and discuss our main results. The proofs follow in the next section.

3.1 Everyone Bids on Everything

Recall that m is the number of distinct items, n is the number of bidders and π the projection of a vector with components indexed by $[m] \sqcup E(G)$ onto its first $[m]$ components. In this section, we consider the complete graph $G = K_m$, and vectors are thus of length $d = m + \binom{m}{2}$.

We begin with auctions in which the seller's bundle contains either 0 or r items of each type for a fixed $r \in \mathbb{N}$. Recall that we assume that each agent is only interested in buying at most one item per type. An important special case is the combinatorial auction where $r = 1$. We show that a competitive equilibrium in this scenario can always be achieved:

Theorem 6 *Let $a^* \in \{0, r\}^m$ and $n \geq r$. If the underlying value graph of all agents is the complete graph, then $\pi^{-1}(a^*) \neq \emptyset$ and for any set of valuations and every $a \in \pi^{-1}(a^*)$ there exists a price $p \in \mathbb{R}^d$ at which a competitive equilibrium exists.*

The proof of this theorem gives an explicit construction of how to split the bundle in question. If $r = 1$, then $a \in \pi^{-1}(a^*)$ is the characteristic vector of a disjoint union of cliques and the procedure in the proof assigns cliques to agents. A choice of $a \in \pi^{-1}(a^*)$ corresponds to a choice of connected components. This construction gives a lot of freedom to the auctioneer: The auctioneer can decide which items are being sold together and is still guaranteed to achieve a competitive equilibrium. The next example shows that even for $r = 1$, the existence of a competitive equilibrium can fail when we do not consider the complete graph as value graph.

Example 7 Let G be the graph consisting of the cycle v_1, v_2, v_3, v_4 together with an additional vertex v_5 and edges v_1v_5, v_4v_5, as shown in Fig. 4. Consider the following 4 edges of the polytope $P(G)$

$$F^1 = \operatorname{conv}\left((0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0), (1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)\right)$$
$$F^2 = \operatorname{conv}\left((0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0), (0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0)\right)$$
$$F^3 = \operatorname{conv}\left((0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0), (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)\right)$$

$$F^4 = \text{conv}\left((0,0,0,0,1,0,0,0,0,0,0), (0,0,0,1,0,0,0,0,0,0,0)\right)$$

For $a^* = (1,1,1,1,1)$, we have $\pi^{-1}(a^*) \cap \sum_{b=1}^{4} F^b = \{(1,1,1,1,1,0,0,0,0,0)\}$ but $\pi^{-1}(a^*) \cap \sum_{b=1}^{4} \text{vert}(F^b) = \emptyset$. Thus, by Corollary 5, there exists a set of graphical valuations such that the packaging a cannot be allocated under a competitive equilibrium for any graphical price p. Hence, the assumption of $G = K_m$ in Theorem 6 is truly necessary.

Next, we loosen the assumption on a^* and allow arbitrary $a^* \in \mathbb{Z}^m \cap [0,n]^m$. We show that again a CE allocating quantities a^* always exists, however we only construct one explicit packaging $a \in \pi^{-1}(a^*)$ at which a competitive equilibrium is guaranteed to exist.

Theorem 7 *Let $a^* \in \{0,1,\dots,n\}^m$. If the underlying graph of all valuations is the complete graph, then there exists an $a \in \pi^{-1}(a^*)$ such that for any set of valuations there exists a price $p \in \mathbb{R}^d$ at which a competitive equilibrium exists.*

The procedure in which the bundle can be split up to achieve a competitive equilibrium is as follows:

If $a^* \in \{0,1\}^m$, then auctioneer sells the entire bundle to one agent, i.e. there is one agent who gets an item of each type $i \in [m]$ such that $a_i^* = 1$. If $a^* \in \{0,1,2\}^m$, then the auctioneer sells the items in two bundles: There is one agent who will be offered one item of each type where $a_i^* > 0$. A second agent will be made an offer for all remaining items, i.e. all items such that $a_i^* = 2$. And so on.

Remark 2 The above described procedure may seem odd for applications in practice. In particular, if $a^* = (1,\dots,1)$, then this procedure proposes to sell the entire set of items to a single agent, and nothing to the remaining agents. We thus emphasize, that Theorem 7 does not give any implications about the number or shape of possible packagings at which a competitive equilibrium exists. The main contribution of this article is to show that the set of packagings at which a competitive equilibrium is guaranteed to exist is non-empty. That is, the auctioneer can always choose at least one packaging under which CE is guaranteed to exist. In practice, the auctioneer might have additional criteria, and may optimize over this set of packagings with respect to their additional criteria (for example the optimal competitive equilibrium from Sect. 2.1). Furthermore, the auctioneer might consider certain packagings as impractical, i.e. would like to only consider certain subsets of the packagings for which a CE is guaranteed to exist. The existence of "practical competitive equilibria" depends on the definition of practical and will be the subject of further studies.

On the other hand, Theorem 6 does imply a lower bound on the number of packagings for which a CE is guaranteed to exist. Here, the result implies that *any* packaging $a \in \pi^{-1}(a^*)$ of a^* can be allocated under a competitive equilibrium, and so the auctioneer has at least $|\pi^{-1}(a^*)|$ many packagings to choose from. A special case of Theorem 6 is $a^* \in \{0,1\}^m$, i.e. for each type there is at most one item to sell in the auction, which is a setup that appears naturally in practice.

3.2 Everything Is Bid on by Someone

We can relax the condition on the valuations by allowing weight vectors $w \in \mathbb{R} \cup \{-\infty\}$. We assume that for every agent $b \in [n]$ the valuation function $\mathrm{val}^b : (\mathbb{R} \cup \{-\infty\})^d \to \mathbb{R} \cup \{-\infty\}$ is of the form $\mathrm{val}^b(a) = \langle w^b, a \rangle$ such that w^b has finite value on the vertices and edges of some clique of K_m and has weight $-\infty$ on all vertices outside the clique. The weights on edges outside the clique are allowed to take any values in $\mathbb{R} \cup \{-\infty\}$. If the vertex set of this clique is the subset $S^b \subseteq [m]$, we say that agent b *bids on the subgraph* K_{S^b}. The *support* of a valuation val^b is the set of vertices and edges where w^b has finite value. A set of valuations $\left\{\mathrm{val}^b \mid b \in [n]\right\}$ is *covering* if every agent $b \in [n]$ bids on a clique K_{S^b} such that $\bigcup_{b\in[n]} S^b = [m]$. A vector $a \in \mathbb{R}^d$ is *compatible* with this covering if for every $ij \in \binom{[m]}{2}$ such that $a_{ij} > 0$ there is a set S^b such that $i, j \in S^b$. We are now ready to state the generalization of Theorem 2 for $a^* \in \{0, 1\}^m$.

