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On Topological Complexity of Billiard Configuration Spaces

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Contents

1 Introduction

The Billiard Configuration Space was initially introduced to study the number of periodic billiard trajectories in a smooth strictly convex domain Ω in euclidean space (Farber & Tabachnikov [17], Blagojević, Harrison, Tabachnikov & Ziegler [7]). The points where a periodic trajectory of length n hits the boundary of Ω live in the Billiard Configuration Space of the boundary of Ω,

$$G(\partial\Omega, n) := \{(x_1, ..., x_n) \in (\partial\Omega)^{\times n} : \ x_i \neq x_{i+1} \text{ for } 1 \leqslant i \leqslant n\},$$

where we set $x_{n+1} := x_1$. In the process of studying topology of the space $G(\partial\Omega, n)$, its local version

$$G(\mathbb{R}^d, n) := \{(x_1, ..., x_n) \in (\mathbb{R}^d)^{\times n} : \ x_i \neq x_{i+1} \text{ for } 1 \leqslant i \leqslant n\},$$

where $x_{n+1} := x_1$, became essential for further calculations [17]. Moreover, estimating the Lusternik–Schnirelmann category of the Billiard Configuration Space $G(S^d, n)$ of the sphere is of great importance for bounding the number of periodic trajectories [7].

A similar concept to that of Lusternik–Schnirelmann category, namely the topological complexity of a space, was introduced by Farber [14]. It is a homotopical invariant of a space which measures how complex a motion planning algorithm on the space must be. In two papers, Farber & Yuzvinsky [16] and Farber & Grant [15] computed topological complexity of the (standard) Configuration Space

$$F(\mathbb{R}^d, n) := \{(x_1, ..., x_n) \in (\mathbb{R}^d)^{\times n} : \ x_i \neq x_j \text{ for } 1 \leqslant i < j \leqslant n\}.$$

In this thesis, we aim to give bounds for topological complexity, as well as calculate the Lusternik–Schnirelmann category, of the Billiard Configuration Space $G(\mathbb{R}^d, n)$.

The thesis is divided into six sections. In Section 2 we provide a cellular model for the Billiard Configuration Space $G(\mathbb{R}^d, n)$ which has the cellular model of the (standard) Configuration Space $F(\mathbb{R}^d, n)$ as a subcomplex. In Section 3 we use these two models to compute the fundamental group of $G(\mathbb{R}^d, n)$.

In Section 4 we provide another cellular model for $G(\mathbb{R}^d, n)$, which is of smaller dimension than the one obtained in Section 2. This dimension coincides with the integral cohomological dimension of $G(\mathbb{R}^d, n)$, which is a fact necessary for our later computations. In Section 5 we provide an introduction to the sectional category of a fibration and to the notion of topological complexity of a space. In Section 6 we first compute the Lusternik–Schnirelmann category of $G(\mathbb{R}^d, n)$. Next, we compute topological complexity of the same space in some cases and provide bounds for the remaining cases. To bound the topological complexity we use obstruction theory and the low dimensional cellular model from Section 4. We believe that fundamental group calculation from Section 3 could be used to further sharpen the bound of topological complexity of $G(\mathbb{R}^d, n)$ using obstruction theory.

In this thesis we provide original results in the setting of Billiard Configuration Space. Therefore, whenever we are using a result from a source, we will reference it without giving a proof in the thesis. However, we will provide proofs of statements from other sources for which we could not find a complete proof elsewhere.

1.1 Some mathematical preliminaries

For $n \geqslant 1$ and $d \geqslant 1$ let

$$F(\mathbb{R}^d, n) := \{(x_1, \dots, x_n) \in (\mathbb{R}^d)^{\times n} : \ x_i \neq x_j \text{ for } 1 \leqslant i < j \leqslant n\}$$

be the *(standard) Configuration Space*. This space is well studied in the literature over the years ([2], [13], [19], [20], [6]). The main space of our interest will be a similarly defined space, namely the *Billiard Configuration Space* defined as

$$G(\mathbb{R}^d, n) := \{(x_1, \dots, x_n) \in (\mathbb{R}^d)^{\times n} : \ x_i \neq x_{i+1} \text{ for } 1 \leqslant i \leqslant n\},$$

where we set $x_{n+1} := x_1$. One can notice that $F(\mathbb{R}^d, n) \subseteq G(\mathbb{R}^d, n)$ and the two spaces are the same if $n \leqslant 3$.

Let $\mathfrak{S}_n$ denote the symmetric group on n elements, that is the set of automorphisms of the set $[n] := \{1, \dots, n\}$. A left action of $\mathfrak{S}_n$ on the n-fold product $X^{\times n}$ of any set X can be defined as

$$\sigma \cdot (x_1, \dots, x_n) := (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}),$$

for any $(x_1, \dots, x_n) \in X^{\times n}$ and any $\sigma \in \mathfrak{S}_n$. In particular, this defines an action of $\mathfrak{S}_n$ on $(\mathbb{R}^d)^{\times n}$ which restricts to a free action on $F(\mathbb{R}^d, n) \subseteq (\mathbb{R}^d)^{\times n}$.

Let $D_n \subseteq \mathfrak{S}_n$ be the dihedral group, i.e., a subgroup generated by the permutations

$$\rho : [n] \to [n], \ i \mapsto i + 1 \quad \text{ and } \quad \theta : [n] \to [n], \ i \mapsto n + 1 - i.$$

In these generators, the dihedral group has a presentation

$$D_n = \langle \rho, \theta \mid \rho^n = 1, \ \theta^2 = 1, \ \rho\theta = \theta\rho^{n-1} \rangle.$$

By restriction, action of D_n on $(\mathbb{R}^d)^{\times n}$ is generated by

$$\rho \cdot (x_1, x_2, \dots, x_n) = (x_n, x_1, \dots, x_{n-1}) \quad \text{and} \quad \theta \cdot (x_1, x_2, \dots, x_n) = (x_n, x_{n-1}, \dots, x_1).$$

Therefore, $G(\mathbb{R}^d, n) \subseteq (\mathbb{R}^d)^{\times n}$ is an invariant space of this action.

2 First CW model

In this section we will define regular $\mathfrak{S}_n$-equivariant CW complexes

$$\mathcal{F}(d,n) \subseteq F(\mathbb{R}^d,n) \quad \text{and} \quad \mathcal{G}(d,n) \subseteq G(\mathbb{R}^d,n)$$

which are equivariant strong deformation retracts of their respective ambient spaces $F(\mathbb{R}^d,n)$ and $G(\mathbb{R}^d,n)$. To do so, we closely follow the work of Blagojević & Ziegler [8], where cellular model $\mathcal{F}(d,n)$ of $F(\mathbb{R}^d,n)$ is constructed. See also [26].

The type of the complex we construct is known as the *Salvetti complex* [28]. See also Fox & Neuwirth [19], Björner & Ziegler [5] and de Concini & Salvetti [12]. We will use the CW model $\mathcal{G}(d,n)$ in Section 3 to compute the fundamental group of $G(\mathbb{R}^d,n)$.

Let us denote by

$$D := \{(x_1,...,x_n) \in (\mathbb{R}^d)^{\times n} : \; x_1 = \cdots = x_n\}$$

the diagonal and

$$W_n^{\oplus d} = \{(x_1,...,x_n) \in (\mathbb{R}^d)^{\times n} : \; x_1 + \cdots + x_n = 0\}$$

its orthogonal complement in $(\mathbb{R}^d)^{\times n}$. The composition

$$(\mathbb{R}^d)^{\times n} \setminus D \longrightarrow_{\mathfrak{S}_n} W_n^{\oplus d} \setminus \{0\} \longrightarrow_{\mathfrak{S}_n} S(W_n^{\oplus d}) \tag{1}$$

is an $\mathfrak{S}_n$-equivariant deformation retraction, where the first map

$$(x_1,...,x_n) \longmapsto (x_1 - \frac{1}{n}(x_1 + \cdots + x_n),..., x_n - \frac{1}{n}(x_1 + \cdots + x_n))$$

is the orthogonal projection along D, the second map $x \mapsto x/\|x\|$ is the normalization. By $S(W_n^{\oplus d})$ we denote the unit sphere in $W_n^{\oplus d}$. Let us denote by

$$\overline{F}(\mathbb{R}^d,n) := F(\mathbb{R}^d,n) \cap S(W_n^{\oplus d}) \quad \text{and} \quad \overline{G}(\mathbb{R}^d,n) := G(\mathbb{R}^d,n) \cap S(W_n^{\oplus d})$$

the induced equivariant strong deformation retracts of $F(\mathbb{R}^d,n)$ and $G(\mathbb{R}^d,n)$, respectively.

The idea is first to construct a CW structure $\mathcal{S}(d,n)$ on the sphere $S(W_n^{\oplus d})$. It will turn out that both $\overline{F}(\mathbb{R}^d,n)$ and $\overline{G}(\mathbb{R}^d,n)$ will be union of relatively open cells of $\mathcal{S}(d,n)$, but will not be subcomplexes of the sphere, because the set of open cells belonging to them will not be closed downwards in the face poset of $\mathcal{S}(d,n)$. However, they will be closed upwards in the poset. Next, a duality argument will be used to show the existence of invariant subcomplexes $\mathcal{F}(d,n)$ and $\mathcal{G}(d,n)$ of the dual cell complex $\mathcal{S}(d,n)^{\text{op}}$ of $\mathcal{S}(d,n)$ with respect to actions of $\mathfrak{S}_n$ and D_n, respectively. They will be consisting of the duals of the open cells of $\mathcal{S}(d,n)$ contained in $\overline{F}(\mathbb{R}^d,n)$ and $\overline{G}(\mathbb{R}^d,n)$, respectively. Finally, $\mathcal{F}(d,n)$ and $\mathcal{G}(d,n)$ will be equivariant strong deformation retracts of $\overline{F}(\mathbb{R}^d,n)$ and $\overline{G}(\mathbb{R}^d,n)$, respectively.

2.1 Stratification of $W_n^{\oplus d}$

Let $(x_1, ..., x_n) \in W_n^{\oplus d}$ be any point. We can associate to it a permutation $\sigma \in \mathfrak{S}_n$ such that

$$x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)},$$

where $\leqslant$ denotes the lexicographic order of the column vectors in $\mathbb{R}^d$. Notice that if $x_{\sigma(i)} = x_{\sigma(i+1)}$, the permutation $\sigma \circ (i, i+1)$ can also be associated to $(x_1, ..., x_n)$ to represent the lexicographic order. Thus, we impose a convention that if $x_{\sigma(i)} = x_{\sigma(i+1)}$ we must have

$$1 \leqslant \sigma(i) < \sigma(i+1) \leqslant n.$$

For each $1 \leqslant i \leqslant n-1$ let

$$r_i := \min\{r \in [d] : \ x_{\sigma(i),r} < x_{\sigma(i+1),r}\} \cup \{d+1\}.$$

Thus $r_i = d+1$ if and only if $x_{\sigma(i)} = x_{\sigma(i+1)}$. Let $\mathbb{r} := (r_1, ..., r_{n-1})$. We will call the pair $(\sigma, \mathbb{r}) \in \mathfrak{S}_n \times [d+1]^{n-1}$ the *combinatorial data* associated to the tuple $(x_1, ..., x_n)$ and write

$$x_{\sigma_1} <_{r_1} \cdots <_{r_{n-1}} x_{\sigma_n}$$

where $\sigma_i := \sigma(i)$ for all $1 \leqslant i \leqslant n$.

Definition 2.1. Let the set

$$C := \{(\sigma, \mathbb{r}) \in \mathfrak{S}_n \times [d+1]^{n-1} : \ r_i = d+1 \implies \sigma_i < \sigma_{i+1}\}$$

parameterize all combinatorial data that can be associated to points in $W_n^{\oplus d}$ and for each $(\sigma, \mathbb{r}) \in C$ let

$$C(\sigma, \mathbb{r}) := \{(x_1, ..., x_n) \in W_n^{\oplus d} : \ x_{\sigma_1} <_{r_1} \cdots <_{r_{n-1}} x_{\sigma_n}\}$$

be the set of all points with combinatorial data $(\sigma, \mathbb{r})$.

The family $\{C(\sigma, \mathbb{r}) : \ (\sigma, \mathbb{r}) \in C\}$ stratifies $W_n^{\oplus d}$. Moreover each $C(\sigma, \mathbb{r})$ is a relatively open cone of dimension

$$d(n-1) - (r_1 - 1) - \cdots - (r_{n-1} - 1),$$

as it is defined by a set of $(r_1 - 1) + \cdots + (r_{n-1} - 1)$ equalities and up to $n-1$ strict inequalities in $W_n^{\oplus d}$. Apart from $C(\mathrm{id}, (d+1, ..., d+1)) = \{\mathbf{0}\}$, for all other $(\sigma, \mathbb{r}) \in C$ the intersection

$$c(\sigma, \mathbb{r}) := S(W_n^{\oplus d}) \cap C(\sigma, \mathbb{r})$$

is homeomorphic to the open disk of dimension $d(n-1) - 1 - (r_1 - 1) - \cdots - (r_{n-1} - 1)$.

Hence, in order to prove that this construction induces a regular CW structure on $S(W_n^{\oplus d})$ it suffices to show that the boundary of each cell is contained in the union of lower dimensional cells. This fact is secured by the following lemma.

Lemma 2.2. *For each $(\sigma, \mathbb{r}) \in C$ the closure of the stratum $C(\sigma, \mathbb{r})$ is a union of strata $C(\tau, \mathbb{s})$, where $(\tau, \mathbb{s}) \in C$ satisfy the following condition. Let $1 \leqslant \mu < \lambda \leqslant n$ be any two indices and let*

$$...\tau_\mu... <_{s_i} ...\tau_\lambda...$$

appear in that order in $(\tau, \mathbb{s})$, where s_i is the minimal such number. Let $\sigma_m = \tau_\mu$ and $\sigma_l = \tau_\lambda$ for some $m, l \in [n]$. Then

- *$...\sigma_m... <_{r_j} ...\sigma_l...$ appear in $(\sigma, \mathbb{r})$ with r_j being the minimal such number and $r_j \leqslant s_i$*

or

- *$...\sigma_l... <_{r_j} ...\sigma_m...$ appear in $(\sigma, \mathbb{r})$ with r_j being the minimal such number and $r_j < s_i$.*

Proof. Each stratum is given by the set of equalities and strict inequalities, so its closure is given by keeping the same set of equalities and replacing the strict by weak inequalities. Therefore, each of the strata described in the statement of the lemma belong to the closure. To prove the opposite, suppose

$$C(\tau, \mathbb{s}) \subseteq \operatorname{cl} C(\sigma, \mathbb{r})$$

and pick $(x_1, ..., x_n) \in C(\tau, \mathbb{s})$. Let $...\tau_\mu... <_{s_i} ...\tau_\lambda...$ appear in that order in $(\tau, \mathbb{s})$ and let $\sigma_m = \tau_\mu$ and $\sigma_l = \tau_\lambda$. Assume moreover that s_i is the smallest such index. Then, the vectors x_{τ_μ} and x_{τ_λ} are equal on the first $s_i - 1$ coordinates and $x_{\tau_\mu, s_i} < x_{\tau_\lambda, s_i}$.

Therefore, if $...\sigma_m... <_{r_j} ...\sigma_l...$ appear in $(\sigma, \mathbb{r})$ and r_j is a minimal such number, it follows that x_{σ_m} and x_{σ_l} are equal the first $r_j - 1$ coordinates and $x_{\sigma_m, r_j} \leqslant x_{\sigma_l, r_j}$. Thus $r_j \leqslant s_i$. On the other hand, if $...\sigma_l... <_{r_j} ...\sigma_m...$ appear in $(\sigma, \mathbb{r})$ and r_j is a minimal such number, then x_{σ_l} and x_{σ_m} are equal the first $r_j - 1$ coordinates and $x_{\sigma_l, r_j} \leqslant x_{\sigma_m, r_j}$. Thus $r_j < s_i$ □

As mentioned before, we will denote the sphere $S(W_n^{\oplus d})$ endowed with such a CW structure by $\mathcal{S}(d, n)$. The $\mathfrak{S}_n$-action on $W_n^{\oplus d}$ is stratum-preserving, since the equality

$$\tau \cdot C(\sigma, \mathbb{r}) = C(\tau\sigma, \mathbb{r})$$

holds for any $\tau \in \mathfrak{S}_n$ and $(\sigma, \mathbb{r}) \in C$. Therefore, $\mathcal{S}(d, n)$ is a regular $\mathfrak{S}_n$-complex.