Theorem 8 *If $\mathcal{V}$ is the collection of sets of covering graphical valuations, then a competitive equilibrium with anonymous graphical pricing is guaranteed to exist at* any *bundle $a^* \in \{0, 1\}^m$ for* any *auction with valuations in $\mathcal{V}$.*

The above theorem immediately follows from this more technical theorem:

Theorem 9 *Let $a^* \in \{0, 1\}^m$ and suppose that every agent $b \in [n]$ bids on a clique K_{S^b} such that $\bigcup_{b\in[n]} S^b = [m]$. Then for any set of valuations $\{\mathrm{val}^b \mid \mathrm{val}^b$ is supported on $K_{S^b}\}$ and any $a \in \pi^{-1}(a^*)$ that is compatible with the covering there exists a price $p \in \mathbb{R}^d$ at which a competitive equilibrium exists.*

The proof of Theorem 9 implies an algorithm to compute an optimal competitive equilibrium.

Algorithm 1

INPUT: The agents' quadratic bids $w^1, \ldots, w^n$.
OUTPUT: An allocation-price pair $((a^1, \ldots, a^n), p)$ which achieves an optimal competitive equilibrium.
1: Check that $w^1, \ldots, w^n$ give a covering of $[m]$.
2: Choose a partition $(V^1, \ldots, V^n)$ of $[m]$ such that w^b is supported on V^b (this defines $a \in \pi^{-1}(a^*)$).
3: Let $\tilde{w}^b$ be the vector where every weight of $-\infty$ in w^b is replaced by $-M$ for some large $M > 0$. Go through all decompositions $a = \sum_{b=1}^n a^b,\ a^b \in \mathrm{vert}(P(K_m))$, check for feasibility and compute the value of

$$\begin{aligned} &\max \langle p, a \rangle \\ &\text{s.t.} \langle p - \tilde{w}^b, a^b \rangle \geq \langle p - \tilde{w}^b, a' \rangle \ \forall a' \in \mathrm{vert}(P(K_m)),\ b \in [n] \end{aligned}$$

4: **return** $((a^1, \ldots, a^n), p)$ at which the linear program above has the maximal value.

4 Proofs

We now proof the results that are stated in the previous sections. First, we give the proofs of the results stated in Sect. 2. Next we prove auxiliary results that are needed for the proofs of the main results, which are stated in Sect. 3. Finally, we give an index that refers the reader to the necessary auxiliary results for the respective main result.

4.1 Convex Results

Proof of Lemma 3: The lifting function on the Cayley polytope $\mathcal{C}(P(G), \dots, P(G)) \subseteq \mathbb{R}^d \times \mathbb{R}^n$ corresponding to the mixed subdivision on nP induced by Val is

$$\overline{\mathrm{Val}} : (a, e_b) \longmapsto \mathrm{val}^b(a) = \langle w^b, a \rangle$$

(cf. [17, Cor. 4.10]). Adding the constant vector w to all weight vectors w^b yields another lifting function of the Cayley polytope

$$\overline{\mathrm{Val}}_w : (a, e_b) \longmapsto \langle w^b + w, a \rangle = \mathrm{val}^b(a) + \left\langle \begin{pmatrix} w \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} a \\ e_b \end{pmatrix} \right\rangle.$$

Thus, $\overline{\mathrm{Val}}_w = \overline{\mathrm{Val}} + \langle \begin{pmatrix} w \\ \mathbf{0} \end{pmatrix}, \cdot \rangle$, i.e. adding w to all weights w^b amounts to adding a linear functional to the lifting function Val. This operation does not change the regular subdivision on $\mathcal{C}(P(G), \dots, P(G))$ and hence leaves the corresponding mixed subdivision of $nP(G)$ unchanged. □

Proof of Proposition 4: First note, that (i) and (ii) are equivalent by the definition of CE in Definition 3. It thus remains to show the equivalence of (ii) and (iii). Before that, recall from Sect. 2.2 that each face F^b of the regular subdivision of $P(G)$ induced by a valuation val^b is given by $F^b = \mathrm{conv}(D(\mathrm{val}^b, p^b))$ for some price $p^b \in \mathbb{R}^d$. The valuations val^b are linear functions on $\mathbb{R}^d$. Therefore, the regular subdivision induced by val^b on $P(G)$ is trivial. The set of lifted points of $P(G)$ by val^b is thus the set of vertices $\mathrm{vert}(P(G))$. Further, each face F of the regular subdivision of $nP(G)$ induced by the aggregate valuation Val (as defined in (1)) corresponds to a set $D(\mathrm{Val}, p)$ for some $p \in \mathbb{R}^d$, where $D(\mathrm{Val}, p)$ is the set of all lifted points in F. Since $D(\mathrm{Val}, p) = \sum_{j=1}^n D(\mathrm{val}^b, p)$, the set of lifted points of $nP(G)$ is

$$\bigcup_{p \in \mathbb{R}^d} D(\mathrm{Val}, p) = \bigcup_{p \in \mathbb{R}^d} \left(\sum_{j=1}^n D(\mathrm{val}^b, p) \right) \subseteq \sum_{b=1}^n \mathrm{vert}(P(G)).$$

We now show $\neg(ii) \iff \neg(iii)$. Explicitly, we show the equivalence of the following statements.

$\neg(ii)$: *There exists a set of valuations* $\{\mathrm{val}^b \mid b \in [n]\}$ *such that for all* $p \in \mathbb{R}^d$ *holds:* $a \notin \sum_{b=1}^n D(\mathrm{val}^b, p)$.