2.2 The duality argument

By Lemma 2.2, there is a poset structure on the set C given by

$$(\tau, \mathbb{s}) \preceq (\sigma, \mathbb{r}) \quad \overset{\text{def}}{\iff} \quad C(\tau, \mathbb{s}) \subseteq \operatorname{cl} C(\sigma, \mathbb{r}),$$

for any $(\tau, \mathbb{s}), (\sigma, \mathbb{r}) \in C$. Poset C has minimum $\hat{0} := (\mathrm{id}, (d+1, \ldots, d+1))$. We will denote this poset with the minimum omitted by

$$S(d, n) := (C \setminus \{\hat{0}\}, \preceq).$$

It represents the face poset of CW complex $\mathcal{S}(d, n)$ and $\hat{0}$ corresponds to the empty cell. Blagojević & Ziegler [8] denoted by $S(d, n)$ the same poset with minimum included.

Let us denote by

$$S(d, n)^{\mathrm{op}} := (C \setminus \{\hat{0}\}, \preceq^{\mathrm{op}})$$

the *opposite poset of* $S(d, n)$. Notice that for each $(\sigma, \mathbb{r}) \in S(d, n)^{\mathrm{op}}$ we have

$$\text{either } C(\sigma, \mathbb{r}) \subseteq F(\mathbb{R}^d, n) \text{ or } C(\sigma, \mathbb{r}) \cap F(\mathbb{R}^d, n) = \varnothing,$$

and similarly for $G(\mathbb{R}^d, n)$. We define $F(d, n)$ and $G(d, n)$ as

$$\begin{aligned} F(d, n) &:= \{(\sigma, \mathbb{r}) \in S(d, n)^{\mathrm{op}} : C(\sigma, \mathbb{r}) \subseteq F(\mathbb{R}^d, n)\} \\ G(d, n) &:= \{(\sigma, \mathbb{r}) \in S(d, n)^{\mathrm{op}} : C(\sigma, \mathbb{r}) \subseteq G(\mathbb{R}^d, n)\}. \end{aligned} \tag{2}$$

Lemma 2.3. *$F(d, n)$ and $G(d, n)$ are subposets of $S(d, n)^{\mathrm{op}}$.*

Proof. The following characterizations hold.

- $(\sigma, \mathbb{r}) \in F(d, n)$ if and only if $1 \leqslant r_1, \ldots, r_{n-1} \leqslant d$.
- $(\sigma, \mathbb{r}) \in G(d, n)$ if and only if whenever $\sigma_j <_{d+1} \ldots <_{d+1} \sigma_k$ occurs, that is whenever we have $r_j = \ldots = r_{k-1} = d + 1$, it follows that $\sigma_j \leqslant \sigma_k$ are not cyclically consecutive, that is $2 \leqslant \sigma_k - \sigma_j \leqslant n - 2$.

From this and Lemma 2.2 it follows that $F(d, n)$ and $G(d, n)$ are subposets of $S(d, n)^{\mathrm{op}}$, since going down in the poset $S(d, n)^{\mathrm{op}}$ does not change any of the two properties. □

Thus, we are motivated to show that $S(d, n)^{\mathrm{op}}$ is a face poset of sphere.

Proposition 2.4. *Poset $S(d, n)^{op}$ is the face poset of CW complex $\mathcal{S}(d, n)^{op}$ on the sphere $S(W_n^{\oplus d})$.*

Proof. First step is to show that $\mathcal{S}(d, n)$ is PL sphere, meaning it has a subdivision which is isomorphic to a subdivision of a boundary of a simplex. To prove this, one can use the same argument as in the proof of [5, Theorem 2.6]. Indeed, each stratum $C(\sigma, \mathbb{r})$ is a relatively open cone, so it can be made a face of some full-dimensional region for a certain hyperplane arrangement. Let $\mathcal{K}$ be a hyperplane arrangement consisting of all such hyperplanes. Family $\mathcal{K}$ divides $W_n^{\oplus d}$ into relatively open cones which define a CW structure on $S(W_n^{\oplus d})$ which is a subdivision of $\mathcal{S}(d, n)$. To prove that this CW structure on $S(W_n^{\oplus d})$ is PL it is enough to show it is polytopal. This is indeed the case for $\mathcal{S}(d, n)$, since its face poset is isomorphic to the boundary poset of the polar of the zonotope associated to hyperplane arrangement $\mathcal{K}$ [4, Theorem 2.2].

Second step is to apply [4, Theorem 4.7.26 (iv)] by which it follows that the opposite poset of the face poset of a PL sphere is also a face poset of the CW sphere. □

Remark 2.5. From the construction in the proof of [4, Theorem 4.7.26 (iv)] about the opposite poset of the face poset of PL sphere, it follows that the opposite CW structure is exactly the dual block complex of $\mathcal{S}(d,n)$ in the sense of Munkres [27, Chapter 64].

This dual block complex of $\mathcal{S}(d,n)$ is defined by having as cells the duals of cells $c(\sigma,\mathbb{r})$ for each $(\sigma,\mathbb{r}) \in S(d,n)$. For each $(\sigma,\mathbb{r}) \in S(d,n)$, the dual cell of $c(\sigma,\mathbb{r})$ is the geometric realization of the order complex $\Delta(S(d,n)_{\geqslant(\sigma,\mathbb{r})})$, which is a subcomplex of the first barycentic subdivision

$$\operatorname{sd}\mathcal{S}(d,n) = \|\Delta S(d,n)\|.$$

Here $\Delta S(d,n)$ denotes the order complex of $S(d,n)$. As explained in [4, Section 4.7], the fact that the dual block complex is indeed a CW complex is provided by the fact that $\mathcal{S}(d,n)$ is PL. See also Hudson [22, Ch. I Sec. 6].

For a cell $c(\sigma,\mathbb{r}) \in \mathcal{S}(d,n)$ let us denote its dual block cell in $\mathcal{S}(d,n)^{\mathrm{op}}$ by $\check{c}(\sigma,\mathbb{r})$. Let us define

$$\mathcal{F}(d,n) \subseteq \mathcal{S}(d,n)^{\mathrm{op}} \quad \text{and} \quad \mathcal{G}(d,n) \subseteq \mathcal{S}(d,n)^{\mathrm{op}}$$

as the CW subcomplexes of $\mathcal{S}(d,n)^{\mathrm{op}}$ corresponding to subposets $F(d,n)$ and $G(d,n)$ of $S(d,n)^{\mathrm{op}}$ defined in (2). In other words, for each dual cell $\check{c}(\sigma,\mathbb{r}) \subseteq \mathcal{S}(d,n)^{\mathrm{op}}$ we have

$$\check{c}(\sigma,\mathbb{r}) \subseteq \mathcal{G}(d,n) \subseteq \mathcal{S}(d,n)^{\mathrm{op}} \quad \text{if and only if} \quad (\sigma,\mathbb{r}) \in G(d,n) \subseteq S(d,n)^{\mathrm{op}}.$$

The cellular $\mathfrak{S}_n$-action of $\mathcal{S}(d,n)$ defines a cellular action on $\mathcal{S}(d,n)^{\mathrm{op}}$ by the rule

$$\tau \cdot \check{c}(\sigma,\mathbb{r}) = \check{c}(\tau\sigma,\mathbb{r}),$$

for each $\tau \in \mathfrak{S}_n$ and each $(\sigma,\mathbb{r}) \in S(d,n)^{\mathrm{op}}$. Therefore $\mathcal{F}(d,n)$ is a $\mathfrak{S}_n$-invariant and $\mathcal{G}(d,n)$ is D_n-invariant subcomplex of $\mathcal{S}(d,n)^{\mathrm{op}}$.

Theorem 2.6. *For $d \geqslant 1$ and $n \geqslant 2$ the following are true.*

(i) Space $F(\mathbb{R}^d,n)$ contains, as an equivariant strong deformation retract, a finite regular $\mathfrak{S}_n$-CW complex $\mathcal{F}(d,n)$ of dimension $(d-1)(n-1)$.

The face poset of $\mathcal{F}(d,n)$ is $F(d,n) \subseteq S(d,n)^{\mathrm{op}}$, which can be described as follows.

- *Cells $\check{c}(\sigma,\mathbb{r})$ of $\mathcal{F}(d,n)$ are indexed by $(\sigma,\mathbb{r}) \in \mathfrak{S}_n \times [d]^{n-1}$ and the dimension of the cell $\check{c}(\sigma,\mathbb{r})$ is*

$$\dim \check{c}(\sigma,\mathbb{r}) = (r_1-1)+\cdots+(r_{n-1}-1).$$

- *The boundary of a cell* $\check{c}(\tau, \mathbb{s}) \subseteq \mathcal{F}(d,n)$ *consists of all cells* $\check{c}(\sigma, \mathbb{r}) \subseteq \mathcal{F}(d,n)$ *which satisfy the following condition. Let* $1 \leqslant \mu < \lambda \leqslant n$ *be any two indices and let* $...\tau_\mu... <_{s_i} ...\tau_\lambda...$ *appear in that order in* $(\tau, \mathbb{s})$*, where* s_i *is the minimal such number. Let* $\sigma_m = \tau_\mu$ *and* $\sigma_l = \tau_\lambda$ *for some* $m, l \in [n]$*. Then*
  - $...\sigma_m... <_{r_j} ...\sigma_l...$ *appear in* $(\sigma, \mathbb{r})$ *with* r_j *being the minimal such number and* $r_j \leqslant s_i$*, or*
  - $...\sigma_l... <_{r_j} ...\sigma_m...$ *appear in* $(\sigma, \mathbb{r})$ *with* r_j *being the minimal such number and* $r_j < s_i$*.*

The $\mathfrak{S}_n$*-action on* $\mathcal{F}(d,n)$ *is given by*

$$\tau \cdot \check{c}(\sigma, \mathbb{r}) = \check{c}(\tau\sigma, \mathbb{r}), \qquad \tau \in \mathfrak{S}_n,\ (\sigma, \mathbb{r}) \in F(d,n).$$

(ii) Space $G(\mathbb{R}^d, n)$ *contains, as an equivariant strong deformation retract, a finite regular* $\mathfrak{D}_n$*-CW complex* $\mathcal{G}(d,n)$ *of dimension* $d(n-1) - 1 - (n \text{ mod } 2)$*.*

The face poset of $\mathcal{G}(d,n)$ *is* $G(d,n) \subseteq S(d,n)^{\text{op}}$*, which can be described as follows.*

- *Cells* $\check{c}(\sigma, \mathbb{r})$ *of* $\mathcal{G}(d,n)$ *are indexed by the set*

$$\{(\sigma, \mathbb{r}) \in \mathfrak{S}_n \times [d+1]^{n-1} : \ (\forall 1 \leqslant j < k \leqslant n)\ r_j = \cdots = r_{k-1} = d+1 \Rightarrow\ 2 \leqslant \sigma_k - \sigma_j \leqslant n-2\}$$

  and the dimension of the cell $\check{c}(\sigma, \mathbb{r})$ *is* $\dim \check{c}(\sigma, \mathbb{r}) = (r_1 - 1) + \cdots + (r_{n-1} - 1)$*.*
- *The boundary of a cell* $\check{c}(\tau, \mathbb{s}) \subseteq \mathcal{G}(d,n)$ *consists of all cells* $\check{c}(\sigma, \mathbb{r}) \subseteq \mathcal{G}(d,n)$ *which satisfy the following condition. Let* $1 \leqslant \mu < \lambda \leqslant n$ *be any two indices and let* $...\tau_\mu... <_{s_i} ...\tau_\lambda...$ *appear in that order in* $(\tau, \mathbb{s})$*, where* s_i *is the minimal such number. Let* $\sigma_m = \tau_\mu$ *and* $\sigma_l = \tau_\lambda$ *for some* $m, l \in [n]$*. Then*
  - $...\sigma_m... <_{r_j} ...\sigma_l...$ *appear in* $(\sigma, \mathbb{r})$ *with* r_j *being the minimal such number and* $r_j \leqslant s_i$*, or*
  - $...\sigma_l... <_{r_j} ...\sigma_m...$ *appear in* $(\sigma, \mathbb{r})$ *with* r_j *being the minimal such number and* $r_j < s_i$*.*

Complex $\mathcal{F}(d,n)$ *is an* $\mathfrak{D}_n$*-invariant subcomplex of* $\mathcal{G}(d,n)$ *and the action of* $\mathfrak{D}_n$ *on* $\mathcal{G}(d,n)$ *is given by*

$$\tau \cdot \check{c}(\sigma, \mathbb{r}) = \check{c}(\tau\sigma, \mathbb{r}), \qquad \tau \in D_n,\ (\sigma, \mathbb{r}) \in G(d,n).$$

Proof. Statements (i) and (ii) follow by analogous reasoning. Since part (i) is proved in Blagojević & Ziegler [8, Theorem 3.13], we will give the proof for part (ii). The composition (1) from the beginning of the section restricts to an equivariant strong deformation retraction from $G(\mathbb{R}^d, n)$ to $\overline{G}(\mathbb{R}^d, n) := G(\mathbb{R}^d, n) \cap S(W_n^{\oplus d})$. Let $\mathcal{H} := \{H_j\}_{1 \leqslant j \leqslant n}$ be a subspace arrangement in $(\mathbb{R}^d)^{\times n}$ defined by

$$H_j := \{(x_1, ..., x_n) \in (\mathbb{R}^d)^{\times n} : x_j = x_{j+1}\}, \qquad 1 \leqslant j \leqslant n,$$

where $x_{n+1} := x_1$. Space $G(\mathbb{R}^d, n)$ is the complement of subspace arrangement $\mathcal{H}$. Let

$$\mathcal{H}(d, n) := S(W_n^{\oplus d}) \cap \bigcup \mathcal{H}.$$

It is a subcomplex of $\mathcal{S}(d, n)$ since it is a union of closed strata not contained in $G(\mathbb{R}^d, n)$, which thus has

$$H(d, n) := S(d, n) \setminus G(d, n) \subseteq S(d, n)$$

as its face poset. To finish the proof, it is enough to show there exists an equivariant deformation retraction from $\overline{G}(\mathbb{R}^d, n) = \mathcal{S}(d, n) \setminus \mathcal{H}(d, n)$ to the subcomplex $\mathcal{G}(d, n) \subseteq \mathcal{S}(d, n)^{\mathrm{op}}$.

As explained in the proof of Proposition 2.4, the CW structure $\mathcal{S}(d, n)^{\mathrm{op}}$ is the dual block structure of $\mathcal{S}(d, n)$ (see Munkres [27, Chapter 64]). Thus $\mathcal{S}(d, n)$ and $\mathcal{S}(d, n)^{\mathrm{op}}$ have the same first barycentric subdivision $\mathrm{sd}\,\mathcal{S}(d, n)$. Moreover, the first barycentric subdivisions $\mathrm{sd}\,\mathcal{H}(d, n)$ and $\mathrm{sd}\,\mathcal{G}(d, n)$ are subcomplexes of $\mathrm{sd}\,\mathcal{S}(d, n)$ on a disjoint set of vertices which together form the set of vertices of $\mathrm{sd}\,\mathcal{S}(d, n)$. Thus, there is a strong deformation retraction

$$\mathcal{S}(d, n) \setminus \mathcal{H}(d, n) \longrightarrow \mathcal{G}(d, n)$$

induced by the map which maps each simplex of $\mathrm{sd}\,\mathcal{S}(d, n)$ which is not totally contained in $\mathrm{sd}\,\mathcal{H}(d, n)$ to its intersection with $\mathrm{sd}\,\mathcal{G}(d, n)$ by projection. This maps is also D_n-equivariant. $\square$

3 Fundamental group

The aim of this section is to compute the fundamental group of the Billiard Configuration Space $G(\mathbb{R}^d, n)$.