$\neg(iii)$: *There exist faces* $F^1, \dots, F^n$ *of* $P(G)$ *such that* $a \in \sum_{b=1}^n F^b$ *and* $a \notin \sum_{b=1}^n \mathrm{vert}(F^b)$.

We begin with $\neg(ii) \implies \neg(iii)$. Suppose there is a set of valuations $\{\mathrm{val}^b \mid b \in [n]\}$ such that for all $p \in \mathbb{R}^d$ holds $a \notin \sum_{b=1}^n D(\mathrm{val}^b, p) = D(\mathrm{Val}, p)$. Since $a \in \pi^{-1}(a^*) \subseteq nP(G)$, a lies in some face of the regular subdivision of $nP(G)$ induced by Val. These faces are in bijection with the distinct sets $D(\mathrm{Val}, p)$, so there exists some $p \in \mathbb{R}^d$ such that $a \in \mathrm{conv}(D(\mathrm{Val}, p))$. The assumption $a \notin D(\mathrm{Val}, p)$ implies that a is not a lifted point. Note that, since Minkowski summation and the operator of forming convex hulls commute,

$$\begin{aligned} a \in \mathrm{conv}(D(\mathrm{Val}, p)) &= \mathrm{conv}\left(\sum_{b=1}^n D(\mathrm{val}^b, p)\right) \\ &= \sum_{b=1}^n \mathrm{conv}(D(\mathrm{val}^b, p)) \\ &= \sum_{b=1}^n F^b \end{aligned}$$

for some faces $F^1, \ldots, F^n$ of $P(G)$. Since $P(G)$ is a 0/1-polytope, for each $b \in [n]$ the vertices $\mathrm{vert}(F^b)$ are precisely the lifted points of F^b in the trivial subdivision induced by val^b, i.e. $\mathrm{vert}(F^b) = D(\mathrm{val}^b, p)$. Therefore

$$\sum_{b=1}^n \mathrm{vert}(F^b) = \sum_{b=1}^n D(\mathrm{val}^b, p) = D(\mathrm{Val}, p),$$

so $\sum_{b=1}^n \mathrm{vert}(F^b)$ is a set of lifted points in the subdivision induced by Val. By assumption, a is not a lifted point, and thus, $a \notin \sum_{b=1}^n \mathrm{vert}(F^b)$.

Next, we show $\neg(iii) \implies \neg(ii)$. Suppose $a \in \sum_{b=1}^n F^b$ but $a \notin \sum_{b=1}^n (\mathrm{vert}(F^b))$. Let w^b denote the outer normal vector of F^b, i.e. the vector such that there exists a constant $\alpha^b \in \mathbb{R}$ with $\langle w^b, x \rangle = \alpha^b$ for all $x \in F^b$ and $\langle w^b, x \rangle \leq \alpha^b$ for all $x \in P(G)$. Set $\mathrm{val}^b(a) = \langle w^b, a \rangle$. We now show that for any $p \in \mathbb{R}^d$ holds $a \notin \sum_{b=1}^n D(\mathrm{val}^b, p) = D(\mathrm{Val}, p)$ by showing that a is not lifted by Val, i.e. we show that $\mathrm{Val}(a) < \widetilde{\mathrm{Val}}(a)$.

We first note, that there exist some points $x^b \in F^b$ such that $\widetilde{\mathrm{Val}}(a) = \sum_{b=1}^n \langle w^b, x^b \rangle$. To see this, consider the face $\tilde{F} = \sum_{b=1}^n F^b$ of the mixed regular subdivision of nP induced by Val, and let $\tilde{a} \in \mathrm{vert}(\tilde{F})$. Since vertices are lifted points and the linear functional w^b is maximized by points in F^b, it follows that there are vertices $\tilde{x}^b \in \mathrm{vert}(F^b)$ such that

$$\widetilde{\mathrm{Val}}(\tilde{a}) = \mathrm{Val}(\tilde{a}) = \sum_{b=1}^n \langle w^b, \tilde{x}^b \rangle.$$

Let I be the index set of vertices of $\tilde{F}$, i.e. $\mathrm{vert}(\tilde{F}) = \{\tilde{a}_i \mid i \in I\}$. We write the point $a = \sum_{i \in I} \lambda_i \tilde{a}_i$ as the convex combination of vertices of $\tilde{F}$. By construction, every such vertex can be written as the sum of vertices $\tilde{a}_i = \sum_{b=1}^n \tilde{x}_i^b,\ \tilde{x}_i{}^b \in \operatorname{vert}(F^b)$. Thus, for the points in $\mathbb{R}^{d+1}$ holds

$$\binom{a}{\widetilde{\mathrm{Val}}(a)} = \sum_{i \in I} \lambda_i \binom{\tilde{a}_i}{\widetilde{\mathrm{Val}}(\tilde{a}_i)} = \sum_{i \in I} \lambda_i \binom{\tilde{a}_i}{\sum_{b=1}^n \langle w^b, \tilde{x}_i^b \rangle}$$
$$= \binom{a}{\sum_{b=1}^n \langle w^b, \sum_{i \in I} \lambda_i \tilde{x}_i^b \rangle} = \binom{a}{\sum_{b=1}^n \langle w^b, x^b \rangle},$$

where $x^b = \sum_{i \in I} \lambda_i \tilde{x}_i^b \in F^b$, and so indeed $\widetilde{\mathrm{Val}}(a) = \sum_{b=1}^n \langle w^b, x^b \rangle$.

Finally, if $\mathrm{Val}(a) > -\infty$, then by definition there exist some $y^1, \ldots, y^n \in \operatorname{vert}(P)$ such that $a = \sum_{b=1}^n y^b$ and $\mathrm{Val}(a) = \sum_{b=1}^n \langle w^b, y^b \rangle$. But since $a \notin \sum_{b=1}^n \operatorname{vert}(F^b)$, there exists no such choice of $y^1, \ldots, y^n$ such that $y^b \in F^b$ and hence

$$\mathrm{Val}(a) = \sum_{b=1}^n \langle w^b, y^b \rangle < \sum_{b=1}^n \langle w^b, x^b \rangle = \widetilde{\mathrm{Val}}(a).$$

□

4.2 Auxiliary Results

For the complete graph $G = K_m$, the polytope $P(K_m)$ is generally known as the *correlation polytope* or *boolean quadric polytope*. It is also affinely equivalent to the well-known cut-polytope [18]. It has been widely studied, yet, the hyperplane description of this polytope remains unknown. We thus make use of a linear relaxation given in [19] describing a superset of the polytope $nP(K_m)$:

(i) $x_{ij} \geq 0$
(ii) $x_i - x_{ij} \geq 0$
(iii) $x_i + x_j - x_{ij} \leq n$
(iv) $x_i + x_{jk} - x_{ij} - x_{ik} \geq 0$
(v) $x_i + x_j + x_k - x_{ij} - x_{ik} - x_{jk} \leq n$

for all $i, j, k \in [m]$. Note that the inequalities (i) – (v) define a bounded polyhedron, containing the polytope $nP(K_m)$. In fact, for $n = 1$ the inequalities (i)–(iii) together with the constraint $x \in \mathbb{Z}^d$ yield the vertices of the correlation polytope, and for $m \leq 3$ the inequalities (i)–(v) suffice to describe the polytope completely ([19, Sect. 2]).

Lemma 10 *Let $a \in \{0, r\}^d$ satisfy inequalities (i), (ii) and (iv) for some $r \in \mathbb{N}$. Then a is the sum of characteristic vectors of pairwise disjoint complete graphs $G_1, \ldots, G_s$. More precisely,*

$$a = \sum_{t=1}^s r\chi^t,$$

where χ^t denotes the characteristic vector of G_t and $s \leq m$.

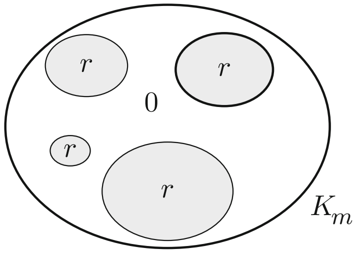


Fig. 5 The vector a is the sum of r copies of characteristic vectors of pairwise disjoint complete graphs $G_1, \dots, G_s$

Proof Since a satisfies inequalities (i) and (ii), $\frac{1}{r}a \in \{0, 1\}^d$ satisfies these inequalities as well. We can therefore interpret $\frac{1}{r}a$ as the characteristic vector of a (non-complete) graph $\hat{G}$, i.e $\frac{1}{r}a = \chi_{\hat{G}}$ and thus $a = r\chi_{\hat{G}}$ where $V(\hat{G}) \subseteq [m]$, as shown in Fig. 5.

Inequality (iv) forbids an induced subgraph consisting of a path of length two (and not a triangle), since otherwise this would imply a set $\{i, j, k\}$ of indices such that

$$a_i = a_j = a_k = a_{ij} = a_{ik} = r, \quad a_{jk} = 0,$$

and thus

$$a_i + a_{jk} - a_{ij} - a_{ik} = -r.$$

Whenever two vertices are connected via a path of length two, they must hence be adjacent. Inductively follows, that whenever two vertices are connected via a path of arbitrary length, they must be adjacent as well. Therefore, $\hat{G}$ is the disjoint union of non-empty complete graphs $G_1, \dots, G_s$ and

$$a = \sum_{t=1}^{s} r\chi^t,$$

where χ^t denotes the characteristic vector of G_t. Further,

$$m \geq |V(\hat{G})| = \sum_{t=1}^{s} |V(G_t)| \geq \sum_{t=1}^{s} 1 = s.$$

□

Proposition 11 *Let $a \in \{0, r\}^d$. Then for all faces $F^1, \dots, F^n$ of $P(K_m)$ containing $a \in \sum_{b=1}^{n} F^b$ in their Minkowski sum, there exist vertices χ^b of F^b such that $a = \sum_{b=1}^{n} \chi^b$.*

Proof If there exist faces $F^1, \dots, F^n$ of $P(K_m)$ such that $a \in \sum_{b=1}^{n} F^b$, then in particular $a \in nP(K_m)$. Thus, the inequalities (i) – (v) hold for a and by Lemma 10 we have

$$a = \sum_{t=1}^{s} r\chi^t,$$

where χ^t is the characteristic vector of a complete graph G_t for each $t \in [s]$, the graphs $G^1, \ldots, G^s$ are pairwise disjoint, and $s \leq m$. Choose $\{k_i \mid i \in [s]\}$ such that $k_i \in V(G_i)$ for each $i \in [s]$ and consider the following inequalities

$$\text{(vi')} \quad \sum_{i\in[s]} x_{k_i} - \sum_{ij\in\binom{[s]}{2}} x_{k_ik_j} \leq 1 \qquad \text{(vi)} \quad \sum_{i\in[s]} x_{k_i} - \sum_{ij\in\binom{[s]}{2}} x_{k_ik_j} \leq n.$$

Each vertex of $P(K_m)$ satisfies the inequality (vi') and so $a \in nP(K_m)$ satisfies (vi). Since $a_{k_i} = r$ and $a_{k_ik_j} = 0$ for all $i, j \in [s]$, we have

$$\sum_{i\in[s]} a_{k_i} - \sum_{ij\in\binom{[s]}{2}} a_{k_ik_j} = s \cdot r - 0 \leq n.$$

We can thus write $a = \sum_{b=1}^n \hat{\chi}^b$, where in this representation we have r copies of χ^t for each $t \in [s]$, and $n - s \cdot r$ copies of the characteristic vector $\chi^0 = \mathbf{0}$ of the empty graph G_0. Our task is now to distribute the vertices $\hat{\chi}^b$ such that $\hat{\chi}^b \in F^b$, $b \in [n]$.

By assumption, $a \in (F^1 + \cdots + F^n)$, i.e. we can find a subset $S^b \subseteq \text{vert}(F^b)$ of vertices of the face F^b such that $\lambda_q > 0$ for all $q \in S^b$ and

$$a = \sum_{b=1}^{n} \sum_{q\in S^b} \lambda_q q, \quad \sum_{q\in S^b} \lambda_q = 1.$$

We now show that $\bigcup_{b=1}^n S^b = \{\hat{\chi}^b \mid b \in [n]\}$ and deduce from this that

$$r\chi^t = \sum_{b=1}^{n} \sum_{\substack{q\in S^b \\ q=\chi^t}} \lambda_q q.$$

Then the following Lemma 12 implies that there is a labelling of the faces such that $\hat{\chi}^b \in F^b$ for all $b \in [n]$.