In the case $d = 1$ notice that $G(\mathbb{R}, n)$ is the complement $\mathbb{R}^n \setminus (H_1 \cup \cdots \cup H_n)$ of union of hyperplanes of the form

$$H_i := \{(x_1, ..., x_n) \in \mathbb{R}^n : \ x_i = x_{i+1}\},$$

for $1 \leqslant i \leqslant n$. This complement is a disjoint union of 2^n open cones and hence has a trivial fundamental group for any choice of base point. Thus, we can focus on the case $d \geqslant 2$. As the next proposition shows, $d = 2$ is the most interesting one.

Proposition 3.1. *Let $n \geqslant 1$ and $d \geqslant 3$ be integers. Then $G(\mathbb{R}^d, n)$ is simply connected.*

Proof. We will use the CW models $\mathcal{F}(d, n)$ and $\mathcal{G}(d, n)$ from Theorem 2.6 because it illustrates the idea we use for $d = 2$ case, even though the claim also follows from simpler considerations. From the work of Fadell and Neuwirth [13] we know that $F(\mathbb{R}^d, n)$ is simply connected for $d \geqslant 3$ (see also [24, Cor. 2.3.4]). Since $\mathcal{F}(d, n) \subseteq \mathcal{G}(d, n)$ is a subcomplex with the full 1-skeleton, $\mathcal{G}(d, n)$ is simply connected as well, due to [21, Ch. 1 Prop. 1.26]. □

In order to provide a presentation of $\pi_1(G(\mathbb{R}^2, n))$, we will first need to recall the Pure braid group presentation of $\pi_1(F(\mathbb{R}^2, n))$ due to Artin [2] and connect it to the CW structure on $\mathcal{F}(2, n)$.

3.1 Configuration space case

Since the fundamental group of a CW complex is naturally isomorphic to the fundamental group of its 2-skeleton, it will be useful to describe the cells and the boundaries cells in the 2-skeleton.

For each $1 \leqslant k \leqslant n - 1$ let I_k denote the set of k-cells of the $(n - 1)$-dimensional cell complex $\mathcal{F}(2, n)$.

- The set of 0-cells is
$$I_0 = \{\check{c}(\sigma, \mathbb{1}) : \ \sigma \in \mathfrak{S}_n\},$$
where $\mathbb{1} := (1, ..., 1) \in [d + 1]^{n-1}$. The base point is set to be $\check{c}(\text{id}, \mathbb{1})$.

- The set of 1-cells is
$$I_1 = \{\check{c}(\sigma, \mathbb{1}_i) : \ \sigma \in \mathfrak{S}_n, \ 1 \leqslant i \leqslant n - 1\},$$
where $\mathbb{1}_i := (1, ..., 2, ..., 1) \in [d + 1]^{n-1}$ has 2 only at position $i \in [n - 1]$.

- The set of 2-cells is

$$I_2 = \{\check{c}(\sigma, \mathbb{1}_{i,j}) : \sigma \in \mathfrak{S}_n, \ 1 \leqslant i < j \leqslant n-1\},$$

  where $\mathbb{1}_{i,j} := (1, ..., 2, ..., 2, ..., 1) \in [d+1]^{n-1}$ has 2 only at (distinct) positions $i, j \in [n-1]$.

For each $1 \leqslant i \leqslant n-1$ let $\tau_i := (i, i+1) \in \mathfrak{S}_n$ denote the transposition. Due to Lemma 2.2 we have the following.

- The boundary of any 1-cell $\check{c}(\sigma, \mathbb{1}_i) \in I_1$, where $1 \leqslant i \leqslant n-1$, consists of vertices $\check{c}(\sigma, \mathbb{1})$ and $\check{c}(\sigma\tau_i, \mathbb{1})$.
- The boundary of any 2-cell $\check{c}(\sigma, \mathbb{1}_{i,j}) \in I_2$, where $1 \leqslant i < j \leqslant n-1$, consists of
  - closures of the six 1-cells

$$\check{c}(\sigma, \mathbb{1}_{i+1}), \ \check{c}(\sigma\tau_{i+1}, \mathbb{1}_i), \ \check{c}(\sigma\tau_{i+1}\tau_i, \mathbb{1}_{i+1}), \ \check{c}(\sigma\tau_i\tau_{i+1}, \mathbb{1}_i), \ \check{c}(\sigma\tau_i, \mathbb{1}_{i+1}), \ \check{c}(\sigma, \mathbb{1}_i)$$

    in the case $j - i = 1$, or
  - closures of the four 1-cells

$$\check{c}(\sigma, \mathbb{1}_j), \ \check{c}(\sigma\tau_j, \mathbb{1}_i), \ \check{c}(\sigma\tau_i, \mathbb{1}_j), \ \check{c}(\sigma, \mathbb{1}_i)$$

    in the case $j - i \geqslant 2$.

We can orient the 1-cells $\check{c}(\sigma, \mathbb{1}_i) \in I_1$ to go from the vertex $\check{c}(\sigma, \mathbb{1})$ to the vertex $\check{c}(\sigma\tau_i, \mathbb{1})$. Thus, with such orientation in place, we can consider each one cell $\check{v}(\sigma, \mathbb{1}_i)$ as a path in $\mathcal{F}(2, n)$.

Let us denote by

$$B(\mathbb{R}^d, n) := F(\mathbb{R}^d, n)/\mathfrak{S}_n$$

the *unordered configuration space* and by $p : F(\mathbb{R}^d, n) \to B(\mathbb{R}^d, n)$ the quotient map, which is a covering map since the group is finite and the action is free. Thus

$$\mathcal{B}(d, n) := \mathcal{F}(d, n)/\mathfrak{S}_n$$

is the homotopically equivalent CW model for $B(\mathbb{R}^d, n)$ which has the quotient CW structure with the unique 0-cell which is also the base point, namely the orbit of $\check{c}(\mathrm{id}, \mathbb{1})$. In what follows we again specify to the case $d = 2$.

Definition 3.2. The *braid group* on n strands is the group

$$B_n := \langle \tau_1, ..., \tau_{n-1} | \ \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}, \ \tau_i\tau_j = \tau_j\tau_i \text{ for } |i-j| \geqslant 2\rangle. \tag{3}$$

Remark 3.3. Artin [2] introduced the notion of *geometric braid*, as a collection of n disjoint arcs $(\gamma_1, ..., \gamma_n)$ connecting two collections of n pairwise distinct points $(x_1, \dots, x_n)$ and $(y_1, ..., y_n)$ placed on the planes $z = 0$ and $z = 1$ in $\mathbb{R}^3$, respectively. In addition, for each $i \in [n]$ and each $t \in [0, 1]$ the point $\gamma_i(t)$ belongs to the plane $z = t$. Concatenation of braids and the shrinking procedure applied to the representatives of homotopy classes of braids nicely fit with the homotopy relation and as such are a well defined group operation on the space of homotopy classes of braids. In fact, he showed that this group has the presentation (3). Generator $\tau_i \in B_n$ and its inverse are depicted in Figure 3.1.

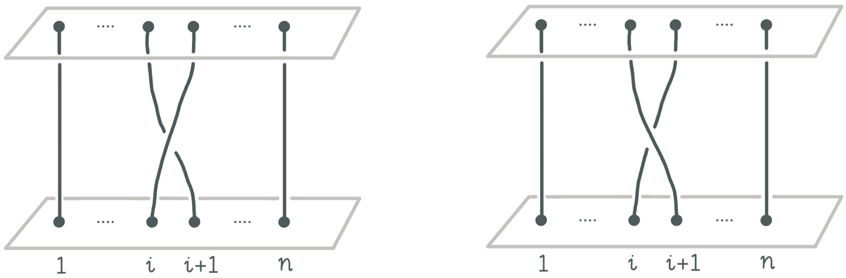


Figure 3.1: Generator $\tau_i \in B_n$ and its inverse τ_i^{-1} as a geometric braid.

The following proposition, due to Artin [2], allows us not to distinguish between B_n and the fundamental group of $B(\mathbb{R}^2, n)$.

Proposition 3.4. *Let $n \geqslant 1$ be an integer. Then there is a group isomorphism*

$$\pi_1(B(\mathbb{R}^2, n)) \xrightarrow{\cong} B_n$$

which maps the orbits of 1-cells $\check{c}(id, \mathbb{1}_i)$ to generators τ_i for each $1 \leqslant i \leqslant n - 1$.

Proof. Fundamental group of a CW complex $\mathcal{B}(2, n)$ is the same as the fundamental group of its 2-skeleton, which is the quotient of the 2-skeleton of $\mathcal{F}(2, n)$. The 0-skeleton of $\mathcal{F}(2, n)$ consists of a single $\mathfrak{S}_n$-orbit, which corresponds to the base point on the quotient, and 1-cells

$$\check{c}(\sigma, \mathbb{1}_i) = \sigma \cdot \check{c}(\mathrm{id}, \mathbb{1}_i)$$

get mapped to loops at the base point of the quotient. Hence, the 1-skeleton of $\mathcal{B}(2, n)$ is the wedge of $n - 1$ circles. Due to [21, Ch. 1 Prop. 1.26], the fundamental group $\pi_1(\mathcal{B}(2, n), \mathrm{pt})$ is isomorphic to the group generated by those loops, subject to relations generated by the boundaries of the 2-cells. The 2-cells in $\mathcal{B}(2, n)$ are the images under the quotient map of 2-cells

$$\check{c}(\sigma, \mathbb{1}_{i,j}) = \sigma \cdot \check{c}(\mathrm{id}, \mathbb{1}_{i,j}),$$

for each $1 \leqslant i < j \leqslant n-1$. The two types of relations in the definition of the braid group B_n correspond exactly the boundary relations from the orbit of the 2-cell $\check{c}(\mathrm{id}, \mathbb{1}_{i,j})$ for $j-i=1$ and $j-i \geqslant 2$, respectively. Thus, the isomorphism follows. □

Lemma 3.5. (Björner & Brenti [3, Example 1.2.3]) *The symmetric group $\mathfrak{S}_n$ has a presentation*

$$\langle \tau_1, ..., \tau_{n-1} | \; \tau_i^2 = 1, \; \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \; \tau_i \tau_j = \tau_j \tau_i \textit{ for } |i-j| \geqslant 2 \rangle \xrightarrow{\cong} \mathfrak{S}_n,$$

with τ_i corresponding to the transposition $(i, i+1) \in \mathfrak{S}_n$ for each $1 \leqslant i \leqslant n-1$.

Proof. Denote by S_n the group on the left hand side. Transpositions generate $\mathfrak{S}_n$ and satisfy relations in the presentation of S_n so the morphism is well defined and surjective. We prove injectivity by inductively showing that S_n is a finite group with $|S_n| \leqslant n!$.

Let $H := \langle \tau_1, ..., \tau_{n-2} \rangle \subseteq S_n$ be a subgroup. Since the group morphism

$$S_{n-1} \longrightarrow H, \quad \tau_i \mapsto \tau_i,$$

is an epimorphism, by the induction hypothesis it follows that

$$|H| \leqslant |S_{n-1}| \leqslant (n-1)!.$$

Due to the relations in S_n, the set cosets S_n/H has exactly n elements, namely

$$S_n/H = \{\tau_1 \cdots \tau_{n-1} H, \; \tau_2 \cdots \tau_{n-1} H, ..., \; \tau_{n-1} H, \; H\}. \tag{4}$$

Therefore S_n is finite with $|S_n| \leqslant n|H| \leqslant n!$. □

In particular, from (4) it follows that for any permutation $\sigma \in \mathfrak{S}_n$ we can fix its presentation in basis $\{\tau_1, ..., \tau_{n-1}\}$, namely

$$\sigma = (\tau_{k_{n-1}} ... \tau_{n-1}) \cdot (\tau_{k_{n-2}} ... \tau_{n-2}) \cdot ... \cdot (\tau_{k_1} ... \tau_1), \tag{5}$$

where $1 \leqslant k_i \leqslant i+1$ for each $1 \leqslant i \leqslant n-1$. Here for every $1 \leqslant i \leqslant n-1$ in each bracket of the form $\tau_{k_i} \cdot ... \cdot \tau_i$ we assume the ascending order of indices from left to right. In particular, if $k_i = i+1$ the product is equal to the identity permutation.

Definition 3.6. The *pure braid group* P_n on n strands is the kernel of the group morphism

$$\phi : B_n \longrightarrow \mathfrak{S}_n, \quad \tau_i \longmapsto \tau_i.$$

The presentation of $\mathfrak{S}_n$ from Lemma 3.5 allows us to see that the morphism ϕ is well defined.

Remark 3.7. We can geometrically interpret ϕ as follows. Each based loop in $\mathcal{B}(2,n)$ is path-homotopic to some based loop γ in its 1-skeleton. Thus, γ lifts to a unique path $\tilde{\gamma}$ on the 1-skeleton of $\mathcal{F}(2,n)$ starting at the base point $\check{c}(\text{id},\mathbb{1}) \in \mathcal{F}(2,n)$. That is, the lift $\tilde{\gamma}$ is a concatenation of (oriented) 1-cells of $\mathcal{F}(2,n)$ and their inverses. Let the endpoint of $\tilde{\gamma}$ be the 0-cell $\check{c}(\sigma,\mathbb{1}) \in \mathcal{F}(2,n)$. Then, $\phi([\gamma]) = \sigma$. Indeed, any (oriented) 1-cell $\check{c}(\pi,\mathbb{1}_i)$ of $\mathcal{F}(2,n)$ starts at $\check{c}(\pi,\mathbb{1})$ and ends at $\check{c}(\pi\tau_i,\mathbb{1})$. Moreover, the orbit of such a 1-cell $\check{c}(\pi,\mathbb{1}_i)$ is a loop in $\mathcal{B}(2,n)$ corresponding to the generator $\tau_i \in B_n$.

From the description in the remark above we see that the composition

$$\pi_1(F(\mathbb{R}^2,n)) \xrightarrow{p_*} \pi_1(B(\mathbb{R}^2,n)) \xrightarrow{\phi} \mathfrak{S}_n$$

is trivial, since loops in the 1-skeleton of $\mathcal{F}(2,n)$ start and end at the base point $\check{c}(\text{id},\mathbb{1})$. Thus, morphism

$$p_* : \pi_1(F(\mathbb{R}^2,n)) \to \pi_1(B(\mathbb{R}^2,n))$$

factors through the pure braid group P_n. Next, we arrive at the presentation of $\pi_1(F(\mathbb{R}^2,n))$.

Proposition 3.8. *The fundamental group of the Configuration Space $F(\mathbb{R}^2,n)$ is a subgroup of the braid group $\pi_1(B(\mathbb{R}^2,n))$ generated by the elements*

$$a_{i,j} := a_{j,i} := (\tau_{j-1}...\tau_{i+1}) \cdot \tau_i^2 \cdot (\tau_{j-1}...\tau_{i+1})^{-1}, \qquad 1 \leqslant i < j \leqslant n.$$

For any $1 \leqslant i < j \leqslant n$ the generator $a_{i,j}$ corresponds to the loop at the vertex $\check{c}(id,\mathbb{1})$ obtained by concatenating oriented 1-cells

$$\check{c}(\tau_{j-1}...\tau_{k+1},\mathbb{1}_k), \qquad k = j-1,...,i$$

the 1-cell $\check{c}(\tau_{j-1}...\tau_i,\mathbb{1}_i)$, and the inverses

$$\check{c}(\tau_{j-1}...\tau_k,\mathbb{1}_k)^{-1}, \qquad k = i+1,...,j-1.$$

Here both products $\tau_{j-1}\cdot...\cdot\tau_{k+1}$ and $\tau_{j-1}\cdot...\cdot\tau_k$ have descending indices from left to right. Moreover, $\pi_1(F(\mathbb{R}^2,n))$ has the following presentation in these generators

$$\begin{aligned}\langle\{a_{ij}\}_{1\leqslant i<j\leqslant n} \mid\ & a_{ij}a_{rs} = a_{rs}a_{ij} && r<s<i<j \text{ or } i<r<s<j\\ & a_{ji}a_{ir}a_{rj} = a_{ir}a_{rj}a_{ji} = a_{rj}a_{ji}a_{ir} && r<i<j\\ & a_{rs}(a_{jr}a_{ji}a_{js}) = (a_{jr}a_{ji}a_{js})a_{rs} && r<i<s<j\rangle. \end{aligned} \qquad (6)$$

Proof. As noted above, we have $\pi_1(F(\mathbb{R}^2,n)) \subseteq P_n \subseteq B_n$. Indices of both subgroups inside B_n are $|\mathfrak{S}_n|$, so the subgroups must be equal. In particular, there is a short exact sequence

$$1 \longrightarrow \pi_1(F(\mathbb{R}^2,n)) \xrightarrow{p_*} \pi_1(B(\mathbb{R}^2,n)) \xrightarrow{\phi} \mathfrak{S}_n \longrightarrow 1.$$

Artin [2] was the first to obtain the presentation on the pure braid group P_n with generators $a_{i,j}$. Later Lee [25, Theorem 1.1] obtained the presentation stated above, which was different from Artin's in the sense that it was positive, i.e., that the relations only included the generators a_{ij} and not their inverses.