We begin with showing $\bigcup_{b=1}^n S^b \subseteq \{\hat{\chi}^b \mid b \in [n]\}$. Let $\hat{q} \in S^b$ for some $b \in [n]$ such that $\hat{q} \neq \mathbf{0}$. Then there is some $i \in [m]$ such that $\hat{q}_i = 1$ and since $\lambda_{\hat{q}} > 0$ we have $a_i = r$. Hence, there is some $t \in [s]$ such that $i \in V(G_t)$. We now show that $\hat{q} = \chi^t$.

If $\hat{q}_j = 1$, then $\hat{q}_{ij} = 1$ and therefore $a_{ij} = r$. This implies that i and j are contained in the same connected component in $\hat{G}$, i.e. $j \in V(G_t)$. Note that if i is an isolated vertex in $\hat{G}$, then the assumption $\hat{q}_j = 1$ immediately leads to a contradiction, so in this case we have indeed $\hat{q} = e_i$.

If $\hat{q}_j = 0$, then $0 = \hat{q}_{ij} < \hat{q}_i$. This implies $a_{ij} = 0$, since otherwise (by (ii))

$$r = a_{ij} = \sum_{b=1}^{n} \sum_{q\in S^b} \lambda_q q_{ij} < \sum_{b=1}^{n} \sum_{p\in S^b} \lambda_q q_i = a_i = r.$$

Hence, $j \notin V(G_t)$ and so $\hat{q} = \chi^t \in \left\{\hat{\chi}^b \,\middle|\, b \in [n]\right\}$.

Next, we show that $\bigcup_{b=1}^n S^b \supseteq \left\{\hat{\chi}^b \,\middle|\, b \in [n]\right\}$. Let $t \in [s]$ and $i \in V(G_t)$. Then $a_i = r$, so there exists some $\hat{q} \in \bigcup_{b=1}^n S^b$ such that $\hat{q}_i = 1$. By the above, this implies $\hat{q} = \chi^t$, and hence, $\left\{\chi^1, \ldots, \chi^s\right\} \subseteq \bigcup_{b=1}^n S^b$. It remains to show that $\mathbf{0} \in \bigcup_{b=1}^n S^b$ if and only if $s \cdot r \leq n$. For $t = 0, \ldots, s$, we define

$$\mu_t = \sum_{b=1}^n \sum_{\substack{q \in S^b \\ q = \chi^t}} \lambda_q.$$

We show that $\mu_0 = n - s \cdot r$. This implies that $\mathbf{0} \in \bigcup_{b=1}^n S^b$ if and only if $s \cdot r < n$, which is equivalent to $\mathbf{0} \in \left\{\hat{\chi}^b \,\middle|\, j \in [n]\right\}$. We can write

$$a = \sum_{b=1}^n \sum_{q \in S^b} \lambda_q q \; = \sum_{t=0}^s \mu_t \chi^t.$$

For $i \in V(G_t)$ we have

$$r = a_i = \sum_{t=0}^s \mu_t \chi_i^t = \mu_t$$

and so

$$s \cdot r + \mu_0 = \sum_{t=0}^s \mu_t = \sum_{t=0}^s \sum_{b=1}^n \sum_{\substack{q \in S^b \\ q = \chi^t}} \lambda_q = \sum_{b=1}^n \sum_{q \in S^b} \lambda_q = \sum_{b=1}^n 1 = n.$$

Thus, $\bigcup_{b=1}^n S^b = \left\{\hat{\chi}^b \,\middle|\, j \in [n]\right\}$.

We finish the proof by showing that

$$r\chi^t = \sum_{b=1}^n \sum_{\substack{q \in S^b \\ q = \chi^t}} \lambda_q q$$

for all $t \in [s]$. Let $i \in V(G_t)$ for some $t \in [s]$. Since $\bigcup_{b=1}^n S^b = \left\{\hat{\chi}^b \,\middle|\, j \in [n]\right\}$ and the graphs $G^1, \ldots, G^s$ are pairwise disjoint, we have that

$$r\chi_i^t = r = a_i = \sum_{b=1}^n \sum_{q \in S^b} \lambda_q q_i = \sum_{b=1}^n \sum_{\substack{q \in S^b \\ q = \chi^t}} \lambda_q q_i.$$

A similar statement holds for the edge ij for $i, j \in V(G^t)$. If $i \notin V(G^t)$ then

$$r\chi_i^t = 0 = \sum_{b=1}^{n} \sum_{\substack{q\in S^b \\ q=\chi^t}} \lambda_q q_i.$$

and similarly for any $i, j \in [m]$ such that ij is not an edge in G^t. Hence, all assumptions of Lemma 12 are fulfilled and Lemma 12 implies that there is a labelling of the faces such that $\hat{\chi}^b \in F^b$ for all $j \in [n]$. □

Lemma 12 *Suppose* $a = \sum_{t=1}^{s} \mu_t \chi^t = \sum_{b=1}^{n} \hat{\chi}^b$*, such that*

- $\{\chi^t \mid t \in [s]\} = \{\hat{\chi}^b \mid b \in [n]\}$,
- χ^t *is the characteristic vector of a complete subgraph* G_t *of* K_m,
- $\mu_t \in \mathbb{N}$ *such that* $\sum_{t=1}^{s} \mu_t = n$.

Further, let a be contained in the Minkowski sum of faces F^b, $b \in [n]$*, i.e.*

$$a = \sum_{b=1}^{n} \sum_{q\in S^b} \lambda_q q, \sum_{q\in S^b} \lambda_q = 1$$

where $S^b \subseteq \text{vert}(F^b)$ *is a subset of vertices of the face* F^b *and* $\lambda_q > 0$. *If for every* $t \in [s]$ *holds*

$$\mu_t \chi^t = \sum_{b=1}^{n} \sum_{\substack{q\in S^b \\ q=\chi^t}} \lambda_q q,$$

then there is a labelling of the faces such that $\chi^b \in F^b$.

Proof First note, that we can write

$$\mu_t \chi^t = \sum_{b=1}^{n} \sum_{\substack{q\in S^b \\ q=\chi^t}} \lambda_q q = \sum_{b=1}^{n} \lambda_t^b \chi^t,$$

where

$$\lambda_t^b = \sum_{\substack{q\in S^b \\ q=\chi^t}} \lambda_q$$

and thus, $0 \leq \lambda_t^b \leq 1$, and $\lambda_t^b > 0$ if and only if $\chi^t \in S^b$. Further, note that this implies

$$\mu_t = \sum_{b=1}^{n} \lambda_t^b \text{ and } \sum_{t=1}^{s} \lambda_t^b = \sum_{q\in S^b} \lambda_q = 1 \text{ for a fixed } b \in [n].$$

We now show that this constitutes a flow in a network. The Max-flow-min-cut theorem (e.g. [16, Th. 7.2]) then implies that an integer flow exists, which will give us the desired assignment of vertices χ^t and faces F^b.

Let $D = (V, E)$ be a directed graph with vertices

$$V = \left\{F^b \mid b \in [n]\right\} \cup \left\{\chi^t \mid t \in [s]\right\} \cup \left\{\hat{v}, \hat{w}\right\},$$

where $\hat{v}$ is the vertex of D representing the source of the network, and $\hat{w}$ represents the sink. The directed edges of D are

$$E = \left\{(\hat{v}, F^b) \mid b \in [n]\right\} \cup \left\{(F^b, \chi^t) \mid b \in [n], t \in [s], \chi^t \in S^b\right\} \\ \cup \left\{(\chi^t, \hat{w}) \mid t \in [s]\right\},$$

and we consider the capacity function $c : E \to \mathbb{R}_{>0}$ such that

$$c(e) = \begin{cases} 1 & e = (\hat{v}, F^b) \text{ for some } b \in [n], \\ 1 & e = (F^b, \chi^t) \text{ for some } b \in [n], t \in [s], \\ \mu_t & e = (\chi^t, \hat{w}) \text{ for some } t \in [s]. \end{cases}$$

We claim that the following function $f : E \to \mathbb{R}_{>0}$ constitutes a maximal flow on D

$$f(e) = \begin{cases} 1 & e = (\hat{v}, F^b) \text{ for some } b \in [n], \\ \lambda_t^b & e = (F^b, \chi^t) \text{ for some } b \in [n], t \in [s], \\ \mu_t & e = (\chi^t, \hat{w}) \text{ for some } t \in [s], \end{cases}$$

i.e. that for all vertices $v \in V \setminus \left\{\hat{v}, \hat{w}\right\}$ holds

$$\sum_{\substack{w \in V \\ e=(v,w)}} f(e) = \sum_{\substack{w \in V \\ e=(w,v)}} f(e).$$

To see this, let first $v = F^b$ for some $b \in [n]$. Then

$$\sum_{\substack{w \in V \\ e=(w,v)}} f(e) = f(\hat{v}, F^b) = 1 \text{ and } \sum_{\substack{w \in V \\ e=(v,w)}} f(e) = \sum_{t=1}^{s} \lambda_t^b = 1.$$

If $v = \chi^t$ for some $t \in [s]$, then

$$\sum_{\substack{w \in V \\ e=(w,v)}} f(e) = \sum_{b=1}^{n} \lambda_t^b = \mu_t \quad \text{and} \quad \sum_{\substack{w \in V \\ e=(v,w)}} f(e) = f(\chi^t, \hat{w}) = \mu_t.$$

This is indeed a flow of maximal size n, since

$$n = \sum_{t=1}^{s} \mu_t = \sum_{t=1}^{s} f(\chi^t, \hat{w}) = \sum_{t=1}^{s} c(\chi^t, \hat{w}),$$

and the flow cannot exceed the capacity. Hence, there exists an integral flow f' of size n. This is a flow such that for each $b \in [n]$ there exists exactly one $t \in [s]$ such that $f'(F^b, \chi^t) = 1$ and has value 0 otherwise. On the other hand, for each $t \in [s]$ there are exactly μ_t facets $F^b, b \in [n]$ such that $f'(F^b, \chi^t) = 1$. Since for each $t \in [s]$ there are exactly μ_t copies of χ^t in the representation $a = \sum_{b=1}^{n} \hat{\chi}^b$, there is a labelling of $\hat{\chi^1}, \dots, \hat{\chi^n}$ such that $f'(F^b, \hat{\chi}^b) = 1$ for all $b \in [n]$. This induces the desired assignment $\left\{(F^b, \hat{\chi}^t) \mid f'(F^b, \chi^t) = 1\right\}$. □

Proposition 13 *Let $a^* \in \{0, 1, \dots, n\}^m$. Then there exists a point $a \in \pi^{-1}(a^*)$ such that the following holds: For all faces $F^1, \dots, F^n$ of $P(K_m)$ containing $a \in \sum_{b=1}^{n} F^b$ in their Minkowski sum, there exist vertices χ^b of F^b such that $a = \sum_{b=1}^{n} \chi^b$.*

Proof We define a as follows:

$$a_i = a_i^* \qquad a_{ij} = \min\left\{a_i^*, a_j^*\right\}.$$

We show that a is the sum of characteristic vectors of complete graphs $G_n \subseteq \cdots \subseteq G_1$. Let $t_0 = 0$ and $t_1 < t_2 < \cdots < t_s$ be the distinct non-zero values of a^*, i.e. $\left\{a_i^* \,\middle|\, i \in [m]\right\} \setminus \{0\} = \left\{t_l \,\middle|\, l \in [s]\right\}$. Let χ^{t_l} be given by

$$\chi_i^{t_l} = \begin{cases} 1, & \text{if } a_i \geq t_l \\ 0, & \text{otherwise} \end{cases} \text{ for } i \in [m], \qquad \chi_{ij}^{t_l} = \chi_i^{t_l}\chi_j^{t_l} \text{ for } ij \in \binom{[m]}{2}.$$