Since $\pi_1(B(\mathbb{R}^2, n))$ is generated by the orbits of the 1-cells $\{\check{c}(\text{id}, \mathbb{1}_i) : 1 \leqslant i \leqslant n-1\}$, it follows that the generators $a_{i,j}$ of $\pi_1(F(\mathbb{R}^2, n))$ are the concatenations of 1-cells as described in the statement of the theorem. ☐

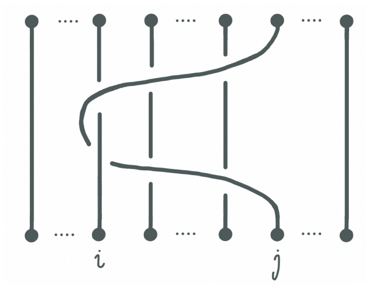


Figure 3.2: Generator $a_{i,j} \in P_n$ as a geometric braid.

3.2 Billiard configuration space case

For $n \leqslant 3$ we have $G(\mathbb{R}^d, n) = F(\mathbb{R}^d, n)$. Thus, in the case $d = 2$ we have the following.

Proposition 3.9. *Let $n \leqslant 3$. Then we have $\pi_1(G(\mathbb{R}^2, n)) = \pi_1(F(\mathbb{R}^2, n))$.*

From now on, we will assume $n \geqslant 4$. Similarly as above, fundamental group of $G(\mathbb{R}^2, n)$ is isomorphic to that of $\mathcal{G}(2, n)$, and also that of its 2-skeleton. Let J_k be the set of the k-cells of $\mathcal{G}(2, n)$. Then

- $J_0 = I_0$ with the base point $\check{c}(\text{id}, \mathbb{1})$, $J_1 = I_1$, and
- $J_2 = I_2 \cup J_2'$, where

$$J_2' := \{\check{c}(\sigma, \mathbb{1}_{i,i}) : \sigma \in \mathfrak{S}_n, \ i \in [n-1] \text{ such that } 1 < \sigma_{i+1} - \sigma_i < n-1\},$$

where $\mathbb{1}_{i,i} := (1, \ldots, 3, \ldots, 1) \in [d+1]^{n-1}$ has 3 only at position $i \in [n-1]$.

The boundary relations of cells in J_1 and I_2 are the same as in the complex $\mathcal{F}(2, n)$, while the boundary of any cell $\check{c}(\sigma, \mathbb{1}_{i,i}) \in J_2'$ is the closure of the two 1-cells

$$\check{c}(\sigma, \mathbb{1}_i) \quad \text{and} \quad \check{c}(\sigma\tau_i, \mathbb{1}_i),$$

for each $1 \leqslant i \leqslant n-1$. For each 2-cell $\check{c}(\sigma, \mathbb{1}_{i,i}) \in J_2'$, let p_σ denote a fixed path from the base point $\check{c}(\mathrm{id}, \mathbb{1})$ to point $\check{c}(\sigma, \mathbb{1})$ on the boundary of the 2-cell $\check{c}(\sigma, \mathbb{1}_{i,i})$. For example, p_σ can be chosen to be the path given by the presentation (5) of $\sigma \in \mathfrak{S}_n$, where 1-cells $\check{c}(-, \mathbb{1}_i)$ are concatenated in the same order as corresponding generators τ_i in presentation of σ, and instead of a dash, there is the unique permutation such that the 1-cells can be concatenated and such that the path starts at $\check{c}(\mathrm{id}, \mathbb{1})$. This choice of a path p_σ associates a loop

$$p_\sigma \cdot \check{c}(\sigma, \mathbb{1}_i) \cdot \check{c}(\sigma\tau_i, \mathbb{1}_i) \cdot p_\sigma^{-1} \tag{7}$$

at the base point $\check{c}(\mathrm{id}, \mathbb{1})$ to the added 2-cell $\check{c}(\sigma, \mathbb{1}_{i,i}) \in J_2'$.

For two elements $a, b \in P_n$ we will write $a \sim b$ if one is a conjugate of the other, that is if $a = cbc^{-1}$ for some $c \in P_n$. Before we proceed to compute $\pi_1(G(\mathbb{R}^2, n))$, it will be useful to have the following results at our disposal.

Lemma 3.10. *For any generator $a_{i,j} \in P_n$, where $1 \leqslant i < j \leqslant n$, and for any $1 \leqslant k \leqslant n-1$ we have the following conjugacy in P_n,*

$$\tau_k \cdot a_{i,j} \cdot \tau_k^{-1} \sim a_{\tau_k(i), \tau_k(j)}.$$

Proof. We split the proof into cases according to how τ_k acts on i and j.

1. If τ_k acts only on i, we have two possibilities. Either $k = i-1$, or $k = i$ and $j > i+1$. In the first case of $k = i$ and $j > i+1$ we have

   $$(\tau_i\tau_{i+1}) \cdot \tau_i^2 \cdot (\tau_i\tau_{i+1})^{-1} = \tau_{i+1}^2,$$

   hence

   $$\tau_i \cdot a_{i,j} \cdot \tau_i^{-1} = (\tau_{j-1}...\tau_{i+2}) \cdot (\tau_i\tau_{i+1}) \cdot \tau_i^2 \cdot (\tau_i\tau_{i+1})^{-1} \cdot (\tau_{j-1}...\tau_{i+2})^{-1} = a_{i+1,j}.$$

   This equality is depicted in Figure 3.3.
   In the second case of $k = i-1$ we have, similarly as above

   $$\tau_{i-1}^{-1} \cdot \tau_i^2 \cdot \tau_{i-1} = \tau_i \cdot \tau_{i-1}^2 \cdot \tau_i^{-1}.$$

   It follows that

   $$\tau_{i-1}^{-1} \cdot a_{i,j} \cdot \tau_{i-1} = (\tau_{j-1}...\tau_{i+1}) \cdot \tau_{i-1}^{-1} \cdot \tau_i^2 \cdot \tau_{i-1} \cdot (\tau_{j-1}...\tau_{i+1})^{-1} = a_{i-1,j}$$

   and therefore

   $$\tau_{i-1} \cdot a_{i,j} \cdot \tau_{i-1}^{-1} = a_{i,i+1} \cdot (\tau_{i-1}^{-1} \cdot a_{i,j} \cdot \tau_{i-1}) \cdot a_{i,i+1}^{-1} \sim a_{i-1,j}.$$

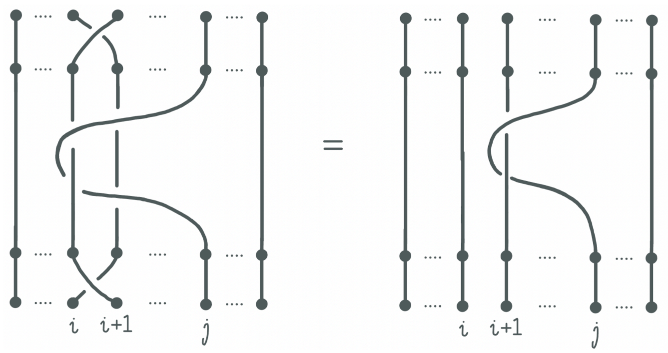


Figure 3.3: Equality $\tau_i \cdot a_{i,j} \cdot \tau_i^{-1} = a_{i+1,j}$ for $j > i+1$.

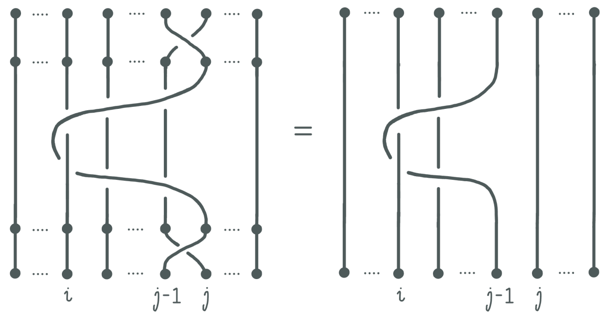


Figure 3.4: Equality $\tau_{j-1}^{-1} \cdot a_{i,j} \cdot \tau_{j-1} = a_{i,j-1}$ for $i < j-1$.

2. If τ_k acts only on j we again have two possibilities. Either $k = j$, or $k = j-1$ and $i < j-1$. In the case $k = j$ we immediately have $\tau_j \cdot a_{i,j} \cdot \tau_j^{-1} = a_{i,j+1}$. In the case $k = j-1$ and $i < j-1$ we immediately have

$$\tau_{j-1}^{-1} \cdot a_{i,j} \cdot \tau_{j-1} = a_{i,j-1}.$$

This equality is depicted in Figure 3.4.
Hence it follows

$$\tau_{j-1} \cdot a_{i,j} \cdot \tau_{j-1}^{-1} = a_{j-1} \cdot (\tau_{j-1}^{-1} \cdot a_{i,j} \cdot \tau_{j-1}) \cdot a_{j-1}^{-1} \sim a_{i,j-1}.$$

3. If $k < i-1$ or $k > j$ then τ_k commutes with

$$a_{i,j} = (\tau_{j-1} ... \tau_{i+1}) \cdot \tau_i^2 \cdot (\tau_{j-1} ... \tau_{i+1})^{-1}$$

and τ_k does not act on i and j, so the claim holds. If $i < k < j-1$ (hence $j - i > 2$), then τ_k does not act on i and j as well. Therefore

$$\begin{aligned}\tau_k(\tau_{j-1} ... \tau_{i+1}) &= (\tau_{j-1} ... \tau_{k+2}) \cdot \tau_k \cdot (\tau_{k+1} ... \tau_{i+1}) = (\tau_{j-1} ... \tau_{k+2}) \cdot (\tau_k \tau_{k+1} \tau_k) \cdot (\tau_{k-1} ... \tau_{i+1}) \\ &= (\tau_{j-1} ... \tau_{k+2}) \cdot (\tau_{k+1} \tau_k \tau_{k+1}) \cdot (\tau_{k-1} ... \tau_{i+1}) = (\tau_{j-1} ... \tau_{i+1}) \tau_{k+1}.\end{aligned}$$

Therefore, we have

$$\tau_k \cdot a_{i,j} \cdot \tau_k^{-1} = (\tau_{j-1} \ldots \tau_{i+1}) \cdot \tau_{k+1} \cdot \tau_i^2 \cdot \tau_{k+1}^{-1} \cdot (\tau_{j-1} \ldots \tau_{i+1})^{-1} = a_{i,j}.$$

4. If τ_k acts on both i and j, then $i = k = j - 1$ and τ_k flips i and $i + 1$. Thus

$$\tau_i \cdot a_{i,i+1} \cdot \tau_i^{-1} = \tau_i \cdot \tau_i^2 \cdot \tau_i^{-1} = a_{i+1,i}$$

With these four cases, the lemma is proven. □

Corollary 3.11. *For any $\sigma \in \mathfrak{S}_n$ and any $1 \leqslant i \leqslant n-1$ we have the following conjugacy in P_n,*

$$\sigma \cdot \tau_i^2 \cdot \sigma^{-1} \sim a_{\sigma(i),\sigma(i+1)},$$

where we abuse the notation on the left hand side and denote by σ the element in B_n with presentation (5).

Proof. Let us say that a permutation is of length k if it is composed of k elements in the presentation (5). We prove the claim by induction on the length of a permutation.

Assume the statement holds for all permutations of length not greater than k, with $k \geqslant 0$, and assume $\sigma \in \mathfrak{S}_n$ is of length $k+1$. Then $\sigma = \tau_m \pi$, for $1 \leqslant m \leqslant n-1$ and $\pi \in \mathfrak{S}_n$ of length at most k. By the induction hypothesis we have

$$\pi \cdot \tau_i^2 \cdot \pi^{-1} = b \cdot a_{\pi(i),\pi(i+1)} \cdot b^{-1}$$

for some $b \in P_n$. Hence by applying Lemma 3.10 in the last step we conclude

$$\begin{aligned}\sigma \cdot \tau_i^2 \cdot \sigma^{-1} &= \tau_m(\pi \cdot \tau_i^2 \cdot \pi^{-1})\tau_m^{-1} = \tau_m(b \cdot a_{\pi(i),\pi(i+1)} \cdot b^{-1})\tau_m^{-1} \\ &= (\tau_m b \tau_m^{-1}) \cdot (\tau_m \cdot a_{\pi(i),\pi(i+1)} \cdot \tau_m^{-1}) \cdot (\tau_m b \tau_m^{-1})^{-1} \\ &\sim \tau_m \cdot a_{\pi(i),\pi(i+1)} \cdot \tau_m^{-1} \sim a_{\sigma(i),\sigma(i+1)}\end{aligned}$$

as desired. □

Proposition 3.12. *Let $n \geqslant 4$ be an integer. Then the fundamental group of $G(\mathbb{R}^2, n)$ is a free abelian group on n generators. The generators can be taken to be the push-forwards of the generators $a_{i,i+1} \in \pi_1(F(\mathbb{R}^2, n))$ in* (6)*, for $1 \leqslant i \leqslant n$, via the inclusion* $\mathrm{inc} : F(\mathbb{R}^2, n) \hookrightarrow G(\mathbb{R}^2, n)$.

Proof. The 2-skeleton of $\mathcal{G}(2, n)$ is obtained from the 2-skeleton of $\mathcal{F}(2, n)$ by attaching only the 2-cells in J_2'. Therefore, by [21, Ch. 1 Prop. 1.26] it follows that the inclusion

$$\mathrm{inc} : \mathcal{F}(2, n) \to \mathcal{G}(2, n)$$

induces a short exact sequence

$$1 \longrightarrow N \longrightarrow \pi_1(\mathcal{F}(2, n)) \xrightarrow{\mathrm{inc}_*} \pi_1(\mathcal{G}(2, n)) \longrightarrow 1, \tag{8}$$

where $N := \ker \mathrm{inc}_*$ is the normal closure of a subgroup of $\pi_1(\mathcal{F}(2,n))$ generated by loops (7) associated to added 2-cells. In more detail, due to the correspondence of 1-cells $\check{c}(\sigma, \mathbb{1}_i)$ of $\mathcal{F}(2,n)$ and the classes of loops $\tau_i \in B_n$, we have

$$N = \mathrm{ncl}_{P_n}\{\sigma\tau_i^2\sigma^{-1} : \ \sigma \in \mathfrak{S}_n, \ 1 \leqslant i \leqslant n-1 \text{ such that } 1 < \sigma_{i+1} - \sigma_i < n-1\}.$$

Here we again abused notation and denote by σ also the braid with presentation (5). Thus, $\pi_1(G(\mathbb{R}^2, n))$ has presentation which is the same as (6) but with additional relation $\sigma\tau_i^2\sigma^{-1} = 1$ for each $\sigma \in \mathfrak{S}_n$ and $1 \leqslant i \leqslant n-1$ for which $1 < \sigma_{i+1} - \sigma_i < n-1$.

Conjugation in P_n of generating elements will not change the normal closure N, and thus by Corollary 3.11 it follows that the added relations are equivalent to $a_{\sigma(i),\sigma(i+1)} = 1$ for each $\sigma \in \mathfrak{S}_n$ and $1 \leqslant i \leqslant n-1$ such that $1 < \sigma_{i+1} - \sigma_i < n-1$. This is further equivalent to $a_{i,j} = 1$ for each $1 \leqslant i < j \leqslant n-1$ with $1 < j-i < n-1$. Thus, the desired presentation follows, since the relations in (6) now amount to elements $a_{1,2}, a_{2,3}, \ldots, a_{n,1}$ commuting. □

4 Second CW model

In this section we will construct another CW model $\mathcal{G}_{d,n}$ for Billiard Configuration Space, again in the spirit of the Salvetti complex [28]. See also Björner & Ziegler [5] and Blagojević & Ziegler [8]. This model will also be a strong deformation retract of $G(\mathbb{R}^d, n)$, but with an important distinction from model $\mathcal{G}(d,n)$ from Section 2. Namely, it will be $(n-1)(d-1)$-dimensional, which is not greater than

$$\dim \mathcal{G}(d,n) = d(n-1) - 1 - (n \text{ mod } 2).$$

This will be particularly useful for calculating topological complexity of $G(\mathbb{R}^d, n)$, since the dimension of CW model used there is important.

Similar to the procedure in Section 2, we first retract $G(\mathbb{R}^d, n)$ to the subspace

$$\overline{G}(\mathbb{R}^d, n) = G(\mathbb{R}^d, n) \cap S(W_n^{\oplus d})$$

via map (1), namely

$$(\mathbb{R}^d)^{\times n} \backslash D \longrightarrow W_n^{\oplus d} \backslash \{0\} \longrightarrow S(W_n^{\oplus d}).$$

The next goal is to stratify the space $W_n^{\oplus d}$ into cones.

To each $(x_1, \ldots, x_n) \in W_n^{\oplus d}$ we assign combinatorial data

$$(\mathbb{r}, \mathbb{s}) \in [d+1]^n \times \{+, -, 0\}^n$$

in the following way. For each $1 \leqslant i \leqslant n$ we are comparing vectors x_i and x_{i+1}, where we set $x_{n+1} := x_1$. They are either equal, in which case we set

$$r_i := d+1 \ \text{ and } \ s_i := 0,$$

or there is a unique $1 \leqslant r \leqslant d$ such that $x_{i,j} = x_{i+1,j}$ for all $1 \leqslant j \leqslant r-1$ and $x_{i+1,r} \neq x_{i,r}$, in which case we set

$$r_i := r \ \text{ and } \ s_i := \text{sign}(x_{i,r} - x_{i+1,r}).$$

More formally, this procedure gives us a function

$$f : W_n^{\oplus d} \longrightarrow [d+1]^n \times \{+, -, 0\}^n$$

assigning to each point its combinatorial data.

Definition 4.1. Let

$$C' := f(W_n^{\oplus d}) \subseteq [d+1]^n \times \{+, -, 0\}^n$$

denote the set of combinatorial data of points in $W_n^{\oplus d}$ and let

$$C'(\mathbb{r}, \mathbb{s}) := f^{-1}((\mathbb{r}, \mathbb{s})) = \{x \in W_n^{\oplus d} : \ (\mathbb{r}, \mathbb{s}) \text{ is the combinatorial data assoc. to } x\}.$$

for each $(\mathbb{r}, \mathbb{s}) \in C'$.

In more detail, the set C' of all possible combinatorial data of points in $W_n^{\oplus d}$ can be described as follows. Assume $(\mathbb{r}, \mathbb{s}) \in [d+1]^n \times \{0, +, -\}^n$ is in C'.

- By definition, we have
$$r_i = d+1 \iff s_i = 0$$
for all $1 \leqslant i \leqslant n$.
- If $J := \{j : r_j = \min_i r_i\}$ denotes the set of all indices for which the minimum of $\mathbb{r}$ is obtained and if we denote this minimum by $r := \min_i r_i < d+1$, then we cannot have that the set of signs $\{s_j : j \in J\}$ has only one element. Indeed, if for example $\{s_j : j \in J\} = \{+\}$ we would have
$$x_{1,r} \leqslant \dots \leqslant x_{n,r} \leqslant x_{1,r}$$
with at least one strict inequality, which is impossible.

Thus, we conclude that $(\mathbb{r}, \mathbb{s}) \in C'$ must satisfy these two conditions, and moreover they characterize C'.

For each $(\mathbb{r}, \mathbb{s}) \in C'$ the set $C'(\mathbb{r}, \mathbb{s})$ is a relatively open cone. To see what the dimension is, let $r := \min_{1 \leqslant i \leqslant n} r_i$. Then $C'(\mathbb{r}, \mathbb{s})$ is given by the set of
$$(r_1 - 1) + \dots + (r_n - 1) - (r - 1)$$
equalities and some strict inequalities. Indeed, for each $1 \leqslant i \leqslant n-1$ we have $r_i - 1$ equalities coming from comparing the coordinates of x_i and x_{i+1}. When we compare x_n and x_1, we already know their first $r-1$ coordinates are the same, so we only need to add $(r_n - 1) - (r - 1)$ additional equalities. Thus, we have
$$\dim C'(\mathbb{r}, \mathbb{s}) = (n-1)d + (\min_i r_i - 1) - (r_1 - 1) - \dots - (r_n - 1).$$
Let us denote by
$$\hat{0} := \big((d+1, \dots, d+1), (0, \dots, 0)\big) \in C'$$
the combinatorial data corresponding to the origin $\{0\} \subseteq W_n^{\oplus d}$.

Definition 4.2. For each $(\mathbb{r}, \mathbb{s}) \in C' \setminus \{\hat{0}\}$ we define
$$c'(\mathbb{r}, \mathbb{s}) := C'(\mathbb{r}, \mathbb{s}) \cap S(W_n^{\oplus d}).$$

As in Section 2, for each $(\mathbb{r}, \mathbb{s}) \in C' \setminus \{\hat{0}\}$ sets $c'(\mathbb{r}, \mathbb{s})$ are non-empty relatively open subsets of the sphere $S(W_n^{\oplus d})$ homeomorphic to open discs of dimensions
$$(n-1)d - 1 + (\min_i r_1 - 1) - (r_1 - 1) - \dots - (r_n - 1).$$
Thus the codimension of $c'(\mathbb{r}, \mathbb{s})$ in the sphere $S(W_n^{\oplus d})$ is
$$\operatorname{codim} c'(\mathbb{r}, \mathbb{s}) = (r_1 - 1) + \dots + (r_n - 1) - (\min_i r_i - 1),$$
for each $(\mathbb{r}, \mathbb{s}) \in C' \setminus \{\hat{0}\}$.

Definition 4.3. On the set $C' \setminus \{\hat{0}\}$ we define a poset structure by letting $(\mathbb{r}, \mathbb{s}) \preceq (\mathbb{r}', \mathbb{s}')$ if and only if we have

- $r_i' \leqslant r_i$ for each $1 \leqslant i \leqslant n$, and
- if $r_i' = r_i$ then $s_i' = s_i$.

Let us denote by $S'(d,n) := (C' \setminus \{\hat{0}\}, \preceq)$ this poset.

Proposition 4.4. *The collection*

$$\{c'(\mathbb{r}, \mathbb{s}) : \ (\mathbb{r}, \mathbb{s}) \in C' \setminus \{\hat{0}\}\}$$

of cells forms a cellular structure $\mathcal{S}'(d,n)$ on the sphere $S(W_n^{\oplus d})$ with face poset $S'(d,n)$. Opposite poset $S'(d,n)^{\text{op}}$ is a face poset of a cellular structure $\mathcal{S}'(d,n)^{\text{op}}$ on the same sphere, with cells $\check{c}'(\mathbb{r}, \mathbb{s})$ of dimension

$$\dim \check{c}'(\mathbb{r}, \mathbb{s}) = (r_1 - 1) + \ldots + (r_n - 1) - (\min_i r_i - 1)$$

for each $(\mathbb{r}, \mathbb{s}) \in S'(d,n)$.

Proof. Notice that $c'(\mathbb{r}, \mathbb{s}) \subseteq \text{cl}(c'(\mathbb{r}', \mathbb{s}'))$ if and only if $(\mathbb{r}, \mathbb{s}) \preceq (\mathbb{r}', \mathbb{s}')$ in $S'(d,n)$. Moreover, in this case we have $\dim c'(\mathbb{r}, \mathbb{s}) \leqslant \dim c'(\mathbb{r}', \mathbb{s}')$. Thus, the cells make a regular CW structure on the sphere with poset $S'(d,n)$.

The reason why $S'(d,n)^{\text{op}}$ is also a face poset of a CW structure on $S(W_n^{\oplus d})$ is analogous to the one made in Proposition 2.4. The cells of the opposite structure are dual block cells, and thus have complementary dimension to the original cells in the sphere. ☐

Similarly to the situation in Section 2, we again have

$$\text{either} \quad C'(\mathbb{r}, \mathbb{s}) \subseteq G(\mathbb{R}^d, n) \quad \text{or} \quad C'(\mathbb{r}, \mathbb{s}) \cap G(\mathbb{R}^d, n) = \varnothing,$$

for each $(\mathbb{r}, \mathbb{s}) \in C'$. Let us define

$$G_{d,n} := \{(\mathbb{r}, \mathbb{s}) \in S'(d,n)^{\text{op}} : \ C'(\mathbb{r}, \mathbb{s}) \subseteq G(\mathbb{R}^d, n)\}.$$

Then for each $(\mathbb{r}, \mathbb{s}) \in C'$ we have

$$(\mathbb{r}, \mathbb{s}) \in G_{d,n} \quad \text{if and only if} \quad 1 \leqslant r_1, \ldots, r_n \leqslant d.$$

We again see that $G_{d,n}$ is closed downwards in the poset $S'(d,n)^{\text{op}}$.

Definition 4.5. Let $n \geqslant 1$ and $d \geqslant 2$ be integers. Define $\mathcal{G}_{n,d} \subseteq \mathcal{S}'(d,n)^{\text{op}}$ to be the CW subcomplex induced by the subposet $G_{n,d} \subseteq S'(n,d)^{\text{op}}$.

Finally, we obtain another CW model of the Billiard Configuration Space.

Theorem 4.6. *Let $d \geqslant 2$ and $n \geqslant 2$ be integers. The space $G(\mathbb{R}^d, n)$ contains, as a strong deformation retract, a finite regular CW complex $\mathcal{G}_{d,n}$ of dimension $(d-1)(n-1)$ with the face poset $G_{d,n} \subseteq S'(d,n)^{\mathrm{op}}$. Moreover, the following facts hold.*

- *The boundary of a cell $\check{c}'(\mathbb{r}, \mathbb{s}) \subseteq \mathcal{G}_{d,n}$ consists of cells $\check{c}'(\mathbb{r}', \mathbb{s}') \subseteq \mathcal{G}_{d,n}$ which satisfy:*
  - *$r'_i \leqslant r_i$ for all $1 \leqslant i \leqslant n$, and*
  - *if $r'_i = r_i$ for some $1 \leqslant i \leqslant n$, then $s'_i = s_i$.*
- *The dimension of the cell $\check{c}'(\mathbb{r}, \mathbb{s})$ is*

$$\dim \check{c}'(\mathbb{r}, \mathbb{s}) = (r_1 - 1) + \cdots + (r_n - 1) - (\min_i r_i - 1).$$

Moreover, $\mathcal{G}_{d,n} \subseteq G(\mathbb{R}^d, n)$ is D_n-invariant subspace, where $D_n = \langle \rho, \theta \rangle \subseteq \mathfrak{S}_n$ is the dihedral group. The action is cellular and for a cell $\check{c}'(\mathbb{r}, \mathbb{s})$ it is generated by

$$\rho \cdot \check{c}'\big((r_1, \ldots, r_n), (s_1, \ldots, s_n)\big) = \check{c}'\big((r_n, r_1, \ldots, r_{n-1}), (s_n, s_1, \ldots, s_{n-1})\big)$$

and

$$\theta \cdot \check{c}'\big((r_1, \ldots, r_n), (s_1, \ldots, s_n)\big) = \check{c}'\big((r_n, \ldots, r_1), (-s_n, \ldots, -s_1)\big),$$

where $\rho : [n] \to [n]$ maps $i \mapsto i+1$ and $\theta : [n] \to [n]$ maps $i \mapsto n - i$.

Proof. The proof that $\mathcal{G}_{d,n}$ is a D_n-equivariant strong deformation retract is analogous to the proof of Theorem 2.6. As for the dimension of the complex $\mathcal{G}_{d,n}$, notice that for each cell $\check{c}'(\mathbb{r}, \mathbb{s}) \subseteq \mathcal{G}_{d,n}$ we have $r_i \leqslant d$ for all $1 \leqslant i \leqslant n$. Thus, if $r_j = \min_i r_i$ we have

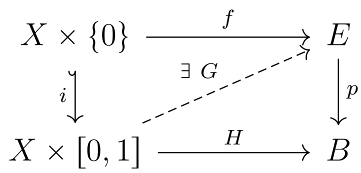
$$\dim \check{c}'(\mathbb{r}, \mathbb{s}) = \sum_{i \in [n] - \{j\}} (r_i - 1) \leqslant (n-1)(d-1),$$

and the equality is obtained when $\mathbb{r} = (d, \ldots, d)$ and $\mathbb{s} = (+, \ldots, +, -)$. □

5 Sectional category of a fibration

The main reference for this section is paper by Schwarz [29], where the notion of a sectional category is referred to as the *genus of a fibration*.

Definition 5.1. A continuous map $p : E \longrightarrow B$ is called a *Hurewitz fibration* if for any space X and any two maps $f : X \times \{0\} \longrightarrow E$ and $H : X \times [0,1] \longrightarrow B$ such that the square

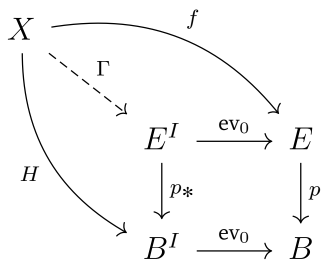


commutes, there is a map $G : X \times [0,1] \longrightarrow E$ such that the two triangles commute, that is, such that $G \circ i = f$ and $p \circ G = H$.

The existence of such a map G is called the *Homotopy lifting property of the map* p. In this thesis we will only deal with Hurewitz fibrations (as opposed to *Serre fibrations*), so for the sake of brevity we will denote them just by the term *fibrations*.

If two points $b_1, b_2 \in B$ are in the same path component of B, then the fibers $p^{-1}(b_1)$ and $p^{-1}(b_2)$ are homotopically equivalent [21, Proposition 4.60]. By *the fiber* of a fibration $p : E \to B$ we will denote the fiber $F := p^{-1}(b)$ of a fixed, and usually implicitly assumed, base point $b \in B$.

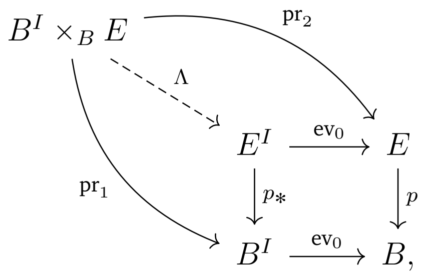
Remark 5.2. The existence of a lift $G : X \times I \to E$ is equivalent to the existence a dashed arrow



making the diagram commute. Here B^I and E^I denote the path spaces of B and E, while the map $H : X \to B^I$ corresponds to the map $H : X \times I \to B$ under the Exponential law [1, Section 1.3]. Due to the universal property of the pull back

$$B^I \times_B E := \{(\gamma, e) \in B^I \times_B E : \gamma(0) = p(e)\},$$

existence of a dashed arrow $\Gamma : X \to E^I$ for any X is equivalent to existence of a *path* *lifting map* $\Lambda : B^I \times_B E \to E^I$ making the diagram



(9)

commute. Here $\mathrm{pr}_1 : B^I \times_B E \to B^I$ and $\mathrm{pr}_2 : B^I \times_B E \to E$ are the projections on the first and second coordinate, respectively [1, Ex. 4.3.13].