Then χ^{t_l} is the characteristic vector of a complete graph G_{t_l} and for a non-zero $a_i = t_k$ we have

$$a_i = t_k = \sum_{l=1}^{k}(t_l - t_{l-1}) = \sum_{l=1}^{k}(t_l - t_{l-1})\chi_i^{t_l} = \sum_{l=1}^{s}(t_l - t_{l-1})\chi_i^{t_l}.$$

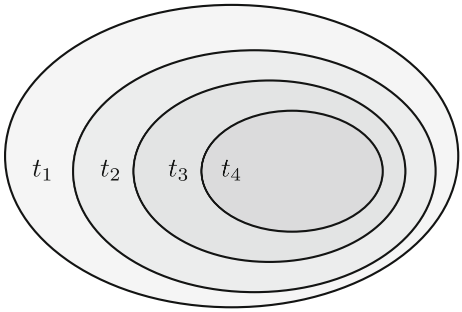
An analogous statement holds for $a_{ij} = \min\{a_i, a_j\}$ and thus

$$a = \sum_{l=1}^{s}(t_l - t_{l-1})\chi^{t_l}$$

for complete graphs $G_{t_s} \subsetneq \cdots \subsetneq G_{t_1}$, as illustrated in Fig. 6. Taking $t_l - t_{l-1}$ copies of G_{t_l} and $n - t_s$ copies of the empty graph, we can write

$$a = \sum_{b=1}^{n} \hat{\chi}^b$$

Fig. 6 The vector a is the sum of characteristic vectors of nested complete graphs $G_{t_s} \subsetneq \cdots \subsetneq G_{t_1}$



for graphs $G_n \subseteq \cdots \subseteq G_1$, where $G_b = G_{t_l}$ for any $b \in [n]$ such that $t_{l-1}+1 \leq b \leq t_l$ and $\hat{\chi}^b$ denotes the characteristic vector of G_b. Thus, a is the sum of m vertices of $P(K_m)$ and is therefore contained in $\pi^{-1}(a^*) \cap nP(K_m)$.

Suppose $a \in (F^1 + \cdots + F^n)$, i.e.

$$a = \sum_{b=1}^{n} \sum_{q \in S^b} \lambda_q q, \quad \sum_{q \in S^b} \lambda_q = 1,$$

where $S^b \subseteq \mathrm{vert}(F^b)$ is a subset of vertices of the face F^b and $\lambda_q > 0$ for all $q \in S^b$. Note that all vertices of the polytope $P(K_m)$ satisfy inequalities (i)–(v) for $n = 1$. Similar to the proof of Proposition 11, we now show that $\bigcup_{b=1}^{n} S^b = \left\{\chi^b \,\middle|\, b \in [n]\right\}$ and deduce from this that

$$(t_l - t_{l-1})\chi^{t_l} = \sum_{b=1}^{n} \sum_{\substack{q \in S^b \\ q = \chi^{t_l}}} \lambda_q q.$$

Then Lemma 12 implies that there is a labelling of the faces such that $\hat{\chi}^b \in F^b$ for all $b \in [n]$.

We begin with $\bigcup_{b=1}^{n} S^b \subseteq \left\{\hat{\chi}^b \,\middle|\, b \in [n]\right\}$. Let $\hat{q} \in S^b$ be a non-zero vector and $i \in [m]$ such that $a_i = \min_{j \in [m]} \left\{a_j \,\middle|\, \hat{q}_j = 1\right\}$. Note that since $\lambda_{\hat{q}} > 0$ and $\hat{q}_i = 1$, we have $a_i > 0$, so $a_i = t_k$ for some non-zero value. We show that $\hat{q} = \chi^{t_k}$. Let $j \in [m]$. If $a_j < t_k$, then $\hat{q}_j = 0$ by the choice of i and thus $\hat{q}_{ij} = 0$ by (ii). If $a_j \geq t_k$, then $a_{ij} = a_i = t_k$ by the definition of a and so $\hat{q}_{ij} = \hat{q}_i = 1$, since otherwise (by (ii))

$$t_k = a_{ij} = \sum_{b=1}^{n} \sum_{q \in S^b} \lambda_q q_{ij} < \sum_{b=1}^{n} \sum_{q \in S^b} \lambda_q q_i = a_i = t_k. \tag{3}$$

Hence, we have $\hat{q}_j = 1$ and so $\hat{q} = \chi^{t_k}$. Let $\hat{q} = \mathbf{0}$ be the zero vector and $i \in [m]$ such that $a_i = t_s$ attains the maximum value. Since $\hat{q}_i = 0$, $\lambda_{\hat{q}} > 0$ and $q_i \in \{0, 1\}^d$

for all $q \in \bigcup_{b=1}^n S^b$, we have

$$t_s = a_i = \sum_{b=1}^{n} \sum_{q \in S^b} \lambda_q q_i < \sum_{b=1}^{n} \sum_{q \in S^b} \lambda_q = \sum_{b=1}^{n} 1 = n \tag{4}$$

and therefore the number of characteristic vectors of the empty graph is $n - t_s > 0$, i.e. $\hat{q} = \mathbf{0} \in \left\{\hat{\chi}^b \,\middle|\, b \in [n]\right\}$.

We now show $\bigcup_{b=1}^n S^b \supseteq \left\{\hat{\chi}^b \,\middle|\, b \in [n]\right\}$. Let t_k be a non-zero value of a. We show that there exists some $q \in \bigcup_{b=1}^n S^b$ such that $q = \chi^{t_k}$. Let $i, j \in [m]$ such that $a_i = t_k, a_j = t_{k-1}$. Since $a_i > a_j$ and $\bigcup_{b=1}^n S^b \subseteq \left\{\hat{\chi}^b \,\middle|\, b \in [n]\right\}$, this implies

$$\left\{\chi^{t_l} \,\middle|\, t_l > t_{k-1}, i \in V(G_{t_l})\right\} \cap \bigcup_{b=1}^{n} S^b \neq \emptyset.$$

Note that $i \in V(G_{t_l})$ implies that $a_i \geq t_l$. Thus, since otherwise $a_i > t_k$,

$$\left\{\chi^{t_l} \,\middle|\, t_l = t_k, i \in V(G_{t_l})\right\} \cap \bigcup_{b=1}^{n} S^b \neq \emptyset.$$

Suppose $\mathbf{0} \in \left\{\chi^b \,\middle|\, b \in [n]\right\}$, i.e. there exists a characteristic vector of the empty graph and so $t_s < n$. Let $i \in [m]$ such that $a_i = t_s$. Inequality (4) implies that there exists a $\hat{q} \in \bigcup_{b=1}^n S^b$ such that $\hat{q}_i = 0$. By the above, we know that $\hat{q} \in \left\{\hat{\chi}^b \,\middle|\, b \in [n]\right\}$, which are characteristic vectors of nested graphs. In particular, all non-empty graphs contain the graph G_{t_s} and hence the vertex i. Since $\hat{q}$ is a characteristic vector of one of the nested graphs graphs, it must be the empty graph, i.e. $\hat{q} = \mathbf{0}$. This shows that $\bigcup_{b=1}^n S^b = \left\{\hat{\chi}^b \,\middle|\, b \in [n]\right\}$.

Finally, we now show

$$(t_l - t_{l-1})\chi^{t_l} = \sum_{b=1}^{n} \sum_{\substack{q \in S^b \\ q = \chi^{t_l}}} \lambda_q q$$

for all $l \in [s]$. Let $i \in [m]$ such that $a_i = t_l$ for some $l \in [s]$. Then $i \in V(G_{t_k})$ for all $k \leq l$ and $i \notin V(G_{t_k})$ for all $k > l$. As $\bigcup_{b=1}^n S^b = \left\{\hat{\chi}^b \,\middle|\, b \in [n]\right\}$ and the graphs $G_{t_1}, \ldots, G_{t_s}$ are nested, we have that

$$t_l \chi_i^{t_k} = t_l = a_i = \sum_{b=1}^{n} \sum_{q \in S^b} \lambda_q q_i = \sum_{b=1}^{n} \sum_{\substack{q \in S^b \\ q \in \{\chi^{t_1}, \ldots, \chi^{t_l}\}}} \lambda_q q_i.$$

for all $k \leq l$. Hence,

$$(t_l - t_{l-1})\chi_i^{t_l} = (t_l \chi_i^{t_l}) - (t_{l-1}\chi_i^{t_{l-1}}) = \sum_{b=1}^{n} \sum_{\substack{q \in S^b \\ q = \chi^{t_l}}} \lambda_q q_i .$$

A similar statement holds for the edge ij for $i, j \in V(G_{t_l})$. If $i \notin V(G_{t_l})$ then

$$(t_l - t_{l-1})\chi_i^{t_l} = 0 = \sum_{b=1}^{n} \sum_{\substack{q \in S^b \\ q = \chi^{t_l}}} \lambda_q q_i .$$

and similarly for any $i, j \in [m]$ such that ij is not an edge in G_{t_l}. Hence, all assumptions of Lemma 12 are fulfilled and it implies that there is a labelling of the faces such that $\hat{\chi}^b \in F^b$ for all $b \in [n]$. □

4.3 Proofs of Main Results

Proof of Theorem 2: This is a reformulation of Theorem 6 for $r = 1$. □

Proof of Theorem 6: Let $a^* \in \{0, r\}^m$, $n \geq r$, $a \in \pi^{-1}(a^*)$. By Proposition 11, for any faces $F^1, \ldots, F^n$ of $P(K_m)$ such that $a \in \sum_{b=1}^n F^b$ holds that $a \in \sum_{b=1}^n \mathrm{vert}(F^b)$. Proposition 4 implies that this holds if and only if for any set of valuations there exists an allocation and price $p \in \mathbb{R}^d$ at which a competitive equilibrium exists. □

Proof of Theorem 7: Let $a^* \in \{0, 1, \ldots, n\}^m$. Then by Proposition 13, there exists a point $a \in \pi^{-1}(a^*)$, such that for any faces $F^1, \ldots F^n$ of $P(K_m)$ containing $a \in \sum_{b=1}^n F^b$ holds that $a \in \sum_{b=1}^n \mathrm{vert}(F^b)$. Hence, by Corollary 5, for any set of valuations there exists an allocation and price at which a competitive equilibrium exists. □

Proof of Theorems 8 and 9: Let $\overline{\mathrm{val}}^b(a) = \langle \overline{w}^b, a \rangle$, where $\overline{w}^b \in \mathbb{R}^d$ is the vector in which we replace every entry of w^b that has value $-\infty$ by the value $-M$ for a sufficiently large $M \in \mathbb{R}$. By assumption, the cliques $K_{S^b}, b \in [n]$ form a covering of the graph K_m. Thus, we can find a partition of the vertices $[m] = V^1 \sqcup \cdots \sqcup V^n$ such that $V^b \subseteq S^b$.

We consider the graph consisting of unions of cliques, where the vertex sets of the cliques are the sets $V^1, \ldots, V^n$. Let $a \in \{0, 1\}^d$ be the characteristic vector of this graph. According to Theorem 6, given the valuations $\overline{\mathrm{val}}^b$, $b \in [n]$, we can find a price $p \in \mathbb{R}^d$ and an allocation $a = \sum_{b=1}^n a^b$ such that $a^b \in D(\overline{\mathrm{val}}^b, p)$.

More precisely, each a^b is the characteristic vector of the union of a subset of the cliques $V^1, \ldots, V^b$. Furthermore, if M is big enough then $a^b \in D(\overline{\mathrm{val}}^b, p)$ implies that the graph corresponding to a^b is contained the graph S^b that agent b has bid on (that is because if a^b contains a vertex where $\overline{w}^b$ has value $-M$, then $\langle \overline{w}^b - p, a^b \rangle < 0 = \langle \overline{w}^b - p, \mathbf{0} \rangle$). Thus, also the value $\langle w^b - p, a^b \rangle$ is finite and we have

$$\langle \overline{w}^b - p, a^b \rangle = \langle w^b - p, a^b \rangle.$$

Since in general we always have

$$\langle \overline{w}^b - p, a \rangle \leq \langle w^b - p, a \rangle$$

for any $a \in P(K_m)$, it follows that $a^b \in D(\text{val}^b, p)$ for the original valuations, for all $b \in [n]$, so competitive equilibrium at a always exists. □

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