Definition 5.3. Let X be a topological space. Then by

$$CX := X \times [0,1]/X \times \{0\}$$

we denote the *cone of* X. Moreover, let

$$C'X := X \times [0,1)/X \times \{0\}.$$

We can see X sitting inside the cone CX as $X \times \{1\} \subseteq CX$, so we have $C'X = CX \setminus X$. Next, we give a definition due to Schwarz [29, Ch. I Sec. 5].

Definition 5.4. Let $X_1, ..., X_k$ be topological spaces. We define their *join* $X_1 * ... * X_k$ to be the space

$$X_1 * ... * X_k := (CX_1 \times \cdots \times CX_k) \setminus (C'X_1 \times \cdots \times C'X_k).$$

Thus, a point in the join $X_1 * \cdots * X_k$ is of the form

$$\big([x_1, t_1], ..., [x_k, t_k]\big) \in CX_1 \times \cdots \times CX_k$$

such that

$$\max\{t_i : 1 \leqslant i \leqslant k\} = 1,$$

where $x_i \in X_i$ and $t_i \in [0,1]$ for each $1 \leqslant i \leqslant k$.

Remark 5.5. This definition coincides (up to homeomorphism) with the notion of join which can be found in the literature, which for $k = 2$ has the form

$$X *' Y := X \times Y \times I/(x, y, 0) \sim (x', y, 0), (x, y, 1) \sim (x, y', 1).$$

For example, we have the homeomorphism [29, Ch. I. Sec. 5]

$$X * Y \xrightarrow{\approx} X *' Y$$

mapping $([x,t],[y,1]) \in X * Y$ to $(x, y, t/2) \in X *' Y$ and $([x,1],[y,t]) \in X * Y$ to $(x, y, 1 - t/2) \in X *' Y$.

Definition 5.6. Let $f : X \longrightarrow Y$ be a continuous map. Then by

$$M_f := (X \times [0,1] \sqcup Y)/(x,0) \sim f(x)$$

we denote the *mapping cylinder* of f. Moreover, let

$$M'_f := (X \times [0,1) \sqcup Y)/(x,0) \sim f(x).$$

Cone of a space is a special case of a mapping cylinder, namely when $Y = \{\mathrm{pt}\}$. For a map $f : X \longrightarrow Y$ we have a natural map

$$M(f) : M_f \longrightarrow Y, \quad [x,t] \mapsto f(x), \quad y \mapsto y.$$

Next definition is due to Schwarz [29, Ch. II, Def. 1 & Def. 1'].

Definition 5.7. Suppose we have a fibration $F_i \to E_i \xrightarrow{p_i} B$ over the same base space for each $1 \leqslant i \leqslant k$. Then we define the space $E_1 + \ldots + E_k$ to be the pullback of the diagram

$$\begin{array}{ccc} E_1 + \ldots + E_k & \longrightarrow & \left(\prod_{i=1}^k M_{p_i}\right) \setminus \left(\prod_{i=1}^k M'_{p_i}\right) \\ & & \big\downarrow \\ \Big\downarrow {\scriptstyle p_1 + \ldots + p_k} & & \prod_{i=1}^k M_{p_i} \\ & & \big\downarrow {\scriptstyle \prod_i M(p_i)} \\ B & \xrightarrow{\quad \Delta \quad} & \prod_{i=1}^k B, \end{array} \tag{10}$$

where $\Delta : B \to \prod_{i=1}^k B$ is the diagonal map and

$$p_1 + \ldots + p_k : E_1 + \ldots + E_k \to B$$

is the induced map.

The map

$$p_1 + \ldots + p_k : E_1 + \ldots + E_k \longrightarrow B$$

is a fibration if $p_1, \ldots, p_k$ are as well and if B is path connected [29, Ch. II Sec. 1]. Since we are not aware of the full proof of this statement, we include it for the sake of completeness. We split the proof into a lemma and a corollary.

Lemma 5.8. *Let $F \to E \xrightarrow{p} B$ be a fibration. Then*

$$\begin{array}{ccc} CF & \longrightarrow & M_p \\ & & \big\downarrow {\scriptstyle M(p)} \\ & & B \end{array}$$

is a fibration. Moreover, if

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f} & M_p \\ {\scriptstyle i}\downarrow & \overset{G}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{H} & B \end{array}$$

is a commutative square, then the lift G can be chosen such that for any $x \in X$ the following implication holds:

$$f(x,0) \in E \times \{1\} \subseteq M_p \implies G(x,I) \subseteq E \times \{1\} \subseteq M_p. \tag{11}$$

Proof. Let $\Lambda : B^I \times_B E \to E^I$ be a path lifting map (9) for fibration p. In view of Remark 5.2 it is enough to construct a path lifting map

$$\Theta : B^I \times_B M_p \longrightarrow M_p^I \tag{12}$$

for which the path $\Theta\big(\gamma, [e,1]\big) \in M_p^I$ lies completely in $E \times \{1\} \subseteq M_p$.

Mapping cylinder M_p is a push out of the diagram

$$\begin{array}{ccc} E \times \{0\} & \hookrightarrow & E \times I \\ {\scriptstyle p}\downarrow & & \downarrow \\ B & \longrightarrow & M_p. \end{array}$$

Moreover, the top horizontal map is a closed cofibration over B and maps $p : E \times \{0\} \to B$, $\mathrm{id} : B \to B$ and $p \circ \mathrm{pr}_1 : E \times I \to B$, from the three spaces making the push out diagram, are fibrations. Therefore, by [10, Proposition 1.3] it follows that the induced map

$$M(p) : M_p \longrightarrow B$$

is a fibration. Moreover, the path lifting map (9)

$$\Gamma : B^I \times_B M_p \longrightarrow M_p^I$$

of fibration $M(p)$ is given by

$$\Gamma\big(\gamma, x\big)(s) = \begin{cases} \big[\Lambda(e,\gamma)(s), \max\{0, t - s/2\}\big], & x = [e,t] \in M_p \text{ for some } (e,t) \in E \times I \\ \gamma(s) & x = b \in M_p \text{ for some } b \in B, \end{cases}$$

for any $(\gamma, x) \in M_p$ and $s \in I$. Let us define

$$\widetilde{M}_p := \big((E \times [0,2]) \sqcup B\big)/\sim,$$

where relation $\sim$ is generated by $(e, 0) \sim p(e)$ for each $e \in E$. We will use this model for the mapping cylinder instead of M_p to define the path lifting map (12)

$$\Theta : B^I \times_B \widetilde{M}_p \longrightarrow \widetilde{M}_p^I$$

with the property that the path $\Theta\big(\gamma, [e, 2]\big)$ is completely contained in $E \times \{2\} \subseteq \widetilde{M}_p$.

Path lifting map Γ is defined on points $(x, \gamma) \in B^I \times_B \widetilde{M}_p$ when $x \in B \subseteq \widetilde{M}_p$ or when $x = [e, t] \in \widetilde{M}_p$ for $e \in E$ and $t \in [0, 1]$. This is the 'lower half' of $B^I \times_B \widetilde{M}_p$. Now we define a map

$$\Gamma' : B^I \times_B \big(E \times [1, 2]\big) \longrightarrow \big(E \times [1/2, 2]\big)^I \hookrightarrow \widetilde{M}_p^I$$

on the 'upper half' of $B^I \times_B \widetilde{M}_p$ to map by the rule

$$\Gamma'(e, t, \gamma)(s) = \big[\Lambda(e, \gamma)(s), t - s + ts/2\big],$$

for $(e, t) \in E \times [1, 2]$, $\gamma \in B^I$ and $s \in I$. Notice that the time coordinate of Γ' satisfies

$$t - s + ts/2 = (2 - t)(1 - s/2) + (t - 1)2 \in [1 - s/2,\ 2] \subseteq [1/2,\ 2]$$

for $t \in [1, 2]$ and $s \in [0, 1]$.

Map Γ' on $B^I \times_B (E \times \{1\})$ coincides with map Γ on $B^I \times_B (E \times \{1\}) \subseteq M_p \times_B B^I$, so they can be glued along the intersection of their domains to form a continuous path lifting map

$$\Theta : B^I \times_B \widetilde{M}_p \longrightarrow \widetilde{M}_p^I.$$

On an element $(\gamma, [e, 2])$ on top level $B^I \times_B (E \times \{2\}) \subseteq B^I \times_B \widetilde{M}_p$ map Θ equals

$$\Theta\big(\gamma, [e, 2]\big)(s) = \Gamma'(\gamma, [e, 2])(s) = \big[\Lambda(\gamma, e)(s), 2\big] \in B^I \times_B (E \times \{2\}),$$

for every $s \in I$, so the path $\Theta\big(\gamma, [e, 2]\big)$ indeed stays on the top level as wanted.

□

Corollary 5.9. *Assume $F_i \to E_i \xrightarrow{p_i} B$ is a family of fibrations, where $1 \leqslant i \leqslant k$. Then the right vertical map in (10) is a fibration*

$$\begin{array}{ccc} F_1 * \ldots * F_k & \longrightarrow & E_1 + \ldots + E_k \\ & & \big\downarrow{\scriptstyle p_1 + \ldots + p_k} \\ & & B \end{array}$$

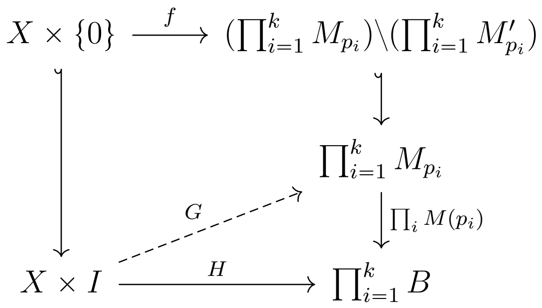
with fiber $F_1 * \ldots * F_k$.

Proof. Since the pull back of a fibration is a fibration [1, Prop. 4.3.11], it is enough to show that the left vertical composition in (10) is fibration.

By Lemma 5.8 it follows that $M(p_i) : M_{p_i} \to B$ is a fibration with fiber CF_i for each $1 \leqslant i \leqslant k$. Hence, the product map

$$\prod_{i=1}^{k} M(p_i) : \prod_{i=1}^{k} M_{p_i} \longrightarrow \prod_{i=1}^{k} B$$

is also a fibration with fiber $\prod_{i=1}^{k} CF_i$. Let



be a commutative diagram with $f = (f_1, ..., f_k)$ and $H = (H_1, ..., H_k)$. Denote by

$$G := (G_1, ..., G_k) : X \times I \longrightarrow \prod_{i=1}^{k} M_{p_i}$$

the lift of H, where G_i is a lift of H_i satisfying (11). In order to show that the composition of the two vertical maps on the right is a fibration, we want to show that G factors through

$$\Big(\prod_{i=1}^{k} M_{p_i}\Big) \setminus \Big(\prod_{i=1}^{k} M'_{p_i}\Big) \subseteq \prod_{i=1}^{k} M_{p_i}.$$

Let $(x, t) \in X \times I$ be any point. Since the image of f avoids $\prod_{i=1}^{k} M'_{p_i}$ it follows that

$$f_j(x, 0) \in E_j \times \{1\} \subseteq M_{p_j}$$

for at least one $1 \leqslant j \leqslant k$. Hence by (11) it follows that $G_j(x, I) \subseteq E_j \times \{1\} \subseteq M_{p_j}$, so we have

$$G(x, I) \cap \prod_{i=1}^{k} M'_{p_i} = \varnothing.$$

Therefore, the desired factorisation holds.

The fiber of this fibration at the diagonal point in B^k is

$$\Big(\prod_{i=1}^{k} CF_i\Big) \setminus \Big(\prod_{i=1}^{k} C'F_i\Big) = F_1 * \cdots * F_k,$$

which finishes the proof. $\square$

For a fibration $F \to E \xrightarrow{p} B$ and an integer k, we will denote the *k-fold sum* of fibration p by

$$\begin{array}{ccc} F^{*k} & \longrightarrow & E_k \\ & & \downarrow{\scriptstyle p_k} \\ & & B. \end{array}$$

Now we state the central definition of this section.

Definition 5.10. (Schwarz [29, Definition 5]) Let $F \to E \xrightarrow{p} B$ be a fibration. The *sectional category* of fibration p, denoted by $\mathrm{seccat}(p)$, is the minimal integer $k \in \mathbb{N} \cup \{\infty\}$ such that there is an open cover of the base space by k open sets,

$$B = U_1 \cup \cdots \cup U_k,$$

such that the restricted fibration $p|_{U_i}$ admits a cross section for each $1 \leqslant i \leqslant k$.

The following result transforms the question of finding k partial sections of fibration to the existence of a global section of the k-fold sum fibration.

Proposition 5.11. (Schwarz [29, Proposition 2]) *Let $F \to E \xrightarrow{p} B$ be a fibration. Then $\mathrm{seccat}(p) \leqslant k$ if and only if the k-fold sum fibration $F^{*k} \to E_k \xrightarrow{p_k} B$ admits a global section.*

For a topological space X we will denote by $\mathrm{conn}(X)$ the connectivity of X. Next we state an upper bound for sectional category of a fibration over particularly nice base space.

Corollary 5.12. (Schwarz, [29, Theorem 5]) *Let $F \to E \xrightarrow{p} B$ be a fibration. Assume B is a finite-dimensional CW complex. Then*

$$\mathrm{seccat}(p) \leqslant \left\lceil \frac{\dim B + 1}{\mathrm{conn}(F) + 2} \right\rceil.$$

Proof. Let k be the number on the right hand side. By Proposition 5.11 we know $\mathrm{seccat}(p) \leqslant k$ if and only if the k-fold sum $F^{*k} \to E_k \xrightarrow{p_k} B$ admits a section. The connectivity of the fiber F^{*k} follows from formula of the connectivity of a join (Whitehead [30]), namely

$$\mathrm{conn}(F^{*k}) \geqslant k\,\mathrm{conn}(F) + 2(k-1) = k(\mathrm{conn}(F) + 2) - 2.$$

Thus, there is no obstruction for existence of a section on skeleton of B of dimension $\mathrm{conn}(F^{*k}) + 1 \geqslant k(\mathrm{conn}(F) + 2) - 1 \geqslant \dim B$. □

The above corollary gives an upper bound on the sectional category of a fibration. Next theorem gives a way to obtain a lower bound.

Theorem 5.13. (Schwarz [29, Theorem 4]) *Let* $F \to E \xrightarrow{p} B$ *be a fibration. Assume there exist* k *cohomology classes*

$$\xi_1 \in H^*(B; \mathcal{A}_1), ..., \xi_k \in H^*(B; \mathcal{A}_k),$$

where $\mathcal{A}_1, ..., \mathcal{A}_k$ *are some local systems of coefficients on* B*, such that*

(i) $p^*\xi_i = 0 \in H^*(E; p^*\mathcal{A}_i)$ *for all* $1 \leqslant i \leqslant k$*, and*

(ii) $\xi_1 \cup \cdots \cup \xi_k \neq 0 \in H^*(B; \mathcal{A}_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathcal{A}_k)$*.*

Then $\mathrm{seccat}(p) \geqslant k + 1$.

5.1 Topological complexity and Lusternik–Schnirelmann category

In this section we define the notion of topological complexity of a topological space, as introduced by Farber [14].

Definition 5.14. Let (X, x_0) be a pointed and path connected topological space. The *topological complexity* of X, denoted by $\mathrm{TC}(X)$, is defined as the sectional category of the fibration

$$\begin{array}{ccc} \Omega X & \longrightarrow & PX \\ & & \downarrow p \\ & & X \times X. \end{array}$$

Here

$$PX := \{\gamma : [0,1] \to X : \ \gamma \text{ continuous}\}$$

denotes the *path space of* X,

$$\Omega X := \{\gamma : [0,1] \to X : \ \gamma \text{ continuous and } \gamma(0) = \gamma(1) = x_0\}$$

denotes the *loop space*, and $p := \mathrm{ev}_0 \times \mathrm{ev}_1$ is the product of evaluations at the end points.

Lemma 5.15. *For any space* X *the map*

$$p : PX \longrightarrow X \times X, \quad \gamma \longmapsto (\gamma(0), \gamma(1))$$

from Definition 5.14 is a fibration.

Proof. Let Y be a space and $f : Y \times \{0\} \to E$ and $H : Y \times I \to B$ continuous maps such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & PX \\ \downarrow{\scriptstyle id \times 0} & & \downarrow{\scriptstyle p} \\ Y \times I & \xrightarrow{H} & X \times X. \end{array}$$

commutes. Map $f : Y \to PX$ corresponds to the map

$$f' : Y \times I \to X, \ (y,t) \mapsto f(y)(t)$$

due to the Exponential law [1, Section 1.3]. Similarly, the lift $G : Y \times I \to PX$ corresponds to a map $G' : Y \times I \times I \longrightarrow X$. Map G' can be defined as

$$G'(y,t,s) := \begin{cases} \mathrm{pr}_1 \circ H(y, t-3s) & 0 \leqslant s \leqslant \frac{t}{3} \\ f'(y, \frac{3s-t}{3-2t}) & \frac{t}{3} \leqslant s \leqslant 1 - \frac{t}{3} \\ \mathrm{pr}_2 \circ H(y, 3s+t-3) & 1 - \frac{t}{3} \leqslant s \leqslant 1. \end{cases}$$

It follows that

$$G(y,0)(s) = G'(y,0,s) = f'(y,s) = f(y)(s)$$

and

$$p \circ G(y,t) = (G'(y,t,0), G'(y,t,1)) = H(y,t)$$

hold for any $y \in Y$ and any $t, s \in I$. □

The map $X \to PX$ assigning a constant path at each point is homotopy equivalence. The homotopy inverse is evaluation at the origin, $\mathrm{ev}_0 : PX \to X$, and homotopy continuously shrinks the path to its origin.

The k-fold sum of fibration p will be denoted by

$$\begin{array}{ccc} (\Omega X)^{*k} & \longrightarrow & P_k X \\ & & \downarrow{\scriptstyle p_k} \\ & & X \times X. \end{array}$$

Example 5.16. (Farber [14, Theorem 1]) For a topological space X we have $\mathrm{TC}(X) = 1$ if and only if X is a contractible space. Indeed, if X and contractible and $h : X \times I \to X$ is homotopy $h_0 = \mathrm{id}_X$ and $h_1 = \mathrm{const}$, then section

$$s : X \times X \to PX$$

of p may be constructed as follows. For $(x,y) \in X \times X$ we set $s(x,y)$ to be the concatenation of paths $t \mapsto h_t(x)$ and $t \mapsto h_{1-t}(y)$.

On the other hand, if $s : X \times X \to PX$ is a section of p and $x_0 \in X$ is the base point, the map

$$h : X \times I \to X, \qquad h_t(x) := s(x, x_0)(t)$$

is homotopy $h : \mathrm{id}_X \simeq x_0$.

In the example above we see that topological complexity of a space homotopic to a point is the same as topological complexity of a point. Even more is true, namely topological complexity is homotopy invariant.

Proposition 5.17. (Farber [14, Theorem 3]) *Let X and Y be topological spaces and suppose that $X \simeq Y$. Then we have* $\mathrm{TC}(X) = \mathrm{TC}(Y)$.

Another well-studied notion of complexity of a space is the notion of Lusternik-Schnirelmann category. In fact, it can be seen as a sectional category of a certain fibration.

Definition 5.18. Let (X, x_0) be a pointed and path connected topological space. The *Lusternik-Schnirelmann category* of X, denoted by $\mathrm{cat}(X)$, is defined as the sectional category of the fibration

$$\begin{array}{ccc} \Omega X & \longrightarrow & P_*X \\ & & \downarrow q \\ & & X \end{array} \tag{13}$$

where

$$P_*X := \{\gamma : ([0,1], 0) \to (X, x_0) : \ \gamma \text{ continuous}\}$$

is the *based path space of X* and $q := \mathrm{ev}_1$.

Notice that the space P_*X is contractible, as all paths can be continuously deformed to the constant path at the base point. Thus from the long exact sequence of the fibration (13) we have the following.

Lemma 5.19. *Let (X, x_0) be pointed and path connected topological space. Then we have*

$$\pi_k(\Omega X, x_0) \cong \pi_{k+1}(X, x_0)$$

for each $k \geqslant 0$. In particular, $\mathrm{conn}(X) = \mathrm{conn}(\Omega X) + 1$.

Since both topological complexity and Lusternik–Schnirelmann category are sectional categories of fibrations, we have the following upper bounds due to connectivity of the loop space and the Corollary 5.12.

Proposition 5.20. *Let X be finite dimensional CW complex. Then*

$$\mathrm{TC}(X) \leqslant \left\lceil \frac{2\dim X + 1}{\mathrm{conn}\, X + 1} \right\rceil \quad \textit{and} \quad \mathrm{cat}(X) \leqslant \left\lceil \frac{\dim X + 1}{\mathrm{conn}\, X + 1} \right\rceil.$$

Next, we provide a lower bound for the topological complexity and Lusternik–Schnirelmann category.

Proposition 5.21. *Let X be a topological space. Assume there exist cohomology classes*

$$\xi_1 \in H^*(X \times X; \mathcal{A}_1), \dots, \xi_k \in H^*(X \times X; \mathcal{A}_k),$$

for some local systems of coefficients $\mathcal{A}_1, \dots, \mathcal{A}_k$ on $X \times X$, such that

(i) $\xi_i|_X = 0 \in H^*(X; \mathcal{A}_i|_X)$ *for all* $1 \leqslant i \leqslant k$*, and*

(ii) $\xi_1 \cup \cdots \cup \xi_k \neq 0 \in H^*(X \times X; \mathcal{A}_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathcal{A}_k)$.

Then $\mathrm{TC}(X) \geqslant k + 1$.

Proof. We apply Theorem 5.13 to fibration $\Omega X \to PX \xrightarrow{p} X \times X$. Since the composition $X \to PX \xrightarrow{p} X \times X$ is the diagonal mapping $d : X \to X \times X$, with the first map being homotopy equivalence, we conclude that a cohomology class $\xi \in H^*(X \times X; \mathcal{A})$ satisfies $p^*\xi = 0 \in H^*(PX; p^*\mathcal{A})$ if an only if it satisfies $d^*\xi = 0 \in H^*(X; d^*\mathcal{A})$. □

Proposition 5.22. *Let X be a topological space. Assume there exist cohomology classes*

$$\xi_1 \in H^{d_1}(X; \mathcal{A}_1), \dots, \xi_k \in H^{d_k}(X; \mathcal{A}_k),$$

for some local systems of coefficients $\mathcal{A}_1, \dots, \mathcal{A}_k$ on X, such that

(i) $d_i \geqslant 1$ *for all* $1 \leqslant i \leqslant k$*, and*

(ii) $\xi_1 \cup \cdots \cup \xi_k \neq 0 \in H^{d_1 + \cdots + d_k}(X; \mathcal{A}_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathcal{A}_k)$.

Then $\mathrm{cat}(X) \geqslant k + 1$.

Proof. We apply Theorem 5.13 to fibration $\Omega X \to P_* X \xrightarrow{p} X$. Since $P_* X \simeq \mathrm{pt}$, the condition $q^*\xi_i = 0$ is satisfied if $d_i \geqslant 1$. □

Example 5.23. (Farber [14, Theorem 8]) We have $\mathrm{TC}(S^1) = 2$. More generally,

$$\mathrm{TC}(S^n) = \begin{cases} 2, & n \text{ is odd}, \\ 3, & n \text{ is even}. \end{cases}$$

Let n be odd. Due to Example 5.16, we have $2 \leqslant \mathrm{TC}(S^n)$ since S^n is not contractible. To prove $\mathrm{TC}(S^n) \leqslant 2$ we will construct two open sets U, V of $S^n \times S^n$ which cover it, and define sections of p on both of them. Let

$$U := \{(x, y) \in S^n \times S^n : x \neq -y\}.$$

For each $(x, y) \in U$ there is a unique geodesic from x to y, namely the part of a great circle containing x and y. This choice yields a continuous section

$$s_U : U \longrightarrow PS^n, \quad (x, y) \longmapsto (t \mapsto x\cos(\pi t/2) + y\sin(\pi t/2)).$$

For the other open set we choose

$$V := \{(x, y) \in S^n \times S^n : \ x \neq y\}.$$

Since n is odd, there is a non-zero vector field X on S^n. Without loss of generality we may assume X is of unit length. Section $s_V : V \to PS^n$ is defined as follows. For $(x, y) \in V$ we set $s_V(x, y)$ to be concatenation of paths

$$t \mapsto x\cos(\pi t/2) + (-y)\sin(\pi t/2) \quad \text{and} \quad t \mapsto (-y)\cos(\pi t) + X(y)\sin(\pi t).$$

The first path goes from x to $-y$ along the unique geodesic, and the second path goes from $-y$ to y along the unique geodesic which passes through $X(y) \in S^n$. The second path is well defined since $X(y)$ is orthogonal to both y and $-y$.

Let n be even. Since S^n is n-dimensional and $(n-1)$-connected, from Proposition 5.20 it follows that $\mathrm{TC}(S^n) \leqslant 3$. On the other hand, there is a graded $\mathbb{Z}$-algebra isomorphism

$$H^*(S^n) \cong \mathbb{Z}[\xi]/(\xi^2),$$

where $\deg \xi = n$. In this case, the cross-product

$$\times : H^*(S^n) \otimes H^*(S^n) \xrightarrow{\cong} H^*(S^n \times S^n)$$

is an isomorphism [21, Theorem 3.15]. Let $\Delta : X \to X \times X$ be the diagonal map. Then the composition of the cross product and the restriction to the diagonal is the cup product, $\Delta^* \circ \times = \cup$. Hence, the class

$$\bar{\xi} := \xi \otimes 1 - 1 \otimes \xi \in H^n(S^n)^{\otimes 2}$$

satisfies $(\Delta^* \circ \times)(\bar{\xi}) = \cup(\bar{\xi}) = \xi \cup 1 - 1 \cup \xi = 0 \in H^n(S^n)$ and

$$\bar{\xi} \cup \bar{\xi} = -((-1)^n + 1)\xi \otimes \xi = (-2)\xi \otimes \xi \neq 0 \in H^n(S^n)^{\otimes 2}.$$

Thus, by Proposition 5.21 we have $\mathrm{TC}(S^n) \geqslant 3$.

6 Topological complexity of Billiard Configuration Space

The aim of this section is to bound topological complexity of the Billiard Configuration Space

$$G(\mathbb{R}^d, n) := \{(x_1, ..., x_n) \in (\mathbb{R}^d)^{\times n} : \; x_i \neq x_{i+1} \text{ for all } 1 \leqslant i \leqslant n\},$$

where $x_{n+1} := x_1$. First, it will be crucial for us to know integral cohomology of $G(\mathbb{R}^d, n)$.

Theorem 6.1. (Farber & Tabachnikov [17, Proposition 2.2]) *Let $d \geqslant 2$ and $n \geqslant 2$ be integers. Then there is a graded-commutative $\mathbb{Z}$-algebra isomorphism*

$$H^*(G(\mathbb{R}^d, n); \mathbb{Z}) \cong \mathbb{Z}[e_1, ..., e_n]/I,$$

where $e_1, .., e_n$ are of degree $d-1$ and $I \subseteq \mathbb{Z}[e_1, ..., e_n]$ is an ideal generated by n elements $e_1^2, ..., e_n^2$ of degree $2(d-1)$ and the homogeneous element

$$e_1 ... e_{n-1} + \varepsilon e_2 ... e_n + \varepsilon^2 e_3 ... e_n e_1 + \cdots + \varepsilon^{n-1} e_n e_1 ... e_{n-2} \tag{14}$$

of degree $(n-1)(d-1)$, where $\varepsilon := (-1)^{(n-1)(d-1)}$.

For each $S \in 2^{[n]}$ let

$$e_S := \prod_{i \in S} e_i \in H^{|S|(d-1)}(G(\mathbb{R}^d, n)),$$

where the product is taken in ascending order of indices. The cohomology ring of $G(\mathbb{R}^d, n)$ is non-trivial only in dimensions $k(d-1)$ for $0 \leqslant k \leqslant n-1$. Moreover, for $0 \leqslant k \leqslant n-2$, as abelian groups we have

$$H^{k(d-1)}(G(\mathbb{R}^d, n)) \cong \bigoplus_{S \in \binom{[n]}{k}} \mathbb{Z}\langle e_S \rangle \tag{15}$$

In the top dimension however, due to relation (14), we have

$$H^{(n-1)(d-1)}(G(\mathbb{R}^d, n)) \cong \bigoplus_{S \in \binom{[n]}{n-1}, n \in S} \mathbb{Z}\langle e_S \rangle. \tag{16}$$

Corollary 6.2. *Let $d \geqslant 2$ and $n \geqslant 2$ be integers. Then $G(\mathbb{R}^d, n)$ is $(d-2)$-connected.*

Proof. For $d \geqslant 3$ by Proposition 3.1 it follows that $G(\mathbb{R}^d, n)$ is simply connected. Thus, the claim follows by Theorem 6.1 and Hurewitz theorem. □

Remark 6.3. Let $\mathcal{G}_{d,n} \subseteq G(\mathbb{R}^d, n)$ be cellular model from Section 4. The cross product

$$\times : H^*(\mathcal{G}_{d,n}) \otimes H^*(\mathcal{G}_{d,n}) \xrightarrow{\cong} H^*(\mathcal{G}_{d,n} \times \mathcal{G}_{d,n})$$

is a graded ring isomorphism, since $\mathcal{G}_{d,n}$ is a finite dimensional CW complex with free integral homology groups of finite rank [21, Theorem 3.15]. Due to this isomorphism, we will sometimes for convenience not make the distinction between the two. Post composing the cross-product with the diagonal map gives the cup product map. Thus, in view of Proposition 5.21, lower bound $\mathrm{TC}(\mathcal{G}_{d,n}) \geqslant k+1$ would follow if we there are classes $v_1, \ldots, v_k \in H^*(\mathcal{G}_{d,n})$ such that $\overline{v}_1 \cdot \ldots \cdot \overline{v}_k \neq 0 \in H^*(\mathcal{G}_{d,n})^{\otimes 2}$, where we define

$$\overline{v} := v \otimes 1 - 1 \otimes v \in H^*(\mathcal{G}_{d,n})^{\otimes 2} \tag{17}$$

for any class $v \in H^*(\mathcal{G}_{d,n})$

Before we continue further, let us compute Lusternik–Schnirelmann category of Billiard Configuration Space.

Theorem 6.4. *Let $n \geqslant 1$ and $d \geqslant 2$ be integers. Then we have*

$$\mathrm{cat}(G(\mathbb{R}^d, n)) = n.$$

Proof. Since Lusternik–Schnirelmann category is homotopy invariant [23], we can compute it for CW model $\mathcal{G}_{d,n} \simeq G(\mathbb{R}^d, n)$ from Section 4. Since $\mathcal{G}_{d,n}$ is $(d-2)$-connected and $(n-1)(d-1)$-dimensional, by Proposition 5.20 we have

$$\mathrm{cat}(\mathcal{G}_{2,n}) \leqslant \left\lceil \frac{(n-1)(d-1)+1}{d-1} \right\rceil = n.$$

On the other hand, generators $e_1, \ldots, e_{n-1} \in H^{d-1}(\mathcal{G}_{2,n})$ have non-trivial product

$$e_1 \ldots e_{n-1} \neq 0 \in H^{(n-1)(d-1)}(\mathcal{G}_{2,n}),$$

so by Proposition 5.22 we have $n \leqslant \mathrm{cat}(\mathcal{G}_{d,n})$. □

Let us know compute topological complexity of $G(\mathbb{R}^d, n)$ in the case when $n \geqslant 3$ is odd.

Theorem 6.5. *For $d \geqslant 3$ odd and $n \geqslant 1$ integer we have*

$$\mathrm{TC}(G(\mathbb{R}^d, n)) = 2n - 1.$$

Proof. Topological complexity is homotopy invariant by Proposition 5.17. Thus it is enough to show $\mathrm{TC}(\mathcal{G}_{d,n}) = 2n-1$. Since $\mathcal{G}_{d,n}$ is $(n-1)(d-1)$-dimensional and $(d-2)$-connected, from Proposition 5.20 we obtain

$$\mathrm{TC}(\mathcal{G}_{d,n}) \leqslant \left\lceil \frac{2(n-1)(d-1)+1}{d-1} \right\rceil = 2n - 1.$$

The lower bound $2n - 1 \leqslant \mathrm{TC}(\mathcal{G}_{d,n})$ is obtained from Proposition 5.21. Indeed, in view of the Remark 6.3, it is enough to find a non-trivial product of $2n - 2$ classes in $H^*(\mathcal{G}_{d,n})^{\otimes 2}$ of the form (17). For example,

$$\prod_{i=1}^{n-1} \overline{e}_i \cdot \overline{e}_i = \prod_{i=1}^{n-1} (-2) e_i \otimes e_i = (-2)^{n-1} (e_1 ... e_{n-1}) \otimes (e_1 ... e_{n-1}) \neq 0 \in H^*(\mathcal{G}_{d,n})^{\otimes 2},$$

because $\overline{e}_i \cdot \overline{e}_i = -((-1)^{d-1} + 1)(e_i \otimes e_i) = (-2)(e_i \otimes e_i)$ for d odd. □

By the work of Farber & Yuzvinsky [16], we know that

$$\mathrm{TC}(F(\mathbb{R}^d, n)) = 2n - 1$$

when d is odd. Thus, topological complexity of Billiard and standard Configuration Space coincide in this case. Natural question arises, that is whether the same is true for the case of d even. In the same paper, Farber and Yuzvinsky proved

$$\mathrm{TC}(F(\mathbb{R}^2, n)) = 2n - 2$$

and obtained bounds

$$2n - 2 \leqslant \mathrm{TC}(F(\mathbb{R}^d, n)) \leqslant 2n - 1$$

for $d \geqslant 4$ even. It wasn't until some years later that Farber & Grant [15] the lower bound is tight. Namely, they obtained

$$\mathrm{TC}(F(\mathbb{R}^d, n)) = 2n - 2$$

for d even. In this thesis, besides computing topological complexity of the Billiard Configuration Space in the case when d is odd, we provide a lower and an upper bound for the case of even d. As mentioned in the proof of the next theorem, we postpone the full proof of the upper bound until Subsection 6.1.

Theorem 6.6. *For $d \geqslant 2$ even and $n \geqslant 3$ we have*

$$n + 1 \leqslant \mathrm{TC}(G(\mathbb{R}^d, n)) \leqslant 2n - 2.$$

Proof. Let us first obtain the lower bound. By the homotopy invariance of topological complexity, Proposition 5.17, we will bound $\mathrm{TC}(\mathcal{G}_{d,n})$. The lower bound is obtained by Proposition 5.21 for all even $d \geqslant 2$. Indeed, in this case there is a non-trivial product of n elements in $H^*(\mathcal{G}_{d,n})^{\otimes 2}$ of the form (17), namely

$$\prod_{i=1}^{n} \overline{e}_i = \prod_{i=1}^{n} (e_i \otimes 1 - 1 \otimes e_i) \neq 0 \in H^*(\mathcal{G}_{d,n})^{\otimes 2}. \tag{18}$$

Therefore $n+1 \leqslant \mathrm{TC}(\mathcal{G}_{n,d})$. The product (18) is non-trivial since it is a signed sum of terms of the form

$$\left(\prod_{i\in I} e_i\right) \otimes \left(\prod_{j\in[n]-I} e_j\right) \in H^*(\mathcal{G}_{d,n})^{\otimes 2}, \tag{19}$$

for $I \in 2^{[n]}$. To see non-triviality, notice first that the terms for $I = \varnothing$ or $I = [n]$ are trivial. Due to cohomology relation (14), the term corresponding to $I = \{n\}$ can be expressed as sum

$$e_n \otimes e_1 \ldots e_{n-1} = -\varepsilon(e_n \otimes e_2 e_3 \ldots e_n) - \ldots - \varepsilon^{n-1}(e_n \otimes e_n e_1 \ldots e_{n-2}),$$

all of which are basis elements not of the form (19) since they have e_n on both coordinates. Analogous expression holds for $I = [n-1]$. When expressed like this, the product (18) is a signed sum of distinct additive basis elements of degree n part of $H^*(\mathcal{G}_{d,n})^{\otimes 2}$ and hence non-zero. Notice that the ring $H^*(\mathcal{G}_{d,n})^{\otimes 2}$ has an additive basis the products of basis elements (15) and (16).

As for the upper bound, we first give an argument for $d = 2$. The proof for $d \geqslant 4$ and even we postpone until Subsection 6.1. We have

$$G(\mathbb{R}^2, n) = \mathbb{C}^n - \bigcup \mathcal{H},$$

where $\mathcal{H} := \{H_i : \ 1 \leqslant i \leqslant n\}$ is a central complex hyperplane arrangement

$$H_i := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \ z_i = z_{i+1}\}$$

of rank $n-1$. Thus, the upper bound $\mathrm{TC}(G(\mathbb{R}^2, n)) \leqslant 2(n-1)$ holds due to Farber & Yuzvinsky [16, Theorem 6]. Namely, they proved that the topological complexity of a complement of central complex hyperplane arrangement is not greater than twice the rank of the arrangement, where the rank of a hyperplane arrangement is the dimension of the linear span of normal vectors to the hyperplanes which make the arrangement. $\square$

6.1 Obstruction theory

In this subsection the goal is to obtain the upper bound

$$\mathrm{TC}(\mathcal{G}_{n,d}) \leqslant 2n-2$$

for $d \geqslant 4$ even. We use Obstruction Theory. Our methods are similar to the ones used by Farber & Grant in the case of standard Configuration Space [15]. See also Costa & Farber [11].

Let $d \geqslant 4$, $n \geqslant 2$ and $k \geqslant 1$ be integers and let

$$\begin{array}{ccc} (\Omega\mathcal{G}_{d,n})^{*k} & \longrightarrow & P_k\mathcal{G}_{d,n} \\ & & \big\downarrow {\scriptstyle p_k} \\ & & \mathcal{G}_{d,n}\times\mathcal{G}_{d,n} \end{array} \tag{20}$$

be the k-fold sum of fibration

$$p : P\mathcal{G}_{n,k} \longrightarrow \mathcal{G}_{n,k} \times \mathcal{G}_{n,k}, \ \gamma \longmapsto (\gamma(0), \gamma(1)).$$

In particular, we have $p_1 = p$. Due to Lemma 5.19 we know that the fiber $(\Omega\mathcal{G}_{d,n})^{*k}$ of p_k is $(k(d-1)-2)$-connected and the base is $(d-2)$-connected. In particular, since $d \geqslant 4$, the fiber and the base space are simply connected, so we avoid cohomology with local coefficients. Thus, without mentioning explicitly, all cohomology coefficients will be assumed to be non-local.

Therefore, due to connectedness of the fiber, by the obstruction theory (see [18, Lecture 18]), fibration (20) admits a section over the $(k(d-1)-1)$-skeleton of the base space $\mathcal{G}_{d,n} \times \mathcal{G}_{d,n}$ and the first obstruction to existence of a section is

$$\mathfrak{o}_k \in H^{k(d-1)}\big(\mathcal{G}_{d,n}^{\times 2}; \pi_{k(d-1)-1}((\Omega\mathcal{G}_{d,n})^{*k})\big). \tag{21}$$

Moreover, $\mathfrak{o}_k = 0$ if and only if there is a section on $k(d-1)$-skeleton of $\mathcal{G}_{d,n} \times \mathcal{G}_{d,n}$. We prove the following lemma.

Lemma 6.7. *Let $d \geqslant 4$ be even and $n \geqslant 2$ be an integer. If $k \geqslant n+1$, then the first obstruction class* (21) *to the existence of section of k-fold sum fibration* (20) *vanishes.*

Before giving a proof of the lemma, let us first obtain the upper bound from previous subsection.

Proof of Theorem. 6.6 for $d \geqslant 4$ even. We want to show the upper bound

$$\mathrm{TC}(\mathcal{G}_{d,n}) \leqslant 2n-2,$$

which by Proposition 5.11 is equivalent to the fact that fibration (20) admits a section for $k = 2n-2$. Since $n \geqslant 3$, we can apply Lemma 6.7 for $k = 2n-2$ to conclude that the first obstruction $\mathfrak{o}_{2n-2}$ vanishes,

$$\mathfrak{o}_{2n-2} = 0 \in H^{(2n-2)(d-1)}\big(\mathcal{G}_{d,n}^{\times 2}; \pi_{(2n-2)(d-1)-1}((\Omega\mathcal{G}_{d,n})^{*2n-2})\big).$$

Thus, there is a section of p_{2n-2} over the $(2n-2)(d-1)$-skeleton of $\mathcal{G}_{d,n}^{\times 2}$. This finishes the proof, since the complex $\mathcal{G}_{d,n}^{\times 2}$ is in fact $(2n-2)(d-1)$-dimensional. $\square$

Before proving Lemma 6.7, let us exhibit the following. The loop space $\Omega\mathcal{G}_{d,n}$ is $(d-3)$-connected, Lemma 5.19. By the work of Whitehead [30] it follows that the first non-trivial homotopy group of a join $(\Omega\mathcal{G}_{d,n})^{*k}$ satisfies

$$\pi_{k(d-1)-1}\big((\Omega\mathcal{G}_{d,n})^{*k}, \mathrm{pt}\big) \cong \pi_{d-2}(\Omega\mathcal{G}_{d,n}, \mathrm{pt})^{\otimes k}.$$

Notice that for $d \geqslant 4$ all homotopy groups in this formula are abelian. Let us denote by

$$\mathfrak{o}_1 \in H^{d-1}\big(\mathcal{G}_{d,n}^{\times 2}; \pi_{d-2}(\Omega\mathcal{G}_{d,n})\big),$$

the first obstruction class of fibration $p_1 = p$. Its k-fold cup product lives in

$$\mathfrak{o}_1^{\cup k} \in H^{k(d-1)}\big(\mathcal{G}_{d,n}^{\times 2}; \pi_{d-2}(\Omega\mathcal{G}_{d,n})^{\otimes k}\big) \cong H^{k(d-1)}\big(\mathcal{G}_{d,n}^{\times 2}; \pi_{k(d-1)-1}((\Omega\mathcal{G}_{d,n})^{*k})\big).$$

Moreover, due to Schwarz [29, Theorem 1], this k-fold cup product coincides with the first obstruction to the existence of section of p_k,

$$\mathfrak{o}_1^{\cup k} = \mathfrak{o}_k \in H^{k(d-1)}\big(\mathcal{G}_{d,n}^{\times 2}; \pi_{d-2}(\Omega\mathcal{G}_{d,n})^{\otimes k}\big).$$

The coefficients of the cohomology group satisfy

$$\pi_{d-2}(\Omega\mathcal{G}_{d,n})^{\otimes k} \cong \pi_{d-1}(\mathcal{G}_{d,n})^{\otimes k} \cong H_{d-1}(\mathcal{G}_{d,n})^{\otimes k}$$

due to Lemma 5.19 and Hurewitz isomorphism. Thus, since $H_{d-1}(\mathcal{G}_{d,n})$ is a free abelian group, by the Universal coefficients theorem we may say

$$\mathfrak{o}_1 \in H^{d-1}(\mathcal{G}_{d,n}^{\times 2}) \otimes H_{d-1}(\mathcal{G}_{d,n}) \cong \big(H^*(\mathcal{G}_{d,n})^{\otimes 2}\big)^{(d-1)} \otimes H_{d-1}(\mathcal{G}_{d,n}) \tag{22}$$

and

$$\mathfrak{o}_1^{\cup k} = \mathfrak{o}_k \in H^{k(d-1)}(\mathcal{G}_{d,n}^{\times 2}) \otimes H_{d-1}(\mathcal{G}_{d,n})^{\otimes k} \cong (H^*(\mathcal{G}_{d,n})^{\otimes 2})^{(k(d-1))} \otimes H_{d-1}(\mathcal{G}_{d,n})^{\otimes k}. \tag{23}$$

The isomorphisms in (22) and (23) are the inverse of the cross product isomorphism, Remark 6.3. Now we have all the ingredients to finish the proof of Lemma 6.7.

Proof of Lemma 6.7. By the Universal coefficient theorem, we know that

$$H_{d-1}(\mathcal{G}_{d,n}) \cong H^{d-1}(\mathcal{G}_{d,n}),$$

so let $s_1, \dots, s_n \in H_{d-1}(\mathcal{G}_{d,n})$ be generators, e.g., the ones corresponding to generators $e_1, \dots, e_n \in H^{d-1}(\mathcal{G}_{d,n})$. Obstruction class $\mathfrak{o}_1$ therefore has a presentation in the right hand side of (22) of the form

$$\mathfrak{o}_1 = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} c_{i,j}(e_i \otimes 1) + d_{i,j}(1 \otimes e_i) \right) \otimes s_j \in \big(H^*(\mathcal{G}_{d,n})^{\otimes 2}\big)^{(d-1)} \otimes H_{d-1}(\mathcal{G}_{d,n}),$$

where $c_{i,j}, d_{i,j} \in \mathbb{Z}$. TH fibration

$$p_1 = p : PX \longrightarrow X \times X,\ \gamma \longmapsto (\gamma(0), \gamma(1))$$

admits a section over the diagonal, namely a map which assigns to each point $x \in X \subseteq X \times X$ on the diagonal the constant path at that point. Thus, by naturality of the obstruction class it follows that $\mathfrak{o}_1|_X = 0$. Therefore,

$$\mathfrak{o}_1|_X = \sum_{j=1}^{n}\sum_{i=1}^{n}(c_{i,j} + d_{i,j})(e_i \otimes s_j) = 0 \in H^{d-1}(\mathcal{G}_{d,n}) \otimes H_{d-1}(\mathcal{G}_{d,n}),$$

which means $c_{i,j} + d_{i,j} = 0$ for each $1 \leqslant i, j \leqslant n$. Thus, the obstruction class $\mathfrak{o}_1$ has the form

$$\mathfrak{o}_1 = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{i,j}(\bar{e}_i \otimes s_j) \in \big(H^*(\mathcal{G}_{d,n})^{\otimes 2}\big)^{(d-1)} \otimes H_{d-1}(\mathcal{G}_{d,n}),$$

where we again use the notation $\bar{e}_i = e_i \otimes 1 - 1 \otimes e_i \in H^*(\mathcal{G}_{d,n})^{\otimes 2}$.

Now, due to (23) we have that the class $\mathfrak{o}_k = \mathfrak{o}_1^{\cup k}$ is a sum of elements of the form

$$(\bar{e}_{i_1} ... \bar{e}_{i_k}) \otimes (s_{j_1} ... s_{j_k}) \in (H^*(\mathcal{G}_{d,n})^{\otimes 2})^{(k(d-1))} \otimes H_{d-1}(\mathcal{G}_{d,n})^{\otimes k},$$

which are all zero on the first coordinate $\bar{e}_{i_1} ... \bar{e}_{i_k}$. Indeed, since d is even for each $1 \leqslant i \leqslant n$ we have

$$\bar{e}_i \cdot \bar{e}_i = -((-1)^{d-1} + 1)(e_i \otimes e_i) = 0,$$

so for $k \geqslant n + 1$ two of the indices $i_1, ..., i_k$ will be the same. □

Remark 6.8. Lemma 6.7 says that the first obstruction of fibration p_k vanishes, whenever $d \geqslant 4$ is even and $k \geqslant n + 1$. To prove

$$\mathrm{TC}(G(\mathbb{R}^d, n)) \leqslant k$$

under the same numerical assumptions as above one might attempt to prove that *all* obstructions of p_k vanish. The situation is perhaps more complicated in the case $d = 2$, since obstructions of p_k live in cohomology modules

$$H^{1 \leqslant \bullet \leqslant 2(n-1)}\big(\mathcal{G}_{2,n} \times \mathcal{G}_{2,n}; \pi_{\bullet - 1}((\Omega\mathcal{G}_{2,n})^{*k})\big)$$

with local coefficients. That is why we believe that our computation of fundamental group of $G(\mathbb{R}^2, n) \simeq \mathcal{G}_{2,d}$ from Section 3 can be useful if one attempts to continue along the path of obstruction theory.

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